Some Sharp Weighted Estimates for Multilinear Operators[1]

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Abstract. In this paper, we establish a sharp inequality for some multilinear operators related to certain integral operators. The operators include Calderón-Zygmund singular integral operator, Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator. As application, we obtain the weighted norm inequalities and $L\log L$ type estimate for the multilinear operators.

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1. Introduction

Let T be a singular integral operator. In[1][2][3], Cohen and Gosselin studied the $L^p(p>1)$ boundedness of the multilinear singular integral operator T^A defined by

$$T^A(f)(x)=\int_{R^n}\frac{R_{m+1}(A;x,y)}{|x-y|^m}K(x,y)f(y)dy.$$

In[6], Hu and Yang obtain a variant sharp estimate for the multilinear singular integral operators. The main purpose of this paper is to prove a sharp inequality for some multilinear operators related to certain non-convolution type integral operators. In fact, we shall establish the sharp inequality for the multilinear operators only under certain conditions on the size of the integral operators. The integral operators include Calderón-Zygmund singular integral operator,

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Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator. As applications, we obtain weighted norm inequalities and $L\log L$ type estimates for these multilinear operators.

2. Notations and Results

First, let us introduce some notations(see[6][12-14]). Throughout this paper, Q will denote a cube of R^n with side parallel to the axes. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{x\in Q} \frac{1}{|Q|}\int_Q |f(y) - f_Q|dy,$$

where, and in what follows, $f_Q = |Q|^{-1}\int_Q f(x)dx$. It is well-known that(see[6])

$$f^{\#}(x) = \sup_{x\in Q} \inf_{c\in C} \frac{1}{|Q|}\int_Q |f(y) - c|dy.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$ and $||f||_{BMO} = ||f^{\#}||_{L^\infty}$. For $0 < r < \infty$, we denote $f_r^{\#}$ by

$$f_r^{\#}(x) = [(|f|^r)^{\#}(x)]^{1/r}.$$

Let M be the Hardy-Littlewood maximal operator defined by $M(f)(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |f(y)|dy$, we write $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$; For $k \in N$, we denote by M^k the operator M iterated k times, i.e., $M^1(f)(x) = M(f)(x)$ and $M^k(f)(x) = M(M^{k-1}(f))(x)$ for $k \geq 2$. Let B be a Young function and $\tilde{B}$ be the complementary associated to B, we denote that, for a function f

$$||f||_{B,\ Q} = \inf\left\{\lambda > 0 : \frac{1}{|Q|}\int_Q B\left(\frac{|f(y)|}{\lambda}\right)dy \leq 1\right\}$$

and the maximal function by

$$M_B(f)(x) = \sup_{x\in Q} ||f||_{B,\ Q};$$

The main Young function to be using in this paper is $B(t) = t(1 + log^+t)$ and its complementary $\tilde{B}(t) = expt$, the corresponding maximal denoted by M_{LlogL} and M_{expL}. We have the generalized Hölder's inequality(see[12])

$$\frac{1}{|Q|}\int_Q |f(y)g(y)|dy \leq ||f||_{B,\ Q}||g||_{B,\ Q}$$

and the following inequality (in fact they are equivalent), for any $x \in R^n$,

$$M_{LlogL}(f)(x) \leq CM^2(f)(x)$$

and the following inequalities, for all cubes Q any $b \in BMO(R^n)$,

$$||b-b_Q||_{\exp L,\ Q} \le C||b||_{BMO},\ |b_{2^{k+1}Q}-b_{2Q}| \le 2k||b||_{BMO}.$$

We denote the Muckenhoupt weights by A_p for $1 \le p < \infty$(see[6]).
We are going to consider some integral operators as following.
Let m be a positive integer and A be a function on R^n. We denote that

$$R_{m+1}(A;x,y) = A(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha.$$

Definition 1. Let S and S' be Schwartz space and its dual and $T : S \to S'$ be a linear operator. Suppose there exists a locally integrable function $K(x,y)$ on $R^n \times R^n$ such that

$$T(f)(x) = \int_{R^n} K(x,y)f(y)dy$$

for every bounded and compactly supported function f. The multilinear operator related to the integral operator T is defined by

$$T^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} K(x,y)f(y)dy.$$

Definition 2. Let $F(x,y,t)$ defined on $R^n \times R^n \times [0,+\infty)$. Set

$$F_t(f)(x) = \int_{R^n} F(x,y,t)f(y)dy$$

for every bounded and compactly supported function f and

$$F_t^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A;x,y)}{|x-y|^m} F(x,y,t)f(y)dy.$$

