Multipliers of Improper Similitudes

R. Preeti[1] and J.-P. Tignol[2]

Received: February 11, 2004
Revised: May 5, 2004

Communicated by Alexander Merkurev

Abstract. For a central simple algebra with an orthogonal involution (A,σ) over a field k of characteristic different from 2, we relate the multipliers of similitudes of (A,σ) with the Clifford algebra $C(A,\sigma)$. We also give a complete description of the group of multipliers of similitudes when $\deg A \leq 6$ or when the virtual cohomological dimension of k is at most 2.

2000 Mathematics Subject Classification: 11E72.
Keywords and Phrases: Central simple algebra with involution, hermitian form, Clifford algebra, similitude.

Introduction

A. Weil has shown in [22] how to obtain all the simple linear algebraic groups of adjoint type D_n over an arbitrary field k of characteristic different from 2: every such group is the connected component of the identity in the group of automorphisms of a pair (A,σ) where A is a central simple k-algebra of degree $2n$ and $\sigma\colon A \to A$ is an involution of orthogonal type, i.e., a linear map which over a splitting field of A is the adjoint involution of a symmetric bilinear form. (See [7] for background material on involutions on central simple algebras and classical groups.) Every automorphism of (A,σ) is inner, and induced by an element $g \in A^\times$ which satisfies $\sigma(g)g \in k^\times$. The group of *similitudes* of (A,σ) is defined by that condition,

$$\mathrm{GO}(A,\sigma) = \{g \in A^\times \mid \sigma(g)g \in k^\times\}.$$

[1]The first author gratefully acknowledges the generous support of the Université catholique de Louvain, Belgium and the ETH-Z, Switzerland.
[2]Work supported in part by the European Community's Human Potential Programme under contract HPRN-CT-2002-00287, KTAGS. The second author is supported in part by the National Fund for Scientific Research (Belgium).

The map which carries $g \in \mathrm{GO}(A,\sigma)$ to $\sigma(g)g \in k^\times$ is a homomorphism

$$\mu\colon\ \mathrm{GO}(A,\sigma) \to k^\times$$

called the *multiplier map.* Taking the reduced norm of each side of the equation $\sigma(g)g = \mu(g)$, we obtain

$$\mathrm{Nrd}_A(g)^2 = \mu(g)^{2n},$$

hence $\mathrm{Nrd}_A(g) = \pm\mu(g)^n$. The similitude g is called *proper* if $\mathrm{Nrd}_A(g) = \mu(g)^n$, and *improper* if $\mathrm{Nrd}_A(g) = -\mu(g)^n$. The proper similitudes form a subgroup $\mathrm{GO}_+(A,\sigma) \subset \mathrm{GO}(A,\sigma)$. (As an algebraic group, $\mathrm{GO}_+(A,\sigma)$ is the connected component of the identity in $\mathrm{GO}(A,\sigma)$.)
Our purpose in this work is to study the multipliers of similitudes of a central simple k-algebra with orthogonal involution (A,σ). We denote by $G(A,\sigma)$ (resp. $G_+(A,\sigma)$, resp. $G_-(A,\sigma)$) the group of multipliers of similitudes of (A,σ) (resp. the group of multipliers of proper similitudes, resp. the coset of multipliers of improper similitudes),

$$\begin{aligned} G(A,\sigma) &= \{\mu(g) \mid g \in \mathrm{GO}(A,\sigma)\},\\ G_+(A,\sigma) &= \{\mu(g) \mid g \in \mathrm{GO}_+(A,\sigma)\},\\ G_-(A,\sigma) &= \{\mu(g) \mid g \in \mathrm{GO}(A,\sigma) \setminus \mathrm{GO}_+(A,\sigma)\}. \end{aligned}$$

When A is split ($A = \mathrm{End}_k V$ for some k-vector space V), hyperplane reflections are improper similitudes with multiplier 1, hence

$$G(A,\sigma) = G_+(A,\sigma) = G_-(A,\sigma).$$

When A is not split however, we may have $G(A,\sigma) \neq G_+(A,\sigma)$.
Multipliers of similitudes were investigated in relation with the discriminant $\operatorname{disc}\sigma$ by Merkurjev–Tignol [14]. Our goal is to obtain similar results relating multipliers of similitudes to the next invariant of σ, which is the Clifford algebra $C(A,\sigma)$ (see [7, §8]). As an application, we obtain a complete description of $G(A,\sigma)$ when $\deg A \leq 6$ or when the virtual cohomological dimension of k is at most 2.

To give a more precise description of our results, we introduce some more notation. Throughout the paper, k denotes a field of characteristic different from 2. For any integers n, $d \geq 1$, let μ_{2^n} be the group of 2^n-th roots of unity in a separable closure of k and let $H^d(k, \mu_{2^n}^{\otimes(d-1)})$ be the d-th cohomology group of the absolute Galois group with coefficients in $\mu_{2^n}^{\otimes(d-1)}$ ($= \mathbf{Z}/2^n\mathbf{Z}$ if $d = 1$). Denote simply

$$H^d k = \varinjlim_n H^d(k, \mu_{2^n}^{\otimes(d-1)}),$$

so H^1k and H^2k may be identified with the 2-primary part of the character group of the absolute Galois group and with the 2-primary part of the Brauer group of k, respectively,

$$H^1k = X_2(k), \qquad H^2k = \mathrm{Br}_2(k).$$

In particular, the isomorphism $k^\times/k^{\times 2} \simeq H^1(k, \mathbf{Z}/2\mathbf{Z})$ derived from the Kummer sequence (see for instance [7, (30.1)]) yields a canonical embedding

$$k^\times/k^{\times 2} \hookrightarrow H^1 k. \tag{1}$$

The Brauer class (or the corresponding element in H^2k) of a central simple k-algebra E of 2-primary exponent is denoted by $[E]$.
If K/k is a finite separable field extension, we denote by $N_{K/k}\colon H^dK \to H^dk$ the norm (or corestriction) map. We extend the notation above to the case where $K \simeq k \times k$ by letting $H^d(k \times k) = H^dk \times H^dk$ and

$$N_{(k\times k)/k}(\xi_1, \xi_2) = \xi_1 + \xi_2 \qquad \text{for } (\xi_1, \xi_2) \in H^d(k \times k).$$

Our results use the product

$$\cdot\colon\ k^\times \times H^dk \to H^{d+1}k \qquad \text{for } d = 1 \text{ or } 2$$

induced as follows by the cup-product: for $x \in k^\times$ and $\xi \in H^dk$, choose n such that $\xi \in H^d(k, \mu_{2^n}^{\otimes(d-1)})$ and consider the cohomology class $(x)_n \in H^1(k, \mu_{2^n})$ corresponding to the 2^n-th power class of x under the isomorphism $H^1(k, \mu_{2^n}) = k^\times/k^{\times 2^n}$ induced by the Kummer sequence; let then

$$x \cdot \xi = (x)_n \cup \xi \in H^{d+1}(k, \mu_{2^n}^{\otimes d}) \subset H^{d+1}k.$$

In particular, if $d = 1$ and ξ is the square class of $y \in k^\times$ under the embedding (1), then $x \cdot \xi$ is the Brauer class of the quaternion algebra $(x, y)_k$.
Throughout the paper, we denote by A a central simple k-algebra of even degree $2n$, and by σ an orthogonal involution of A. Recall from [7, (7.2)] that $\operatorname{disc}\sigma \in k^\times/k^{\times 2} \subset H^1k$ is the square class of $(-1)^n \operatorname{Nrd}_A(a)$ where $a \in A^\times$ is an arbitrary skew-symmetric element. Let Z be the center of the Clifford algebra $C(A, \sigma)$; thus, Z is a quadratic étale k-algebra, $Z = k[\sqrt{\operatorname{disc}\sigma}]$, see [7, (8.10)]. The following relation between similitudes and the discriminant is proved in [14, Theorem A] (see also [7, (13.38)]):

Theorem 1. *Let (A, σ) be a central simple k-algebra with orthogonal involution of even degree. For $\lambda \in G(A, \sigma)$,*

$$\lambda \cdot \operatorname{disc}\sigma = \begin{cases} 0 & \text{if } \lambda \in G_+(A, \sigma), \\ [A] & \text{if } \lambda \in G_-(A, \sigma). \end{cases}$$

For $d = 2$ (resp. 3), let $(H^dk)/A$ be the factor group of H^dk by the subgroup $\{0, [A]\}$ (resp. by the subgroup $k^\times \cdot [A]$). Theorem 1 thus shows that for $\lambda \in G(A, \sigma)$

$$\lambda \cdot \operatorname{disc}\sigma = 0 \quad \text{in } (H^2k)/A.$$

Our main results are Theorems 2, 3, 4, and 5 below.

