Automorphism Groups of Shimura Varieties

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Received: April 14, 2003
Revised: February 5, 2006

Communicated by Don Blasius

Abstract. In this paper, we determine the scheme automorphism group of the reduction modulo p of the integral model of the connected Shimura variety (of prime-to-p level) for reductive groups of type A and C. The result is very close to the characteristic 0 version studied by Shimura, Deligne and Milne-Shih.

2000 Mathematics Subject Classification: 11G15, 11G18, 11G25
Keywords and Phrases: Shimura variety, reciprocity law

There are two aspects of the Artin reciprocity law. One is representation theoretic, for example,

$$\mathrm{Hom}_{cont}(\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}), \mathbb{C}^\times) \cong \mathrm{Hom}_{cont}((\mathbb{A}^{(\infty)})^\times/\mathbb{Q}_+^\times, \mathbb{C}^\times)$$

via the identity of L–functions. Another geometric one is:

$$\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \cong GL_1(\mathbb{A}^{(\infty)})/\mathbb{Q}_+^\times.$$

They are equivalent by duality, and the first is generalized by Langlands in non-abelian setting. Geometric reciprocity in non-abelian setting would be via Tannakian duality; so, it involves Shimura varieties.

Iwasawa theory is built upon the geometric reciprocity law. The cyclotomic field $\mathbb{Q}(\mu_{p^\infty})$ is the maximal p–ramified extension of $\mathbb{Q}$ fixed by $\widehat{\mathbb{Z}}^{(p)} \subset \mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}_+^\times$ removing the p–inertia toric factor $\mathbb{Z}_p^\times$. We then try to study arithmetically constructed modules X out of $\mathbb{Q}(\mu_{p^\infty}) \subset \mathbb{Q}^{ab}$. The main idea is to regard X as a module over over the Iwasawa algebra (which is a completed Hecke algebra relative to $\frac{GL_1(\mathbb{A}^{(\infty)})}{GL_1(\widehat{\mathbb{Z}}^{(p)})\mathbb{Q}_+^\times}$), and ring theoretic techniques are used to determine X.

If one wants to get something similar in a non-abelian situation, we really need a scheme whose automorphism group has an identification with $G(\mathbb{A}^{(\infty)})/\overline{Z(\mathbb{Q})}$ for a reductive algebraic group G. If $G = GL(2)_{/\mathbb{Q}}$, the tower $V_{/\mathbb{Q}^{ab}}$ of modular curves has $\mathrm{Aut}(V_{/\mathbb{Q}})$ identified with $GL_2(\mathbb{A}^{(\infty)})/Z(\mathbb{Q})$ as Shimura proved. The decomposition group of (p) is given by $B(\mathbb{Q}_p) \times SL_2(\mathbb{A}^{(p\infty)})/\{\pm 1\}$ for a Borel subgroup B, and I have been studying various arithmetically constructed modules over the Hecke algebra of $\frac{GL_2(\mathbb{A}^{(\infty)})}{GL_2(\widehat{\mathbb{Z}}^{(p)})U(\mathbb{Z}_p)\mathbb{Q}_+^\times}$, relative to the unipotent subgroup $U(\mathbb{Z}_p) \subset B(\mathbb{Z}_p)$ (removing the toric factor from the decomposition group). Such study has yielded a p–adic deformation theory of automorphic forms (see [PAF] Chapter 1 and 8), and it would be therefore important to study the decomposition group at p of a given Shimura variety, which is basically the automorphism group of the mod p Shimura variety.

Iwasawa theoretic applications (if any) are the author's motivation for the investigation done in this paper. However the study of the automorphism group of a given Shimura variety has its own intrinsic importance. As is clear from the construction of Shimura varieties done by Shimura ([Sh]) and Deligne ([D1] 2.4-7), their description of the automorphism group (of Shimura varieties of characteristic 0) is deeply related to the geometric reciprocity laws generalizing classical ones coming from class field theory and is almost equivalent to the existence of the canonical models defined over a canonical algebraic number field. Except for the modulo p modular curves and Shimura curves studied by Y. Ihara, the author is not aware of a single determination of the automorphism group of the integral model of a Shimura variety and of its reduction modulo p, although Shimura indicated and emphasized at the end of his introduction of the part I of [Sh] a good possibility of having a canonical system of automorphic varieties over finite fields described by the adelic groups such as the ones studied in this paper.

We shall determine the automorphism group of mod p Shimura varieties of PEL type coming from symplectic and unitary groups.

1 Statement of the theorem

Let B be a central simple algebra over a field M with a positive involution ρ (thus $\mathrm{Tr}_{B/\mathbb{Q}}(xx^\rho) > 0$ for all $0 \neq x \in B$). Let F be the subfield of M fixed by ρ. Thus F is a totally real field, and either $M = F$ or M is a CM quadratic extension of F. We write O (resp. R) for the integer ring of F (resp. M). We fix an algebraic closure $\mathbb{F}$ of the prime field $\mathbb{F}_p$ of characteristic $p > 0$. Fix a proper subset Σ of rational places including ∞ and p. Let $F_+^\times$ be the subset of totally positive elements in F, and $O_{(\Sigma)}$ denotes the localization of O at Σ (disregarding the infinite place in Σ) and O_Σ is the completion of O at Σ (again disregarding the infinite place). We write $O^\times_{(\Sigma)+} = F_+^\times \cap O_{(\Sigma)}$. We have an exact sequence

$$1 \to B^\times/M^\times \to \mathrm{Aut}_{alg}(B) \to \mathrm{Out}(B) \to 1,$$

and by a theorem of Skolem-Noether, $\mathrm{Out}(B) \subset \mathrm{Aut}(M)$. Here $b \in B^\times$ acts on B by $x \mapsto bxb^{-1}$. Since B is central simple, any simple B-module N is isomorphic each other. Take one such simple B-module. Then $\mathrm{End}_B(N)$ is a division algebra D°. Taking a base of N over D° and identifying $N \cong (D^\circ)^r$, we have $B = \mathrm{End}_{D^\circ}(N) \cong M_r(D)$ for the opposite algebra D of D°. Letting $\mathrm{Aut}_{alg}(D)$ act on $b \in M_r(D)$ entry-by-entry, we have $\mathrm{Aut}_{alg}(D) \subset \mathrm{Aut}_{alg}(B)$, and $\mathrm{Out}(D) = \mathrm{Out}(B)$ under this isomorphism.

Let O_B be a maximal order of B. Let L be a projective O_B–module with a non-degenerate F-linear alternating form $\langle\ ,\ \rangle : L_\mathbb{Q} \times L_\mathbb{Q} \to F$ for $L_A = L \otimes_\mathbb{Z} A$ such that $\langle bx, y\rangle = \langle x, b^\rho y\rangle$ for all $b \in B$. Identifying $L_\mathbb{Q}$ with a product of copies of the column vector space D^r on which $M_r(D)$ acts via matrix multiplication, we can let $\sigma \in \mathrm{Aut}_{alg}(D)$ act component-wise on $L_\mathbb{Q}$ so that $\sigma(bv) = \sigma(b)\sigma(v)$ for all $\sigma \in \mathrm{Aut}_{alg}(D)$.

Let C be the opposite algebra of $C^\circ = \mathrm{End}_B(L_\mathbb{Q})$. Then C is a central simple algebra and is isomorphic to $M_s(D)$, and hence $\mathrm{Out}(C) \cong \mathrm{Out}(D) = \mathrm{Out}(B)$. We write $C_A = C \otimes_\mathbb{Q} A$, $B_A = B \otimes_\mathbb{Q} A$ and $F_A = F \otimes_\mathbb{Q} A$. The algebra C has involution $*$ given by $\langle cx, y\rangle = \langle x, c^* y\rangle$ for $c \in C$, and this involution "$*$" of C extends to an involution again denoted by "$*$" of $\mathrm{End}_\mathbb{Q}(L_\mathbb{Q})$ given by $\mathrm{Tr}_{F/\mathbb{Q}}(\langle gx, y\rangle) = \mathrm{Tr}_{F/\mathbb{Q}}(\langle x, g^* y\rangle)$ for $g \in \mathrm{End}_\mathbb{Q}(L_\mathbb{Q})$. The involution $*$ (resp. ρ) induces the involution $* \otimes 1$ (resp. $\rho \otimes 1$) on C_A (resp. on B_A) which we write as $*$ (resp. ρ) simply. Define an algebraic group $G_{/\mathbb{Q}}$ by

$$G(A) = \left\{ g \in C_A \middle| \nu(g) := gg^* \in (F_A)^\times \right\} \quad \text{for } \mathbb{Q}\text{-algebras } A \tag{1.1}$$

and an extension $\widetilde{G}$ of G by the following subgroup of the opposite group $\mathrm{Aut}^\circ_A(L_A)$ of the A-linear automorphism group $\mathrm{Aut}_A(L_A)$:

$$\widetilde{G}(A) = \left\{ g \in \mathrm{Aut}^\circ_A(L_A) \middle| gC_A g^{-1} = C_A \ \text{ and } \ \nu(g) := gg^* \in (F_A)^\times \right\}. \tag{1.2}$$

Since $C^\circ = \mathrm{End}_B(L_\mathbb{Q})$, we have $B^\circ = \mathrm{End}_C(L_\mathbb{Q})$, and from this we find that $gBg^{-1} = B \Leftrightarrow gCg^{-1} = C$ for $g \in \mathrm{Aut}_\mathbb{Q}(L_\mathbb{Q})$, and if this holds for g, then $gg^* \in F^\times \Leftrightarrow gx^\rho g^{-1} = (gxg^{-1})^\rho$ for all $x \in B$ and $gy^* g^{-1} = (gyg^{-1})^*$ for all $y \in C$. Then G is a normal subgroup of $\widetilde{G}$ of finite index, and $\widetilde{G}(\mathbb{Q})/G(\mathbb{Q}) = \mathrm{Out}_{\mathbb{Q}\text{-alg}}(C, *)$. Here $\mathrm{Out}_{\mathbb{Q}\text{-alg}}(C, *)$ is the outer automorphism group of C commuting with $*$; in other words, it is the quotient of the group of automorphisms of C commuting with $*$ by the group of inner automorphisms commuting with $*$. Thus we have $\mathrm{Out}_{\mathbb{Q}\text{-alg}}(C, *) \subset H^0(\langle * \rangle, \mathrm{Out}_{\mathbb{Q}\text{-alg}}(C)) \subset \mathrm{Out}_{\mathbb{Q}\text{-alg}}(C) \subset \mathrm{Aut}(M/\mathbb{Q})$. All the four groups are equal if $G_{/\mathbb{Q}}$ is quasi-split but are not equal in general. We put $PG = G/Z$ and $P\widetilde{G} = \widetilde{G}/Z$ for the center Z of G.

We write G_1 for the derived group of G; thus, $G_1 = \{g \in G | N_C(g) = \nu(g) = 1\}$ for the reduced norm N_C of C over M. We write $Z^G = G/G_1$ for the cocenter of G. Then $g \mapsto (\nu(g), N_C(g))$ identifies Z^G with a sub-torus of $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m \times \mathrm{Res}_{M/\mathbb{Q}}\mathbb{G}_m$. If $M = F$, G_1 is equal to the kernel of the similitude map $g \mapsto \nu(g)$; so, in this case, we ignore the right factor $\mathrm{Res}_{M/\mathbb{Q}}\mathbb{G}_m$ and regard $Z^G \subset \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$. By a result of Weil ([W]) combined with an observation in [Sh] II (4.2.1), the automorphism group $\mathrm{Aut}_{A\text{-alg}}(C_A, *)$ of the algebra C_A preserving the involution $*$ is given by $P\widetilde{G}(A)$. In other words, we have an exact sequence of $\mathbb{Q}$-algebraic groups

$$1 \to PG(A) \to \mathrm{Aut}_{A\text{-alg}}(C_A, *)(= P\widetilde{G}(A)) \to \mathrm{Out}_{A\text{-alg}}(B_A, \rho) \to 1. \tag{1.3}$$

We write

$$\pi : \widetilde{G}(A) \to \widetilde{G}(A)/G(A) = \mathrm{Out}_{A\text{-alg}}(B_A, \rho) \subset \mathrm{Out}_{A\text{-alg}}(B_A)$$

for the projection.

The automorphism group of the Shimura variety of level away from Σ is a quotient of the following locally compact subgroup of $\widetilde{G}(\mathbb{A}^{(\Sigma)})$:

$$\mathcal{G}^{(\Sigma)} = \left\{ x \in \widetilde{G}(\mathbb{A}^{(\Sigma)}) \middle| \pi(x) \in \mathrm{Out}_{\mathbb{Q}\text{-alg}}(B, \rho) \right\}, \tag{1.4}$$

where we embed $\mathrm{Out}_{\mathbb{Q}\text{-alg}}(B, \rho)$ into $\prod_{\ell \notin \Sigma} \mathrm{Out}_{\mathbb{Q}_\ell\text{-alg}}(B_\ell) = \mathrm{Out}_{\mathbb{A}^{(\Sigma)}\text{-alg}}(B \otimes_{\mathbb{Q}} \mathbb{A}^{(\Sigma)})$ by the diagonal map ($B_\ell = B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$).

We suppose to have an $\mathbb{R}$–algebra homomorphism $h : \mathbb{C} \to C_{\mathbb{R}}$ such that $h(\overline{z}) = h(z)^*$ and

(h1) $(x, y) = \langle x, h(i)y \rangle$ induces a positive definite hermitian form on $L_{\mathbb{R}}$.

We define X to be the conjugacy classes of h under $G(\mathbb{R})$. Then X is a finite disjoint union of copies of the hermitian symmetric domain isomorphic to $G(\mathbb{R})^+/C_h$, where C_h is the stabilizer of h and the superscript "+" indicates the identity connected component of the Lie group $G(\mathbb{R})$. Then the pair (G, X) satisfies the three axioms (see [D1] 2.1.1.1-3) specifying the data for defining the Shimura variety Sh (and its field of definition, the reflex field E; see [Ko] Lemma 4.1). In [D1], two more axioms are stated to simplify the situation: (2.1.1.4-5). These two extra axioms may not hold generally for our choice of (G, X) (see [M] Remark 2.2).

The complex points of Sh are given by

$$Sh(\mathbb{C}) = G(\mathbb{Q}) \backslash \left(G(\mathbb{A}^{(\infty)}) \times X \right) / \overline{Z(\mathbb{Q})}.$$

This variety can be characterized as a moduli variety of abelian varieties up to isogeny with multiplication by B. For each $x \in X$, we have $h_x : \mathbb{C} \to C_{\mathbb{R}}$ given by $z \mapsto g \cdot h(z)g^{-1}$ for $g \in G(\mathbb{R})/C_h$ sending h to x. Then $v \mapsto h_x(z)v$ for $z \in \mathbb{C}$ gives rise to a complex vector space structure on $L_{\mathbb{R}}$, and $\mathbb{X}_x(\mathbb{C}) = L_{\mathbb{R}}/L$ is an abelian variety, because by (h1), $\langle\ ,\ \rangle$ induces a Riemann form on L. The multiplication by $b \in O_B$ is given by $(v \mod L) \mapsto (b \cdot v \mod L)$.

We suppose

(h2) all rational primes in Σ are unramified in $M/\mathbb{Q}$, and Σ contains ∞ and p;

(h3) For every prime $\ell \in \Sigma$, $O_{B,\ell} = O_B \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong M_n(R_\ell)$ for $R_\ell = R \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$;

(h4) For every prime $\ell \in \Sigma$, $\langle\ ,\ \rangle$ induces $L_\ell = L \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong \mathrm{Hom}(L_\ell, O_\ell)$;

(h5) The derived subgroup G_1 is simply connected; so, $G_1(\mathbb{R})$ is of type A (unitary groups) or of type C (symplectic groups).

Let $G(\mathbb{Z}_\Sigma) = \{g \in G(\mathbb{Q}_\Sigma) | g \cdot L_\Sigma = L_\Sigma\}$ for $\mathbb{Q}_\Sigma = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_\Sigma$ and $L_\Sigma = L \otimes_{\mathbb{Z}} \mathbb{Z}_\Sigma$. We define $Sh^{(\Sigma)} = Sh/G(\mathbb{Z}_\Sigma)$. This moduli interpretation (combined with (h1-4)) allows us to have a well defined p–integral model of level away from Σ (see below for a brief description of the moduli problem, and a more complete description can be found in [PAF] 7.1.3). In other words,

$$Sh^{(\Sigma)}(\mathbb{C}) = G(\mathbb{Q})\backslash \left(G(\mathbb{A}^{(\infty)}) \times X\right)/\overline{Z(\mathbb{Q})}G(\mathbb{Z}_\Sigma)$$

has a well defined smooth model over $O_{E,(p)} := O_E \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ which is again a moduli scheme of abelian varieties up to prime-to–Σ isogenies. We write $Sh^{(p)}$ for $Sh^{(\Sigma)}$ when $\Sigma = \{p, \infty\}$. We also write $\mathbb{Q}_\Sigma^{(p)} = \mathbb{Q}_\Sigma/\mathbb{Q}_p$ and $\mathbb{Z}_\Sigma^{(p)} = \mathbb{Z}_\Sigma/\mathbb{Z}_p$.

We have taken full polarization classes under scalar multiplication by $O^\times_{(\Sigma)+}$ in our moduli problem (while Kottwitz's choice in [Ko] is a partial class of multiplication by $\mathbb{Z}^\times_{(\Sigma)+}$). By our choice, the group G is the full similitude group, while Kottwitz choice is a partial rational similitude group. Our choice is convenient for our purpose because G has cohomologically trivial center, and the special fiber at p of the characteristic 0 Shimura variety $Sh/G(\mathbb{Z}_\Sigma)$ gives rise to the mod p moduli of abelian varieties of the specific type we study (as shown in [PAF] Theorem 7.5), while Kottwitz's mod p moduli is a disjoint union of the reduction modulo p of finitely many characteristic 0 Shimura varieties associated to finitely many different pairs (G_i, X_i) with G_i locally isomorphic each other at every place ([Ko] Section 8).

