Projective Bundle Theorem in Homology Theories with Chern Structure

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Received: December 17, 2003
Revised: December 5, 2004

Communicated by Ulf Rehmann

Abstract. Panin and Smirnov deduced the existence of push-forwards, along projective morphisms, in a cohomology theory with cup products, from the assumption that the theory is endowed with an extra structure called orientation. A part of their work is a proof of the Projective Bundle Theorem in cohomology based on the assumption that we have the first Chern class for line bundles. In some examples we have to consider a pair of theories, cohomology and homology, related by a cap product. It would be useful to construct transfer maps (pull-backs) along projective morphisms in homology in such a situation under similar assumptions. In this note we perform the projective bundle theorem part of this project in homology.

Keywords and Phrases: (Co)homology theory, Chern structure, projective bundle, algebraic variety

1. Introduction

Let k be a field and Sm be the category of smooth quasi-projective algebraic varieties over k. Let $\mathcal{P}$ denote the category of pairs (X,U), with $X \in Sm$ and U a Zarisky open in X, where a morphism $(X,U) \to (X',U')$ is a morphism $f : X \to X'$ in Sm such that $f(U) \subset U'$. Sm embeds into $\mathcal{P}$ by $X \mapsto (X,\emptyset)$. For any functor A defined on $\mathcal{P}$, we can compose it with this embedding and write $A(X)$ for $A(X,\emptyset)$.
For $f : (X,U) \to (X',U')$ we will denote by f_A (resp. f^A) the morphism $A(f) : A(X,U) \to A(X',U')$ (resp. $A(f) : A(X',U') \to A(X,U)$) if A is covariant (respectively, contravariant). We will call such maps push-forwards or pull-backs respectively. Note that the rule $(X,U) \mapsto (U,\emptyset)$ defines an endofunctor on $\mathcal{P}$.

Definition. *A homology theory over k with values in an abelian category $\mathcal{M}$ is a covariant functor $A.: \mathcal{P} \to \mathcal{M}$ endowed with a natural transformation $d: A.(X,U) \to A.(U)$ called the boundary homomorphism, subject to the following requirements:*
(h1) (Homotopy invariance) The arrow $p_A : A.(X \times \mathbb{A}^1) \to A.(X)$ induced by the projection $p: X \times \mathbb{A}^1 \to X$ is an isomorphism for any $X \in Sm$.
(h2) (Localization sequence) For any $(X,U) \in \mathcal{P}$, the sequence

$$\ldots \to A.(U) \to A.(X) \to A.(X,U) \xrightarrow{d} A.(U) \to A(X) \to \ldots$$

is exact.
(h3) (Nisnevich excision) Let $(X,U),(X',U') \in \mathcal{P}$, $Z = X - U$, and $Z' = X' - U'$. Then for any étale morphism $f: X' \to X$ such that $f^{-1}(Z) = Z'$ and $f: Z' \to Z$ is an isomorphism, the map $f_A : A.(X',U') \to A.(X,U)$ must be an isomorphism.

These axioms are dual to the axioms of a cohomology theory given in [PS] and [PS1].[1] The objective of [PS1] is to provide simple conditions under which one can construct transfer maps (push-forwards) along projective morphisms in a cohomology theory. This, in its turn, is a prerequisite for the proof of a very general version of the Riemann-Roch Theorem in [Pa]. All the assumptions made in [PS] and [Pa] are true for many particular cohomology theories such as, for instance, K-theory, étale cohomology, higher Chow groups, and the algebraic cobordism theory introduced by Voevodsky in [V]. We therefore get, in a very uniform way, the existence of push-forwards and the Riemann-Roch Theorem in all these theories.
In some situations we have to consider a pair of theories $(A^{\cdot}, A.)$ consisting of a cohomology and a homology theory related by a cap-product. An important example of this is given by motivic cohomology and homology introduced by Suslin and Voevodsky in [SV]. An ultimate goal in such a situation is to obtain a Poincaré duality in the sense of [PY] for the pair $(A^{\cdot}, A.)$. Among the assumptions from which the Poincaré duality is deduced in [PY], there is the assumption of existence of transfer maps in both $A^{\cdot}$ and $A.$. However, the homology part of this, i.e. the verification of existence of transfers (pull-backs) in homology is still lacking. A general objective in this context is to construct transfers along projective morphisms in a homology theory starting from simple assumptions analogous to those made in [PS] for cohomology.
The purpose of this note is to prove the Projective Bundle Theorem in homology (PBTH), which is a part of the whole program aimed towards the existence of transfer maps in homology. In Section 2 we provide definitions and state the main result (PBTH). Its proof is given in Sections 3 and 4.
A similar result was obtained independently by K. Pimenov in a slightly different framework [Pi].

[1] [PS] is a part of [PS1] which has been published already.

Acknowledgements. I wish to thank Ivan Panin for attracting my attention to the problem of existence of pull-backs in homology, for his continuous interest in my work, and for numerous useful remarks. Special thanks are due to the Alexander von Humboldt Foundation (Germany) for sponsoring my stay in Bielefeld in May 2003, where the very initial steps of the project were undertaken. I am grateful to the University of Regina (Canada) where the work was completed.

