On Connecting Orbits of Semilinear Parabolic Equations on S^1

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Abstract. It is well-known that any bounded orbit of semilinear parabolic equations of the form

$$u_t = u_{xx} + f(u, u_x), \quad x \in S^1 = \mathbb{R}/\mathbb{Z}, \quad t > 0,$$

converges to steady states or rotating waves (non-constant solutions of the form $U(x - ct)$) under suitable conditions on f. Let S be the set of steady states and rotating waves (up to shift). Introducing new concepts — the *clusters* and the *structure* of S —, we clarify, to a large extent, the heteroclinic connections within S; that is, we study which $u \in S$ and $v \in S$ are connected heteroclinically and which are not, under various conditions. We also show that $\sharp S \geq N + \sum_{j=1}^{N} [[\sqrt{(f_u(r_j, 0))_+}/(2\pi)]]$ where $\{r_j\}_{j=1}^{N}$ is the set of the roots of $f(\,\cdot\,, 0)$ and $[[y]]$ denotes the largest integer that is strictly smaller than y. In paticular, if the above equality holds or if f depends only on u, the *structure* of S completely determines the heteroclinic connections.

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1. Introduction

We will investigate the global dynamics of semilinear parabolic partial differential equations on $S^1 = \mathbb{R}/\mathbb{Z}$ in $X = C^1(S^1)$

$$\begin{cases} u_t = u_{xx} + f(u, u_x), & x \in S^1, \\ u(x,0) = u_0(x), & x \in S^1. \end{cases} \tag{1.1}$$

The above problem is equivalent to a problem on the interval $[0,1]$ under the periodic boundary conditions $u(0,t) = u(1,t)$, $u_x(0,t) = u_x(1,t)$ for $t > 0$. Under suitable conditions on f, the solutions of (1.1) exist globally in $t > 0$. Thus (1.1) defines a global semiflow Φ_t on X. We will call each solution of (1.1) an *orbit*.

Angenent and Fiedler [AF88] and Matano [Ma88] have shown independently that any solution of (1.1) approaches as $t \to \infty$ to a solution (or a family of solutions) of the form $U(x-ct)$, where c is some real constant. Since $U(x-ct)$ is a solution to (1.1), the function $U(\zeta)$ should satisfy the following equation:

$$\frac{d^2U}{d\zeta^2} + c\frac{dU}{d\zeta} + f\left(U, \frac{dU}{d\zeta}\right) = 0, \quad \zeta \in S^1, \tag{1.2}$$

where $\zeta = x-ct$. Note that $U(\zeta+\theta)$ is a solution to (1.2) for all $\theta \in S^1$ provided that $U(\zeta)$ is a solution. If $c \neq 0$ and if $U(\zeta)$ is not a constant function, then $U(x-ct)$ is a time periodic solution called a *rotating wave* with speed c. If $c = 0$ and if $U(\zeta)$ is not a constant function, then $U(x)$ is called a *standing wave*. Thus steady states consist of both standing waves and constant steady states. By using these terms, the above assertion can be restated that any solution of (1.1) approaches either rotating waves or steady states.

Under suitable conditions on f that will be specified later, (1.1) has the set $\mathcal{A} \subset X$ called the *global attractor*. This set $\mathcal{A}$ is characterized as the maximal compact invariant set and it attracts all the orbits of (1.1).

Matano and Nakamura [MN97] have shown that the global attractor $\mathcal{A}$ of (1.1) consists of rotating waves, standing waves and *connecting orbits* that connect these waves. Therefore, in order to understand the dynamical structure of $\mathcal{A}$ it is important to know which pairs of waves are connected heteroclinically and which pairs are not. The paper [AF88] proves the existence of some connecting orbits for the problem (1.1) by using a topological method. We are interested in finding out a sharper criterion for the existence of connecting orbits.

In this paper we will give a precise lower bound for the number of mutually distinct rotating waves and steady states (Corollary B). If the Morse index of every wave is odd or zero, then certain order relations among waves defined below and the Morse index of all the waves determine which pairs of waves are connected heteroclinically and which pairs are not (Theorem A). In particular, if the actual number of the waves coincides with the lower bound given in Corollary B, then the hypothesis of Theorem A is automatically fulfilled, hence the heteroclinic connections are completely determined (Theorem C). In the special case where f depends only on u, we can completely determine which pairs of waves are connected heteroclinically and which are not (Theorems A and A'), and we will present rather simple and explicit sufficient conditions on f for the hypotheses of Theorem C to be satisfied (Proposition D).

Theorems A and C and Proposition D are proved by using the concepts of *clusters* and the *structure* which we introduce in Section 2. Let S be the set of all the waves. Roughly speaking, a *cluster* is a subset of S consisting of waves sharing certain common features, and S is expressed as a disjoint union of *clusters*. One can show that each *cluster* is a totally ordered set with respect to the following order relation

$$u \rhd v \quad \stackrel{def}{\Longleftrightarrow} \quad R(u) \supset R(v),$$

where $R(u)$ denotes the range of u (see Definition 2.5 and Remark 2.6). We then define the *structure* of S by associating each *cluster* with the sequence of (modified) Morse indices of its elements. Lemmas E, F and F' give fundamental properties of this sequence of modified Morse indices.

Now, many authors study the global attractor of (1.1) for the case where the boundary conditions in (1.1) is replaced by the Dirichlet or the Neumann boundary conditions. We can see [BF89] for the Dirichlet boundary conditions, [FR96] and [Wo02] for the Neumann boundary conditions and [MN97] for periodic boundary conditions. Here we recall the results of [FR96]. In the case of the Neumann boundary conditions on $[0,1]$, the global attractor consists of the steady states and the connecting orbits between these steady states, if all the steady states are hyperbolic. Let $\{U_j(x)\}_{j=1}^n$ $(U_1(0) < U_2(0) < \cdots < U_n(0))$ be the set of all the steady states. Roughly speaking, the permutation that rearranges the sequence $(U_1(1), U_2(1), \ldots, U_n(1))$ in increasing order determines the Morse indices of all the steady states and the zero number of functions $U_j(x) - U_k(x)$ $(1 \leq j < k \leq n)$ (In brief, the zero number of a function, which is defined in Section 2, is the number of the roots of the function). Once these Morse indices and the zero number of the difference of all the pairs among the waves are obtained, then this information tells which steady states are connected and which are not. Wolfrum [Wo02] has simplified the conditions of whether steady states are connected heteroclinically or not using the concept of *k-adjacent*. The concept of *k-adjacent* also uses the zero number of functions $U_j(x) - U_k(x)$ and the value of one of end points $U_j(0)$ (or $U_j(1)$). In the case of the periodic boundary conditions, we cannot use the method of [FR96] because the end points do not exist on S^1, therefore the Morse indices and the zero number of the difference of the pairs cannot be characterized in terms of permutation. Instead the maximum value, the minimum value and the mode of the waves play an important role in determining the Morse index of the waves and the zero number of the difference of the pairs, thereby giving the global picture of their heteroclinic connection.

This paper is organized as follows: In Section 2 we introduce some notation and definitions and state our main results (Theorems A A' and C, Corollary B, Proposition D and Lemmas E, F and F'). Roughly speaking, Corollary B gives a lower bound for the number of the waves in terms of the derivatives of f, and Lemma F is concerned with the modified Morse indices of waves and the structure of *clusters*. Theorems A, A' and C and Proposition D determine the heteroclinic connections among waves under various conditions. In Section 3 we will prove Lemma 3.1 which is the key lemma of this paper. In Section 4 we will show that each *cluster* is a totally ordered set in our order relation. We state the main results of [AF88]. We will prove Theorem C by using the results. In Section 5 we will investigate a sequence of modified Morse indices of waves in each *cluster* and prove Lemmas E and F and Corollary B. In Section 6 we will prove Theorem A, using Lemma F and main results of [AF88]. In Section 7 we consider the case where f depends only on u. We will prove Theorem A' and Lemma F'. In Section 8 we prove Proposition D, which is a special case of Theorems A and A'. We will give rather simple and explicit sufficient conditions on f under which all the *clusters* are *monotone* and *simple*, the meaning of which will be defined in Section 2. The *monotonicity* and *simplicity* of *clusters* automatically determine the Morse index of all the waves and the zero number of the difference of the pairs among the waves, hence their heteroclinic connections.

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2. Notation and Main Theorems

In this paper the nonlinear term f satisfies the following assumptions:

(A1) $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a C^3-function.

(A2) There exists a constant $L_1>0$ such that $u\cdot f(u,0)<0$ for $|u|>L_1$, and the function $f(\,\cdot\,,0)$ has finitely many real roots.

(A3) (i) For any solution $u(x,t)$ to (1.1), $||u(\,\cdot\,,t)||_{C^1(S^1)} := ||u(\,\cdot\,,t)||_{C^0(S^1)} + ||u_x(\,\cdot\,,t)||_{C^0(S^1)}$ remains bounded as $t\to\infty$.

(ii) There exists a constant $L_2>0$ such that

$$||U(\zeta)||_{C^1(\mathbb{R})} := ||U(\zeta)||_{C^0(\mathbb{R})} + ||U_\zeta(\zeta)||_{C^0(\mathbb{R})} < L_2$$

for any periodic solution or constant solution $U(\zeta)$ to the following equation:

$$\frac{d^2U}{d\zeta^2} + c\frac{dU}{d\zeta} + f\left(U, \frac{dU}{d\zeta}\right) = 0, \quad \zeta \in \mathbb{R}, \tag{2.1}$$

where c is an arbitrary real number.

The assumption (A3) (ii) will be needed in Section 3, where we study the bifucation structure of rotating waves and constant steady states. The assumption (A3) is satisfied if the following condition (A3)$'$ holds:

(A3)$'$ For any constant $M_1 > 0$, there exists a constant $L_3 > 0$ such that $f_u(u,p) \leq 0$ for $|u| < M_1$ and $|p| > L_3$.

From (A1), (A2) and (A3) it follows that (1.1) defines a global semiflow Φ_t on X that is dissipative. Here a semiflow Φ_t on X is called *dissipative* if there exists a ball $B \subset X$ which satisfies the following: For any $u_0 \in X$, there exists $t_0 > 0$ such that $\Phi_t(u_0) \in B$ for all $t \geq t_0$ (see [Ma76]).

Hereafter, we assume (A1)+(A2)+(A3)$'$ throughout the present paper.

By the standard parabolic estimates, the mapping Φ_t is a compact mapping for every $t > 0$. This, together with the dissipativity of Φ_t, implies that there is the (nonempty) maximal compact invariant set $\mathcal{A} \subset X$. It is well-known from the general theory of dissipative dynamical systems that $\mathcal{A}$ is connected and attracts all the orbits of (1.1). This set $\mathcal{A}$ is called the global attractor. The Hausdorff dimension of $\mathcal{A}$ of (1.1) is $2\,[M/2] + 1$ where M is the maximal generalized Morse index of the steady states or the rotating waves (see [MN97]).

Let us introduce some definitions and notation. In this paper we denote by S the set of steady states and rotating waves of (1.1). Note that if $U(x - ct)$ is a rotating wave (or a steady state in the case where $c = 0$), then $U(x - ct + \theta)$ is also a rotating wave (or a steady state) for any $\theta \in S^1$. Hereafter we identify $U(\cdot)$ and $U(\cdot + \theta)$. In other words, we will understand S to be the set of equivalence classes, each of which is expressed in the form

$$\Gamma(U) := \{U(x - ct + \theta) |\ \theta \in S^1\},$$

where $U(\zeta)$ is a solution of (1.2). However in order to simplify notation, we write $U(x - ct) \in S$ to mean $[U(x - ct)] \in S$, where $[U(x - ct)]$ denotes the equivalence class to which $U(x - ct)$ belongs. Therefore $u(x,t) \in S$ shall mean that $u(x,t) = U(x - ct + \theta)$ for some $\theta \in S^1$ where $U(\zeta)$ is a solution to (1.2). Furthermore, by a heteroclinic connection from $u(x,t)(:= U(x - ct)) \in S$ to $v(x,t)(:= V(x - \tilde{c}t)) \in S$ we mean that there is an orbit $w(x,t)$ of (1.1) such that

$$\inf_{\theta_1 \in S^1} \|w(x,t) - U(x - ct + \theta_1)\|_{L^\infty(S^1)} \to 0 \ (t \to -\infty),$$
$$\inf_{\theta_2 \in S^1} \|w(x,t) - V(x - \tilde{c}t + \theta_2)\|_{L^\infty(S^1)} \to 0 \ (t \to +\infty).$$

In particular, if U and V are 'hyperbolic' (whose meaning is defined below in this section), then a heteroclinic connection from u to v automatically implies the following stronger convergence:

$$\|w(x,t) - U(x - ct + \theta_1)\|_{L^\infty(S^1)} \to 0 \ (t \to -\infty) \text{ for some } \theta_1 \in S^1,$$
$$\|w(x,t) - V(x - \tilde{c}t + \theta_2)\|_{L^\infty(S^1)} \to 0 \ (t \to +\infty) \text{ for some } \theta_2 \in S^1.$$

The number of the roots of $f(\,\cdot\,,0)$ is finite owing to (A2). Let $\{r_j\}_{j=1}^N$ $(r_1 < r_2 < \cdots < r_N)$ be the roots of $f(\,\cdot\,,0)$ throughout the present paper. All the constant steady states are $u(x,t) = r_j$ $(j \in \{1,2,\ldots,N\})$.