We fix a strict henselization $\mathcal{W} \subset \overline{\mathbb{Q}}$ of $\mathbb{Z}_{(p)}$. Thus $\mathcal{W}$ is an unramified valuation ring with residue field $\mathbb{F} = \overline{\mathbb{F}}_p$. Under these five conditions (h1-5), combining (and generalizing) the method of Chai-Faltings [DAV] for Siegel modular varieties and that of [Ra] for Hilbert modular varieties, Fujiwara ([F] Theorem in §0.4) proved the existence of a smooth toroidal compactification of $Sh^{(p)}_S = Sh^{(p)}/S$ for sufficiently small principal congruence subgroups S with respect to L in $G(\mathbb{A}^{(p)})$ as an algebraic space over a suitable open subscheme of $Spec(O_E)$ containing $Spec(\mathcal{W} \cap E)$. Although our moduli problem is slightly different from the one Kottwitz considered in [Ko], as was done in [PAF] 7.1.3, following [Ko] closely, the p-integral moduli $Sh^{(p)}$ over $O_{E,(p)}$ is proven to be a quasi-projective scheme; so, Fujiwara's algebraic space is a projective scheme (if we choose the toroidal compactification data well). If the reader is not familiar with Fujiwara's work, the reader can take the existence of the smooth toroidal compactification (which is generally believed to be true) as an assumption of our main result.

Since p is unramified in $M/\mathbb{Q}$, $O_{E,(p)}$ is contained in $\mathcal{W}$. We fix a geometrically connected component $V_{/\overline{\mathbb{Q}}}$ of $Sh^{(\Sigma)} \times_E \overline{\mathbb{Q}}$ and write $V_{/\mathcal{W}}$ for the schematic closure of $V_{/\overline{\mathbb{Q}}}$ in $Sh^{(\Sigma)}_{/\mathcal{W}} := Sh^{(\Sigma)}_{/O_{E,(p)}} \otimes_{O_{E,(p)}} \mathcal{W}$. By Zariski's connectedness theorem combined with the existence of a normal projective compactification (either minimal or smooth) of $Sh^{(\Sigma)}_{/\mathcal{W}}$, the reduction $V_{/\mathbb{F}} = V \times_{\mathcal{W}} \mathbb{F}$ is a geometrically irreducible component of $Sh^{(\Sigma)}_{/\mathcal{W}} \otimes_{\mathcal{W}} \mathbb{F}$. The scheme $Sh^{(\Sigma)}_{/S}$ classifies, for any S–scheme T, quadruples $(A, \overline{\lambda}, i, \phi^{(\Sigma)})_{/T}$ defined as follows: A is an abelian scheme of dimension $\frac{1}{2}\operatorname{rank}_{\mathbb{Z}} L$ for which we define the Tate module $\mathcal{T}(A) = \varprojlim_N A[N]$, $\mathcal{T}^{(\Sigma)}(A) = \mathcal{T}(A) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^{(\Sigma)}$, $\mathcal{T}_\Sigma(A) = \mathcal{T}(A) \otimes_{\mathbb{Z}} \mathbb{Z}_\Sigma$ and $V^{(\Sigma)}(A) = \mathcal{T}(A) \otimes_{\mathbb{Z}} \mathbb{A}^{(\Sigma)}$; The symbol i stands for an algebra embedding $i : O_B \hookrightarrow \mathrm{End}(A)$ taking the identity to the identity map on A; $\phi^{(\Sigma)}$ is a level structure away from Σ, that is, an O_B–linear $\phi^{(p)} : L \otimes_{\mathbb{Q}} \mathbb{A}^{(p)} \cong V^{(p)}(A)$ modulo $G(\mathbb{Z}_\Sigma)$, where we require that $\phi^{(p)}_\Sigma : L \otimes_{\mathbb{Z}} \mathbb{Q}^{(p)}_\Sigma \cong \mathcal{T}_\Sigma(A) \otimes_{\mathbb{Z}} \mathbb{Q}^{(p)}_\Sigma$ send $L \otimes_{\mathbb{Z}} \mathbb{Z}^{(p)}_\Sigma$ isomorphically onto $\mathcal{T}_\Sigma(A) \otimes_{\mathbb{Z}} \mathbb{Z}^{(p)}_\Sigma$; $\overline{\lambda}$ is a class of polarizations λ up to scalar multiplication by $i(O^\times_{(\Sigma)+})$ which induces the Riemann form $\langle \cdot, \cdot \rangle$ on L up to scalar multiplication by $O^\times_{(\Sigma)+}$. There is one more condition (cf. [Ko] Section 5 or [PAF] 7.1.1 (det)) specifying the module structure of $\Omega_{A/T}$ over $O_B \otimes_{\mathbb{Z}} \mathcal{O}_T$ (which we do not recall).

The group $G(\mathbb{A}^{(\Sigma)})$ acts on $Sh^{(\Sigma)}$ by $\phi^{(\Sigma)} \mapsto \phi^{(\Sigma)} \circ g$. We can extend the action of $G(\mathbb{A}^{(\Sigma)})$ to the extension $\mathcal{G}^{(\Sigma)}$. Each element $g \in \mathcal{G}^{(\Sigma)}$ with projection $\pi(g) = \sigma_g$ in $\mathrm{Out}_{\mathbb{Q}\text{-alg}}(B, \rho)$ acts also on $Sh^{(\Sigma)}$ by $(A, \overline{\lambda}, i, \phi^{(\Sigma)})_{/T} \mapsto (A, \overline{\lambda}, i \circ \sigma_g, \phi^{(\Sigma)} \circ g)_{/T}$.

The reduced norm map $N_C : C^\times \to M^\times$ extends uniquely to a homomorphism $N_C : \widetilde{G} \to \mathrm{Res}_{M/\mathbb{Q}}\mathbb{G}_m$ of algebraic groups. Indeed, we have an isomorphism $\widetilde{G}(A) \cong \mathrm{Out}_{A\text{-alg}}(C_A, *) \ltimes G(A)$ for a suitable finite extension field A of M, and the norm map N_C factoring through the right factor $G(A)$ descends to $N_C : \widetilde{G}(\mathbb{Q}) \to M^\times$ (which determines

$N_C : \widetilde{G} \to \mathrm{Res}_{M/\mathbb{Q}}\mathbb{G}_m$ independently of the choice of A). The diagonal map $\mu : \widetilde{G} \ni g \mapsto (\nu(g), N_C(g)) \in (\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m \times \mathrm{Res}_{M/\mathbb{Q}}\mathbb{G}_m)$ factors through the cocenter Z^G of G; so, we have a homomorphism $\mu : \widetilde{G} \to Z^G$ of algebraic $\mathbb{Q}$-groups. We write $Z^G(\mathbb{R})^+$ for the identity connected component of $Z^G(\mathbb{R})$ and put $Z^G(\mathbb{Z}_{(\Sigma)})^+ = Z^G(\mathbb{R})^+ \cap \left(O^\times_{(\Sigma)} \times R^\times_{(\Sigma)}\right)$; so, $Z^G(\mathbb{Z}_{(\Sigma)})^+ = O^\times_{(\Sigma)+}$ if $M = F$. Similarly, we identify Z with $\mathrm{Res}_{R/\mathbb{Z}}\mathbb{G}_m$ so that $Z(\mathbb{Z}_{(\Sigma)}) = R^\times_{(\Sigma)}$.

We now state the main result:

Theorem 1.1. *Suppose* (h1-5). *Then the field automorphism group* $\mathrm{Aut}(\mathbb{F}(V)/\mathbb{F})$ *of the function field* $\mathbb{F}(V)$ *over* $\mathbb{F}$ *is given by the stabilizer the connected component* V *(in* $\pi_0(Sh^{(\Sigma)}_{/\mathbb{F}})$*) inside* $\mathcal{G}^{(\Sigma)}/\overline{Z(\mathbb{Z}_{(\Sigma)})}$*. The stabilizer is given by*

$$\mathcal{G}_V = \frac{\left\{g \in \mathcal{G}^{(\Sigma)} \middle| \mu(g) \in \overline{Z^G(\mathbb{Z}_{(\Sigma)})^+}\right\}}{\overline{Z(\mathbb{Z}_{(\Sigma)})}},$$

where $\overline{Z^G(\mathbb{Z}_{(\Sigma)})^+}$ *(resp.* $\overline{Z(\mathbb{Z}_{(\Sigma)})}$*) is the topological closure of* $Z^G(\mathbb{Z}_{(\Sigma)})^+$ *(resp.* $Z(\mathbb{Z}_{(\Sigma)})$*)in* $Z^G(\mathbb{A}^{(\Sigma)})$ *(resp. in* $G(\mathbb{A}^{(\Sigma)})$*). In particular, this implies that the scheme automorphism group* $\mathrm{Aut}(V_{/\mathbb{F}})$ *coincides with the field automorphism group* $\mathrm{Aut}(\mathbb{F}(V)/\mathbb{F})$ *and is given as above.*

This type of theorems in characteristic 0 situation has been proven mainly by Shimura, Deligne and Milne-Shih (see [Sh] II, [D1] 2.4-7 and [MS] 4.13), whose proof uses the topological fundamental group of V and the existence of the analytic universal covering space. Our proof uses the algebraic fundamental group of V and the solution of the Tate conjecture on endomorphisms of abelian varieties over function fields of characteristic p due to Zarhin (see [Z], [DAV] Theorem V.4.7 and [RPT]). The characteristic 0 version of the finiteness theorem due to Faltings (see [RPT]) yields a proof in characteristic 0, arguing slightly more, but we have assumed for simplicity that the characteristic of the base field is positive (see [PAF] for the argument in characteristic 0). We shall give the proof in the following section and prove some group theoretic facts necessary in the proof in the section following the proof. Our original proof was longer and was based on a density result of Chai (which has been proven under some restrictive conditions on G), and Ching-Li Chai suggested us a shorter proof via the results of Zarhin and Faltings (which also eliminated the extra assumptions we imposed). The author is grateful for his comments.

2 Proof of the theorem

We start with

Proposition 2.1. *Suppose* (h1-5). *Let* $\sigma \in \mathrm{Aut}(\mathbb{F}(V)/\mathbb{F})$*. Let* $U \subset V$ *be a connected open dense subscheme on which* $\sigma \in \mathrm{Aut}(\mathbb{F}(V)/\mathbb{F})$ *induces an* *isomorphism $U \cong \sigma(U)$. For $x \in (U \cap \sigma(U))(\mathbb{F})$, the two abelian varieties $\mathbb{X}_x$ and $\mathbb{X}_{\sigma(x)}$ are isogenous over $\mathbb{F}$, where $\mathbb{X}_x$ is the abelian variety sitting over x.*

Proof. We recall the subgroup in the theorem:

$$\mathcal{G}_V = \frac{\left\{g \in \mathcal{G}^{(\Sigma)} \middle| \mu(g) \in \overline{Z^G(\mathbb{Z}_{(\Sigma)})^+}\right\}}{\overline{Z(\mathbb{Z}_{(\Sigma)})}}.$$

By characteristic 0 theory in [D] Theorem 2.4 or [MS] p.929 (or [PAF] 7.2.3), the action of $\widetilde{G}(\mathbb{A}^{(\Sigma)})/\overline{Z(\mathbb{Z}_{(\Sigma)})}$ on $\pi_0(Sh^{(\Sigma)}_{/\overline{\mathbb{Q}}})$ factors through the homomorphism $\widetilde{G}(\mathbb{A}^{(\Sigma)})/\overline{Z(\mathbb{Z}_{(\Sigma)})} \to Z^G(\mathbb{A}^{(\Sigma)})/\overline{Z^G(\mathbb{Z}_{(\Sigma)})^+}$ induced by μ. The idele class group of cocenter $Z^G(\mathbb{A}^{(\Sigma)})/\overline{Z^G(\mathbb{Z}_{(\Sigma)})^+}$ acts on $\pi_0(Sh^{(\Sigma)}_{/\overline{\mathbb{Q}}})$ faithfully.

Since each geometrically connected component of $Sh^{(\Sigma)}$ is defined over the field $\mathcal{K}$ of fractions of $\mathcal{W}$, by the existence of a normal projective compactification (either smooth toroidal or minimal) over $\mathcal{W}$ (and Zariski's connectedness theorem), we have a bijection between geometrically connected components over $\mathcal{K}$ and over the residue field $\mathbb{F}$ induced by reduction modulo p. Then the stabilizer in $\widetilde{G}(\mathbb{A}^{(\Sigma)})/\overline{Z(\mathbb{Z}_{(\Sigma)})}$ of V in $\pi_0(Sh^{(\Sigma)}_{/\mathbb{F}})$ is given by $\mathcal{G}_V$.

The scheme theoretic automorphism group $\mathrm{Aut}(V/\mathbb{F})$ is a subgroup of the field automorphism group $\mathrm{Aut}(\mathbb{F}(V)/\mathbb{F})$. By a generalization due to N. Jacobson of the Galois theory (see [IAT] 6.3) to field automorphism groups, the Krull topology of $\mathrm{Aut}(\mathbb{F}(V)/\mathbb{F})$ is defined by a system of open neighborhoods of the identity, which is made up of the stabilizers of subfields of $\mathbb{F}(V)$ finitely generated over $\mathbb{F}$. For an open compact subgroups K in $G(\mathbb{A}^{(\Sigma)})/\overline{Z(\mathbb{Z}_{(\Sigma)})}$, we consider the image V_K of V in $Sh^{(\Sigma)}/K$. Then we have $V_K = V/\overline{K}_V$ for $\overline{K}_V = K \cap \mathcal{G}_V$, and $\overline{K}_V$ is isomorphic to the scheme theoretic Galois group $\mathrm{Gal}(V/V_{K/\mathbb{F}})$, which is in turn isomorphic to $\mathrm{Gal}(\mathbb{F}(V)/\mathbb{F}(V_K))$. Since all sufficiently small open compact subgroups of $\mathcal{G}_V$ are of the form $\overline{K}_V$ for open compact subgroups K of $G(\mathbb{A}^{(\Sigma)})/\overline{Z(\mathbb{Z}_{(\Sigma)})}$, the $\overline{K}_V$'s for open compact subgroups K of $G(\mathbb{A}^{(\Sigma)})/\overline{Z(\mathbb{Z}_{(\Sigma)})}$ give a fundamental system of open neighborhoods of the identity of $\mathrm{Aut}(\mathbb{F}(V)/\mathbb{F})$ under the Krull topology. In other words, the scheme theoretic automorphism group $\mathrm{Aut}(V_{/\mathbb{F}})$ is an open subgroup of $\mathrm{Aut}(\mathbb{F}(V)/\mathbb{F})$. If we choose K sufficiently small depending on $\sigma \in \mathrm{Aut}(\mathbb{F}(V)/\mathbb{F})$, we have ${}^{\sigma}\overline{K}_V = \sigma\overline{K}_V\sigma^{-1}$ still inside $\mathcal{G}_V$ in $G(\mathbb{A}^{(\Sigma)})/\overline{Z(\mathbb{Z}_{(\Sigma)})}$. We write U_K (resp. $U_{^\sigma K}$) for the image of U (resp. of $\sigma(U)$) in V_K (resp. in $V_{^\sigma K}$).

By the above description of the stabilizer of V, the image of $G_1(\mathbb{A}^{(\Sigma)})$ in the scheme automorphism group $\mathrm{Aut}(Sh^{(\Sigma)}_{/\mathbb{F}})$ is contained in the stabilizer $\mathrm{Aut}(V_{/\mathbb{F}})$ and hence in the field automorphism group $\mathrm{Aut}(\mathbb{F}(V)/\mathbb{F})$. Let $\overline{G}_1(\mathbb{A}^{(\Sigma)})$ be the image of $G_1(\mathbb{A}^{(\Sigma)})$ in $\mathrm{Aut}(\mathbb{F}(V)/\mathbb{F})$ (so $\overline{G}_1(\mathbb{A}^{(\Sigma)})$ is isomorphic to the quotient of $G_1(\mathbb{A}^{(\Sigma)})$ by the center of $G_1(\mathbb{Z}_{(\Sigma)})$). We take a sufficiently small open compact subgroup S of $G(\mathbb{A}^{(\Sigma)})$. We write $S_1 = G_1(\mathbb{A}^{(\Sigma)}) \cap S$ and $\overline{S}_1$ for the image of S_1 in $\overline{G}_1(\mathbb{A}^{(\Sigma)})$ with $\overline{G}_1(\mathbb{A}^{(\Sigma)})$. Shrinking S if necessary, we may assume that $S_1 \cong \overline{S}_1$, that V/V_S is étale, that $\mathrm{Gal}(V/V_{S/\mathbb{F}}) = \overline{S}_1$, that $S = \prod_\ell S_\ell$ with $S_\ell = S \cap G(\mathbb{Q}_\ell)$ for primes $\ell \notin \Sigma$ and that ${}^\sigma\overline{S}_1 := \sigma\overline{S}_1\sigma^{-1} \subset \overline{G}_1(\mathbb{A}^{(\Sigma)})$. We identify $Sh^{(p)}/\widetilde{S} = Sh^{(\Sigma)}/S$ for $\widetilde{S} = S \times G(\mathbb{Z}_\Sigma^{(p)})$, where $\mathbb{Z}_\Sigma^{(p)} = \prod_{\ell \in \Sigma - \{p,\infty\}} \mathbb{Z}_\ell$. Since $S_1 \cong \overline{S}_1$, we hereafter identify the two groups.

Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{W}$ and we write $\kappa = O_E/(O_E \cap \mathfrak{m})$ for the reflex field E. Since $Sh^{(\Sigma)}_{/\mathbb{F}}$ is a scalar extension relative to $\mathbb{F}/\kappa$ of the model $Sh^{(\Sigma)}_{/\kappa}$ defined over the finite field κ, the Galois group $\mathrm{Gal}(\mathbb{F}/\kappa)$ acts on the underlying topological space of $Sh^{(\Sigma)}/S$. Since $\pi_0((Sh^{(\Sigma)}/S)_{/\overline{\mathbb{Q}}})$ is finite, $\pi_0((Sh^{(\Sigma)}/S)_{/\mathbb{F}})$ is finite, and we have therefore a finite extension $\mathbb{F}_q$ of the prime field $\mathbb{F}_p$ such that $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ gives the stabilizer of V_S in $\pi_0((Sh^{(\Sigma)}/S)_{/\mathbb{F}})$. We may assume that (as varieties) U_S and $U_{{}^\sigma S}$ are defined over $\mathbb{F}_q$ and that $U_S \times_{\mathbb{F}_q} \mathbb{F}$ and $U_{{}^\sigma S} \times_{\mathbb{F}_q} \mathbb{F}$ are irreducible. Since $\sigma \in \mathrm{Hom}(U_S, U_{{}^\sigma S})$, the Galois group $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ acts on σ by conjugation. By further extending $\mathbb{F}_q$ if necessary, we may assume that σ is fixed by $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$, $x \in U_S(\mathbb{F}_q)$ and $\sigma(x) \in U_{{}^\sigma S}(\mathbb{F}_q)$. Thus σ descends to an isomorphism $\sigma_S : U_S \cong U_{{}^\sigma S}$ defined over $\mathbb{F}_q$.

Let $\mathbb{X}_{S/\mathbb{F}_q} \to U_{S/\mathbb{F}_q}$ be the universal abelian scheme with the origin $\mathbf{0}$. We write $(\mathbb{X}_x, \mathbf{0}_x)$ for the fiber of $(\mathbb{X}_S, \mathbf{0})$ over x and fix a geometric point $\overline{x} \in V(\mathbb{F})$ above x. The prime-to-p part $\pi_1^{(p)}(\mathbb{X}_x, \mathbf{0}_{\overline{x}})$ of $\pi_1(\mathbb{X}_x, \mathbf{0}_{\overline{x}})$ is canonically isomorphic to the prime to-p part $\mathcal{T}^{(p)}(\mathbb{X}_{\overline{x}/\mathbb{F}})$ of the Tate module $\mathcal{T}(\mathbb{X}_{\overline{x}/\mathbb{F}})$, and the p-part of $\pi_1(\mathbb{X}_x, \mathbf{0}_{\overline{x}})$ is the discrete p-adic Tate module of $\mathbb{X}_{x/\mathbb{F}}$ which is the inverse limit of the reduced part of $\mathbb{X}_x[p^n](\mathbb{F})$ (e.g. [ABV] page 171). We can make the quotient $\pi_1^{\{p\}}(\mathbb{X}_{S/\mathbb{F}}, \mathbf{0}_{\overline{x}})$ by the image of the p-part of $\pi_1(\mathbb{X}_x, \mathbf{0}_{\overline{x}})$. Then we have the following exact sequence ([SGA] 1.X.1.4):

$$\mathcal{T}^{(p)}(\mathbb{X}_{\overline{x}}) \xrightarrow{i} \pi_1^{\{p\}}(\mathbb{X}_{S/\mathbb{F}_q}, \mathbf{0}_{\overline{x}}) \to \pi_1(U_{S/\mathbb{F}_q}, \overline{x}) \to 1.$$

This sequence is split exact, because of the zero section $\mathbf{0} : U_S \to \mathbb{X}_S$. The multiplication by N: $\mathbb{X} \to \mathbb{X}$ (for N prime to p) is an irreducible étale covering, and we conclude that $\mathcal{T}^{(p)}(\mathbb{X}_x)$ injects into $\pi_1^{\{p\}}(\mathbb{X}_{S/\mathbb{F}}, \mathbf{0}_{\overline{x}})$. We make the quotient $\pi_1^{\Sigma}(\mathbb{X}_{S/\mathbb{F}_q}, \mathbf{0}_{\overline{x}}) = \pi_1^{\{p\}}(\mathbb{X}_{S/\mathbb{F}_q}, \mathbf{0}_{\overline{x}})/i(\mathcal{T}_\Sigma^{(p)}(\mathbb{X}_{\overline{x}}))$, and we get a split short exact sequence:

$$0 \to \mathcal{T}^{(\Sigma)}(\mathbb{X}_{\overline{x}}) \to \pi_1^{\Sigma}(\mathbb{X}_{S/\mathbb{F}_q}, \mathbf{0}_{\overline{x}}) \to \pi_1(U_{S/\mathbb{F}_q}, \overline{x}) \to 1. \tag{2.1}$$

By this exact sequence, $\pi_1(U_{S/\mathbb{F}_q}, \overline{x})$ acts by conjugation on $\mathcal{T}^{(\Sigma)}(\mathbb{X}_{\overline{x}})$. Recall that we have chosen S sufficiently small so that $V \twoheadrightarrow V_S$ is étale. We have a canonical surjection $\pi_1(U_{K/\mathbb{F}_q}, \overline{x}) \twoheadrightarrow \mathrm{Gal}(U/U_S)$. We write $S_V = \mathrm{Gal}(U/U_S)$, which is an extension of $\overline{S}_V$ by $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ generated by the Frobenius automorphism over $\mathbb{F}_q$. Since $\mathbb{X}_{\overline{x}}[N]$ for all integers N outside Σ gets trivialized over

U, the action of $\pi_1(U_{S/\mathbb{F}_q}, \overline{x})$ on $\mathcal{T}^{(\Sigma)}(\mathbb{X}_{\overline{x}})$ factors through $\pi_1(U_{S/\mathbb{F}_q}, \overline{x}) \twoheadrightarrow S_V$.

We now have another split exact sequence:

$$0 \to \mathcal{T}^{(\Sigma)}(\mathbb{X}_{\sigma(\overline{x})}) \to \pi_1^{\Sigma}(\mathbb{X}_{{}^{\sigma}S/\mathbb{F}_q}, \mathbf{0}_{\sigma(\overline{x})}) \to \pi_1(U_{{}^{\sigma}S/\mathbb{F}_q}, \sigma(\overline{x})) \to 1. \tag{2.2}$$

Again the action of $\pi_1(U_{{}^{\sigma}S/\mathbb{F}_q}, \sigma(\overline{x}))$ on $\mathcal{T}^{(\Sigma)}(\mathbb{X}_{\sigma(\overline{x})})$ factors through $\mathrm{Gal}(U/U_{{}^{\sigma}S}) = {}^{\sigma}S_V$. We fix a path in $U_{{}^{\sigma}S}$ from $\sigma(\overline{x})$ to $\overline{x}$ and lift it to a path from $\sigma(\mathbf{0}_{\overline{x}})$ to $\mathbf{0}_{\overline{x}}$ in $\mathbb{X}_{{}^{\sigma}S}$, which induces canonical isomorphisms ([SGA] V.7):

$$\iota^{\sigma} : \pi_1^{\Sigma}(\mathbb{X}_{{}^{\sigma}S/\mathbb{F}}, \mathbf{0}_{\sigma(\overline{x})}) \cong \pi_1^{\Sigma}(\mathbb{X}_{{}^{\sigma}S/\mathbb{F}}, \mathbf{0}_{\overline{x}}) \text{ and } \iota_{\sigma} : \pi_1(U_{{}^{\sigma}S}, \mathbf{0}_{\sigma(\overline{x})}) \cong \pi_1(U_{{}^{\sigma}S}, \overline{x}).$$

The isomorphism ι^{σ} in turn induces an isomorphism $\iota : \mathcal{T}^{(\Sigma)}(\mathbb{X}_{\sigma(\overline{x})}) \to \mathcal{T}^{(\Sigma)}(\mathbb{X}_{\overline{x}})$ of ${}^{\sigma}S_1$–modules.

We want to have the following commutative diagram

$$\begin{array}{ccccc}
\mathcal{T}^{(\Sigma)}(\mathbb{X}_{\overline{x}}) & \xrightarrow{\hookrightarrow} & \pi_1^{\Sigma}(\mathbb{X}_{S/\mathbb{F}}, \mathbf{0}_{\overline{x}}) & \xrightarrow{\twoheadrightarrow} & \pi_1(U_{S/\mathbb{F}}, \overline{x}) \\
{\scriptstyle ?}\downarrow & & \downarrow{\scriptstyle ?} & & {\scriptstyle \sigma_*}\downarrow \\
\mathcal{T}^{(\Sigma)}(\mathbb{X}_{\sigma(\overline{x})}) & \xrightarrow{\hookrightarrow} & \pi_1^{\Sigma}(\mathbb{X}_{{}^{\sigma}S/\mathbb{F}}, \mathbf{0}_{\sigma(\overline{x})}) & \xrightarrow{\twoheadrightarrow} & \pi_1(U_{{}^{\sigma}S/\mathbb{F}}, \sigma(\overline{x})) \\
{\scriptstyle \iota}\downarrow & & {\scriptstyle \iota^{\sigma}}\downarrow & & \downarrow{\scriptstyle \iota_{\sigma}} \\
\mathcal{T}^{(\Sigma)}(\mathbb{X}_{\overline{x}}) & \xrightarrow[\hookrightarrow]{} & \pi_1^{\Sigma}(\mathbb{X}_{{}^{\sigma}S/\mathbb{F}}, \mathbf{0}_{\overline{x}}) & \xrightarrow[\twoheadrightarrow]{} & \pi_1(U_{{}^{\sigma}S/\mathbb{F}}, \overline{x}),
\end{array}$$

and we will find homomorphisms of topological groups fitting into the spot indicated by "?". In other words, we ask if we can find a linear endomorphism $\mathcal{L} \in \mathrm{End}_{\mathbb{A}^{(\Sigma)}}(\mathcal{T}^{(\Sigma)}(\mathbb{X}_{\overline{x}}) \otimes \mathbb{Q})$ such that $\mathcal{L}(s \cdot v) = {}^{\sigma}s \cdot \mathcal{L}(v)$ for all $s \in S_1$ and $v \in \mathcal{T}^{(\Sigma)}(\mathbb{X}_{\overline{x}})$, where ${}^{\sigma}s = \sigma s \sigma^{-1}$ is the image of $\iota_{\sigma}(\sigma_*(s))$ in ${}^{\sigma}S_1$ for any lift $s \in \pi_1(V_S, \overline{x})$ inducing $s \in S_1$. Since $\mathrm{Hom}(G_1(\mathbb{Z}_\ell), G_1(\mathbb{Z}_{\ell'}))$ is a singleton made of the zero-map (taking the entire $G_1(\mathbb{Z}_\ell)$ to the identity of $G_1(\mathbb{Z}_{\ell'})$) if two primes ℓ and ℓ' are large and distinct (see Section 3 (S3) for a proof of this fact), $s \mapsto {}^{\sigma}s$ sends $S_{1,\ell}$ into ${}^{\sigma}S_{1,\ell}$ for almost all primes ℓ, where $S_{1,\ell} = G_1(\mathbb{Q}_\ell) \cap S_\ell$. If we shrink S further if necessary for exceptional finitely many primes, we achieve that S_ℓ is ℓ–profinite for exceptional ℓ and the logarithm $\log_\ell : S_{1,\ell} \to Lie(S_{1,\ell})$ given by $\log_\ell(s) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(s-1)^n}{n}$ is an ℓ-adically continuous isomorphism. Then by a result of Lazard [GAN] IV.3.2.6 (see Section 3 (S1)), σ induces by $\log_\ell \circ \sigma = [\sigma]_\ell \circ \log_\ell$ an automorphism $[\sigma]_\ell$ of the Lie algebra $Lie(S_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ over $\mathbb{Q}_\ell$. Note that

$$Lie(S_{1,\ell}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = \left\{ x \in C_\ell \middle| \rho(x) = -x \text{ and } \mathrm{Tr}(x) = 0 \right\},$$

where $\mathrm{Tr} : C_\ell \to M_\ell$ is the reduced trace map. Extending scalar to a finite Galois extension of $K/\mathbb{Q}_\ell$, $Lie(S_\ell) \otimes_{\mathbb{Z}_\ell} K$ becomes split semi-simple over K, and therefore $[\sigma]_\ell$ is induced by an element of $P\widetilde{G}(K)$ (the Lie algebra version of (S2) in the following section), which implies by Galois descent that $[\sigma]_\ell$ is induced by an element of $P\widetilde{G}(\mathbb{Q}_\ell)$. Thus for all $\ell \notin \Sigma$, $s \mapsto {}^\sigma s$ sends $S_{1,\ell}$ into ${}^\sigma S_{1,\ell}$ and that the isomorphism: $s \mapsto {}^\sigma s$ is induced by an element $\overline{\mathcal{L}}$ of the group fitting into the middle term of the exact sequence (1.3):

$$1 \to PG(\mathbb{A}^{(\Sigma)}) \to P\widetilde{G}(\mathbb{A}^{(\Sigma)}) \to \mathrm{Aut}(M_{\mathbb{A}}^{(\Sigma)}/\mathbb{A}^{(\Sigma)}).$$

The element $\overline{\mathcal{L}}$ in $\mathrm{Aut}(G_1(\mathbb{A}^{(\Sigma)}))$ is in turn induced by an endomorphism $\mathcal{L} \in \mathrm{End}_{\mathbb{A}^{(\Sigma)}}(\mathcal{T}^{(\Sigma)}(\mathbb{X}_{\overline{x}}) \otimes \mathbb{Q})$. Define $g(\sigma) = \iota^{-1} \circ \mathcal{L}$. Then $g(\sigma)$ is an element of $\mathrm{Hom}_{\overline{S}_1}(\mathcal{T}^{(\Sigma)}\mathbb{X}_x, \mathcal{T}^{(\Sigma)}\mathbb{X}_{\sigma(x)})$ invertible in $\mathrm{Hom}_{\overline{S}_1}(\mathcal{T}^{(\Sigma)}\mathbb{X}_x, \mathcal{T}^{(\Sigma)}\mathbb{X}_{\sigma(x)}) \otimes_{\mathbb{Z}} \mathbb{Q}$, and $g(\sigma)$ is S_1–linear in the sense that $g(\sigma)(sx) = {}^\sigma s \cdot g(\sigma)(x)$ for all $s \in \overline{S}_1$. Though $\mathcal{L}$ may depends on the choice of the path from x to $\sigma(x)$, the isomorphism $g(\sigma)$ (modulo the centralizer of S_1) is independent of the choice of the path; so, we will forget about the path hereafter. Applying this argument to $\Sigma = \{p, \infty\}$, we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{T}^{(p)}(\mathbb{X}_x) \otimes_{\mathbb{Z}} \mathbb{A}^{(p)} \arrow[r, "g(\sigma)"] & \mathcal{T}^{(p)}(\mathbb{X}_{\sigma(x)}) \otimes_{\mathbb{Z}} \mathbb{A}^{(p)} \\
L \otimes_{\mathbb{Z}} \mathbb{A}^{(p)} \arrow[u, "\phi_x^{(p)}"', "\wr"] \arrow[r, "g_\sigma"'] & L \otimes_{\mathbb{Z}} \mathbb{A}^{(p)} \arrow[u, "\phi_{\sigma(x)}^{(p)}"', "\wr"]
\end{tikzcd} \tag{2.3}$$

for $g_\sigma \in \widetilde{G}(\mathbb{A}^{(p)})$. Thus $g_\sigma^{(\Sigma)}$ has the projection $\pi(g_\sigma^{(\Sigma)}) \in \mathrm{Out}_{\mathbb{A}^{(\Sigma)}\text{-alg}}(B_{\mathbb{A}}^{(\Sigma)}, \rho)$.

