Matsumoto K-Groups Associated to Certain Shift Spaces

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Abstract. In [24] Matsumoto associated to each shift space (also called a subshift) an Abelian group which is now known as Matsumoto's K_0-group. It is defined as the cokernel of a certain map and resembles the first cohomology group of the dynamical system which has been studied in for example [2], [28], [13], [16] and [11] (where it is called the dimension group).

In this paper, we will for shift spaces having a certain property (∗), show that the first cohomology group is a factor group of Matsumoto's K_0-group. We will also for shift spaces having an additional property (∗∗), describe Matsumoto's K_0-group in terms of the first cohomology group and some extra information determined by the left special elements of the shift space.

We determine for a broad range of different classes of shift spaces if they have property (∗) and property (∗∗) and use this to show that Matsumoto's K_0-group and the first cohomology group are isomorphic for example for finite shift spaces and for Sturmian shift spaces.

Furthermore, the ground is laid for a description of the Matsumoto K_0-group as an *ordered* group in a forthcoming paper.

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1 Introduction

Invariants for symbolic dynamical systems in the form of Abelian groups have a fruitful history. Important examples are the dimension group defined by Krieger in [19] and [20], and the Bowen-Franks group defined in [1] by Bowen and Franks.

In [24] Matsumoto generalized the definition of dimension groups and Bowen-Franks groups to the whole class of shift spaces and introduced what is now known as Matsumoto's K-groups.

In another direction, Putnam [29], Herman, Putnam and Skau [16], Giordano, Putnam and Skau [15], Durand, Host and Skau [11] and Forrest [13] studied what they called the dimension group (it is not the same as Krieger's or Matsumoto's dimension group) for Cantor minimal systems. The same group has for a broader class of topological dynamical systems been studied in [2], [28] and [27] where it is shown that it is the first cohomology group of the standard suspension of the dynamical system in question.

It turns out that Matsumoto's K_0-group and the first cohomology group are closely related. We will for shift spaces having a certain property $(*)$, show that the first cohomology group is a factor group of Matsumoto's K_0-group, and we will also for shift spaces having an additional property $(**)$, describe Matsumoto's K_0-group in terms of the first cohomology group and some extra information determined by the left special elements of the shift space.

We will for a broad range of different classes of shift spaces, which includes shift of finite types, finite shift spaces, Sturmian shift spaces, substitution shift spaces and Toeplitz shift spaces, determine if they have property $(*)$ and property $(**)$. This will allow us to show that Matsumoto's K_0-group and the first cohomology group are isomorphic for example for finite shift spaces and for Sturmian shift spaces and to describe Matsumoto's K_0-group for substitution shift spaces in such a way that we in [8] can for every shift space associated with a aperiodic and primitive substitution present Matsumoto's K_0-group as a stationary inductive limit of a system associated to an integer matrix defined from combinatorial data which can be computed in an algorithmic way (cf. [6], [7]).

Since both Matsumoto's K_0-group and the first cohomology group are K_0-groups of certain C^*-algebras they come with a natural (pre)order structure. All the results presented in this paper hold not just in the category of Abelian groups, but also in the category of preordered groups. Since we do not know how to prove this without involving C^*-algebras we have decided to defer this to [9], where we also show that Matsumoto's K_0-group with order is a finer invariant than Matsumoto's K_0-group without order.

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2 Preliminaries and notation

Throughout this paper $\mathbb{Z}$ will denote the set of integers, $\mathbb{N}_0$ will denote the set of non-negative integers and $-\mathbb{N}$ will denote the negative integers.
The symbol Id will always denote the identity map. For a map ϕ between two sets X and Y, we will by $\phi^\star$ denote the map which maps a function f on Y to the function $f \circ \phi$ on X.
Let $\mathfrak{a}$ be a finite set of symbols, and let $\mathfrak{a}^\sharp$ denote the set of finite, nonempty words with letters from $\mathfrak{a}$. Thus with ϵ denoting the *empty word*, $\epsilon \notin \mathfrak{a}^\sharp$. By $|\mu|$ we denote the *length* of a finite word μ (i.e. the number of letters in μ). The length of ϵ is 0.

2.1 Shift spaces

We equip

$$\mathfrak{a}^{\mathbb{Z}}, \mathfrak{a}^{\mathbb{N}_0}, \mathfrak{a}^{-\mathbb{N}}$$

with the product topology from the discrete topology on $\mathfrak{a}$. We will strive to denote elements of $\mathfrak{a}^{\mathbb{Z}}$ by z, elements of $\mathfrak{a}^{\mathbb{N}_0}$ by x and elements of $\mathfrak{a}^{-\mathbb{N}}$ by y. If $x \in \mathfrak{a}^{\mathbb{N}_0}$ and $y \in \mathfrak{a}^{-\mathbb{N}}$, then we will by $y.x$ denote the element z of $\mathfrak{a}^{\mathbb{Z}}$ where

$$z_n = \begin{cases} y_n & \text{if } n < 0, \\ x_n & \text{if } n \geq 0. \end{cases}$$

We define $\sigma : \mathfrak{a}^{\mathbb{Z}} \to \mathfrak{a}^{\mathbb{Z}}$, $\sigma_+ : \mathfrak{a}^{\mathbb{N}_0} \to \mathfrak{a}^{\mathbb{N}_0}$, and $\sigma_- : \mathfrak{a}^{-\mathbb{N}} \to \mathfrak{a}^{-\mathbb{N}}$ by

$$(\sigma(z))_n = z_{n+1} \qquad (\sigma_+(x))_n = x_{n+1} \qquad (\sigma_-(y))_n = y_{n-1}.$$

Such maps we will refer to as *shift maps*.
A *shift space* is a closed subset of $\mathfrak{a}^{\mathbb{Z}}$ which is mapped into itself by σ. We shall refer to such spaces by "$\underline{\mathsf{X}}$".
With the obvious restriction maps

$$\pi_+ : \underline{\mathsf{X}} \to \mathfrak{a}^{\mathbb{N}_0} \qquad \pi_- : \underline{\mathsf{X}} \to \mathfrak{a}^{-\mathbb{N}}$$

we get

$$\sigma_+ \circ \pi_+ = \pi_+ \circ \sigma \qquad \sigma_- \circ \pi_- = \pi_- \circ \sigma^{-1}.$$

We denote $\pi_+(\underline{\mathsf{X}})$, respectively $\pi_-(\underline{\mathsf{X}})$, by $\underline{\mathsf{X}}^+$, respectively $\underline{\mathsf{X}}^-$, and notice that $\sigma_+(\underline{\mathsf{X}}^+) = \underline{\mathsf{X}}^+$ and $\sigma_-(\underline{\mathsf{X}}^-) = \underline{\mathsf{X}}^-$. For $z \in \mathfrak{a}^{\mathbb{Z}}$ and $n \in \mathbb{Z}$, we write

$$z_{[n,\infty[} = \pi_+(\sigma^n(z)) \text{ and } z_{]-\infty,n[} = \pi_-(\sigma^n(z)).$$

The *language* of a shift space is the subset of $\mathfrak{a}^\sharp \cup \{\epsilon\}$ given by

$$\mathcal{L}(\underline{\mathsf{X}}) = \{z_{[n,m]} \mid z \in \underline{\mathsf{X}}, n \leq m \in \mathbb{Z}\}$$

where the interval subscript notation should be self-explanatory. A compactness argument shows that an element $z \in \mathfrak{a}^{\mathbb{Z}}$ (respectively $z \in \mathfrak{a}^{\mathbb{N}_0}$, $z \in \mathfrak{a}^{-\mathbb{N}}$)

is in $\underline{X}$ (respectively $\underline{X}^+$, $\underline{X}^-$) if and only if $z_{[n,m]} \in \mathcal{L}(\underline{X})$ for all $n < m \in \mathbb{Z}$ (respectively $n < m \in \mathbb{N}_0$, $n < m \in -\mathbb{N}$) (cf. [21, Corollary 1.3.5 and Theorem 6.1.21]).

We say that shift spaces are *conjugate*, denoted by "$\simeq$", when they are homeomorphic via a map which intertwines the relevant shift maps. The concept of conjugacy also makes sense for the "one-sided" shift spaces $\underline{X}^+$. If $\underline{X}^+ \simeq \underline{Y}^+$, then we say that $\underline{X}$ and $\underline{Y}$ are *one-sided conjugate*. It is not difficult to see that $\underline{X}^+ \simeq \underline{Y}^+ \Rightarrow \underline{X} \simeq \underline{Y}$ (cf. [21, §13.8]).

Finally we want to draw attention to a third kind of equivalence between shift spaces, called *flow equivalence*, which we denote by $\cong_f$. We will not define it here (see [26], [14], [2] or [21, §13.6] for the definition), but just notice that $\underline{X} \simeq \underline{Y} \Rightarrow \underline{X} \cong_f \underline{Y}$.

A *flow invariant* of a shift space $\underline{X}$ is a mapping associating to each shift space another mathematical object, called the *invariant*, in such a way that flow equivalent shift spaces give isomorphic invariants. In the same way, a *conjugacy invariant* of $\underline{X}$, respectively $\underline{X}^+$, is a mapping associating to each shift space an invariant in such a way that conjugate, respectively one-sided conjugate, shift spaces give isomorphic invariants.

Since $\underline{X} \simeq \underline{Y} \Rightarrow \underline{X} \cong_f \underline{Y}$, a flow invariant of $\underline{X}$ is also a conjugacy invariant of $\underline{X}$, and since $\underline{X}^+ \simeq \underline{Y}^+ \Rightarrow \underline{X} \simeq \underline{Y}$, a conjugacy invariant of $\underline{X}$ is also a conjugacy invariant of $\underline{X}^+$.

2.2 Special elements

We say (cf. [17]) that $z \in \underline{X}$ is *left special* if there exists $z' \in \underline{X}$ such that

$$z_{-1} \neq z'_{-1} \qquad \pi_+(z) = \pi_+(z').$$

It follows from [4, Proposition 2.4.1] (cf. [3, Theorem 3.9]) that a sufficient condition for a shift space $\underline{X}$ to have a left special element is that $\underline{X}$ is infinite. Conversely, the following proposition shows that this condition is necessary.

PROPOSITION 2.1. *Let $\underline{X}$ be a finite shift space. Then $\underline{X}$ contains no left special element.*

Proof: Since $\underline{X}$ is finite, every $z \in \underline{X}$ is periodic. Hence if $\pi_+(z) = \pi_+(z')$, then $z = z'$. □

We say that the left special word z is *adjusted* if $\sigma^{-n}(z)$ is not left special for any $n \in \mathbb{N}$, and that z is *cofinal* if $\sigma^n(z)$ is not left special for any $n \in \mathbb{N}$. Thinking of left special words as those which are not deterministic from the right at index -1, the adjusted and cofinal left special words are those where this is the *leftmost* and *rightmost* occurrence of nondeterminacy, respectively.

Let $z, z' \in \underline{X}$. If there exist an n and an M such that $z_m = z'_{n+m}$ for all $m > M$ then we say that z and z' are *right shift tail equivalent* and write $z \sim_r z'$. We will denote the right shift tail equivalence class of z by $\mathbf{z}$.

2.3 The first cohomology group

The first cohomology group (cf. [2]) of a shift space $\underline{\mathsf{X}}$ is the group

$$C(\underline{\mathsf{X}},\mathbb{Z})/(\mathrm{Id}-(\sigma^{-1})^{\star})(C(\underline{\mathsf{X}},\mathbb{Z})).$$

Notice that usually σ is used instead of σ^{-1}, but for our purpose it is more natural to use σ^{-1}, and we of course get the same group. The group $C(\underline{\mathsf{X}},\mathbb{Z})/(\mathrm{Id}-(\sigma^{-1})^{\star})(C(\underline{\mathsf{X}},\mathbb{Z}))$ is the first Čech cohomology group of the standard suspension of $(\underline{\mathsf{X}},\sigma)$ (cf. [27, IV.15. Theorem]). It is also isomorphic to the homotopy classes of continuous maps from the standard suspension of $(\underline{\mathsf{X}},\sigma)$ into the circle (cf. [27, page 60]).
It is proved in [2, Theorem 1.5] that $C(\underline{\mathsf{X}},\mathbb{Z})/(\mathrm{Id}-(\sigma^{-1})^{\star})(C(\underline{\mathsf{X}},\mathbb{Z}))$ is a flow invariant of $\underline{\mathsf{X}}$ and thus also a conjugacy invariant of $\underline{\mathsf{X}}$ and $\underline{\mathsf{X}}^{+}$.

2.4 Past equivalence and Matsumoto's K_0-group

Let $\underline{\mathsf{X}}$ be a shift space. For every $x\in\underline{\mathsf{X}}^{+}$ and every $k\in\mathbb{N}$ we set

$$\mathcal{P}_k(x)=\{\mu\in\mathcal{L}(\underline{\mathsf{X}})\mid \mu x\in\underline{\mathsf{X}}^{+},\ |\mu|=k\},$$

and define for every $l\in\mathbb{N}$ an equivalence relation $\sim_l$ on $\underline{\mathsf{X}}^{+}$ by

$$x\sim_l x' \iff \mathcal{P}_l(x)=\mathcal{P}_l(x').$$

Likewise we let for every $x\in\underline{\mathsf{X}}^{+}$

$$\mathcal{P}_\infty(x)=\{y\in\underline{\mathsf{X}}^{-}\mid y.x\in\underline{\mathsf{X}}\},$$

and define an equivalence relation $\sim_\infty$ on $\underline{\mathsf{X}}^{+}$ by

$$x\sim_\infty x' \iff \mathcal{P}_\infty(x)=\mathcal{P}_\infty(x').$$

The set

$$\mathcal{ND}_\infty(\underline{\mathsf{X}}^{+})=\{x\in\underline{\mathsf{X}}^{+}\mid \exists k\in\mathbb{N}:\#\mathcal{P}_k(x)>1\}$$

then consists exactly of all words on the form $z_{[n,\infty[}$ where z is left special and $n\in\mathbb{N}_0$.
Following Matsumoto ([23]), we denote by $[x]_l$ the equivalence class of x and refer to the relation as *l-past equivalence.*
Obviously the set of equivalence classes of the l-past equivalence relation $\sim_l$ is finite. We will denote the number of such classes $m(l)$ and enumerate them $\mathcal{E}_s^l$ with $s\in\{1,\dots,m(l)\}$. For each $l\in\mathbb{N}$, we define an $m(l+1)\times m(l)$-matrix I^l by

$$(\mathsf{I}^l)_{rs}=\begin{cases}1 & \text{if } \mathcal{E}_r^{l+1}\subseteq\mathcal{E}_s^l\\ 0 & \text{otherwise,}\end{cases}$$

and note that I^l induces a group homomorphism from $\mathbb{Z}^{m(l)}$ to $\mathbb{Z}^{m(l+1)}$. We denote by $\mathbb{Z}_{\underline{\mathsf{X}}}$ the group given by the inductive limit

$$\varinjlim(\mathbb{Z}^{m(l)}, \mathsf{I}^l).$$

For a subset $\mathcal{E}$ of $\underline{\mathsf{X}}^+$ and a finite word μ we let $\mu\mathcal{E} = \{\mu x \in \underline{\mathsf{X}}^+ \mid x \in \mathcal{E}\}$. For each $l \in \mathbb{N}$ and $a \in \mathfrak{a}$ we define an $m(l+1) \times m(l)$-matrix

$$(\mathsf{L}_a^l)_{rs} = \begin{cases} 1 & \text{if } \emptyset \neq a\mathcal{E}_r^{l+1} \subseteq \mathcal{E}_s^l \\ 0 & \text{otherwise,} \end{cases}$$

and letting $\mathsf{L}^l = \sum_{a \in \mathfrak{a}} \mathsf{L}_a^l$ we get a matrix inducing a group homeomorphism from $\mathbb{Z}^{m(l)}$ to $\mathbb{Z}^{m(l+1)}$. Since one can prove that $\mathsf{L}^{l+1} \circ \mathsf{I}^l = \mathsf{I}^{l+1} \circ \mathsf{L}^l$, a group endomorphism λ on $\mathbb{Z}_{\underline{\mathsf{X}}}$ is induced.