Theorem 2. *Suppose A is split by Z. There exists an element $\gamma(\sigma) \in H^2k$ such that $\gamma(\sigma)_Z = [C(A,\sigma)]$ in H^2Z. For $\lambda \in G(A,\sigma)$,*

$$\lambda \cdot \gamma(\sigma) = 0 \quad \textit{in } (H^3k)/A.$$

Remark 1. In the conditions of the theorem, the element $\gamma(\sigma) \in H^2k$ is not uniquely determined if $Z \not\simeq k \times k$. Nevertheless, if $\lambda \cdot \operatorname{disc}\sigma = 0$ in $(H^2k)/A$, then $\lambda \cdot \gamma(\sigma) \in (H^3k)/A$ is uniquely determined. Indeed, if γ, $\gamma' \in H^2k$ are such that $\gamma_Z = \gamma'_Z$, then there exists $u \in k^\times$ such that $\gamma' = \gamma + u \cdot \operatorname{disc}\sigma$, hence

$$\lambda \cdot \gamma' = \lambda \cdot \gamma + \lambda \cdot u \cdot \operatorname{disc}\sigma.$$

The last term vanishes in $(H^3k)/A$ since $\lambda \cdot \operatorname{disc}\sigma = 0$ in $(H^2k)/A$.

The proof of Theorem 2 is given in Section 1. It shows that in the split case, where $A = \operatorname{End}_k V$ and σ is adjoint to some quadratic form q on V, we may take for $\gamma(\sigma)$ the Brauer class of the full Clifford algebra $C(V,q)$. Note that the statement of Theorem 2 does not discriminate between multipliers of proper and improper similitudes, but Theorem 1 may be used to distinguish between them. Slight variations of the arguments in the proof of Theorem 2 also yield the following result on multipliers of *proper* similitudes:

Theorem 3. *Suppose the Schur index of A is at most 4. If $\lambda \in G_+(A,\sigma)$, then there exists $z \in Z^\times$ such that $\lambda = N_{Z/k}(z)$ and*

$$N_{Z/k}\bigl(z \cdot [C(A,\sigma)]\bigr) = 0 \quad \textit{in } (H^3k)/A.$$

The proof is given in Section 1. Note however that the theorem holds without the hypothesis that $\operatorname{ind} A \leq 4$, as follows from Corollaries 1.20 and 1.21 in [12]. Using the Rost invariant of Spin groups, these corollaries actually yield an explicit element z as in Theorem 3 from any proper similitude with multiplier λ.

Remark 2. The element $N_{Z/k}\bigl(z \cdot [C(A,\sigma)]\bigr) \in (H^3k)/A$ depends only on $N_{Z/k}(z)$ and not on the specific choice of $z \in Z$. Indeed, if z, $z' \in Z^\times$ are such that $N_{Z/k}(z) = N_{Z/k}(z')$, then Hilbert's Theorem 90 yields an element $u \in Z^\times$ such that, denoting by ι the nontrivial automorphism of Z/k,

$$z' = zu\iota(u)^{-1},$$

hence

$$N_{Z/k}\bigl(z' \cdot [C(A,\sigma)]\bigr) = \\ N_{Z/k}\bigl(z \cdot [C(A,\sigma)]\bigr) + N_{Z/k}\bigl(u \cdot [C(A,\sigma)]\bigr) - N_{Z/k}\bigl(\iota(u) \cdot [C(A,\sigma)]\bigr).$$

Since $N_{Z/k} \circ \iota = N_{Z/k}$ and since the properties of the Clifford algebra (see [7, (9.12)]) yield

$$[C(A,\sigma)] - \iota[C(A,\sigma)] = [A]_Z,$$

it follows that

$$N_{Z/k}\big(u\cdot[C(A,\sigma)]\big)-N_{Z/k}\big(\iota(u)\cdot[C(A,\sigma)]\big)=N_{Z/k}\big(u\cdot[A]_Z\big).$$

By the projection formula, the right side is equal to $N_{Z/k}(u)\cdot[A]$. The claim follows.

Remark 3. Theorems 2 and 3 coincide when they both apply, i.e., if A is split by Z (hence $\operatorname{ind} A=1$ or 2), and $\lambda\in G_+(A,\sigma)$. Indeed, if $\lambda=N_{Z/k}(z)$ and $\gamma(\sigma)_Z=[C(A,\sigma)]$ then the projection formula yields

$$N_{Z/k}\big(z\cdot[C(A,\sigma)]\big)=\lambda\cdot\gamma(\sigma).$$

Remarkably, the conditions in Theorems 1 and 2 turn out to be sufficient for λ to be the multiplier of a similitude when $\deg A\le 6$ or when the virtual cohomological 2-dimension[3] of k is at most 2.

THEOREM 4. *Suppose $n\le 3$, i.e.,* $\deg A\le 6$.

- *If A is not split by Z , then every similitude is proper,*

$$G(A,\sigma)=G_+(A,\sigma),\qquad G_-(A,\sigma)=\varnothing.$$

Moreover, for $\lambda\in k^\times$, we have $\lambda\in G(A,\sigma)$ if and only if there exists $z\in Z^\times$ such that $\lambda=N_{Z/k}(z)$ and

$$N_{Z/k}\big(z\cdot[C(A,\sigma)]\big)=0\quad\textit{in } (H^3k)/A.$$

- *If A is split by Z, let $\gamma(\sigma)\in H^2k$ be as in Theorem 2. For $\lambda\in k^\times$, we have $\lambda\in G(A,\sigma)$ if and only if*

$$\lambda\cdot\operatorname{disc}\sigma=0\ \textit{in } (H^2k)/A\qquad\textit{and}\qquad\lambda\cdot\gamma(\sigma)=0\ \textit{in } (H^3k)/A.$$

The proof is given in Section 2.

Note that if $\deg A=2$, then A is necessarily split by Z and we may choose $\gamma(\sigma)=0$, hence Theorem 4 simplifies to

$$\lambda\in G(A,\sigma)\quad\text{if and only if}\quad\lambda\cdot\operatorname{disc}\sigma=0\text{ in }(H^2k)/A,$$

a statement which is easily proved directly. (See [14, p. 15] or [7, (12.25)].)

If $\deg A=4$, multipliers of similitudes can also be described up to squares as reduced norms from a central simple algebra E of degree 4 such that $[E]=\gamma(\sigma)$ if A is split by Z (see Corollary 4.5) or as norms of reduced norms of $C(A,\sigma)$ if A is not split by Z (see Corollary 2.1).

[3]The authors are grateful to Parimala for her suggestion to investigate the case of low cohomological dimension.

For the next statement, recall that the virtual cohomological 2-dimension of k (denoted $\mathrm{vcd}_2\,k$) is the cohomological 2-dimension of $k(\sqrt{-1})$. If v is an ordering of k, we let k_v be a real closure of k for v and denote simply by $(A,\sigma)_v$ the algebra with involution $(A\otimes_k k_v, \sigma\otimes \mathrm{Id}_{k_v})$.

Theorem 5. *Suppose* $\mathrm{vcd}_2\,k\le 2$, *and A is split by Z. For $\lambda\in k^\times$, we have $\lambda\in G(A,\sigma)$ if and only if*

$$\lambda>0 \quad \textit{at every ordering } v \textit{ of } k \textit{ such that } (A,\sigma)_v \textit{ is not hyperbolic},$$
$$\lambda\cdot\operatorname{disc}\sigma=0 \textit{ in } (H^2k)/A \qquad \textit{and} \qquad \lambda\cdot\gamma(\sigma)=0 \textit{ in } (H^3k)/A.$$

The proof is given in Section 3.

1 Proofs of Theorems 2 and 3

Theorems 2 and 3 are proved by reduction to the split case, which we consider first. We thus assume $A=\mathrm{End}_k V$ for some k-vector space V of dimension $2n$, and σ is adjoint to a quadratic form q on V. Then $\operatorname{disc}\sigma=\operatorname{disc}q$ and $C(A,\sigma)$ is the even Clifford algebra $C(A,\sigma)=C_0(V,q)$. We denote by $C(V,q)$ the full Clifford algebra of q, which is a central simple k-algebra, and by I^mk the m-th power of the fundamental ideal Ik of the Witt ring Wk.

Lemma 1.1. *For $\lambda\in k^\times$, the following conditions are equivalent:*

(a) $\lambda\cdot\operatorname{disc}q=0$ *in* H^2k *and* $\lambda\cdot[C(V,q)]=0$ *in* H^3k;

(b) $\langle\lambda\rangle\cdot q\equiv q \bmod I^4k$.