Let H be a Banach space of functions $h : [0,+\infty) \to R$. For each fixed $x \in R^n$, we view $F_t(f)(x)$ and $F_t^A(f)(x)$ as a mapping from $[0,+\infty)$ to H. Then, the multilinear operators related to F_t is defined by

$$S^A(f)(x) = ||F_t^A(f)(x)||;$$

We also define that $S(f)(x) = ||F_t(f)(x)||$.
Note that when $m = 0$, T^A and S^A are just the commutators of T, S and A. While when $m > 0$, it is non-trivial generalizations of the commutators. It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-5][7]). The main purpose of this paper is to prove a sharp inequality for the multilinear operators T^A and S^A. We shall prove the following theorems in Section 3.
Theorem 1. Let $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$. Suppose that T is the same as in Definition 1 such that T is bounded on $L^p(w)$ for all $w \in A_p$

with $1 < p < \infty$ and weak bounded of $(L^1(w), L^1(w))$ for all $w \in A_1$. If T^A satisfies the following size condition:

$$|T^A(f)(x) - T^A(f)(x_0)| \le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M^2(f)(x)$$

for any cube $Q = Q(x_0, d)$ with $suppf \subset (2Q)^c$, $x \in Q = Q(x_0, d)$. Then for any $0 < r < 1$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $x \in R^n$,

$$(T^A(f))_r^\#(x) \le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M^2(f)(x).$$

THEOREM 2. Let $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$. Suppose that S is the same as in Definition 2 such that S is bounded on $L^p(w)$ for all $w \in A_p$, $1 < p < \infty$ and weak bounded of $(L^1(w), L^1(w))$ for all $w \in A_1$. If S^A satisfies the following size condition:

$$||F_t^A(f)(x) - F_t^A(f)(x_0)|| \le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M^2(f)(x)$$

for any cube $Q = Q(x_0, d)$ with $suppf \subset (2Q)^c$, $x \in Q = Q(x_0, d)$. Then for any $0 < r < 1$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $x \in R^n$,

$$(S^A(f))_r^\#(x) \le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M^2(f)(x).$$

From the theorems, we get the following

COROLLARY. Let $D^\alpha A \in BMO(R^n)$ for all α with $|\alpha| = m$. Suppose that T^A, T and S^A, S satisfy the conditions of Theorem 1 and Theorem 2.

(A). If $w \in A_p$ for $1 < p < \infty$. Then T^A and S^A are all bounded on $L^p(w)$, that is

$$||T^A(f)||_{L^p(w)} \le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} ||f||_{L^p(w)}$$

and

$$||S^A(f)||_{L^p(w)} \le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} ||f||_{L^p(w)}.$$

(B). If $w \in A_1$. Then there exists a constant $C > 0$ such that for each $\lambda > 0$,

$$w(\{x \in R^n : |T^A(f)(x)| > \lambda\}) \le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \int_{R^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda}\right)\right) w(x)dx$$

and

$$w(\{x \in R^n : |S^A(f)(x)| > \lambda\}) \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \int_{R^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+ \left(\frac{|f(x)|}{\lambda}\right)\right) w(x)dx.$$

3. PROOF OF THEOREM

To prove the theorems, we need the following lemmas.

LEMMA 1 (Kolmogorov, [6, p.485]). Let $0 < p < q < \infty$ and for any function $f \geq 0$. We define that, for $1/r = 1/p - 1/q$

$$||f||_{WL^q} = \sup_{\lambda>0} \lambda |\{x \in R^n : f(x) > \lambda\}|^{1/q}, N_{p,q}(f) = \sup_E ||f\chi_E||_{L^p}/||\chi_E||_{L^r},$$

where the sup is taken for all measurable sets E with $0 < |E| < \infty$. Then

$$||f||_{WL^q} \leq N_{p,q}(f) \leq (q/(q-p))^{1/p} ||f||_{WL^q}.$$

LEMMA 2([12, p.165]) Let $w \in A_1$. Then there exists a constant $C > 0$ such that for any function f and for all $\lambda > 0$,

$$w(\{y \in R^n : M^2 f(y) > \lambda\}) \leq C\lambda^{-1} \int_{R^n} |f(y)|(1 + \log^+(\lambda^{-1}|f(y)|))w(y)dy.$$

LEMMA 3.([3, p.448]) Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for all α with $|\alpha| = m$ and some $q > n$. Then

$$|R_m(A; x, y)| \leq C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz\right)^{1/q},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

PROOF OF THEOREM 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|} \int_Q |T^A(f)(x) - C_0|^r dx\right)^{1/r} \leq CM^2(f).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{\tilde{Q}} x^\alpha$, then $R_m(A; x, y) = R_m(\tilde{A}; x, y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned} T^A(f)(x) &= \int_{R^n} \frac{R_{m+1}(A; x, y)}{|x-y|^m} K(x, y) f(y) dy \\ &= \int_{R^n} \frac{R_{m+1}(A; x, y)}{|x-y|^m} K(x, y) f_2(y) dy \end{aligned}$$

$$+\int_{R^n}\frac{R_m(\tilde{A};x,y)}{|x-y|^m}K(x,y)f_1(y)dy$$
$$-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{R^n}\frac{K(x,y)(x-y)^\alpha}{|x-y|^m}D^\alpha\tilde{A}(y)f_1(y)dy$$