Remark 2.1. If $f_u(r_j,0) \neq 0$ for all $j \in \{1,2,\ldots,N\}$, then N is odd because of (A2). Moreover $u(x,t) = r_j$ $(j \in \{1,3,5,\ldots,N\})$ is a stable constant steady state, while $u(x,t) = r_j$ $(j \in \{2,4,6,\ldots,N-1\})$ is an unstable constant steady state (see Remark 2.8 below).

The zero number is a powerful tool to analyze nonlinear single reaction-diffusion equations in one space dimension:

$$z(w) := \sharp \left\{x \mid w(x) = 0,\ x \in S^1\right\} \quad \text{for } w \in X,$$

where $\sharp Y$ denotes the number of elements of the set Y. It is well-known that $z(w(\,\cdot\,,t))$ is a non-increasing function of t if w is a solution of a one-dimensional linear parabolic equation (see [Ma82], [Ni62] and [St36]). Furthermore, the following proposition holds:

PROPOSITION 2.2 (Angenent and Fiedler [AF88] and Angenent [An88]). *Let $a(x,t)$ and $b(x,t)$ be C^2-functions in $(x,t) \in S^1 \times (0,\tau)$ $(\tau > 0)$. Let $w(x,t) \in X$ be a solution to the following equations:*

$$w_t = w_{xx} + a(x,t)w_x + b(x,t)w, \qquad (x,t) \in S^1 \times (0,\tau).$$

Then $z(w(\,\cdot\,,t))$ is finite for every $t \in (0,\tau)$ and is non-increasing in t. Moreover $z(w(\,\cdot\,,t))$ drops at each $t = t_0$ when the function $x \longmapsto w(x,t_0)$ has a multiple zero.

Remark 2.3. Angenent and Fiedler [AF88] have proved Proposition 2.2 in the case where $a(x,t)$ and $b(x,t)$ are real analytic functions. Angenent [An88] has relaxed this analyticity assumption.

Using the moving frame with speed c, we can rewrite (1.1) as follows:

$$u_t = u_{\zeta\zeta} + cu_\zeta + f(u,u_\zeta), \tag{2.2}$$

where $\zeta = x - ct$. Let $U(x-ct) \in S$. The wave $U(\zeta)(= U(x-ct))$ is a steady state of (2.2). In order to analyze the stability of $U(\zeta)$, we define the linearized operator of (2.2) at $U(\zeta)$ by

$$L_U w = w_{\zeta\zeta} + cw_\zeta + f_u(U,U_\zeta)w + f_p(U,U_\zeta)w_\zeta, \quad \zeta \in S^1,$$

provided that U is a non-constant steady state of (2.2). Here f_p denotes the derivative of f with respect to the second variable. If U is a constant steady state of (2.2), then we define the linearized operator by

$$L_U w = w_{\zeta\zeta} + f_u(U,0)w + f_p(U,0)w_\zeta, \quad \zeta \in S^1.$$

By the standard spectral theory for ordinary differential operators of the second order, the spectrum of L_U consists of eigenvalues of finite multiplicity and has no accumulation point except ∞. Let $\{\lambda_n\}_{n=0}^{\infty}$ be the eigenvalues of L_U that are repeated according to their algebraic multiplicity. We define the Morse index of $U \in S$ by $i(U) := \sharp\{\lambda_n |\ \mathrm{Re}(\lambda_n) > 0\}$. By a Sturm-Liouville type theorem (see [AF88] and [MN97]), we have

$$\mathrm{Re}(\lambda_0) > \mathrm{Re}(\lambda_1) \geq \mathrm{Re}(\lambda_2) > \mathrm{Re}(\lambda_3) \geq \cdots \geq \mathrm{Re}(\lambda_{2j}) > \mathrm{Re}(\lambda_{2j+1}) \geq \cdots .$$

Moreover if U is a non-constant steady state, we can see

$$i(U) \in \{z(U_\zeta),\ z(U_\zeta) - 1\} \tag{2.3}$$

(see [AF88] and [MN97]). Note that $z(U_\zeta)$ is even and $z(U_\zeta)-1$ is odd since U_ζ is a periodic function of ζ. We can see the Morse index of the constant steady states by easy calculations (see Remark 2.8 below).

Next we define the hyperbolicity of $U \in S$. Because of translation equivariance of the equation (1.1), each rotating wave and each non-constant steady state form a one-dimensional manifold that is homeomorphic to S^1. This equivariance has to be taken into account when we define the hyperbolicity of those solutions.

DEFINITION 2.4.

(i) *Let u be a (non-constant) rotating wave ($c \neq 0$) or a non-constant steady state ($c = 0$). We say u is hyperbolic if 0 is the only eigenvalue of L_u on the imaginary axis and if 0 is a simple eigenvalue.*

(ii) *Let u be a constant steady state (i.e. $u(x,t) = r_j$). We say u is hyperbolic if there is no eigenvalue of L_u on the imaginary axis.*

DEFINITION 2.5. *Let $u(x,t)$ be a solution of (1.1). We define*

$$R(u(\,\cdot\,,t)) := \left\{ y \in \mathbb{R} \;\middle|\; \min_{x\in S^1} u(x,t) \leq y \leq \max_{x\in S^1} u(x,t) \right\}.$$

Remark 2.6. If $u \in S$, then $R(u(\,\cdot\,,t))$ is independent of t. Hereafter we simply write $R(u)$ if $u \in S$.

DEFINITION 2.7. *For $u \in S$, we define its "modified Morse index" by*

$$I(u) := \begin{cases} z(u_x) & \textit{if } u \textit{ is not a constant steady state;} \\ i(u) + 1 & \textit{if } u \textit{ is an unstable constant steady state;} \\ 0 & \textit{if } u \textit{ is a stable constant steady state.} \end{cases}$$

Remark 2.8. One can calculate the Morse index of the constant steady states. Let u be a constant steady state (*i.e.* $u(x) = r_j$). Then

$$i(u) = \begin{cases} 2\left[\dfrac{\sqrt{f_u(r_j,0)}}{2\pi}\right] + 1 & \text{if } f_u(r_j,0) > 0; \\ 0 & \text{if } f_u(r_j,0) \leq 0, \end{cases}$$

where $[y]$ denotes the largest integer not exceeding y. If all the constant steady states are hyperbolic, then $i(u) = 2[\sqrt{f_u(r_j,0)}/(2\pi)]+1$ for $j \in \{2,4,6,\ldots,N-1\}$ and $i(u) = 0$ for $j \in \{1,3,5,\ldots,N\}$. Thus r_j $(j \in \{1,3,5,\ldots,N\})$ is stable and r_j $(j \in \{2,4,6,\ldots,N-1\})$ is unstable.

Note that $I(u)$ is always a non-negative even integer. From (2.3) it follows that

$$i(u) \le I(u) \le i(u) + 1.$$

Therefore $I(u)$ is a good approximation of the real Morse index $i(u)$. Clearly, $I(u) = i(u)$ if and only if $i(u)$ is even.

While the modified Morse index $I(u)$ is easily computable from Definition 2.7 and Remark 2.8, the real Morse index $i(u)$ is not always easily to determine. This is the reason why we introduce the notion modified Morse index.

Now we can define the *cluster.*

Definition 2.9. *Let $1 \le k \le l \le N$. We define the clusters by*

$$C_{kl} := \{u \in S|\ S_{kl} \subset R(u),\ (\{r_1, r_2, \ldots, r_N\} \backslash S_{kl}) \cap R(u) = \emptyset\},$$

where $S_{kl} := \{r_k, r_{k+1}, \ldots, r_l\}$.

It is not difficult to see that

$$C_{kl} \cap C_{k'l'} = \emptyset \quad \text{if } (k,l) \ne (k',l'),$$

$$S = \bigcup_{1 \le k \le l \le N} C_{kl}.$$

Furthermore one can see that, if k or l is odd, then

$$C_{kk} = \{r_k\} \ \text{ and } \ C_{kl} = \emptyset \ \ (k \ne l).$$

The concept of *clusters* will be useful in the phase plane analysis as we will see in Section 6.

Definition 2.10. *Let C_{kl} be a cluster. We define*

$$R(C_{kl}) := \bigcup_{u \in C_{kl}} R(u).$$

Definition 2.11. *Let $u, v \in S$. We define the order relation of S as follows:*

$$u \rhd v \quad \overset{def}{\Longleftrightarrow} \quad R(u) \supset R(v).$$

Let $u, v, w \in S$. If $u \rhd v$, then we say v is smaller than u in the order $\rhd$, and u is bigger than v in the order $\rhd$. If there is no w such that $u \rhd w \rhd v$, then we say that u is the smallest wave in the order $\rhd$ that satisfies $u \rhd v$.

We have either $R(u) \supset R(v)$ or $R(v) \supset R(u)$ provided that $R(u) \cap R(v) \ne \emptyset$. This will be shown in Corollary 4.2 in Section 4. Consequently we have either $u \rhd v$ or $v \rhd u$ if $u, v \in C_{kl}$. Thus C_{kl} is a totally ordered set. Hereafter, we number the elements of each $C_{kl} = \left\{u_1^{kl}, u_2^{kl}, \ldots, u_{m_{kl}}^{kl}\right\}$ (with $m_{kl} := \sharp C_{kl}$) in such a way that

$$u_1^{kl} \lhd u_2^{kl} \lhd \cdots \lhd u_{m_{kl}}^{kl}.$$

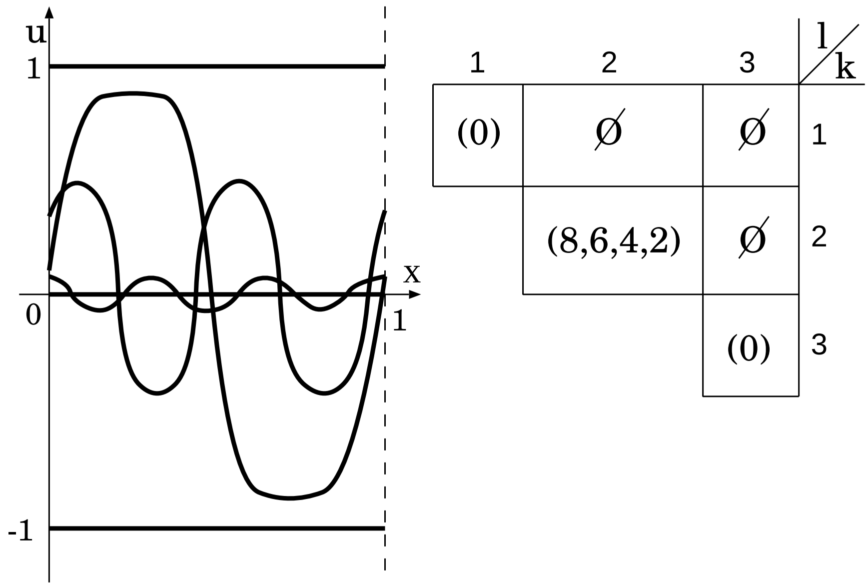

FIGURE 1. Wave profiles (left) and the *structure* of S (right) for equation (2.4). The three horizontal lines indicate constant steady states.

We call C_{kl} a *monotone cluster* if $I(u_1^{kl}) > I(u_2^{kl}) > \cdots > I(u_{m_{kl}}^{kl})$. The *cluster* C_{kk} is called a *simple cluster*. We call C_{kk} a *trivial cluster* provided that $\sharp C_{kk} = 1$. Note that C_{kk} always contains the constant steady state r_k, but it may contain other elements under certain circumstances.

Next we define an order relation among *clusters* in S.

DEFINITION 2.12. *Let $C_{k_1 l_1}, C_{k_2 l_2}$ be clusters. We define the order relation $\rhd$ as follows:*

$$C_{k_1 l_1} \rhd C_{k_2 l_2} \quad \overset{def}{\Longleftrightarrow} \quad k_1 \le k_2 \text{ and } l_1 \ge l_2.$$

Let $C_{k_1 l_1}, C_{k_2 l_2}$ be *clusters*. If $C_{k_1 l_1} \rhd C_{k_2 l_2}$, then we say $C_{k_2 l_2}$ is smaller than $C_{k_1 l_1}$ in the order $\rhd$.

We define the *structure* of S.

DEFINITION 2.13. *Let $C_{kl} := \{u_1^{kl}, u_2^{kl}, \ldots, u_{m_{kl}}^{kl}\}$ (with $m_{kl} := \sharp C_{kl}$) be a cluster. We call*

$$J_{kl} := (I(u_1^{kl}), I(u_2^{kl}), \ldots, I(u_{m_{kl}}^{kl}))$$

the sequence of modified Morse indices. We call

$$(J_{kl})_{1 \le k \le l \le N}$$

the structure of S.