Proof. For $\alpha_1,\ldots,\alpha_m\in k^\times$, let

$$\langle\!\langle \alpha_1,\ldots,\alpha_m\rangle\!\rangle=\langle 1,-\alpha_1\rangle\otimes\cdots\otimes\langle 1,-\alpha_m\rangle.$$

Let $e_2\colon I^2k\to H^2k$ be the Witt invariant and $e_3\colon I^3k\to H^3k$ be the Arason invariant. By a theorem of Merkurjev [9] (resp. of Merkurjev–Suslin [13] and Rost [17]), we have $\ker e_2=I^3k$ and $\ker e_3=I^4k$. Therefore, the lemma follows if we prove

$$\lambda\cdot\operatorname{disc}q=0 \quad\text{if and only if}\quad \langle\!\langle\lambda\rangle\!\rangle\cdot q\in I^3k, \tag{2}$$

and that, assuming that condition holds,

$$e_3(\langle\!\langle\lambda\rangle\!\rangle\cdot q)=\lambda\cdot[C(V,q)]. \tag{3}$$

Let $\delta\in k^\times$ be such that $\operatorname{disc}q=(\delta)_1\in H^1(k,\mathbf{Z}/2\mathbf{Z})\subset H^1k$. Then

$$q\equiv\langle\!\langle\delta\rangle\!\rangle \bmod I^2k, \tag{4}$$

hence

$$e_2(\langle\!\langle\lambda\rangle\!\rangle\cdot q)=e_2(\langle\!\langle\lambda,\delta\rangle\!\rangle)=\lambda\cdot\operatorname{disc}q,$$

proving (2). Now, assuming $\lambda\cdot\operatorname{disc} q = 0$, we have $\langle\!\langle\lambda,\delta\rangle\!\rangle = 0$ in Wk, hence

$$\langle\!\langle\lambda\rangle\!\rangle\cdot q = \langle\!\langle\lambda\rangle\!\rangle\cdot(q\perp\langle\!\langle\delta\rangle\!\rangle).$$

By (4), we have $q\perp\langle\!\langle\delta\rangle\!\rangle\in I^2k$, hence

$$e_3(\langle\!\langle\lambda\rangle\!\rangle\cdot q) = \lambda\cdot e_2(q\perp\langle\!\langle\delta\rangle\!\rangle). \tag{5}$$

The computation of Witt invariants in [8, Chapter 5] yields

$$e_2(q\perp\langle\!\langle\delta\rangle\!\rangle) = [C(V,q)] + (-1)\cdot\operatorname{disc} q. \tag{6}$$

Since $\lambda\cdot\operatorname{disc} q = 0$ by hypothesis, (3) follows from (5) and (6). □

Proof of Theorem 2. If A is split, then using the same notation as in Lemma 1.1 we may take $\gamma(\sigma) = [C(V,q)]$, and Theorem 2 readily follows from Lemma 1.1. For the rest of the proof, we may thus assume A is not split, hence $\operatorname{disc}\sigma\neq 0$ since Z is assumed to split A. Let $G = \{\mathrm{Id},\iota\}$ be the Galois group of Z/k. The properties of the Clifford algebra (see for instance [7, (9.12)]) yield

$$[C(A,\sigma)] - \iota[C(A,\sigma)] = [A]_Z = 0.$$

Therefore, $[C(A,\sigma)]$ lies in the subgroup $(\operatorname{Br} Z)^G$ of $\operatorname{Br} Z$ fixed under the action of G. The "Teichmüller cocycle" theory [6] (or the spectral sequence of group extensions, see [19, Remarque, p. 126]) yields an exact sequence

$$\operatorname{Br} k \to (\operatorname{Br} Z)^G \to H^3(G, Z^\times).$$

Since G is cyclic, $H^3(G,Z^\times) = H^1(G,Z^\times)$. By Hilbert's Theorem 90, $H^1(G,Z^\times) = 1$, hence $(\operatorname{Br} Z)^G$ is the image of the scalar extension map $\operatorname{Br} k\to\operatorname{Br} Z$, and there exists $\gamma(\sigma)\in\operatorname{Br} k$ such that $\gamma(\sigma)_Z = [C(A,\sigma)]$. Then, by [7, (9.12)],

$$2\gamma(\sigma) = N_{Z/k}\big([C(A,\sigma)]\big) = \begin{cases} 0 & \text{if } n \text{ is odd,}\\ [A] & \text{if } n \text{ is even,}\end{cases} \tag{7}$$

hence $4\gamma(\sigma) = 0$. Therefore, $\gamma(\sigma)\in\operatorname{Br}_2(k) = H^2k$.
Note that $\operatorname{ind} A = 2$, since A is split by the quadratic extension Z/k, hence A is Brauer-equivalent to a quaternion algebra Q. Let X be the conic associated with Q; the function field $k(X)$ splits A. Since Theorem 2 holds in the split case, we have

$$\lambda\cdot\gamma(\sigma)\in\ker\big(H^3k\to H^3k(X)\big).$$

By a theorem of (Arason–) Peyre [16, Proposition 4.4], the kernel on the right side is the subgroup $k^\times\cdot[A]\subset H^3k$, hence

$$\lambda\cdot\gamma(\sigma) = 0 \quad \text{in } (H^3k)/A.$$

□

Proof of Theorem 3. Suppose first A is split, and use the same notation as in Lemma 1.1. If $\lambda \in G(A,\sigma)$, then $\langle\lambda\rangle \cdot q \simeq q$ and Lemma 1.1 yields

$$\lambda \cdot \operatorname{disc} q = 0 \text{ in } H^2k \qquad \text{and} \qquad \lambda \cdot [C(V,q)] = 0 \text{ in } H^3k.$$

The first equation implies that $\lambda = N_{Z/k}(z)$ for some $z \in Z^\times$. Since

$$[C(A,\sigma)] = [C_0(V,q)] = [C(V,q)]_Z,$$

the projection formula yields

$$N_{Z/k}\big(z \cdot [C(A,\sigma)]\big) = N_{Z/k}(z) \cdot [C(V,q)] = \lambda \cdot [C(V,q)] = 0,$$

proving the theorem if A is split.
If A is not split, we extend scalars to the function field $k(X)$ of the Severi–Brauer variety of A. For $\lambda \in G_+(A,\sigma)$, there still exists $z \in Z^\times$ such that $\lambda = N_{Z/k}(z)$, by Theorem 1. Since Theorem 3 holds in the split case, we have

$$N_{Z/k}\big(z \cdot [C(A,\sigma)]\big) \in \ker\big(H^3k \to H^3k(X)\big),$$

and Peyre's theorem concludes the proof. (Note that applying Peyre's theorem requires the hypothesis that $\operatorname{ind} A \le 4$.) □

2 Algebras of low degree

We prove Theorem 4 by considering separately the cases $\operatorname{ind} A = 1$, 2, and 4.

2.1 Case 1: A is split

Let $A = \operatorname{End}_k V$, $\dim V \le 6$, and let σ be adjoint to a quadratic form q on V. Since $C(A,\sigma) = C_0(V,q)$, we may choose $\gamma(\sigma) = [C(V,q)]$. The equations

$$\lambda \cdot \operatorname{disc} \sigma = 0 \text{ in } (H^2k)/A \qquad \text{and} \qquad \lambda \cdot \gamma(\sigma) = 0 \text{ in } (H^3k)/A$$

are then equivalent to

$$\lambda \cdot \operatorname{disc} q = 0 \text{ in } H^2k \qquad \text{and} \qquad \lambda \cdot [C(V,q)] = 0 \text{ in } H^3k,$$

hence, by Lemma 1.1, to $\langle\langle\lambda\rangle\rangle \cdot q \in I^4k$. Since $\dim q = 6$, the Arason–Pfister Hauptsatz [8, Chapter 10, Theorem 3.1] shows that this relation holds if and only if $\langle\langle\lambda\rangle\rangle \cdot q = 0$, i.e., $\lambda \in G(V,q) = G(A,\sigma)$, and the proof is complete.

