Generic Observability of Dynamical Systems

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Abstract. We deal with a certain observation mapping defined by means of weighted measurments on a dynamical system and give necessary and sufficient conditions, under which this mapping is generically an injective immersion.

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1 Introduction

The observability problem of nonlinear dynamical systems has been an interesting subject and active field of research throughout the last decades. In the present work we consider time invariant systems of the form*

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x). \end{aligned}$$

The first equation describes a real dynamic process. Its state $x(t)$ at time t is assumed to be element of a smooth second countable (hence paracompact) n-dimensional manifold M called state space. The dynamics of the system is given by the vector field f on M. The output function h is a mapping from the state space into the reals and stands for a measuring device. The second equation describes the output, which contains partial information of the state. The output y is the only measurable quantity. There is a very broad variety of systems, which can be described in this way. We call the triple (M, f, h) or simply the pair (f, h) a system. The system is called

*We use the customary abbreviations: Time t denotes the natural coordinate on $\mathbb{R}$ and the first equation is identified with its local representative.

C^r iff M, f and h are C^r. We denote the set of C^r vector fields on M by $\mathbb{X}^r(M)$.

In many applications it is of essential importance to know the state of the system at any time t. But measurement of the entire state (e.g. all its coordinates) is often impossible or very difficult. For instance because of high costs or technical reasons. In most cases one has only partial information from measurement of the output y, mathematically being described by the second equation. So the issue is to get the state (as well as distinguish different states) by using only output measurement*. If this is possible, then the system is said to be observable. Hence, the issue of the observability problem is the following. Find criteria on the system, such that by means of information from the output trajectory, it is possible to distinguish different states as well as to reconstruct the states. There is no uniform or canonical definition of observability in the literature. The weakest and most natural definition is the following.

DEFINITION 1.1 The system (f, h) is said to be observable (or distinguishable) if for each $(x, x') \in M \times M$ with $x \neq x'$ there exists a time $t_0 = t_0(x, x')$ such that $h(\Phi_{t_0}(x)) \neq h(\Phi_{t_0}(x'))$.

However, the above definition is not well-suited for the treatment of the observability problem. Therefore one seeks to establish a stonger notion of observability as follows. Consider a mapping Θ (in the sequel called observation mapping), which maps the state space into some finite dimensional Euclidian space, assigning states to data derived from the output trajectories on some observation time interval J. Then decide the observability of the system by means of injectivity of Θ. Moreover the following natural question arises. Is observability in this sense (, i.e. with respect to Θ) generic? The latter is the main issue of the present work. In the control theory the noatation of observer is generally standing for another system having the output (and input in the controlled case) of the original system as input and generating an output which is an asymptotic estimate of the original system state.

Beside the task of reconstruction of the state, observability has also application in the theory of chaos and turbulence in the following sense. Suppose an observable system has a global attractor. Then using an observation mapping one can get a homeomorphic picture of the attractor or at least information about some of its characteristic properties. Examples can be found in [RT] and [T]. We consider a certain observation mapping introduced in [KE] and [E]. We derive necessary and sufficient conditions for genericity of observability and local observability with respect to this mapping. Basically, there are two other well-knwon approaches to the observability task: sampling and high-gain approach. In his classical work [T], Takens proved genericity results similar to

*It is worth to mention that often the information by output measurements underly some errors leading to the problem of stabiliy.

ours for these approaches. About the same time Aeyels [Ay] achieved sharper results concerning the sampling approach. Further results on high-gain approach can be found in [GK], [GHK] and [J]. Particularly, for a comrehensive and intensive investigation of deterministic observation theory and applications including many deep results concerning high-gain approach we refer to [GK]. Our approach has the disadvantage that a suitable linear filter has to be constructed. On the other side in contrast to high-gain approach we do not need to restrict ourselves to smooth systems. For more on our approach with applications we refer to [N].