2. Definitions and the Main Result

Let $A.$ be a homology theory satisfying (h1-h3) and let $A^{\cdot}$ be a cohomology theory in the sense of [PS, Def. 2.0.1]. The latter means that $A^{\cdot}$ is a contravariant functor $\mathcal{P} \to \mathcal{M}$ equiped with a natural transformation $d : A^{\cdot}(U) \to A^{\cdot}(X, U)$ and satisfying the dual set of axioms that we will call (c1-c3). All the general properties of a cohomology theory deduced from (c1-c3) in [PS, Sect. 2.2] have their duals for a homology theory, obtained by inverting the arrows. In particular, the Mayer-Vietoris exact sequence in homology and the localization sequence for a triple can be deduced from (h1-h3).
We will use the "(co)homology with support" notation $A_Z(X) = A(X, U)$, where $Z = X - U$, for both $A.$ and $A^{\cdot}$. For simplicity, we will assume that $A.$ and $A^{\cdot}$ take their values in the category $\mathcal{A}b$ of abelian groups. From now on we will often write just A for the homology groups, while keeping the upper dot in the cohomology notation.

2.1. Product structures. We will assume that $A^{\cdot}$ is a ring cohomology theory in the sense of [PS, Sect. 2.4]. This, in particular, means that $A^{\cdot}$ is equiped with cup-products

$$\cup : A^{\cdot}_Z(X) \times A^{\cdot}_{Z'}(X) \to A^{\cdot}_{Z \cap Z'}(X)$$

that are functorial with respect to pull-backs and satisfy the following properties:
(cup1)(associativity) $(a \cup b) \cup c = a \cup (b \cup c)$ in $A^{\cdot}_{Z_1 \cap Z_2 \cap Z_3}(X)$ for any $a \in A^{\cdot}_{Z_1}(X)$, $b \in A^{\cdot}_{Z_2}(X), c \in A^{\cdot}_{Z_3}(X)$.
(cup2) The absolute cohomology groups $A^{\cdot}(X)$ become associative unitary rings; the pull-back maps $f^A : A^{\cdot}(X) \to A^{\cdot}(Y)$ are homomorphisms of such rings for all $f : Y \to X$.
(cup3) The groups $A^{\cdot}_Z(X)$ become two-sided unitary modules over $A^{\cdot}(X)$ for all X and closed $Z \subset X$.
We say that $a \in A^{\cdot}_Z(X)$ is a central element if $a \cup b = b \cup a$ for any $b \in A^{\cdot}(X)$. We say that a is universally central if $f^A(a) \in A^{\cdot}_{Z'}(X')$ is central for any $f : (X', X' - Z') \to (X, X - Z)$ in $\mathcal{P}$. Note that the notion of a ring cohomology theory also requires compatibility of cup-products with boundary maps, which implies compatibility of cup-products with Mayer-Vietoris arguments, etc.
We will also assume that A is a left unitary module over $A^{\cdot}$ in the sense that we have cap-products

$$\cap : A^{\cdot}_Z(X) \times A_{Z \cap Z'}(X) \to A_{Z'}(X)$$

satisfying the properties:
(cap1) $(a \cup b) \cap c = a \cap (b \cap c)$ in $A_{Z_3}(X)$ for any $a \in A^{\cdot}_{Z_1}(X)$, $b \in A^{\cdot}_{Z_2}(X)$, $c \in A_{Z_1 \cap Z_2 \cap Z_3}(X)$.
(cap2) $1 \cap a = a$ whenever defined.
(cap3) Let U and U' (resp. V and V') be Zarisky opens in X (resp. in Y). Let $Z = X - U$, $Z' = X - U'$, $T = Y - V$, $T' = Y - V'$. Then for any $f : (Y, V, V') \to (X, U, U')$ and any $a \in A^{\cdot}_Z(X)$, $b \in A_{T \cap T'}(Y)$, we have $f_A(f^A(a) \cap b) = a \cap f_A(b)$ in $A_{Z'}(X)$.
(cap4) (compatibility with boundary maps)

Chern structure. We will assume that $A^{\cdot}$ is equiped with a Chern structure in the sense of [PS, Def. 3.2.1], i.e., to any $X \in Sm$ and any line bundle L over X there is assigned a universally central element $c(L) \in A^{\cdot}(X)$ called the (first) Chern class of L, subject to the requirements:
(ch1) Functoriality with respect to pull-backs; $c(L) = c(L')$ if $L \cong L'$ over X.
(ch2) $c(\mathbf{1}_X) = 0 \in A^{\cdot}(X)$, where $\mathbf{1}_X$ denotes the trivial line bundle $X \times \mathbb{A}^1$ over X, for any X.
(ch3) For any $X \in Sm$, let $\xi = c(\mathcal{O}_{X \times \mathbb{P}^1}(-1)) \in A^{\cdot}(X \times \mathbb{P}^1)$, where $\mathcal{O}_{X \times \mathbb{P}^1}(-1) = p^*(\mathcal{O}_{\mathbb{P}^1}(-1))$, $\mathcal{O}_{\mathbb{P}^1}(-1)$ denotes the tautological line bundle over $\mathbb{P}^1$, and $p : X \times \mathbb{P}^1 \to \mathbb{P}^1$ is the projection. Define the maps $f^{(0)}, f^{(1)} : A^{\cdot}(X) \to A^{\cdot}(X \times \mathbb{P}^1)$ by $f^{(0)} = p^A$ and $f^{(1)} = (\xi \cup -) \circ p^A$. Then the map $(f^{(0)}, f^{(1)}) : A^{\cdot}(X) \oplus A^{\cdot}(X) \to A^{\cdot}(X \times \mathbb{P}^1)$ is an isomorphism.
In the homology, define the maps $f_0, f_1 : A.(X \times \mathbb{P}^1) \to A.(X)$ by $f_0 = p_A$ and $f_1 = p_A \circ (\xi \cap -)$. We will say that we have an *extended Chern structure* (extended to homology) if
(ch4) The map $(f_0, f_1) : A.(X \times \mathbb{P}^1) \to A.(X) \oplus A.(X)$ is an isomorphism for any $X \in Sm$.
The axioms (ch3) and (ch4) can be considered as a dim $= 1$ case of the PBTC and PBTH accordingly. Our goal is to show that the extended Chern structure on $(A^{\cdot}, A.)$ implies the following general version of PBTH for A.