then

$$\begin{aligned}
&\left|T^A(f)(x)-T^A(f_2)(x_0)\right| \\
\leq\ &\left|T\left(\frac{R_m(\tilde{A};x,\cdot)}{|x-\cdot|^m}f_1\right)(x)\right|+\sum_{|\alpha|=m}\frac{1}{\alpha!}\left|T\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m}D^\alpha\tilde{A}f_1\right)(x)\right| \\
&+\left|T^A(f_2)(x)-T^A(f_2)(x_0)\right| \\
:=\ &I(x)+II(x)+III(x),
\end{aligned}$$

thus,

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q\left|T^A(f)(x)-T^A(f_2)(x_0)\right|^r dx\right)^{1/r} \\
\leq\ &\left(\frac{C}{|Q|}\int_Q I(x)^r dx\right)^{1/r}+\left(\frac{C}{|Q|}\int_Q II(x)^r dx\right)^{1/r}+\left(\frac{C}{|Q|}\int_Q III(x)^r dx\right)^{1/r} \\
:=\ &I+II+III.
\end{aligned}$$

Now, let us estimate I, II and III, respectively. First, for $x\in Q$ and $y\in\tilde{Q}$, using Lemma 3, we get

$$R_m(\tilde{A};x,y)\leq C|x-y|^m\sum_{|\alpha|=m}||D^\alpha A||_{BMO},$$

thus, by Lemma 1 and the weak type (1,1) of T, we get

$$\begin{aligned}
I\ &\leq\ C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|Q|^{-1}\frac{||T(f_1)\chi_Q||_{L^r}}{||\chi_Q||_{L^{r/(1-r)}}} \\
&\leq\ C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|Q|^{-1}||T(f_1)||_{WL^1} \\
&\leq\ C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}|\tilde{Q}|^{-1}\int_{\tilde{Q}}|f(y)|dy\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M(f)(\tilde{x});
\end{aligned}$$

For II, similar to the proof of I, we get

$$\begin{aligned}
II\ &\leq\ C\sum_{|\alpha|=m}|Q|^{-1}\frac{||T(D^\alpha\tilde{A}f_1)\chi_Q||_{L^r}}{||\chi_Q||_{L^{r/(1-r)}}}\leq C\sum_{|\alpha|=m}|Q|^{-1}||T(D^\alpha\tilde{A}f_1)||_{WL^1} \\
&\leq\ C\sum_{|\alpha|=m}|\tilde{Q}|^{-1}\int_{\tilde{Q}}|D^\alpha\tilde{A}(y)||f(y)|dy\leq C\sum_{|\alpha|=m}||D^\alpha A||_{\exp L,\tilde{Q}}||f||_{LlogL,\tilde{Q}} \\
&\leq\ C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M_{L\log L}(f)(\tilde{x})\leq C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M^2(f)(\tilde{x});
\end{aligned}$$

For III, using Hölder' inequality and the size condition of T, we have

$$III \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M^2(f)(\tilde{x}).$$

This completes the proof of Theorem 1.

Proof of Theorem 2. It is only to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q |S^A(f)(x) - C_0|^r dx\right)^{1/r} \leq CM^2(f).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q}$ and $\tilde{A}(x)$ be the same as the proof of Theorem 1. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n\setminus\tilde{Q}}$,

$$\begin{aligned} F_t^A(f)(x) &= \int_{R^n} \frac{R_m(\tilde{A}; x, y)}{|x-y|^m} F(x,y,t) f_1(y) dy \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{F(x,y,t)(x-y)^\alpha}{|x-y|^m} D^\alpha \tilde{A}(y) f_1(y) dy \\ &\quad + \int_{R^n} \frac{R_{m+1}(A; x, y)}{|x-y|^m} F(x,y,t) f_2(y) dy, \end{aligned}$$

then

$$\begin{aligned} & |S^A(f)(x) - S^A(f_2)(x_0)| = \big| ||F_t^A(f)(x)|| - ||F_t^A(f_2)(x_0)|| \big| \\ \leq \;& ||F_t^A(f)(x) - F_t^A(f_2)(x_0)|| \\ \leq \;& \left\| F_t\left(\frac{R_m(\tilde{A}; x, \cdot)}{|x-\cdot|^m} f_1\right)(x)\right\| + \sum_{|\alpha|=m}\frac{1}{\alpha!}\left\| F_t\left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m} D^\alpha \tilde{A} f_1\right)(x)\right\| \\ & + ||F_t^A(f_2)(x) - F_t^A(f_2)(x_0)|| \\ := \;& J(x) + JJ(x) + JJJ(x), \end{aligned}$$

thus,

$$\begin{aligned} & \left(\frac{1}{|Q|}\int_Q |S^A(f)(x) - S^A(f_2)(x_0)|^r dx\right)^{1/r} \\ \leq \;& \left(\frac{C}{|Q|}\int_Q J(x)^r dx\right)^{1/r} + \left(\frac{C}{|Q|}\int_Q JJ(x)^r dx\right)^{1/r} + \left(\frac{C}{|Q|}\int_Q JJJ(x)^r dx\right)^{1/r} \\ := \;& J + JJ + JJJ. \end{aligned}$$

