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# The Asymptotic Behavior of the Solutions of $(\Delta+\lambda)u=0$ in a Domain with the Unbounded Boundary

By

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## 1. Introduction

We shall consider the equation

$$(1.1) \qquad (\Delta+\lambda)u=0$$

in an unbounded domain $\Omega$ in the Euclidean $n$-space $E^n(n\geqq 2)$, with the boundary condition

$$(1.2) \qquad u|_{\Gamma}=0,$$

where $\Gamma$ is the boundary of $\Omega$, and $\lambda$ is a positive constant. Let $\Omega(L)=\Omega_\cap\{(x_1,\cdots,x_n)\in E^n: x_1>L\}$. We shall assume that $\Gamma$ is smooth $(C^1)$, and that there are positive numbers $C$, $N$ and $l(l\leqq 1)$ such that the following (1.3) and (1.4) hold for at least one of the connected components of $\Omega(N)$, say $\Omega_1(N)$.

$$(1.3) \qquad \Omega_1(N)\subset\{x_1,\cdots,x_n)\in E^n: (x_2^2+\cdots+x_n^2)^{\frac{1}{2}}<Cx_1^l\}$$

$$(1.4) \qquad \mathbf{n}(p)\cdot\mathbf{a}(p)\leqq 0 \qquad \text{for } p\in\Gamma_\cap\partial\Omega_1(N)$$

where $\mathbf{n}(p)$ is the outer unit normal to $\Gamma$ at $p=(x_1,\cdots,x_n)$ and $\mathbf{a}(p)$ is the vector $\mathbf{a}(p)=(x_1, lx_2,\cdots, lx_n)$. Our purpose in this paper is to prove the following.

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**Theorem 1.1.** *Let $\Omega$ and $\lambda$ be as above. If $u$ is a non-trivial solution of* (1.1) *and* (1.2), *then*

$$\lim_{t\to\infty} t^{\varepsilon}\int_{P_t}(u^2+|\nabla u|^2)dS=\infty \tag{1.5}$$

*for any $\varepsilon>0$, where $P_t$ is the section of $\Omega_1(N)$ by the hyperplane $x_1=t$.*

If $\Omega$ lies in the half-space $x_1>1$, and (1.4) holds on the whole of $\Gamma$ with $l=0$. (1.5) is a part of the well known results by Rellich [1]. Jones [2] (Theorem 9) has treated the problem in the case of $l=1$. We can find in Agmon [3] (Theorem 11) an extension of Jones' result, and, when $l=1$, our Theorem 1.1 is also included in Agmon's theorem. So the proof of Theorem 1.1 must be carried out for $0<l<1$, and it will be done in the framework developed by Roze [4] and Eidus [5].

In §2, introducing a curvilinear coordinate system for the convenience of calculations, we shall give some preliminary lemmas. In §3, it will be shown that a solution which does not satisfy (1.5) decreases, in a sense, like an exponential function in $\Omega_1(N)$, and in §4, it will turn out that such solution is the trivial solution.

In consequence of Theorem 1.1 it is easy to see that the self-adjoint realization of $-\Delta$ in $L^2(\Omega)$ with the Dirichlet boundary condition has no positive point eigenvalues. A short remark on the spectrum will be given in the final §5.

## 2. Preliminaries

In the sequel the conditions of the Theorem 1.1 are always assumed. Let us introduce a curvilinear coordinate system $(X_1, \cdots, X_n)$ in $E_+^n=\{(x_1, \cdots, x_n): x_1>0\}$ as follows;

$$\begin{cases} X_1=\{x_1^2+l(x_2^2+\cdots+x_n^2)\}^{\frac{1}{2}} & (X_1>0), \\ X_2=\tan^{-1}\{(x_2^2+\cdots+x_n^2)^{\frac{1}{2}}/x_1^l\} & (0\leqq X_2<\frac{\pi}{2}), \end{cases} \tag{2.1}$$

and $X_3, \cdots, X_n$ are the parameters which are suitably chosen on the sphere $S^{n-2}=\{(x_2, \cdots, x_n): x_2^2+\cdots+x_n^2=1\}$. (For example, we may put $x_2=$

$\cos X_3$, $x_3=\sin X_3 \cos X_4, \cdots, x_{n-1}=\sin X_3 \cdots \sin X_{n-1} \cos X_n$, $x_n=\sin X_3 \cdots \sin X_n$, $0 \leqq X_i \leqq \pi$ for $3 \leqq i \leqq n-1$ and $0 \leqq X_n \leqq 2\pi$.) Let $(ds)^2=\sum_{i=1}^{n}(dx_i)^2 = \sum_{i,j=1}^{n} g_{ij}dX_i dX_j$ be the ordinary Euclidean metric. Then we can see $g_{11}=X_1^2/\{X_1^2+(l^2-l)r^2\}$, $g_{12}=g_{21}=0$, $g_{22}=x_1^{2l+2}\sec^4 X_2/\{X_1^2+(l^2-l)r^2\}$ and $g_{ij}=r^2\tilde{g}_{ij}$ for $i, j \geqq 3$, where $r=(x_2^2+\cdots+x_n^2)^{\frac{1}{2}}$ and $\sum_{i,j=3}^{n} \tilde{g}_{ij}dX_i dX_j$ is the metric on the sphere $S^{n-2}$ induced from $E^{n-1}=\{(x_2, \cdots, x_n)\}$. Put $(g^{ij})=(g_{ij})^{-1}$ and $G=\det(g_{ij})$. Then

$$
(2.2) \quad \begin{cases} \Delta f \equiv \sum_{i=1}^{n} \partial^2 f/\partial x_i^2=(1/\sqrt{G})\sum_{i,j=1}^{n}(g^{ij}\sqrt{G}f_{X_j})_{X_i} \\ |\nabla f|^2 \equiv \sum_{i=1}^{n} |\partial f/\partial x_i|^2= \sum_{i,j=1}^{n} g^{ij}f_{X_i}\bar{f}_{X_j} \end{cases}
$$

for a smooth function $f$, where $f_{X_i}=\partial f/\partial X_j$.