2.2 Case 2: $\operatorname{ind} A = 2$

Let Q be a quaternion (division) algebra Brauer-equivalent to A. We represent A as $A = \operatorname{End}_Q U$ for some 3-dimensional (right) Q-vector space. The involution σ is then adjoint to a skew-hermitian form h on U (with respect to the conjugation involution on Q), which defines an element in the Witt group

$W^{-1}(Q)$. Let X be the conic associated with Q. The function field $k(X)$ splits Q, hence Morita equivalence yields an isomorphism

$$W^{-1}\big(Q \otimes k(X)\big) \simeq Wk(X).$$

Moreover, Dejaiffe [4] and Parimala–Sridharan–Suresh [15] have shown that the scalar extension map

$$W^{-1}(Q) \to W^{-1}\big(Q \otimes k(X)\big) \simeq Wk(X) \tag{8}$$

is injective. Let (V, q) be a quadratic space over $k(X)$ representing the image of (U, h) under (8). We may assume $\dim V = \deg A \leq 6$ and σ is adjoint to q after scalar extension to $k(X)$. An element $\lambda \in k^\times$ lies in $G(V, q)$ if and only if $\langle\langle\lambda\rangle\rangle \cdot q = 0$; by the injectivity of (8), this condition is also equivalent to $\langle\langle\lambda\rangle\rangle \cdot h = 0$ in $W^{-1}(Q)$, i.e., to $\lambda \in G(A, \sigma)$. Therefore,

$$G(V, q) \cap k^\times = G(A, \sigma). \tag{9}$$

Suppose first A is not split by Z. Theorem 1 then shows that every similitude of (A, σ) is proper, and it only remains to show that if $\lambda = N_{Z/k}(z)$ for some $z \in Z^\times$ such that

$$N_{Z/k}\big(z \cdot [C(A, \sigma)]\big) = 0 \quad \text{in } (H^3k)/A,$$

then $\lambda \in G(A, \sigma)$. Extending scalars to $k(X)$, we derive from the last equation by the projection formula

$$N_{Z(X)/k(X)}(z) \cdot [C(V, q)] = 0 \quad \text{in } H^3k(X).$$

Therefore, by Lemma 1.1, $\langle\lambda\rangle \cdot q \equiv q \bmod I^4k(X)$, i.e.,

$$\langle\langle\lambda\rangle\rangle \cdot q \in I^4k(X).$$

Since $\dim q \leq 6$, the Arason–Pfister Hauptsatz implies $\langle\langle\lambda\rangle\rangle \cdot q = 0$, hence $\lambda \in G(V, q)$ and therefore $\lambda \in G(A, \sigma)$ by (9). Theorem 4 is thus proved when $\operatorname{ind} A = 2$ and A is not split by Z.

Suppose next A is split by Z. In view of Theorems 1 and 2, it suffices to show that if $\lambda \in k^\times$ satisfies

$$\lambda \cdot \operatorname{disc} \sigma = 0 \text{ in } (H^2k)/A \qquad \text{and} \qquad \lambda \cdot \gamma(\sigma) = 0 \text{ in } (H^3k)/A,$$

then $\lambda \in G(A, \sigma)$. Again, extending scalars to $k(X)$, the conditions become

$$\lambda \cdot \operatorname{disc} q = 0 \text{ in } H^2k(X) \qquad \text{and} \qquad \lambda \cdot [C(V, q)] = 0 \text{ in } H^3k(X).$$

By Lemma 1.1, these equations imply $\langle\langle\lambda\rangle\rangle \cdot q \in I^4k(X)$, hence $\langle\langle\lambda\rangle\rangle \cdot q = 0$ by the Arason–Pfister Hauptsatz since $\dim q \leq 6$. It follows that $\lambda \in G(V, q)$, hence $\lambda \in G(A, \sigma)$ by (9).

2.3 Case 3: $\operatorname{ind} A = 4$

Since $\deg A \le 6$, this case arises only if $\deg A = 4$, i.e., A is a division algebra. This division algebra cannot be split by the quadratic k-algebra Z, hence all the similitudes are proper, by Theorem 1. Theorem 3 shows that if $\lambda \in G(A,\sigma)$, then there exists $z \in Z^\times$ such that $\lambda = N_{Z/k}(z)$ and $N_{Z/k}\big(z \cdot [C(A,\sigma)]\big) = 0$ in $(H^3k)/A$, and it only remains to prove the converse.

Let $z \in Z^\times$ be such that $N_{Z/k}\big(z \cdot [C(A,\sigma)]\big) = u \cdot [A]$ for some $u \in k^\times$. Since by [7, (9.12)], $N_{Z/k}\big([C(A,\sigma)]\big) = [A]$, it follows that

$$N_{Z/k}\big(u^{-1}z \cdot [C(A,\sigma)]\big) = 0 \quad \text{in } H^3k. \tag{10}$$

Since $\deg A = 4$, the Clifford algebra $C(A,\sigma)$ is a quaternion algebra over Z. Let

$$C(A,\sigma) = (z_1, z_2)_Z.$$

Suppose first $\operatorname{disc}\sigma \neq 0$, i.e., Z is a field. Let $s\colon Z \to k$ be a k-linear map such that $s(1) = 0$, and let $s_*\colon WZ \to Wk$ be the corresponding (Scharlau) transfer map. By [2, Satz 3.3, Satz 4.18], Equation (10) yields

$$s_*\big(\langle\!\langle u^{-1}z, z_1, z_2\rangle\!\rangle\big) \in I^4k.$$

However, the form $s_*\big(\langle\!\langle u^{-1}z, z_1, z_2\rangle\!\rangle\big)$ is isotropic since $\langle\!\langle u^{-1}z, z_1, z_2\rangle\!\rangle$ represents 1 and $s(1) = 0$. Moreover, its dimension is 2^4, hence the Arason–Pfister Hauptsatz implies

$$s_*\big(\langle\!\langle u^{-1}z, z_1, z_2\rangle\!\rangle\big) = 0 \quad \text{in } Wk.$$

It follows that

$$s_*\big(\langle u^{-1}z\rangle \cdot \langle\!\langle z_1, z_2\rangle\!\rangle\big) = s_*\big(\langle\!\langle z_1, z_2\rangle\!\rangle\big),$$

hence the form on the left side is isotropic. Therefore, the form $\langle u^{-1}z\rangle \cdot \langle\!\langle z_1, z_2\rangle\!\rangle$ represents an element $v \in k^\times$. Then $v^{-1}u^{-1}z$ is represented by $\langle\!\langle z_1, z_2\rangle\!\rangle$, which is the reduced norm form of $C(A,\sigma)$, hence $z \in k^\times \operatorname{Nrd}(C(A,\sigma)^\times)$, and

$$N_{Z/k}(z) \in k^{\times 2} N_{Z/k}\big(\operatorname{Nrd}(C(A,\sigma)^\times)\big).$$

By [7, (15.11)], the group on the right is $G_+(A,\sigma)$. We have thus proved $N_{Z/k}(z) \in G(A,\sigma)$, and the proof is complete when Z is a field.

Suppose finally $\operatorname{disc}\sigma = 0$, i.e., $Z \simeq k \times k$. Then $C(A,\sigma) \simeq C' \times C''$ for some quaternion k-algebras $C' = (z_1', z_2')_k$ and $C'' = (z_1'', z_2'')_k$, and [7, (15.13)] shows

$$G(A,\sigma) = \operatorname{Nrd}(C'^\times)\operatorname{Nrd}(C''^\times).$$

We also have $z = (z', z'')$ for some z', $z'' \in k^\times$, and (10) becomes

$$u^{-1}z' \cdot [C'] + u^{-1}z'' \cdot [C''] = 0 \quad \text{in } H^3k.$$

It follows that

$$\langle\!\langle u^{-1}z', z_1', z_2'\rangle\!\rangle \simeq \langle\!\langle u^{-1}z'', z_1'', z_2''\rangle\!\rangle.$$

By [2, Lemma 1.7], there exists $v \in k^\times$ such that

$$\langle\langle u^{-1}z', z_1', z_2'\rangle\rangle \simeq \langle\langle v, z_1', z_2'\rangle\rangle \simeq \langle\langle v, z_1'', z_2''\rangle\rangle \simeq \langle\langle u^{-1}z'', z_1'', z_2''\rangle\rangle,$$

hence $v^{-1}u^{-1}z' \in \operatorname{Nrd}(C')$ and $v^{-1}u^{-1}z'' \in \operatorname{Nrd}(C'')$. Therefore,

$$N_{Z/k}(z) = z'z'' \in \operatorname{Nrd}(C'^\times)\operatorname{Nrd}(C''^\times),$$

and the proof of Theorem 4 is complete.
To finish this section, we compare the descriptions of $G_+(A,\sigma)$ for $\deg A = 4$ or 6 in [7] with those which follow from Theorem 4 (and Remark 3).