Example 2.14. Let us investigate the *structure* of the waves of the following equation:

$$u_t = u_{xx} + 500(u - u^3), \quad x \in S^1. \tag{2.4}$$

Clearly, there are three constant steady states. Let $r_1 = 1$, $r_2 = 0$ and $r_3 = -1$. The nonlinear term depends only on u. Thus all the waves are standing waves (see Remark 2.18). A simple calculation reveals that the nonlinear term satisfies the hypothesis of Proposition D below. Thus we can see that all the clusters are *simple* and *monotone*, using Proposition D. Since all the *clusters* are *simple*, there are precisely three *clusters*: C_{11}, C_{22} and C_{33} (where $r_1 \in C_{11}$, $r_2 \in C_{22}$ and $r_3 \in C_{33}$). We can see that C_{11} and C_{33} are *trivial clusters*, using (i) of Lemma F. Furthermore r_1 and r_3 are stable (see Remark 2.1) and $I(r_1) = I(r_3) = 0$ (see Definition 2.7 and Remark 2.8). The *cluster* C_{22} is *monotone*. Thus Theorem C below tells us that the derivative of the nonlinear term at $u = r_2$ gives $\sharp C_{22} = 4$, because

$$3 < \frac{\sqrt{\frac{d}{du}\left\{500(u - u^3)\right\}\big|_{u=r_2}}}{2\pi} < 4.$$

Therefore C_{22} has three non-constant standing waves and one constant steady state. The profile of the waves are as shown in Figure 1. We denote by u_1^{22} the constant steady state in C_{22} and by u_2^{22}, u_3^{22} and u_4^{22} the non-constant standing waves. We can assume that $u_1^{22} \lhd u_2^{22} \lhd u_3^{22} \lhd u_4^{22}$, because all the *clusters* are totally ordered sets. Since C_{22} is *monotone*, we can see by (ii) and (v) of Lemma F that $I(u_1^{22}) = 8$, $I(u_2^{22}) = 6$, $I(u_3^{22}) = 4$ and $I(u_4^{22}) = 2$. Therefore the *structure* of S is as shown in the table in Figure 1.

We introduce some more notation to state main theorems. Let $u \in S$ and let $C(u)$ be the *cluster* containing u. Define

$$u_+ := \inf\{w |\ w > u,\ w \text{ is a constant steady state}\},$$
$$u_- := \sup\{w |\ w < u,\ w \text{ is a constant steady state}\},$$

and for each integer $n \geq 0$, define u_n to be the smallest wave in the order $\rhd$ that satisfies the following: $I(u_n) = 2n$, $u_n \rhd u$, and $u_n \in C(u)$. That is,

$$u_n = \min_{\rhd}\{v \in C(u) |\ v \rhd u,\ I(v) = 2n\}.$$

Lemma F below tells us that such u_n exists for $n \in \{1, 2, \ldots, [i(u)/2]\}$.

Roughly speaking u_+ is the constant steady state that is just above u in the usual order, and u_- is the constant steady state that is just below u in the usual order.

THEOREM A. *Suppose that all the elements of S are hyperbolic. Then*

(i) *If the wave u is not a stable constant steady state, then u connects to u_+, u_- and u_n for all $n \in \{1, 2, \ldots, I(u)/2 - 1\}$.*

(ii) *Furthermore if $i(u)$ is odd, then u does not connect to any other waves. Therefore the structure of S determines completely which $u \in S$ and $v \in S$ are connected and which are not, if the Morse index of every wave is odd or zero.*

Remark 2.15. The statement (i) of Theorem A is obtained by Angenent-Fiedler [AF88] (see Proposition 6.3 of the present paper).

THEOREM A'. *Suppose that f is dependent only on u, say $f = g(u)$, and that all the waves are hyperbolic. Let u be a wave whose Morse index $i(u)$ is even. Then u connects only to u_+, u_-, u_n ($n \in \{1, 2, \cdots, I(u)/2\}$), and every $v \in S$ that satisfies the following: $v \lhd u$, $I(v) \leq I(u)$, and there is no wave w such that $u \rhd w \rhd v$, $I(u) = I(w)$, and $u \neq w \neq v$.*

Remark 2.16. The *structure* of S tells us the modified Morse index of every wave. In the case where f depends only on u, we can know the (real) Morse index of every wave by using Lemmas F and F' stated below. Thus we see by Theorems A and A' that the heteroclinic connections are determined by the *structure* of S provided that f depends only on u.

COROLLARY B.

$$\sharp S \geq N + \sum_{j=1}^{N} \left[\left[\frac{\sqrt{(f_u(r_j, 0))_+}}{2\pi} \right]\right],$$

where $[[y]]$ denotes the largest integer that is strictly smaller than y (i.e. $[[y]] = -[-y] - 1$) and $(y)_+ := \max\{y, 0\}$.

Remark 2.17. The hyperbolicity of the solutions is not assumed in Corollary B.

THEOREM C. *Suppose that all $u \in S$ are hyperbolic. Then the following two conditions are equivalent:*

(a)

$$\sharp S = N + \sum_{j=1}^{N} \left[\left[\frac{\sqrt{(f_u(r_j, 0))_+}}{2\pi} \right]\right], \tag{2.5}$$

where $(y)_+ := \max\{y, 0\}$.

(b) *all the clusters are simple and monotone.*

Moreover, under these conditions, $i(u) = I(u) - 1 = (z(u_x) - 1)$ is odd for any non-constant $u \in S$. Thus the hypotheses of Theorem A *are satisfied. The conclusions of* Theorem A *hold. Specifically the structure of S is uniquely determined by the sequence $[[\sqrt{(f_u(r_j, 0))_+}/(2\pi)]]$ ($j = 1, 2, \ldots, N$). The global picture of heteroclinic connections in S is also uniquely determined as shown in* Figure 9.

In the case where f is dependent only on u, say $f = g(u)$, we introduce other two assumptions (A4) and $(A5)_j$ below. Let

$$G(u) = \int_0^u g(r)dr. \tag{2.6}$$

(A4) There exists an odd constant k such that $G(r_1) \le G(r_3) \le \cdots \le G(r_k) \ge G(r_{k+2}) \ge \cdots \ge G(r_N)$, $G(r_2) \le G(r_4) \le \cdots \le G(r_{k-1})$ and $G(r_{k+1}) \ge G(r_{k+3}) \ge \cdots \ge G(r_{N-1})$.

If $k = 1$ or $k = n$, then the second or the third inequalities in (A4) are not assumed respectively. We will see in Section 8 that C_{kl} $(k \neq l)$ is empty provided that (A4) holds. Thus every *cluster* is *simple* (see Figures 11 and 12). We impose the other assumption: For $j \in \{2, 4, 6, \dots, N-1\}$,

$(\mathbf{A5})_j$ $g(u)/|u|$ is decreasing for $u \in (r_{j-1}, r_j) \cup (r_j, r_{j+1})$.

The condition $(\text{A5})_j$ guarantees that C_{jj} is *monotone* (Lemma 8.1). Hence we obtain the following:

PROPOSITION D. *Suppose that f is dependent only on u, say $f = g(u)$, and that all the waves are hyperbolic. If* (A4) *holds and if* $(\text{A5})_j$ *holds for all even $j \in \{2, 4, 6, \dots, N-1\}$, then the hypotheses of* Theorem C *are satisfied. Thus the conclusions of* Theorems A, A' *and* C *hold.*

Remark 2.18. The equation (1.1) does not have rotating waves in the case where the nonlinear term f depends only on u. For the details, see the beginning of Section 7.

The next lemma is concerned with the structure of each *cluster*.

LEMMA E (Cluster lemma 1). *Suppose that all $u \in S$ are hyperbolic. Let $1 \le k \le l \le N$. Let $C_{kl} = \{u_1^{kl}, u_2^{kl}, \dots, u_{m_{kl}}^{kl}\}$ $(m_{kl} = \sharp C_{kl})$ be a cluster and let $J_{kl} = (I(u_1^{kl}), I(u_2^{kl}), \dots, I(u_{m_{kl}}^{kl}))$ be the corresponding sequence of modified Morse indices. Then the following hold:*

- (i) *If k or l is odd and if $k \neq l$, then $C_{kl} = \emptyset$.*
- (ii) *If k is odd, then $\sharp J_{kk} = 1$. Thus C_{kk} is a trivial cluster. Moreover $I(u_1^{kk}) = 0$.*

LEMMA F (Cluster lemma 2). *Under the same hypotheses of Lemma E, the following hold:*

- (i) *Every $I(u)$ is an even integer, and $I(u_n^{kl}) - I(u_{n+1}^{kl})$ is equal to -2, 0 or 2 for all $n \in \{1, 2, \dots, m_{kl} - 1\}$.*
- (ii) *If $I(u_{n_1-1}^{kl}) < I(u_{n_1}^{kl}) = \cdots = I(u_{n_2}^{kl}) < I(u_{n_2+1}^{kl})$ $(2 \le n_1 \le n_2 \le m_{kl} - 1)$ or if $I(u_{n_1-1}^{kl}) > I(u_{n_1}^{kl}) = \cdots = I(u_{n_2}^{kl}) > I(u_{n_2+1}^{kl})$ $(2 \le n_1 \le n_2 \le m_{kl} - 1)$, then $n_2 - n_1$ is even.*
- (iii) *If $I(u_{n_1-1}^{kl}) < I(u_{n_1}^{kl}) = \cdots = I(u_{n_2}^{kl}) > I(u_{n_2+1}^{kl})$ $(2 \le n_1 \le n_2 \le m_{kl} - 1)$ or if $I(u_{n_1-1}^{kl}) > I(u_{n_1}^{kl}) = \cdots = I(u_{n_2}^{kl}) < I(u_{n_2+1}^{kl})$ $(2 \le n_1 \le n_2 \le m_{kl} - 1)$, then $n_2 - n_1$ is odd.*
- (iv) *If C_{kl} is not trivial, that is, if $\sharp J_{kl} \ge 2$, then $I(u_{m_{kl}}^{kl}) = 2$.*
- (v) *If $k \neq l$, and if $C_{kl} \neq \emptyset$, then $I(u_1^{kl}) = 2$.*

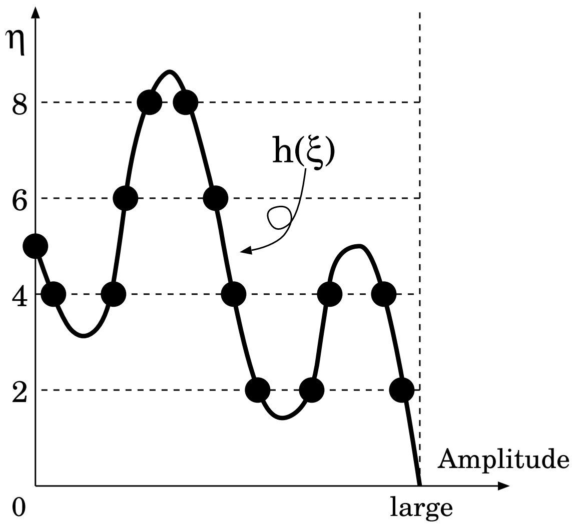

Figure 2. An example of $h(\xi)$ mentioned in Remark 2.19. In this case, the sequence of modified Morse indices is $(6, 4, 4, 6, 8, 8, 6, 4, 2, 2, 4, 4, 2)$.

Remark 2.19. In view of Lemma F, the sequence of modified Morse indices $J_{kk} = \left(I(u_j^{kk})\right)_{j=1}^{m_{kk}}$ may better be illustrated as the intersection points between the graph of a function $\eta = h(\xi)$ where $1/h(\xi)$ is the time-map and the horizontal lines $\eta = 2, 4, 6, 8, \cdots$ (see Figure 2). The time-map is used in Section 3 (see the definition of $T(a)$ in the statement of Lemma 3.1). This function h satisfies that $h'(\xi) \neq 0$ whenever $h(\xi)$ is an even integer, and that $h(\xi) = 0$ if ξ is large.

Lemma F' *Suppose that f is dependent only on u, say $f = g(u)$, and that all the waves are hyperbolic. Let $\{u_{b_1}^{kl}, u_{b_2}^{kl}, \ldots, u_{b_n}^{kl}\}$ ($b_1 < b_2 < \cdots < b_n$) be the non-constant waves in a cluster C_{kl} whose modified Morse indices are the same number (i.e. $I(u_{b_1}^{kl}) = I(u_{b_2}^{kl}) = \cdots = I(u_{b_n}^{kl})$). Then $i(u_{b_{n-2j}}^{kl}) = I(u_{b_{n-2j}}^{kl}) - 1$ ($j \in \{0, 1, \ldots, [(n-1)/2]\}$) and $i(u_{b_{n-2j-1}}^{kl}) = I(u_{b_{n-2j-1}}^{kl})$ ($j \in \{0, 1, \ldots, [(n-2)/2]\}$).*

3. Proof of the Key Lemma

We will also prove three lemmas which are used in the proof of main theorems. One of these lemmas (Lemma 3.1) is the key to the present paper.