2 Main Results

Definition 2.1 Let $0 < \tau < \infty$, (A, b) be a stable and controllable m-dimensional linear filter and (M, f, h) a C^r-system with flow Φ. We call the mapping $M \ni x \mapsto \Theta_{f,h} := \int_{-\tau}^{0} e^{-At} b h(\Phi_t(x)) dt \in \mathbb{R}^m$ the observation mapping and the number m observation dimension. If $\Theta_{f,h}$ is injective we call the system (M, f, h) Θ-observable or simply observable.
Furthermore we call the system locally observable if $\Theta_{f,h}$ is an immersion.
The indistinguishable subset of $M \times M$ is given by

$$\Omega_{f,h} := \{(x, x') \in M \times M \setminus \Delta_M : \Theta_{f,h}(x) = \Theta_{f,h}(x')\}.$$

The time interval $I := [-\tau, 0]$ has the physical interpretation of observation interval. We treat for simplicity of notation and in view of the physical interpretation of the observation data as history of the output mapping, the case $I \subset \mathbb{R}_{\leq 0}$, $0 \in I$. For unbounded intervals modifications are needed, which we give explicitly for the case $I = \mathbb{R}_{\leq 0}$.

The key point in the proof of the following genericity results is to show that zero is a regular value of the observation mapping. Proving this we then apply transversality density and openness theorems to get locally our statements, which then will be globalized to the whole state space. A Baire argument then yields the final results. Particularly for $\tau < \infty$ in any C^r neighborhood of the system there is a system which is both observable and locally observable with respect to Θ. An appropriate statement is also valid for $\tau = \infty$ in the C^1 topology.

If the state space is not compact, it is more suitable to consider the so called strong or Whitney (C^r-)topology on $C^r(M, \mathbb{R})$ and $\mathbb{X}^r(M)$. The reason is that in this topology one has more control on the behavior of the functions and vector fields at infinity. Note that density in this topology is a stronger property than in the compact-open (also called weak) topology. For instance, roughly speaking, a sequence of output functions h_j converges in Whitney topology to h iff there exists a compact set K such that $h_j = h$ outside of K except for finitely many j and all the derivatives up to order k converge uniformly on K.

The case of smooth vector fields is similar. The well known fact that $C^r(M,\mathbb{R})$ and $\mathbb{X}^r(M)$ are Baire spaces in the Whitney topology (proofs can be found in in [H, 2.4.4] and [P]) is of basic importance for our results. A residual subset of $C^r(M,\mathbb{R})$ or $\mathbb{X}^r(M)$ is dense. *From now on the spaces of C^r functions as well as vector fields on M are equipped with their C^r Whitney topology, unless otherwise indicated.* If the state space M is compact and $r<\infty$, then $C^r(M,\mathbb{R}^k)$ and $\mathbb{X}^r(M)$ endowed with the compact-open topology are Banach spaces (while in general Fréchet spaces), their Whitney and compact-open topology coincide and the flow Φ of each vector field $f \in \mathbb{X}^r(M)$ is defined globally on $M\times\mathbb{R}$.

For some pairs in $\mathbb{X}^r(M) \times C^r(M,\mathbb{R})$ there is no possibility to distinguish or locally distinguish the states. For genericity results it is an essential fact that the complement of the set of such pairs is residual. Let $Sing(f) := \{x \in M : f(x) = 0_x \in T_xM\}$ denote the set of singularities of the vector field f (equilibria of the system), where 0_x is the zero of T_xM. In the sequel we shall often omit the subscript x and write simply $f(x) = 0$ for $x \in Sing(f)$.
Recall that a singularity $x_0 \in Sing(f)$ is called simple iff the principial part of the linearization of f at x_0, i.e. the linear mapping $d_{x_0}f : T_{x_0}M \to T_{x_0}M$, does not have zero as an eigenvalue.
We denote the set of C^r vector fields, whose singularities are all simple, by $\mathbb{X}^r_0(M)$. It is well known that a simple singularity is isolated and $\mathbb{X}^r_0(M)$ is an open and dense subset of $\mathbb{X}^r(M)$ (for a proof we refer to [PD, 3.3]).