Projective Bundle Theorem. *Let X be a smooth quasiprojective variety over k and E a vector bundle over X of rank $n+1$. Assume that the pair of theories $(A^{\cdot}, A.)$ is endowed with a product structure and an extended Chern structure. Denote $\mathbb{P}(E)$ the projectivisation of E, $\mathcal{O}(-1)$ the tautological line bundle over $\mathbb{P}(E)$, and let $\xi = c(\mathcal{O}(-1)) \in A^{\cdot}(\mathbb{P}(E))$ be its Chern class. For $0 \le i \le n$, denote $f_i = f_{n,i}$ the composite map $A.(\mathbb{P}(E)) \xrightarrow{\xi^i \cap -} A.(\mathbb{P}(E)) \xrightarrow{p_A} A.(X)$, where $p : \mathbb{P}(E) \to X$ is the natural projection. Then the map*

$$F_n := (f_0, f_1, \ldots, f_n) : A.(\mathbb{P}(E)) \to A.(X) \oplus A.(X) \oplus \ldots \oplus A.(X)$$

is an isomorphism.

A crucial reason for which we cannot consider the theory $A.$ separately and must rather work with the pair $(A^{\cdot}, A.)$ is that ξ lives in the cohomology. However, everything works smoothly along the same guidelines as in [PS, Sect. 3.3].

3. Proof: Part I

Localizing and applying the Mayer-Vietoris, we reduce the situation to the case of a trivial bundle $E \cong X \times \mathbb{A}^{n+1}$, $\mathbb{P}(E) \cong X \times \mathbb{P}^n$. Next we can reduce it to the case $X = pt$. We leave it to the reader to check that $X \times -$ can be inserted throughout the proof. Thus we want to prove that the map

$$(f_0, \ldots, f_n) : A.(\mathbb{P}^n) \to A.(pt) \oplus \ldots \oplus A.(pt)$$

is an isomorphism.
We proceed by induction on n. Choose homogeneous coordinates $[x_0 : \ldots : x_n]$ in $\mathbb{P}^n$ and introduce the following notation:
(i) $0 = [1 : 0 : \ldots : 0]$ the distinguished point;
(ii) for $0 \leq i \leq n$, $\mathbb{P}^{n-1}_i$ is the projective hyperplane $x_i = 0$;
(iii) for $1 \leq i \leq n$, $\mathbb{P}^1_i$ is the projective axis on which all $x_j = 0$ for $j \neq 0, i$;
(iv) $\mathbb{A}^n_i = \mathbb{P}^n - \mathbb{P}^{n-1}_i$ for $0 \leq i \leq n$; we will often write just $\mathbb{A}^n$ for $\mathbb{A}^n_0$;
(v) $\mathbb{A}^1_i = \mathbb{P}^1_i \cap \mathbb{A}^n$ and $\mathbb{A}^{n-1}_i = \mathbb{P}^{n-1}_i \cap \mathbb{A}^n$ for $1 \leq i \leq n$.
Consider the localization sequence of the pair $(\mathbb{P}^n, \mathbb{P}^n - 0)$:

$$\ldots \to A(\mathbb{P}^n - 0) \xrightarrow{u_A} A(\mathbb{P}^n) \xrightarrow{v_A} A_0(\mathbb{P}^n) \to \ldots , \tag{3.1}$$

where $u : \mathbb{P}^n - 0 \to \mathbb{P}^n$ and $v : (\mathbb{P}^n, \emptyset) \to (\mathbb{P}^n, \mathbb{P}^n - 0)$ are the natural maps. Note that $\mathbb{P}^n - 0$ can be considered as a line bundle over $\mathbb{P}^{n-1}_0$, with the projection map $t : \mathbb{P}^n - 0 \to \mathbb{P}^{n-1}_0$ given by $[x_0 : x_1 : \ldots : x_n] \mapsto [0 : x_1 : \ldots : x_n]$. Denote by $s : \mathbb{P}^{n-1}_0 \to \mathbb{P}^n - 0$ the inclusion map, then by (h1), $s_A : A(\mathbb{P}^{n-1}_0) \to A(\mathbb{P}^n - 0)$ and $t_A : A(\mathbb{P}^n - 0) \to A(\mathbb{P}^{n-1}_0)$ are inverse isomorphisms. Let $u' : \mathbb{P}^{n-1}_0 \to \mathbb{P}^n$ be the inclusion map, then $u' = us$ and $u'_A = u_A s_A$. Consider the diagram