Consider the relative Frobenius map $\pi_S : U_S \to U_S$ over $\mathbb{F}_q$. Since $\sigma : U_S \cong U_{{}^\sigma S}$ is defined over $\mathbb{F}_q$ by our choice, σ satisfies $\sigma_S \circ \pi_S = \pi_{{}^\sigma S} \circ \sigma_S$. If $X \to U_S$ is an étale irreducible covering, $X \times_{U_S, \pi_S} U_S \to U_S$ is étale irreducible, and $\pi_S : U_S \to U_S$ induces an endomorphism $\pi_{S,*} : \pi_1(U_S, \overline{x}) \to \pi_1(U_S, \overline{x})$. We have a diagram:

$$\begin{tikzcd}
\mathcal{T}^{(\Sigma)}(\mathbb{X}_{\sigma(\overline{x})}) \arrow[r, hook] \arrow[d, "g(\sigma)^{-1}"'] & \pi_1^{\Sigma}(\mathbb{X}_{{}^\sigma S/\mathbb{F}}, \mathbf{0}_{\sigma(\overline{x})}) \arrow[r, two heads] \arrow[d, "g(\sigma)^{-1} \ltimes \sigma_{S,*}^{-1}"] & \pi_1(U_{{}^\sigma S/\mathbb{F}}, \sigma(\overline{x})) \arrow[d, "\sigma_{S,*}^{-1}"] \\
\mathcal{T}^{(\Sigma)}(\mathbb{X}_{\overline{x}}) \arrow[r, hook] \arrow[d, "\pi_x"'] & \pi_1^{\Sigma}(\mathbb{X}_{S/\mathbb{F}}, \mathbf{0}_{\overline{x}}) \arrow[r, two heads] \arrow[d, "\pi_x \ltimes \pi_{S,*}"] & \pi_1(U_{S/\mathbb{F}}, \overline{x}) \arrow[d, "\pi_{S,*}"] \\
\mathcal{T}^{(\Sigma)}(\mathbb{X}_{\overline{x}}) \arrow[r, hook] \arrow[d, "g(\sigma)"'] & \pi_1^{\Sigma}(\mathbb{X}_{S/\mathbb{F}}, \mathbf{0}_{\overline{x}}) \arrow[r, two heads] \arrow[d, "g(\sigma) \ltimes \sigma_{S,*}"] & \pi_1(U_{S/\mathbb{F}}, \overline{x}) \arrow[d, "\sigma_{S,*}"] \\
\mathcal{T}^{(\Sigma)}(\mathbb{X}_{\sigma(\overline{x})}) \arrow[r, hook] & \pi_1^{\Sigma}(\mathbb{X}_{{}^\sigma S/\mathbb{F}}, \mathbf{0}_{\sigma(\overline{x})}) \arrow[r, two heads] & \pi_1(U_{{}^\sigma S/\mathbb{F}}, \sigma(\overline{x})),
\end{tikzcd}$$

where π_x is the relative Frobenius endomorphism of $\mathbb{X}_x$ over $\mathbb{F}_q$. The middle horizontal three squares of the above diagram are commutative, because $(\pi_x \ltimes$

$\pi_{S,*}$) is induced by the relative Frobenius endomorphism of $\mathbb{X}_{S/\mathbb{F}_q}$. The top and the bottom three squares are commutative by construction; so, the entire diagram is commutative. In short, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{T}^{(\Sigma)}(\mathbb{X}_{\sigma(\overline{x})}) & \xrightarrow{\hookrightarrow} & \pi_1^{\Sigma}(\mathbb{X}_{{}^\sigma S/\mathbb{F}}, \mathbf{0}_{\sigma(\overline{x})}) & \xrightarrow{\twoheadrightarrow} & \pi_1(U_{{}^\sigma S/\mathbb{F}}, \sigma(\overline{x})) \\ {\scriptstyle g(\sigma)\pi_x g(\sigma)^{-1}}\downarrow & & \downarrow & & {\scriptstyle \pi_{{}^\sigma S,*}}\downarrow \\ \mathcal{T}^{(\Sigma)}(\mathbb{X}_{\sigma(\overline{x})}) & \xrightarrow[\hookrightarrow]{} & \pi_1^{\Sigma}(\mathbb{X}_{{}^\sigma S/\mathbb{F}}, \mathbf{0}_{\sigma(\overline{x})}) & \xrightarrow[\twoheadrightarrow]{} & \pi_1(U_{{}^\sigma S/\mathbb{F}}, \sigma(\overline{x})), \end{array}$$

because $\sigma_S \pi_S \sigma_S^{-1} = \pi_{{}^\sigma S}$. Since $\pi_{\sigma(x)}$ also gives a similar commutative diagram:

$$\begin{array}{ccccc} \mathcal{T}^{(\Sigma)}(\mathbb{X}_{\sigma(\overline{x})}) & \xrightarrow{\hookrightarrow} & \pi_1^{\Sigma}(\mathbb{X}_{{}^\sigma S/\mathbb{F}}, \mathbf{0}_{\sigma(\overline{x})}) & \xrightarrow{\twoheadrightarrow} & \pi_1(U_{{}^\sigma S/\mathbb{F}}, \sigma(\overline{x})) \\ {\scriptstyle \pi_{\sigma(x)}}\downarrow & & \downarrow & & \downarrow{\scriptstyle \pi_{{}^\sigma S,*}} \\ \mathcal{T}^{(\Sigma)}(\mathbb{X}_{\sigma(\overline{x})}) & \xrightarrow[\hookrightarrow]{} & \pi_1^{\Sigma}(\mathbb{X}_{{}^\sigma S/\mathbb{F}}, \mathbf{0}_{\sigma(\overline{x})}) & \xrightarrow[\twoheadrightarrow]{} & \pi_1(U_{{}^\sigma S/\mathbb{F}}, \sigma(\overline{x})), \end{array}$$

we find out that $g(\sigma)\pi_x g(\sigma)^{-1}\pi_{\sigma(x)}^{-1}$ commutes with the action of ${}^\sigma\overline{S}_1$, and hence it is in the center of $\mathrm{Aut}_R(\mathcal{T}^{(\Sigma)}(\mathbb{X}_{\sigma(\overline{x})}))$. In other words, $g(\sigma)\pi_x g(\sigma)^{-1} = z\pi_{\sigma(x)}$ for $z \in (\widehat{R}^{(\Sigma)})^\times$. Taking the determinant with respect to $\bigwedge^g \mathcal{T}_\ell(\mathbb{X}_{\sigma(\overline{x})})$ for the rank $g = \mathrm{rank}_{R_\ell} \mathcal{T}_\ell(\mathbb{X}_{\sigma(\overline{x})})$ with a prime $\ell \notin \Sigma$, we find that $\det(\pi_x) = z^g \det(\pi_{\sigma(x)})$. Since $\det(\pi_x) = N(\pi_x)^r$ with a positive integer r for the reduced norm map $N : B \to M$, we find that $\det(\pi_x) = \det(\pi_{\sigma(x)})$, and hence z is a g–th root of unity in $(\widehat{R}^{(\Sigma)})^\times$ (purity of the Weil number π_x). Then $g(\sigma) \in \mathrm{Hom}(\mathcal{T}^{(p)}\mathbb{X}_x, \mathcal{T}^{(p)}\mathbb{X}_{\sigma(x)})$ satisfies $g(\sigma) \circ \pi_x^g = \pi_{\sigma(x)}^g \circ g(\sigma)$, and hence $g(\sigma)$ is an isogeny of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_{q^g})$–modules. Then by a result of Tate ([T]),

$$\mathrm{Hom}_{\mathrm{Gal}(\mathbb{F}/\mathbb{F}_{q^g})}(\mathcal{T}^{(\Sigma)}\mathbb{X}_x, \mathcal{T}^{(\Sigma)}\mathbb{X}_{\sigma(x)}) = \mathrm{Hom}(\mathbb{X}_{x/\mathbb{F}_{q^g}}, \mathbb{X}_{\sigma(x)/\mathbb{F}_{q^g}}) \otimes_{\mathbb{Z}} \mathbb{A}^{(\Sigma)},$$

we find that $\mathbb{X}_x$ and $\mathbb{X}_{\sigma(x)}$ are isogenous over $\mathbb{F}_{q^g}$. □

We have a canonical projection $\mathrm{Aut}_{\text{top group}}(\overline{G}_1(\mathbb{A}^{(\Sigma)})) \to \mathrm{Out}_{\mathbb{A}^{(\Sigma)}\text{-alg}}(B_{\mathbb{A}}^{(\Sigma)}, \rho)$ (induced by π) whose kernel is given by $PG(\mathbb{A}^{(\Sigma)})$. Thus $\sigma \in \mathrm{Aut}(\mathbb{F}(V)_{/\mathbb{F}})$ has projection $\pi(g_\sigma^{(\Sigma)})$ (for g_σ in (2.3)) in

$$\mathrm{Out}_{\mathbb{A}^{(\Sigma)}\text{-alg}}(B_{\mathbb{A}}^{(\Sigma)}, \rho) \subset \mathrm{Aut}_{\mathbb{A}^{(\Sigma)}\text{-alg}}(M_{\mathbb{A}}^{(\Sigma)}) = \prod_{\ell \notin \Sigma} \mathrm{Aut}_{\mathbb{Q}_\ell\text{-alg}}(M_\ell)$$

which will be written as $\sigma_B = \pi(g_\sigma^{(\Sigma)})$.

Corollary 2.2. *If* $\sigma \in \mathrm{Aut}(\mathbb{F}(V)/\mathbb{F})$, *we have* $\sigma_B \in \mathrm{Out}_{\mathbb{Q}\text{-}alg}(B, \rho)$, *where the group* $\mathrm{Out}_{\mathbb{Q}\text{-}alg}(B, \rho)$ *is diagonally embedded into* $\prod_{\ell \notin \Sigma} \mathrm{Aut}_{\mathbb{Q}_\ell\text{-}alg}(M_\ell)$.

Proof. The element $g(\sigma) = \iota^{-1} \circ \mathcal{L} \in \mathrm{Hom}_S(\mathcal{T}^{(\Sigma)}\mathbb{X}_x, \mathcal{T}^{(\Sigma)}\mathbb{X}_{\sigma(x)})$ in the proof of Proposition 2.1 acts on $G_1(\mathbb{A}^{(\Sigma)})$ by conjugation of $g_\sigma \in \widetilde{G}(\mathbb{A}^{(\Sigma)})$ in (2.3); so, its projection $\pi(g_\sigma)$ in $\mathrm{Out}_{\mathbb{A}^{(\Sigma)}\text{-alg}}(B^{(\Sigma)}_{\mathbb{A}},\rho)$ inside $\mathrm{Aut}(M^{(\Sigma)}_{\mathbb{A}}/\mathbb{A}^{(\Sigma)})$ is given by σ_B. By the proof of Proposition 2.1, $g(\sigma)$ is induced by $\xi\in\mathrm{Hom}_{O_B}(\mathbb{X}_x,\mathbb{X}_{\sigma(x)})$ modulo $Z(\mathbb{Z}_{(\Sigma)})S$. Choose a rational prime q outside Σ. We have $B_q\cap\mathrm{End}(\mathbb{X}_x)=O_B$. Note that $b\mapsto g(\sigma)^{-1}\circ b\circ g(\sigma)$ sends B_q into itself and that the conjugation by ξ sends $B_q\cap(\mathrm{End}(\mathbb{X}_x)\otimes_{\mathbb{Z}}\mathbb{Q})=B$ into itself. Since the image of the conjugation by $g(\sigma)$ in $\mathrm{Out}_{\mathbb{Q}_q\text{-alg}}(B_q)$ and the image of the conjugation by ξ in $\mathrm{Out}(\mathrm{End}(\mathbb{X}_x)\otimes_{\mathbb{Z}}\mathbb{Q})$ coincide, we conclude $\sigma_B\in\mathrm{Out}_{\mathbb{Q}\text{-alg}}(B)$. Since $\sigma_B\in\mathrm{Out}_{\mathbb{A}^{(\Sigma)}\text{-alg}}(B^{(\Sigma)}_{\mathbb{A}},\rho)\cap\mathrm{Out}_{\mathbb{Q}\text{-alg}}(B)$, we get $\sigma_B\in\mathrm{Out}_{\mathbb{Q}\text{-alg}}(B,\rho)$. □

Corollary 2.3. *For the generic point η of V_S, $\mathbb{X}_\eta$ and $\mathbb{X}_{\sigma(\eta)}$ are isogenous. In particular, if σ_B is the identity in $\mathrm{Out}_{\mathbb{Q}\text{-}alg}(B,\rho)$, we find $a_S\in G(\mathbb{A}^{(\Sigma)})$ inducing σ on $\mathbb{F}(V_S)$ for all sufficiently small open compact subgroups S of $G(\mathbb{A}^{(\Sigma)})$.*

Proof. We choose S sufficiently small as in the proof of Proposition 2.1. We replace q in the proof of Proposition 2.1 by q^g at the end of the proof in order to simplify the symbols.

Suppose that σ_S induces $U_S\cong U_{\sigma S}$ for an open dense subscheme $U_S\subset V_S$. Again we use the exact sequence:$0\to\mathcal{T}^{(\Sigma)}\mathbb{X}_{\overline{\eta}}\to\pi_1^{\Sigma}(\mathbb{X}_{/\mathbb{F}_q},\mathbf{0}_{\overline{\eta}})\to\pi_1(U_{S/\mathbb{F}_q},\overline{\eta})\to 1$. By the same argument as above, we find $g_\eta(\sigma)\in\mathrm{Hom}_{\pi_1(U_{S/\mathbb{F}},\overline{\eta})}(\mathcal{T}^{(\Sigma)}\mathbb{X}_\eta,\mathcal{T}^{(\Sigma)}\mathbb{X}_{\sigma(\eta)})$. Since $\mathbb{X}_\eta[\ell^\infty]$ gets trivialized over U for a prime $\ell\notin\Sigma$, fixing a path from η to x for a closed point $x\in U_S(\mathbb{F}_q)$ and taking its image from $\sigma(\eta)$ to $\sigma(x)$, we may identify $\pi(U_{S/\mathbb{F}_q},\overline{x})$ (resp. $\mathcal{T}^{(\Sigma)}\mathbb{X}_x$ and $\mathcal{T}^{(\Sigma)}\mathbb{X}_{\sigma(x)}$) with the Galois group $\mathrm{Gal}(\mathbb{F}(\widetilde{U})/\mathbb{F}_q(U_S))$ for the universal covering $\widetilde{U}$ (resp. with the generic Tate modules $\mathcal{T}^{(\Sigma)}\mathbb{X}_\eta$ and $\mathcal{T}^{(\Sigma)}\mathbb{X}_{\sigma(\eta)}$). By the universality, $\sigma:U\cong{}^{\sigma}U$ extends to $\widetilde{\sigma}:\widetilde{U}\cong{}^{\sigma}\widetilde{U}$. Writing D_x for the decomposition group of the closed point $x\in U_S(\mathbb{F}_q)$, the points $x:Spec(\mathbb{F}_q)\hookrightarrow U_S$ and $\sigma(x):Spec(\mathbb{F}_q)\hookrightarrow U_{\sigma S}$ induce isomorphisms $D_x\cong\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)\cong D_{\sigma(x)}=\widetilde{\sigma}D_x\widetilde{\sigma}^{-1}$ (choosing the extension $\widetilde{\sigma}$ suitably) and splittings: $\mathrm{Gal}(\mathbb{F}(\widetilde{U})/\mathbb{F}_q(U_S))=D_x\ltimes\mathrm{Gal}(\mathbb{F}(\widetilde{U})/\mathbb{F}(U_S))$ and $\mathrm{Gal}(\mathbb{F}({}^{\sigma}\widetilde{U})/\mathbb{F}_q(U_{\sigma S}))=D_{\sigma(x)}\ltimes\mathrm{Gal}(\mathbb{F}({}^{\sigma}\widetilde{U})/\mathbb{F}(U_{\sigma S}))$. The morphism $g(\sigma):\mathcal{T}^{(\Sigma)}\mathbb{X}_x\to\mathcal{T}^{(\Sigma)}\mathbb{X}_{\sigma(x)}$ induces a morphism $g_\eta(\sigma):\mathcal{T}^{(\Sigma)}\mathbb{X}_\eta\to\mathcal{T}^{(\Sigma)}\mathbb{X}_{\sigma(\eta)}$ satisfying $g_\eta(\sigma)(sx)={}^{\sigma}s\cdot g_\eta(\sigma)(x)$ for all $s\in S_V$. Thus $g_\eta(\sigma)$ is a morphism of $\pi_1(U_{S/\mathbb{F}_q},\overline{\eta})$–modules (not just that of $\pi_1^{\Sigma}(U_{S/\mathbb{F}},\overline{\eta})$–modules). Then by a result of Zarhin (see [RPT] Chapter VI, [Z] and also [ARG] Chapter II), $\mathbb{X}_{\eta/\mathbb{F}_q(V_S)}$ and $\mathbb{X}_{\sigma(\eta)/\mathbb{F}_q(V_{\sigma S})}$ are isogenous. Here we note that the field $\mathbb{F}_q(V_S)=\mathbb{F}_q(U_S)$ is finitely generated over $\mathbb{F}_p$ (which has to be the case in order to apply Zarhin's result). Thus we can find an isogeny $\alpha_\eta:\mathbb{X}_\eta\to\mathbb{X}_{\sigma(\eta)}$, which extends to an isogeny $\mathbb{X}_S\to\sigma^*\mathbb{X}_{\sigma S}=\mathbb{X}_{\sigma S}\times_{U_{\sigma S},\sigma}U_S$ over U_S. We write $\alpha:\mathbb{X}_S\to\mathbb{X}_{\sigma S}$ for the composite of the above isogeny with the projection $\mathbb{X}_{\sigma S}\times_{U_{\sigma S},\sigma}U_S\to\mathbb{X}_{\sigma S}$.

We then have the commutative diagram:

$$\begin{tikzcd} \mathbb{X}_S \arrow[r, "\alpha"] \arrow[d] & \mathbb{X}_{^\sigma S} \arrow[d] \\ U_S \arrow[r, "\sigma"'] & U_{^\sigma S}. \end{tikzcd}$$

Assume that $\sigma_B = 1$. Then α is B-linear. Suppose we have another B-linear isogeny $\alpha' : \mathbb{X}_S \to \sigma^*\mathbb{X}_{^\sigma S}$ inducing $g_\eta(\sigma)$. Then $\alpha^{-1}\alpha'$ commutes with the action of $\mathrm{Gal}(\mathbb{F}(\widetilde{U})/\mathbb{F}(U_S))$ and hence with the action of S. Thus we find $\xi = \phi^{-1}\alpha^{-1}\alpha'\phi \in \mathrm{End}_S(L_{\mathbb{A}^{(\Sigma)}})$ for the level structure $\phi = \phi^{(\Sigma)} : L_{\mathbb{A}^{(\Sigma)}} \to V^{(\Sigma)}(\mathbb{X}_\eta)$. This implies B-linear ξ commutes with the action of C, and hence in the center of $B \subset \mathrm{End}_C(L_{\mathbb{Q}})$. We thus find $\xi \in Z(\mathbb{Q})G(\mathbb{Z}_\Sigma^{(p)})S$. We consider the commutative diagram similar to (2.3):

$$\begin{tikzcd} \mathcal{T}^{(p)}(\mathbb{X}_\eta) \otimes_{\mathbb{Z}} \mathbb{A}^{(p)} \arrow[r, "\alpha_\eta"] & \mathcal{T}^{(p)}(\mathbb{X}_{\sigma(\eta)}) \otimes_{\mathbb{Z}} \mathbb{A}^{(p)} \\ L \otimes_{\mathbb{Z}} \mathbb{A}^{(p)} \arrow[r, "g_\sigma"'] \arrow[u, "\phi_\eta^{(p)}", "\wr"'] & L \otimes_{\mathbb{Z}} \mathbb{A}^{(p)}. \arrow[u, "\phi_{\sigma(\eta)}^{(p)}", "\wr"'] \end{tikzcd} \tag{2.4}$$

The prime-to-Σ component of g_σ^{-1} eventually gives a_S in the corollary. By the above fact, $g_\sigma^{(\Sigma)}$ is uniquely determined in $\widetilde{G}(\mathbb{A}^{(\Sigma)})/Z(\mathbb{Q})S$.