COROLLARY 2.1. *Suppose* $\deg A = 4$. *If* $\operatorname{disc}\sigma \neq 0$, *then*

$$\begin{aligned} G_+(A,\sigma) &= k^{\times 2} N_{Z/k}\big(\operatorname{Nrd}(C(A,\sigma)^\times)\big) \\ &= \{N_{Z/k}(z) \mid N_{Z/k}\big(z\cdot[C(A,\sigma)]\big) = 0 \text{ in } (H^3k)/A\}. \end{aligned}$$

If $\operatorname{disc}\sigma = 0$, *then* $C(A,\sigma) \simeq C' \times C''$ *for some quaternion* k*-algebras* C', C'', *and*

$$\begin{aligned} G_+(A,\sigma) &= \operatorname{Nrd}(C'^\times)\operatorname{Nrd}(C''^\times) \\ &= \{z'z'' \mid z'\cdot[C'] + z''\cdot[C''] = 0 \text{ in } (H^3k)/A\}. \end{aligned}$$

Proof. See [7, (15.11)] for the case $\operatorname{disc}\sigma \neq 0$ and [7, (15.13)] for the case $\operatorname{disc}\sigma = 0$. □

COROLLARY 2.2. *Suppose* $\deg A = 6$. *If* $\operatorname{disc}\sigma \neq 0$, *let* ι *be the nontrivial automorphism of the field extension* Z/k *and let* $\underline{\sigma}$ *be the canonical (unitary) involution of* $C(A,\sigma)$. *Let also*

$$\operatorname{GU}(C(A,\sigma),\underline{\sigma}) = \{g \in C(A,\sigma) \mid \underline{\sigma}(g)g \in k^\times\}.$$

Then

$$\begin{aligned} &G_+(A,\sigma) = \\ &\quad\{N_{Z/k}(z) \mid z\iota(z)^{-1} = (\underline{\sigma}(g)g)^{-2}\operatorname{Nrd}(g) \text{ for some } g \in \operatorname{GU}(C(A,\sigma),\underline{\sigma})\} \\ &\qquad\qquad = \{N_{Z/k}(z) \mid N_{Z/k}\big(z\cdot[C(A,\sigma)]\big) = 0 \text{ in } (H^3k)/A\}. \end{aligned}$$

If $\operatorname{disc}\sigma = 0$, *then* $C(A,\sigma) \simeq C \times C^{\mathrm{op}}$ *for some central simple* k*-algebra* C *of degree* 4, *and*

$$\begin{aligned} G_+(A,\sigma) &= k^{\times 2}\operatorname{Nrd}(C^\times) \\ &= \{z \in k^\times \mid z\cdot[C] = 0 \text{ in } (H^3k)/A\}. \end{aligned}$$

Proof. See [7, (15.31)] for the case $\operatorname{disc}\sigma \neq 0$ and [7, (15.34)] for the case $\operatorname{disc}\sigma = 0$. In the latter case, Theorem 3 shows that $G_+(A,\sigma)$ consists of products $z'z''$ where z', $z'' \in k^\times$ are such that

$$z' \cdot [C] + z'' \cdot [C^{\mathrm{op}}] = 0 \quad \text{in } (H^3k)/A.$$

However, $[C^{\mathrm{op}}] = -[C]$, and $2[C] = [A]$ by [7, (9.15)], hence

$$z' \cdot [C] + z'' \cdot [C^{\mathrm{op}}] = z'z'' \cdot [C] \quad \text{in } (H^3k)/A.$$

Note that the equation

$$k^{\times 2}\operatorname{Nrd}(C^\times) = \{z \in k^\times \mid z \cdot [C] = 0 \text{ in } (H^3k)/A\}$$

can also be proved directly by a theorem of Merkurjev [11, Proposition 1.15]. □

3 Fields of low virtual cohomological dimension

Our goal in this section is to prove Theorem 5. Together with Theorem 2, the following lemma completes the proof of the "only if" part:

Lemma 3.1. *If $\lambda \in G(A,\sigma)$, then $\lambda > 0$ at every ordering v such that $(A,\sigma)_v$ is not hyperbolic.*

Proof. If $(A,\sigma)_v$ is not hyperbolic, then A_v is split, by [18, Chapter 10, Theorem 3.7]. We may thus represent $A_v = \operatorname{End}_{k_v} V$ for some k_v-vector space V, and $\sigma \otimes \operatorname{Id}_{k_v}$ is adjoint to a non-hyperbolic quadratic form q. If $\lambda \in G(A,\sigma)$, then $\lambda \in G(V,q)$, hence

$$\langle\lambda\rangle \cdot q \simeq q.$$

Comparing the signatures of each side, we obtain $\lambda > 0$. □

For the "if" part, we use the following lemma:

Lemma 3.2. *Let F be an arbitrary field of characteristic different from 2. If $\operatorname{vcd}_2 F \leq 3$, then the torsion part of the 4-th power of IF is trivial,*

$$I_t^4 F = 0.$$

Proof. Our proof uses the existence of the cohomological invariants $e_n\colon I^nF \to H^n(F,\mu_2)$, and the fact that $\ker e_n = I^{n+1}F$, proved for fields of virtual cohomological 2-dimension at most 3 by Arason–Elman–Jacob [3].
Suppose first $-1 \notin F^{\times 2}$. From $\operatorname{vcd}_2 F \leq 3$, it follows that $H^n(F(\sqrt{-1}),\mu_2) = 0$ for $n \geq 4$, hence the Arason exact sequence

$$H^n(F(\sqrt{-1}),\mu_2) \xrightarrow{N} H^n(F,\mu_2) \xrightarrow{(-1)_1\cup} H^{n+1}(F,\mu_2) \to H^{n+1}(F(\sqrt{-1}),\mu_2)$$

(see [2, Corollar 4.6] or [7, (30.12)]) shows that the cup-product with $(-1)_1$ is an isomorphism $H^n(F,\mu_2) \simeq H^{n+1}(F,\mu_2)$ for $n \geq 4$. If $q \in I_t^4 F$, there is an integer ℓ such that $2^\ell q = 0$, hence the 4-th invariant $e_4(q) \in H^4(F,\mu_2)$ satisfies

$$\underbrace{(-1)_1 \cup \cdots \cup (-1)_1}_{\ell} \cup e_4(q) = 0 \quad \text{in } H^{\ell+4}(F,\mu_2).$$

Since $(-1)_1\cup$ is an isomorphism, it follows that $e_4(q) = 0$, hence $q \in I_t^5 F$. Repeating the argument with e_5, e_6, ..., we obtain $q \in \bigcap_n I^n F$, hence $q = 0$ by the Arason–Pfister Hauptsatz [8, p. 290].

If $-1 \in F^{\times 2}$, then the hypothesis implies that $H^n(F,\mu_2) = 0$ for $n \geq 4$, hence for $q \in I^4 F$ we get successively $e_4(q) = 0$, $e_5(q) = 0$, etc., and we conclude as before. □

Proof of Theorem 5. As observed above, the "only if" part follows from Theorem 2 and Lemma 3.1. The proof of the "if" part uses the same arguments as the proof of Theorem 2 in the case where $\operatorname{ind} A = 2$.

We first consider the split case. If $A = \operatorname{End}_k V$ and σ is adjoint to a quadratic form q on V, then we may choose $\gamma(\sigma) = C(V,q)$, and the conditions

$$\lambda \cdot \operatorname{disc}\sigma = 0 \text{ in } (H^2 k)/A \qquad \text{and} \qquad \lambda \cdot \gamma(\sigma) = 0 \text{ in } (H^3 k)/A$$

imply, by Lemma 1.1, that $\langle\langle\lambda\rangle\rangle \cdot q \in I^4 k$. Moreover, for every ordering v on k, the signature $\operatorname{sgn}_v(\langle\langle\lambda\rangle\rangle \cdot q)$ vanishes, since $\lambda > 0$ at every v such that $\operatorname{sgn}_v(q) \neq 0$. Therefore, by Pfister's local-global principle [8, Chapter 8, Theorem 4.1], $\langle\langle\lambda\rangle\rangle \cdot q$ is torsion. Since the hypothesis on k implies, by Lemma 3.2, that $I_t^4 k = 0$, we have $\langle\langle\lambda\rangle\rangle \cdot q = 0$, hence $\lambda \in G(V,q) = G(A,\sigma)$. Note that Lemma 3.2 yields $I_t^4 k = 0$ under the weaker hypothesis $\operatorname{vcd}_2 k \leq 3$. Therefore, the split case of Theorem 5 holds when $\operatorname{vcd}_2 k \leq 3$.

Now, suppose A is not split. Since A is split by Z, it is Brauer-equivalent to a quaternion algebra Q. Let $k(X)$ be the function field of the conic X associated with Q. This field splits A, hence there is a quadratic space (V,q) over $k(X)$ such that $A \otimes k(X)$ may be identified with $\operatorname{End}_{k(X)} V$ and $\sigma \otimes \operatorname{Id}_{k(X)}$ with the adjoint involution with respect to q. As in Section 2 (see Equation (9)), we have

$$G(V,q) \cap k^\times = G(A,\sigma).$$

Therefore, it suffices to show that the conditions on λ imply $\lambda \in G(V,q)$.