$$\begin{array}{ccc} A(\mathbb{P}^{n-1}_0) & \xrightarrow{u'_A} & A(\mathbb{P}^n) \\ {\scriptstyle F_{n-1}}\downarrow & & \downarrow{\scriptstyle F_n} \\ \bigoplus_{i=0}^{n-1} A(pt) & \xrightarrow{a_{n-1,n}} & \bigoplus_{i=0}^{n} A(pt) \end{array} \tag{3.2}$$

where $a_{n-1,n}$ maps each summand of $\bigoplus_{i=0}^{n-1} A(pt)$ to the same summand in $\bigoplus_{i=0}^{n} A(pt)$ as the identity map, the last summand in the latter group is therefore not being covered. We claim that the diagram commutes. For it suffices to prove that the diagram

$$\begin{array}{ccc} A(\mathbb{P}^{n-1}_0) & \xrightarrow{u'_A} & A(\mathbb{P}^n) \\ {\scriptstyle f_{n-1,i}}\downarrow & & \downarrow{\scriptstyle f_{n,i}} \\ A(pt) & \xrightarrow{1} & A(pt) \end{array}$$

commutes for every $0 \leq i \leq n-1$ and that $f_{n,n}u'_A = 0$. The first assertion follows from the commutativity of the diagram

$$\begin{tikzcd} A(\mathbb{P}_0^{n-1}) \arrow[r, "u'_A"] \arrow[d, "\xi_{n-1}^i \cap -"'] & A(\mathbb{P}^n) \arrow[d, "\xi_n^i \cap -"] \\ A(\mathbb{P}_0^{n-1}) \arrow[r, "u'_A"] & A(\mathbb{P}^n) \end{tikzcd}$$

which commutes by (cap3) since the restriction of $\mathcal{O}_{\mathbb{P}^n}(-1)$ to $\mathbb{P}_0^{n-1}$ is isomorphic to $\mathcal{O}_{\mathbb{P}_0^{n-1}}(-1)$ and $(u')^A(\xi_n) = \xi_{n-1}$. The same diagram with $i = n$ implies that the composition $f_{n,n}u'_A$ vanishes as $\xi_{n-1}^n = 0$. (See [PS, Sect. 3.3] for a standard argument that proves $\xi_{n-1}^n = 0$.)
Now consider the map $a_{n,n-1} : \bigoplus_{i=0}^n A(pt) \to \bigoplus_{i=0}^{n-1} A(pt)$ that identically maps the ith summand to the ith summand for all $0 \leq i \leq n-1$ and vanishes on the nth summand. As $a_{n,n-1}a_{n-1,n} = 1$, the commutativity of (3.2) implies $F_{n-1} = a_{n,n-1}a_{n-1,n}F_{n-1} = a_{n,n-1}F_n u'_A$. By the inductional hypothesis F_{n-1} is an isomorphism, whence u'_A is a split monomorphism, and so is u_A as s_A is an isomorphism. This has two important consequences:
(i) (3.1) is in fact a split short exact sequence;
(ii) the map $f_{n,n} : A(\mathbb{P}^n) \to A(pt)$ factors uniquely through v_A.
Denote by $g : A_0(\mathbb{P}^n) \to A(pt)$ the factoring map: $f_{n,n} = gv_A$. The diagram

$$\begin{tikzcd} 0 \arrow[r] & A(\mathbb{P}_0^{n-1}) \arrow[r, "u'_A"] \arrow[d, "F_{n-1}"] & A(\mathbb{P}^n) \arrow[r, "v_A"] \arrow[d, "F_n"] & A_0(\mathbb{P}^n) \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & \bigoplus_{i=0}^{n-1} A(pt) \arrow[r, "a_{n-1,n}"] & \bigoplus_{i=0}^{n} A(pt) \arrow[r] & A(pt) \arrow[r] & 0 \end{tikzcd}$$

shows that we will be done as soon as it is proved that g is an isomorphism.
For $1 \leq i \leq n$, consider the cohomology localization sequence of the pair $(\mathbb{P}^n, \mathbb{A}_i^n)$:

$$A^{\cdot}_{\mathbb{P}_i^{n-1}}(\mathbb{P}^n) \xrightarrow{v_i^A} A^{\cdot}(\mathbb{P}^n) \xrightarrow{u_i^A} A^{\cdot}(\mathbb{A}_i^n)\,, \tag{3.3}$$

where $u_i : \mathbb{A}_i^n \to \mathbb{P}^n$ and $v_i : (\mathbb{P}^n, \emptyset) \to (\mathbb{P}^n, \mathbb{A}_i^n)$ are the natural maps. As $A^{\cdot}(\mathbb{A}_i^n) \cong A^{\cdot}(pt)$ by (c1), this is a split short exact sequence, the splitting for u_i^A given by $1 \mapsto 1$. The element $\xi_n \in A^{\cdot}(\mathbb{P}^n)$ maps to zero via u_i^A as the restriction of $\mathcal{O}(-1)$ to $\mathbb{A}_i^n$ is isomorphic to the trivial line bundle. Thus ξ_n comes from a uniquely determined element $\bar{\xi}_{n,i} \in A^{\cdot}_{\mathbb{P}_i^{n-1}}(\mathbb{P}^n)$. Note that $\mathbb{P}_1^{n-1} \cap \ldots \cap \mathbb{P}_n^{n-1} = \{0\}$ and consider the diagram