Note that $\sigma^*(\mathbb{X}_{^\sigma S}, \overline{\lambda}_{^\sigma S}, i_{^\sigma S}, \phi_{\sigma(\eta)}^{(p)} \circ g_\sigma^{(\Sigma)})_{/U_S}$ is a quadruple classified by $Sh_{/S}^{(\Sigma)} = Sh^{(p)}/G(\mathbb{Z}_\Sigma^{(p)})S$. By the universality of $Sh_{/S}^{(\Sigma)}$ (proven under (h2-4)), we have a morphism $\tau : U_S \to U_S \subset Sh^{(\Sigma)}/S$ with a prime-to-Σ and B-linear isogeny $\beta : \sigma^*\mathbb{X}_{^\sigma S} \to \tau^*\mathbb{X}_S$ over U_S. Identifying $\mathrm{Gal}(U/U_S)$ with a subgroup S_V of $S \subset G(\mathbb{A}^{(\Sigma)})$, the actions of $s \in S_V$ on $\phi_1 = \phi_{\sigma(\eta)}^{(\Sigma)} \circ g_\sigma$ and on $\phi_2 = \phi_\eta^{(\Sigma)}$ have identical effect: $s \circ \phi_j = \phi_j \circ s$ $(j = 1, 2)$. Thus the effect of τ (and $\beta\alpha_\eta$) on $\mathcal{T}^{(\Sigma)}(\mathbb{X}_\eta)$ commutes with the action of S_V, and the action of B-linear $\beta\alpha_\eta$ on the Tate module $\mathcal{T}^{(\Sigma)}(\mathbb{X}_\eta)$ commutes with the action of S_V. Therefore it is in the center $Z(\mathbb{Q})$. Thus the isogeny α between $\sigma^*(\mathbb{X}_{^\sigma S})$ and $\mathbb{X}_S$ can be chosen (after modification by a central element) to be a prime-to-Σ isogeny. This τ could be non-trivial without the three assumptions (h2-4), and if this is the case, the action of τ is induced by an element of $G(\mathbb{Q}_\Sigma^{(p)})$ normalizing $G(\mathbb{Z}_\Sigma^{(p)})$. Under (h2-4), τ is determined by its effect on $\mathcal{T}^{(\Sigma)}(\mathbb{X}_\eta)$ and is the identity map (see the following two paragraphs), and we may assume that α is a prime-to-Σ isogeny (after modifying by an element of $Z(\mathbb{Q})$). Thus $g_\sigma^{(\Sigma)}$ is uniquely determined in $\mathcal{G}_V$ modulo S.

We add here a few words on this point related to the universality of $Sh^{(\Sigma)}$. Without the assumptions (h2-4), the effect of σ on the restriction of each level structure $\phi^{(p)}$ to $L_{\mathbb{A}^{(\Sigma)}}$ may not be sufficient to uniquely determine σ. In other words, in the definition of our moduli problem, we indeed have the datum of $\phi^{(p)}_{\Sigma}$ modulo $G(\mathbb{Z}^{(p)}_{\Sigma})$, which we cannot forget. To clarify this, take a (characteristic 0) geometrically connected component V_0 of $Sh_{/E}$ whose image $V_0^{(\Sigma)}$ in $Sh^{(\Sigma)}_{/E} = Sh/G(\mathbb{Z}_{\Sigma})$ giving rise to $V_{/\mathbb{F}}$ after extending scalar to $\mathcal{W}$ and then taking reduction modulo p. By the description of $\pi_0(Sh_{/\overline{\mathbb{Q}}})$ at the beginning of the proof of Proposition 2.1, the stabilizer in $G(\mathbb{A}^{(\infty)})/\overline{Z(\mathbb{Q})}$ of $V_{0/\overline{\mathbb{Q}}} \in \pi_0(Sh_{/\overline{\mathbb{Q}}})$ is given by

$$\mathcal{G}_0 = \frac{\left\{g \in G(\mathbb{A}^{(\infty)}) \middle| \mu(g) \in \overline{Z^G(\mathbb{Q})^+}\right\}}{\overline{Z(\mathbb{Q})}}.$$

Inside this group, every element $g \in G(\mathbb{A}^{(\infty)})$ inducing an automorphism of $V_0^{(\Sigma)}$ has its ℓ-component g_ℓ for a prime $\ell \in \Sigma$ in the normalizer of $G(\mathbb{Z}_\ell)$. Under (h2-4), as is well known, the normalizer of $G(\mathbb{Z}_\ell)$ in $G(\mathbb{Q}_\ell)$ is $Z(\mathbb{Q}_\ell)G(\mathbb{Z}_\ell)$ (see [Ko] Lemma 7.2, which is one of the key points of the proof of the universality of $Sh^{(\Sigma)}$). By (h2-4), $G(\mathbb{Q}_\ell)$ is quasi-split over $\mathbb{Q}_\ell$, and we have the Iwasawa decomposition $G(\mathbb{Q}_\ell) = P_0(\mathbb{Q}_\ell)G(\mathbb{Z}_\ell)$ for $\ell \in \Sigma$ with a minimal parabolic subgroup P_0 of G, from which we can easily prove that the normalizer of $G(\mathbb{Z}_\ell)$ is $Z(\mathbb{Q}_\ell)G(\mathbb{Z}_\ell)$. An elementary proof of the Iwasawa decomposition (for a unitary group or a symplectic group acting on M_ℓ^r keeping a skew-hermitian form relative to M_ℓ/F_ℓ) can be found in [EPE] Section 5, particularly pages 36-37. By (h3-4), $G(\mathbb{Q}_\ell)$ is isomorphic to a unitary or symplectic group acting on $M_\ell^n = \varepsilon L_{\mathbb{Q}_\ell}$ for an idempotent ε (for example, $\varepsilon = \mathrm{diag}[1, 0, \dots, 0]$ fixed by ρ) of $O_{B_\ell} \cong M_n(R_\ell)$ with respect to the skew-hermitian form on $\varepsilon L_{\mathbb{Q}_\ell}$ induced by $\langle\cdot,\cdot\rangle$; so, the result in [EPE] Section 5 applies to our case.

Suppose that $g \in G(\mathbb{A}^{(\infty)})$ preserves the quotient $V_0^{(\Sigma)}$ of V_0. If Σ is finite, we can therefore choose $\xi \in Z(\mathbb{Q})$ so that $(\xi g)_\ell$ is in $G(\mathbb{Z}_\ell)$ for all $\ell \in \Sigma$, and the action of $(\xi g)^{(\Sigma)} \in \mathcal{G}_V$ on $V_0^{(\Sigma)}$ induces the action of g. Suppose that Σ is infinite. Since α_η is a prime-to-Σ isogeny, $(g_\sigma)_\Sigma$ is contained in $G(\mathbb{Z}^{(p)}_{\Sigma})$. Thus σ is induced by $(g_\sigma^{(\Sigma)})^{-1}$ even if Σ is infinite. This fact can be also shown in a group theoretic way as in the case of finite Σ: Modifying g by an element in $G(\mathbb{Z}_\Sigma)$, we may assume that $g_\ell \in Z(\mathbb{Q}_\ell)$ for all $\ell \in \Sigma$. Taking an increasing sequence of finite sets Σ_i so that $\Sigma = \bigcup_i \Sigma_i$ and choosing $\xi_i \in Z(\mathbb{Q})$ so that the action of $(\xi_i g)^{(\Sigma_i)}$ induces the action of g on $V_0^{(\Sigma_i)}$, we find $\xi_i g \in G(\mathbb{Z}_{\Sigma_i})G(\mathbb{A}^{(\Sigma_i)})$ whose action on $V_0^{(\Sigma)}$ is identical to that of g. We write $\mathcal{F}_i$ for the closed subset of elements in $G(\mathbb{Z}_{\Sigma_i})G(\mathbb{A}^{(\Sigma_i)})$ whose action on $V_0^{(\Sigma)}$ is identical to that of g. In the locally compact group $G(\mathbb{A}^{(\infty)})$, the filter $\{\mathcal{F}_i\}_i$ has a nontrivial intersection $\bigcap_i \mathcal{F}_i \neq \emptyset$. Thus the action of g on $V_0^{(\Sigma)}$ is represented by an element in $G(\mathbb{A}^{(\Sigma)})$. In other words, an element of $G(\mathbb{A}^{(\infty)})$ in the stabilizer of the connected component $V_0^{(\Sigma)} \in \pi_0(Sh^{(\Sigma)}_{/\overline{\mathbb{Q}}})$ is represented by an element in the group $G(\mathbb{A}^{(\Sigma)})$ without Σ-component. Since $\pi_0(Sh^{(\Sigma)}_{/\mathbb{Q}})$ is in bijection with $\pi_0(Sh^{(\Sigma)}_{/\mathbb{F}})$, any element in the stabilizer in $\widetilde{G}(\mathbb{A}^{(p)})$ of $V \in \pi_0(Sh^{(\Sigma)}_{/\mathbb{F}})$ is represented by an element in $G(\mathbb{A}^{(\Sigma)})$. By this fact, under (h2-4), the effect of σ on $\phi^{(p)}_{\Sigma}$ is determined by $g^{(\Sigma)}_{\sigma}$ outside Σ. Thus we can really forget about the Σ-component.

Writing the prime-to-Σ level structures of $\mathbb{X}_\eta$ and $\mathbb{X}_{\sigma(\eta)}$ as $\phi^{(\Sigma)}_\eta$ and $\phi^{(\Sigma)}_{\sigma(\eta)}$, respectively, we now find that $\alpha_\eta \circ \phi^{(\Sigma)}_\eta = \phi^{(\Sigma)}_{\sigma(\eta)} \circ a_S^{-1}$ for $a_S^{-1} = g^{(\Sigma)}_\sigma \in \widetilde{G}(\mathbb{A}^{(\Sigma)})$. Since the effect of σ on $\mathcal{T}^{(\Sigma)}(\mathbb{X}_\eta)$ determines σ, we have $a_S^{-1}(\sigma(\eta)) = \eta$, which implies that $a_S = \sigma$ on the Zariski open dense subset U_S of V_S, and hence, they are equal on the entire V_S. $\square$

By the smoothness of $Sh^{(\Sigma)}$ over $\mathcal{W}$, Zariski's connectedness theorem (combined with the existence of a projective compactification normal over $\mathcal{W}$), we have a bijection $\pi_0(Sh^{(\Sigma)}_{/\mathbb{F}}) \cong \pi_0(Sh^{(\Sigma)}_{/\mathbb{Q}})$ as described at the beginning of the proof of Proposition 2.1. Since our group G has cohomologically trivial center (cf., [MS] 4.12), the stabilizer of $V_0^{(\Sigma)} \in \pi_0(Sh^{(\Sigma)}_{/\mathbb{Q}})$ in $\frac{G(\mathbb{A}^{(\Sigma)})}{Z(\mathbb{Z}_{(\Sigma)})}$ has a simple expression given by the subgroup $\mathcal{G}_V$ in the theorem (see [D1] 2.1.6, 2.1.16, 2.6.3 and [MS] Theorem 4.13), and the above corollaries finish the proof of the theorem because σ on V is then induced by $a = \lim_{S\to 1} a_S$ in $\frac{G(\mathbb{A}^{(\Sigma)})}{Z(\mathbb{Z}_{(\Sigma)})}$. Since a fixes $V \in \pi_0(Sh^{(\Sigma)}_{/\mathbb{F}})$, we conclude $a \in \mathcal{G}_V$. The description of the stabilizer of V in the theorem necessitates the strong approximation theorem (which follows from noncompactness of $G_1(\mathbb{R})$ combined with simply connectedness of G_1: [Kn]).

3 Automorphism groups of quasi-split classical groups

In the above proof of the theorem, we have used the following facts:

(S) *For an open compact subgroups $S, S' \subset G_1(\mathbb{A}^{(\Sigma)})$, if $\sigma : S \cong S'$ is an isomorphism of groups, replacing S by an open subgroup and replacing S' accordingly by the image of σ, σ is induced by the conjugation by an element $g(\sigma) \in \widetilde{G}(\mathbb{A}^{(\Sigma)})$ as in* (1.2).

We may modify σ by $g \in \widetilde{G}(\mathbb{A}^{(\Sigma)})$ so that $\sigma_B = 1$. Then this assertion (S) follows from the following three assertions for σ with $\sigma_B = 1$:

(S1) *For open subgroups S_ℓ and ${S'}_\ell$ of $G_1(\mathbb{Q}_\ell)$ (for every prime ℓ), an isomorphism $\sigma_\ell : S_\ell \cong {S'}_\ell$ is induced by conjugation $s \mapsto g_\ell(\sigma) s g_\ell(\sigma)^{-1}$ for $g_\ell(\sigma) \in G(\mathbb{Q}_\ell)$ after replacing S_ℓ by an open subgroup of S_ℓ and replacing S'_ℓ by the image of the new S_ℓ;*

(S2) *For a prime ℓ at which $G_{1/\mathbb{Z}_\ell}$ is smooth quasi-split (so for a sufficiently large rational prime ℓ), we have*

$$\mathrm{Aut}(G_1(\mathbb{Z}_\ell)) = \mathrm{Aut}(M_\ell/\mathbb{Q}_\ell) \ltimes PG(\mathbb{Z}_\ell)$$

and

$$\mathrm{Aut}(G_1(\mathbb{Q}_\ell)) = \mathrm{Aut}(M_\ell/\mathbb{Q}_\ell) \ltimes PG(\mathbb{Q}_\ell),$$

(S3) *For sufficiently large distinct rational primes p and ℓ, any group homomorphism $\phi : G_1(\mathbb{Z}_p) \to G_1(\mathbb{Z}_\ell))$ is trivial* (that is, $\mathrm{Ker}(\phi) = G_1(\mathbb{Z}_p)$).

The assertion (S1) follows directly from a result of Lazard on ℓ–adic Lie groups (see [GAN] IV.3.2.6), because the automorphism of the Lie algebra of S (hence of S') are all inner up to the automorphism of the field (in our case). The assertion (S2) for finite fields is an old theorem of Steinberg (see [St] 3.2), and as remarked in [CST] in the comments (in page 587) on [St], (S2) for general infinite fields follows from a very general result in [BT] 8.14. Since the paper [BT] is a long paper and treats only algebraic groups over an infinite field (not over a valuation ring like $\mathbb{Z}_\ell$), for the reader's convenience, we will give a self-contained proof of (S2) restricting ourselves to unitary groups and symplectic groups.

Since the assertions (S3) concerns only sufficiently large primes, we may always assume

(QS) *$G_1(\mathbb{Z}_p)$ and $G_1(\mathbb{Z}_\ell)$ are quasi-split.*

We now prove the assertion (S3). Let $\phi : G_1(\mathbb{Z}_p) \to G_1(\mathbb{Z}_\ell)$ be a homomorphism. Since $G_1(\mathbb{Z}_p)$ is quasi split, $G_1(\mathbb{Z}_p)$ is generated by unipotent elements (see Proposition 3.1), and its unipotent radical U is generated by an additive subgroup U_α corresponding to a simple root α.

If $G_1 = SL(n)_{/\mathbb{Z}}$, for example, we may assume that U_α is made of diagonal matrices

$$\mathrm{diag}[1_j, \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right), 1_{n-j-2}] := \begin{pmatrix} 1_j & 0 & 0 \\ 0 & \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right) & 0 \\ 0 & 0 & 1_{n-j-2} \end{pmatrix}$$

with $u \in \mathbb{Z}_p$ for an index j (with $1 \leq j \leq n$), where 1_j is the $j \times j$ identity matrix.

In general, U and U_α are p-profinite. We consider the normalizer $N(U_\alpha)$ and the centralizer $Z(U_\alpha)$ of U_α in G_1. Then by conjugation, $N(U_\alpha)/Z(U_\alpha)$ acts on U_α. Since ϕ is a group homomorphism, the quotient $N(U_\alpha)/Z(U_\alpha)$ keeps acting on the image $\phi(U_\alpha)$ in $G_1(\mathbb{Z}_\ell)$ through conjugation by elements in $\phi(N(U_\alpha))$. If $p \neq \ell$, every element of $\phi(U_\alpha)$ is semi-simple (because unipotent radical of $G_1(\mathbb{Z}_\ell)$ is ℓ-profinite). Thus the centralizer (resp. the normalizer) of $\phi(U_\ell)$ is given by $Z(\mathbb{Z}_\ell)$ (resp. $N(\mathbb{Z}_\ell)$) for a reductive subgroup Z (resp. N)

of G_1. Then $N(\mathbb{Z}_\ell)/Z(\mathbb{Z}_\ell)$ is a finite subgroup of the Weyl group W_1 of $G_{1/\mathbb{C}}$ which is independent of ℓ.

For example, if $G_1 = SL(n)_{/\mathbb{Z}}$, and if $\phi(U_\alpha)$ is made of diagonal matrices $\mathrm{diag}[\zeta_1 1_{m_1}, \zeta_2 1_{m_2}, \dots, \zeta_r 1_{m_r}]$ for generically distinct ζ_j, Z is given by the subgroup

$$SL(n) \cap (GL(m_1) \times GL(m_2) \times \cdots \times GL(m_r)),$$

where $GL(m_1) \times GL(m_2) \times \cdots \times GL(m_r)$ is embedded in $GL(n)$ diagonally. The quotient N/Z in this case is isomorphic to the subgroup of permutation matrices preserving Z.

If $\phi(U_\alpha)$ is nontrivial, the image of $N(U_\alpha)/Z(U_\alpha)$ in $\mathrm{Aut}(\phi(U_\alpha))$ grows at least on the order of p as p grows. In the above example of $G_1 = SL(n)_{/\mathbb{Z}}$, if $\phi(U_\alpha) \cong \mathbb{Z}/p^m\mathbb{Z}$ for $m > 0$, all elements in $\mathrm{Aut}(\phi(U_\alpha)) \cong (\mathbb{Z}/p^m\mathbb{Z})^\times$ come from $N(U_\alpha)/Z(U_\alpha) \cong \mathbb{Z}_p^\times$. This is impossible if $p \gg |W_1|$. Thus $\phi(U_\alpha) = 1$. Since $G_1(\mathbb{Z}_p)$ for large enough p is generated by U_α for all simple roots α, ϕ has to be trivial for p large enough. □

Since we assume to have the strong approximation theorem, we need to assume that G_1 is simply connected; so, we may restrict ourselves to symplectic and unitary groups (those groups of types A and C). We shall give a detailed exposition of how to prove (S2) for general linear groups and split symplectic groups and give a sketch for quasi split unitary groups.