If v is an ordering of k such that $(A,\sigma)_v$ is hyperbolic, then q_w is hyperbolic for any ordering w of $k(X)$ extending v, since hyperbolic involutions remain hyperbolic over scalar extensions. Therefore, $\lambda > 0$ at every ordering w of $k(X)$ such that q_w is not hyperbolic. Moreover, the conditions

$$\lambda \cdot \operatorname{disc}\sigma = 0 \text{ in } (H^2 k)/A \qquad \text{and} \qquad \lambda \cdot \gamma(\sigma) = 0 \text{ in } (H^3 k)/A$$

imply

$$\lambda \cdot \operatorname{disc} q = 0 \text{ in } H^2 k(X) \qquad \text{and} \qquad \lambda \cdot [C(V,q)] = 0 \text{ in } H^3 k(X).$$

Since X is a conic, Proposition 11, p. 93 of [20] implies

$$\operatorname{vcd}_2 k(X) = 1 + \operatorname{vcd}_2 k \leq 3.$$

As Theorem 5 holds in the split case over fields of virtual cohomological 2-dimension at most 3, it follows that $\lambda \in G(V, q)$. $\square$

Remark. The same arguments show that if $\operatorname{vcd}_2 k \leq 2$ and $\operatorname{ind} A = 2$, then $G_+(A, \sigma)$ consists of the elements $N_{Z/k}(z)$ where $z \in Z^\times$ is such that

$$N_{Z/k}\big(z \cdot [C(A, \sigma)]\big) = 0 \quad \text{in } (H^3 k)/A.$$

4 Examples

In this section, we give an explicit description of the element $\gamma(\sigma)$ of Theorem 2 in some special cases. Throughout this section, we assume the algebra A is not split, and is split by Z (hence Z is a field and $\operatorname{disc} \sigma \neq 0$). Our first result is easy:

Proposition 4.1. *If A is split by Z and σ becomes hyperbolic after scalar extension to Z, then we may choose $\gamma(\sigma) = 0$.*

Proof. Let ι be the nontrivial automorphism of Z/k. Since Z is the center of $C(A, \sigma)$,

$$C(A, \sigma) \otimes_k Z \simeq C(A, \sigma) \times {}^{\iota}C(A, \sigma). \tag{11}$$

On the other hand, $C(A, \sigma) \otimes_k Z \simeq C(A \otimes_k Z, \sigma \otimes \mathrm{Id}_Z)$, and since σ becomes hyperbolic over Z, one of the components of $C(A \otimes_k Z, \sigma \otimes \mathrm{Id}_Z)$ is split, by [7, (8.31)]. Therefore,

$$[C(A, \sigma)] = [{}^{\iota}C(A, \sigma)] = 0 \qquad \text{in } \operatorname{Br} Z.$$

$\square$

Corollary 4.2. *In the conditions of Proposition 4.1, if $\deg A \leq 6$ or $\operatorname{vcd}_2 k \leq 2$, then*

$$G_+(A, \sigma) = \{\lambda \in k^\times \mid \lambda \cdot \operatorname{disc} \sigma = 0 \text{ in } H^2 k\}$$

and

$$G_-(A, \sigma) = \{\lambda \in k^\times \mid \lambda \cdot \operatorname{disc} \sigma = [A] \text{ in } H^2 k\}.$$

Proof. This readily follows from Proposition 4.1 and Theorem 2 or 5. $\square$

To give further examples where $\gamma(\sigma)$ can be computed, we fix a particular representation of A as follows. Since A is assumed to be split by Z, it is Brauer-equivalent to a quaternion k-algebra Q containing Z. We choose a quaternion basis 1, i, j, ij of Q such that $Z = k(i)$. Let $A = \operatorname{End}_Q U$ for some right Q-vector space U, and let σ be the adjoint involution of a skew-hermitian form h on U with respect to the conjugation involution on Q. For x, $y \in U$, we decompose

$$h(x,y) = f(x,y) + jg(x,y) \qquad \text{with } f(x,y),\ g(x,y) \in Z.$$

It is easily verified that f (resp. g) is a skew-hermitian (resp. symmetric bilinear) form on U viewed as a Z-vector space. (See [18, Chapter 10, Lemma 3.1].) We have

$$A \otimes_k Z = (\operatorname{End}_Q U) \otimes_k Z = \operatorname{End}_Z U.$$

Moreover, for x, $y \in U$ and $\varphi \in \operatorname{End}_Q U$, the equation

$$h\bigl(x, \varphi(y)\bigr) = h\bigl(\sigma(\varphi)(x), y\bigr)$$

implies

$$g\bigl(x, \varphi(y)\bigr) = g\bigl(\sigma(\varphi)(x), y\bigr),$$

hence $\sigma \otimes_k \mathrm{Id}_Z$ is adjoint to g.

Proposition 4.3. *With the notation above,*

$$[C(A,\sigma)] = [C(U,g)] \qquad \textit{in } \operatorname{Br} Z.$$

Proof. Since $\sigma \otimes \mathrm{Id}_Z$ is the adjoint involution of g,

$$C(A \otimes_k Z, \sigma \otimes \mathrm{Id}_Z) \simeq C_0(U,g). \tag{12}$$

Now, $\operatorname{disc}\sigma$ is a square in Z, hence $C_0(U,g)$ decomposes into a direct product

$$C_0(U,g) \simeq C' \times C'' \tag{13}$$

where C', C'' are central simple Z-algebras Brauer-equivalent to $C(U,g)$. The proposition follows from (11), (12), and (13). □

To give an explicit description of g, consider an h-orthogonal basis $(e_1, \dots, e_n)$ of U. In the corresponding diagonalization of h,

$$h \simeq \langle u_1, \dots, u_n \rangle,$$

each $u_\ell \in Q$ is a pure quaternion, since h is skew-hermitian. Let $u_\ell^2 = a_\ell \in k^\times$ for $\ell = 1, \dots, n$. Then

$$\operatorname{disc}\sigma = (-1)^n \operatorname{Nrd}(u_1) \dots \operatorname{Nrd}(u_n) = a_1 \dots a_n,$$

so we may assume $i^2 = a_1 \dots a_n$. Write

$$u_\ell = \mu_\ell i + j v_\ell \qquad \text{where } \mu_\ell \in k \text{ and } v_\ell \in Z. \tag{14}$$

Each $e_\ell Q$ is a 2-dimensional Z-vector space, and we have a g-orthogonal decomposition

$$U = e_1 Q \oplus \dots \oplus e_n Q.$$

If $v_\ell = 0$, then $g(e_\ell, e_\ell) = 0$, hence $e_\ell Q$ is hyperbolic. If $v_\ell \neq 0$, then $(e_\ell, e_\ell u_\ell)$ is a g-orthogonal basis of $e_\ell Q$, which yields the following diagonalization of the restriction of g:

$$\langle v_\ell, -a_\ell v_\ell \rangle.$$

Therefore,

$$g = g_1 + \cdots + g_n \tag{15}$$

where

$$g_\ell = \begin{cases} 0 & \text{if } v_\ell = 0, \\ \langle v_\ell \rangle \langle 1, -a_\ell \rangle & \text{if } v_\ell \neq 0. \end{cases} \tag{16}$$

We now consider in more detail the cases $n = 2$ and $n = 3$.

4.1 Algebras of degree 4

Suppose $\deg A = 4$, i.e., $n = 2$, and use the same notation as above. If $v_1 = 0$, then squaring each side of (14) yields $a_1 = \mu_1^2 a_1 a_2$, hence $a_2 \in k^{\times 2}$, a contradiction since Q is assumed to be a division algebra. The case $v_2 = 0$ leads to the same contradiction. Therefore, we necessarily have $v_1 \neq 0$ and $v_2 \neq 0$. By (15) and (16),

$$g = \langle v_1 \rangle \langle 1, -a_1 \rangle + \langle v_2 \rangle \langle 1, -a_2 \rangle,$$

hence by [8, p. 121],

$$\begin{aligned} [C(A, \sigma)] &= (a_1, v_1)_Z + (a_2, v_2)_Z + (a_1, a_2)_Z \\ &= (a_1, -v_1 v_2)_Z. \end{aligned} \tag{17}$$

Since the division algebra Q contains the pure quaternions u_1, u_2 and i with $u_1^2 = a_1$, $u_2^2 = a_2$ and $i^2 = a_1 a_2$, we have a_1, a_2, $a_1 a_2 \notin k^{\times 2}$ and we may consider the field extension

$$L = k(\sqrt{a_1}, \sqrt{a_2}).$$

We identify Z with a subfield of L by choosing in L a square root of $a_1 a_2$, and denote by ρ_1, ρ_2 the automorphisms of L/k defined by

$$\begin{aligned} \rho_1(\sqrt{a_1}) &= -\sqrt{a_1}, & \qquad \rho_2(\sqrt{a_1}) &= \sqrt{a_1}, \\ \rho_1(\sqrt{a_2}) &= \sqrt{a_2}, & \qquad \rho_2(\sqrt{a_2}) &= -\sqrt{a_2}. \end{aligned}$$