DEFINITION 2.2 We call $x_0 \in M$ a Θ-simple singularity iff there exists a cotangent vector $v \in T^*_{x_0}M$ such that the linear system $(T_{x_0}M, d_{x_0}f, v^T)$ is observable, i.e., the linear mapping

$$\int_{-\tau}^{0} e^{-tA} b v e^{t d_{x_0} f} dt : T_{x_0}M \to \mathbb{R}^m$$

is injective. In this case we say that v^T is a Θ-cocyclic covector of $d_{x_0}f$.
We denote by $\mathbb{X}^r_1(M)$ the set of C^r vector fields on M, whose singularities are Θ-simple. Moreover we set $\mathbb{X}^r_{0,1}(M) := \mathbb{X}^r_0(M) \cap \mathbb{X}^r_1(M)$.

In the limiting case $\tau = \infty$ dense orbits as well as nontrivial recurrence cause difficulties and special considerations are necessary. We investigate this case under the assumption that M is compact. Appropriate results in the noncompact case can be similarly deriven if we further restrict (in order to achieve well-definedness of the observation mapping) the systems to be globally Lipschitzian. Furthermore, in order to ensure differentiability of the observation mapping, if $\tau = \infty$ we restrict the vector fields to the open set

$$\mathbb{X}^r(M,a) := \{f \in \mathbb{X}^r(M) : \sup_{x\in M} \|d^j_x f\| < a \text{ for all } j = 1, ..., r\}$$

Denoting the set of critical elements (equilibria and closed orbits) of a vector field f by $\mathcal{C}(f)$ and the union of the negative limit sets by $\mathcal{L}_-(f)$, we set

$$\mathbb{X}^r_-(M) := \{f \in \mathbb{X}^r(M) : \mathcal{L}_-(f) \subset \mathcal{C}(f)\}.$$

We denote that some ineteresting classes of vector fields like Morse-Smale fields* are contained in $\mathbb{X}^r_-(M)$. Particularly, since the set consisting of Morse-Smale vector fields is open and nonempty in $X^r(M)$, the interior $int(\mathbb{X}^r_-(M))$ of $\mathbb{X}^r_-(M)$ is a Baire space (in the induced topology). Furthermore note that the limit set of a gradient field consists only of the critical points of the potential function. Therefore $\mathbb{X}^r_-(M)$ also contains the set of gradient fields. Moreover we set

$$\mathbb{X}^r_2(M) := \mathbb{X}^r(M,a) \cap int(\mathbb{X}^r_-(M)).$$

It is well known that on a compact manifold C^1-generically the nonwandering set of a smooth vector field coincides with the closure of the set of its periodic points. This statement called general density theorem is a consequence of Pugh's closing lemma, which ensures that a nonwandering point can be made periodic by a small C^1-petrubation in a neighbourhood of the point. See also [Pu], [AR, 7.3.6] and the references given there*. Particularly $\{f \in \mathbb{X}^1(M) : \mathcal{L}_-(f) \subset \overline{\mathcal{C}(f)}\}$ is a residual subset of $\mathbb{X}^1(M)$.

We set for $y \in E^r_{a,\tau} := \{y \in C^r([-\tau,0],\mathbb{R}) : \int_{-\tau}^0 e^{at}|y(t)|dt < \infty\}$

$$P_\tau y := \int_{-\tau}^0 e^{-At} by(t) dt$$

and for fixed τ simply write P instead of P_τ.

Lemma 2.1 Let $r, \tau \leq \infty$.Furthermore let $q_0 \in \mathbb{R}^m$ and $T, \delta > 0$. Then the followings hold.
a) There exists a function $y \in C^r(\mathbb{R},\mathbb{R})$ being compactly supported in $]-\delta, 0[$ and satisfying $P_\tau y = q_0$.
b) There exists a T-periodic function $y \in C^r(\mathbb{R},\mathbb{R})$ with $P_\tau y = q_0$ and $supp(y|_{[-(k+1)T,0]}) \subset]-(k+1)\delta, 0[$ for all $k \in \mathbb{Z}$.