THEOREM 2.2 (CF. [24], [25, THEOREM]). *Let $\underline{\mathsf{X}}$ be a shift space. The group*

$$K_0(\underline{\mathsf{X}}) = \mathbb{Z}_{\underline{\mathsf{X}}}/(\mathrm{Id} - \lambda)\mathbb{Z}_{\underline{\mathsf{X}}},$$

called Matsumoto's K_0-group, *is a conjugacy invariant of $\underline{\mathsf{X}}$ and $\underline{\mathsf{X}}^+$, and a flow invariant of $\underline{\mathsf{X}}$.*

2.5 THE SPACE $\Omega_{\underline{\mathsf{X}}}$

We will now give an alternative description of $K_0(\underline{\mathsf{X}})$. The group $K_0(\underline{\mathsf{X}})$ is defined by taking a inductive limits of $\mathbb{Z}^{m(l)}$, where $\mathbb{Z}^{m(l)}$ could be thought of as $C(\underline{\mathsf{X}}^+/\sim_l, \mathbb{Z})$.

We will now do things in different order. First we will take the projective limit of $\underline{\mathsf{X}}^+/\sim_l$ and then look at the continuous functions from the projective limit to $\mathbb{Z}$.

Since $\sim_l$ is coarser than $\sim_{l+1}$, there is a projection π_l of $\underline{\mathsf{X}}^+/\sim_{l+1}$ onto $\underline{\mathsf{X}}^+/\sim_l$.

DEFINITION 2.3 (CF. [23, PAGE 682]). *Let $\underline{\mathsf{X}}$ be a shift space. We then define $\Omega_{\underline{\mathsf{X}}}$ to be the compact topological space given by the projective limit*

$$\varprojlim(\underline{\mathsf{X}}^+/\sim_l, \pi_l).$$

We will identify $\Omega_{\underline{\mathsf{X}}}$ with the closed subspace

$$\{([x_n]_n)_{n \in \mathbb{N}_0} \mid \forall n \in \mathbb{N}_0 : x_{n+1} \sim_n x_n\}$$

of $\prod_{l=0}^{\infty} \underline{\mathsf{X}}^+/\sim_l$, where $\prod_{l=0}^{\infty} \underline{\mathsf{X}}^+/\sim_l$ is endowed with the product of the discrete topologies.

Notice that if we identify $C(\underline{\mathsf{X}}^+/\sim_l, \mathbb{Z})$ with $\mathbb{Z}^{m(l)}$, then I^l is the map induced by π_l, so $C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})$ can be identified with $\mathbb{Z}_{\underline{\mathsf{X}}}$.

If $([x_n]_n)_{n \in \mathbb{N}_0} \in \Omega_{\underline{\mathsf{X}}}$, then

$$\{([x'_n]_n)_{n \in \mathbb{N}_0} \in \Omega_{\underline{\mathsf{X}}} \mid x'_1 \sim_1 x_1\}$$

is a clopen subset of $\Omega_{\underline{X}}$, and if $a \in \mathcal{P}_1(x_1)$, then $([ax'_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{X}}$ for every $([x'_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{X}}$ with $x'_1 \sim_1 x_1$, and the map

$$([x'_n]_n)_{n\in\mathbb{N}_0} \mapsto ([ax'_n]_n)_{n\in\mathbb{N}_0}$$

is a continuous map on $\{([x'_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{X}} \mid x'_1 \sim_1 x_1\}$. This allows us to define a map $\lambda_{\underline{X}} : C(\Omega_{\underline{X}}, \mathbb{Z}) \to C(\Omega_{\underline{X}}, \mathbb{Z})$ in the following way:

DEFINITION 2.4. *Let $\underline{X}$ be a shift space, $h \in C(\Omega_{\underline{X}}, \mathbb{Z})$ and $([x_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{X}}$. Then we let*

$$\lambda_{\underline{X}}(h)(([x_n]_n)_{n\in\mathbb{N}_0}) = \sum_{a\in\mathcal{P}_1(x_1)} h([ax_n]_{n\in\mathbb{N}_0}).$$

Under the identification of $C(\Omega_{\underline{X}}, \mathbb{Z})$ and $\mathbb{Z}_{\underline{X}}$, $\lambda_{\underline{X}}$ is equal to λ, thus we have the following proposition:

PROPOSITION 2.5. *Let $\underline{X}$ be a shift space. Then $K_0(\underline{X})$ and*

$$C(\Omega_{\underline{X}}, \mathbb{Z})/(\mathrm{Id} - \lambda_{\underline{X}})(C(\Omega_{\underline{X}}, \mathbb{Z}))$$

are isomorphic as groups.

3 PROPERTY (*) AND (**)

We will introduce the properties $(*)$ and $(**)$ and show that they are invariant under flow equivalence and thus under conjugacy. At the end of the section, we will for various examples of shift spaces determine if they have property $(*)$ and $(**)$.

DEFINITION 3.1. *We say that a shift space $\underline{X}$ has* property $(*)$ *if for every $\mu \in \mathcal{L}(\underline{X})$ there exists an $x \in \underline{X}^+$ such that $\mathcal{P}_{|\mu|}(x) = \{\mu\}$.*

DEFINITION 3.2. *We say that a shift space $\underline{X}$ has* property $(**)$ *if it has property $(*)$ and if the number of left special words of $\underline{X}$ is finite, and no such left special word is periodic.*

Since flow equivalence is generated by conjugacy and symbolic expansion (cf. [25, Lemma 2.1] and [26]), it is, in order to prove the following proposition, enough to check that $(*)$ and $(**)$ are invariant under symbolic expansion and conjugacy.

PROPOSITION 3.3. *The properties $(*)$ and $(**)$ are invariant under flow equivalence.*

EXAMPLE 3.4. It follows from Proposition 2.1 that if a shift space $\underline{X}$ is finite, then it contains no left special element, and thus has property $(**)$.

EXAMPLE 3.5. An infinite shift of finite type does not have property $(*)$.

Proof: Let $\underline{X}$ be a shift of finite type. This means (cf. [21, Chapter 2]) that there is a $k \in \mathbb{N}_0$ such that

$$\underline{X} = \{z \in \mathfrak{a}^{\mathbb{Z}} \mid \forall n \in \mathbb{Z} : z_{[n,n+k]} \in \mathcal{L}(\underline{X})\}.$$

Suppose that $\underline{X}$ has property $(*)$. Let $\mathcal{L}(\underline{X})_k = \{\mu \in \mathcal{L}(\underline{X}) \mid |\mu| = k\}$, and notice that if $\mu, \nu, \omega \in \mathcal{L}(\underline{X})_k$ and $\mu\nu, \nu\omega \in \mathcal{L}(\underline{X})$, then $\mu\nu\omega \in \mathcal{L}(\underline{X})$.
Let $\mu \in \mathcal{L}(\underline{X})_k$. Then there is a $x \in \underline{X}^+$ such that $\mathcal{P}_{|\mu|}(x) = \{\mu\}$. Let $\mu' = x_{[0,k[}$, and suppose that $\nu \in \mathcal{L}(\underline{X})_k$ and $\nu\mu' \in \mathcal{L}(\underline{X})$. Then $\nu x \in \underline{X}^+$, so ν must be equal to μ. Thus there is for every $\mu \in \mathcal{L}(\underline{X})_k$ a $\mu' \in \mathcal{L}(\underline{X})_k$ such that

$$\nu \in \mathcal{L}(\underline{X})_k \wedge \nu\mu' \in \mathcal{L}(\underline{X}) \iff \nu = \mu.$$

Since $\mathcal{L}(\underline{X})_k$ is finite and the map $\mu \mapsto \mu'$ is injective, there is for every $\nu \in \mathcal{L}(\underline{X})_k$ a $\mu \in \mathcal{L}(\underline{X})_k$ such that $\nu = \mu'$. Hence there is for every $\mu \in \mathcal{L}(\underline{X})_k$ a unique $\mu' \in \mathcal{L}(\underline{X})_k$ such that $\mu\mu' \in \mathcal{L}(\underline{X})$ and a unique $\mu'' \in \mathcal{L}(\underline{X})_k$ such that $\mu''\mu \in \mathcal{L}(\underline{X})$. Thus every $z \in \underline{X}$ is determined by $z_{[0,k[}$, but since $\mathcal{L}(\underline{X})_k$ is finite, this implies that $\underline{X}$ is finite. □

Example 3.6. An infinite minimal shift space (cf. [21, §13.7]) $\underline{X}$ has property $(**)$ precisely when the number of left special words of $\underline{X}$ is finite.

Proof: Since no elements in such a shift space is periodic, we only need to prove that property $(*)$ follows from finiteness of the number of left special elements. Let $\mu \in \mathcal{L}(\underline{X})$ and pick any $x \in \underline{X}^+$. Since $\underline{X}^+$ is infinite and minimal, x is not periodic, and since the set of left special words is finite there exists $N \in \mathbb{N}$ such that $\sigma^n(x)$ is not left special for any $n \geq N$. Since $\underline{X}^+$ is minimal there exists a $k \geq N$ such that $x_{[k+1,k+|\mu|]} = \mu$. Hence $\mathcal{P}_{|\mu|}(\sigma^{k+|\mu|+1}(x)) = \{\mu\}$. □

Example 3.7. If z is a non-periodic, non-regular Toeplitz sequence (cf. [32, pp. 97 and 99]), then the shift space

$$\overline{\mathcal{O}(z)} = \overline{\{\sigma^n(z) \mid n \in \mathbb{Z}\}},$$

where $\overline{X}$ denotes the closure of X, has property $(*)$.

Proof: Let $\mu \in \mathcal{L}(\overline{\mathcal{O}(z)})$. Since $\overline{\mathcal{O}(z)}$ is minimal (cf. [32, page 97]), there is an $m \in \mathbb{N}$ such that $z_{[-m-|\mu|,-m[} = \mu$. We claim that $\mathcal{P}_{|\mu|}(z_{[-m,\infty[}) = \{\mu\}$.
Assume that $z' \in \overline{\mathcal{O}(z)}$ and $z'_{[-m,\infty[} = z_{[-m,\infty[}$. Then $\pi(z') = \pi(z)$, where π is the factor map of $\overline{\mathcal{O}(z)}$ onto its maximal equicontinuous factor $(G, \hat{1})$ (cf. [32, Theorem 2.2]), because since $z'_{[-m,\infty[} = z_{[-m,\infty[}$, the distance between $\sigma^n(z')$ and $\sigma^n(z)$, and thus the distance between $\hat{1}^n(\pi(z'))$ and $\hat{1}^n(\pi(z))$, goes to 0 as n goes to infinity, but since $\hat{1}$ is equicontinuous, this implies that $\pi(z') = \pi(z)$. Since z is a Toeplitz sequence, it follows from [32, Corollary 2.4]) that $z' = z$. Thus $\mathcal{P}_{|\mu|}(z_{[-m,\infty[}) = \{\mu\}$. □

The following example shows that property $(**)$ does not follow from property $(*)$.

EXAMPLE 3.8. We will construct a non-regular Toeplitz sequence $z \in \{0,1\}^{\mathbb{Z}}$ such that the shift space

$$\overline{\mathcal{O}(z)} = \overline{\{\sigma^n(z) \mid n \in \mathbb{Z}\}}$$

has infinitely many left special elements and thus does not have property $(**)$. We will construct z by using the technique introduced by Susan Williams in [32, Section 4]. We will use the same notation as in [32, Section 4]. We let Y be the full 2-shift $\{0,1\}^{\mathbb{Z}}$ and defined $(p_i)_{i\in\mathbb{N}}$ recursively by setting $p_1 = 3$ and $p_{i+1} = 3^{r_i+i}p_i$ for $i \in \mathbb{N}$, where r_i is as defined in [32, Section 4]. We then have that

$$\frac{p_i\beta_{r_i}}{p_{i+1}} = \frac{2^{r_i}}{3^{r_i+i}} < 3^{-i},$$

so

$$\sum_{i=1}^{\infty} \frac{p_i\beta_{r_i}}{p_{i+1}}$$

converges, and z is non-regular by [32, Proposition 4.1].

CLAIM. *The shift space $\overline{\mathcal{O}(z)}$ has infinitely many left special elements.*

Proof: Let D be as defined on [32, page 103]. If

$$g \in \pi(\{z' \in D \mid -1 \in \mathrm{Aper}(z')\}),$$

$y, y' \in Y$, $y_{[0,\infty[} = y'_{[0,\infty[}$ and $y_{-1} \neq y'_{-1}$, then $\phi(g,y)_{[0,\infty[} = \phi(g,y')_{[0,\infty[}$ and $\phi(g,y)_{-1} \neq \phi(g,y')_{-1}$, where ϕ is the map define on [32, page 103]. Thus $\phi(g,y)$ and $\phi(g,y')$ are left special elements, and since

$$\pi(\{z' \in D \mid -1 \in \mathrm{Aper}(z')\}) \times \{y \in Y \mid y \text{ is left special}\}$$

is infinite and contained in $\pi(D) \times Y$, on which ϕ is $1-1$, $\overline{\mathcal{O}(z)}$ has infinitely many left special elements. □

4 THE FIRST COHOMOLOGY GROUP IS A FACTOR OF $K_0(\underline{\mathsf{X}})$

We will now show that if a shift space $\underline{\mathsf{X}}$ has property $(*)$, then the first cohomology group is a factor group of $K_0(\underline{\mathsf{X}})$.

Suppose that a shift space $\underline{\mathsf{X}}$ has property $(*)$. We can then define a map $\iota_{\underline{\mathsf{X}}}$ from $\underline{\mathsf{X}}^-$ into $\Omega_{\underline{\mathsf{X}}}$ in the following way: For each $y \in \underline{\mathsf{X}}^-$ and each $n \in \mathbb{N}_0$ we choose an $x_n \in \underline{\mathsf{X}}^+$ such that $\mathcal{P}_n(x_n) = \{y_{[-n,-1]}\}$. Then $([x_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{\mathsf{X}}}$, and we denote this element by $\iota_{\underline{\mathsf{X}}}(y)$. The map $\iota_{\underline{\mathsf{X}}}$ is obviously injective and continuous.

We denote the map

$$(\iota_{\underline{\mathsf{X}}} \circ \pi_-)^\star : C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z}) \to C(\underline{\mathsf{X}}, \mathbb{Z})$$

by κ.