Thus, $Z \subset L$ is the subfield of $\rho_1 \circ \rho_2$-invariant elements. Let $j^2 = b$. Then (14) yields

$$a_1 = \mu_1^2 a_1 a_2 + b N_{Z/k}(v_1), \qquad a_2 = \mu_2^2 a_1 a_2 + b N_{Z/k}(v_2),$$

hence $N_{Z/k}(-v_1 v_2) = a_1 a_2 b^{-2}(1 - \mu_1^2 a_2)(1 - \mu_2^2 a_1)$ and

$$\frac{-v_1 v_2}{\rho_1(-v_1 v_2)} = \frac{-v_1 v_2}{\rho_2(-v_1 v_2)} = \frac{a_1 a_2}{b^2 \rho_1(-v_1 v_2)^2}(1 - \mu_1^2 a_2)(1 - \mu_2^2 a_1).$$

Since $L = Z(\sqrt{a_1}) = Z(\sqrt{a_2})$, it follows that $1 - \mu_1^2 a_2$ and $1 - \mu_2^2 a_1$ are norms from L/Z. Therefore, the preceding equation yields

$$\frac{-v_1v_2}{\rho_1(-v_1v_2)} = \frac{-v_1v_2}{\rho_2(-v_1v_2)} = N_{L/Z}(\ell) \qquad \text{for some } \ell \in L^\times.$$

Since $N_{Z/k}(-v_1v_2\rho_1(-v_1v_2)^{-1}) = 1$, we have $N_{L/k}(\ell) = 1$. By Hilbert's Theorem 90, there exists $b_1 \in L^\times$ such that

$$\rho_1(b_1) = b_1 \qquad \text{and} \qquad b_1\rho_2(b_1)^{-1} = \ell\rho_1(\ell). \tag{18}$$

Set $b_2 = -v_1v_2\rho_1(\ell)b_1^{-1}$. Computation yields

$$\rho_2(b_2) = b_2 \qquad \text{and} \qquad \rho_1(b_2)b_2^{-1} = \ell\rho_2(\ell). \tag{19}$$

Define an algebra E over k by

$$E = L \oplus Lr_1 \oplus Lr_2 \oplus Lr_1r_2$$

where the multiplication is defined by

$$\begin{aligned} &r_1x = \rho_1(x)r_1, \qquad r_2x = \rho_2(x)r_2 \qquad \text{for } x \in L,\\ &r_1^2 = b_1, \qquad r_2^2 = b_2, \qquad \text{and} \qquad r_1r_2 = \ell r_2r_1. \end{aligned}$$

Since b_1, b_2 and ℓ satisfy (18) and (19), the algebra E is a crossed product, see [1]. It is thus a central simple k-algebra of degree 4.

PROPOSITION 4.4. *With the notation above, we may choose* $\gamma(\sigma) = [E] \in \operatorname{Br} k$.

Proof. The centralizer C_EZ of Z in E is $L \oplus Lr_1r_2$. Computation shows that

$$(r_1r_2)^2 = -v_1v_2.$$

Since conjugation by r_1r_2 maps $\sqrt{a_1} \in L$ to its opposite, it follows that

$$C_EZ = (a_1, -v_1v_2)_Z.$$

Since $[C_EZ] = [E]_Z$, the proposition follows from (17). □

COROLLARY 4.5. *Let*

$$E_+ = C_EZ = \{x \in E^\times \mid xz = zx \text{ for all } z \in Z\}$$

and

$$E_- = \{x \in E^\times \mid xz = \rho_1(z)x \text{ for all } z \in Z\}.$$

Then

$$G_+(A,\sigma) = k^{\times 2}\operatorname{Nrd}_E(E_+) \qquad \text{and} \qquad G_-(A,\sigma) = k^{\times 2}\operatorname{Nrd}_E(E_-).$$

Proof. As observed in the proof of Proposition 4.4, $C_E Z \simeq C(A,\sigma)$. Since, by [5, Corollary 5, p. 150],

$$\operatorname{Nrd}_E(x) = N_{Z/k}(\operatorname{Nrd}_{C_E Z} x) \qquad \text{for } x \in C_E Z,$$

the description of $G_+(A,\sigma)$ above follows from [7, (15.11)] (see also Corollary 2.1).

To prove $k^{\times 2}\operatorname{Nrd}_E(E_-) \subset G_-(A,\sigma)$, it obviously suffices to prove $\operatorname{Nrd}_E(E_-) \subset G_-(A,\sigma)$. From the definition of E, it follows that $r_1 \in E_-$. By [10, p. 80],

$$\operatorname{Nrd}_E(r_1)\cdot[E] = 0 \qquad \text{in } H^3 k. \tag{20}$$

Let $L_1 \subset L$ be the subfield fixed under ρ_1. We have $r_1^2 = b_1 \in L_1$, hence

$$\operatorname{Nrd}_E(r_1) = N_{L_1/k}(b_1).$$

On the other hand, the centralizer of L_1 is

$$C_E L_1 = L \oplus L r_1 \simeq (a_1 a_2, b_1)_{L_1},$$

hence

$$[N_{L_1/k}(C_E L_1)] = \big(a_1 a_2, N_{L_1/k}(b_1)\big)_k = \operatorname{Nrd}_E(r_1)\cdot \operatorname{disc}\sigma \quad \text{in } H^2 k. \tag{21}$$

Since $[C_E L_1] = [E_{L_1}]$, we have $[N_{L_1/k}(C_E L_1)] = 2[E]$. But $2[E] = 2\gamma(\sigma) = [A]$ by (7), hence (21) yields

$$\operatorname{Nrd}_E(r_1)\cdot\operatorname{disc}\sigma = [A] \qquad \text{in } H^2 k. \tag{22}$$

From (20), (22) and Theorems 1, 2 it follows that $\operatorname{Nrd}_E(r_1) \in G_-(A,\sigma)$.

Now, suppose $x \in E_-$. Then $r_1 x \in E_+$, hence $\operatorname{Nrd}_E(r_1 x) \in G_+(A,\sigma)$ by the first part of the corollary. Since

$$G_+(A,\sigma)G_-(A,\sigma) = G_-(A,\sigma)$$

it follows that

$$\operatorname{Nrd}_E(x) \in \operatorname{Nrd}_E(r_1)G_+(A,\sigma) = G_-(A,\sigma).$$

We have thus proved $k^{\times 2}\operatorname{Nrd}_E(E_-) \subset G_-(A,\sigma)$.

To prove the reverse inclusion, consider $\lambda \in G_-(A,\sigma)$. Since

$$G_-(A,\sigma)G_-(A,\sigma) = G_+(A,\sigma),$$

we have $\lambda\operatorname{Nrd}_E(r_1) \in G_+(A,\sigma)$, hence by the first part of the corollary,

$$\lambda\operatorname{Nrd}_E(r_1) \in k^{\times 2}\operatorname{Nrd}_E(E_+).$$

It follows that

$$\lambda \in k^{\times 2}\operatorname{Nrd}_E(r_1 E_+) = k^{\times 2}\operatorname{Nrd}_E(E_-).$$

□

4.2 Algebras of degree 6

Suppose $\deg A = 6$, i.e., $n = 3$, and use the same notation as in the beginning of this section. If σ (i.e., h) is isotropic, then h is Witt-equivalent to a rank 1 skew-hermitian form, say $\langle u\rangle$. Hence $i^2 = \operatorname{disc}\sigma = u^2 \in k^\times$. Hence we may assume that h is Witt-equivalent to the rank 1 skew-hermitian form $\langle \mu i\rangle$ for some $\mu \in k^\times$. This implies that the form g is hyperbolic and $C(U,g)$ is split. Hence we may choose $\gamma(\sigma) = 0$. By Theorem 4, we then have $\lambda \in G(A,\sigma)$ if and only if $\lambda.\operatorname{disc}\sigma = 0$ in $(H^2k)/A$. If σ becomes isotropic over Z, the form g is isotropic, hence we may choose a diagonalization of h

$$h \simeq \langle u_1, u_2, u_3\rangle$$

such that $g(u_3,u_3) = 0$, i.e., in the notation of (14), $u_3 = \mu_3 i$. Raising each side to the square, we obtain

$$a_3 = \mu_3^2 a_1 a_2 a_3,$$

hence $a_1 \equiv a_2 \bmod k^{\times 2}$. It follows that u_2 is conjugate to a scalar multiple of u_1, i.e., there exists $x \in Q^\times$ and $\theta \in k^\times$ such that

$$u_2 = \theta x u_1 x^{-1} = \theta \operatorname{Nrd}_Q(x)^{-1} x u_1 \overline{x}.$$