Now we give some lemmas specifying the asymptotic properties of $g^{ij}$ and $G$, which will play important roles in the following sections.

**Lemma 2.1.** *$g^{11}\to 1$, $X_1 g^{11}_{X_1}/g^{11}\to 0$, $X_1 g^{ij}_{X_1}/g^{ij}\to 2l$ ($i=j=2$ or $i, j \geqq 3$) and $X_1 G_{X_1}/G \to 2(n-1)l$ when $X_1\to\infty$. These convergences are uniform in $X_2 \in [0, \theta]$ for any $\theta<\frac{\pi}{2}$.*

*Remark.* Because of the condition (1.3), there is a number $\theta<\frac{\pi}{2}$, such that $x_2<\theta$ for any point in $\Omega_1(N)$.

*Proof of Lemma* 2.1. In the case of $l=1$, the proof is easy. If $0<l<1$, then $r/X_1 \to 0$ ($X_1\to\infty$) uniformly when $X_2$ varies in $[0, \theta]$, because, $r=x_1^l \tan X_2$ and $x_1 \leqq X_1$. From this $g^{11}\to 1$ is obvious since $g^{11}=\{X_1^2+(l^2-l)r^2\}/X_1^2$. The other convergences can be proved also easily by straightforward calculations if we use the facts that $x_{X_1}=x_1X_1/\{X_1^2+(l^2-l)r^2\}$, $r_{X_1}=lrX_1/\{X_1^2+(l^2-l)r^2\}$, $G=g_{11}g_{22}r^{2(n-2)} \det(\tilde{g}_{ij})$, and $\tilde{g}_{ij}$ are independent of $X_1$ and $X_2$. Q.E.D.

**Lemma 2.2.** *For any real $\delta$, we have $X_1^{1-\delta}(\sqrt{G}X_1^\delta)/\sqrt{G}\to\delta+(n-1)l$, $X_1^{1-\delta}(g^{11}\sqrt{G}X_1^\delta)_{X_1}/\sqrt{G}\to\delta+(n-1)l$, and $X_1^{1-\delta}(g^{ij}\sqrt{G}X_1^\delta)_{X_1}/g^{ij}\sqrt{G}\to \delta+(n-3)l$ for $i=j=2$ or $i, j\geqq 3$, when $X_1\to\infty$. These convergences are* *uniform in* $X_2\in[0,\ \theta]$ *for any* $\theta<\frac{\pi}{2}$.

*Proof.* Lemma 2.1. and direct calculations lead us to this lemma.

Q.E.D.

Let $\Omega_{AB}$, $\Omega_A$ and $S_A$ be the subsets of $\Omega_1(N)$ characterized by $A<X_1<B$, $A<X_1<\infty$ and $X_1=A$ respectively, and put $\Gamma_{AB}=\partial\Omega_{AB}-(S_A\cup S_B)$ (the 'side' of $\Omega_{AB}$). If $u$ is a solution of (1.1) and (1.2), then $v=X^m u$ $(m\geqq 0)$ satisfies

$$\Delta V-\frac{2m}{X_1}g^{11}V_{X_1}+(M+\lambda)V=0 \qquad (\text{in } \Omega_A) \tag{2.3}$$

and

$$V|_{\Gamma_A}=0 \tag{2.4}$$

for $A>\inf_{\Omega_1(N)} X_1\equiv N_0$, where $M=(m^2+m)g^{11}/X_1^2-m(g^{11}\sqrt{G})_{X_1}/X_1\sqrt{G}$.

**Lemma 2.3.** $X_1^2M-g^{11}m^2\to m(1-(n-1)l)$ *uniformly when* $X_1\to\infty$ *in* $\Omega_1(N)$, *and there exist positive constants* $C_1$ *and* $N_1$ *which are independent of* $m\geqq 0$ *such that the inequalities* $M\geqq 0$ *and* $XM_{X_1}+2(m/X_1)^2\leqq mC_1/X_1^2$ *hold in* $\Omega_A$ *for* $A>N_1$.

*Proof.* It is easy to prove that $g^{11}_{X_1X_1}=o(1/X_1^2)$ and $G_{X_1X_1}=O(1/X_1^2)$ as $X_1\to\infty$ in $\Omega_1(N)$. Using these facts and Lemmas 2.1–2, we have the lemma.

Q.E.D.

The next two lemmas are concerned with the solutions of (2.3) and (2.4).

**Lemma 2.4.** *Let* $v$ *be a real valued solution of* (2.3) *and* (2.4). *Then*

$$\int_{\Omega_{AB}}\psi|\nabla v|^2d\Omega=\left\{\int_{S_B}-\int_{S_A}\right\}\sqrt{g^{11}}\psi vv_{X_1}dS \tag{2.5}$$

$$-\int_{\Omega_{AB}}\left(\psi'+\frac{2m\psi}{X_1}\right)g^{11}vv_{X_1}d\Omega+\int_{\Omega_{AB}}\psi(M+\lambda)v^2d\Omega,$$

where $B>A>N_0$, and $\psi$ is smooth and depends only on $X_1$.