Proposition 4.1. *Let* $\underline{\mathsf{X}}$ *be a shift space which has property* $(*)$. *Then there is a surjective group homomorphism* $\bar{\kappa}$ *from* $C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z})/(\mathrm{Id}-\lambda_{\underline{\mathsf{X}}})(C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z}))$ *to* $C(\underline{\mathsf{X}},\mathbb{Z})/(\mathrm{Id}-(\sigma^{-1})^\star)(C(\underline{\mathsf{X}},\mathbb{Z}))$ *which makes the following diagram commute:*

$$\begin{CD}
C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z}) @>{\kappa}>> C(\underline{\mathsf{X}},\mathbb{Z}) \\
@VVV @VVV \\
C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z})/(\mathrm{Id}-\lambda_{\underline{\mathsf{X}}})(C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z})) @>{\bar{\kappa}}>> C(\underline{\mathsf{X}},\mathbb{Z})/(\mathrm{Id}-(\sigma^{-1})^\star)(C(\underline{\mathsf{X}},\mathbb{Z}))
\end{CD}$$

Proof: Let q be the quotient map from $C(\underline{\mathsf{X}},\mathbb{Z})$ to

$$C(\underline{\mathsf{X}},\mathbb{Z})/(\mathrm{Id}-(\sigma^{-1})^\star)(C(\underline{\mathsf{X}},\mathbb{Z})).$$

We will show that 1) $q\circ\kappa$ is surjective and 2) $(\mathrm{Id}-\lambda_{\underline{\mathsf{X}}})(C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z}))\subseteq\ker(q\circ\kappa)$. This will prove the existence and surjectivity of $\bar{\kappa}$.

1) *$q\circ\kappa$ is surjective:* Given $f\in C(\underline{\mathsf{X}},\mathbb{Z})$. Our goal is to find a function $g\in C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z})$ which is mapped to $q(f)$ by $q\circ\kappa$.
Since f is continuous, there are $k,m\in\mathbb{N}$ such that

$$z_{[-k,m]}=z'_{[-k,m]}\Rightarrow f(z)=f(z').$$

Thus

$$z_{[-k-m-1,-1]}=z'_{[-k-m-1,-1]}\Rightarrow f\circ\sigma^{-(m+1)}(z)=f\circ\sigma^{-(m+1)}(z').$$

Define a function g from $\Omega_{\underline{\mathsf{X}}}$ to $\mathbb{Z}$ by

$$g(([x_n]_n)_{n\in\mathbb{N}_0})=\begin{cases} f\circ\sigma^{-(m+1)}(z) & \text{if } \mathcal{P}_{k+m+1}(x_{k+m+1})=\{z_{[-k-m-1,-1]}\},\\ 0 & \text{if } \#\mathcal{P}_{k+m+1}(x_{k+m+1})>1.\end{cases}$$

Then $g\in C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z})$, and $g\circ\iota_{\underline{\mathsf{X}}}\circ\pi_-=f\circ\sigma^{-(m+1)}$, so $q\circ\kappa(g)=q(f)$.

2) *$(\mathrm{Id}-\lambda_{\underline{\mathsf{X}}})(C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z}))\subseteq\ker(q\circ\kappa)$:* Let $g\in C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z})$ and $y\in\underline{\mathsf{X}}^-$. Then $\lambda_{\underline{\mathsf{X}}}(g)(\iota_{\underline{\mathsf{X}}}(y))=g(\iota_{\underline{\mathsf{X}}}(\sigma_-(y)))$, so

$$\kappa(\lambda_{\underline{\mathsf{X}}}(g))=g\circ\iota_{\underline{\mathsf{X}}}\circ\pi_-\circ\sigma^{-1},$$

which shows that $(\mathrm{Id}-\lambda_{\underline{\mathsf{X}}})(g)\in\ker(q\circ\kappa)$. □

The following corollary now follows from Proposition 2.5:

Corollary 4.2. *Let* $\underline{\mathsf{X}}$ *be a shift space which has property* $(*)$. *Then* $C(\underline{\mathsf{X}},\mathbb{Z})/(\mathrm{Id}-(\sigma^{-1})^\star)(C(\underline{\mathsf{X}},\mathbb{Z}))$ *is a factor group of* $K_0(\underline{\mathsf{X}})$.

5 K_0 of shift spaces having property $(**)$

We saw in the last section that if a shift space $\underline{X}$ has property $(*)$, then the first cohomology group is a factor group of $K_0(\underline{X})$. This stems from the fact that property $(*)$ causes an inclusion of $\underline{X}^-$ into $\Omega_{\underline{X}}$, and thus a surjection of $C(\Omega_{\underline{X}}, \mathbb{Z})$ onto $C(\underline{X}^-, \mathbb{Z})$. We will now for shift spaces having property $(**)$ describe K_0 in terms of the first cohomology group and some extra information determined by the left special elements of the shift space.
We will first define the group $\mathcal{G}_{\underline{X}}$ which is a subgroup of the external direct product of $C(\underline{X}^-, \mathbb{Z})$ and an infinite product of copies of $\mathbb{Z}$, and isomorphic to $C(\Omega_{\underline{X}}, \mathbb{Z})$. Next, we will define the group $G_{\underline{X}}$ which is the external direct product of $C(\underline{X}, \mathbb{Z})$ and an infinite sum of copies of $\mathbb{Z}$, and has a factor group which is isomorphic to $K_0(\underline{X})$. We will round off by relating this with the fact that the first cohomology group is a factor group of $K_0(\underline{X})$ and look at some examples.

Lemma 5.1. *Let $\underline{X}$ be a shift space which has property $(*)$. Then*

$$\iota_{\underline{X}}(\underline{X}^-) = \{([x_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{X}} \mid \forall n \in \mathbb{N}_0 : \#\mathcal{P}_n(x_n) = 1\}.$$

Proof: Clearly

$$\iota_{\underline{X}}(\underline{X}^-) \subseteq \{([x_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{X}} \mid \forall n \in \mathbb{N}_0 : \#\mathcal{P}_n(x_n) = 1\}.$$

Suppose $([x_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{X}}$ and $\mathcal{P}_n(x_n) = \{\mu_n\}$ for every $n \in \mathbb{N}_0$. Let for every $n \in \mathbb{N}$, y_{-n} be the first letter of μ_n. Since $y_{[-n,-1]} = \mu_n$ for every $n \in \mathbb{N}$, $y \in \underline{X}^-$, and clearly $\iota_{\underline{X}}(y) = ([x_n]_n)_{n\in\mathbb{N}_0}$. □

Denote by $\mathcal{I}_{\underline{X}}$ the set $\mathcal{ND}_\infty(\underline{X}^+)/\sim_\infty$ (cf. Section 2.4). We will now define a map $\phi_{\underline{X}}$ from $\mathcal{I}_{\underline{X}}$ to $\Omega_{\underline{X}}$. We see that for $x \in \mathcal{ND}_\infty(\underline{X}^+)$, $([x]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{X}}$, and we notice that $x \sim_\infty \tilde{x}$, if and only if $([x]_n)_{n\in\mathbb{N}_0} = ([\tilde{x}]_n)_{n\in\mathbb{N}_0}$. So if we let

$$\phi_{\underline{X}}([x]_\infty) = ([x]_n)_{n\in\mathbb{N}_0},$$

then $\phi_{\underline{X}}$ is a well-defined and injective map from $\mathcal{I}_{\underline{X}}$ to $\Omega_{\underline{X}}$.

Lemma 5.2. *Let $\underline{X}$ be a shift space which has property $(*)$. Then $\iota_{\underline{X}}(\underline{X}^-) \cap \phi_{\underline{X}}(\mathcal{I}_{\underline{X}}) = \emptyset$, and if $\underline{X}$ has property $(**)$, then $\iota_{\underline{X}}(\underline{X}^-) \cup \phi_{\underline{X}}(\mathcal{I}_{\underline{X}}) = \Omega_{\underline{X}}$.*

Proof: If $([x_n]_n)_{n\in\mathbb{N}_0} \in \iota_{\underline{X}}(\underline{X}^-)$, then according to Lemma 5.1, $\#\mathcal{P}_n(x_n) = 1$ for every $n \in \mathbb{N}_0$, and if $([x_n]_n)_{n\in\mathbb{N}_0} \in \phi_{\underline{X}}(\mathcal{I}_{\underline{X}})$, then $\#\mathcal{P}_n(x_n) > 1$ for some $n \in \mathbb{N}_0$. Hence $\iota_{\underline{X}}(\underline{X}^-) \cap \phi_{\underline{X}}(\mathcal{I}_{\underline{X}}) = \emptyset$.
Suppose that $\underline{X}$ has property $(**)$. If $([x_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{X}} \setminus \iota_{\underline{X}}(\underline{X}^-)$, then according to Lemma 5.1, there is an $n \in \mathbb{N}_0$ such that $\#\mathcal{P}_n(x_n) > 1$, and since there only are finitely many left special words, $[x_n]_n$ must be finite. Since $[x_k]_k \neq \emptyset$ and $[x_{k+1}]_{k+1} \subseteq [x_k]_k$ for every $k \in \mathbb{N}_0$, this implies that $\bigcap_{k\in\mathbb{N}_0}[x_k]_k$ is not empty. Let $x \in \bigcap_{k\in\mathbb{N}_0}[x_k]_k$. Since $\#\mathcal{P}_n(x) = \#\mathcal{P}_n(x_n) > 1$, $x \in \mathcal{ND}_\infty(\underline{X}^+)$, and since $([x_n]_n)_{n\in\mathbb{N}_0} = \phi_{\underline{X}}([x]_\infty)$, we have that $([x_n]_n)_{n\in\mathbb{N}_0} \in \phi_{\underline{X}}(\mathcal{I}_{\underline{X}})$. □

5.1 The group $\mathcal{G}_{\underline{\mathsf{X}}}$

We will from now on assume that $\underline{\mathsf{X}}$ has property (∗∗). Let for every function $h : \Omega_{\underline{\mathsf{X}}} \to \mathbb{Z}$,

$$\gamma_{\underline{\mathsf{X}}}(h) = (h \circ \iota_{\underline{\mathsf{X}}}, (h(\phi_{\underline{\mathsf{X}}}(\mathsf{i})))_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}).$$

It follows from Lemma 5.2 that $\gamma_{\underline{\mathsf{X}}}$ is a bijective correspondence between functions from $\Omega_{\underline{\mathsf{X}}}$ to $\mathbb{Z}$ and pairs $(g, (\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}})$, where g is a function from $\underline{\mathsf{X}}^-$ to $\mathbb{Z}$ and each α_{i} is an integer.

Lemma 5.3. *Let g be a function from $\underline{\mathsf{X}}^-$ to $\mathbb{Z}$ and let for every $\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}$, α_{i} be an integer. Then $(g, (\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}) \in \gamma_{\underline{\mathsf{X}}}(C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z}))$ if and only if there is an $N \in \mathbb{N}_0$ such that*

1. $\forall y, y' \in \underline{\mathsf{X}}^- : y_{[-N,-1]} = y'_{[-N,-1]} \Rightarrow g(y) = g(y')$,
2. $\forall x, x' \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+) : [x]_N = [x']_N \Rightarrow \alpha_{[x]_\infty} = \alpha_{[x']_\infty}$,
3. $\forall x \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+),\ y \in \underline{\mathsf{X}}^- : \mathcal{P}_N(x) = \{y_{[-N,-1]}\} \Rightarrow \alpha_{[x]_\infty} = g(y)$.

Proof: A function from $\Omega_{\underline{\mathsf{X}}}$ to $\mathbb{Z}$ is continuous if and only if there is an $N \in \mathbb{N}_0$ such that

$$[x_N]_N = [x'_N]_N \Rightarrow h(([x_n]_n)_{n\in\mathbb{N}_0}) = h(([x'_n]_n)_{n\in\mathbb{N}_0}),$$

for $([x_n]_n)_{n\in\mathbb{N}_0}, ([x'_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{\mathsf{X}}}$, and since we have that if $y, y' \in \underline{\mathsf{X}}^-$, and $([x_n]_n)_{n\in\mathbb{N}_0} = \iota_{\underline{\mathsf{X}}}(y)$ and $([x'_n]_n)_{n\in\mathbb{N}_0} = \iota_{\underline{\mathsf{X}}}(y')$, then

$$[x_N]_N = [x'_N]_N \iff y_{[-N,-1]} = y'_{[-N,-1]},$$

and if $x \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$, $y \in \underline{\mathsf{X}}^-$ and $([x'_n]_n)_{n\in\mathbb{N}_0} = \iota_{\underline{\mathsf{X}}}(y)$, then

$$[x]_N = [x'_N]_N \iff \mathcal{P}_N(x) = \{y_{[-N,-1]}\},$$

the conclusion follows. □

Definition 5.4. *Let $\underline{\mathsf{X}}$ be a shift space which has property (∗∗). We denote $\gamma_{\underline{\mathsf{X}}}(C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z}))$ by $\mathcal{G}_{\underline{\mathsf{X}}}$, and we let for every function $g : \underline{\mathsf{X}}^- \to \mathbb{Z}$ and $(\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}} \in \mathbb{Z}^{\mathcal{I}_{\underline{\mathsf{X}}}}$,*

$$\mathcal{A}_{\underline{\mathsf{X}}}(g, (\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}) = (g \circ \sigma_-, (\tilde{\alpha}_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}),$$

where

$$\tilde{\alpha}_{[x]_\infty} = \sum_{\substack{x' \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+) \\ \sigma_+(x') = x}} \alpha_{[x']_\infty} + \sum_{\substack{z \in \underline{\mathsf{X}} \\ z_{[0,\infty[} \notin \mathcal{ND}_\infty(\underline{\mathsf{X}}^+) \\ z_{[1,\infty[} = x}} g(\pi_-(z)).$$

Lemma 5.5. *The map* $\mathcal{A}_{\underline{\mathsf{X}}}$ *maps* $\mathcal{G}_{\underline{\mathsf{X}}}$ *into* $\mathcal{G}_{\underline{\mathsf{X}}}$*, and the following diagram commutes:*

$$\begin{array}{ccc} C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z}) & \xrightarrow{\gamma_{\underline{\mathsf{X}}}} & \mathcal{G}_{\underline{\mathsf{X}}} \\ \downarrow{\scriptstyle \lambda_{\underline{\mathsf{X}}}} & & \downarrow{\scriptstyle \mathcal{A}_{\underline{\mathsf{X}}}} \\ C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z}) & \xrightarrow{\gamma_{\underline{\mathsf{X}}}} & \mathcal{G}_{\underline{\mathsf{X}}} \end{array}$$

Proof: Let $h \in C(\Omega_{\underline{\mathsf{X}}},\mathbb{Z})$ and $([x_n]_n)_{n\in\mathbb{N}_0} \in \Omega_{\underline{\mathsf{X}}}$. Then

$$\lambda_{\underline{\mathsf{X}}}(h)(([x_n]_n)_{n\in\mathbb{N}_0}) = \sum_{a\in\mathcal{P}_1(x_1)} h([ax_n]_{n\in\mathbb{N}_0}).$$

We will show that $\lambda_{\underline{\mathsf{X}}}(h)(([x_n]_n)_{n\in\mathbb{N}_0}) = \gamma_{\underline{\mathsf{X}}}^{-1}\circ\mathcal{A}_{\underline{\mathsf{X}}}\circ\gamma_{\underline{\mathsf{X}}}(h)(([x_n]_n)_{n\in\mathbb{N}_0})$. It will then follow that $\mathcal{A}_{\underline{\mathsf{X}}} = \gamma_{\underline{\mathsf{X}}}\circ\lambda_{\underline{\mathsf{X}}}\circ\gamma_{\underline{\mathsf{X}}}^{-1}$, and thus that $\mathcal{A}_{\underline{\mathsf{X}}}$ maps $\mathcal{G}_{\underline{\mathsf{X}}}$ into $\mathcal{G}_{\underline{\mathsf{X}}}$, and the diagram commutes.