$$\begin{tikzcd} A^{\cdot}_{\mathbb{P}_1^{n-1}}(\mathbb{P}^n) \oplus A^{\cdot}_{\mathbb{P}_2^{n-1}}(\mathbb{P}^n) \oplus \ldots \oplus A^{\cdot}_{\mathbb{P}_n^{n-1}}(\mathbb{P}^n) \arrow[r, "\cup"] \arrow[d, "v_1^A \oplus \ldots \oplus v_n^A"'] & A^{\cdot}_0(\mathbb{P}^n) \arrow[d, "v^A"] \\ A^{\cdot}(\mathbb{P}^n) \oplus A^{\cdot}(\mathbb{P}^n) \oplus \ldots \oplus A^{\cdot}(\mathbb{P}^n) \arrow[r, "\cup"] & A^{\cdot}(\mathbb{P}^n) \end{tikzcd}$$

which commutes since $\cup$ is compatible with pull-backs. It follows that the element

$$\bar{th}_n := \bar{\xi}_{n,1} \cup \bar{\xi}_{n,2} \cup \ldots \cup \bar{\xi}_{n,n} \in A_0^{\cdot}(\mathbb{P}^n)$$

satisfies $v^A(\bar{th}_n) = \xi_n^n$.
Now apply (cap3), with $X = Y = \mathbb{P}^n$, $U = U' = \mathbb{P}^n - 0$, $V = V' = \emptyset$ and $f = v$, to $a = \bar{th}_n$ and any $b \in A(\mathbb{P}^n)$ and get the commutativity of the diagram

$$\begin{array}{ccccc}
A(\mathbb{P}^n) & \xrightarrow{\xi_n^n \cap -} & A(\mathbb{P}^n) & \longrightarrow & A(pt) \\
{\scriptstyle v_A}\downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
A_0(\mathbb{P}^n) & \xrightarrow{\bar{th}_n \cap -} & A(\mathbb{P}^n) & \longrightarrow & A(pt)
\end{array}$$

The composition of the top arrows is $f_{n,n}$. As g is the unique arrow satisfying $f_{n,n} = gv_A$, we can conclude that g equals the composition of the bottom arrows.
Let $j : \mathbb{A}^n \to \mathbb{P}^n$ denote the inclusion map, and let $j_i : (\mathbb{A}^n, \mathbb{A}^n - \mathbb{A}_i^{n-1}) \to (\mathbb{P}^n, \mathbb{P}^n - \mathbb{P}_i^{n-1})$, with $1 \leq i \leq n$, and $\tilde{j} : (\mathbb{A}^n, \mathbb{A}^n - 0) \to (\mathbb{P}^n, \mathbb{P}^n - 0)$ denote the corresponding maps of pairs. Define $\xi_{n,i} \in A^{\cdot}_{\mathbb{A}_i^{n-1}}(\mathbb{A}^n)$ to be the image of $\bar{\xi}_{n,i}$ under the map $j_i^A : A^{\cdot}_{\mathbb{P}_i^{n-1}}(\mathbb{P}^n) \to A^{\cdot}_{\mathbb{A}_i^{n-1}}(\mathbb{A}^n)$. The diagram

$$\begin{array}{ccc}
A^{\cdot}_{\mathbb{P}_1^{n-1}}(\mathbb{P}^n) \oplus \ldots \oplus A^{\cdot}_{\mathbb{P}_n^{n-1}}(\mathbb{P}^n) & \xrightarrow{\cup} & A_0^{\cdot}(\mathbb{P}^n) \\
{\scriptstyle j_1^A \oplus \ldots \oplus j_n^A}\downarrow & & \downarrow{\scriptstyle \tilde{j}^A} \\
A^{\cdot}_{\mathbb{A}_1^{n-1}}(\mathbb{A}^n) \oplus \ldots \oplus A^{\cdot}_{\mathbb{A}_n^{n-1}}(\mathbb{A}^n) & \xrightarrow{\cup} & A_0^{\cdot}(\mathbb{A}^n)
\end{array}$$

shows that the element

$$th_n := \xi_{n,1} \cup \ldots \cup \xi_{n,n} \in A_0^{\cdot}(\mathbb{A}^n)$$

satisfies $\tilde{j}^A(\bar{th}_n) = th_n$.
Consider the diagram

$$\begin{array}{ccccc}
A_0(\mathbb{A}^n) & \xrightarrow{th_n \cap -} & A(\mathbb{A}^n) & \xrightarrow{\sim} & A(pt) \\
{\scriptstyle \tilde{j}_A}\downarrow & & \downarrow{\scriptstyle j_A} & & \downarrow{\scriptstyle 1} \\
A_0(\mathbb{P}^n) & \xrightarrow{\bar{th}_n \cap -} & A(\mathbb{P}^n) & \longrightarrow & A(pt)
\end{array}$$

that commutes by (cap3). Recall that our current goal is to prove that g, which equals the composition of the bottom arrows in the diagram, is an isomorphism. As $\tilde{j}_A$ is an isomorphism by excision and $A(\mathbb{A}^n) \to A(pt)$ is an isomorphism by homotopy invariance, it now suffices to prove that $th_n \cap - : A_0(\mathbb{A}^n) \to A(\mathbb{A}^n)$ is an isomorphism. This will be done in the next section.