*Proof.* Multiply (2.3) by $\psi v$ and integrate over $\Omega_{AB}$. Using (2.4) we have (2.5). Q.E.D.

If we put $m=0$ in (2.5),

$$\int_{\Omega_{AB}}\psi|\nabla u|^2 d\Omega=\left\{\int_{S_B}-\int_{S_A}\right\}\sqrt{g^{11}}\psi u u_{X_1}dS \tag{2.6}$$
$$-\int_{\Omega_{AB}}g^{11}\psi' u u_{X_1}d\Omega+\lambda\int_{\Omega_{AB}}\psi u^2 d\Omega.$$

**Lemma 2.5.** *Let $v$ be a real valued solution of (2.3) and (2.4). For any $\delta>0$, $\eta>0$ and $m\geqq 0$, we can find a real $N_2=N_2(\delta,\eta)$ which is independent of m such that the inequality*

$$\left\{\int_{S_B}-\int_{S_A}\right\}X_1^{\delta}\left\{g^{11}v_{X_1}^2-\frac{|\nabla v|^2}{2}+\frac{(M+\lambda)}{2}v^2\right\}\sqrt{g_{11}}\,dS \tag{2.7}$$
$$-(2m+\delta-l)\int_{\Omega_{AB}}X^{\delta-1}g^{11}v_X^2 d\Omega$$
$$+\frac{1}{2}\int_{\Omega_{AB}}X_1^{\delta-1}[(\delta+(n-3)l+\eta)|\nabla v|^2$$
$$-\{(\delta+(n-1)l-\eta)(M+\lambda)+X_1 M_{X_1}\}v^2]d\Omega$$
$$\geqq 0$$

*holds for $B>A>N_2$.*

*Proof.* We multiply (2.3) by $X_1^{\delta}v_{X_1}$ and integrate over $\Omega_{AB}(A>N_0)$. Integrating by parts, we have

$$\left\{\int_{S_B}-\int_{S_A}\right\}X_1^{\delta}\left\{g^{11}v_{X_1}^2-\frac{|\nabla v|^2}{2}+\frac{(M+\lambda)}{2}v^2\right\}\sqrt{g_{11}}\,dS \tag{2.8}$$
$$-2m\int_{\Omega_{AB}}X_1^{\delta-1}g^{11}v_{X_1}^2 d\Omega-\int_{AB}\left\{\delta X_1^{\delta-1}g^{11}\sqrt{G}\right.$$
$$\left.-\frac{1}{2}\left(g^{11}\sqrt{G}X_1^{\delta}\right)_{X_1}\right\}v_{X_1}^2\frac{d\Omega}{\sqrt{G}}+\frac{1}{2}\int_{\Omega_{AB}}\Big[\sum_{i,j\geqq 2}(g^{ij}\sqrt{G}X_1^{\delta})_{X_1}v_{X_i}v_{X_j}$$

$$-\{(M+\lambda)\sqrt{G}\,X_1^{\delta}\}_{X_1}v^2]\frac{d\Omega}{\sqrt{G}}$$

$$=-\frac{1}{2}\int_{\Gamma_{AB}} X_1^{\delta}|\nabla v|^2\sqrt{g_{11}}(\boldsymbol{n}\cdot\boldsymbol{X}_1)dS,$$

where $\boldsymbol{X}_1$ is the vector $\boldsymbol{X}_1=\frac{a}{|a|}=(x_1, lx_2,\cdots, lx_n)/(x_1^2+l^2x_2^2+\cdots+l^2x_n^2)^{\frac{1}{2}}$. Here we have used the fact that $v_{X_1}(\nabla v\cdot\boldsymbol{n})=\sqrt{g_{11}}|\nabla v|^2(\boldsymbol{X}_1\cdot\boldsymbol{n})$ on $\Gamma_{AB}$, which follows from the boundary condition (2.4). In view of the condition (1.4), the right side of (2.8) is non-negative. In consequence of Lemma 2.2, for any $\eta>0$, we can take $N_2'(\delta, \eta)$ such that the inequalites

$$\delta g^{11}X_1^{\delta-1}-\frac{1}{2}(g^{11}\sqrt{G}\,X_1^{\delta})_{X_1}/\sqrt{G}\geqq\frac{1}{2}(\delta-(n-1)l-\eta)g^{11}X_1^{\delta-1},$$

$$(g^{ij}\sqrt{G}\,X_1^{\delta})_{X_1}/\sqrt{G}\leqq(\delta+(n-3)l+\eta)g^{ij}X_1^{\delta-1}\,(i, j\geqq 2),$$

$$(\sqrt{G}\,X_1^{\delta})_{X_1}\geqq(\delta+(n-1)l-\eta)X_1^{\delta-1}$$

hold if $X_1>N_2'(\delta, \eta)$. Thus we have the inequality (2.7) for $B>A>N_2(\delta, \eta)=\max(N_0, N_1, N_2'(\delta, \eta))$ by Lemma 2.3. Q.E.D.

## 3. On a Solution Which Does Not Satisfy (1.5)

In this and following sections, we use the abbreviations $X$, $f_X$ and $\gamma$ which stand for $X_1$, $f_{X_1}=\partial f/\partial X_1$ and $g^{11}$ respectively.