Assume first that $([x_n]_n)_{n\in\mathbb{N}_0} \in \iota_{\underline{\mathsf{X}}}(\underline{\mathsf{X}}^-)$. Then $\#\mathcal{P}_1(x_1) = 1$ and

$$\iota_{\underline{\mathsf{X}}}(\sigma_-(\iota_{\underline{\mathsf{X}}}^{-1}(([x_n]_n)_{n\in\mathbb{N}_0}))) = [ax_n]_{n\in\mathbb{N}_0},$$

where $a\in\mathcal{P}_1(x_1)$. Thus

$$\lambda_{\underline{\mathsf{X}}}(h)(([x_n]_n)_{n\in\mathbb{N}_0}) = h(([ax_n]_n)_{n\in\mathbb{N}_0}) = \gamma_{\underline{\mathsf{X}}}^{-1}\circ\mathcal{A}_{\underline{\mathsf{X}}}\circ\gamma_{\underline{\mathsf{X}}}(h)(([x_n]_n)_{n\in\mathbb{N}_0}).$$

Now assume that $([x_n]_n)_{n\in\mathbb{N}_0} \in \phi_{\underline{\mathsf{X}}}(\mathcal{I}_{\underline{\mathsf{X}}})$ and choose $x\in\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$ such that $\phi_{\underline{\mathsf{X}}}([x]_\infty) = ([x_n]_n)_{n\in\mathbb{N}_0}$. We claim that

$$\sum_{a\in\mathcal{P}_1(x_1)} h([ax_n]_{n\in\mathbb{N}_0}) \quad = \sum_{\substack{x'\in\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)\\ \sigma_+(x')=x}} h(\phi_{\underline{\mathsf{X}}}([x']_\infty)) \;+ \sum_{\substack{z\in\underline{\mathsf{X}}\\ z_{[0,\infty[}\notin\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)\\ z_{[1,\infty[}=x}} h(\iota_{\underline{\mathsf{X}}}(z_{[-\infty,-1]})). \quad (1)$$

To see this let $a\in\mathcal{P}_1(x_1)$. Assume first that $([ax_n]_n)_{n\in\mathbb{N}_0}\in\iota_{\underline{\mathsf{X}}}(\underline{\mathsf{X}}^-)$, and let z be the element of $\mathfrak{a}^{\mathbb{Z}}$ satisfying $z_{]-\infty,0[} = \iota_{\underline{\mathsf{X}}}^{-1}(([ax_n]_n)_{n\in\mathbb{N}_0})$, $z_0 = a$, and $z_{[1,\infty[} = x$. Then $z\in\underline{\mathsf{X}}$, $z_{[0,\infty[}\notin\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$, $z_{[1,\infty[} = x$, and $\iota_{\underline{\mathsf{X}}}(z_{]-\infty,-1]}) = [ax_n]_{n\in\mathbb{N}_0}$. Let us then assume that $([ax_n]_n)_{n\in\mathbb{N}_0}\in\phi_{\underline{\mathsf{X}}}(\mathcal{I}_{\underline{\mathsf{X}}})$. Then $ax\in\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$, $\sigma_+(ax)=x$, and $\phi_{\underline{\mathsf{X}}}([ax]_\infty) = [ax_n]_{n\in\mathbb{N}_0}$.

If on the other hand z is an element of $\underline{\mathsf{X}}$ which satisfies $z_{[0,\infty[}\notin\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$, and $z_{[1,\infty[}=x$, then $z_0\in\mathcal{P}_1(x_1)$, and $\iota_{\underline{\mathsf{X}}}(z_{]-\infty,-1]}) = ([z_0x_n]_n)_{n\in\mathbb{N}_0}$, and if $x'\in\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$ and $\sigma_+(x')=x$, then $x'_0\in\mathcal{P}_1(x_1)$, and $\phi_{\underline{\mathsf{X}}}([x']_\infty) = [x'_0x_n]_{n\in\mathbb{N}_0}$.

Thus (1) holds, and

$$\begin{aligned}\lambda_{\underline{\mathsf{X}}}(h)(([x_n]_n)_{n\in\mathbb{N}_0}) &= \sum_{a\in\mathcal{P}_1(x_1)} h([ax_n]_{n\in\mathbb{N}_0}) \\ &= \sum_{\substack{x'\in\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)\\ \sigma_+(x')=x}} h(\phi_{\underline{\mathsf{X}}}([x']_\infty)) + \sum_{\substack{z\in\underline{\mathsf{X}}\\ z_{[0,\infty[}\notin\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)\\ z_{[1,\infty[}=x}} h(\iota_{\underline{\mathsf{X}}}(z_{[-\infty,-1]})) \\ &= \gamma_{\underline{\mathsf{X}}}^{-1}\circ\mathcal{A}_{\underline{\mathsf{X}}}\circ\gamma_{\underline{\mathsf{X}}}(h)(([x_n]_n)_{n\in\mathbb{N}_0}).\end{aligned}$$

□

The following corollary now follows from Proposition 2.5:

Corollary 5.6. *Let* $\underline{\mathsf{X}}$ *be a shift space which has property* $(**)$. *Then* $K_0(\underline{\mathsf{X}})$ *and*

$$\mathcal{G}_{\underline{\mathsf{X}}}/(\mathrm{Id}-\mathcal{A}_{\underline{\mathsf{X}}})\mathcal{G}_{\underline{\mathsf{X}}}$$

are isomorphic as groups.

5.2 The space $\mathcal{I}_{\underline{\mathsf{X}}}$

In order to get a better understanding of the group $\mathcal{G}_{\underline{\mathsf{X}}}$ and the map $\mathcal{A}_{\underline{\mathsf{X}}}$, we will now try to describe $\mathcal{I}_{\underline{\mathsf{X}}}$ in the case where $\underline{\mathsf{X}}$ has properties $(**)$. For that we will need the concept of right shift tail equivalence (cf. section 2.2).
Denote the set of those right shift tail equivalence classes of $\underline{\mathsf{X}}$ which contains a left special element by $\mathcal{J}_{\underline{\mathsf{X}}}$. Notice that it is finite. Let for every $\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}$, $M_{\mathbf{j}}$ be the set of adjusted left special elements belonging to $\mathbf{j}$. Notice that there only is a finite – but positive – number of elements in $M_{\mathbf{j}}$.
Let us take a closer look at $\pi_+(\mathbf{j})$. It is clear that

$$\pi_+(\mathbf{j}) = \{z_{[n,\infty[} \mid z\in M_{\mathbf{j}}, n\in\mathbb{Z}\},$$

and it follows from the definition of adjusted left special elements that $z_{[n,\infty[}\in\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$ if and only if $n\geq 0$. It follows from the definition of adjusted left special elements and the fact that $\underline{\mathsf{X}}$ contains no periodic left special elements that if $z,z'\in M_{\mathbf{j}}$ and $n,n'<0$, then

$$z_{[n,\infty[} = z'_{[n',\infty[} \iff z=z'\wedge n=n'.$$

Contrary to this, it might happen that $z_{[n,\infty[} = z'_{[n',\infty[}$ for $z\neq z'$ if $n,n'\geq 0$. In fact, it turns out that $\mathbf{j}$ has a "common tail".

Definition 5.7. *Let* $\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}$. *An* $x\in\underline{\mathsf{X}}^+$ *such that there for every* $z\in\mathbf{j}$ *is an* $n\in\mathbb{Z}$ *such that* $z_{[n,\infty[}=x$ *is called a common tail of* $\mathbf{j}$.

Lemma 5.8. *Let* z *be a left special element and* $n\in\mathbb{Z}$. *Then* $z_{[n,\infty[}$ *is a common tail of* $\mathbf{z}$ *if and only if* $\sigma^m(z)$ *is not left special for any* $m>n$.

Proof: Assume that $\sigma^m(z)$ is not left special for any $m > n$, and let $z' \in \mathbf{z}$. Then there are $k, k' \in \mathbb{Z}$ such that $z_{[k,\infty[} = z'_{[k',\infty[}$, and since $\sigma^m(z)$ is not left special for any $m > n$, $z_{[n,\infty[} = z'_{[n-k+k',\infty[}$ if $k > n$. If $k \leq n$, then obviously $z_{[n,\infty[} = z'_{[n-k+k',\infty[}$. Thus $z_{[n,\infty[}$ is a common tail of $\mathbf{z}$.
Assume now that there is an $m > n$ such that $\sigma^m(z)$ is left special. Then there is a $z' \in \underline{\mathsf{X}}$ such that $z_{[m,\infty[} = z'_{[m,\infty[}$, but $z_{m-1} \neq z'_{m-1}$. This implies that $z' \in \mathbf{z}$, so if $z_{[n,\infty[}$ is a common tail of $\mathbf{z}$, then there is a $k \in \mathbb{Z}$ such that $z'_{[k,\infty[} = z_{[n,\infty[}$, and since $z_{m-1} \neq z'_{m-1}$, $k \neq n$. But we then have for all $i \geq m$ that

$$z_i = z'_{i+k-n} = z_{i+k-n},$$

which cannot be true, since there are no periodic left special words in $\underline{\mathsf{X}}$. □

The reason for introducing the concept of common tails is illustrated by the following lemma.

LEMMA 5.9. *If x is a common tail of a $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$, then in the notation of Definition 5.4,*

$$\tilde{\alpha}_{[\sigma_+^{n+1}(x)]_\infty} = \alpha_{[\sigma_+^n(x)]_\infty}$$

for every $n \in \mathbb{N}_0$.

Proof: It follows from Lemma 5.8 that $\mathcal{P}_1(\sigma_+^{n+1}(x)) = \{x_n\}$. Thus there is no $z \in \underline{\mathsf{X}}$ such that $z_{[0,\infty[} \notin \mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$ and $z_{[1,\infty[} = \sigma_+^{n+1}(x)$, and the only $x' \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$ such that $\sigma_+(x') = \sigma_+^{n+1}(x)$ is $\sigma_+^n(x)$. Hence $\tilde{\alpha}_{[\sigma_+^{n+1}(x)]_\infty} = \alpha_{[\sigma_+^n(x)]_\infty}$. □

DEFINITION 5.10. *An $x \in \underline{\mathsf{X}}^+$ is called isolated if there is a $k \in \mathbb{N}_0$ such that $[x]_k = \{x\}$.*

LEMMA 5.11. *Every $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ has an isolated common tail.*

Proof: Let z be the cofinal left special element of $\mathbf{j}$. Then $z_{[0,\infty[}$, and thus $z_{[n,\infty[}$ for every $n \in \mathbb{N}_0$, is a common tail by Lemma 5.8. Since there only are finitely many left special words, $[z_{[0,\infty[}]_1$ is finite. Hence there is an $n \in \mathbb{N}$ such that

$$x \in [z_{[0,\infty[}]_1 \wedge x_{[0,n]} = z_{[0,n]} \Rightarrow x = z_{[0,\infty[}.$$

Thus $[z_{[n,\infty[}]_{n+1} = \{z_{[n,\infty[}\}$ and therefore $z_{[n,\infty[}$ is an isolated common tail. □

REMARK 5.12. In [22] Matsumoto introduced the condition (I) for shift spaces, which is a generalization of the condition (I) for topological Markov shifts in the sense of Cuntz and Krieger (cf. [10]).
A shift space $\underline{\mathsf{X}}$ satisfies condition (I) if and only if $\underline{\mathsf{X}}^+$ has no isolated elements (cf. [22, Lemma 5.1]). Thus, it follows from Lemma 5.11 that a shift space which has property (**) does not satisfy condition (I).

Let $\underline{\mathsf{X}}$ be a shift space which has property (**). Choose once and for all, for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ an isolated common tail $x^{\mathbf{j}}$ and a $z^{\mathbf{j}} \in \underline{\mathsf{X}}$ such that $\pi_+(z^{\mathbf{j}}) = x^{\mathbf{j}}$.

Remark 5.13. Notice that $\sigma_+^n(x^{\mathbf{j}})$ is isolated for every $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ and every $n \in \mathbb{N}_0$, because if $[x^{\mathbf{j}}]_k = \{x^{\mathbf{j}}\}$, then $[\sigma_+^n(x^{\mathbf{j}})]_{k+n} = \{\sigma_+^n(x^{\mathbf{j}})\}$.

Let z be an adjusted left special element of $\underline{\mathsf{X}}$. Since $x^{\mathbf{z}}$ is a common tail of $\mathbf{z}$, there exists an $n_z \in \mathbb{N}_0$ such that $z_{[n_z,\infty[} = x^{\mathbf{z}}$. We let

$$K_{\underline{\mathsf{X}}} = \{[z_{[n,\infty[}]_\infty \mid z \text{ is an adjusted left special element of } \underline{\mathsf{X}},\ 0 \leq n < n_z\},$$

and we let for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$,

$$K_{\mathbf{j}} = \{[z_{[n,\infty[}]_\infty \mid z \in M_{\mathbf{j}},\ 0 \leq n \leq n_z\}.$$

We notice that

$$K_{\underline{\mathsf{X}}} = \bigcup_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}} \left(K_{\mathbf{j}} \setminus \{x^{\mathbf{j}}\}\right).$$

The following lemma shows that

$$K_{\underline{\mathsf{X}}} \cup \bigcup_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}} \bigcup_{n \in \mathbb{N}_0} \{[\sigma_+^n(x^{\mathbf{j}})]_\infty\}$$

is a partition of $\mathcal{I}_{\underline{\mathsf{X}}}$.

Lemma 5.14.