Since $\langle u_1\rangle \simeq \langle x u_1 \overline{x}\rangle$, we may let $\nu = -\theta \operatorname{Nrd}(x)^{-1} \in k^\times$ to obtain

$$h \simeq \langle u_1, -\nu u_1, \mu_3 i\rangle.$$

If $v_1 = 0$, then g is hyperbolic, hence we may choose $\gamma(\sigma) = 0$ by Proposition 4.1. If $v_1 \neq 0$, then (15) and (16) yield

$$g = \langle v_1\rangle\langle 1, -a_1\rangle + \langle -\nu v_1\rangle\langle 1, -a_1\rangle = \langle v_1\rangle\langle\langle a_1, \nu\rangle\rangle.$$

The Clifford algebra of g is the quaternion algebra $(a_1,\nu)_Z$, hence we may choose

$$\gamma(\sigma) = (a_1,\nu)_k.$$

Suppose finally that σ does not become isotropic over Z, hence v_1, v_2, $v_3 \neq 0$. Then

$$g = \langle v_1\rangle\langle 1, -a_1\rangle + \langle v_2\rangle\langle 1, -a_2\rangle + \langle v_3\rangle\langle 1, -a_3\rangle$$

and, by Proposition 4.3,

$$[C(A,\sigma)] = (a_1,v_1)_Z + (a_2,v_2)_Z + (a_3,v_3)_Z + (a_1,a_2)_Z + (a_1,a_3)_Z + (a_2,a_3)_Z.$$

Since $Z = k(\sqrt{a_1a_2a_3})$, the right side simplifies to

$$[C(A,\sigma)] = (a_1, v_1v_3)_Z + (a_2, v_2v_3)_Z + (a_1,a_2)_Z + (a_1a_2, -1)_Z. \tag{23}$$

By [7, (9.16)], $N_{Z/k}C(A,\sigma)$ is split, hence

$$\big(a_1, N_{Z/k}(v_1v_3)\big)_k = \big(a_2, N_{Z/k}(v_2v_3)\big)_k \qquad \text{in } \operatorname{Br} k.$$

By the "common slot lemma" (see for instance [2, Lemma 1.7]), there exists $\alpha \in k^\times$ such that

$$\big(a_1, N_{Z/k}(v_1v_3)\big)_k = \big(\alpha, N_{Z/k}(v_1v_3)\big)_k = \big(\alpha, N_{Z/k}(v_2v_3)\big)_k = \big(a_2, N_{Z/k}(v_2v_3)\big)_k.$$

Then

$$\big(\alpha a_1, N_{Z/k}(v_1v_3)\big)_k = \big(\alpha a_2, N_{Z/k}(v_2v_3)\big)_k = \big(\alpha, N_{Z/k}(v_1v_2)\big)_k = 0.$$

By [21, (2.6)], there exist β_1, β_2, $\beta_3 \in k^\times$ such that

$$(\alpha a_1, v_1v_3)_Z = (\alpha a_1, \beta_1)_Z, \qquad (\alpha a_2, v_2v_3)_Z = (\alpha a_2, \beta_2)_Z,$$
$$(\alpha, v_1v_2)_Z = (\alpha, \beta_3)_Z.$$

Since

$$(a_1, v_1v_3)_Z + (a_2, v_2v_3)_Z = (\alpha a_1, v_1v_3)_Z + (\alpha a_2, v_2v_3)_Z + (\alpha, v_1v_2)_Z,$$

it follows from (23) that

$$[C(A,\sigma)] = (\alpha a_1, \beta_1)_Z + (\alpha a_2, \beta_2)_Z + (\alpha, \beta_3)_Z + (a_1, a_2)_Z + (a_1a_2, -1)_Z.$$

We may thus take

$$\begin{aligned}\gamma(\sigma) &= (a_1, \beta_1)_k + (a_2, \beta_2)_k + (\alpha, \beta_1\beta_2\beta_3)_k + (a_1, a_2)_k + (a_1a_2, -1)_k \\ &= (a_1, -a_2\beta_1)_k + (a_2, -\beta_2)_k + (\alpha, \beta_1\beta_2\beta_3)_k.\end{aligned}$$

References

[1] S. A. Amitsur and D. Saltman, Generic Abelian crossed products and p-algebras, J. Algebra 51 (1978), no. 1, 76–87. MR0491789 (58 #10988)

[2] J. Kr. Arason, Cohomologische invarianten quadratischer Formen, J. Algebra 36 (1975), no. 3, 448–491. MR0389761 (52 #10592)

[3] J. Kr. Arason, R. Elman and B. Jacob, Fields of cohomological 2-dimension three, Math. Ann. 274 (1986), no. 4, 649–657. MR0848510 (87m:12006)

[4] I. Dejaiffe, Formes antihermitiennes devenant hyperboliques sur un corps de déploiement, C. R. Acad. Sci. Paris Sér. I Math. 332 (2001), no. 2, 105–108. MR1813765 (2001m:11054)

[5] P. K. Draxl, *Skew fields*, Cambridge Univ. Press, Cambridge, 1983. MR0696937 (85a:16022)

[6] S. Eilenberg and S. MacLane, Cohomology and Galois theory. I. Normality of algebras and Teichmüller's cocycle, Trans. Amer. Math. Soc. 64 (1948), 1–20. MR0025443 (10,5e)

[7] M.-A. Knus et al., *The book of involutions*, Amer. Math. Soc., Providence, RI, 1998. MR1632779 (2000a:16031)

[8] T. Y. Lam, *The algebraic theory of quadratic forms*, W. A. Benjamin, Inc., Reading, Mass., 1973. MR0396410 (53 #277)

[9] A. S. Merkurjev, On the norm residue symbol of degree 2, Dokl. Akad. Nauk SSSR 261 (1981), no. 3, 542–547. MR0638926 (83h:12015)

[10] A. S. Merkurjev, K-theory of simple algebras, in *K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)*, 65–83, Proc. Sympos. Pure Math., Part 1, Amer. Math. Soc., Providence, RI. MR1327281 (96f:19004)

[11] A. S. Merkurjev, Certain K-cohomology groups of Severi-Brauer varieties, in *K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)*, 319–331, Proc. Sympos. Pure Math., Part 2, Amer. Math. Soc., Providence, RI. MR1327307 (96g:19004)

[12] A. S. Merkurjev, R. Parimala and J.-P. Tignol, Invariants of quasi-trivial tori and the Rost invariant, Algebra i Analiz 14 (2002) 110–151; St. Petersburg Math. J. 14 (2003) 791–821.

[13] A. S. Merkurjev and A. A. Suslin, Norm residue homomorphism of degree three, Izv. Akad. Nauk SSSR Ser. Mat. 54 (1990), no. 2, 339–356; translation in Math. USSR-Izv. 36 (1991), no. 2, 349–367. MR1062517 (91f:11083)

[14] A. S. Merkurjev and J.-P. Tignol, The multipliers of similitudes and the Brauer group of homogeneous varieties, J. Reine Angew. Math. 461 (1995), 13–47. MR1324207 (96c:20083)

[15] R. Parimala, R. Sridharan and V. Suresh, Hermitian analogue of a theorem of Springer, J. Algebra 243 (2001), no. 2, 780–789. MR1850658 (2002g:11043)

[16] E. Peyre, Products of Severi-Brauer varieties and Galois cohomology, in *K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)*, 369–401, Proc. Sympos. Pure Math., Part 2, Amer. Math. Soc., Providence, RI. MR1327310 (96d:19008)

[17] M. Rost, On Hilbert Satz 90 for K_3 for degree-two extensions, preprint, Regensburg 1986.
`http://www.mathematik.uni-bielefeld.de/~rost/K3-86.html`

[18] W. Scharlau, *Quadratic and Hermitian forms*, Springer, Berlin, 1985. MR0770063 (86k:11022)

[19] J.-P. Serre, *Corps locaux*, Actualités Sci. Indust., No. 1296. Hermann, Paris, 1962. MR0150130 (27 #133)

[20] J-P. Serre, *Cohomologie galoisienne*, Fifth edition, Springer, Berlin, 1994. MR1324577 (96b:12010)

[21] J.-P. Tignol, Corps à involution neutralisés par une extension abélienne élémentaire, in *The Brauer group (Sem., Les Plans-sur-Bex, 1980)*, 1–34, Lecture Notes in Math., 844, Springer, Berlin. MR0611863 (82h:16013)

[22] A. Weil, Algebras with involutions and the classical groups, J. Indian Math. Soc. (N.S.) 24 (1960), 589–623 (1961). MR0136682 (25 #147)

R. Preeti
Université de Rennes 1
F-35042 Rennes, France
preeti.raman@math.univ-rennes1.fr

J.-P. Tignol
Université catholique de Louvain
B-1348 Louvain-la-Neuve, Belgium
tignol@math.ucl.ac.be