**Lemma 3.1.** *Let $u$ be a solution of* (1.1) *and* (1.2). *If*

$$\liminf_{t\to\infty} t^{\delta}\int_{S_t}(|u|^2+|\nabla u|^2)dS=0 \tag{3.1}$$

*for some $\delta>0$, then*

$$\int_{\Omega_1(N)} X^m(|u|^2+|\nabla u|^2)d\Omega<\infty \tag{3.2}$$

*for any $m\geqq 0$.*

*Proof.* We may assume that $u$ is real valued. If we put $m=0$ in Lemma 2.5, we have

$$(3.3)\qquad \left\{\int_{S_B}-\int_{S_B}\right\}X^{\delta}\left\{\gamma u_X^2-\frac{|\nabla u|^2}{2}+\lambda u^2\right\}\frac{dS}{\sqrt{\gamma}}\geqq$$

$$(\delta-l)\int_{\Omega_{AB}}X^{\delta-1}\lambda u_X^2\,d\Omega-\frac{1}{2}\int_{\Omega_{AB}}X^{\delta-1}\{(\delta+(n-3)l+\eta)|\nabla u|^2-$$

$$\lambda(\delta+(n-1)l-\eta)u^2\}\,d\Omega,$$

for $B>A>N_2(\delta,\eta)$. On the other hand, taking $X^{\delta-1}$ as $\psi$ in (2.6), we see

$$(3.4)\qquad \left\{\int_{S_B}-\int_{S_A}\right\}X^{\delta-1}uu_X\sqrt{\gamma}\,dS$$

$$=\int_{\Omega_{AB}}X^{\delta-1}(|\nabla u|^2-\lambda u^2)\,d\Omega+\int_{\Omega_{AB}}\gamma(\delta-1)X^{\delta-2}uu_X\,d\Omega.$$

From (3.3) and (3.4), we have, for $A>N_2$,

$$(3.5)\qquad \left\{\int_{S_B}-\int_{S_A}\right\}X^{\delta}\left\{\gamma u_X^2-\frac{|\nabla u|^2}{2}+\lambda u^2\right\}\frac{dS}{\sqrt{\gamma}}$$

$$+\frac{(n-1)}{2}l\left\{\int_{S_B}-\int_{S_A}\right\}X^{\delta-1}uu_X\sqrt{\gamma}\,dS\geqq(\delta-l)\int_{\Omega_{AB}}X^{\delta-1}\gamma u_X^2\,d\Omega$$

$$-\frac{1}{2}\int_{\Omega_{AB}}X^{\delta-1}\{(\delta-2l+\eta)|\nabla u|^2-\lambda(\delta-\eta)u^2\}\,d\Omega$$

$$+\frac{(n-1)}{2}(\delta-1)l\int_{\Omega_{AB}}\gamma X^{\delta-2}uu_X\,d\Omega$$

$$=\frac{1}{2}\int_{\Omega_{AB}}X^{\delta-1}\{(\delta-2\eta)|\nabla u|^2+\lambda(\delta-2\eta)u^2\}\,d\Omega$$

$$+\left(l-\delta+\frac{\eta}{2}\right)\int_{\Omega_{AB}}X^{\delta-1}|\nabla u|^2\,d\Omega+(\delta-l)\int_{\Omega_{AB}}X^{\delta-1}\gamma u_X^2\,d\Omega$$

$$+\frac{\eta\lambda}{2}\int_{\Omega_{AB}}X^{\delta-1}u^2\,d\Omega+\frac{(n-1)(\delta-1)l}{2}\int_{\Omega_{AB}}\gamma X^{\delta-2}uu_X\,d\Omega.$$

Without loss of generality, we may assume $\delta<l$. So we can consider that

$$\left|(\delta-l)\int_{\Omega_{AB}}X^{\delta-1}\gamma u_X^2\,d\Omega\right|\leqq(l-\delta)\int_{\Omega_{AB}}X^{\delta-1}|\nabla u|^2\,d\Omega.$$

Moreover, $|(\delta-1)(n-1)uu_X/X|\leqq\eta(\gamma u_X^2+\lambda u^2)$ for sufficiently large $X$, say $X>N_3(\eta, \lambda)$. Thus, passing to the limit for $B\to\infty$, it follows from (3.5) that

$$\int_{S_A} X^{\delta}(|\nabla u|^2-\lambda u^2)\frac{dS}{\sqrt{\gamma}}-(n-1)l\int_{S_A} X^{\delta-1}uu_X\sqrt{\gamma}\,dS \tag{3.6}$$

$$\geqq\frac{\delta}{2}\int X^{\delta-1}(|\nabla u|^2+\lambda u^2)d\Omega$$

for $A>N_4(\delta, \lambda)\equiv\max\left\{N_2\left(\delta, \frac{\delta}{4}\right), N_3\left(\frac{\delta}{4}, \lambda\right)\right\}$.

We integrate (3.6) with respect to $A$ from $\xi_0$ to $\xi_1$ ($\xi_1>\xi_0>N_4$). Using $|uu_X|<(u^2+|\nabla u|^2)/2$ and (2.6) in which we replace $\psi$ by $X^{\delta}$, we have

$$\frac{\delta}{2}\int_{\xi_0}^{\xi_1}d\xi\int_{\Omega_\xi} X^{\delta-1}\{|\nabla u|^2+\lambda u^2\}\,d\Omega\leqq C_2\int_{\Omega_{\xi_0\xi_1}} X^{\delta-1}(|\nabla u|^2+u^2)d\Omega \tag{3.7}$$

$$+\left\{\int_{S_{\xi_1}}-\int_{S_{\xi_0}}\right\}X^{\delta}uu_X\sqrt{\gamma}\,dS,$$

where $C_2=C_2(\delta)$ is some positive constant which is independent of $\xi_0$ and $\xi_1$. By (3.1) and

$$\int_{\xi_0}^{\xi_1}d\xi\int_{\Omega_\xi} X^{\delta-1}(|\nabla u|^2+\lambda u^2)d\Omega$$

$$=\int_{\Omega_{\xi_0\xi_1}}(X-\xi_0)X^{\delta-1}(|\nabla u|^2+\lambda u^2)d\Omega+(\xi_1-\xi_0)\int_{\Omega_{\xi_1}} X_1^{\delta-1}(|\nabla u|^2+\lambda u^2)d\Omega$$