1. $K_{\underline{\mathsf{X}}} \cup \{[\sigma_+^n(x^{\mathbf{j}})]_\infty \mid \mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}},\ n \in \mathbb{N}_0\} = \mathcal{I}_{\underline{\mathsf{X}}}$,
2. $K_{\underline{\mathsf{X}}} \cap \{[\sigma_+^n(x^{\mathbf{j}})]_\infty \mid \mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}},\ n \in \mathbb{N}_0\} = \emptyset$,
3. *the map* $(\mathbf{j}, n) \mapsto [\sigma_+^n(x^{\mathbf{j}})]_\infty$, *from* $\mathcal{J}_{\underline{\mathsf{X}}} \times \mathbb{N}_0$ *to* $\mathcal{I}_{\underline{\mathsf{X}}}$ *is injective.*

Proof: Let $x \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$. Then there is an adjusted left special word z and an $n \in \mathbb{N}_0$ such that $x = z_{[n,\infty[}$. If $n \geq n_z$, then

$$x = z_{[n,\infty[} = z^{\mathbf{z}}_{[n-n_z,\infty[},$$

and if $n < n_z$, then $[x]_\infty = [z_{[n,\infty[}]_\infty \in K_{\underline{\mathsf{X}}}$. Thus

$$K_{\underline{\mathsf{X}}} \cup \{[\sigma_+^n(x^{\mathbf{j}})]_\infty \mid \mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}},\ n \in \mathbb{N}_0\} = \mathcal{I}_{\underline{\mathsf{X}}}.$$

Assume that $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$, $n \in \mathbb{N}_0$ and $[\sigma_+^n(x^{\mathbf{j}})]_\infty \in K_{\underline{\mathsf{X}}}$. Since $\sigma_+^n(x^{\mathbf{j}})$ is isolated, this implies that there exist an adjusted left special element z and $0 \leq m < n_z$ such that $\sigma_+^n(x^{\mathbf{j}}) = z_{[m,\infty[}$. But then

$$z_{[m,\infty[} = \sigma_+^n(x^{\mathbf{j}}) = z_{[n_z+n,\infty[}$$

which cannot be true since there are no periodic left special words in $\underline{\mathsf{X}}$. Thus

$$K_{\underline{\mathsf{X}}} \cap \{[\sigma_+^n(x^{\mathbf{j}})]_\infty \mid \mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}},\ n \in \mathbb{N}_0\} = \emptyset.$$

Assume that $[\sigma_+^{n_1}(x^{\mathbf{j}_1})]_\infty = [\sigma_+(x^{\mathbf{j}_2})]_\infty$. Since $\sigma_+^{n_1}(x^{\mathbf{j}_1})$ is isolated, $\sigma_+^{n_1}(x^{\mathbf{j}_1})$ must be equal to $\sigma_+^{n_2}(x^{\mathbf{j}_2})$. This implies that $z^{\mathbf{j}_1}$ and $z^{\mathbf{j}_2}$ are right shift tail equivalent, so $\mathbf{j}_1 = \mathbf{j}_2$, and since there are no periodic left special words in $\underline{\mathsf{X}}$, n_1 and n_2 must be equal. □

Remark 5.13 shows that if $[x]_\infty \in \{[\sigma_+^n(x^{\mathbf{j}})]_\infty \mid \mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}},\ n \in \mathbb{N}_0\}$, then x is isolated. Although it can happen that x is not isolated if $[x]_\infty \in K_{\underline{\mathsf{X}}}$, the following lemma shows that we anyway can separate $K_{\underline{\mathsf{X}}}$ from $\{[\sigma_+^n(x^{\mathbf{j}})]_\infty \mid \mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}},\ n \in \mathbb{N}_0\}$.

Lemma 5.15. *There exists an $N_{K_{\underline{\mathsf{X}}}} \in \mathbb{N}_0$ such that if $[x]_\infty \in K_{\underline{\mathsf{X}}}$, then $\#\mathcal{P}_{N_{K_{\underline{\mathsf{X}}}}}(x) > 1$ and*

$$[x]_{N_{K_{\underline{\mathsf{X}}}}} = [x']_{N_{K_{\underline{\mathsf{X}}}}} \Rightarrow [x]_\infty = [x']_\infty$$

for every $x' \in \underline{\mathsf{X}}^+$.

Proof: Since $K_{\underline{\mathsf{X}}}$ is a finite set, it is enough to find for each adjusted left special word $z \in \underline{\mathsf{X}}$ and each $0 \le n < n_z$, an $m \in \mathbb{N}_0$ such that $\#\mathcal{P}_m(z_{[n,\infty[}) > 1$ and $[z_{[n,\infty[}]_m = [x]_m \Rightarrow [z_{[n,\infty[}]_\infty = [x]_\infty$ for every $x \in \underline{\mathsf{X}}^+$.
If z is an adjusted left special element and $0 \le n < n_z$, then $\#\mathcal{P}_{n+1}(z_{[n,\infty[}) > 1$, and since there only is a finite number of left special element in $\underline{\mathsf{X}}$, $[z_{[n,\infty[}]_{n+1}$ is finite, so there exists an $m \in \mathbb{N}_0$ such that $\#\mathcal{P}_m(z_{[n,\infty[}) > 1$ and $[z_{[n,\infty[}]_m = [x]_m \Rightarrow [z_{[n,\infty[}]_\infty = [x]_\infty$ for every $x \in \underline{\mathsf{X}}^+$. □

We have now described the space $\mathcal{I}_{\underline{\mathsf{X}}}$ is such great detail that we are able to rephrase the condition of Lemma 5.3 for when a pair $(g, (\alpha_{\mathsf{i}})_{\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}})$ belongs to $\mathcal{G}_{\underline{\mathsf{X}}}$ into a condition which is more readily checkable.

Lemma 5.16. *Let g be a function from $\underline{\mathsf{X}}^-$ to $\mathbb{Z}$ and let for every $\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}$, α_{i} be an integer. Then $(g, (\alpha_{\mathsf{i}})_{\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}}) \in \mathcal{G}_{\underline{\mathsf{X}}}$ if and only if g is continuous and there exists an $N \in \mathbb{N}_0$ such that $\alpha_{[\sigma_+^n(x^{\mathbf{j}})]_\infty} = g(z^{\mathbf{j}}_{]-\infty,n[})$ for all $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ and all $n > N$.*

Proof: Assume that $(g, (\alpha_{\mathsf{i}})_{\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}}) \in \mathcal{G}_{\underline{\mathsf{X}}}$. Then there exists by Lemma 5.3 an $N \in \mathbb{N}_0$ such that

1. $\forall y, y' \in \underline{\mathsf{X}}^- : y_{[-N,-1]} = y'_{[-N,-1]} \Rightarrow g(y) = g(y')$,
2. $\forall x, x' \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+) : [x]_N = [x']_N \Rightarrow \alpha_{[x]_\infty} = \alpha_{[x']_\infty}$,
3. $\forall x \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+),\ y \in \underline{\mathsf{X}}^- : \mathcal{P}_N(x) = \{y_{[-N,-1]}\} \Rightarrow \alpha_{[x]_\infty} = g(y)$.

It follows from 1. that g is continuous, and since $\mathcal{P}_N(\sigma_+^n(x^{\mathbf{j}})) = \{z^{\mathbf{j}}_{[n-N,n-1]}\}$ for every $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ and all $n > N$, it follows from 3. that $\alpha_{[\sigma_+^n(x^{\mathbf{j}})]_\infty} = g(z^{\mathbf{j}}_{]-\infty,n[})$.
Assume now that g is continuous and there exists an $N \in \mathbb{N}_0$ such that $\alpha_{[\sigma_+^n(x^{\mathbf{j}})]_\infty} = g(z^{\mathbf{j}}_{]-\infty,n[})$ for all $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ and all $n > N$. Since g is continuous there is an $M \in \mathbb{N}_0$ such that $y_{[-M,-1]} = y'_{[-M,-1]} \Rightarrow g(y) = g(y')$ for all

$y, y' \in \underline{\mathsf{X}}^-$, and since $\sigma_+^n(x^{\mathbf{j}})$ is isolated for every $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ and every $n \in \mathbb{N}_0$ (cf. Remark 5.13), there is for each $0 \le n \le \max\{M, N\}$ a $k_n^{\mathbf{j}} \in \mathbb{N}$ such that $[\sigma_+^n(x^{\mathbf{j}})]_{k_n^{\mathbf{j}}} = \{\sigma_+(x^{\mathbf{j}})\}$, and by increasing $k_n^{\mathbf{j}}$ if necessary, we may (and will) assume that $\#\mathcal{P}_{k_n^{\mathbf{j}}}(\sigma_+(x^{\mathbf{j}})) > 1$. Let

$$N' = \max\left(\{k_n^{\mathbf{j}} \mid \mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, 0 \le n \le \max\{M, N\}\} \cup \{N_{K_{\underline{\mathsf{X}}}}, M, N\}\right),$$

where $N_{K_{\underline{\mathsf{X}}}}$ is as in Lemma 5.15. We claim that

1. $\forall y, y' \in \underline{\mathsf{X}}^- : y_{[-N',-1]} = y'_{[-N',-1]} \Rightarrow g(y) = g(y')$,
2. $\forall x, x' \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+) : [x]_{N'} = [x']_{N'} \Rightarrow \alpha_{[x]_\infty} = \alpha_{[x']_\infty}$,
3. $\forall x \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+), y \in \underline{\mathsf{X}}^- : \mathcal{P}_{N'}(x) = \{y_{[-N',-1]}\} \Rightarrow \alpha_{[x]_\infty} = g(y)$,

which implies that $(g, (\alpha_{\mathsf{i}})_{\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}}) \in \mathcal{G}_{\underline{\mathsf{X}}}$. 1. follows from the fact that $N' \ge M$. Notice that if

$$[x]_\infty \in K_{\underline{\mathsf{X}}} \cup \{[\sigma_+^n(x^{\mathbf{j}})]_\infty \mid \mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}},\ 0 \le n \le \max\{M, N\}\},$$

then $[x]_{N'} = [x']_{N'} \Rightarrow [x]_\infty = [x']_\infty$. This takes care of 2. in the case where $[x]_\infty \in K_{\underline{\mathsf{X}}} \cup \{[z^{\mathbf{j}}_{[n,\infty[}]_\infty \mid \mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}},\ 0 \le n \le \max\{M, N\}\}$.
Since

$$\begin{aligned}[\sigma_+^n(x^{\mathbf{j}})]_{N'} = [\sigma_+^{n'}(x^{\mathbf{j}'})]_{N'} &\Rightarrow z^{\mathbf{j}}_{[n-M,n-1]} = z^{\mathbf{j}'}_{[n'-M,n'-1]} \\ &\Rightarrow \alpha_{[\sigma_+^n(x^{\mathbf{j}})]_\infty} = g(z^{\mathbf{j}}_{]-\infty,n[}) = g(z^{\mathbf{j}'}_{]-\infty,n'[}) = \alpha_{[\sigma_+^{n'}(x^{\mathbf{j}'})]_\infty},\end{aligned}$$

for $\mathbf{j}, \mathbf{j}' \in \mathcal{J}_{\underline{\mathsf{X}}}$ and $n, n' > \max\{M, N\}$, 2. and 3. hold, and $(g, (\alpha_{\mathsf{i}})_{\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}}) \in \mathcal{G}_{\underline{\mathsf{X}}}$. □

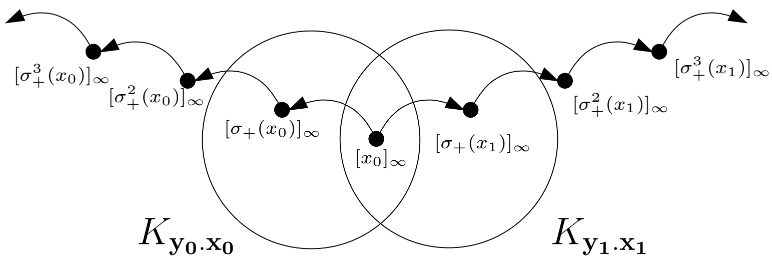
We will now look at $\mathcal{I}_{\underline{\mathsf{X}}}$ for three examples. First let $\underline{\mathsf{X}}$ be the shift space associated with the Morse substitution (see for example [12])

$$0 \mapsto 01, \quad 1 \mapsto 10.$$

The shift space $\underline{\mathsf{X}}$ is minimal and has 4 left special elements:

$$y_0.x_0 \quad y_0.x_1 \quad y_1.x_0 \quad y_1.x_1$$

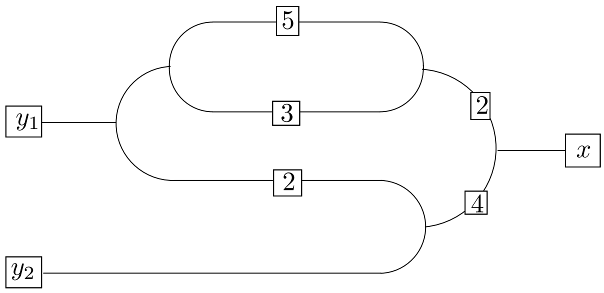
where y_0, y_1 are the fixpoints in $\underline{\mathsf{X}}^-$ of the substitution ending with 0 respectively 1, and x_0, x_1 are the fixpoints in $\underline{\mathsf{X}}^+$ of the substitution beginning with 0 respectively 1. Thus it follows from Example 3.6 that $\underline{\mathsf{X}}$ has property $(**)$. We see that $\mathcal{J}_{\underline{\mathsf{X}}}$ consists of 2 elements: $\mathbf{y_0.x_0}$ and $\mathbf{y_1.x_1}$. Notice that although all of the 4 left special elements are cofinal (and adjusted) neither x_0 nor x_1 are isolated, because $[x_0]_\infty = [x_1]_\infty$, but $\sigma_+(x_0)$ and $\sigma_+(x_1)$ are, so we can choose $\sigma(y_0.x_0)$ and $\sigma(y_1.x_1)$ as $z^{\mathbf{y_0.x_0}}$ and $z^{\mathbf{y_1.x_1}}$ respectively. We then have that $K_{\underline{\mathsf{X}}} = \{[x_0]_\infty\}$, and that the whole of $\mathcal{I}_{\underline{\mathsf{X}}}$ looks like this:



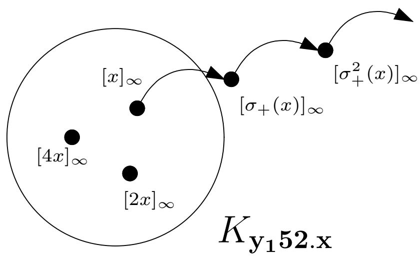
where an arrow from a to b means that in Definition 5.4, $\tilde{\alpha}_b = \alpha_a$. We notice further that $\tilde{\alpha}_{[x_0]_\infty} = g(\sigma_-(y_0)) + g(\sigma_-(y_1))$.
Our second example is the shift space associated to the substitution

$$1 \mapsto 123514, \quad 2 \mapsto 124, \quad 3 \mapsto 13214, \quad 4 \mapsto 14124, \quad 5 \mapsto 15214.$$

The shift space $\underline{\mathsf{X}}$ is minimal and has 8 left special elements (4 adjusted and 4 cofinal) as illustrated on this figure:



where $x \in \underline{\mathsf{X}}^+$ and $y_1, y_2 \in \underline{\mathsf{X}}^-$. Thus it follows from 3.6 that $\underline{\mathsf{X}}$ has property $(**)$. The set $\mathcal{J}_{\underline{\mathsf{X}}}$ consists of one element $\mathbf{y_1 52.x}$, and since x is isolated, we can choose $y_1 52.x$ as $z^{\mathbf{y_1 52.x}}$. We then have that $K_{\underline{\mathsf{X}}} = \{[2x]_\infty, [4x]_\infty\}$, and that the whole of $\mathcal{I}_{\underline{\mathsf{X}}}$ looks like this:

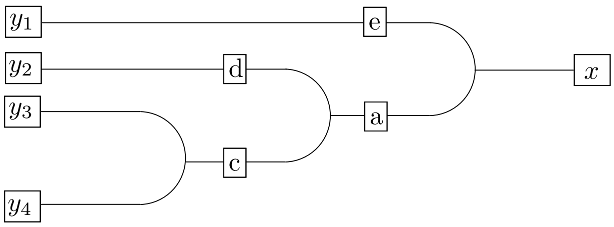


where an arrow from a to b means that in Definition 5.4, $\tilde{\alpha}_b = \alpha_a$. We notice further that $\tilde{\alpha}_{[x]_\infty} = \alpha_{[2x]_\infty} + \alpha_{[4x]_\infty}$, $\tilde{\alpha}_{[2x]_\infty} = 2g(y_1)$ and $\tilde{\alpha}_{[4x]_\infty} = g(y_1) + g(\sigma_-(y_2))$.