Write $\chi : G \to Z^G = G/G_1$ for the projection map for the cocenter Z^G. In the following subsection, we assume that the base field K is either a number field or a nonarchimedean field of characteristic 0 (often a p-adic field). When K is nonarchimedean, we suppose that the classical group G is defined over $\mathbb{Z}_p$, and if K is a number field, G is defined over $\mathbb{Q}$. We write O for the maximal compact ring of K if K is nonarchimedean (so, O is the p-adic integer ring if K is p-adic). We equip the natural locally compact topology (resp. the discrete topology) on $G(A)$ for $A = K$ or O if K is a local field (resp. a number field). Then we define, for $A = K$ and O, $\mathrm{Aut}_\chi(G(A))$ by the group of continuous automorphisms of the group $G(A)$ which preserve χ up to automorphisms induced on Z^G by the field automorphisms of K. Thus

$$\mathrm{Aut}_\chi(G(A)) = \{\sigma \in \mathrm{Aut}(G(A)) | \chi(\sigma(g)) = \tau(\chi(g)) \text{ for } \exists\, \tau \in \mathrm{Aut}(K)\}.$$

For a subgroup H with $G_1(A) \subset H \subset G(A)$ and a section s of $\chi : H \to Z^G(A)$, we write $\mathrm{Aut}_s(H)$ for the group of continuous automorphisms of H preserving s up to field automorphisms and inner automorphisms (in [PAF] 4.4.3, the symbol $\mathrm{Aut}_{\det}(GL_2(A))$ means the group Aut_s here for a section s of $\det : GL(2) \to \mathbb{G}_m$).

3.1 General linear groups

Let $L_j \subset \mathbf{P}^{n-1}$ be the hyperplane of the projective space $\mathbf{P}^{n-1}(K)$ defined by the vanishing of the j–th homogeneous coordinate x_j. We start with the following well known fact:

Proposition 3.1. *Let P be the maximal parabolic subgroup of $GL(n)$ fixing the infinity hyperplane L_n of $\mathbf{P}^{n-1}$. For an infinite field K, $SL_n(K)$ is generated by conjugates of $U_P(K)$, where U_P is the unipotent radical of P.*

Proof. Let H be a subgroup generated by all conjugates of U_P. Thus H is a normal subgroup of $SL_n(K)$. Since $SL_n(K)$ is almost simple, we find that $SL_n(K) = H$. □

Proposition 3.2. *For an open compact subgroup S of $SL_n(K)$ for a p–adic local field K, the unipotent radical U of a Borel subgroup B_1 of $SL_n(K)$ and S generate $SL_n(K)$. Similarly S and a Borel subgroup B of $GL_n(K)$ generate $GL_n(K)$.*

Proof. We may assume that U is upper triangular. Thus $U \supset U_P$ for the maximal parabolic subgroup P in Proposition 3.1. We consider the subgroup H generated by S and U_P. The group U_P acts transitively on the affine space $\mathbf{A}^{n-1}(K) = \mathbf{P}^{n-1}(K) - L_n$. For any $g \in S - B_1$ for the upper triangular Borel subgroup B_1, gUg^{-1} acts transitively on $\mathbf{P}^{n-1}(K) - g(L_n)$. Note that $\bigcap_{g\in S} g(L_n)$ is empty, because intersection of n transversal hyperplanes is empty. Thus we find that H acts transitively on $\mathbf{P}^{n-1}(K)$. Since $\mathbf{P}^{n-1}(K)$ is in bijection with the set of all unipotent subgroups conjugate to U_P in $SL_n(K)$, H contains all conjugates of U_P in $SL_n(K)$; so, $H = SL_n(K)$ by Proposition 3.1. From this, generation of $GL_n(K)$ by B and S is clear because $GL_n(K) = B \cdot SL_n(K)$. □

In this case of GL_n, we have $\chi = \det$ and $Z^G = \mathbb{G}_m$. Thus, for $A = K$ or O, $\mathrm{Aut}_{\det}(GL_n(A))$ is the automorphism group of the group $GL_n(A)$ preserving the determinant map up to field automorphisms of K, that is, $\sigma \in \mathrm{Aut}_{\det}(GL_n(A))$ satisfies $\det(\sigma(g)) = \tau(\det(g))$ for a field automorphism $\tau \in \mathrm{Aut}(K)$. More generally, for a subgroup $H \subset GL_n(A)$ containing $SL_n(A)$, we define $\mathrm{Aut}_{\det}(H)$ for the automorphism group of H preserving $\det : H \to A^\times$ up to field automorphisms of K. Fixing a section $s : A^\times \to GL_n(A)$ of the determinant map, that is, $\det(s(x)) = x$, we recall $\mathrm{Aut}_s(GL_n(A)) = \{\sigma \in \mathrm{Aut}(GL_n(A)) | \sigma(s(x)) = g \cdot s(\tau(x))g^{-1}\}$ for some $g \in GL_n(A)$ and $\tau \in \mathrm{Aut}(K)$. Similarly, we define $\mathrm{Aut}_s(H)$ for a section s of the determinant map $\det : H \to A^\times$. We write $Z(SL_n(A))$ for the center of $SL_n(A)$, which is the finite group $\mu_n(A)$ of n-th roots of unity.

We now prove (S2) for $SL_n(A)$:

PROPOSITION 3.3. *If $A = K$, assume that K is either a local field of characteristic 0 or a number field. If $A = O$, assume that K is a local field of characteristic 0. Then we have*

1. *The continuous automorphism groups* $\mathrm{Aut}(PGL_n(A))$, $\mathrm{Aut}(PSL_n(A))$ *and* $\mathrm{Aut}(SL_n(A))$ *are all canonically isomorphic to*

$$\begin{cases} (\mathrm{Aut}(K) \times \langle J \rangle) \ltimes PGL_n(A) & \text{if } n \geq 3, \\ \mathrm{Aut}(K) \ltimes PGL_n(A) & \text{if } n = 2, \end{cases}$$

where $J(x) = w_0{}^t x^{-1} w_0^{-1}$ *for* $w_0 = (\delta_{i,n+1-j}) \in GL(n)$ *and* $\mathrm{Aut}(K)$ *is the continuous field automorphism group of* K.

2. *If* $H \supset SL_n(A)$ *is a subgroup of* $GL_n(A)$*, we have*

$$\mathrm{Aut}_s(H) = \{g \in \mathrm{Aut}(PGL_n(A)) | \tau_g(\det(H)) = \det(H)\},$$

where τ_g *indicates the projection of* g *to* $\mathrm{Aut}(K)$.

3. *We have a canonical split exact sequence*

$$1 \to \mathrm{Hom}(A^\times, Z(SL_n(A))) \to \mathrm{Aut}_{\det}(GL_n(A)) \to \mathrm{Aut}(SL_n(A)) \to 1.$$

In other words, for $\sigma \in \mathrm{Aut}_{\det}(GL_n(A))$*, there exists* $g \in GL_n(A)$ *and* $\tau \in \mathrm{Aut}(K)$ *with* $\sigma(x) = \zeta(\det(x)) g\tau(x) g^{-1}$ *for* $\zeta \in \mathrm{Hom}(A^\times, Z(SL_n(A)))$.

If K is a local field, we put the natural locally compact topology on the group, and if K is a global field, we put the discrete topology on the group. We shall give a computational proof for GL_n, because it describes well the mechanism of how an automorphism is determined entry by entry (of the matrices involved).

Proof. We first deal with the case where A is the field K. We first study $PGL_n(K)$. We have an exact sequence:

$$1 \to PGL_n(K) \xrightarrow{i} \mathrm{Aut}(PGL_n(K)) \to \mathrm{Out}(PGL_n(K)) \to 1,$$

where $i(x)(g) = xgx^{-1}$. We write B (resp. U) for the upper triangular Borel subgroup (resp. the upper triangular unipotent subgroup) of $GL_n(K)$. Their image in $PGL_n(K)$ will be denoted by $\overline{B}$ and $\overline{U}$.

Let $\mathcal{A}$ be a subgroup of $GL_n(K)$ isomorphic to the additive group K; so, we have an isomorphism $a : K \cong \mathcal{A}$. Consider the image $a(1)$ of $1 \in K$ in $\mathcal{A}$. Replacing K by a finite extension containing an eigenvalue α of $a(1)$, let $V_\alpha \subset K^n$ be the eigenspace of $a(1)$ with eigenvalue α. Then $a(\frac{1}{m})$ acts on V_α and $a(\frac{1}{m})^m = a(1) = \alpha \in \mathrm{End}(V_\alpha)$. Thus we have an algebra homomorphism: $K[x]/(x^m - \alpha) \to \mathrm{End}_K(V_\alpha)$ for all $0 < m \in \mathbb{Z}$. If K is a p–adic local field, $\bigcap_m (K^\times)^{m!} = \{1\}$. By using this, we find $\bigcap_m (K^\times)^m = \{1\}$ for a number field

K. Thus if K is a non-archimedean local field or a number field, we find that α has to be 1. Thus $\mathcal{A}$ is made up of commuting unipotent elements; so, by conjugation, we can embed $\mathcal{A}$ into U.

Since $\overline{U} \cong U$ is generated by unipotent subgroups isomorphic to K, by the above argument, $\sigma(\overline{U})$ for $\sigma \in \mathrm{Aut}(PGL_n(K))$ is again a unipotent subgroup of $PGL_n(K)$. Since $\overline{B}$ is the normalizer of $\overline{U}$, again $\sigma(\overline{B})$ is the normalizer of $\sigma(\overline{U})$; so, $\sigma(\overline{B})$ is a Borel subgroup. We find $g \in GL_n(K)$ such that $\sigma(\overline{B}) = g\overline{B}g^{-1}$. Thus we may assume that σ fixes $\overline{B}$. Applying the same argument to $\overline{U}$, we may assume that σ fixes $\overline{U}$. Since we have a unique filtration:

$$\overline{U} = U_1 \supset U_2 \supset U_3 \supset \cdots \supset U_{n-1} \supset \{1\} = U_n$$

with $[U_j, U_j] = U_{j+1}$ and $U_j/U_{j+1} \cong K^{n-j}$, σ preserves this filtration. We fix an isomorphism $a_j : K^{n-j} \hookrightarrow U_j$ given by

$$a_j(\alpha_1, \dots, \alpha_{n-j}) = 1 + \alpha_1 E_{1,j+1} + \alpha_2 E_{2,j+2} + \cdots + \alpha_{n-j} E_{n-j,n},$$

where $E_{i,j}$ is the matrix having non-zero entry 1 only at the (i,j)-spot. Then a_j induces $K^{n-j} \cong U_j/U_{j+1}$. Since $\sigma([u,u']) = [\sigma(u), \sigma(u')]$ for $u, u' \in U_j$ and $[\sigma(u), \sigma(u')] \mod U_{j+2}$ is uniquely determined by the cosets uU_{j+1} and $u'U_{j+1}$, $\sigma : \overline{U} \cong \overline{U}$ is uniquely determined by $\sigma_1 : U_1/U_2 \cong U_1/U_2$ induced by σ.

Each subquotient U_j/U_{j+1} is a K-vector space and is a direct sum of one-dimensional eigenspaces under the conjugate action of $\overline{T} := \overline{B}/\overline{U}$. Define an isomorphism $t : (K^\times)^n/K^\times \cong T$ by $t(\alpha_1, \dots, \alpha_n) = \mathrm{diag}[\alpha_1, \dots, \alpha_n]$, and we write $\alpha_j(t) = \alpha_j$ if $t = \mathrm{diag}[\alpha_1, \dots, \alpha_n]$. Then $U_{ij} \subset \overline{U}$ $(j > i)$ generated by $u_{ij} = 1 + E_{i,j}$ is the eigen-subgroup (isomorphic to one dimensional vector space over K) on which $t \in \overline{T}$ acts via the multiplication by $\chi_{ij}(t) = \alpha_i\alpha_j^{-1}(t)$. The automorphism σ also induces an automorphism $\overline{\sigma}$ of $T = \overline{B}/\overline{U}$. Thus σ permutes the eigen-subgroups U_{ij} of $\overline{U}$.

Let $k = \mathbb{Q}$ if K is a number field and $k = \mathbb{Q}_p$ if K is a p-adic field. Then σ induces a k-linear automorphism on U_j/U_{j+1} for all j. We first assume $K = k$. Write $\sigma(a_1(1, \dots, 1)) = a_1(\alpha_1, \dots, \alpha_{n-1}) \mod U_2$. Solve $a_j a_{j+1}^{-1} = \alpha_j$ for $j = 1, \dots, n-1$. Then changing σ by $x \mapsto t\sigma(x)t^{-1}$ for $t = \mathrm{diag}[a_1, a_2, \dots, a_n]$, we may assume that $\sigma_1(a_1(\mathbf{1})) \equiv a_1(\mathbf{1}) \mod U_2$ for $\mathbf{1} = (1, 1, \dots, 1) \in K^{n-1}$. Further by conjugating σ by an element in $\overline{U}$, we may assume that $\sigma(a_1(\mathbf{1})) = a_1(\mathbf{1})$. Thus $\sigma(a_1(r \cdot \mathbf{1})) = a_1(r \cdot \mathbf{1})$ for all $r \in \mathbb{Q}$. By taking commutators of $a_1(r \cdot \mathbf{1})$, we have a nontrivial element in U_j/U_{j+1} fixed by σ for all j. In particular, σ fixes $U_{n-1} \cong k$ and hence fixes the character χ_{1n}. If $n \geq 3$, looking at $U_{n-2}/U_{n-1} = U_{1,n-1} \oplus U_{2,n}$, we conclude that σ either interchanges the two eigenspaces or fixes each. If σ interchange the two, replacing σ by $\sigma \circ J$ for the automorphism $J = J_n$ of $GL_n(K)$ given by $J(x) = w_0{}^t x^{-1} w_0^{-1}$, we may assume σ fix each $\overline{T}$-eigenspace of U_{n-2}/U_{n-1}. By the commutator relation $[u_{ij}, u_{jk}] = u_{ik}$ if $i < j < k$, we conclude that

σ has to fix all eigen-subgroups U_{ij} of $\overline{U}$. Since $tu_{ij}t^{-1} = \chi_{ij}(t)u_{ij}$ and σ commutes with the multiplication by $\chi_{ij}(t) \in k$ on U_{ij}, we find by the K-linearity of σ that $\chi_{ij}(\overline{\sigma}(t))\sigma(u_{ij}) = \sigma(\chi_{ij}(t)u_{ij}) = \chi_{ij}(t)\sigma(u_{ij})$. Thus σ acts trivially on $\overline{T}$. Since $a_1(\mathbf{1})$ has non-trivial projection to all $\overline{T}$-eigenspaces in U_1/U_2, we conclude that $\sigma(u_{i,i+1}) = u_{i,i+1}$. Then $\overline{U}$ is fixed by σ again by the commutator relation $[u_{ij}, u_{jk}] = u_{ik}$ if $i < j < k$. Thus, modifying σ further by an inner automorphism and J, we may assume that σ fixes B element-by-element.

Now we assume that $K \supsetneq k$. Modifying σ as above by composing an inner automorphism and the action of J if necessary, we assume that σ preserves the eigen subgroups U_{ij} for all $i < j$. We are going to show that $\chi_{ij} \circ \overline{\sigma} = \widetilde{\sigma} \circ \chi_{ij}$ (for all $i < j$) for a continuous field automorphism $\widetilde{\sigma}$ of K. Since σ induces a k-linear automorphism of $U_{ij} \cong K$ and $\chi_{ij}(t) \in \mathrm{Im}(\chi_{ij}) = K^\times$ acts on U_{ij} through the multiplication by $\chi_{ij}(t) \in K^\times$, we find an automorphism $\widetilde{\sigma}_{ij} \in \mathrm{Aut}(K)$ of the field K such that $\chi_{ij} \circ \overline{\sigma} = \widetilde{\sigma}_{ij} \circ \chi_{ij}$. This field automorphism $\widetilde{\sigma} = \widetilde{\sigma}_{ij}$ does not depend on (i,j) by the commutator relation $[u_{ij}, u_{jk}] = u_{ik}$ for all $i < j < k$. Thus modifying σ further by an element of $\mathrm{Aut}(K)$, we may assume that σ fixes B.

We are going to prove that σ inducing the identity map on B is the identity on the entire group. For the moment, we suppose that K is p–adic. Then by [GAN] IV.3.2.6, for a sufficiently small open compact subgroup $S \subset PGL_n(K)$, $\sigma : S \cong \sigma(S)$ induces an automorphism Φ_σ of the Lie algebra $\mathfrak{G}_{\mathbb{Q}_p}$ of $PGL_n(K)$ over $\mathbb{Q}_p$. Since $\dim_{\mathbb{Q}_p} \mathfrak{G}_{\mathbb{Q}_p} = \dim_{\mathbb{Q}_p} \mathfrak{G}_K$ for the Lie algebra $\mathfrak{G}_K$ of $PGL_n(K)$ over K, we find that $\mathrm{Aut}_K(\mathfrak{G}_K) \subset \mathrm{Aut}_{\mathbb{Q}_p} \mathfrak{G}_{\mathbb{Q}_p}$ has the same dimension over $\mathbb{Q}_p$ as a Lie group over $\mathbb{Q}_p$ (cf. [BLI] VIII.5.5). Thus $\Phi_\sigma \in \mathfrak{G}_K$ is induced by $g \in GL_n(K)$ through the adjoint action (cf. [BLI] VIII.13). Since σ fixes B, we find that g commutes with B and, hence, g is in the center. Therefore, shrinking S further if necessary, we conclude $\sigma = 1$ on S and on B. Since B and S generate $PGL_n(K)$ (see Proposition 3.2), we find σ is the identity map over entire $PGL_n(K)$. This shows that, under the condition that $n \geq 3$,

$$\mathrm{Out}(PGL_n(K)) \cong \mathrm{Aut}(K) \times \langle J \rangle \quad \text{and} \quad \mathrm{Aut}(PGL_n(K)) = (\mathrm{Aut}(K) \times \langle J \rangle) \ltimes PGL_n(K)$$

if K is a local p–adic field. If $n = 2$, we need to remove the factor $\langle J \rangle$ from the above formula. If $K = \mathbb{R}$ or $\mathbb{C}$, the above fact is well known (see [BLI] III.10.2).