In this section we assume that all the waves are hyperbolic in order to simplify notation. The number of all the constant steady states N is odd owing to the hyperbolicity and (A2) (see Remark 2.1). Using (A2), we can see that the

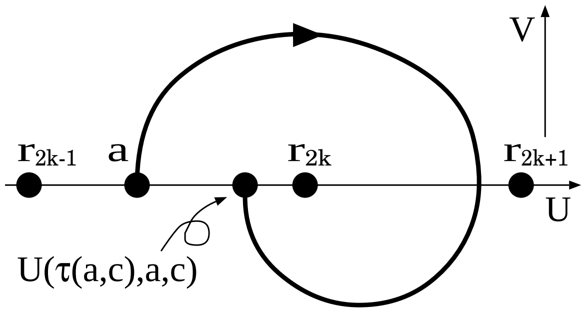

Figure 3. The picture denotes the arc that starts from the point $(a, 0)$ at the time 0 and arrives at a point $(b, 0)$ $(r_{2k-1} < b < r_{2k})$ at a certain positive time. The U-coordinate of the arrival point is denoted by $U(\tau(a, c), a, c)$ whose meaning is specified below. The arc deforms with respect to a and c. If we select suitable a and c, then the arrival point coincides with the starting point (*i.e.* $U(\tau(a, c), a, c) = a$), which means the arc is a closed orbit.

following hold:

$$f_u(r_{2k}, 0) < 0 \quad \text{if } k \in \{1, 2, \ldots, [N/2]\};$$
$$f_u(r_{2k-1}, 0) > 0 \quad \text{if } k \in \{1, 2, \ldots, [N/2] + 1\}.$$

Let us introduce some notation. Let $u(x, t) = U(\zeta) \in S$. The wave $U(\zeta)$ should satisfy the following equation and periodic boundary conditions:

$$\begin{cases} U_{\zeta\zeta} + cU_\zeta + f(U, U_\zeta) = 0, \quad \zeta \in (0, 1), \\ U(0) = U(1),\ U_\zeta(0) = U_\zeta(1). \end{cases} \tag{3.1}$$

We transform the equation of (3.1) into the normal form:

$$\begin{cases} \dfrac{dU}{d\zeta} = V \\ \dfrac{dV}{d\zeta} = -cV - f(U, V). \end{cases} \tag{3.2}$$

Let U-axis and V-axis be the horizontal and vertical axes of the phase plane respectively. First, we note that no closed orbit appears near the points $(r_{2k-1}, 0)$ $(k \in \{1, 2, \ldots, [N/2] + 1\})$, since there points are saddle points. In what follows we will construct closed orbits in a neighborhood of the points $(r_{2k}, 0)$ $(k \in \{1, 2, \ldots, [N/2]\})$ on the phase plane.

In order to explain our idea suppose that there is an orbit as shown in Figure 3. This orbit starts from the point $(a, 0)$, passes the segment $(r_{2k}, r_{2k+1}) \times \{0\}$, and arrives at a point on the segement $(r_{2k-1}, r_{2k}) \times \{0\}$.

Let $(b, 0)$ be the arrival point. As we will see in the proof of Lemma 3.1, the value of b depends continuously on a and c as far as the orbit remains within the band domain $r_{2k-1} < U < r_{2k+1}$. Hereafter by *the arc corresponding to* (a, c) we shall mean the portion of the orbit of (3.2) starting at $(a, 0)$ and ending at a point on the segment $(r_{2k-1}, r_{2k}) \times \{0\}$ as shown in Figure 3.

Let $\tau(a, c)$ be the arrival time of this arc; that is $\tau(a, c)$ is the smallest positive time τ such that $U(\tau(a, c), a, c) \in (r_{2k-1}, r_{2k})$ and $U_\zeta(\tau(a, c), a, c) = 0$ where $U(\zeta, a, c)$ denotes the solution of (3.2) with initial data $U(0) = a$, $V(0) = 0$, and U_ζ denotes the derivative of U with respect to the first variable. Clearly the arc forms a closed orbit of (3.2) if and only if

$$a = U(\tau(a, c), a, c). \tag{3.3}$$

Furthermore this closed orbit represents a solution of (3.1) if and only if

$$\tau(a, c) = \frac{1}{n}$$

for some $n \in \{1, 2, \ldots\}$.

The following lemma shows that there is a continuous family of closed orbits corresponding to varying choice of a and c.

Lemma 3.1. *For each* r_{2k} $(k = 1, 2, \ldots, [N/2])$, *there exists a constant* $\underline{a}$ *with* $r_{2k-1} \leq \underline{a} < r_{2k}$ *and a function* $c = c(a) \in C^1((\underline{a}, r_{2k}))$ *such that the following hold.*

(i) *For each* $a \in (\underline{a}, r_{2k})$, *the relation* (3.3) *holds if and only if* $c = c(a)$.

(ii) *Let* $T(a)$ *be the period of the closed orbit obtained in* (i), *that is,* $T(a) = \tau(a, c(a))$. *Then*

$$\lim_{a \to \underline{a}} T(a) = \infty, \quad \lim_{a \to r_{2k}} T(a) = \frac{2\pi}{\sqrt{f_u(r_{2k}, 0)}}.$$

Proof. We begin with the outline of the proof. The proof consists of three steps. In Step 1 we will show by using the bifurcation theory that there exists a family of closed orbits of (3.2) near the point $(r_{2k}, 0)$. Thus $c(a)$ can be defined near $a = r_{2k}$. In Step 2 we will show that whenever (a_0, c_0) satisfies (3.3), a C^1-function $c(a)$ can be defined in a neighborhood of a_0 such that $c(a_0) = c_0$. We will use the implicit function theorem to show that. In Step 3 we will expand the domain of the function $c(a)$. We will define the infimum $\underline{a}$ such that $c(a)$ can be defined on the interval $(\underline{a}, r_{2k})$. We will prove $\lim_{a \to r_{2k}} T(a) = 2\pi/\sqrt{f_u(r_{2k}, 0)}$ where $T(a)$ is the period of the closed orbit corresponding to $(a, c(a))$. We will also prove $\lim_{a \to \underline{a}} T(a) = \infty$.

Step 1: We linearize (3.2) at the point $(r_{2k}, 0)$:

$$\frac{d}{d\zeta}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -f_u(r_{2k}, 0) & -c - f_p(r_{2k}, 0) \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix}, \tag{3.4}$$

where f_u and f_p indicate derivatives of f with respect to the first and the second variable respectively. Let $\nu_\pm$ be the eigenvalues of the above matrix. Then we have

$$\mathrm{Re}(\nu_\pm) = -\frac{c + f_p(r_{2k}, 0)}{2}, \ \mathrm{Im}(\nu_\pm) = \pm\sqrt{-f_u(r_{2k}, 0) + \left(\frac{c + f_p(r_{2k}, 0)}{2}\right)^2}.$$

We regard c as a parameter. If $c = -f_p(r_{2k}, 0)$, then the matrix is non-singular, has the pair of simple pure imaginary eigenvalues $\pm i\mu$ ($\mu > 0$), and has no eigenvalue of the form $\pm ik\mu$ ($k \in \mathbb{N},\ k \neq 1$). Moreover we can easily see that

$$\left.\frac{d\mathrm{Re}(\nu_\pm)}{dc}\right|_{c=-f_p(r_{2k},0)} = -\frac{1}{2} < 0.$$

Therefore, (see for example Theorem 2.6 of [AP93] (Section 7, page 144)) a Hopf bifurcation occurs at $c = -f_p(r_{2k}, 0)$. Thus there are closed orbits encircling the point $(r_{2k}, 0)$ on the phase plane that have any small amplitude.

Step 2: From Step 1, we assume that there is a closed orbit corresponding to (a_0, c_0) on the phase plane. The continuity of the arc with respect to a and c guarantees that there is a constant $\varepsilon > 0$ such that the arc corresponding to (a, c) exists as shown in Figure 3 provided that $|a - a_0| < \varepsilon$ and $|c - c_0| < \varepsilon$. Since the solution $U(\zeta)$ to (3.2) with initial data $U(0) = a$, $U_\zeta(0) = 0$ depends on a and c continuously, we write $U = U(\zeta, a, c)$. Let $F(\,\cdot\,,\,\cdot\,)$ be a function as follows:

$$F(a, c) := U(\tau(a, c), a, c) - a, \tag{3.5}$$

where $\tau(a, c)$ which is defined in the first part of Section 3 is the arrival time of *the arc corresponding to* (a, c). From (3.3), the arc corresponding to (a, c) is a closed orbit if and only if $F(a, c) = 0$. We will prove that there exists a C^1-function $c(a)$ in a neighborhood of a_0 that satisfies $F(a, c(a)) = 0$. First we see by the assumption that $F(a_0, c_0) = 0$. Second we see that $U(\zeta, a, c)$ is a C^2-function of ζ, a and c by the general theory of ordinary differential equations. Using the equation

$$U_\zeta(\tau(a + \Delta a, c), a + \Delta a, c) - U_\zeta(\tau(a, c), a, c) = 0,$$

where Δa is a small number and the definition of the derivative, we can show that $\tau(a, c)$ is of class C^1. Thus $F(a, c)$ is of class C^1. Third we will show that $F_c(a_0, c_0) \neq 0$ where

$$F_c(a, c) = U_\zeta(\tau(a, c), a, c)\tau_c(a, c) + U_c(\tau(a, c), a, c).$$

Since $U_\zeta(\tau(a_0, c_0), a_0, c_0) = 0$, we obtain

$$F_c(a_0, c_0) = U_c(\tau(a_0, c_0), a_0, c_0).$$

We will prove in Lemma 3.2 below that

$$U_c(\tau(a_0, c_0), a_0, c_0) \neq 0. \tag{3.6}$$

Now we assume that Lemma 3.2 holds. Then the implicit function theorem says that there is a C^1-function $c(a)$ that satisfies $F(a, c(a)) = 0$ for $a \in (a_0 - \tilde{\varepsilon}, a_0 + \tilde{\varepsilon})$ where $\tilde{\varepsilon}$ (> 0) is so small that $|c_0 - c(a)| < \varepsilon$ and $|a_0 - a| < \varepsilon$ for $a \in (a_0 - \tilde{\varepsilon}, a_0 + \tilde{\varepsilon})$.

We will see in Lemma 3.2 that $U(\tau(a, c), a, c)$ is non-decreasing in c and (3.6) holds. Thus $U(\tau(a, c), a, c)$ is increasing in c. For each fixed a, if there exists c satisfying (3.3), then c is uniquely determined. The function $c(a)$ is

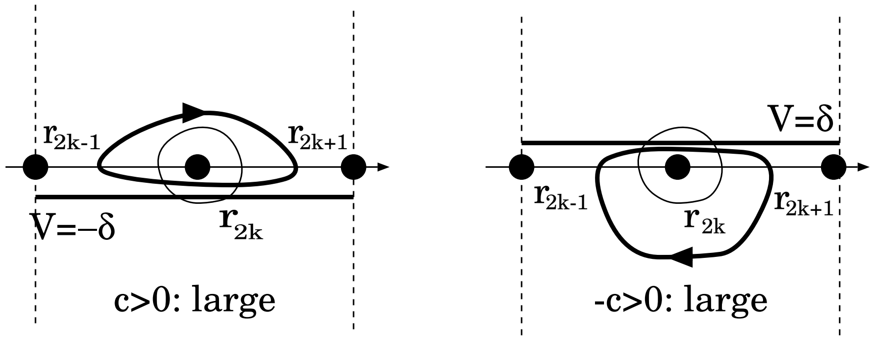

FIGURE 4. The phase planes of (3.2) for two extreme cases. Circles in each picture are the closed orbit corresponding to (a_0, c_0). If c is large, then the arc corresponding to (a, c) $(a < a_0)$ cannot pass the segment $(r_{2k-1}, r_{2k+1}) \times \{-\delta\}$ and the arrival point is in the inside of the closed orbit (the left picture). If $-c$ is large, then the arc corresponding to (a, c) does not pass the segment $(r_{2k}, r_{2k+1}) \times \{0\}$ (the right picture).

uniquely determined. This means that there is no closed orbit corresponding to (a_0, c_1) $(c_1 \neq c_0)$ when there is a closed orbit corresponding to (a_0, c_0).

Step 3: Hereafter we suppose that there exists the closed orbit corresponding to (a_0, c_0). We define $\underline{a}$ as follows:

$$\underline{a} := \inf\{a \in \mathbb{R} |\ c = c(\xi) \text{ can be defined for all } \xi \in (a, a_0)\}.$$

Note that there is a closed orbit corresponding to $(a, c(a))$ for all $a \in (\underline{a}, a_0)$. We will show by contradiction that the family of closed orbit corresponding to $(a, c(a))$ $(a \in (\underline{a}, a_0))$ is not uniformly away from two points $(r_{2k-1}, 0)$ and $(r_{2k+1}, 0)$. We assume that the family is uniformly away from two points.

We will show that there exists a constant $c^* > 0$ such that the following holds: if $|c| > c^*$, then a closed orbit starting from the point $(a, 0)$ $(a < a_0)$ does not exists.

For any $\delta > 0$, there is a constant c (> 0) such that $-cV - f(U, V) > 0$ on the segment $(r_{2k-1}, r_{2k+1}) \times \{-\delta\}$. The segment should intersect the closed orbit corresponding to (a_0, c_0) provided that δ is small. If there is a closed orbit corresponding to (a, c) $(a < a_0)$, then it should intersect the other closed orbit and this contradicts to Lemma 4.1. Similarly, if $-c$ (> 0) is large, then there should not exist closed orbits corresponding to (a, c) $(a < a_0)$.