Proof: Ad a) Let $\epsilon := min\{\delta, \tau\}$. The mapping $L^1([-\epsilon, 0]) \ni y \mapsto K(y) := \int_{-\epsilon}^0 e^{-At} by(t)dt \in \mathbb{R}^m$ is linear, continuous and because of the controllability of (A,b) surjective. C^r_ϵ is a dense linear subspace of $L^1([-\epsilon,0])$. Therefore $R := K(C^r_\epsilon)$ is a dense linear subspace of $\mathbb{R}^m$ and consequently $R = \mathbb{R}^m$. Therefore there exists a function $y_0 \in C^r_\epsilon$ having the property $K(y_0) = q_0$. The trivial extension of y_0 on $\mathbb{R}$ is obviously the desired function.
Ad b) If $\tau \leq T$, the assertion follows directly from part a). We prove the result for $\tau > T$ using sampling. Assume first $\tau < \infty$ and let $N := max\{k \in \mathbb{N} : NT \leq \tau\}$. Due to the stability of A the series $\sum_{k=0}^\infty e^{kTA}$

*Recall that Morse-Smale vector fields are structurally stable.
*It is still unknown whether the C^k-closing lemma with $k \geq 2$ fails in general.

converges to $(I-e^{TA})^{-1}$ and $S_N := \sum_{k=0}^{N} e^{kTA} = (I-e^{TA})^{-1}(I-e^{(N+1)TA})$ is invertible. Let us write $\epsilon := min\{\tau - T, \delta\}$. According to part a) there exists $\tilde{y}_0 \in C^r_\epsilon$ with $P_T\tilde{y}_0 = S_N^{-1}q_0$ (as well as $(I-e^{TA})q_0$ in case $\tau = \infty$). Let y_0 denote the trivial extension of $\tilde{y}_0$ on $[-T,0]$. Then we have $P_Ty_0 = S_N^{-1}q_0$. Let y denote the T-periodic extension of y_0 on $\mathbb{R}$. Consequently

$$\begin{aligned}\int_{-\tau}^{0} e^{-At}by(t)dt &= \sum_{k=0}^{N}\int_{-(1+k)T}^{-kT} e^{-At}by_0(t+kT)dt\\ &= \sum_{k=0}^{N} e^{kTA}\int_{-T}^{0} e^{-At}by_0(t)dt\\ &= q_0,\end{aligned}$$

which yields immediately the desired conclusion. □

Let π denote the canonical projection of the tangent bundle TM:

$$\pi : TM \to M,\ \pi(v) := x \text{ for } v \in T_xM.$$

Let M be endowed with a Riemannian metric. Denoting the induced norm on the tangent spaces by $|.|$, the unit tangent bundle T_1M is given by

$$T_1M := \bigcup_{x\in M}\{v \in T_xM : |v| = 1\}.$$

Recall that T_1M is a $(2n-1)$-dimensional C^{r-1} submanifold of TM. It is compact, if M is compact.
Let K be a subset of M. We set $T_0K := K \times K \setminus \Delta_K$ and denote the restriction of T_1M to K with T_1K, i.e., $T_1K := \{v \in T_1M : \pi(v) \in K\}$. Recall that if K is an s-dimensional submanifold, then T_1K has dimension $n+s-1$.

Definition 2.3 We define the τ-history of K by the flow Φ of the vector field f to be the closure of

$$\Phi(K;\tau) := \{\Phi_t(x) : -\tau < t \le 0, x \in K\}.$$

We denote

$$\Delta\Theta_{f,h}(x,x') := \Theta_{f,h}(x) - \Theta_{f,h}(x') \text{ for } x,x' \in M.$$

Let V be an open subset of M containing the τ-history of K and L be the closure of V. Then we denote

$$\mathcal{H}_0(L;K) := \{h \in C^r(L,\mathbb{R}) : \text{ zero is a regular value of } \Delta\Theta_{f,h}|\Lambda_0K\},$$

and

$$\mathcal{H}_1(L;K) := \{h \in C^{r+1}(L,\mathbb{R}) : \text{ zero is a regular value of } d\Theta_{f,h}|\Lambda_1K\}.$$