(3.7) implies

$$\frac{\delta}{2}\int_{\Omega_{\xi_0}}(X-\xi_0)X^{\delta-1}(|\nabla u|^2+\lambda u^2)d\Omega\leqq C_2\int_{\Omega_{\xi_0}} X^{\delta-1}(|\nabla u|^2+u^2)d\Omega$$

$$-\int_{S_{\xi_0}} X^{\delta}uu_X\sqrt{\gamma}\,dS$$

($\xi_0>N_4$). Integrating this inequality with respect to $\xi_0$ from $\xi_1$ to $\infty$ ($\xi_1>N_4$), we find

$$\int_{\Omega_{\xi_1}}(X-\xi_1)^2X^{\delta-1}(|\nabla u|^2+\lambda u^2)d\Omega\leqq C_3\int_{\Omega_{\xi_1}} X^{\delta}(|\nabla u|^2+u^2)d\Omega,$$

where $C_3$ does not depend on $\xi_1$. Repeating this process, we arrive at (3.2). Q.E.D.

**Lemma 3.2.** *Under the assumption of Lemma* 3.1,

$$\lim_{t\to\infty} e^{2\alpha t}\int_{S_t} |u|^2 dS=0 \tag{3.8}$$

for $\alpha<\sqrt{\lambda l/(1-l)}$. If $l=1$, $\alpha$ may be taken arbitrarily.

*Proof.* We may assume that $u$ is real valued. Put $v=X^m u$. In Lemma 2.5, we replace $\delta$ by $l$ and let $B\to\infty$. Then, by Lemma 3.1, we have

$$\begin{aligned}(3.9)\quad & -\int_{S_A} X^l\Big(\gamma v_X^2-\frac{|\nabla v|^2}{2}+\frac{M+\lambda}{2}v^2\Big)\frac{dS}{\sqrt{\gamma}}-2m\int_{\Omega_A} X^{l-1}\gamma v_X^2 d\Omega \\ & +\frac{1}{2}\int_{\Omega_A} X^{l-1}[\{(n-2)l+\eta\}|\nabla v|^2-\{(nl-\eta)(M+\lambda)+XM_X\}v^2]d\Omega \\ & \geqq 0\end{aligned}$$

for $A>N_2(l, \eta)$. On the other hand, taking $X^{l-1}$ as $\psi$ in (2.5) we see

$$\begin{aligned}(3.10)\quad & \int_{\Omega_A} X^{l-1}|\nabla v|^2 d\Omega=-\int_{S_A} X^{l-1}\sqrt{\gamma}\, vv_X dS \\ & -\int_{\Omega_A} X^{l-2}\{(l-1)+2m\}\gamma vv_X d\Omega+\int_{\Omega_A} X^{l-1}(M+\lambda)v^2 d\Omega.\end{aligned}$$

From (3.9) and (3.10), we have

$$\begin{aligned}(3.11)\quad & \int_{S_A} X^l\Big\{\gamma v_X^2-\frac{|\nabla v|^2}{2}+\frac{1}{2}(M+\lambda)v^2\Big\}\frac{dS}{\sqrt{\gamma}} \\ & +\frac{(n-2)l+\eta}{2}\int_{S_A} X^{l-1}\sqrt{\gamma}\, vv_X dS+2m\int_{\Omega_A} X^{l-1}\gamma v_X^2 d\Omega \\ & +\frac{1}{2}(l-1+2m)\{(n-2)l+\eta)\}\int_{S_A} X^{l-2}\gamma vv_X d\Omega \\ & +\frac{1}{2}\int_{\Omega_A} X^{l-1}\{2(l-\eta)(M+\lambda)+XM_X\}v^2 d\Omega\leqq 0\end{aligned}$$

for $A>N_2$. Note that the fourth term of (3.11) is estimated as follows;

$$\left|\frac{1}{2}(l-1+2m)\{(n-2)l+\eta\}\int_{\Omega_A}\cdots d\Omega\right|\leqq 2m\int_{\Omega_A}X^{l-1}\gamma v_X^2 d\Omega$$

$$+\frac{C_4}{2}m\int_{\Omega_A}X^{l-3}v^2 d\Omega$$

where $C_4$ is a positive constant independent of $A>N_0$ and $m\geqq 1$. Thus we have the inequality

$$(3.12)\qquad \int_{S_A}X^l\left\{\gamma v_X^2-\frac{|\nabla v|^2}{2}+\frac{(M+\lambda)}{2}v^2\right\}\frac{dS}{\sqrt{\gamma}}$$

$$-\frac{(n-2)l+\eta}{2}\int_{S_A}X^{l-1}\sqrt{\gamma}\,|vv_X|\,dS+\frac{1}{2}\int_{\Omega_A}X^{l-1}\{2(l-\varepsilon)(M+\lambda)$$

$$+XM_X-mC_4/X^2\}v^2 d\Omega\leqq 0$$

for $A>N_2$. Using the equality

$$|\nabla v|^2=X^{2m}|\nabla u|^2+2mX^{2m-1}\gamma uu_X+m^2X^{2m-2}\gamma u^2,$$

the first term of (3.12) can be written in the form

$$\int_{S_A}X^l\left\{\gamma v_X^2+\frac{X^{2m}}{2}(M-\gamma m^2/X^2)u^2-mX^{2m-1}\gamma uu_X\right\}\frac{dS}{\sqrt{\gamma}}$$

$$+\frac{1}{2}\int_{S_A}X^{2m+l}(-|\nabla u|^2-\lambda u^2)\frac{dS}{\sqrt{\gamma}}.$$