If the closed orbit corresponding to (a, c) $(a < a_0)$ exists, then $c = c(a)$ is bounded.

Let $\{a_m\}_{m=1}^{\infty}$ be a sequence that satisfies the following:

$$a_m > \underline{a}, \quad a_m \to \underline{a} \ \text{ as } \ m \to \infty.$$

Since $c(a)$ is bounded, then there exists a constant c_* such that the following holds:

$$c(a_m) \to c_* \quad \text{as} \quad m \to \infty.$$

We consider the arc corresponding to $(\underline{a}, c_*)$. Let $(U(\zeta), U_\zeta(\zeta))$ be a closed orbit with period T_1. Then $U(\zeta)$ satisfies (2.1). From Lemma 3.3 below, there is a constant $M > 0$ such that $||U_\zeta(\zeta)|| \le M$. Thus any closed orbit is bounded on the phase plane.

Because of the continuity of arcs with respect to a and c, the boundedness of arcs, and the assumption that the family of closed orbit is uniformly away from the two points, the arrival point of the arc corresponding to $(\underline{a}, c_*)$ exists. Thus $U(\tau(\underline{a}, c_*), \underline{a}, c_*)$ can be defined. Using the continuity of $U(\tau(a,c), a, c)$ with respect to a and c, we can obtain a contradiction if we assume that $U(\tau(\underline{a}, c_*), \underline{a}, c_*) \neq \underline{a}$. Thus we see that

$$U(\tau(\underline{a}, c_*), \underline{a}, c_*) = \underline{a}.$$

This implies that there exists a closed orbit that contains $(\underline{a}, 0)$ on the phase plane. This is a contradiction because of the definition of $\underline{a}$ and Step 2. Thus the family is not uniformly away from the two points $(r_{2k-1}, 0)$ and $(r_{2k+1}, 0)$. This means that $\underline{a} = r_{2k-1}$ or the shortest distance of the closed orbit corresponding to $(a, c(a))$ and the point $(r_{2k+1}, 0)$ goes to zero as $a \to \underline{a}$.

We will show that $c(a)$ can be defined in $(\underline{a}, r_{2k})$. We define $\bar{a}$ as follows:

$$\bar{a} := \sup\{a \in \mathbb{R} |\ c = c(\xi) \text{ can be defined for all } \xi \in (\underline{a}, \bar{a})\}.$$

Suppose $\bar{a} < r_{2k}$. From Step 1 we can find $\tilde{a}$ with $\bar{a} < \tilde{a} < r_{2k}$ so that there is a closed orbit that contains $(\tilde{a}, 0)$ on the phase plane. Since there are closed orbits with any small amplitude encircling the point $(r_{2k}, 0)$. The function $c(a)$ can be defined at some $\tilde{a}$ for $\tilde{a} \in (\bar{a}, r_{2k})$. Using Step 2, we can expand the domain of $c(a)$ to the left. Since $c(a)$ is unique, this contradicts to the definition of $\bar{a}$. Thus $\bar{a} = r_{2k}$.

Since $c(a)$ is unique and continuous, there is precisely one closed orbit that contains the point $(a, 0)$. Thus the limit $\lim_{a \to r_{2k}} T(a)$ should coincide with the limit in the statements of Theorem 2.6 in Section 7 of [AP93]. We have

$$\lim_{a \to r_{2k}} T(a) = \frac{2\pi}{\sqrt{f_u(r_{2k}, 0)}}.$$

Hereafter we will show that $\lim_{a \to \underline{a}} T(a) = \infty$ in the case where the shortest distance of the family of periodic orbits and the point $(r_{2k+1}, 0)$ goes to zero. First, we consider the linearized eigenvalue problem of (3.2) at the point $(r_{2k+1}, 0)$. Let λ_1, λ_2 be the eigenvalues. Then we have

$$\lambda_1 = \frac{1}{2}\left\{ -(c + f_p(r_{2k+1}, 0)) - \sqrt{(c + f_p(r_{2k+1}, 0))^2 - 4 f_u(r_{2k+1}, 0)} \right\},$$
$$\lambda_2 = \frac{1}{2}\left\{ -(c + f_p(r_{2k+1}, 0)) + \sqrt{(c + f_p(r_{2k+1}, 0))^2 - 4 f_u(r_{2k+1}, 0)} \right\}.$$

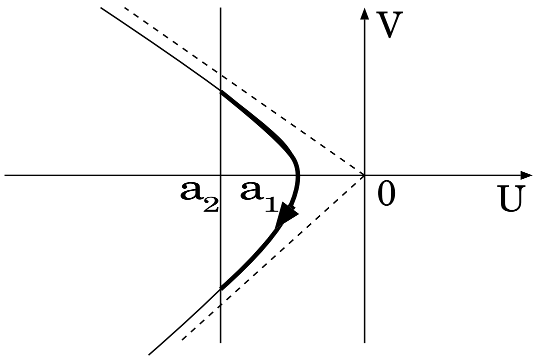

FIGURE 5. This picture indicates the phase plane displayed in the new coordinate. The thick curved arrow $(\tilde{U}, \tilde{V})$ is the arc that we observe. Two dashed lines are directions of the two eigenvectors of the matrix. The time required for traveling through the thick part of the arc diverges as $a_1 \to 0$.

Because $f_u(r_{2k+1}, 0) < 0$, we have $\lambda_1 < 0 < \lambda_2$. Thus the equilibrium point $(r_{2k+1}, 0)$ on the phase plane is hyperbolic. Then the Grobman-Hartman theorem says that there is a local homeomorphism Ψ such that $\phi_t \circ \Psi = \Psi \circ \tilde{\phi}_t$ and $\Psi(0,0) = (r_{2k+1}, 0)$ where $\phi_t, \tilde{\phi}_t$ are the semiflows on $\mathbb{R}^2$ formed by (3.2) and (3.4) respectively.

We can see that the time required for traveling through a neighborhood of the origin diverges as the shortest distance of the arc and the origin tends to zero. We omit the details of the proof of this fact.

We consider arcs of (3.2) in a neighborhood $(r_{2k+1}, 0)$. For each arc corresponding to (a, c), there is an orbit of (3.4) that is mapped to the arc by Ψ. Since $c(a)$ is bounded, the time which needs the orbit of (3.4) to pass a neighborhood of the origin uniformly diverges. Thus the time which needs the arc corresponding to $(a, c(a))$ to pass a neighborhood of the origin diverges as $a \to \underline{a}$. This means

$$\lim_{a \to \underline{a}} T(a) = \infty. \tag{3.7}$$

We can prove (3.7) similarly in the case where $\underline{a} = r_{2k-1}$. The proof is completed. □

LEMMA 3.2. *Let $F(a, c)$ be the function defined by* (3.5)*. If there is a closed orbit corresponding to* (a_0, c_0) *on the phase plane, then* $F_c(a_0, c_0) \neq 0$.

Proof. We use the notation used in the proof of Lemma 3.1. We assume that a closed orbit corresponding to (a_0, c_0) exists. Differentiating $F(a, c) = U(\tau(a, c), a, c) - a$ with respect to c yields

$$F_c(a, c) = U_\zeta(\tau(a, c), a, c)\tau_c(a, c) + U_c(\tau(a, c), a, c).$$

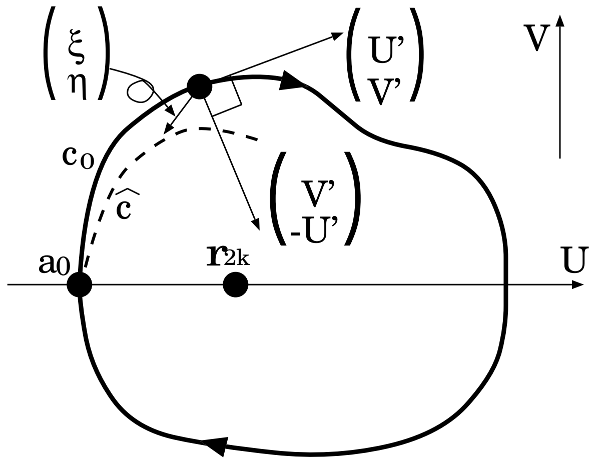

FIGURE 6. The thick closed curve represents the closed orbit corresponding to (a_0, c_0) whose starting and arrival points are $(a_0, 0)$. The dashed curve represents the arc corresponding to $(a_0, \hat{c})$ $(\hat{c} > c_0)$ whose starting point is also $(a_0, 0)$. The short arrow represents the vector (ξ, η). This picture indicates that the vector (ξ, η) points toward the interior of the closed orbit.

We have

$$F_c(a_0, c_0) = U_c(\tau(a_0, c_0), a_0, c_0),$$

because $U_\zeta(\tau(a_0, c_0), a_0, c_0) = 0$. We have to show that $U_c(\tau(a_0, c_0), a_0, c_0) \neq 0$. Let $\hat{c}$ $(> c_0)$ be a real number that is close to c_0. Using the vector $\begin{pmatrix} V \\ -c_0V - f(U,V) \end{pmatrix}$, we can see by [Du53] that the arc corresponding to (a_0, c_0) does not intersect with the arc corresponding to $(a_0, \hat{c})$ in spite that all assumptions of [Du53] are not satisfied on $\{V = 0\}$. The continuity of the arc corresponding to (a, c) with respect to c, togather with the above fact, tells us that the point $(U(\zeta, a_0, \hat{c}), V(\zeta, a_0, \hat{c}))$ $(\zeta > 0)$ is in the domain surrounded by the closed orbit corresponding to (a_0, c_0). This means that $U(\tau(a, c), a, c)$ is non-decreasing in c. We define ξ and η as follows:

$$\xi(\zeta) := U_c(\zeta, a_0, c_0), \quad \eta(\zeta) := V_c(\zeta, a_0, c_0),$$

where U_c is a derivative of U with respect to the third variable. Let $G(\zeta)$ be the inner product of $\begin{pmatrix} V_\zeta \\ -U_\zeta \end{pmatrix}$ and $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$. Namely $G(\zeta) = \xi(\zeta)V_\zeta(\zeta) - \eta(\zeta)U_\zeta(\zeta)$. Then we have $G(\zeta) \geq 0$, because the vector $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ points toward the interior of the closed orbit (see Figure 6).

Differentiating (3.2) with respect to c yields

$$\begin{cases} \dfrac{d\xi}{d\zeta} = \eta \\ \dfrac{d\eta}{d\zeta} = -v - c\eta - f_u(U(\zeta), V(\zeta))\xi - f_p(U(\zeta), V(\zeta))\eta. \end{cases} \tag{3.8}$$

Using (3.2) and (3.8), we can express $G(\zeta)$, $G_\zeta(\zeta)$, $G_{\zeta\zeta}(\zeta)$ and $G_{\zeta\zeta\zeta}(\zeta)$ with ξ, η, c, V and derivatives of f as follows:

$$\begin{aligned} G(\zeta) =& -(c\xi V + \xi f + \eta V), \\ G_\zeta(\zeta) =& (c\xi V + \xi f + \eta V)(c + f_p) + V^2, \\ G_{\zeta\zeta}(\zeta) =& (c\xi V + \xi f + \eta V)\left\{f_{up}V - (cV + f)f_{pp} - (c + f_p)^2\right\} \\ & - 3cV^2 - V^2 f_p - 2Vf, \\ G_{\zeta\zeta\zeta}(\zeta) =& (c\xi V + \xi f + \eta V)\left[(cV + f)^2 f_{ppp} + \{4(c + f_p)(cV + f) - Vf_u\} f_{pp}\right. \\ & - (4cV + 3Vf_p + f_p)f_{up} - V(3cV + 1)f_{upp} \\ & \left. + V^2 f_{uup} - V^2 + (c + f_p)^3\right] \\ & - V^2(Vf_{up} - cVf_{pp}ff_{pp}) - 2V(Vf_u - cVf_p - ff_p) \\ & + 2(cV + f)(3cV + ff_pV + f). \end{aligned}$$

We suppose that $U_c(\tau(a_0, c_0), a_0, c_0) = \xi(\tau(a_0, c_0)) = 0$. Since $V(\tau(a_0, c_0), a_0, c_0) = 0$, we obtain $G(\tau) = G_\zeta(\tau) = G_{\zeta\zeta}(\tau) = 0$ and $G_{\zeta\zeta\zeta}(\tau) = 2f^2 > 0$ where $\tau = \tau(a_0, c_0)$. Therefore, there is a small constant $\delta > 0$ such that $G(P - \delta) < 0$. This is a contradiction, because $G(\zeta) \geq 0$. □

LEMMA 3.3. *There is a constant $M > 0$ such that* $\sup_{\zeta\in\mathbb{R}} |U_\zeta(\zeta)| \leq M$ *for any closed orbit* $(U(\zeta), U_\zeta(\zeta))$ *of* (3.2) .

Proof. Let $(U(\zeta), U_\zeta(\zeta))$ be a closed orbit of (3.2) with some c. Then $U(\zeta)$ satisfies (2.1). Thus from (A3) there is a constant $L_2 > 0$ such that $||U(\zeta)||_{C^1(S^1)} < L_2$ for any periodic solution or constant solution. The lemma is proved. □

Lemma 3.2 completes the proof of Lemma 3.1.