Now, similar to the proof of Theorem 1, we have

$$J \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |f(x)| dx \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M(f)(\tilde{x})$$

and

$$\begin{aligned} JJ &\le C\sum_{|\alpha|=m}|Q|^{-1}\frac{||S(D^\alpha\tilde{A}f_1)\chi_Q||_{L^r}}{||\chi_Q||_{L^{r/(1-r)}}}\le C\sum_{|\alpha|=m}|Q|^{-1}||S(D^\alpha\tilde{A}f_1)||_{WL^1}\\ &\le C\sum_{|\alpha|=m}|\tilde{Q}|^{-1}\int_{\tilde{Q}}|D^\alpha\tilde{A}(y)||f(y)|dy\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M^2(f)(\tilde{x}); \end{aligned}$$

For JJJ, using the size condition of S, we have

$$JJJ\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M^2(f)(\tilde{x}).$$

This completes the proof of Theorem 2.
From Theorem 1, 2 and the weighted boundedness of T and S, we may obtain the conclusion of Corollary(a).
From Theorem 1, 2 and Lemma 2, we may obtain the conclusion of Corollary(b).

4. Applications

In this section we shall apply the Theorem 1, 2 and Corollary of the paper to some particular operators such as the Calderón-Zygmund singular integral operator, Littlewood-Paley operator, Marcinkiewicz operator and Bochner-Riesz operator.

Application 1. Calderón-Zygmund singular integral operator.
Let T be the Calderón-Zygmund operator(see[6][14][15]), the multilinear operator related to T is defined by

$$T^A(f)(x)=\int\frac{R_{m+1}(A;x,y)}{|x-y|^m}K(x,y)f(y)dy.$$

Then it is easily to see that T satisfies the conditions in Theorem 1 and Corollary. In fact, it is only to verify that T^A satisfies the size condition in Theorem 1, which has done in [6](see also [12][13]). Thus the conclusions of Theorem 1 and Corollary hold for T^A.

Application 2. Littlewood-Paley operator.
Let $\varepsilon>0$ and ψ be a fixed function which satisfies the following properties:
(1) $\int_{R^n}\psi(x)dx=0$,
(2) $|\psi(x)|\le C(1+|x|)^{-(n+1)}$,
(3) $|\psi(x+y)-\psi(x)|\le C|y|^\varepsilon(1+|x|)^{-(n+1+\varepsilon)}$ when $2|y|<|x|$;
The multilinear Littlewood-Paley operator is defined by(see[8])

$$g_\psi^A(f)(x)=\left(\int_0^\infty|F_t^A(f)(x)|^2\frac{dt}{t}\right)^{1/2},$$

where

$$F_t^A(f)(x)=\int_{R^n}\frac{R_{m+1}(A;x,y)}{|x-y|^m}\psi_t(x-y)f(y)dy$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ for $t > 0$. We write $F_t(f) = \psi_t * f$. We also define that

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2},$$

which is the Littlewood-Paley operator(see [15]);
Let H be a space of functions $h : [0, +\infty) \to R$, normed by $||h|| = \left(\int_0^\infty |h(t)|^2 dt/t\right)^{1/2} < \infty$. Then, for each fixed $x \in R^n$, $F_t^A(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H, and it is clear that

$$g_\psi(f)(x) = ||F_t(f)(x)|| \text{ and } g_\psi^A(f)(x) = ||F_t^A(f)(x)||.$$

It is known that g_ψ is bounded on $L^p(w)$ for all $w \in A_p$, $1 < p < \infty$ and weak $(L^1(w), L^1(w))$ bounded for all $w \in A_1$. Thus it is only to verify that g_ψ^A satisfies the size condition in Theorem 2. In fact, we write, for a cube $Q = Q(x_0, d)$ with $suppf \subset (\tilde{Q})^c$, $x \in Q = Q(x_0, d)$,

$$\begin{aligned} & F_t^A(f)(x) - F_t^A(f)(x_0) \\ = & \int_{R^n} \left(\frac{\psi_t(x-y)}{|x-y|^m} - \frac{\psi_t(x_0-y)}{|x_0-y|^m}\right) R_m(\tilde{A}; x, y) f(y) dy \\ & + \int_{R^n} \frac{\psi_t(x_0-y)}{|x_0-y|^m} (R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)) f(y) dy \\ & - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left(\frac{(x-y)^\alpha \psi_t(x-y)}{|x-y|^m} - \frac{(x_0-y)^\alpha \psi_t(x_0-y)}{|x_0-y|^m}\right) D^\alpha \tilde{A}(y) f(y) dy \\ := & I_1 + I_2 + I_3. \end{aligned}$$