In the sequel we set $i = 0, 1$ as well as $\mathcal{H}_i(K) := \mathcal{H}_i(M; K)$ and $\mathcal{H}_i := \mathcal{H}_i(M)$. If $K' \subset K$ and for an output function h, zero is a regular value of the mapping $\Delta\Theta_{f,h}$ on $\Lambda_0 K$, then it is also a regular value of the restricted mapping $\Delta\Theta_{f,h}|_{\Lambda_0 K'}$, that is, $\mathcal{H}_0(K) \subset \mathcal{H}_0(K')$. Similarly $\mathcal{H}_1(K) \subset \mathcal{H}_1(K')$ holds.

In the following, if $f \in \mathbb{X}^r_{0,1}(M)$, then $\mathbb{H}_0(L)$ stands for the set of output functions $h \in C^r(L, \mathbb{R})$ such that

$$h(x_0) \neq h(x_0') \text{ for all } (x_0, x_0') \in \Lambda_0(L \cap Sing(f)),$$

and $\mathbb{H}_1(L)$ for those $h \in C^{r+1}(L, \mathbb{R})$ such that

$$d_{x_0}h \text{ is a } \Theta\text{-cocyclic covector of } d_{x_0}f \text{ for all } x_0 \in L \cap Sing(f).$$

Note that if L is compact, then the number of singularities of f in L is finite and for finite r, as an immediate application of the transversality openness and density theorems, it follows that $\mathbb{H}_i(L)$ is an open and dense subset of the Banach space $C^{r+i}(L, \mathbb{R})$.

Lemma 2.2 Assume that $i < r < \infty$ and $f \in \mathbb{X}^r_{0,1}(M)$ is complete. Let S be a C^r submanifold of M such that the τ-history of S is contained in an open subset V of M with compact closure $L := \bar{V}$. Consider the mappings $F^i : \mathbb{H}_i(L) \times \Lambda_i S \to \mathbb{R}^m$ defined by

$$F^0(h, x, x') := \Delta\Theta_{f,h}(x, x')$$

and

$$F^1(h, v) := d_{\pi(v)}\Theta_{f,h}(v).$$

Then the following holds.
a) Zero is a regular value of F^0.
b) Zero is a regular value of F^1.
Suppose moreover that $f \in X^r_2(M)$. Then the assertions also hold for $\tau = \infty$.

Proof: Ad a) Let $W_0 := \{(h, x, x') \in \mathbb{H}_0(L) \times T_0 S : F^0(h, x, x') = 0\}$. We have to show that the function F^0 is submersive on W_0. Since $\mathbb{R}^m$ is finite dimensional, it sufficies to prove that the linear mapping $d_{(h,x,x')}F^0 : T_{(h,x,x')}(\mathbb{H}_0(L) \times T_0 S) \to \mathbb{R}^m$ is surjective for all $(h, x, x') \in W_0$. Fix $(h, x, x') \in W_0$ and $q_0 \in \mathbb{R}^m$. According to the condition $h \in \mathbb{H}_0(L)$ we see that x and x' cannot be both equilibrium points. Therefore we assume without loss of generality that x is not an equilibrium point. Since $\mathbb{H}_0(L)$ is open and $\frac{d}{ds}|_{s=0}F^0(h + sg, x, x') = \Delta\Theta_{f,g}(x, x') = F^0(g, x, x')$, it is sufficient to show the existence of an output function $g \in C^r(L, \mathbb{R})$ satisfying $F^0(g, x, x') = q_0$. We use the fact that the flow through a point of the state space M maps each closed finite time interval on a closed subset of M and define a suitable mapping g on an appropriate closed subset of the state space and then extend it to L.