4. Proof: Part II

First we will obtain another description for the elements $\xi_{n,i}$ and th_n. Consider the short exact sequnce

$$0 \to A^{\cdot}_0(\mathbb{P}^1) \to A^{\cdot}(\mathbb{P}^1) \to A^{\cdot}(\mathbb{P}^1 - 0) \to 0\,,$$

which is the one-dimensional version of (3.3). The element $\xi_1 = c(\mathcal{O}(-1)) \in A^{\cdot}(\mathbb{P}^1)$ maps to zero and comes therefore from a uniquely determined element $\bar{t} \in A^{\cdot}_0(\mathbb{P}^1)$. Let $t \in A^{\cdot}_0(\mathbb{A}^1)$ denote its image under the restriction map $A^{\cdot}_0(\mathbb{P}^1) \to A^{\cdot}_0(\mathbb{A}^1)$. (As the one-dimensional case plays a distinguished role, we change the notation and denote these elements by $\bar{t}$ and t.) If we think of $\mathbb{P}^1$ and $\mathbb{A}^1$ as coordinate axes $\mathbb{P}^1_i$ and $\mathbb{A}^1_i$ in $\mathbb{P}^n$ and $\mathbb{A}^n$ accordingly, $1 \leq i \leq n$, then we will denote the corresponding elements by $\bar{t}_i \in A^{\cdot}_0(\mathbb{P}^1_i)$ and $t_i \in A^{\cdot}_0(\mathbb{A}^1_i)$. Denote by $pr_i : \mathbb{A}^n \to \mathbb{A}^1_i$ the projection to the i-th coordinate and consider the map $pr_i^A : A^{\cdot}_0(\mathbb{A}^1_i) \to A^{\cdot}_{\mathbb{A}^{n-1}_i}(\mathbb{A}^n)$. It is proved in [PS] that $pr_i^A(t_i) = \xi_{n,i}$, and we can therefore rewrite th_n in the form

$$th_n = pr_1^A(t_1) \cup pr_2^A(t_2) \cup \ldots \cup pr_n^A(t_n). \tag{4.1}$$

(NB: In [PS] th_n is defined by the above formula and then it is proved that $pr_i^A(t_i)$ can be replaced by $\xi_{n,i}$, with a different notation.)
To proceed further we first need to prove a technical lemma which is the homology counterpart of [PS, Lemma 3.3.2]. Let $Y \in Sm$ and $Z \subset Y$ be a closed subset. Let $p : Y \times \mathbb{A}^1 \to Y$ and $pr : Y \times \mathbb{A}^1 \to \mathbb{A}^1$ denote the projections. Consider the map $pr^A : A^{\cdot}_0(\mathbb{A}^1) \to A^{\cdot}_{Y\times 0}(Y \times \mathbb{A}^1)$ and the image $pr^A(t)$ of $t \in A^{\cdot}_0(\mathbb{A}^1)$ under this map. The cap-product

$$\cap : A^{\cdot}_{Y\times 0}(Y \times \mathbb{A}^1) \times A_{Z\times 0}(Y \times \mathbb{A}^1) \to A_{Z\times\mathbb{A}^1}(Y \times \mathbb{A}^1)$$

induces the map $pr^A(t) \cap - : A_{Z\times 0}(Y \times \mathbb{A}^1) \to A_{Z\times\mathbb{A}^1}(Y \times \mathbb{A}^1)$.

Lemma. *The map $pr^A(t) \cap - : A_{Z\times 0}(Y \times \mathbb{A}^1) \to A_{Z\times\mathbb{A}^1}(Y \times \mathbb{A}^1)$ is an isomorphism.*

Proof. As $p_A : A_{Z\times\mathbb{A}^1}(Y \times \mathbb{A}^1) \to A_Z(Y)$ is an isomorphism by (h1), the assertion of the lemma is equivalent to the claim that the composed map

$$T := p_A \circ (pr^A(t) \cap -) : A_{Z\times 0}(Y \times \mathbb{A}^1) \to A_Z(Y)$$

is an isomorphism. It is this claim that we will actually prove.
We will make use of the localization sequence of the triple $(Y \times \mathbb{P}^1, Y \times \mathbb{P}^1 - Z \times 0, (Y - Z) \times \mathbb{P}^1)$:
(4.2)

$$\ldots \to A_{Z\times\mathbb{A}^1_\infty}(Y \times \mathbb{P}^1 - Z \times 0) \xrightarrow{\alpha_A} A_{Z\times\mathbb{P}^1}(Y \times \mathbb{P}^1) \xrightarrow{\beta_A} A_{Z\times 0}(Y \times \mathbb{P}^1) \to \ldots\,,$$

where $\mathbb{A}^1_\infty := \mathbb{P}^1 - 0$ and α and β are the corresponding inclusion maps of pairs.