By Lemma 3 and the following inequality(see[14])

$$|b_{Q_1} - b_{Q_2}| \le C \log(|Q_2|/|Q_1|)||b||_{BMO}, for Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $y \in 2^{k+1}Q \setminus 2^k Q$ with $k \ge 1$,

$$\begin{aligned} |R_m(\tilde{A}; x, y)| & \le C|x-y|^m \sum_{|\alpha|=m} (||D^\alpha A||_{BMO} + |(D^\alpha A)_{\tilde{Q}(x,y)} - (D^\alpha A)_{\tilde{Q}}|) \\ & \le Ck|x-y|^m \sum_{|\alpha|=m} ||D^\alpha A||_{BMO}. \end{aligned}$$

Note that $|x - y| \sim |x_0 - y|$ for $x \in Q$ and $y \in R^n \setminus Q$. By the condition on ψ and Minkowski' inequality , we obtain

$$\begin{aligned} ||I_1|| \le & C \int_{R^n} \frac{|R_m(\tilde{A}; x, y)||f(y)|}{|x_0-y|^m} \\ & \left[\int_0^\infty \left(\frac{t|x-x_0|}{|x_0-y|(t+|x_0-y|)^{n+1}} + \frac{t|x-x_0|^\varepsilon}{(t+|x_0-y|)^{n+1+\varepsilon}}\right)^2 \frac{dt}{t}\right]^{1/2} dy \end{aligned}$$

$$
\begin{aligned}
&\le C\int_{(2Q)^c}\left(\frac{|x-x_0|}{|x_0-y|^{m+n+1}}+\frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{m+n+\varepsilon}}\right)|R_m(\tilde{A};x,y)||f(y)|dy\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\\
&\qquad \sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}k\left(\frac{|x-x_0|}{|x_0-y|^{n+1}}+\frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{n+\varepsilon}}\right)|f(y)|dy\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}|f(y)|dy\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M(f)(x);
\end{aligned}
$$

For I_2, by the formula (see [3]):

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x,x_0)(x-y)^\beta$$

and Lemma 3, we have

$$|R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)|\le C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\beta|}|x-y|^{|\beta|}||D^\alpha A||_{BMO},$$

similar to the estimates of I_1, we get

$$
\begin{aligned}
||I_2|| &\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^{\infty}\int_{2^{k+1}\setminus 2^kQ}\frac{k|x-x_0|}{|x_0-y|^{n+1}}|f(y)|dy\\
&\le C||D^\alpha A||_{BMO}\sum_{k=1}^{\infty}k2^{-k}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}|f(y)|dy\\
&\le C||D^\alpha A||_{BMO}M(f)(x);
\end{aligned}
$$

For I_3, similar to the proof of I_1, we obtain

$$
\begin{aligned}
||I_3|| &\le C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}\int_{2^{k+1}\setminus 2^kQ}\left(\frac{|x-x_0|}{|x_0-y|^{n+1}}+\frac{|x-x_0|^{\varepsilon}}{|x_0-y|^{n+\varepsilon}}\right)|D^\alpha\tilde{A}(y)||f(y)|dy\\
&\le C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|D^\alpha\tilde{A}(y)||f(y)|dy\\
&\le C\sum_{|\alpha|=m}\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})\\
&\qquad\left(||D^\alpha A||_{\exp L,2^{k+1}Q}||f||_{LlogL,2^{k+1}Q}+||D^\alpha A||_{BMO}M(f)(x)\right)\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}(M_{L\log L}(f)(x)+M(f)(x))\\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M^2(f)(x).
\end{aligned}
$$

From the above estimates, we know that Theorem 2 and Corollary hold for g_ψ^A.
Application 3. Marcinkiewicz operator.
Let Ω be homogeneous of degree zero on R^n and $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$ for $0 < \gamma \le 1$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x-y|^\gamma$. The multilinear Marcinkiewicz operator is defined by(see[9])

$$\mu_\Omega^A(f)(x) = \left(\int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_t^A(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} \frac{R_{m+1}(A;x,y)}{|x-y|^m} f(y)dy,$$

we write that

$$F_t(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

We also define that

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

which is the Marcinkiewicz operator(see [16]);
Let H be a space of functions $h : [0, +\infty) \to R$, normed by $||h|| = \left(\int_0^\infty |h(t)|^2 dt/t^3\right)^{1/2} < \infty$. Then, it is clear that

$$\mu_\Omega(f)(x) = ||F_t(f)(x)|| \text{ and } \mu_\Omega^A(f)(x) = ||F_t^A(f)(x)||.$$