Let γ and γ' denote the τ-histories of the points x and x' respectively and $Z := \gamma \cup \gamma'$. We define g on Z. We first treat the case $\tau < \infty$.
Case 1: Both orbits are critical elements. Let T denote the period of x. In view of lemma 2.1 there exists a T-periodic function $y \in C^k(\mathbb{R}, \mathbb{R})$, which satisfies the condition $P_\tau y = q_0$. We set $g(\Phi_{t+kT}(x)) = y(t)$ for $0 \leq t \leq T, k \in \mathbb{Z}$ and $g = 0$ else.
Case 2: One of the integral curves, say $\Phi(x)$, is injective and the other one is periodic or an equilibrium point. According to Lemma 2.1 there is a function $y \in C^k([-\tau, 0], \mathbb{R})$ with compact support such that $P_\tau y = q_0$. We define g on γ by $g(\Phi_t(x)) = y(t)$ for $-\tau \leq t \leq 0$ and $g = 0$ else. If $\Phi(x')$ is injective and x is periodic, we just set $g = 0$ on γ and define g on γ' such that $P_\tau(g \circ \Phi(x'))) = -q_0$.
Case 3: Both integral curves are injective. In this case we define g on γ as in case 2 and on γ' by $g = 0$.
If $\tau = \infty$, then because of eventual presence of dense orbits g cannot be simply defined on a part of Z and then trivially extended. Hence the construction of g becomes a little more delicate. Assuming $f \in X_2$ ensures then that the recurrence is trivial and the previous procedure also works. If both orbits are critical elements, i.e. in case 1, then everything remains the same as for finite observation time. Problems could arise in case 2 or 3 if at least one of the integral curves is injective and the past half of one of the orbits, say the one through x', belongs to the negative limit set of the other orbit or itself, i.e., $\{\Phi_t(x') : t \leq 0\} \subset \alpha(x) \cup \alpha(x')$. But this can according to the assumption $f \in \mathbb{X}_2^r(M)$ only occur if x' is periodic or an equilibrium point. We proceed as in case 2 of finite τ and find again in view of Lemma 2.1 a function $y \in C^r(\mathbb{R}, \mathbb{R})$ compactly supported in an interval $[-\epsilon, 0]$ with $\epsilon > 0$ and set $g(\Phi_t(x)) := y(t)$ for all t and $g = 0$ else.
In all cases we have defined a C^r function on the closed subset Z of the state space M with the property that $Z \subset L$ and $P_\tau(g \circ \Phi(x) - g \circ \Phi(x')) = q_0$. According to the smooth Tietze extension theorem, there exists a C^r extension of the function g to L. This function denoted again by g is obviously the desired function, which satisfies $F^0(g, x, x') = q_0$.

Ad b) Denote $W_1 := \{(h, v) \in \mathbb{H}_1(L) \times T_1S : F^1(h, v) = 0\}$. We fix $(h, v) \in W_1$, set $x_0 := \pi(v)$ and show that F^1 is submersive at (h, v). Fix $q_0 \in \mathbb{R}^m$. Since $\frac{d}{ds}|_{s=0} F^1(h + sg, v) = F^1(g, v)$ for arbitrary $g \in C^r(M, \mathbb{R})$, it suffices to prove the existence of a function $g \in C^r(L, \mathbb{R})$ with $F^1(g, v) = q_0$ locally and extend it L. If x_0 would be an equilibrium point, then in view of the assumption $h \in \mathbb{H}_1(L)$, the linear system $(T_{x_0}M, d_{x_0}f, d_{x_0}h)$ would be Θ-observable and consequently $F^1(h, v) \neq 0$ in contradiction to the assumption $(h, v) \in W_1$. Therefore we may assume that x_0 is not an equilibrium point. Hence, in view of the straightening-out theorem there is a local chart (U, ψ) at x_0 such that $\psi(U) = U' \times] - \epsilon, -\epsilon[$ with $\epsilon > 0$, U' an open subset of $\mathbb{R}^{n-1}$, $\psi(x_0) = 0$ and the vector field f has the local representative $(z, t) \mapsto e_n$. Here e_n denotes the nth standard base vector in $\mathbb{R}^n$. Denote the induced coordinate function on T_1M at v by $\widehat{\psi}$. Since $v \neq 0$ we can and do assume that v has the local representative $\eta = (\eta_1, \eta_2)^T$ with $\eta_1 \in \mathbb{R}^{n-1}$, $\eta_2 \in \mathbb{R}$, $\eta \neq 0$, i.e. $v = \frac{\partial}{\partial z}(x_0)\eta_1 + \frac{\partial}{\partial t}(x_0)\eta_2$. Furthermore for $t \in]-\epsilon, -\epsilon[$, the local representative of $d_{x_0}\Phi_t$ reads as $\begin{bmatrix} Id & 0 \\ 0 & 1 \end{bmatrix}$ and subsequently that of $d_{x_0}h \circ \Phi_t v$ as $\nabla h(0,t)\eta$.