4. PREPARATION FOR THE PROOF OF THEOREM A

In this section we will show that every *cluster* is a totally ordered set in the order $\rhd$ (Corollary 4.2). We will show that $z(u - v) = z(v_x)$ provided that $u, v \in S$ and $v \rhd u$ (Lemma 4.4). The two lemmas are used to prove Theorem A.

The following Lemma 4.1 is a generalized version of Corollary 4.2 below.

LEMMA 4.1. *Let $(u(x), u_x(x))$, $(v(x), v_x(x))$ be closed orbits on the phase plane. Then the two closed orbits does not intersect.*

We can prove Lemma 4.1 by contradiction. We omit the proof.

Using a phase plane analysis and Lemma 4.1, we immediately obtain the following corollary.

Corollary 4.2 (Matano and Nakamura [MN97]). *Let* $u, v \in S$. *If* $R(u) \cap R(v) \neq \emptyset$, *then* $\mathrm{Int}(R(v)) \supset R(u)$ *or* $\mathrm{Int}(R(u)) \supset R(v)$ *where* $\mathrm{Int}(R(u))$ *indicates the set consists of the interior points of* $R(u)$.

Remark 4.3. Let $u, v \in S$ $(u \neq v)$. By Corollary 4.2, we can see that $u \rhd v$ means that $\mathrm{Int}(R(u)) \supset R(v)$.

Let $u, v \in S$. By using Corollary 4.2, we have either $u \rhd v$ or $v \rhd u$ provided that $R(u) \cap R(v) \neq \emptyset$.

Corollary 4.2 and the definition of the *clusters* show that every *cluster* is a totally ordered set. Thus we can number the elements of each *cluster* $\{u_1^{kl}, u_2^{kl}, \ldots, u_{m_{kl}}^{kl}\}$ in such a way that

$$u_1^{kl} \lhd u_2^{kl} \lhd \cdots \lhd u_{m_{kl}}^{kl}.$$

Lemma 4.4 (Matano and Nakamura [MN97]). *Let* $u, v \in S$. *If* $v \rhd u$, *then* $z(u - v) = z(v_x)$.

5. Proof of Corollary B and Lemmas E and F

In this section we will prove Corollary B and Lemmas E and F by using Lemma 5.1 and the results in Sections 3 and 4.

Let $c = c(a)$ be the function defined in the statement of Lemma 3.1, and let $T = T(a)$ be the period of the closed orbit corresponding to $(a, c(a))$ defined in the statement of Lemma 3.1.

Lemma 5.1. *Let* $u \in S$ *be the closed orbit corresponding to* $(a_0, c(a_0))$ *in Section* 3. *If* u *is hyperbolic, then* $\partial_a T(a)|_{a=a_0} \neq 0$.

Proof. We will prove the lemma by contradiction. We assume that $\partial_a T(a)|_{a=a_0} = 0$. Let $u(x,t) = U(\zeta)$ $(\zeta = x - ct)$ be a rotating wave or a steady state. We can suppose that $U(0) = a$ and $U_\zeta(0) = 0$ without loss of generality. The function $U = U(\zeta, a, c(a))$ defined in Section 3 satisfies $U_{\zeta\zeta} + c(a)U_\zeta + f(U, U_\zeta) = 0$. Differentiating the equation with respect to a gives

$$\partial_{\zeta\zeta}(U_a + c_a U_c) + c\partial_\zeta(U_a + c_a U_c) + f_u \cdot (U_a + c_a U_c) + f_p \partial_\zeta(U_a + c_a U_c) = -c_a U_\zeta.$$

Let $\varphi(\zeta) = U_a(\zeta) + c_a U_c(\zeta)$. The function $\varphi(\zeta)$ satisfies the following equation:

$$\varphi_{\zeta\zeta} + c\varphi_\zeta + f_u \varphi + f_p \varphi_\zeta = -c_a U_\zeta, \quad \zeta \in S^1. \tag{5.1}$$

If $c_a(a_0) = 0$, then $\alpha \cdot U_\zeta(\zeta)$ $(\alpha \in \mathbb{R})$ are the solutions to (5.1) because of the hyperbolicity of $U(\zeta)$. If $c_a(a_0) \neq 0$, then (5.1) has no solution. Because 0 is a simple eigenvalue of the following problem:

$$\varphi_{\zeta\zeta} + c\varphi_\zeta + f_u \varphi + f_p \varphi_\zeta = \lambda\varphi, \quad \zeta \in S^1.$$

Case 1: $c_a(a_0) = 0$
Differentiating $U(0, a, c(a)) = U(T(a), a, c(a))$ with respect to a gives

$$(5.2)\quad \begin{aligned} &U_a(0, a, c(a)) + c_a(a)U_c(0, a, c(a)) \\ &= \partial_a T(a)U_\zeta(T(a), a, c(a)) + U_a(T(a), a, c(a)) + c_a(a)U_c(T(a), a, c(a)). \end{aligned}$$

Substituting $\partial_a T(a)|_{a=a_0} = 0$ and $c_a(a_0) = 0$ for (5.2) gives $U_a(0, a_0, c(a_0)) = U_a(T(a_0), a_0, c(a_0))$. Since $U_a(\,\cdot\,, a_0, c(a_0))$ is a periodic function and the period $T(a_0)$ is equal to $1/n$ for some $n \in \{1, 2, \cdots\}$, we have $U_a(0, a_0, c(a_0)) = U_a(1, a_0, c(a_0))$. Since $\varphi(\zeta) = U_a(\zeta)$, we have

$$(5.3)\qquad \varphi(0) = \varphi(1).$$

We differentiate $U_\zeta(0, a, c(a)) = U_\zeta(T(a), a, c(a))$ with respect to a, and substitute a_0 for it. Then we obtain

$$U_{\zeta a}(0, a_0, c(a_0)) = U_{\zeta a}(T(a_0), a_0, c(a_0)) + \partial_a T(a)|_{a=a_0} U_{\zeta\zeta}(T(a_0), a_0, c(a_0)).$$

Since $\varphi_\zeta(\zeta) = U_{\zeta a}(\zeta, a_0, c(a_0))$, we have

$$\varphi_\zeta(0) = \varphi_\zeta(T(a_0)) + \partial_a T(a)|_{a=a_0} U_{\zeta\zeta}(T(a_0), a_0, c(a_0)).$$

Since $\partial_a T(a)|_{a=a_0} = 0$ and $\varphi_\zeta(T(a_0)) = \varphi_\zeta(1)$, we have

$$(5.4)\qquad \varphi_\zeta(0) = \varphi_\zeta(1).$$

Using (5.3) and (5.4), we can see that $\varphi(\zeta)(= U_a(\zeta))$ satisfies (5.1) and periodic boundary conditions. By the hyperbolicity of $u(x, t)(= U(\zeta))$, we see that $\varphi(\zeta) = \alpha \cdot U_\zeta(\zeta)$ ($\alpha \in \mathbb{R}$) are the solutions to (5.1). On the other hand $\varphi(0) = U_a(0) = 1$. It contradicts that $U_\zeta(0) = 0$. We can see that $\partial_a T(a)|_{a=a_0} \neq 0$.

Case 2: $c_a(a_0) \neq 0$
Using the assumption of contradiction $\partial_a T(a)|_{a=a_0} = 0$, we can obtain the following two equalities in a similar way of Case 1:

$$(5.5)\qquad \varphi(0) = \varphi(1), \quad \varphi_\zeta(0) = \varphi_\zeta(1).$$

Using (5.5), we can see that $\varphi(\zeta)$ satisfies (5.1) and periodic boundary conditions. The function $\varphi(\zeta)$ is a non-trivial solution to (5.1). This is a contradiction. Therefore, we obtain $\partial_a T(a)|_{a=a_0} \neq 0$. □

Hereafter, we consider the structure of each *cluster*. We divide the *clusters* in two types. One is a type of *clusters* that contain a constant steady state, and the other is a type of *clusters* that do not have a constant steady state.

First, we consider the type of *clusters* that have a constant steady state. Since the *cluster* C_{kl} has a constant steady state, we can see that $k = l$ by using a phase plane analysis. If k is odd, then $\sharp C_{kk} = 1$ and the element of C_{kk} is a stable constant steady state. If k is even, then $\sharp C_{kk} \geq 1$ and C_{kk} has precisely one unstable constant steady state.

Second, we consider the type of *clusters* C_{kl} that do not have a constant steady state. By observing the phase plane, we see that $l \geq k+2$, and k and l are even. If $u(x, t) = U(x - ct)$ is an element of C_{kl} that satisfies $(U(0), U_\zeta(0)) = (a, 0)$ and $U(\zeta) \leq a$, then we can deform the closed orbit $(U(\zeta), U_\zeta(\zeta))$ on the phase

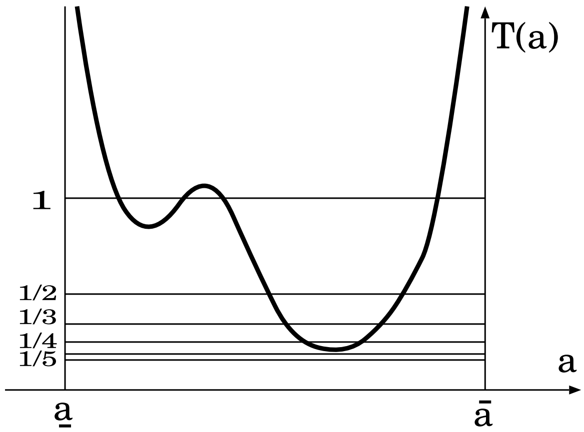

FIGURE 7. The picture shows the graph of $T(a)$ in the case of C_{kl} $(k \neq l)$. Each of the intersections of the curve and the lines corresponds to a rotating wave. In this case, the sequence of modified Morse indices is $(2, 2, 2, 4, 6, 8, 8, 6, 4, 2)$.

plane by using a similar way of Step 2 and Step 3 in the proof of Lemma 3.1, and enlarge the domain of $c = c(a)$. Let $(\underline{a}, \bar{a})$ be the maximal connected domain of the function $c = c(a)$. The closed orbit that corresponds to $(a, c(a))$ approaches $(r_{2k-1}, 0)$ or $(r_{2l+1}, 0)$ as $a \to \underline{a}$. Since $c(a)$ is bounded, the function $T(a)$ diverges to $+\infty$ as $a \to \underline{a}$. The function $T(a)$ diverges to $+\infty$ as $a \to \bar{a}$, because the closed orbit approaches $(r_{2k}, 0)$, …, $(r_{2l-1}, 0)$ or $(r_{2l}, 0)$, and $c(a)$ is bounded. Hence the graph of $T(a)$ is as shown in Figure 7.

Proof of Lemma E. The statements (i) and (ii) are easily understood by observing a phase plane. □

Proof of Lemma F. We can see that (i), (iv) and (v) follow from Figures 7 and 8. Lemma 3.1 implies (ii) and (iii). □

Proof of Corollary B. Since $S = \bigcup_{1 \le k \le l \le N} C_{kl}$, we obtain the following:

$$\begin{aligned} \sharp S &= \sum_{1 \le k \le l \le N} \sharp C_{kl} \\ &\ge \sum_{j=1}^{N} \sharp C_{jj} \\ &\underset{\text{by Figure 8}}{\ge} N + \sum_{j=1}^{N} \left[\left[\frac{\sqrt{(f_u(r_j, 0))_+}}{2\pi} \right]\right]. \end{aligned}$$

□

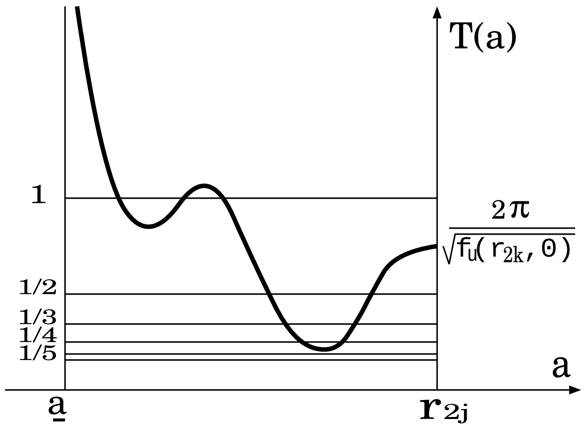

FIGURE 8. The picture indicates the graph of $T(a)$ in $(\underline{a}, r_{2k})$. Each of the intersections of the curve and the lines corresponds to a rotating wave. The sequence of modified Morse indices is easily computed from this picture. In this case, the sequence of modified Morse indices is $(4, 4, 6, 8, 8, 6, 4, 2, 2, 2)$.

Remark 5.2. If $\sharp S$ attains the lower bound, then every *cluster* is *simple* and *monotone*. If every *cluster* is *simple*, then the equality in the first inequality in the proof of Corollary B holds. If every *cluster* is *monotone*, then the equality in the second inequality in the proof of Corollary B holds. Therefore, $\sharp S$ attains the lower bound if and only if every *cluster* is *simple* and *monotone*.

6. PROOF OF THEOREMS A AND C

In this section we will prove Theorems A and C by using Lemma 6.1, Lemma F and the main results of [AF88]. A simple example is given at the end of this section.