Multiplying this by $A^{2-2m-l}$ and integrating with respect to $A$ from $\xi$ to $\infty$ ($\xi>N_2$), we have, by Lemma 3.1,

$$\int_{\Omega_\xi}X^{2-2m}\gamma v_X^2 d\Omega+\frac{1}{2}\int_{\Omega_\xi}(X^2M-\gamma m^2)u^2 d\Omega+(1-m)\int_{\Omega_\xi}X\gamma uu_X d\Omega$$

$$+\frac{1}{2}\int_{S_\xi}X^2\sqrt{\gamma}\,uu_X dS$$

$$=\int_{\Omega_\xi}X^{2-2m}\gamma v_X^2 d\Omega+\frac{1}{2}\int_{\Omega_\xi}\{X^2M-\gamma m^2+(m-1)(X\gamma\sqrt{G})_X/\sqrt{G}\}u^2 d\Omega$$

$$+\frac{m-1}{2}\int_{S_\xi}X\sqrt{\gamma}\,u^2 dS+\frac{1}{2}\int_{S_\xi}X^2\sqrt{\gamma}\,uu_X dS.$$

Here we have used (2.6) with $\psi=X^2$. Thus we have from (3.12)

$$(3.13)\qquad \int_{\Omega_\xi}\{X^2M-\gamma m^2+(m-1)(X\gamma\sqrt{G})_X/\sqrt{G}-(nl-2l+\eta)^2/4\}u^2d\Omega$$
$$+\int_{S_\xi}X^2\sqrt{\gamma}\,uu_XdS+(m-1)\int_{S_\xi}X\sqrt{\gamma}\,u^2dS$$
$$+\int_\xi^\infty A^{2-2m-l}dA\int_{\Omega_A}X^{l-1}\{2(l-\eta)(M+\lambda)+XM_X-mC_4/X^2\}v^2d\Omega$$
$$\leqq 0$$

for $\xi>N_2$. (Note that $\{(n-2)l+\eta\}X^{1-2m}|vv_X|\leqq X^{2-2m}v_X^2+\frac{1}{4}\{(n-2)l+\eta\}^2X^{-2m}v^2$).

Put

$$\Phi(\xi)=\int_{S_\xi}X^2\sqrt{\gamma}\,u^2dS.$$

Then

$$(3.14)\qquad \frac{1}{2}\frac{d\Phi}{d\xi}=\int_{S_\xi}\{X^2\sqrt{\gamma}\,uu_X+(X^2\gamma\sqrt{G})_Xu^2\sqrt{\gamma}G\}dS.$$

By Lemma 2.2, we can choose $C_5>0$ such that $(\gamma X^2\sqrt{G_X}/\sqrt{G})<(C_5-1)\gamma X$ for $X>N_0$. (3.13) and (3.14) give

$$(3.15)\qquad \int_{\Omega_\xi}\{X^2M-\gamma m^2+(m-1)(X\gamma\sqrt{G})_X/\sqrt{G}-(nl-2l+\eta)^2/4\}u^2d\Omega$$
$$+\frac{1}{2}\frac{d\Phi}{d\xi}+(m-C_5)\frac{1}{\xi}\Phi$$
$$+\int_\xi^\infty A^{2-2m-l}dA\int_{\Omega_A}X^{l-1}\{2(l-\eta)(M+\lambda)+XM_X-mC_4/X^2\}v^2d\Omega$$
$$\leqq 0.$$

The coefficient of $u^2$ in the first integral of (3.15) tends to $2m-1-(n-1)l-(nl-2l+\eta)^2/4$ as $X\to\infty$. See Lemmas 2.1, 2.2 and 2.3. So it is positive if $m>C_6>\{1+(n-1)l\}/2+(nl-2l+\eta)^2/8$ and $X$ is sufficiently large, say $X>N_5$. We can take $N_5$ independently of $m$, at least, when $m>C_6$. If we put

$$(3.16)\qquad 2(l-\eta)(M+\lambda)+XM_X-\frac{C_4}{X^2}$$

$$=2(l-\eta-1)m^2/X^2+2(l-\eta)\lambda-mh,$$

then, by Lemma 2.3, we can take a positive constant $C_7$ such that $|h|<C_7/X^2$ for $X>N_0$. Now if we put in (3.15) and (3.16) $m=m(\xi,\ \eta)=\xi\sqrt{\lambda(l-\eta)/\left(1-l+\frac{3}{2}\eta\right)}$, then there exists positive $N_6(\eta)$ such that $m>C_6$, $m>C_7/\eta$ for $\xi>N_6$. Note that

$$(3.16)=2(l-\eta-1)(l-\eta)\lambda\xi^2/(1-l+\frac{3}{2}\eta)X^2+2(l-\eta)\lambda-mh$$

$$\geqq 2\lambda(l-\eta)\left(1-\frac{\xi^2}{X^2}\right)+(m\eta-C_7)m/X^2$$

$$>0$$

if $X>\xi>N_6$. Taking $\eta$ sufficiently small we may assume $m(\xi,\ \eta)/\xi>\alpha$. Moreover, for such $\eta$, we can take $N_7(\eta)\ (>N_6(\eta))$ so that $(m(\xi,\ \eta)-C_5)/\xi>\alpha$ for $\xi>N_7$. Thus we have from (3.15) the differential inequality

$$\frac{d\Phi}{d\xi}+2\alpha\Phi\leqq 0$$

for large $\xi$. This proves the lemma. Q.E.D.