By shrinking U if necessary, we can (in the case $\tau = \infty$ on account of the assumption that $f \in \mathbb{X}_2^r(M)$ if the point x_0 is recurrent, then it is periodic) and do assume that the intersection of U and the τ-history of x_0 is connected. According to lemma 2.1 there exists a function $\hat{y} \in C^{r+1}(\mathbb{R}, \mathbb{R})$ with derivative y being supported (on each period, if x_0 has period $> \tau$) in $[-\epsilon, 0]$ such that $P_\epsilon y(t)dt = q_0$. Obviously there exists a function $g \in C^r(U' \times]-\epsilon, \epsilon[, \mathbb{R})$ being compactly supported, with $\nabla g(0,t)\eta = y(t)$. For instance define $\tilde{g}(z,t) = |\eta|^{-2}(z^T\eta_1 y(t) + \eta_2\hat{y}(t))$. The trivial extension of $\tilde{g} \circ \psi$ to L is the desired function g. □

Sometimes in applications one is interested in or limited to observation restricted to a subset of the state space. It can be for instance because of technical or physical reasons, or if it happens that all the information needed can be evaluated from measurements on a certain subset. The latter case being perhaps the most important one, occurs if the subset under observation is an attractor. Other subsets invariant under the flow can also be of interest. Therefore we state our genericity results for observations of subsets of the state space as well.

Lemma 2.3 Suppose that K is a subset of an s-dimensional C^r submanifold of M denoted by S, $\tau < \infty$ and $f \in \mathbb{X}_{0,1}^r(M)$ is complete. Then the following holds.
a) Assume that $m \geq n + s - r$ and $r \geq 2$. Then $\mathcal{H}_1(K)$ is residual. If K is closed, then $\mathcal{H}_1(K)$ is also open.
b) Assume that $m \geq n + s + 1 - r$. Then $\mathcal{H}_0(K)$ is residual. If K is closed, then $\mathcal{H}_0(K)$ contains an open set.
Suppose moreover that M is compact and $f \in \mathbb{X}_2^r(M)$. Then the assertions hold also in the case $\tau = \infty$.

Proof: Assume first $r < \infty$, K is compact, U is a chart domain of S, which contains K and has compact closure. By compactness of $\overline{U}$ and finiteness of τ in case of finite observation time and because of compactness of M in case $\tau = \infty$, the τ-history of U is compact. By local compactness there is an open set $V \subset M$ with compact closure $L := \overline{V}$ such that V contains the τ-history of U.

Local density: We prove residuality with respect to $\mathbb{H}_i(L)$. Since the latter is open and dense in $C^r(L, \mathbb{R})$, density of $\mathcal{H}_i(L; U)$ is also then shown with respect to $C^r(L, \mathbb{R})$.
According to the previous lemma zero is a regular value of the evaluation map-

ping $F^i : \mathbb{H}_i(L) \times T_iU \to \mathbb{R}^m$ defined by

$$F^0(h,x,x') := \Delta\Theta_{f,h}(x,x')$$

and

$$F^1(h,v) := d_{\pi(v)}\Theta_{f,h}(v).$$

Therefore according to the transversality density theorem (see for instance [AR, 19.1]) $\mathcal{H}_i(L;U)$ is a residual subset of $\mathbb{H}_i(L)$, and hence dense in $C^r(L,\mathbb{R})$.
Local Openness: Compactness of K implies that T_1K is also compact in T_1U. According to the transversality openness theorem $\mathcal{H}_1(L;K)$ is open (with respect to the compact-open and by compactness of L also with respect to the Whitney topology) in $C^r(L,\mathbb{R})$. These conclusions do not work for $\mathcal{H}_0(L;K)$. Instead let Λ be a compact subset of T_0U. Then in view of transversality openness theorem