Now, we will verify that μ_Ω^A satisfies the size condition in Theorem 2. In fact, for a cube $Q = Q(x_0, d)$ with $suppf \subset (2Q)^c$, $x \in Q = Q(x_0, d)$, we have

$$\begin{aligned}
&||F_t^A(f)(x) - F_t^A(f)(x_0)|| \\
\le\ & \left(\int_0^\infty \left|\int_{|x-y|\le t} \frac{\Omega(x-y)R_m(\tilde{A};x,y)}{|x-y|^{m+n-1}} f(y)dy \right.\right. \\
&\qquad \left.\left. - \int_{|x_0-y|\le t} \frac{\Omega(x_0-y)R_m(\tilde{A};x_0,y)}{|x_0-y|^{m+n-1}} f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2} \\
&+ \sum_{|\alpha|=m} \left(\int_0^\infty \left|\int_{|x-y|\le t} \left(\frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^{m+n-1}} \right.\right.\right. \\
&\qquad \left.\left.\left. - \int_{|x_0-y|\le t} \frac{\Omega(x_0-y)(x_0-y)^\alpha}{|x_0-y|^{m+n-1}}\right) D^\alpha \tilde{A}(y) f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\leq \left(\int_0^\infty \left[\int_{|x-y|\leq t,\ |x_0-y|>t} \frac{|\Omega(x-y)||R_m(\tilde{A};x,y)|}{|x-y|^{m+n-1}}|f(y)|dy\right]^2 \frac{dt}{t^3}\right)^{1/2} \\
&\quad + \left(\int_0^\infty \left[\int_{|x-y|>t,\ |x_0-y|\leq t} \frac{|\Omega(x_0-y)||R_m(\tilde{A};x_0,y)|}{|x_0-y|^{m+n-1}}|f(y)|dy\right]^2 \frac{dt}{t^3}\right)^{1/2} \\
&\quad + \left(\int_0^\infty \left[\int_{|x-y|\leq t, |x_0-y|\leq t} \left|\frac{\Omega(x-y)R_m(\tilde{A};x,y)}{|x-y|^{m+n-1}}\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. - \frac{\Omega(x_0-y)R_m(\tilde{A};x_0,y)}{|x_0-y|^{m+n-1}}\right| |f(y)|dy\right]^2 \frac{dt}{t^3}\right)^{1/2} \\
&\quad + \sum_{|\alpha|=m} \left(\int_0^\infty \left|\int_{|x-y|\leq t} \left(\frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^{m+n-1}}\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. - \int_{|x_0-y|\leq t} \frac{\Omega(x_0-y)(x_0-y)^\alpha}{|x_0-y|^{m+n-1}}\right) D^\alpha \tilde{A}(y) f(y) dy\right|^2 \frac{dt}{t^3}\right)^{1/2} \\
&:= J_1 + J_2 + J_3 + J_4
\end{aligned}$$

and

$$\begin{aligned}
J_1 &\leq C \int_{R^n} \frac{|f(y)||R_m(\tilde{A};x,y)|}{|x-y|^{m+n-1}} \left(\int_{|x-y|\leq t<|x_0-y|} \frac{dt}{t^3}\right)^{1/2} dy \\
&\leq C \int_{R^n} \frac{|f(y)||R_m(\tilde{A};x,y)|}{|x-y|^{m+n-1}} \left(\frac{1}{|x-y|^2} - \frac{1}{|x_0-y|^2}\right)^{1/2} dy \\
&\leq C \int_{(2Q)^c} \frac{|f(y)||R_m(\tilde{A};x,y)|}{|x-y|^{m+n-1}} \frac{|x_0-x|^{1/2}}{|x-y|^{3/2}} dy \\
&\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \sum_{k=1}^\infty k 2^{-k/2} |2^{k+1}Q|^{-1} \int_{2^{k+1}Q} |f(y)|dy \\
&\leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M(f)(x),
\end{aligned}$$

similarly, we have $J_2 \leq C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M(f)(x)$;
For J_3, by the following inequality (see [16]):

$$\left|\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-1}}\right| \leq C\left(\frac{|x-x_0|}{|x_0-y|^n} + \frac{|x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\right),$$

we gain

$$\begin{aligned} J_3 &\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\int_{(2Q)^c}\left(\frac{|x-x_0|}{|x_0-y|^n}+\frac{|x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\right) \\ &\qquad\qquad \left(\int_{|x_0-y|\le t, |x-y|\le t}\frac{dt}{t^3}\right)^{1/2}|f(y)|dy \\ &\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}\sum_{k=1}^\infty k(2^{-k}+2^{-\gamma k})M(f)(x) \\ &\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M(f)(x); \end{aligned}$$

For J_4, similar to the proof of J_1, J_2 and J_3, we obtain

$$\begin{aligned} ||J_4|| &\le C\sum_{|\alpha|=m}\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\left(\frac{|x-x_0|}{|x_0-y|^{n+1}}\right. \\ &\qquad \left.+\frac{|x-x_0|^{1/2}}{|x_0-y|^{n+1/2}}+\frac{|x-x_0|^\gamma}{|x_0-y|^{n+\gamma}}\right)|D^\alpha\tilde{A}(y)||f(y)|dy \\ &\le C\sum_{|\alpha|=m}\sum_{k=1}^\infty k(2^{-k}+2^{-k/2}+2^{-\gamma k})\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|D^\alpha\tilde{A}(y)||f(y)|dy \\ &\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M^2(f)(x). \end{aligned}$$