**Lemma 3.3.** *Under the assumption of Lemma* 3.1,

$$(3.17)\qquad \int_{\Omega_1(N)}e^{2\alpha X}(u^2+|\nabla u|^2)d\Omega<\infty$$

*for* $\alpha<\sqrt{\lambda l/(1-l)}$. *If* $l=1$, $\alpha$ *may be taken arbitrarily.*

*Proof.* We assume that $u$ is real valued. Replace $\psi$ in (2.6) by $e^{2\alpha X}$, then we have

$$(3.18)\qquad \int_{\Omega_{AB}}e^{2\alpha X}|\nabla u|^2d\Omega=\left\{\int_{S_A}-\int_{S_B}\right\}\sqrt{\gamma}\,e^{2\alpha X}uu_XdS$$

$$-2\alpha\int_{\Omega_{AB}}\gamma e^{2\alpha X}uu_Xd\Omega+\lambda\int_{\Omega_{AB}}e^{2\alpha X}u^2d\Omega.$$

Next note that

$$(3.19)\qquad \int_{\Omega_{AB}} \gamma e^{2\alpha X} u u_X \, d\Omega = -\frac{1}{2}\int_{\Omega_{AB}} (\sqrt{\gamma}\, e^{2\alpha X})_X \sqrt{\gamma}\, u^2 d\Omega$$

$$+\frac{1}{2}\left\{\int_{S_B} - \int_{S_A}\right\}\sqrt{\gamma}\, e^{2\alpha X} u^2 dS.$$

In view of Lemma 3.2, the limit of (3.19) exists when $B\to\infty$. Hence

$$\liminf_{B\to\infty}\left|\int_{S_B}\sqrt{\gamma}\, e^{2\alpha X} u u_X dS\right| = 0.$$

Thus the limit of (3.18) exists when $B\to\infty$. Q.E.D.

## 4. Proof of Theorem 1.1

If the assertion of Theorem 1.1 is not true, there exists some $\delta>0$, and

$$\liminf_{t\to\infty} t^{\delta}\int_{P_t}(|u|^2+|\nabla u|^2)dS=0.$$

This is nothing but (3.1) of Lemma 3.1. Thus, for the proof of Theorem 1.1, it suffices to show the following assertion.

*Let $u$ be a solution of* (1.1) *and* (1.2). *If $u$ satisfies* (3.1), *then $u\equiv 0$ on the whole of $\Omega$.*

First note that

$$(4.1)\qquad \int_{\Omega_1(N)} X^k e^{mX^\beta}(|u|^2+|\nabla u|^2)\, d\Omega<\infty$$

for any $m>0$, $k>0$ and $\beta<1$. This is a direct consequence of Lemma 3.3.

Put $v=e^{mX^\beta}u$, then

$$(4.2)\qquad \Delta v-2m\beta X^{\beta-1}\gamma v_X+(L+\lambda)v=0,$$

where

(4.3) $$L=m^2\beta^2X^{2\beta-2}-m\beta(\beta-1)X^{\beta-2}-m\beta X^{\beta-1}(\gamma\sqrt{G})_X/\sqrt{G}.$$

We multiply (4.2) by $X^k v_X$ and integrate over $\Omega_A$. From (4.1) we have

(4.4) $$-\int_{S_A} X^k\left\{\gamma v_X^2-\frac{|\nabla v|^2}{2}+\frac{(L+\lambda)}{2}v^2\right\}\frac{dS}{\sqrt{\gamma}}$$
$$-\int_{\Omega_A}(2m\beta X^{\beta+k-1}+(k-l)X^{k-1})\gamma v_X^2\,d\Omega+$$
$$\frac{1}{2}\int_{\Omega_A} X^{k-1}[(k+(n-3)l+\eta)|\nabla v|^2-\{(k+(n-1)l-\eta)(L+\lambda)$$
$$+XL_X\}v^2]d\Omega$$
$$\geqq 0$$

for $\eta>0$ and sufficiently large $A$, say $A>N_8(\eta)$. (See the proof of Lemma 2.5.) If we put $k=(3-n)l-\eta$ in (4.4),

(4.5) $$\int_{S_A} X^{(3-n)l-\eta}\left\{\gamma v_X^2-\frac{|\nabla v|^2}{2}+\frac{(L+\lambda)}{2}v^2\right\}\frac{dS}{\sqrt{\gamma}}$$
$$\leqq -\int_{\Omega_A}\{2m\beta X^{\beta+(3-n)l-\eta-1}+((2-n)l-\eta)X^{(3-n)l-\eta-1}\}\gamma v_X^2\,d\Omega$$
$$-\int_{\Omega_A} X^{(3-n)l-\eta-1}\{(l-\eta)(L+\lambda)+XL_X\}v^2\,d\Omega=I_1+I_2.$$

There is $N_9(\eta)$ ($>N_8(\eta)$) such that $I_1\leqq 0$ for $A>N_9$. Assuming $m\geqq 1$, and $\frac{1}{2}<\beta<1$, we can take $N_9$ independently of $m$ and $\beta$.