LEMMA 6.1 (Blocking lemma). *Let $v, w \in S$ ($w \rhd v$ and $I(w) < I(v)$). If there exists a wave $\bar{v} \in S$ such that $w \rhd \bar{v} \rhd v$ and $I(\bar{v}) = I(w)$, then v does not connect to w.*

The proof of Lemma 6.1 is essentially the same as the explanation after Definition 1.6 of [FR96].

Remark 6.2. Lemma 6.1 is called the *zero number blocking* (see Definition 1.6 of [FR96]).

We will use the following proposition to prove Theorem A.

PROPOSITION 6.3 (Angenent and Fiedler [AF88]). *Let $u \in S$ with $i(u) > 0$ be hyperbolic. Then*

(i) *The wave u connects to u_+ and u_-.*

(ii) *For any* $n \in \mathbb{N}$, $0 < 2n \leq i(u)$, *there exists a wave* $u^{(n)} \in S$ *such that* $u_- < u^{(n)} < u_+$, $z(u^{(n)} - u) = 2n$, *and* u *connects to* $u^{(n)}$.

We are in a position to prove Theorem A.

Proof of Theorem A. Let v be a wave in C_{kl} ($k \leq l$) and let w be a wave in C_{mn} ($m \leq n$). We prove whether v connects to w or not. When there is a connecting orbit $u = u(t)$ that connects v and w, we can suppose that $I(w) \leq I(v)$, because $i(w) + 1 \leq z(u - v) \leq i(v)$ (see Lemma 3.7 in [AF88]). If $I(v) = 0$, then there is no connecting orbit starting from v. Thus we assume that $I(v) > 0$. We can see that k and l are odd, using a phase plane analysis.

There are two cases in general terms. In one case, w belongs to the same *cluster* as v (*i.e.* $(m, n) = (k, l)$). In the other case, w belongs to another *cluster* which does not include v (*i.e.* $(m, n) \neq (k, l)$). First, we consider the case where $w \in C_{mn}$ ($(m, n) \neq (k, l)$).

Case 1: $(m, n) \neq (k, l)$

We can divide the case into four more cases.

Case 1-1: $(m, n) \in \{(k-1, k-1), (l+1, l+1)\}$

Since both $k - 1$ and $l + 1$ are even, the cluster C_{mn} has precisely one wave (This wave is a stable constant steady state). We can see that v connects to w by (i) of Theorem 6.3, because $w = v_+$ or $w = v_-$.

Case 1-2: $(m, n) \notin \{(k-1, k-1), (l+1, l+1)\}$ *and* $R(C_{kl}) \cap R(C_{mn}) = \emptyset$

There is a wave $\bar{w} \in S$ ($I(\bar{w}) = 0$) between v and w in the usual order (*i.e.* $v(x) < \bar{w}(x) < w(x)$ or $w(x) < \bar{w}(x) < v(x)$). We assume that there is a connecting orbit $u(t)$ that connects v and w. The function $z(u(t) - \bar{w}(t))$ is not non-increasing in t. This is a contradiction. Therefore, the wave v does not connect to any $w \in C_{mn}$. Namely the wave v does not connect to any wave of the *above clusters* and *below clusters* in the usual order except for the two *clusters* of Case 1-1.

Case 1-3: $C_{kl} \rhd C_{mn}$

We see that $i(v) \in \{I(v), I(v) - 1\}$ generally. We have

$$i(v) = I(v) - 1,$$

in the case that $i(v)$ is odd. We suppose that there is a connecting orbit $u(t)$ that connects v and w. Then

$$z(u - v) \leq i(v), \tag{6.1}$$

(see Lemma 3.7 in [AF88]). Lemma 4.4 tells us that (6.1) contradicts that $z(u(t) - v(t)) = I(v)$ for large $t > 0$. The wave v does not connect to any $w \in C_{mn}$. Namely v does not connect to any wave of the *clusters* that is smaller than C_{kl} in the order $\rhd$.

Case 1-4: $C_{mn} \rhd C_{kl}$

There is a $\bar{w} \in S$ ($I(\bar{w}) = 0$) such that $R(v) \cap R(\bar{w}) = \emptyset$ and $w \rhd \bar{w}$. We suppose that there is a connecting orbit $u(t)$ which connects v and w. The function $z(u(t) - \bar{w}(t))$ is not non-increasing in t. This is a contradiction. Therefore, v does not connect to any $w \in C_{mn}$.

The Case 1 can be summarized as follows: If v connects to w in another *cluster*, then w should be v_+ or v_-.

Case 2: $(m, n) = (k, l)$
Let w be another wave of the same *cluster* C_{kl}. We divide this case in two more cases.

Case 2-1: $v \rhd w$
We suppose that there is a connecting orbit $u(t)$ that connects v and w. We can see that

$$I(v) = z(u(t) - v(t)) \leq i(v) \text{ for large } t,$$

(see Lemma 3.7 in [AF88]), because $v \rhd w$. Thus if $i(v)$ is odd (*i.e.* $i(v) = I(v) - 1$), then we obtain a contradiction. The wave u does not connect to w provided that $i(v)$ is odd.

Case 2-2: $w \rhd v$
Owing to Theorem 6.3, the wave v connects to w that attains the following minimum for each d ($d = 2, 4, 6, \ldots, I(v) - 2$):

$$\min_{I(w)=2d, w \rhd v} |R(w)|,$$

where $|R(u)| := \max_{x \in S^1} u(x,t) - \min_{x \in S^1} u(x,t)$. Suppose $i(v)$ is odd. The wave v, however, does not connect to any other w, because Lemma F tells us that there exists a wave $\bar{w}$ such that $w \rhd \bar{w} \rhd v$ and $I(w) = I(\bar{w})$. Thus we can see by Lemma 6.1 that the zero number blocking occurs.

The Case 2 can be summarized as follows. The wave v connects to $I(v)/2 - 1$ different waves that are bigger than v in the order $\rhd$ in the same *cluster*. The wave does not connect to any other wave in the same *cluster* provided that $i(v)$ is odd.

The Case 1 and the Case 2 cover all the combinations of v and w. Thus the proof is completed. □

Proof of Theorem C. We show that the hypotheses of Theorem C satisfy those of Theorem A.

Every *cluster* is *simple* and *monotone* if and only if $\sharp S$ attains the lower bound (see Remark 5.2).

We will show that the Morse index of every wave is odd or zero. Suppose that there is a wave $u \in S$ whose Morse index is even and not zero. Using Proposition 6.3, we can see that there exists a wave $v \in S$ such that $I(u) = I(v)$ and u connects to v heteroclinically. However, u and v are not in the same *cluster*, because the *cluster* is *monotone*. Thus v belongs to another *cluster*. However, there is no heteroclinic connection, because every *cluster* is *simple* and there should be a stable steady state between u and v in the usual order. This is a contradiction. Therefore all the hypotheses of Theorem A are satisfied. □

Example 6.4. Figure 9 shows the profile of every $u \in S$ and the diagram that shows which $u \in S$ and $v \in S$ are connected heteroclinically and which are not when $\{r_j\}_{j=1}^5$ are the roots of $f(\,\cdot\,, 0)$, $[[\sqrt{f_u(r_2, 0)}/(2\pi)]] = 2$, $[[\sqrt{f_u(r_4, 0)}/(2\pi)]] = 3$, $\sharp S = 10$, and all $u \in S$ are hyperbolic.

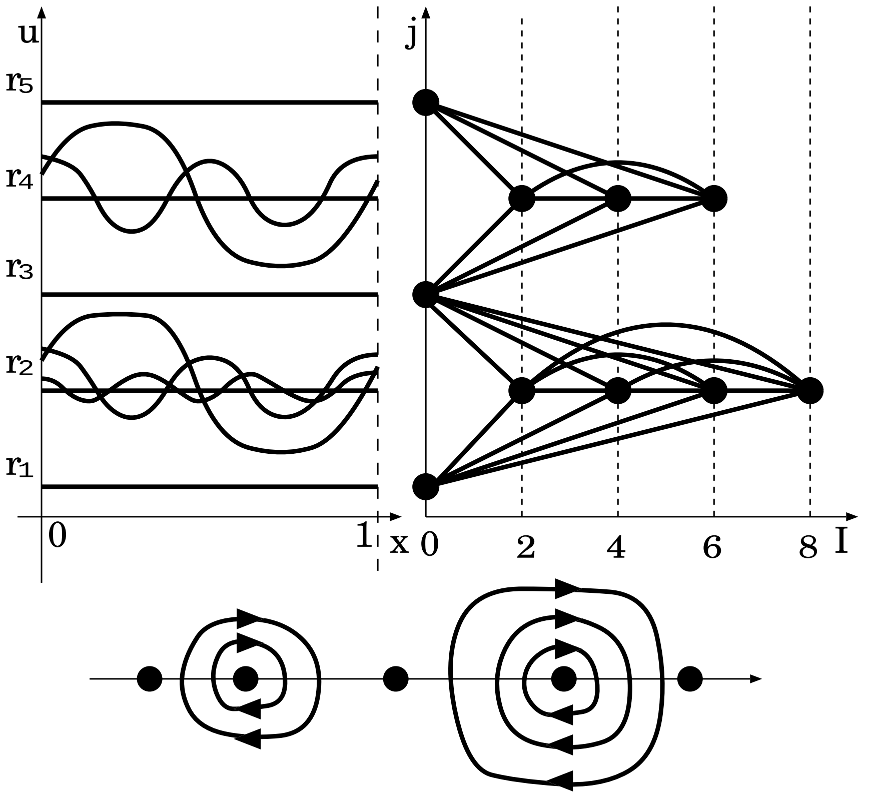

FIGURE 9. In the left figure, the thick curves and the lines indicate the profile of all the waves that move to the right or the left at each constant speed. In the right figure, the horizontal axis indicates the modified Morse index and the vertical axis indicates the suffix of C_{jj}. The points mean elements of S. The thick curves and the lines represent the connecting orbits. The lower figure shows closed orbits and equibrium points in the uu_x-plane. Note that they do not necessarily correspond to the same value of c.

Remark 6.5. If there is a wave $v \in S$ such that $i(v)(\neq 0)$ is even, then we cannot determine by the method used in the proof of Theorem A whether v connects to waves that are smaller than v in the order $\rhd$ or not.

Remark 6.6. We have shown Theorem A by using the *structure* and the results of [AF88]. This means that the results of [AF88] that looks a partial answer is a complete answer in some sense when the Morse index of every wave is odd or zero.

7. Proof of Theorem A' and Lemma F'

In this section we will study the case where the nonlinear term f depends only on u, and establish a sufficient condition that guarantees that all the *clusters* are *simple* and *monotone*.

We will use a character g to denote the nonlinear term (*i.e.* $f(u, u_x) = g(u)$). In this case (1.1) is written as follows:

$$\begin{cases} u_t = u_{xx} + g(u), & x \in S^1, \\ u(x,0) = u_0(x), & x \in S^1. \end{cases} \tag{7.1}$$

Matano [Ma88] showed that (1.1) does not have rotating waves provided that $f(u,p) = f(u,-p)$. Since the nonlinear term g depends only on u and satisfies this property, the equation (7.1) does not have rotating waves.

We consider the following Neumann problem:

$$\begin{cases} u_t = u_{xx} + g(u), & x \in (0, 1/2), \\ u_x(0) = 0 = u_x(1/2). \end{cases} \tag{7.2}$$

Let $u(x)$ be a wave of (7.1). Then there exists $\theta (\in S^1)$ such that $u_x(\theta) = 0$ and $u(x) \leq u(\theta)$ for all $x \in S^1$. We can see by a phase plane analysis that $u_x(\theta + 1/2) = 0$. Therefore $u(x + \theta)$ $(0 < x < 1/2)$ is a steady state of (7.2). Let $\tilde{u}(x)$ denotes $u(x + \theta)$. Thus $\tilde{u}(x)$ is a steady state of (7.2).

Next, let $v(x)$ be a non-constant steady state of (7.2) that satisfies $v(x) \leq v(0)$. Then $u(x)$ is a standing wave of (7.1) where

$$u(x) = \begin{cases} v(x), & 0 \leq x \leq 1/2; \\ v(1-x), & 1/2 \leq x \leq 1. \end{cases}$$

We can identify any wave u of (7.1) with a steady state $\tilde{u}$ of (7.2), and by *the steady state associated with u of* (7.2) we shall mean $\tilde{u}$. In short $\tilde{u} = v$.

Let v, w be steady states of (7.1) and let $\tilde{v}$, $\tilde{w}$ be steady states associated with v, w respectively. Suppose that a heteroclinic orbit $\tilde{u}(x,t)$ of (7.2) that connects $\tilde{v}$ and $\tilde{w}$ exists. Then $u(x,t)$ is a solution of (7.1) where

$$u(x,t) = \begin{cases} \tilde{u}(x,t), & 0 \leq x \leq 1/2; \\ \tilde{u}(1-x,t), & 1/2 \leq x \leq 1. \end{cases}$$

Moreover $u(\,\cdot\,,t) \to v(x)$ $(t \to -\infty)$ and $u(\,\cdot\,,t) \to w(x)$ $(t \to \infty)$. Thus $u(x,t)$ is a connecting orbit of (7.1) that connects v and w. In short, v connects to w if $\tilde{v}$ connects to $\tilde{w}$. We will use this fact to prove the existence of connecting orbits in the proof of Theorem A'.