Thus, Theorem 2 and Corollary hold for μ_Ω^A.
Application 4. Bochner-Riesz operator.
Let $B_t^\delta(f)\hat{}(\xi)=(1-t^2|\xi|^2)_+^\delta\hat{f}(\xi)$. Denote

$$B_{\delta,\,t}^A(f)(x)=\int_{R^n}\frac{R_{m+1}(A;x,y)}{|x-y|^m}B_t^\delta(x-y)f(y)dy,$$

where $B_t^\delta(z)=t^{-n}B^\delta(z/t)$ for $t>0$. The maximal multilinear Bochner-Riesz operator is defined by(see[9])

$$B_{\delta,*}^A(f)(x)=\sup_{t>0}|B_{\delta,\,t}^A(f)(x)|.$$

We also define

$$B_*^\delta(f)(x)=\sup_{t>0}|B_t^\delta(f)(x)|,$$

which is the maximal Bochner-Riesz operator (see [10][11]).
Let H be the space of functions $h(t)$ such that $||h||=\sup_{t>0}|h(t)|<\infty$, where $h(t)$ maps $[0,+\infty)$ to H. Then it is clear that

$$B_*^\delta(f)(x)=||B_t^\delta(f)(x)|| \text{ and } B_{\delta,*}^A(f)(x)=||B_{\delta,\,t}^A(f)(x)||.$$

Now, we will verify that $B^A_{\delta,*}$ satisfies the size condition in Theorem 2. In fact, for a cube $Q = Q(x_0, d)$ with $suppf \subset (2Q)^c$, $x \in Q = Q(x_0, d)$, we have

$$B^{\tilde{A}}_{t,\delta}(f)(x) - B^{\tilde{A}}_{t,\delta}(f)(x_0) = \int_{R^n} \left[\frac{B^\delta_t(x-y)}{|x-y|^m} - \frac{B^\delta_t(x_0-y)}{|x_0-y|^m}\right] R_m(\tilde{A}; x, y) f(y) dy$$
$$+ \int_{R^n} \frac{B^\delta_t(x_0-y)}{|x_0-y|^m}[R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)] f(y) dy$$
$$- \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left(\frac{B^\delta_t(x-y)(x-y)^\alpha}{|x-y|^m} - \frac{B^\delta_t(x_0-y)(x_0-y)^\alpha}{|x_0-y|^m}\right) D^\alpha \tilde{A}(y) f(y) dy$$
$$= L_1 + L_2 + L_3.$$

Consider the following two cases:

Case 1. $0 < t \le d$. In this case, notice that (see [11])

$$|B^\delta(z)| \le c(1 + |z|)^{-(\delta+(n+1)/2)},$$

we obtain

$$\begin{aligned}
|L_1| &\le Ct^{-n} \int_{R^n\setminus\tilde{Q}} \frac{|f(y)||R_m(\tilde{A}; x, y)|}{|x_0 - y|^m}(1 + |x - y|/t)^{-(\delta+(n+1)/2)} dy \\
&\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} t^{-n} \sum_{k=0}^\infty k \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} |f(y)|(1 + |x - y|/t)^{-(\delta+(n+1)/2)} dy \\
&\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} (t/d)^{\delta-(n-1)/2} \sum_{k=1}^\infty k 2^{k((n-1)/2-\delta)} M(f)(x) \\
&\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M(f)(x), \\
|L_2| &\le Ct^{-n} \int_{R^n\setminus\tilde{Q}} \frac{|f(y)||R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)|}{|x_0 - y|^m}(1 + |x - y|/t)^{-(\delta+(n+1)/2)} dy \\
&\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} t^{-n} \\
&\qquad \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|x - x_0||f(y)|}{|x_0 - y|}(1 + |x - y|/t)^{-(\delta+(n+1)/2)} dy \\
&\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M(f)(x), \\
|L_3| &\le C \sum_{|\alpha|=m} t^{-n} \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} |f(y)||D^\alpha \tilde{A}(y)|(1 + |x_0 - y|/t)^{-(\delta+(n+1)/2)} dy \\
&\le C \sum_{|\alpha|=m} (t/d)^{\delta-\frac{n-1}{2}} \sum_{k=0}^\infty 2^{k(\frac{n-1}{2}-\delta)} \frac{1}{|2^{k+1}\tilde{Q}|} \int_{2^{k+1}\tilde{Q}} |f(y)||D^\alpha A(y) - (D^\alpha A)_{\tilde{Q}}| dy \\
&\le C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} M^2(f)(x).
\end{aligned}$$