Suppose now that K is a number field. Write O for the integer ring of K. Take a prime $\mathfrak{p}$ such that $O_\mathfrak{p} \cong \mathbb{Z}_p$. Since σ fixes B, for the diagonal torus T, σ fixes its normalizer $N(T)$. Since $N(T) = W \ltimes T$, we find that $\sigma(w) = tw$ for an element $t \in T$. Since $PGL_n(K) = \bigsqcup_{w \in W} BwB$, we find that σ is continuous with respect to the $\mathfrak{p}$–adic topology. Thus σ induces $\mathrm{Aut}(PGL_n(K_\mathfrak{p}))$ fixing B, and we find that $\sigma = 1$, which shows again $\mathrm{Out}(PGL_n(K)) \cong (\mathrm{Aut}(K) \times \langle J \rangle)$

and $\mathrm{Aut}(PGL_n(K)) = (\mathrm{Aut}(K) \times \langle J\rangle) \ltimes PGL_n(K)$ for a number field K.

We can apply the same argument to $\mathrm{Aut}(PSL_n(K))$ and $\mathrm{Aut}(SL_n(K))$. Modifying σ by inner automorphisms, J and an element in $\mathrm{Aut}(K)$, we may assume that σ leaves $\overline{B}_1$ fixed. Then by the same argument as above, we conclude $\sigma = 1$ and hence we find that, if $n > 2$,

$$\mathrm{Aut}(SL_n(K)) = \mathrm{Aut}(PSL_n(K)) = (\mathrm{Aut}(K) \times \langle J\rangle) \ltimes PGL_n(K).$$

If $n = 2$, again we need to remove the factor $\langle J\rangle$ from the above formulas.

Now we look at $\mathrm{Aut}_s(H)$ for a section s of $\det : H \to K^\times$. Since $\sigma \in \mathrm{Aut}_{\det}(H)$ preserves the section s up to field and inner automorphisms, modifying σ by such an automorphism, we may assume that σ fixes $\mathrm{Im}(s)$. Then σ is determined by its restriction to $SL_n(K) \subset H$ and, hence, comes from and element in $\mathrm{Aut}(K) \ltimes PGL_2(K)$ preserving H.

To see the last assertion (3) for $A = K$, we consider the restriction map

$$\mathrm{Res} : \mathrm{Aut}_{\det}(GL_n(K)) \to \mathrm{Aut}(SL_n(K)).$$

Since $\mathrm{Aut}(SL_n(K))$ acts naturally on $GL_n(K)$ by the result already proven, the homomorphism Res is surjective. Take $\sigma \in \mathrm{Ker}(\mathrm{Res})$, and fix a section $s : K^\times \to GL_n(K)$ of the determinant map. Then for $x \in SL_n(K)$, we have $s(a)xs(a)^{-1} = \sigma(s(a)xs(a)^{-1}) = \sigma(s(a))x\sigma(s(a))^{-1}$, because $\mathrm{Res}(\sigma)$ is the identity map. Thus $\sigma(s(a))s(a)^{-1}$ commutes with $SL_n(K)$. Taking the determinant of $\sigma(s(a))s(a)^{-1}$, we find that $\sigma(s(a))s(a)^{-1} \in Z(SL_n(K))$ and $a \mapsto \zeta(a) = \sigma(s(a))s(a)^{-1}$ is a homomorphism of the group $K^\times$ into $Z(SL_2(K))$.

For any $g \in GL_n(K)$, we can write uniquely $g = s(\det(g))u$ with $u \in SL_n(K)$. For a homomorphism $\zeta : K^\times \to Z(SL_n(K))$,

$$\sigma(g) = \sigma(s(\det(g))u) = \zeta(\det(g))s(\det(g))u = \zeta(\det(g))g$$

gives an endomorphism of $GL_n(K)$. It is an automorphism because σ induces the identity on $SL_n(K)$ and $K^\times = \det(GL_n(K))$. Thus we get the desired exact sequence.

We now assume $A = O$. Since the argument is the same as in the case of the field K, we only indicates some essential points. Let $U(O) = U \cap SL_n(O)$ for the subgroup U of upper unipotent matrices. Since $\mathbf{P}^{n-1}(O) = \mathbf{P}^{n-1}(K)$, all Borel subgroups of $SL_n(O)$ are conjugate each other. Since $B_1(O) = SL_n(O) \cap B$ is a semi-direct product of $T_1(O)$ and $U(O)$, all unipotent subgroups are conjugate each other. By the same argument in the case of the field, we may assume that $\sigma \in \mathrm{Aut}(PGL_n(O))$ leaves $U(O)$ stable. Writing b_j for $(0, \dots, \overset{j}{b_j}, 0, \dots, 0) \in$

O^{n-1} with $b_j \in O$, we have $t(\alpha)a_1(b_j)t(\alpha)^{-1} = a_1(\alpha_j b_j \alpha_{j+1}^{-1})$ for $a_1 : O^{n-1} \cong U(O)/[U(O), U(O)]$ and $t : (O^\times)^n \cong T(O)$ as in the proof of Proposition 3.3. Then applying σ to the above formula, we see that $\overline{\sigma}$ preserves the coordinates α_j, and $\overline{\sigma}(t(\alpha))\sigma(a_1(b_j))\overline{\sigma}(t(\alpha)^{-1}) = \sigma(a_1(\alpha_j b_j \alpha_{j+1}^{-1}))$ for $\alpha = (\alpha_j) \in (O^\times)^n$. If $\sigma(a_1(e_j)) \in \mathfrak{m}$ for $e_j = (0, \dots, \overset{j}{1}, 0, \dots, 0) \in O^{n-1}$, then

$$\overline{\sigma}(t(\alpha))\sigma(a_1(e_j))\overline{\sigma}(t(\alpha))^{-1} = \sigma(a_1(\alpha e_j \alpha^{-1}))$$

has entry in $\mathfrak{m}$ at $(j, j+1)$. However $\sigma(U/U_2) = U/U_2 \cong O^{n-1}$ by $a_1(b) \leftrightarrow b$, we find $O \subset \mathfrak{m}$, a contradiction. Thus we have $\sigma(a_1(\mathbf{1})) = a_1(\alpha)$ for an element in $\alpha \in (O^\times)^{n-1}$. Then $t(\alpha) \in GL_n(O)$,and modifying σ by the conjugation of $t(\alpha)$, we may assume that $\sigma(a_1(\mathbf{1})) = a_1(\mathbf{1})$. Then proceeding in exactly the same way in the case of the field, we find that

$$\mathrm{Aut}(PGL_n(O)) = \begin{cases} (\mathrm{Aut}(O) \times \langle J \rangle) \ltimes PGL_n(O) & \text{if } n \geq 3, \\ \mathrm{Aut}(O) \ltimes PGL_2(O) & \text{if } n = 2. \end{cases}$$

From this, again we obtain the desired result for all other automorphism groups listed in the proposition. □

Let $M = K \oplus K$ be a semi-simple algebra with involution $c(x, y) = (y, x)$. Then we can realize $SL_n(K)$ as a special unitary group with respect to the hermitian form $(u, v) = \mathrm{Tr}({}^t u^c w_0 v)$ $(u, v \in M^n)$:

$$G_1(K) = \left\{ \alpha \in SL_n(M) \middle| (\alpha u, \alpha v) = (u, v) \right\}.$$

Indeed $SL_n(K) \cong G_1(K)$ by $x \mapsto (x, J(x))$. Then we have $\mathrm{Aut}(M) \cong \mathrm{Aut}(K) \times \langle J \rangle$, and the results in Propositions 3.3 for $A = O$ and K can be restated as

$$\mathrm{Aut}(G_1(A)) = \mathrm{Aut}(M) \ltimes PG(A)$$

for the unitary group G with respect to $(\cdot, \cdot)$. We used in the proof of the theorem this version of the result in this section when K is a completion of the totally real field F at a prime $\mathfrak{l}$ splitting in the CM field M; so, $M_\mathfrak{l} = K_\mathfrak{l} \oplus K_\mathfrak{l}$ and c is induced by complex conjugation c of M.

3.2 Symplectic groups

We start with a general fact valid for quasi-split almost simple connected groups G_1 not necessarily a symplectic group.

Proposition 3.4. *Let K be a p–adic local field. Let S be an open subgroup of $G_1(K)$ of a classical almost simple connected group G_1 quasi-split over K. Let P_0 be a minimal parabolic subgroup of G_1 defined over K with unipotent radical U. Then S and $U(K)$ generate $G_1(K)$.*

The proof is similar to that of Proposition 3.2. Here is a sketch. Taking the universal covering of G_1, we may assume that G_1 is simply connected and is given by a Chevalley group G_1 inside $GL(n)$ for an appropriate n defined over O. Thus $G_1(K)$ is almost simple. We may assume that $P_0 = B \cap G_1$ for the upper-triangular Borel subgroup B of $GL(n)$. Thus G_1 acts on the projective space $\mathbf{P}^{n-1}$ through the embedding $G_1 \subset GL(n)$. Take the stabilizer $P \subset G_1$ of the infinity hyperplane L_n of $\mathbf{P}^{n-1}$. Then P is a maximal parabolic subgroup of G_1 containing P_0. Since $G_1(K)$ is almost simple, $G_1(K)$ is generated by conjugates of the unipotent radical $U_P(K)$ of P. The flag variety $\mathbf{P} = G_1/P$ is an irreducible closed subscheme of $\mathbf{P}^{n-1}$, and $U_P(K)$ acts transitively on $\mathbf{P} - L_n$. Since $\mathbf{P}$ is covered by finitely many affine open subschemes of the form $\mathbf{P} - g(L_n)$ (on which gU_Pg^{-1} acts transitively), the subgroup H generated by S and $U_P(K)$ acts transitively on $\mathbf{P}$ and hence contains all conjugates of $U_P(K)$. This shows that $G_1(K)$ is generated by S and $U_P(K)$. $\square$

Let I_n be the antidiagonal $I_n = (\delta_{n+1-i,j}) \in M_n(\mathbb{Q})$ and $J_n = \left(\begin{smallmatrix} 0 & -I_g \\ I_g & 0 \end{smallmatrix}\right)$ for $n = 2g$ is an anti-diagonal alternating matrix. In this subsection, we deal with the split symplectic group defined over $\mathbb{Q}$ given by

$$G(A) = GSp_{2g}(A) = \left\{\alpha \in GL_n(A) \middle| \alpha J_{2g}{}^t\alpha = \nu(\alpha) J_{2g} \text{ for } \nu(\alpha) \in A^\times\right\},$$

and $G_1 = Sp_{2g} = \mathrm{Ker}(\nu)$. We write Z for the center of GSp_{2g}. We write B for the upper triangular Borel subgroup of GSp_{2g}. We write U for the unipotent radical of B. For the diagonal torus T, we have $B = T \ltimes U$, and B is the normalizer of $U(K)$ in $GSp_{2g}(K)$ for a field extension K of $\mathbb{Q}$. We take a standard parabolic subgroup $P \supset B$ of $GSp_{2g}(K)$ with unipotent radical U_P contained in U.

In this symplectic case, $\chi : G \to Z^G$ is the similitude map $\nu : GSp_{2n} \to \mathbb{G}_m$; so, we have $\mathrm{Aut}_\nu(GSp_{2n}(A))$ and $\mathrm{Aut}_s(H)$ for a subgroup H with $Sp_{2n}(A) \subset H \subset GSp_{2n}(A)$ and a section s of $\nu : H \to K^\times$. Here $A = K$ or O.

Proposition 3.5. *Let K be a local or global field of characteristic* 0. *Then we have*

1. $\mathrm{Aut}(Sp_{2g}(K)) = \mathrm{Aut}(PGSp_{2n}(K)) = \mathrm{Aut}(K) \ltimes PGSp_{2g}(K)$, *where we define* $PGSp_{2g}(K) = GSp_{2n}(K)/Z(K)$.
2. *For a section s of $\nu : H \to K^\times$ for a closed subgroup H with $Sp_{2n}(K) \subset H \subset GSp_{2n}(K)$,* $\mathrm{Aut}_s(H)$*is given by*

$$\{(\tau, g) \in \mathrm{Aut}(K) \ltimes PGSp_{2g}(K) \big| \tau(\nu(H)) = \nu(H)\}.$$

3. *We have a canonical split exact sequence*

$$1 \to \mathrm{Hom}(K^\times, Z(Sp_{2n}(K))) \to \mathrm{Aut}_\nu(GSp_{2n}(K)) \to \mathrm{Aut}(Sp_{2n}(K)) \to 1,$$

where $Z(Sp_{2n}(K))$ is the center $\{\pm 1\}$ of $Sp_{2n}(K)$.

We describe here a shorter argument proving the assertion (1) for GSp_{2g} (than the computational one for $GL(n)$) using the theory of root systems (although this is just an interpretation of the computational argument in terms of a slightly more sophisticated language). The assertions (2) and (3) follow from the assertion (1) by the same argument as in the case of $GL_n(K)$.

Proof. By [BLI] III.10.2, we may assume that K is either p–adic local or a number field. Write simply $B = B(K)$, $U = U(K)$ and $T = T(K)$. Let $\sigma \in \mathrm{Aut}(G(K))$. In the same manner as in the case of $GL(n)$, we verify that σ sends unipotent elements to unipotent elements. Write $\mathfrak{N} = \log(U)$ which is a maximal nilpotent subalgebra of the Lie algebra $\mathfrak{G}$ of $Sp_{2g}(K)$.

Let $k = \mathbb{Q}$ if K is global and $k = \mathbb{Q}_p$ if K is a p–adic field. Since $\sigma(U)$ is generated by unipotent matrices, we have $\log(\sigma(U))$ (which we write $\sigma(\mathfrak{N})$) is a nilpotent subalgebra of $\mathfrak{G}$, and $\dim_k \sigma(\mathfrak{N}) = \dim_k(\mathfrak{N})$. Thus $\sigma(\mathfrak{N})$ is a maximal nilpotent subalgebra of $\mathfrak{G}$; so, it is a conjugate of $\mathfrak{N}$ by $a \in Sp_{2g}(K)$. This implies $\sigma(\mathfrak{N}) = a\mathfrak{N}a^{-1}$. Conjugating back by a, we may assume that $\sigma(\mathfrak{N}) = \mathfrak{N}$. Then $\sigma(U) = U$ and hence $\sigma(B) = B$ because B is the normalizer of U. Thus σ induces an automorphism $\overline{\sigma}$ of $B/U \cong T$. We have weight spaces $\mathfrak{N}_\alpha$ and $\mathfrak{N} = \bigoplus_\alpha \mathfrak{N}_\alpha$. From this, we conclude that σ permutes $\mathfrak{N}_\alpha$: $\sigma(\mathfrak{N}_\alpha) = \mathfrak{N}_{\alpha\circ\overline{\sigma}}$.

Suppose $K = k$. Then σ is K–linear; in particular, σ induces a permutation of roots which has to give rise to a K–linear automorphism of the Lie algebra $\mathfrak{G}$. Modifying σ by the action of Weyl group (conjugation by a permutation matrix), we find that the permutation has to be trivial or an outer automorphism of the Dynkin diagram of Sp_{2g} (e.g. [Tt] 3.4.2 or [BLI] VIII.13). Since the Dynkin diagram of Sp_{2g} does not have any non-trivial automorphism, we find that the permutation is the identity map. Since on $\mathfrak{N}_\alpha$, T acts by a character $\alpha : T \to K^\times$, we find that $\alpha(t) = \alpha(\overline{\sigma}(t))$; so, $\overline{\sigma}$ is also the identity map.

We now assume that $K \neq k$. For the set of simple roots Δ of T with respect to $\mathfrak{N}$, $\bigoplus_{\alpha\in\Delta} \mathfrak{N}_\alpha \hookrightarrow \mathfrak{N}$ induces an isomorphism $\bigoplus_{\alpha\in\Delta} \mathfrak{N}_\alpha \cong \mathfrak{N}/[\mathfrak{N}, \mathfrak{N}]$. In other words, $\{\mathfrak{N}_\alpha | \alpha \in \Delta\}$ generates $\mathfrak{N}$ over K. The K–vector space structure of $\mathfrak{N}$ induces an embedding $i_1 : K \hookrightarrow \mathrm{End}_k(\mathfrak{N})$ of k–algebras. Since σ induces $\sigma_\mathfrak{N} \in \mathrm{End}_k(\mathfrak{N})$, we have another embedding $i_2 = \sigma_\mathfrak{N}^{-1} i_1 \sigma_\mathfrak{N}$ of K into $\mathrm{End}_k(\mathfrak{N})$. In $\mathrm{End}_k(\mathfrak{N}/[\mathfrak{N}, \mathfrak{N}])$, the subalgebra A_1 generated by $i_1(K)$ and the action of T is a maximal commutative k–subalgebra. The subtorus T_0 given by the connected component of

$$\{t \in T | \alpha(t) = \beta(t) \quad \text{for all } \alpha, \beta \in \Delta\}$$

is dimension 1 and acts on $\mathfrak{N}/[\mathfrak{N}, \mathfrak{N}]$ by scalar multiplication. This property characterizes T_0. Since the fact that T_0 acts by scalar multiplication on $\mathfrak{N}/[\mathfrak{N}, \mathfrak{N}]$ does not change after applying $\overline{\sigma}$, we have $\overline{\sigma}(T_0) = T_0$. The image of $i_j(K)$ in $\mathrm{End}_k(\mathfrak{N}/[\mathfrak{N}, \mathfrak{N}])$ is generated over k by the action of T_0; so, they coincide. Since $\sigma \in \mathrm{Aut}_k(\mathfrak{N})$ is an automorphism of the Lie algebra, the action of σ on $\mathfrak{N}/[\mathfrak{N},\mathfrak{N}]$ determines the action of σ on $\mathfrak{N}$. In particular, we conclude $i_1(K) = i_2(K)$, and we can think of $\tau = i_2^{-1} \circ i_1 \in \mathrm{Aut}(K)$. Hence, σ is τ–linear for $\tau \in \mathrm{Aut}(K)$ (that is, $\sigma(\xi v) = \xi^\tau \sigma(v)$ for $\xi \in K$). Thus modifying by τ, we may assume that $\sigma : \mathfrak{N} \to \mathfrak{N}$ is K–linear. Then σ induces a permutation of roots which has to give rise to an automorphism of the Lie algebra of Sp_{2g}. Then by the same argument as in the case where $K = k$, we conclude that σ induces identity map on B/U and U. Taking T to be diagonal, we may assume that $\sigma(T) = u_\sigma T u_\sigma^{-1}$ for $u_\sigma \in U$. Thus by modifying σ by the inner automorphism of u_σ, we may assume that σ is the identity on B.