We give two lemmas about (7.2) without proofs.

Lemma 7.1. *Let $\{u_1^{kl}, u_2^{kl}, \ldots, u_{m_{kl}}^{kl}\}$ be a cluster and let $\{\tilde{u}_1^{kl}, \tilde{u}_2^{kl}, \ldots, \tilde{u}_{m_{kl}}^{kl}\}$ be the set of steady states of* (7.2) *associated with the waves of the cluster. Let $\{u_{b_1}^{kl}, u_{b_2}^{kl}, \ldots, u_{b_n}^{kl}\}$ $(b_1 < b_2 < \cdots < b_n)$ be the waves whose Morse indices are the same number (i.e. $I(u_{b_1}^{kl}) = I(u_{b_2}^{kl}) = \cdots = I(u_{b_n}^{kl})$). Then $i(\tilde{u}_{b_{n-2j}}^{kl}) =$*

$I(u^{kl}_{b_{n-2j}})/2$ for $j \in \{0, 1, \dots, [(n-1)/2]\}$, and $i(\tilde{u}^{kl}_{b_{n-2j-1}}) = I(u^{kl}_{b_{n-2j-1}})/2 + 1$ for $j \in \{0, 1, \dots, [(n-2)/2]\}$.

Proof. In the case of the Dirichlet problem, we can find the proof in Lemma 2.1 of [BF89]. We can prove the lemma in a similar way. □

Lemma 7.2. *Let u, v, w be waves and let $\tilde{u}$, $\tilde{v}$ be the steady states associated with u, v. If $i(u)$ is even, then the steady state $\tilde{u}$ connects to every $\tilde{v}$ that satisfies the following: $u \rhd v$, and there is no wave w such that $u \rhd w \rhd v$ and $I(u) = I(w)$.*

Proof. In the case of the Neumann problem, the problem of the heteroclinic connections are completely determined by [FR96]. We can prove the lemma by using Lemma 7.1, Definition 1.6 of [FR96] and Lemma 1.7 of [FR96]. □

Proof of Lemma F'. If $b_{n-2j} > 1$, then there exists v ($\lhd u^{kl}_{b_{n-2j}}$) such that v blocks the connections from $u^{kl}_{b_{n-2j}}$ to all the wave that are smaller than $u^{kl}_{b_{n-2j}}$ in the order $\rhd$. This means that $i(u^{kl}_{b_{n-2j}}) = I(u^{kl}_{b_{n-2j}}) - 1$. If $b_{n-2j} = 1$, then $k = l$. There also exists a wave v that satisfies the above conditions (the wave v may be a constant steady state). Thus $i(u^{kl}_{b_{n-2j}}) = I(u^{kl}_{b_{n-2j}}) - 1$. In short $i(u^{kl}_{b_{n-2j}}) = I(u^{kl}_{b_{n-2j}}) - 1$ for $j \in \{0, 1, \dots, [(n-1)/2]\}$.

We consider whether $i(u^{kl}_{b_{n-2j-1}}) = I(u^{kl}_{b_{n-2j-1}}) - 1$ or $i(u^{kl}_{b_{n-2j-1}}) = I(u^{kl}_{b_{n-2j-1}})$. If $n - 2j - 1 > 1$, then $\tilde{u}^{kl}_{b_{n-2j-1}}$ connects to $\tilde{u}^{kl}_{b_{n-2j-2}}$. Thus $u^{kl}_{b_{n-2j-1}}$ connects to $u^{kl}_{b_{n-2j-2}}$. This means that $i(u^{kl}_{b_{n-2j-2}}) = I(u^{kl}_{b_{n-2j-1}})$. If $n-2j-1 = 1$, then there exists a wave $\tilde{v}$ such that the following hold: $v \lhd u^{kl}_{b_1}$ and $\tilde{u}^{kl}_{b_1}$ connects to $\tilde{v}$. Thus $u^{kl}_{b_1}$ connects to v. Hence $i(u^{kl}_{b_1}) = I(u^{kl}_{b_1})$. In short $i(u^{kl}_{b_{n-2j-1}}) = I(u^{kl}_{b_{n-2j-1}})$ for $j \in \{0, 1, \dots, [(n-2)/2]\}$.

□

Proof of Theorem A'. Let u be a non-constant wave whose Morse index is even. In Theorem A we have identified waves that are connected by u and that satisfy $z(u - v) \le I(u) - 2$. Thus we have to check whether u connects to v or not, in the case where $z(u - v) = I(u)$.

Case 1: $v \rhd u$

Let w be a wave that satisfies the following: w is the smallest wave in the order $\rhd$ that satisfies $w \rhd u$ and $I(u) = I(w)$. Because of Lemma F', w exists in the *cluster* to which u belongs, and $i(w) = I(w) - 1$. Let $\tilde{u}$ and $\tilde{w}$ be steady states of (7.2) associated with u and w respectively. We can see that $\tilde{u}$ connects to $\tilde{w}$ (see Case 2-1 in the proof of Lemma F'). Thus u connects to w. There is no other wave that is connected by u, because w blocks other connections (see Lemma 6.1).

Case 2: $v \lhd u$

Since $v \lhd u$, it is automatically satisfied that $z(u - v) = I(u)$. If there is a wave w such that $u \rhd w \rhd v$ and $I(u) = I(w)$, then u does not connect to v because w blocks the connection (see Lemma 6.1). On the other hand, if there

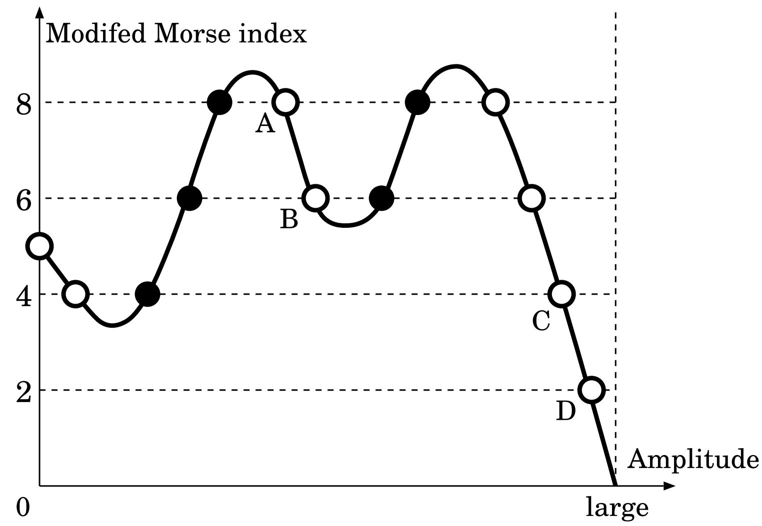

FIGURE 10. Each black point indicates a wave whose Morse index is even (*i.e.* $i(u_j^{kl}) = I(u_j^{kl})$) and each white point indicates a wave whose Morse index is odd (*i.e.* $i(u_j^{kl}) = I(u_j^{kl}) - 1$). The point A connects only to B, C, D and two constant steady states.

is no such wave, then u connects to v because $\tilde{u}$ connects to $\tilde{v}$ (see Lemma 7.2). Therefore the theorem is proved. □

Example 7.3. Let $J_{kl} = \big(I(u_j^{kl})\big)_{j=1}^{m_{kl}}$ $(k \neq l)$ be a sequence of modified Morse indices. Figure 10 represents the sequence of modified Morse indices J_{kl} (see Remark 2.19). Since $k \neq l$, we see by (v) of Lemma F that $I(u_{m_{kl}}^{kl}) = 2$. If $i(u)$ is odd, all connections toward a smaller wave in the order $\rhd$ (*i.e.* toward the left in Figure 10) are blocked. If $i(u)$ is even, the connections to a smaller wave in the order $\rhd$ are not necessarily blocked.

8. PROOF OF PROPOSITION D

In this section we consider the case where the nonlinear term does not depend on u_x (see (7.1)). We will use the notation used in Section 7.

We will show a sufficient condition that guarantees *clusters* to be *monotone*. The following lemma is well-known:

LEMMA 8.1. *Suppose $g(\cdot)$ has exactly three roots $\{r_i\}_{i=1}^3$ and $r_1 < r_2 = 0 < r_3$. If $g(u)/|u|$ is decreasing for $u \in (r_1, 0) \cup (0, r_3)$, then there are only three monotone clusters.*

The proof of Lemma 8.1 is essentially the same as that of Theorem 5.2 of [CI74]. We omit the proof.

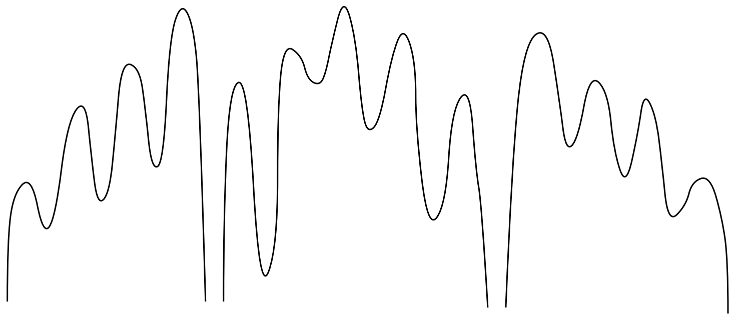

Figure 11. The graph of $G(r)$; $k = 1$ (left), $1 < k < n$ (center) and $k = n$ (right).

We will prove Proposition D after we state some definitions and notation. Hereafter, we assume that every wave of S is hyperbolic. Hence $g'(r_j) \neq 0$ for all $j \in \{1, 2, \ldots, N\}$. The point $G(r_j)$ ($j \in \{1, 3, 5, \ldots, N\}$) is a local maximum point and $G(r_j)$ ($j \in \{2, 4, 6, \ldots, N-1\}$) is a local minimum point where $G(r)$ is defined by (2.6).

First, we define a set of intervals

$$W(r) := \{\rho \mid G(\rho) < r\}.$$

We impose the following condition of the function G:

(**A6**) Let I be a bounded connected component of $W(r)$ for $r \in \mathbb{R}$. Let $J = \{r_k, r_{k+1}, \ldots, r_{l-1}, r_l\}$ $(1 \leq k \leq l \leq N)$. If $I \supset J$, then $\sharp J = 1$.

The closed curves described as $\{(u, v) \mid v^2 + 2G(u) = \text{constant}\}$ on the phase plane are candidates of steady state solutions of (7.1). If (A6) holds, then C_{kl} ($k \neq l$) is empty. Therefore, when (A6) holds, there is only one possibility which is the condition (A4) in Section 2. When (A4) is satisfied, the graph of $G(r)$ looks like one of Figure 11.

Example 8.2. If the graph of $G(r)$ is as shown in the center of Figure 11, the corresponding phase portrait is as shown in Figure 12.

If (A4) holds, then every cluster is *simple*. If $(A5)_j$ holds, then C_{jj} is *monotone*. Now we can prove Proposition D.

Proof of Proposition D. If (A4) holds, then (A6) holds. Thus every C_{kl} ($k \neq l$) is empty. Namely all the *clusters* are *simple*. After all $S = \bigcup_{j=1}^{N} C_{jj}$. Since every wave is hyperbolic, the *cluster* C_{jj} ($j \in \{1, 3, 5, \ldots, N\}$) has precisely one wave which is the stable constant steady state (see Remark 2.1). The condition $(A5)_j$ tells us that the *cluster* C_{jj} ($j \in \{2, 4, 6, \ldots, N-1\}$) is *monotone*. Thus

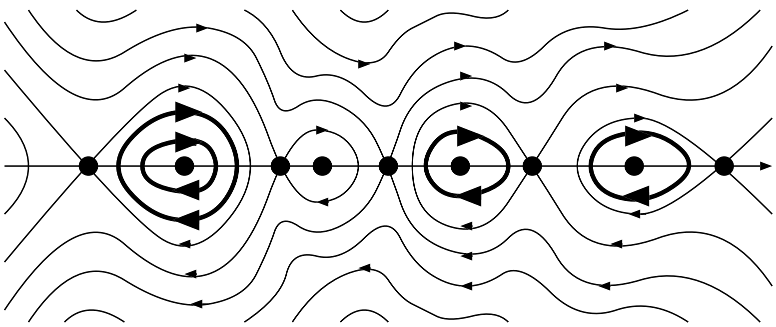

FIGURE 12. The picture indicates the phase portrait when $G(p)$ is as shown in the center of Figure 11. The thick closed curves indicate the closed orbits which correspond to standing waves, and the points indicate equilibrium points which correspond to constant steady states.

every *cluster* is *monotone*. Therefore, all the hypotheses of Theorem C are satisfied. The proof is completed. □

After completing this work, the author has been informed about the paper [FRW04] written by Fiedler, Rocha and Wolfrum. They have given the necessary and sufficient conditions whether any pair of waves is connected heteroclinically or not, and the method to calculate the Morse index of waves (*i.e.* the method to decide whether $i(u) = I(u)$ or $i(u) = I(u) - 1$).

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