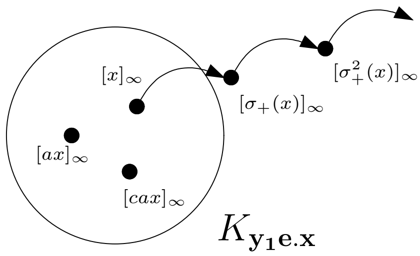
The third example is the shift space associated to the substitution

$$a \mapsto adbac, \quad b \mapsto aedbbc, \quad c \mapsto ac, \quad d \mapsto adac, \quad e \mapsto aecadbac.$$

The shift space $\underline{\mathsf{X}}$ is minimal and has 9 left special elements (1 which is both adjusted and cofinal, 3 which are adjusted but not cofinal, 3 which are cofinal but not adjusted, and 2 which are neither adjusted nor cofinal) as illustrated on this figure:



where $x \in \underline{\mathsf{X}}^+$ and $y_1, y_2, y_3, y_4 \in \underline{\mathsf{X}}^-$. Thus it follows from 3.6 that $\underline{\mathsf{X}}$ has property (∗∗). The set $\mathcal{J}_{\underline{\mathsf{X}}}$ consists of one element $\mathbf{y_1 e.x}$, and since x is isolated, we can choose $y_1 e.x$ as $z^{\mathbf{y_1 e.x}}$. We then have that $K_{\underline{\mathsf{X}}} = \{[cax]_\infty, [ax]_\infty\}$, and that the whole of $\mathcal{I}_{\underline{\mathsf{X}}}$ looks like this:



where an arrow from a to b means that in Definition 5.4, $\tilde{\alpha}_b = \alpha_a$. We notice further that $\tilde{\alpha}_{[x]_\infty} = \alpha_{[ax]_\infty} + g(y_1)$, $\tilde{\alpha}_{[ax]_\infty} = \alpha_{[cax]_\infty} + g(y_2)$ and $\tilde{\alpha}_{[cax]_\infty} = g(\sigma_-(y_3)) + g(\sigma_-(y_4))$.

5.3 $K_0(\underline{\mathsf{X}})$ is a factor of $G_{\underline{\mathsf{X}}}$

We are now ready to define the group $G_{\underline{\mathsf{X}}}$ which has a factor which is isomorphic to $\mathcal{G}_{\underline{\mathsf{X}}}/(\mathrm{Id} - \mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}})$.

Loosely speaking, the idea is to simplify $\mathcal{G}_{\underline{\mathsf{X}}}$ in three ways. First we collapse for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$, $K_{\mathbf{j}}$ to one point, which makes it possible to replace $\mathcal{I}_{\underline{\mathsf{X}}}$ by $\mathcal{J}_{\underline{\mathsf{X}}} \times \mathbb{N}_0$, secondly we replace the condition of Lemma 5.16 for when a pair belongs to $\mathcal{G}_{\underline{\mathsf{X}}}$, by the condition that the corresponding sequence in $\mathcal{J}_{\underline{\mathsf{X}}} \times \mathbb{N}_0$ is eventually 0, and thirdly, we replace $\underline{\mathsf{X}}^-$ by $\underline{\mathsf{X}}$. The resulting group $G_{\underline{\mathsf{X}}}$ is of course not necessarily isomorphic to $\mathcal{G}_{\underline{\mathsf{X}}}$, but it turns out that we can still define a map $A_{\underline{\mathsf{X}}} : G_{\underline{\mathsf{X}}} \to G_{\underline{\mathsf{X}}}$ such that $G_{\underline{\mathsf{X}}}/(\mathrm{Id} - A_{\underline{\mathsf{X}}})\big(G_{\underline{\mathsf{X}}}\big)$ is isomorphic to $\mathcal{G}_{\underline{\mathsf{X}}}/(\mathrm{Id} - \mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}})$.

Definition 5.17. *Let $\underline{\mathsf{X}}$ be a shift space which has property $(**)$. Denote by $G_{\underline{\mathsf{X}}}$ the group $C(\underline{\mathsf{X}},\mathbb{Z})\oplus\sum_{n\in\mathbb{N}_0}\mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}$, let $A_{\underline{\mathsf{X}}}$ be the map from $G_{\underline{\mathsf{X}}}$ to itself defined by*

$$(f,(a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}},n\in\mathbb{N}_0})\mapsto(f\circ\sigma^{-1},(\tilde{a}_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}},n\in\mathbb{N}_0}),$$

where $\tilde{a}_0^{\mathbf{j}}=\sum_{z\in M_{\mathbf{j}}}f(\sigma^{-1}(z))-f(\sigma^{-1}(z^{\mathbf{j}}))$, and $\tilde{a}_n^{\mathbf{j}}=a_{n-1}^{\mathbf{j}}$ for $n>0$, and let ψ be the map from $\mathcal{G}_{\underline{\mathsf{X}}}$ to $G_{\underline{\mathsf{X}}}$ defined by

$$(g,(\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}})\mapsto(g\circ\pi_-,(a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}},n\in\mathbb{N}_0}),$$

where for each $j\in\mathbf{j}$, $a_0^{\mathbf{j}}=\sum_{\mathsf{i}\in K_{\mathbf{j}}}\alpha_{\mathsf{i}}-g(\pi_-(z^{\mathbf{j}}))$ and $a_n^{\mathbf{j}}=\alpha_{[\sigma_+^n(x^{\mathbf{j}})]_\infty}-g(z^{\mathbf{j}}_{]-\infty,n[})$ for $n>0$.

Remark 5.18. It directly follows from Lemma 5.16 that ψ in fact maps $\mathcal{G}_{\underline{\mathsf{X}}}$ into $G_{\underline{\mathsf{X}}}$.

Proposition 5.19. *Let $\underline{\mathsf{X}}$ be a shift space which has property $(**)$. Then there is an isomorphism*

$$\bar{\psi}:\mathcal{G}_{\underline{\mathsf{X}}}/(\mathrm{Id}-\mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}})\to G_{\underline{\mathsf{X}}}/(\mathrm{Id}-A_{\underline{\mathsf{X}}})\left(G_{\underline{\mathsf{X}}}\right)$$

which makes the following diagram commute:

$$\begin{array}{ccc}\mathcal{G}_{\underline{\mathsf{X}}} & \xrightarrow{\quad\psi\quad} & G_{\underline{\mathsf{X}}}\\ \downarrow & & \downarrow\\ \mathcal{G}_{\underline{\mathsf{X}}}/(\mathrm{Id}-\mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}}) & \xrightarrow{\bar{\psi}} & G_{\underline{\mathsf{X}}}/(\mathrm{Id}-A_{\underline{\mathsf{X}}})\left(G_{\underline{\mathsf{X}}}\right)\end{array}$$

We will postpone the proof of proposition 5.19 to section 5.5, and instead state our main theorem which immediately follows from Proposition 5.19 and Corollary 5.6.

Theorem 5.20. *Let $\underline{\mathsf{X}}$ be a shift space which has property $(**)$. Then $K_0(\underline{\mathsf{X}})$ and*

$$G_{\underline{\mathsf{X}}}/(\mathrm{Id}-A_{\underline{\mathsf{X}}})\left(G_{\underline{\mathsf{X}}}\right)$$

are isomorphic as groups.

5.4 Examples

Example 5.21. Let $\underline{\mathsf{X}}$ be a finite shift space. Then $K_0(\underline{\mathsf{X}})$ and

$$C(\underline{\mathsf{X}},\mathbb{Z})/(\mathrm{Id}-(\sigma^{-1})^\star)(C(\underline{\mathsf{X}},\mathbb{Z}))$$

are isomorphic as groups.

Proof: We saw in Example 3.4, that a finite shift space has property $(**)$ and has no left special elements. Thus $\mathcal{J}_{\underline{\mathsf{X}}} = \emptyset$, so $G_{\underline{\mathsf{X}}} = C(\underline{\mathsf{X}}, \mathbb{Z})$ and $A_{\underline{\mathsf{X}}} = (\sigma^{-1})^\star$ and it follows from Theorem 5.20, that $K_0(\underline{\mathsf{X}})$ and

$$C(\underline{\mathsf{X}}, \mathbb{Z})/(\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}, \mathbb{Z}))$$

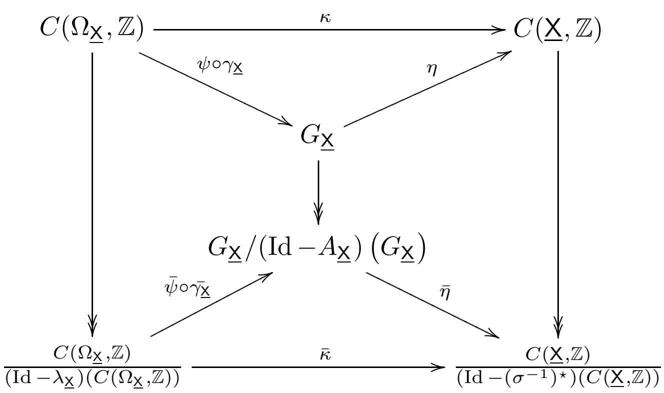
are isomorphic as groups. □

Let η be the canonical projection from $G_{\underline{\mathsf{X}}}$ to $C(\underline{\mathsf{X}}, \mathbb{Z})$. We tie things up with the following proposition:

PROPOSITION 5.22. *Let $\underline{\mathsf{X}}$ be a shift space which has property $(**)$. Then there is a surjective group homomorphism*

$$\bar{\eta} : G_{\underline{\mathsf{X}}}/(\mathrm{Id} - A_{\underline{\mathsf{X}}})\left(G_{\underline{\mathsf{X}}}\right) \to C(\underline{\mathsf{X}}, \mathbb{Z})/(\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}, \mathbb{Z}))$$

which makes the following diagram commute:



where $\bar{\gamma}_{\underline{\mathsf{X}}}$ is the map from $C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z})/(\mathrm{Id} - \lambda_{\underline{\mathsf{X}}})(C(\Omega_{\underline{\mathsf{X}}}, \mathbb{Z}))$ to $\mathcal{G}_{\underline{\mathsf{X}}}/(\mathrm{Id} - \mathcal{A}_{\underline{\mathsf{X}}})\mathcal{G}_{\underline{\mathsf{X}}}$ induced by $\gamma_{\underline{\mathsf{X}}}$.

Proof: Since

$$\eta \circ A_{\underline{\mathsf{X}}} = (\sigma^{-1})^\star \circ \eta,$$

η induces a map from $G_{\underline{\mathsf{X}}}/(\mathrm{Id} - A_{\underline{\mathsf{X}}})\left(G_{\underline{\mathsf{X}}}\right)$ to $C(\underline{\mathsf{X}}, \mathbb{Z})/(\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}, \mathbb{Z}))$. It is easy to check that this map makes the diagram commute. □

COROLLARY 5.23. *Let $\underline{\mathsf{X}}$ be a shift space which has property $(**)$ and only has two left special words. Then $\bar{\eta}$ is an isomorphism from $G_{\underline{\mathsf{X}}}/(\mathrm{Id} - A_{\underline{\mathsf{X}}})\left(G_{\underline{\mathsf{X}}}\right)$ to $C(\underline{\mathsf{X}}, \mathbb{Z})/(\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}, \mathbb{Z}))$. Thus $K_0(\underline{\mathsf{X}})$ and*

$$C(\underline{\mathsf{X}}, \mathbb{Z})/(\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}, \mathbb{Z}))$$

are isomorphic as groups.

Proof: If $\underline{\mathsf{X}}$ only has two left special words, z_1 and z_2, then they must necessarily be right shift tail equivalent, so $\mathcal{J}_{\underline{\mathsf{X}}} = \{\mathbf{j}\}$, where $\mathbf{j} = \mathbf{z_1} = \mathbf{z_2}$. We also have that $z_{1[0,\infty[} = z_{2[0,\infty[}$ is an isolated common tail of $\mathbf{j}$, so we can choose z_2 to be $z^{\mathbf{j}}$. The set $M_{\mathbf{j}}$ is equal to $\{z_1, z_2\}$, so for any $(h, (b^{\mathbf{j}}_n)_{n\in\mathbb{N}_0}) \in G_{\underline{\mathsf{X}}}$ is

$$A_{\underline{\mathsf{X}}}((h, (b^{\mathbf{j}}_n)_{n\in\mathbb{N}_0})) = (h \circ \sigma^{-1}, (\tilde{b}^{\mathbf{j}}_n)_{n\in\mathbb{N}_0}),$$

where $\tilde{b}^{\mathbf{j}}_0 = h(\sigma^{-1}(z_1))$, and $\tilde{b}^{\mathbf{j}}_n = b^{\mathbf{j}}_{n-1}$ for $n > 0$.
Suppose that $(f, (a^{\mathbf{j}}_n)_{n\in\mathbb{N}_0}) \in G_{\underline{\mathsf{X}}}$ and that

$$\eta((f, (a^{\mathbf{j}}_n)_{n\in\mathbb{N}_0})) \in (\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}, \mathbb{Z})).$$

Then there is a $\tilde{f} \in C(\underline{\mathsf{X}}, \mathbb{Z})$ such that $f = \tilde{f} - \tilde{f} \circ \sigma^{-1}$. Since $(a^{\mathbf{j}}_n)_{n\in\mathbb{N}_0} \in \sum_{n\in\mathbb{N}_0} \mathbb{Z}$, there is an $N \in \mathbb{N}_0$ such that $a^{\mathbf{j}}_n = 0$ for $n > N$. Let

$$c = -\tilde{f}(\sigma^{-1}(z_1)) - \sum_{n=0}^{N} a^{\mathbf{j}}_n$$

and $h \in C(\underline{\mathsf{X}}, \mathbb{Z})$ the function $\tilde{f}$ plus the constant c, and let $b^{\mathbf{j}}_n = \sum_{i=0}^{n} a^{\mathbf{j}}_i + h(\sigma^{-1}(z_1))$ for $n \in \mathbb{N}_0$. Then $b^{\mathbf{j}}_n = 0$ for $n > N$, so $(h, (b^{\mathbf{j}}_n)_{n\in\mathbb{N}_0}) \in G_{\underline{\mathsf{X}}}$, and

$$(f, (a^{\mathbf{j}}_n)_{n\in\mathbb{N}_0}) = (\mathrm{Id} - A_{\underline{\mathsf{X}}})((h, (b^{\mathbf{j}}_n)_{n\in\mathbb{N}_0})) \in (\mathrm{Id} - A_{\underline{\mathsf{X}}})\left(G_{\underline{\mathsf{X}}}\right),$$

which prove that $\bar{\eta}$ is injective and thus an isomorphism. □

EXAMPLE 5.24. As noted in [12], a Sturmian shift space $\underline{\mathsf{X}}_\alpha$, $\alpha \in [0,1]\backslash\mathbb{Q}$ is minimal and has two special words. Thus it follows from Example 3.6 and Corollary 5.23 that $K_0(\underline{\mathsf{X}}_\alpha)$ and

$$C(\underline{\mathsf{X}}_\alpha, \mathbb{Z})/(\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}_\alpha, \mathbb{Z}))$$

are isomorphic as groups.
In [31] it is shown that

$$C(\underline{\mathsf{X}}_\alpha, \mathbb{Z})/(\mathrm{Id} - (\sigma^{-1})^\star)(C(\underline{\mathsf{X}}_\alpha, \mathbb{Z}))$$

is isomorphic to $\mathbb{Z} + \mathbb{Z}\alpha$ as an *ordered* group. Thus it follows that $K_0(\underline{\mathsf{X}}_\alpha)$ and $\mathbb{Z} + \mathbb{Z}\alpha$ are isomorphic as groups.
In [9, Corollary 5.2] we prove that $K_0(\underline{\mathsf{X}}_\alpha)$ with the order structure mentioned in the Introduction is isomorphic to $\mathbb{Z} + \mathbb{Z}\alpha$.