Suppose that K is p–adic, then σ sends an open compact subgroup S to $\sigma(S)$, which induces an endomorphism of the Lie algebra of $Sp_{2g}(O)$ for the p–adic integer ring O and induces the identity map on the Lie algebra of B; in particular, σ is a O–linear map on the Lie algebra. Since an automorphism of the Lie algebra is inner induced by conjugation by an element $g \in Sp_{2g}(K)$, we have $gbg^{-1} = b$ for $b \in B$. Since the centralizer of B is the center of G, we find that σ is the identity on S.
Since S and U generate $G_1 = Sp_{2g}$, we find that σ is the identity over G. This proves the desired result for G_1 and p–adic fields K.

We can proceed in exactly the same way as in the case of $GL(n)$ when K is a number field and conclude the result. □

We then get the following integral analogue in a manner similar to Proposition 3.3:

Proposition 3.6. *If K is a finite extension of $\mathbb{Q}_p$ for $p > 2$ with integer ring O, we have*

$$\mathrm{Aut}(Sp_n(O)) = \mathrm{Aut}(PGSp_n(O)) = \mathrm{Aut}(K) \ltimes PGL_n(O),$$

and a canonical split exact sequence

$$1 \to \mathrm{Hom}(O^\times, Z(Sp_{2n}(O))) \to \mathrm{Aut}_\nu(GSp_n(O)) \to \mathrm{Aut}(Sp_{2n}(O)) \to 1,$$

where $Z(Sp_{2n}(O)) = \{\pm 1\}$ is the center of $Sp_{2n}(O)$.

3.3 Quasi-split unitary groups

Let M/K be a p–adic quadratic extension with p–adic integer rings R/O, and consider the quasi split unitary group

$$G(K) = \left\{\alpha \in GL_n(M) \middle| \alpha I_n {}^t\alpha^c = \nu(\alpha) I_n\right\} \quad \text{and} \quad G(O) = GL_n(R) \cap G(K),$$

where c is the generator of $\mathrm{Gal}(M/K)$, $\nu : G \to K^\times$ is the similitude map, $I_n = w_0$ if n is odd and $I_{2m} = \left(\begin{smallmatrix} 1_m & 0 \\ 0 & -1_m \end{smallmatrix}\right) w_0$ if $n = 2m$ is even. We may assume that $n \geq 3$ because in the case of $n = 2$, we have $PG \cong PGL_2$ (so, the desired result in this case has been proven already in 3.1).

We write G_1 for the derived group of G. Thus

$$G_1(K) = \{g \in G(K) | \det(g) = \nu(g) = 1\}.$$

Define the cocenter $Z^G = G/G_1$ and write $\mu : G \to Z^G$ for the projection. We may identify μ with $\det \times \nu$ and Z^G with its image in $\mathrm{Res}_{M/\mathbb{Q}_p}\mathbb{G}_m \times \mathrm{Res}_{K/\mathbb{Q}_p}\mathbb{G}_m$. We consider $\mathrm{Aut}_\mu(G(A))$ for $A = O$ and K made up of group automorphisms σ of $G(A)$ satisfying $\mu \circ \sigma = \mu$. We suppose that the nontrivial automorphism c of M over K is induced by an order 2 automorphism of the Galois closure M^{gal} of $M/\mathbb{Q}_p$ in the center of $\mathrm{Gal}(M^{gal}/\mathbb{Q}_p)$.

Write $\mathfrak{G}_A$ for the Lie algebra of $G_1(A)$ for $A = K$, O, M and R. Since $G_1(M) \cong SL_n(M)$, by Proposition 3.3, the Lie algebra automorphism group $\mathrm{Aut}(\mathfrak{G}_M)$ is isomorphic to $(\mathrm{Aut}(M) \times \langle J\rangle) \ltimes PG(M)$. Since $\mathfrak{G}_K \otimes_K M = \mathfrak{G}_M$, any automorphism of $\mathfrak{G}_K$ extends to an automorphism of $\mathfrak{G}_M$; so, $\mathrm{Aut}_K(\mathfrak{G}_K) \subset \mathrm{Aut}(\mathfrak{G}_M)$, and by this inclusion sends $\sigma \in \mathrm{Aut}(M) \subset \mathrm{Aut}(\mathfrak{G}_K)$ to an element $(\sigma, 1) \in (\mathrm{Aut}(M) \times \langle J\rangle)$. By this fact, at the level of the Lie algebra, all automorphisms of $\mathfrak{G}_A$ for $A = O$ and K are inner up to automorphism of M, and we have $\mathrm{Aut}(\mathfrak{G}_A) = \mathrm{Aut}(A) \ltimes PG(A)$.

We now study the automorphism group of the p-adic Lie group $G(K)$ and $G(O)$.

Proposition 3.7. *Let $A = O$ or K for a p-adic field K. We assume that $p > 2$ and $K/\mathbb{Q}_p$ is unramified if $A = O$. Then we have*

1. $\mathrm{Aut}(G_1(A)) = \mathrm{Aut}(PG(A)) = \mathrm{Aut}(M) \ltimes PG(A)$,
2. *We have a canonical split exact sequence:*

$$1 \to \mathrm{Hom}(Z^G(A), Z(G_1(A))) \to \mathrm{Aut}_\mu(G(A)) \to \mathrm{Aut}(G_1(A)) \to 1,$$

where $Z(G_1(A))$ is the center of $G_1(A)$ and is isomorphic to $\mu_n(A)$.

Proof. We start with a brief sketch of the argument. A standard Borel subgroup of G (i.e., a standard minimal parabolic subgroup) is given by the subgroup B made up of all upper triangular matrices. We consider the subgroup $U \subset B$ made up of upper unipotent matrices. If $\sigma \in \mathrm{Aut}_\mu(G(K))$, $\sigma(U)$ is again generated by unipotent elements. Thus by [B] 6.5, $\sigma(U)$ is a conjugate of U in $G(K)$. Then by the same argument in the case of $GL(n)$, modifying σ by an element in $\mathrm{Aut}(M) \ltimes G(K)$, we may assume that σ fixes B. Again by the same argument as in the case of $GL(n)$, we conclude that $\sigma = 1$. Thus $\mathrm{Aut}(PG(K))$ and $\mathrm{Aut}(G_1(K))$ are given by $\mathrm{Aut}(M) \ltimes PG(K)$, and further assuming that p is odd and unramified in $M/\mathbb{Q}_p$, $\mathrm{Aut}(PG(O))$ and $\mathrm{Aut}(G_1(O))$ are given by $\mathrm{Aut}(M) \ltimes PG(O)$. If $\sigma \in \mathrm{Aut}_\mu(G(A))$, then we can write $\sigma(g) = h_\sigma\tau(g)h_\sigma^{-1}$ with a unique $h_\sigma \in PG(A)$ and $\tau \in \mathrm{Aut}(M)$ for all $x \in G_1(A)$. Since $\mathrm{Aut}(PG(A)) = \mathrm{Aut}(M) \ltimes PG(A)$, we find $\sigma(g) = \zeta(g)h_\sigma\tau(g)h_\sigma^{-1}$ with $\zeta(g) \in Z(G(A))$ for all $g \in G(A)$. Applying μ and noting that $\mu(\sigma(g)) = \tau(\mu(g))$, we find $\tau(\mu(g)) = \mu(\zeta(g))\tau(\mu(g))$; so, $\zeta(g) \in Z(G_1(A))$. Since σ is an automorphism, $\zeta : G(A) \to Z(G_1(A))$ is a homomorphism. By our assumption on p, $G_1(A)$ is the derived group of the topological group $G(A)$, and hence, ζ factors through $Z^G(A) = G(A)/G_1(A)$, since $Z(G_1(A))$ is abelian. This shows that the assertion (2) follows from the assertion (1).

Let us fill in the proof of the assertion (1) with some more details, assuming first for simplicity that $n = 3$. In this case, by computation, we have

$$U(K) = \left\{ u(x,y) = \left(\begin{smallmatrix} 1 & x & y \\ 0 & 1 & -x^c \\ 0 & 0 & 1 \end{smallmatrix}\right) \in GL_3(M) \middle| xx^c + (y + y^c) = 0 \right\}.$$

The diagonal torus $T(K) \subset G$ is made of $t(a,b) = \mathrm{diag}[a, b, a^{-c}bb^c]$ for $a \in M^\times$ and $b \in M^\times$. Thus writing $\mathfrak{N} = \log(U(K))$, we have $\mathfrak{N}/[\mathfrak{N},\mathfrak{N}] \cong M$ by $u(x,y) \mapsto x$ and $\mathfrak{N}/[\mathfrak{N},\mathfrak{N}]$ is a one-dimensional vector space over M (so, it is two-dimensional over the field of definition K) on which $t(a,b)$ acts through the multiplication by ab^c: $t(a,b)u(x,y)t(a,b)^{-1} = u(ab^cx, (ab^{-1})(ab^{-1})^cy)$. By the above argument in the general case, we may assume that $\sigma(B) = B$ for $\sigma \in \mathrm{Aut}_\mu(G(K))$. Then we have $\sigma(u(1,y)) = u(a,y')$ for $a \in M^\times$ and $t(a,1)^{-1}\sigma(u(1,y))t(a,1) = u(1,y'')$ for some $y', y'' \in M$. Thus modifying σ by an inner automorphism of an element in $T(K)$ and identifying $\mathfrak{N}/[\mathfrak{N},\mathfrak{N}]$ with M by $u(x,*) \mod [\mathfrak{N},\mathfrak{N}] \mapsto x$, we find that σ induces an automorphism of the field $M = \mathfrak{N}/[\mathfrak{N},\mathfrak{N}]$ and the same automorphism on $T(K) = B(K)/U(K)$ coordinate-wise. Thus again modifying σ by an element in $\mathrm{Aut}(K)$ and by an inner automorphism of an element of $U(K)$, we may assume that σ induces the identity map on B. We then conclude that σ is the identity map on $G(K)$ by the same argument as in the case of $GL(n)$ and $GSp(2n)$.

Next, we suppose that $n = 4$. Again by computation, we have

$$U(K) = \left\{ u(w,y,x,z) = \left(\begin{smallmatrix} 1 & w & x & z \\ 0 & 1 & y & x^c - yw^c \\ 0 & 0 & 1 & -w^c \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right) \in GL_4(M) \middle| \begin{smallmatrix} y = y^c \text{ and} \\ z^c - z = xw^c - wx^c \end{smallmatrix} \right\}.$$

The diagonal torus $T(K) \subset G$ is made of $t(a,b,\nu) = \mathrm{diag}[a\nu, b\nu, b^{-c}, a^{-c}]$ for $a, b \in M^\times$ and $\nu \in K^\times$. We have

$$t(a,b,\nu)u(w,y,x,z)t(a,b,\nu)^{-1} = u(ab^{-1}w, bb^c\nu y, ab^c\nu x, aa^c\nu z).$$

Thus writing $\mathfrak{N} = \log(U(K))$, we have $\mathfrak{N}/[\mathfrak{N},\mathfrak{N}] \cong M \oplus K$ by $u(w,y,x,z) \mapsto (w,y)$ and $\mathfrak{N}/[\mathfrak{N},\mathfrak{N}]$ is a three-dimensional vector space over K on which $t(a,b,\nu)$ acts through $(w,y) \mapsto (ab^{-1}w, bb^c\nu y)$. By the above argument at the level of the Lie algebra, we may assume that $\sigma(B) = B$ for $\sigma \in \mathrm{Aut}_\mu(G(K))$ and that σ preserves the root space decomposition $M \oplus K$ of $\mathfrak{N}/[\mathfrak{N},\mathfrak{N}]$. We write σ_M (resp. σ_K) for the $\mathbb{Q}_p$-linear map induced by σ on M (resp. K). Thus modifying σ by conjugation of $t \in T(K)$, we may assume that $\sigma(1,1) = (1,1)$ for $(1,1) \in M \oplus K$. Writing the character of T giving the action of T on M (resp. K) by χ_M (resp. χ_K), we have $\chi_M(t(a,b,\nu)) = ab^{-1}$ and $\chi_K(t(a,b,\nu)) = bb^c\nu$. Then on M (resp. on K), the action of T gives rise to the multiplication by elements of $M^\times = \mathrm{Im}(\chi_M)$ (resp. in $K^\times = \mathrm{Im}(\chi_K)$), which is preserved by σ_M (resp. by σ_K); that is, we have $\sigma_M(\chi_M(t)w) = \chi_M(\sigma(t))\sigma_M(w)$ and $\sigma_K(\chi_K(t)y) = \chi_K(\sigma(t))\sigma_K(y)$. Since σ fixes $(1,1)$, we find that $\sigma_M \in \mathrm{Aut}(M)$ and $\sigma_K \in \mathrm{Aut}(K)$ satisfying $\chi_M \circ \sigma = \sigma_M \circ \chi_M$ and $\chi_K \circ \sigma = \sigma_K \circ \chi_K$. By the commutator relation $[u(w,y,*,*), u(w',y',*,*)] = u(0,0,wy'-w'y,*)$, we find that $\sigma_M|_K = \sigma_K$. Then modifying σ by the element $\sigma_M \in \mathrm{Aut}(M)$, we find that σ fixes $\overline{T} = B/U$, and again modifying σ by the conjugation by an element in $U(K)$, we bring σ to preserve $T \subset B$; so, σ induces the identity map on B. Out of this, we conclude that σ is the identity map on $G(K)$ by the same argument as in the case of $GL(n)$ and $GSp(2n)$, because σ coincides with an inner automorphism on an open neighborhood of the identity in $G(K)$ (by the argument at the level of the Lie algebra).

In the general case of $n > 4$, if $n = 2m+1$ is odd, we may identify $\mathfrak{N}/[\mathfrak{N},\mathfrak{N}] = M^m$ as M-vector space by $(u_{i,j}) \in \mathfrak{N} \mapsto (u_{1,2},\dots,u_{m,m+1})$. On the j-th factor, $t = \mathrm{diag}[a_1,\dots,a_n] \in T(K)$ acts through the multiplication by $a_j a_{j+1}^{-1} \in M^\times$. If $n = 2m$ is even, $\mathfrak{N}/[\mathfrak{N},\mathfrak{N}] \cong M^{m-1} \oplus K$ by sending upper unipotent matrices $(u_{i,j}) \in U(K)$ to $(u_{1,2},\dots,u_{m,m+1})$. On the first j-th factors with $j < m$, $t = \mathrm{diag}[\nu a_1,\dots,\nu a_m, a_m^{-c},\dots,a_1^{-c}] \in T(K)$ ($a_j \in M^\times$ and $\nu \in K^\times$ acts through the multiplication by $a_j a_{j+1}^{-1} \in M^\times$ and on the m-th factor, t acts through the multiplication by $\nu a_m a_m^c \in K^\times$). More generally, writing $\mathfrak{N}_j = [\mathfrak{N}_{j-1},\mathfrak{N}_{j-1}]$ starting with $\mathfrak{N}_1 = [\mathfrak{N},\mathfrak{N}]$, $\mathfrak{N}_j/\mathfrak{N}_{j+1}$ is a $T(K)$-module under the conjugation action. We go in the same way as in the case of $GL(n)$: modifying σ by an inner automorphism of an element of T, we may assume that σ fixes $\mathbf{1} = (1,\dots,1) \in M^m$ if $n = 2m+1$ and that if $n = 2m$, σ fixes $\mathbf{1} = (1,\dots,1,1) \in M^{m-1} \oplus K$. Once σ is normalized in this way, we see that σ on B/U and σ on $\mathfrak{N}/[\mathfrak{N},\mathfrak{N}]$ are the action of an element of $\mathrm{Aut}(M)$ coordinate-wise. The action of $\mathrm{Aut}(M)$ is faithful if $n \geq 3$ because we have a factor M in $\mathfrak{N}/[\mathfrak{N},\mathfrak{N}]$. We modify σ therefore by an element of $\mathrm{Aut}(M)$; then, σ is $T(K)$-linear on $\mathfrak{N}/[\mathfrak{N},\mathfrak{N}]$. Once this is established, we verify, using commutator relations, that σ commutes with the conjugation action of T on $\mathfrak{N}_j/\mathfrak{N}_{j+1}$ for all j. Then modifying again by conjugation of an element of $U(K)$, we conclude that σ is the identity on B, and the rest is the same as the proof in the case of $GL(n)$ and $GSp(2n)$.

If p is odd and unramified in $M/\mathbb{Q}_p$, the nilpotent Lie algebra $\mathfrak{N}_{/O}$ is the direct sum of its root spaces as $T(O)$-modules. Then the above argument done over the field K can be checked word-by-word over O, and we get the same assertion for $A = O$. $\square$

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