Consider the inclusion $i : (Y \times \mathbb{A}^1_\infty, (Y - Z) \times \mathbb{A}^1_\infty) \to (Y \times \mathbb{P}^1 - Z \times 0, (Y - Z) \times \mathbb{P}^1)$. One checks that i satisfies the excision conditions (Zarisky version), whence $i_A : A_{Z\times\mathbb{A}^1_\infty}(Y \times \mathbb{A}^1_\infty) \to A_{Z\times\mathbb{A}^1_\infty}(Y \times \mathbb{P}^1 - Z \times 0)$ is an isomorphism.
Let $\tilde{p} : (Y \times \mathbb{P}^1 - Z \times 0, (Y - Z) \times \mathbb{P}^1) \to (Y, Y - Z)$ and $p' = \tilde{p}i : (Y \times \mathbb{A}^1_\infty, (Y - Z) \times \mathbb{A}^1_\infty) \to (Y, Y - Z)$ be the projections. As p'_A is an isomorphism by (h1), $p'_A = \tilde{p}_A i_A$ shows that $\tilde{p}_A : A_{Z\times\mathbb{A}^1_\infty}(Y \times \mathbb{P}^1 - Z \times 0) \to A_Z(Y)$ is an isomorphism too.
Let $\bar{p} : (Y \times \mathbb{P}^1, (Y - Z) \times \mathbb{P}^1) \to (Y, Y - Z)$ denote the projection. Then $\tilde{p} = \bar{p}\alpha$ and $\tilde{p}_A = \bar{p}_A\alpha_A$, which implies that α_A is a split monomorphism. It follows that β_A is surjective and (4.2) is a short exact sequence.
Let $\bar{pr} : Y \times \mathbb{P}^1 \to \mathbb{P}^1$ denote the projection. Consider the diagram

$$\begin{tikzcd} A^{\cdot}_0(\mathbb{P}^1) \arrow[r] \arrow[d, "\bar{pr}^A"'] & A^{\cdot}(\mathbb{P}^1) \arrow[d, "\bar{pr}^A"] \\ A^{\cdot}_{Y\times 0}(Y \times \mathbb{P}^1) \arrow[r] & A^{\cdot}(Y \times \mathbb{P}^1) \end{tikzcd} \tag{4.3}$$

and the cap-products

$$\cap : A^{\cdot}_{Y\times 0}(Y \times \mathbb{P}^1) \times A_{Z\times 0}(Y \times \mathbb{P}^1) \to A_{Z\times\mathbb{P}^1}(Y \times \mathbb{P}^1)$$
$$\cap : A^{\cdot}(Y \times \mathbb{P}^1) \times A_{Z\times\mathbb{P}^1}(Y \times \mathbb{P}^1) \to A_{Z\times\mathbb{P}^1}(Y \times \mathbb{P}^1)\,.$$

The element $\bar{pr}^A(\bar{t})$ maps to $\bar{pr}^A(\xi_1)$ via the bottom arrow in (4.3). By (cap3), it follows that the diagram

$$\begin{tikzcd} A_{Z\times\mathbb{P}^1}(Y \times \mathbb{P}^1) \arrow[r, "\beta_A"] \arrow[d, "\bar{pr}^A(\xi_1)\cap -"'] & A_{Z\times 0}(Y \times \mathbb{P}^1) \arrow[d, "\bar{pr}^A(\bar{t})\cap -"] \\ A_{Z\times\mathbb{P}^1}(Y \times \mathbb{P}^1) \arrow[r, "1"] & A_{Z\times\mathbb{P}^1}(Y \times \mathbb{P}^1) \end{tikzcd} \tag{4.4}$$

commutes. Denote $\bar{T} = \bar{p}_A \circ (\bar{pr}^A(\bar{t}) \cap -)$, $f^Z_0 = \bar{p}_A$, and $f^Z_1 = \bar{p}_A \circ (\bar{pr}^A(\xi_1) \cap -)$. We are now prepared to consider the diagram

$$\begin{tikzcd} A_{Z\times\mathbb{A}^1_\infty}(Y \times \mathbb{P}^1 - Z \times 0) \arrow[r, "\alpha_A"] \arrow[d, "\tilde{p}_A"'] & A_{Z\times\mathbb{P}^1}(Y \times \mathbb{P}^1) \arrow[r, "\beta_A"] \arrow[d, "\binom{f^Z_0}{f^Z_1}"] & A_{Z\times 0}(Y \times \mathbb{P}^1) \arrow[d, "\bar{T}"] \\ A_Z(Y) \arrow[r, "\binom{1}{0}"] & A_Z(Y) \oplus A_Z(Y) \arrow[r, "(0{,}1)"] & A_Z(Y) \end{tikzcd}$$

where the rows are short exact sequences, with undisplayed zeros on both sides. The right square commutes since (4.4) commutes. To prove the commutativity on the left we must check that $f_1^Z \alpha_A = 0$ (recall that $\bar{p}_A \alpha_A = \tilde{p}_A$). Consider the diagram