EXAMPLE 5.25. It is proved in [30, pp. 90 and 107] that if τ is an aperiodic and primitive substitution, then the associated shift space X_τ is minimal and only has a finite number of left special words. Thus by Example 3.6, X_τ has property $(**)$. It follows from [6, Proposition 3.5] that if τ furthermore is proper and elementary, then $\pi_+(z)$ is isolated for every left special word z. Thus $K_0(\mathsf{X}_\tau)$ is isomorphic to the cokernel of the map

$$A_\tau(f,[(a_0^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\mathsf{X}_\tau}},(a_1^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\mathsf{X}_\tau}},\dots]) =$$
$$\left(f\circ\sigma^{-1},\left[\left(\left(\sum_{z\in M_{\mathbf{j}}} f(\sigma^{-1}(z))\right)-f(\sigma^{-1}(z^{\mathbf{j}}))\right)_{\mathbf{j}\in\mathcal{J}_{\mathsf{X}_\tau}},(a_0^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\mathsf{X}_\tau}},(a_1^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\mathsf{X}_\tau}},\dots\right]\right)$$

defined on

$$G_\tau = C(\underline{\mathsf{X}}_\tau,\mathbb{Z})\oplus\sum_{i=0}^{\infty}\mathbb{Z}^{\mathcal{J}_{\mathsf{X}_\tau}},$$

where $\mathcal{J}_{\mathsf{X}_\tau}$ and $M_{\mathbf{j}}$ are as defined in section 5.2, and $z^{\mathbf{j}}$ is a cofinal special element belonging to the right shift tail equivalence class $\mathbf{j}$.
In the notation of [8],

$$\mathcal{J}_{\mathsf{X}_\tau}=\{\widetilde{\mathsf{y}}^1,\widetilde{\mathsf{y}}^2,\dots,\widetilde{\mathsf{y}}^{\mathsf{n}_\tau}\},\ M_{\widetilde{\mathsf{y}}^{\mathsf{j}}}=\{\mathsf{y}_1^j,\mathsf{y}_2^j,\dots,\mathsf{y}_{p_j+1}^j\}\ \text{ and } z^{\widetilde{\mathsf{y}}^{\mathsf{j}}}=\widetilde{\mathsf{y}}^j.$$

In [8], this is used for every aperiodic and primitive (but not necessarily proper or elementary) substitution τ, to present $K_0(\mathsf{X}_\tau)$ as a stationary inductive limit of a system associated to an integer matrix defined from combinatorial data which can be computed in an algorithmic way (cf. [6] and [7]).

5.5 THE PROOF OF PROPOSITION 5.19

In order to prove Proposition 5.19, we will define maps and groups as indicated on the diagram:

$$\begin{array}{ccc}
\mathcal{G}_{\underline{\mathsf{X}}} & \twoheadrightarrow & \mathcal{G}_{\underline{\mathsf{X}}}/(\mathrm{Id}-\mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}}) \\
\downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \overline{\psi}_1} \\
\psi_1(\mathcal{G}_{\underline{\mathsf{X}}}) & \twoheadrightarrow & \psi_1(\mathcal{G}_{\underline{\mathsf{X}}})/(\mathrm{Id}-A_1)(\psi_1(\mathcal{G}_{\underline{\mathsf{X}}})) \\
\downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \overline{\psi}_2} \\
C(\underline{\mathsf{X}}^-,\mathbb{Z})\oplus\sum_{n\in\mathbb{N}_0}\mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}} & \twoheadrightarrow & \dfrac{C(\underline{\mathsf{X}}^-,\mathbb{Z})\oplus\sum_{n\in\mathbb{N}_0}\mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}}{(\mathrm{Id}-A_2)\left(C(\underline{\mathsf{X}}^-,\mathbb{Z})\oplus\sum_{n\in\mathbb{N}_0}\mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}\right)} \\
\downarrow{\scriptstyle \psi_3} & & \downarrow{\scriptstyle \overline{\psi}_3} \\
G_{\underline{\mathsf{X}}} & \twoheadrightarrow & G_{\underline{\mathsf{X}}}/(\mathrm{Id}-A_{\underline{\mathsf{X}}})\left(G_{\underline{\mathsf{X}}}\right)
\end{array}$$

such that the diagram commutes, $\psi_3 \circ \psi_2 \circ \psi_1 = \psi$, and $\overline{\psi}_1$, $\overline{\psi}_2$ and $\overline{\psi}_3$ are isomorphisms.
Let $\psi_1 : \mathcal{G}_{\underline{\mathsf{X}}} \to C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \prod_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}$ be the map defined by

$$(g, (\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}) \mapsto (g, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}),$$

where for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$, $a_0^{\mathbf{j}} = \sum_{\mathsf{i}\in K_{\mathbf{j}}} \alpha_{\mathsf{i}}$, and $a_n^{\mathbf{j}} = \alpha_{[\sigma_+^n(x^{\mathbf{j}})]_\infty}$ for $n \in \mathbb{N}$.

LEMMA 5.26. *Let* $(g, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \in C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \prod_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}$. *Then* $(g, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \in \psi_1(\mathcal{G}_{\underline{\mathsf{X}}})$ *if and only if*

$$\exists N \in \mathbb{N}_0 \forall \mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}} \forall n > N : a_n^{\mathbf{j}} = g(z^{\mathbf{j}}_{]-\infty,n[}).$$

Proof: The forward implication directly follows from Lemma 5.16.
Assume that $(g, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \in C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \prod_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}$ and there exists an $N \in \mathbb{N}_0$ such that $a_n^{\mathbf{j}} = g(z^{\mathbf{j}}_{]-\infty,n[})$ for all $n > N$. We let $\alpha_{\mathsf{i}} = 0$ for each $\mathsf{i} \in K_{\underline{\mathsf{X}}}$, and we let for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ and each $n \in \mathbb{N}_0$, $\alpha_{[z^{\mathbf{j}}_{[n,\infty[}]_\infty} = a_n^{\mathbf{j}}$. It then follows from Lemma 5.16 that $(g, (\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}) \in \mathcal{G}_{\underline{\mathsf{X}}}$, and since $\psi_1(g, (\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}) = (g, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0})$, we have that $(g, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \in \psi_1(\mathcal{G}_{\underline{\mathsf{X}}})$.
$\square$

Let $A_1 : C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \prod_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}} \to C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \prod_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}$ be the map defined by

$$(g, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \mapsto (g \circ \sigma_-, (\tilde{a}_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}),$$

where for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$, $\tilde{a}_0^{\mathbf{j}} = \sum_{z\in M_{\mathbf{j}}} g(\sigma_-(\pi_-(z)))$, and $\tilde{a}_n^{\mathbf{j}} = a_{n-1}^{\mathbf{j}}$ for $n \in \mathbb{N}$.
It follows from Lemma 5.26 that A_1 maps $\psi_1(\mathcal{G}_{\underline{\mathsf{X}}})$ into itself. Thus $(\mathrm{Id} - A_1)\psi_1(\mathcal{G}_{\underline{\mathsf{X}}})$ is a subgroup of $\psi_1(\mathcal{G}_{\underline{\mathsf{X}}})$, and we can form the quotient $\psi_1(\mathcal{G}_{\underline{\mathsf{X}}})/(\mathrm{Id} - A_1)\psi_1(\mathcal{G}_{\underline{\mathsf{X}}})$. Let

$$q : \psi_1(\mathcal{G}_{\underline{\mathsf{X}}}) \mapsto \psi_1(\mathcal{G}_{\underline{\mathsf{X}}})/(\mathrm{Id} - A_1)\psi_1(\mathcal{G}_{\underline{\mathsf{X}}})$$

be the quotient map. We then have:

LEMMA 5.27. $\ker(q \circ \psi_1) = (\mathrm{Id} - \mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}})$.

Proof: Assume $(g, (\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}) \in \ker(q \circ \psi_1)$. That means that

$$(g, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) = \psi_1(g, (\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}) \in (\mathrm{Id} - A_1)\psi_1(\mathcal{G}_{\underline{\mathsf{X}}}).$$

Thus there exists $(\tilde{g}, (\tilde{a}_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \in \psi_1(\mathcal{G}_{\underline{\mathsf{X}}})$ such that $g = \tilde{g} - \tilde{g} \circ \sigma_-$, and such that for every $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$,

$$a_0^{\mathbf{j}} = \sum_{\mathsf{i}\in K_{\mathbf{j}}} \alpha_{\mathsf{i}} = \tilde{a}_0^{\mathbf{j}} - \sum_{z\in M_{\mathbf{j}}} \tilde{g}(\sigma_-(\pi_-(z))),$$

and $a_n^{\mathbf{j}} = \alpha_{[\sigma_+^n(x^{\mathbf{j}})]_\infty} = \tilde{a}_n^{\mathbf{j}} - \tilde{a}_{n-1}^{\mathbf{j}}$ for all $n \in \mathbb{N}$.

Now let $\mathsf{i} \in \bigcup_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}} K_{\mathbf{j}}$. Choose $x_{\mathsf{i}} \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+)$ such that $[x_{\mathsf{i}}]_\infty = \mathsf{i}$. Then there is, for each $z \in \underline{\mathsf{X}}$ which satisfies $\pi_+(z) = x_{\mathsf{i}}$, a unique $m_z \in \mathbb{N}_0$ such that $\sigma^{-m_z}(z)$ is an adjusted left special word. We let

$$L_{\mathsf{i}} = \{[z_{[-m,\infty[}]_\infty \mid \pi_+(z) = x_{\mathsf{i}},\ 0 \le m \le m_z\} \subseteq \mathcal{I}_{\underline{\mathsf{X}}},$$
$$B_{\mathsf{i}} = \{\sigma^{-m_z}(z) \mid \pi_+(z) = x_{\mathsf{i}}\} \subseteq \underline{\mathsf{X}},$$

and

$$\tilde{\alpha}_{\mathsf{i}} = \sum_{\mathsf{i}'\in L_{\mathsf{i}}} \alpha_{\mathsf{i}'} + \sum_{z\in B_{\mathsf{i}}} \tilde{g}(\sigma_-(\pi_-(z))),$$

and we let for $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ and $n \in \mathbb{N}$, $\tilde{\alpha}_{[\sigma_+^n(x^{\mathbf{j}})]_\infty} = \tilde{a}_n^{\mathbf{j}}$.

Since $(\tilde{g}, (\tilde{a}_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \in \psi_1(\mathcal{G}_{\underline{\mathsf{X}}})$, $\tilde{g}$ is continuous and there exists by Lemma 5.26, an $N \in \mathbb{N}_0$ such that for all $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ and all $n > N$, $\tilde{\alpha}_{[\sigma_+^n(x^{\mathbf{j}})]_\infty} = \tilde{a}_n^{\mathbf{j}} = \tilde{g}(z^{\mathbf{j}}_{]-\infty,n[})$, so $(\tilde{g}, (\tilde{\alpha}_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}) \in \mathcal{G}_{\underline{\mathsf{X}}}$ by Lemma 5.16.

Let $(\tilde{\tilde{g}}, (\tilde{\tilde{\alpha}}_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}) = \mathcal{A}_{\underline{\mathsf{X}}}(\tilde{g}, (\tilde{\alpha}_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}})$. Then $\tilde{\tilde{g}} = \tilde{g} \circ \sigma_-$, and by lemma 5.9,

$$\tilde{\tilde{\alpha}}_{[\sigma_+^{n+1}(x^{\mathbf{j}})]_\infty} = \tilde{\alpha}_{[\sigma_+^n(x^{\mathbf{j}})]_\infty} = \tilde{a}_n^{\mathbf{j}},$$

for $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ and $n \in \mathbb{N}$.

Now let $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$. Then $L_{[x^{\mathbf{j}}]_\infty} = K_{\mathbf{j}}$ and $B_{[x^{\mathbf{j}}]_\infty} = M_{\mathbf{j}}$, so

$$\begin{aligned}\tilde{\tilde{\alpha}}_{[\sigma_+(x^{\mathbf{j}})]_\infty} &= \tilde{\alpha}_{[x^{\mathbf{j}}]_\infty} \\ &= \sum_{\mathsf{i}\in K_{\mathbf{j}}} \alpha_{\mathsf{i}} + \sum_{z\in M_{\mathbf{j}}} \tilde{g}(\sigma_-(\pi_-(z))) \\ &= a_0^{\mathbf{j}} + \sum_{z\in M_{\mathbf{j}}} \tilde{g}(\sigma_-(\pi_-(z))) \\ &= \tilde{a}_0^{\mathbf{j}}.\end{aligned}$$

If $[x]_\infty \in K_{\mathbf{j}}$, then $L_{[x]_\infty}$ is the disjoint union of $L_{[x']_\infty}$, where $[x']_\infty \in \mathcal{I}_{\underline{\mathsf{X}}}$ and $\sigma_+(x') = x$, and $\{[x]_\infty\}$, and $B_{[x]_\infty}$ is the disjoint union of $B_{[x']_\infty}$, where $[x']_\infty \in \mathcal{I}_{\underline{\mathsf{X}}}$ and $\sigma_+(x') = x$, and $\{\sigma(z) \mid z \in \underline{\mathsf{X}}, z_{[0,\infty[} \notin \mathcal{ND}_\infty(\underline{\mathsf{X}}^+), z_{[1,\infty[} = x\}$.