Case 2. $t > d$. In this case, we choose δ_0 such that $(n-1)/2 < \delta_0 < \min(\delta, (n+1)/2)$, notice that (see [11])

$$|B^\delta(x-y) - B^\delta(x_0-y)| \le C|x-x_0|(1+|x-y|)^{-(\delta+(n+1)/2)},$$

similar to the proof of Case 1, we obtain

$$\begin{aligned}
|L_1| &\le Ct^{-n} \int_{R^n\setminus\tilde{Q}} \frac{|f(y)||R_m(\tilde{A};x,y)|}{|x_0-y|^{m+1}}|x_0-x|(1+|x_0-y|/t)^{-(\delta_0+(n+1)/2)}dy \\
&\quad +Ct^{-n-1} \int_{R^n\setminus\tilde{Q}} \frac{|f(y)||R_m(\tilde{A};x,y)|}{|x_0-y|^{m}}|x_0-x|(1+|x_0-y|/t)^{-(\delta_0+(n+1)/2)}dy \\
&\le C\sum_{|\alpha|=m} ||D^\alpha A||_{BMO}(d/t)^{(n+1)/2-\delta_0}\sum_{k=1}^\infty k2^{k((n-1)/2-\delta_0)}M(f)(x) \\
&\le C\sum_{|\alpha|=m} ||D^\alpha A||_{BMO}M(f)(x), \\
|L_2| &\le Ct^{-n}\int_{R^n\setminus\tilde{Q}} \frac{|f(y)||R_m(\tilde{A};x,y)-R_m(\tilde{A};x_0,y)|}{|x_0-y|^m}(1+|x_0-y|/t)^{-(\delta_0+(n+1)/2)}dy \\
&\le C\sum_{|\alpha|=m} ||D^\alpha A||_{BMO}(d/t)^{(n+1)/2-\delta_0}\sum_{k=1}^\infty 2^{k((n-1)/2-\delta_0)}M(f)(x) \\
&\le C\sum_{|\alpha|=m} ||D^\alpha A||_{BMO}M(f)(x), \\
|L_3| &\le C\sum_{|\alpha|=m}(d/t)^{(n+1)/2-\delta_0}\sum_{k=0}^\infty 2^{k((n-1)/2-\delta_0)}\frac{1}{|2^{k+1}\tilde{Q}|}\int_{2^{k+1}\tilde{Q}}|f(y)||D^\alpha\tilde{A}(y)|dy \\
&\le C\sum_{|\alpha|=m}\sum_{k=1}^\infty k2^{k((n-1)/2-\delta_0)}\frac{1}{|2^{k+1}\tilde{Q}|}\int_{2^{k+1}\tilde{Q}}|f(y)||D^\alpha A(y)-(D^\alpha A)_{\tilde{Q}}|dy \\
&\le C\sum_{|\alpha|=m}||D^\alpha A||_{BMO}M^2(f)(x).
\end{aligned}$$

Thus, Theorem 2 and Corollary hold for $B^A_{\delta,*}$.

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References

[1] J.Cohen, A sharp estimate for a multilinear singular integral on R^n, Indiana Univ. Math. J., 30(1981), 693-702.

[2] J.Cohen and J.Gosselin, On multilinear singular integral operators on R^n, Studia Math., 72(1982), 199-223.

[3] J.Cohen and J.Gosselin, A BMO estimate for multilinear singular integral operators, Illinois J. Math., 30(1986), 445-465.

[4] R.Coifman and Y.Meyer, Wavelets, Calderón-Zygmund and multilinear operators, Cambridge Studies in Advanced Math.48, Cambridge University Press, Cambridge , 1997.

[5] Y.Ding and S.Z.Lu, Weighted boundedness for a class rough multilinear operators, Acta Math. Sinica, 17(2001), 517-526.

[6] J.Garcia-Cuerva and J.L.Rubio de Francia, Weighted norm inequalities and related topics, North-Holland Math.16, Amsterdam, 1985.

[7] G.Hu and D.C.Yang, A variant sharp estimate for multilinear singular integral operators, Studia Math., 141(2000), 25-42.

[8] L.Z. Liu, Weighted weak type estimates for commutators of Littlewood-Paley operator, Japanese J. of Math., 29(1)(2003), 1-13.

[9] L.Z. Liu, The continuity of commutators on Triebel-Lizorkin spaces, Integral Equations and Operator Theory, 49(2004), 65-75.

[10] L.Z. Liu and S.Z. Lu, Weighted weak type inequalities for maximal commutators of Bochner-Riesz operator, Hokkaido Math. J., 32(1)(2003), 85-99.

[11] S.Z.Lu, Four lectures on real H^p spaces, World Scientific, River Edge, NI, 1995.

[12] C.Perez, Endpoint estimate for commutators of singular integral operators, J. Func. Anal., 128(1995), 163-185.

[13] C.Perez and G. Pradolini, Sharp weighted endpoint estimates for commutators of singular integral operators, Michigan Math. J., 49(2001), 23-37.

[14] E.M.Stein, Harmonic Analysis: real variable methods, orthogonality and oscillatory integrals, Princeton Univ. Press, Princeton NJ, 1993.

[15] A.Torchinsky, The real variable methods in harmonic analysis, Pure and Applied Math., 123, Academic Press, New York, 1986.

[16] A.Torchinsky and S.Wang, A note on the Marcinkiewicz integral, Colloq. Math., 60/61(1990), 235-24.

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