$$\begin{tikzcd}
A_{Z\times\mathbb{A}^1_\infty}(Y\times\mathbb{A}^1_\infty) \arrow[r, "\alpha_A i_A"] \arrow[d, "i^A(\bar{pr}^A(\xi_1))\cap -"'] & A_{Z\times\mathbb{P}^1}(Y\times\mathbb{P}^1) \arrow[d, "\bar{pr}^A(\xi_1)\cap -"] \\
A_{Z\times\mathbb{A}^1_\infty}(Y\times\mathbb{A}^1_\infty) \arrow[r, "\alpha_A i_A"] & A_{Z\times\mathbb{P}^1}(Y\times\mathbb{P}^1)
\end{tikzcd}$$

which commutes by (cap3). But $i^A(\bar{pr}^A(\xi_1)) \in A^{\cdot}(Y\times\mathbb{A}^1_\infty)$ vanishes as $\mathcal{O}(-1)$ restricted to $\mathbb{A}^1$ is trivial. Thus $(\bar{pr}^A(\xi_1)\cap -)\alpha_A i_A = 0$. As i_A is an isomorphism, $(\bar{pr}^A(\xi_1)\cap -)\alpha_A = 0$, whence $f_1^Z\alpha_A = 0$ and the big diagram commutes. Now we claim that the arrow $\begin{pmatrix} f_0^Z \\ f_1^Z \end{pmatrix}$ is an isomorphism. The absolute (without supports) version of this is postulated in (ch4). The 'with supports' version can be deduced from (ch4) by applying the five-lemma to obvious localization sequences. Recall that $\tilde{p}_A$ is an isomorphism and conclude that $\bar{T}$ is an isomorphism.

To complete the proof, consider the diagram

$$\begin{tikzcd}
A_{Z\times 0}(Y\times\mathbb{A}^1) \arrow[r, "pr^A(t)\cap -"] \arrow[d] & A_{Z\times\mathbb{A}^1}(Y\times\mathbb{A}^1) \arrow[r] \arrow[d] & A_Z(Y) \arrow[d, "1"] \\
A_{Z\times 0}(Y\times\mathbb{P}^1) \arrow[r, "\bar{pr}^A(\bar{t})\cap -"] & A_{Z\times\mathbb{P}^1}(Y\times\mathbb{P}^1) \arrow[r] & A_Z(Y)
\end{tikzcd}$$

where $\mathbb{A}^1$ now denotes $\mathbb{P}^1 - \infty$ (as opposed to $\mathbb{A}^1_\infty$). It commutes by (cap3) as $\bar{pr}^A(\bar{t})$ maps to $pr^A(t)$ via the map $A^{\cdot}_{Y\times 0}(Y\times\mathbb{P}^1) \to A^{\cdot}_{Y\times 0}(Y\times\mathbb{A}^1)$. The top composition is T, the bottom one is $\bar{T}$, and the left vertical arrow is an isomorphism by excision. We conclude that T is an isomorphism. The lemma is proved.

Define $\mathbb{A}^{(k)}$ to be the k-dimensional affine subspace of $\mathbb{A}^n$ given by $x_1 = \ldots = x_{n-k} = 0$, for $0 \le k \le n$. Then $\mathbb{A}^{(k+1)} \cap \mathbb{A}^{n-1}_{n-k} = \mathbb{A}^{(k)}$. By (4.1) and (cap1), the map $th_n \cap - : A_0(\mathbb{A}^n) \to A(\mathbb{A}^n)$ can be decomposed as

$$A_0(\mathbb{A}^n) \xrightarrow{pr_n^A(t_n)\cap -} A_{\mathbb{A}^{(1)}}(\mathbb{A}^n) \xrightarrow{pr_{n-1}^A(t_{n-1})\cap -} A_{\mathbb{A}^{(2)}}(\mathbb{A}^n) \xrightarrow{pr_{n-2}^A(t_{n-2})\cap -} \\ \ldots \xrightarrow{pr_1^A(t_1)\cap -} A(\mathbb{A}^n)\,.$$

A generic step of this decomposition is a map

$$pr_{n-k}^A(t_{n-k}) : A_{\mathbb{A}^{(k)}}(\mathbb{A}^n) \to A_{\mathbb{A}^{(k+1)}}(\mathbb{A}^n)\,. \tag{4.5}$$

In the notation of the lemma, put $Y = \mathbb{A}^{n-1}_{n-k}$, $Z = \mathbb{A}^{(k)}$, and think of $\mathbb{A}^1$ as $\mathbb{A}^1_{n-k}$. Then $Y\times\mathbb{A}^1$ can be identified with $\mathbb{A}^n$, $Z\times\mathbb{A}^1$ with $\mathbb{A}^{(k+1)}$, $pr^A(t)$ becomes $pr_{n-k}^A(t_{n-k})$, and we get that (4.5) is an isomorphism. The theorem is proved.

Applying (h2) we obtain

Corollary (PBTH with supports). *If Z is a closed subvariety in a smooth X, E is a vector bundle over X of rank $n+1$, and E_Z is its restriction to Z, then the map*

$$F_n = (f_0, \dots , f_n) : A_{\mathbb{P}(E_Z)}(\mathbb{P}(E)) \to A_Z(X) \oplus \dots \oplus A_Z(X)$$

defined the same way as F_n in PBTH is an isomorphism.

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