Hence

$$\begin{aligned}\tilde{\tilde{\alpha}}_{[x]_\infty} &= \sum_{\substack{[x']_\infty\in\mathcal{I}_{\underline{\mathsf{X}}}\\ \sigma_+(x')=x}} \tilde{\alpha}_{[x']_\infty} + \sum_{\substack{z\in\underline{\mathsf{X}}\\ z_{[0,\infty[}\notin\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)\\ z_{[1,\infty[}=x}} \tilde{g}(z_{]-\infty,-1]})\\ &= \sum_{\substack{[x']_\infty\in\mathcal{I}_{\underline{\mathsf{X}}}\\ \sigma_+(x')=x}} \left(\sum_{\mathsf{i}\in L_{[x']_\infty}} \alpha_{\mathsf{i}} + \sum_{z\in B_{[x']_\infty}} \tilde{g}(\sigma_-(\pi_-(z)))\right) + \sum_{\substack{z\in\underline{\mathsf{X}}\\ z_{[0,\infty[}\notin\mathcal{ND}_\infty(\underline{\mathsf{X}}^+)\\ z_{[1,\infty[}=x}} \tilde{g}(z_{]-\infty,-1]})\\ &= \sum_{\mathsf{i}\in L_{[x]_\infty}} \alpha_{\mathsf{i}} - \alpha_{[x]_\infty} + \sum_{z\in B_{[x]_\infty}} \tilde{g}(\sigma_-(\pi_-(z)))\\ &= \tilde{\alpha}_{[x]_\infty} - \alpha_{[x]_\infty}.\end{aligned}$$

So

$$\tilde{g} - \tilde{\tilde{g}} = \tilde{g} - \tilde{g}\circ\sigma_- = g,$$

and for $\mathsf{i}\in\bigcup_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}} K_{\mathbf{j}}$,

$$\tilde{\alpha}_{\mathsf{i}} - \tilde{\tilde{\alpha}}_{\mathsf{i}} = \tilde{\alpha}_{\mathsf{i}} - \tilde{\alpha}_{\mathsf{i}} + \alpha_{\mathsf{i}} = \alpha_{\mathsf{i}},$$

and for $\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}$ and $n\in\mathbb{N}$

$$\tilde{\alpha}_{[\sigma_+^n(x^{\mathbf{j}})]_\infty} - \tilde{\tilde{\alpha}}_{[\sigma_+^n(x^{\mathbf{j}})]_\infty} = \tilde{a}_n^{\mathbf{j}} - \tilde{a}_{n-1}^{\mathbf{j}} = a_n^{\mathbf{j}} = \alpha_{[\sigma_+^n(x^{\mathbf{j}})]_\infty}.$$

Thus

$$\begin{aligned}(g,(\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}) &= (\tilde{g},(\tilde{\alpha}_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}) - (\tilde{\tilde{g}},(\tilde{\tilde{\alpha}}_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}})\\ &= (\mathrm{Id}-\mathcal{A}_{\underline{\mathsf{X}}})(\tilde{g},(\tilde{\alpha}_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}}) \in (\mathrm{Id}-\mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}}),\end{aligned}$$

which shows that $\ker(q\circ\psi_1)\subseteq(\mathrm{Id}-\mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}})$.
Now let $(g,(\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}})\in\mathcal{G}_{\underline{\mathsf{X}}}$. We will find an element $(g,(a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}},n\in\mathbb{N}_0})\in\psi_1(\mathcal{G}_{\underline{\mathsf{X}}})$ such that

$$\psi_1((\mathrm{Id}-\mathcal{A}_{\underline{\mathsf{X}}})(g,(\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}})) = (\mathrm{Id}-A_1)(g,(a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}},n\in\mathbb{N}_0}).$$

This will show that $(\mathrm{Id}-\mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}})\subseteq\ker(\rho\circ\psi_1)$.
Let for each $\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}$ and every $n\in\mathbb{N}_0$, $a_n^{\mathbf{j}}=\alpha_{[\sigma_+^n(x^{\mathbf{j}})]_\infty}$. It then follows from Lemma 5.16 and 5.26 that $(g,(a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}},n\in\mathbb{N}_0})\in\psi_1(\mathcal{G}_{\underline{\mathsf{X}}})$.
Now,

$$(\mathrm{Id}-A_1)(g,(a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}},n\in\mathbb{N}_0}) = (g-g\circ\sigma_-,(a_n^{\mathbf{j}}-\tilde{a}_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}},n\in\mathbb{N}_0}),$$

where for each $\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}$, $\tilde{a}_0^{\mathbf{j}}=\sum_{z\in M_{\mathbf{j}}} g(\sigma_-(\pi_-(z)))$, and $\tilde{a}_n^{\mathbf{j}}=a_{n-1}^{\mathbf{j}}$ for $n\in\mathbb{N}$, and

$$\psi_1((\mathrm{Id}-\mathcal{A}_{\underline{\mathsf{X}}})(g,(\alpha_{\mathsf{i}})_{\mathsf{i}\in\mathcal{I}_{\underline{\mathsf{X}}}})) = (g-g\circ\sigma_-,(b_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}},n\in\mathbb{N}_0}),$$

where by Lemma 5.9

$$b_n^{\mathbf{j}} = \alpha_{[\sigma_+^n(x^{\mathbf{j}})]_\infty} - \alpha_{[\sigma_+^{n-1}(x^{\mathbf{j}})]_\infty} = a_n^{\mathbf{j}} - a_{n-1}^{\mathbf{j}} = a_n^{\mathbf{j}} - \tilde{a}_n^{\mathbf{j}}$$

for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$ and every $n \in \mathbb{N}$, and

$$\begin{aligned} b_0^{\mathbf{j}} = & \sum_{\mathsf{i} \in K_{\mathbf{j}}} \alpha_{\mathsf{i}} - \sum_{[x]_\infty \in K_{\mathbf{j}}} \left(\sum_{\substack{x' \in \mathcal{ND}_\infty(\underline{\mathsf{X}}^+) \\ \sigma_+(x') = x}} \alpha_{[x']_\infty} + \sum_{\substack{z \in \underline{\mathsf{X}} \\ z_{[0,\infty[} \notin \mathcal{ND}_\infty(\underline{\mathsf{X}}^+) \\ z_{[1,\infty[} = x}} g(\pi_-(z)) \right) \\ = & \alpha_{x^{\mathbf{j}}} - \sum_{z \in M_{\mathbf{j}}} g(\sigma_-(\pi_-(z))) \\ = & a_0^{\mathbf{j}} - \tilde{a}_0^{\mathbf{j}} \end{aligned}$$

for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$. Thus $\psi_1((\mathrm{Id} - \mathcal{A}_{\underline{\mathsf{X}}})(g, (\alpha_{\mathsf{i}})_{\mathsf{i} \in \mathcal{I}_{\underline{\mathsf{X}}}})) = (\mathrm{Id} - A_1)(g, (a_n^{\mathbf{j}})_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \in \mathbb{N}_0})$. □

It follows from the previous lemma, that $\psi_1 : \mathcal{G}_{\underline{\mathsf{X}}} \to \psi_1(\mathcal{G}_{\underline{\mathsf{X}}})$ induces an isomorphism $\overline{\psi}_1$ from $\mathcal{G}_{\underline{\mathsf{X}}}/(\mathrm{Id} - \mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}})$ to $\psi_1(\mathcal{G}_{\underline{\mathsf{X}}})/(\mathrm{Id} - A_1)(\psi_1(\mathcal{G}_{\underline{\mathsf{X}}}))$ such that the diagram

$$\begin{array}{ccc} \mathcal{G}_{\underline{\mathsf{X}}} & \twoheadrightarrow & \mathcal{G}_{\underline{\mathsf{X}}}/(\mathrm{Id} - \mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}}) \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \overline{\psi}_1} \\ \psi_1(\mathcal{G}_{\underline{\mathsf{X}}}) & \twoheadrightarrow & \psi_1(\mathcal{G}_{\underline{\mathsf{X}}})/(\mathrm{Id} - A_1)(\psi_1(\mathcal{G}_{\underline{\mathsf{X}}})) \end{array}$$

commutes.

Let for every $(g, (a_n^{\mathbf{j}})_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \in \mathbb{N}_0}) \in \psi_1(\mathcal{G}_{\underline{\mathsf{X}}})$,

$$\psi_2(g, (a_n^{\mathbf{j}})_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \in \mathbb{N}_0}) = (g, (a_n^{\mathbf{j}} - g(z_{]-\infty,n[}^{\mathbf{j}}))_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \in \mathbb{N}_0}).$$

It follows from Lemma 5.26 that ψ_2 is an isomorphism from $\psi_1(\mathcal{G}_{\underline{\mathsf{X}}})$ to $C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n \in \mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}$.

Let $A_2 : C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n \in \mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}} \to C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n \in \mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}$ be the map given by

$$(g, (a_n^{\mathbf{j}})_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \in \mathbb{N}_0}) \mapsto (g \circ \sigma_-, (\hat{a}_n^{\mathbf{j}})_{\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}, n \in \mathbb{N}_0}),$$

where for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$,

$$\hat{a}_0^{\mathbf{j}} = \sum_{z \in M_{\mathbf{j}}} g(\pi_-(\sigma^{-1}(z))) - g(\pi_-(\sigma^{-1}(z^{\mathbf{j}}))),$$

and $\hat{a}_n^{\mathbf{j}} = a_{n-1}^{\mathbf{j}}$ for $n \in \mathbb{N}$.

Then $\psi_2 \circ A_1 = A_2 \circ \psi_2$, so ψ_2 induces an isomorphism

$$\overline{\psi}_2 : \psi_1(\mathcal{G}_{\underline{\mathsf{X}}})/(\mathrm{Id} - A_1)(\psi_1(\mathcal{G}_{\underline{\mathsf{X}}})) \to \frac{C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}}{(\mathrm{Id} - A_2)\left(C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}\right)}$$

such that the diagram

$$\begin{array}{ccc}
\psi_1(\mathcal{G}_{\underline{\mathsf{X}}}) & \twoheadrightarrow & \psi_1(\mathcal{G}_{\underline{\mathsf{X}}})/(\mathrm{Id} - A_1)(\psi_1(\mathcal{G}_{\underline{\mathsf{X}}})) \\
\downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \overline{\psi}_2} \\
C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}} & \twoheadrightarrow & \frac{C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}}{(\mathrm{Id} - A_2)\left(C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}\right)}
\end{array}$$

commutes.
Let $\psi_3 : C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}} \to G_{\underline{\mathsf{X}}}$ be the map defined by

$$(g, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \mapsto (g \circ \pi_-, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}).$$

We then have:

Lemma 5.28. $\psi_3 \circ A_2 = A \circ \psi_3$.

Proof: Let $(g, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \in C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}$. Then

$$\begin{aligned}
A \circ \psi_3(g, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) &= A(g \circ \pi_-, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \\
&= (g \circ \pi_- \circ \sigma^{-1}, (\tilde{a}_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}),
\end{aligned}$$

where for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$,

$$\tilde{a}_0^{\mathbf{j}} = \sum_{z\in M_{\mathbf{j}}} g \circ \pi_-(\sigma^{-1}(z)) - g \circ \pi_-(\sigma^{-1}(z^{\mathbf{j}})),$$

and $\tilde{a}_n^{\mathbf{j}} = a_{n-1}^{\mathbf{j}}$ for $n \in \mathbb{N}$, and

$$\begin{aligned}
\psi_3 \circ A_2(g, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) &= \psi_3(g \circ \sigma_-, (\hat{a}_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \\
&= (g \circ \sigma_- \circ \pi_-, (\hat{a}_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}),
\end{aligned}$$

where for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$,

$$\hat{a}_0^{\mathbf{j}} = \sum_{z\in M_{\mathbf{j}}} g(\pi_-(\sigma^{-1}(z))) - g(\pi_-(\sigma^{-1}(z^{\mathbf{j}}))) = \tilde{a}_0^{\mathbf{j}},$$

and $\hat{a}_n^{\mathbf{j}} = a_{n-1}^{\mathbf{j}} = \tilde{a}_n^{\mathbf{j}}$ for $n \in \mathbb{N}$, and since $\pi_- \circ \sigma^{-1} = \sigma_- \circ \pi_-$, we have that $\psi_3 \circ A_2(g, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) = A \circ \psi_3(g, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0})$. □

It follows from the previous lemma that $\psi_3 : C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}} \to G_{\underline{\mathsf{X}}}$ induces an injective map

$$\overline{\psi}_3 : \frac{C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}}{(\mathrm{Id} - A_2)\left(C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}\right)} \to G_{\underline{\mathsf{X}}}/(\mathrm{Id} - A_{\underline{\mathsf{X}}})\left(G_{\underline{\mathsf{X}}}\right)$$

such that the diagram

$$\begin{array}{ccc}
C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}} & \twoheadrightarrow & \frac{C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}}{(\mathrm{Id} - A_2)\left(C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}\right)} \\
\downarrow \psi_3 & & \downarrow \overline{\psi}_3 \\
G_{\underline{\mathsf{X}}} & \twoheadrightarrow & G_{\underline{\mathsf{X}}}/(\mathrm{Id} - A_{\underline{\mathsf{X}}})\left(G_{\underline{\mathsf{X}}}\right)
\end{array}$$

commutes. We will now show that $\overline{\psi}_3$ in fact is an isomorphism.

LEMMA 5.29. *The map $\overline{\psi}_3$ is surjective.*

Proof: In order to prove that $\overline{\psi}_3$ is surjective, it is enough to show that for every $(f, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \in G_{\underline{\mathsf{X}}}$, there is a $(g, (\tilde{a}_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \in C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}$ such that

$$(f, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) - \psi_3(g, (\tilde{a}_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \in (\mathrm{Id} - A_{\underline{\mathsf{X}}})\left(G_{\underline{\mathsf{X}}}\right).$$

Since f is continuous, there are $k, m \in \mathbb{N}$ such that $z_{[-k,m]} = z'_{[-k,m]} \Rightarrow f(z) = f(z')$. Thus

$$z_{[-k-m-1,-1]} = z'_{[-k-m-1,-1]} \Rightarrow f \circ \sigma^{-(m+1)}(z) = f \circ \sigma^{-(m+1)}(z').$$

Hence there is an $g \in C(\underline{\mathsf{X}}^-, \mathbb{Z})$ such that $g \circ \pi_- = f \circ \sigma^{-(m+1)}$. We let for each $\mathbf{j} \in \mathcal{J}_{\underline{\mathsf{X}}}$,

$$\tilde{a}_n^{\mathbf{j}} = \sum_{z\in M_{\mathbf{j}}} f \circ \sigma^{n-m-1}(z) - f \circ \sigma^{n-m-1}(z^{\mathbf{j}})$$

for $0 \le n \le m$, and $\tilde{a}_n^{\mathbf{j}} = a_{n-(m+1)}^{\mathbf{j}}$ for $n > m$. Since $(f, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \in G_{\underline{\mathsf{X}}}$, $(g, (\tilde{a}_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) \in C(\underline{\mathsf{X}}^-, \mathbb{Z}) \oplus \sum_{n\in\mathbb{N}_0} \mathbb{Z}^{\mathcal{J}_{\underline{\mathsf{X}}}}$, and it is easy to check that

$$\psi_3(g, (\tilde{a}_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) = A_{\underline{\mathsf{X}}}^{m+1}(f, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}).$$

Thus

$$\begin{gathered}
(f, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) - \psi_3(g, (\tilde{a}_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) = \\
(f, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) - A_{\underline{\mathsf{X}}}^{m+1}(f, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0}) = \\
\sum_{k=0}^{m} (\mathrm{Id} - A_{\underline{\mathsf{X}}})(A_{\underline{\mathsf{X}}}^k(f, (a_n^{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}_{\underline{\mathsf{X}}}, n\in\mathbb{N}_0})) \in (\mathrm{Id} - A_{\underline{\mathsf{X}}})\left(G_{\underline{\mathsf{X}}}\right).
\end{gathered}$$

□

We now have that $\bar{\psi} = \overline{\psi}_3 \circ \overline{\psi}_2 \circ \overline{\psi}_1$ is an isomorphism and since $\psi = \psi_3 \circ \psi_2 \circ \psi_1$, the diagram

$$\begin{array}{ccc} \mathcal{G}_{\underline{\mathsf{X}}} & \xrightarrow{\psi} & G_{\underline{\mathsf{X}}} \\ \downarrow & & \downarrow \\ \mathcal{G}_{\underline{\mathsf{X}}}/(\mathrm{Id} - \mathcal{A}_{\underline{\mathsf{X}}})(\mathcal{G}_{\underline{\mathsf{X}}}) & \xrightarrow{\bar{\psi}} & G_{\underline{\mathsf{X}}}/(\mathrm{Id} - A_{\underline{\mathsf{X}}})\left(G_{\underline{\mathsf{X}}}\right) \end{array}$$

commutes, and we are done with the proof of Proposition 5.19.

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