Next note that

$$L\geqq m^2\beta^2X^{2\beta-2}-mC_8(\beta)X^{\beta-2},$$
$$XL_X\leqq m^2\beta^2(2\beta-2)X^{2\beta-2}+mC_9(\beta)X^{\beta-2},$$

where $C_8(\beta)$ and $C_9(\beta)$ are constants which are independent of $m$.

Now let us assume $\eta$ is small so that $l-\eta>0$. If we take $\beta(<1)$ near to 1, then $l-\eta>2(1-\beta)$, and hence

$$(l-\eta)(L+\lambda)+XL_X$$

$$\geqq m^2\beta^2(l-\eta+2\beta-2)X^{2\beta-2}-m\{(l-\eta)C_8(\beta)-C_9(\beta)\}X^{\beta-2}$$

$$\geqq 0$$

for large $m$ and $X$, say $m\geq C_{10}$ and $X\geq N_{10}(\eta)$ $(>N_9)$, where $N_{10}$ is independent of $m$ $(>C_{10})$. Thus we have from (4.5)

$$\text{(4.6)} \qquad \int_{S_A} X^{(3-n)l-\eta}\left\{\gamma v_X^2-\frac{|\nabla v|^2}{2}+\frac{(L+\lambda)}{2}v^2\right\}\frac{dS}{\sqrt{\gamma}}\leqq I_1+I_2\leqq 0$$

for $A>N_{10}$ and $m>C_{10}$.

On the other hand, if we put

$$\int_{S_A} X^{(3-n)l-\eta}\left\{\gamma v_X^2-\frac{|\nabla v|^2}{2}+\frac{(L+\lambda)}{2}v^2\right\}\frac{dS}{\sqrt{\gamma}}$$

$$=m^2M_1(u,\ A)+mM_2(u,\ A)+M_3(u,\ A),$$

(where $M_1$, $M_2$ and $M_3$ are independent of $m$,) then it is easy to see $M_1(u, A)>0$ when $u\not\equiv 0$ on $S_A$. Note that $v=e^{mX^\beta}u$. If we assume $M_1(u, A)>0$ for some $A>N_{10}$,

$$\int_{S_A} X^{(3-n)l-\eta}\left\{\nu v_X^2-\frac{|\nabla v|^2}{2}+\frac{(L+\lambda)}{2}v^2\right\}\frac{dS}{\sqrt{\gamma}}>0$$

for sufficiently large $m$. This contradicts (4.6), hence we see $u\equiv 0$ on $\Omega_{N_{10}}$. The unique continuation theorem for the second order elliptic equations enables us to conclude $u\equiv 0$ on the whole of $\Omega$. The proof of Theorem 1.1 is now complete.

## 5. On the Spectrum of $-\Delta$

This final brief section concerns the spectrum of $-\Delta$ in $\Omega$ with the Dirichlet boundary condition.

Let $L$ be the operator in $L^2(\Omega)$ with the domain $D(L)=\{f: f\in D^1_{L^2}, \Delta f\in L^2(\Omega)\}$, and $Lu=-\Delta u$, where $D^1_{L^2}$ is the completion of $C_0^\infty(\Omega)$ with regard to the norm

$$\|f\|_1=\left\{\int_\Omega(|f|^2+|\nabla f|^2)d\Omega\right\}^{\frac{1}{2}}$$

Then $L$ is a non-negative self-adjoint operator in $L^2(\Omega)$.

**Theorem 5.1.** *Under the assumption on $\Omega$ in* §1, *$L$ has no point eigenvalues. Moreover, the continuous spectrum of $L$ fills up the non-negative half of the real axis.*

*Proof.* $L$ is a non-negative operator, and so it has no negative eigenvalues. Let $u \in D(L)$, and $Lu=0$. Integrate $uL\bar{u}$ over $\Omega$, we have

$$\int_{\Omega} |\nabla u|^2 d\Omega = 0.$$

Hence $u=$ constant. By the Diriclet condition, $u=0$, and so $\lambda=0$ cannot be an eigenvalue of $L$. If $u \in D(L)$, then

$$\liminf_{t\to 0} t\int_{P_t} (|u|^2 + |\nabla u|^2) dS = 0.$$

This shows that the non-existence of positive eigenvalues is a consequence of Theorem 1.1.

Next let us prove the latter half of the theorem. That is, we must prove that any non-negative real number $\lambda$ belongs to the spectrum of $L$.

Let $\varphi = \varphi(X_1, \cdots, X_n)$ be a function which is in $C_0^\infty(\Omega_{N_0\, N_0+1})$, and $\varphi_m = \varphi(X_1/m, X_2, \cdots, X_n)$. Put

$$\Phi_m = e^{i\sqrt{\lambda}X_1}\varphi_m/\nu_m,$$

where $\nu_m = ||\varphi_m||_{L^2(\Omega)}$. It is not difficult to show that

$$||L\Phi_m - \lambda\Phi_m||_{L^2(\Omega)} \to 0 \qquad (m \to \infty).$$

Taking subsequence if necessary, we may assume supp $\Phi_i \cap$ supp $\Phi_j = \phi$ $(i \neq j)$ ($\Phi_i$ and $\Phi_j$ $(i \neq j)$ are orthogonal), where supp $\Phi$ denotes the support of $\Phi$. This shows that $\lambda$ is in the spectrum of $L$, because, if not, $\{\Phi_m\}$ would tend to 0, which is impossible, however, on account of $||\Phi_m||_{L^2(\Omega)} = 1$.

Q.E.D.

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