$$H'_0(L;\Lambda) := \{h \in C^r(L,\mathbb{R}) : \text{ zero is a regular value of } \Delta\Theta_{f,h} \text{ on } \Lambda\}$$

is open in $C^r(L,\mathbb{R})$.
Since the assertions are proved for r finite and sufficiently large, they also hold for $r=\infty$. Hence we assume from now on that $r \leq \infty$.
Globalization: This part of the proof is basicly standard. Therefore we give an outline and refer to [H, 2.2] for details. Since $K \subset U$, we have $\mathcal{H}_1(L;U) \subset \mathcal{H}_1(L;K)$. Hence $\mathcal{H}_1(L;K)$ is also dense in $C^r(L,\mathbb{R})$. Likewise one gets density of $\mathcal{H}'_0(L;\Lambda)$ in $C^r(L,\mathbb{R})$ from density of $\mathcal{H}_0(L;U)$, since $\Lambda \subset T_0U$ and subsequently $\mathcal{H}_0(L;U) \subset \mathcal{H}'_0(L;\Lambda)$. Using a bump function we now prove that $\mathcal{H}_1(K)$ is dense in $C^r(M,\mathbb{R})$. Fix $g_0 \in C^r(M,\mathbb{R})$. Since $\mathcal{H}_1(L;K)$ is dense in $C^r(L,\mathbb{R})$, there is a sequence $\{h_j\}$ in $\mathcal{H}_1(L;K)$ converging in the compact-open topology to $g_0|_L$. Since $K \subset U$, there is a C^r-function $\rho : M \to [0,1]$ with compact support in L, such that $\rho = 1$ on an open neighborhood of K. The sequence $\rho h_j + (1-\rho)g_0$ converges to g_0 with respect to the Whitney topology. Therefore $\mathcal{H}_1(K)$ is a dense subset of $C^r(M,\mathbb{R})$. A similar argument shows that $\mathcal{H}'_0(\Lambda)$ is a dense subset of $C^r(M,\mathbb{R})$.
We now drop the assumption that $K \subset U$. Let J be a countable indexing set, $\{U_j\}$ with $j \in J$ be a covering of S with chart domains U_j. Furthermore let $\{K_j\}$ be a subordinate family of compact sets such that $K = \bigcup_J K_j$ and $K_j \subset U_j$. We can and do assume that there is a compact covering of M denoted by $\{L_j\}$ such that the interior of L_j contains the τ-history of U_j.
Openness statements: Suppose that K is closed. Then it is also paracompact and the covering can be assumed to be locally finite. Since $\{L_j\}$ covers M, it holds that $\mathcal{H}_1(K) = \{h \in C^r(M,\mathbb{R}) : h|_{L_j} \in \mathcal{H}_1(L_j;K_j) \text{ for all} j \in J\}$. Hence local finiteness of the covering implies that $\mathcal{H}_1(K)$ is open. Simliarily it follows that the set $\{h \in C^r(M,\mathbb{R}) : \text{ zero is a regular value of } \Delta\Theta_{f,h} \text{ on } K \times K\}$ is open. The latter is contained in $\mathcal{H}_0(K)$.
Residuality statements: We now drop the assumption that K is closed. By the preceding arguments $\mathcal{H}_1(K_j)$ is open and dense. Therefore $\mathcal{H}_1(K) = \bigcap_j \mathcal{H}_1(K_j)$

is residual (and in view of the Baire property of $C^r(M,\mathbb{R})$ also dense). Taking a compact covering $\{\Lambda_j\}$ of T_0K a similar argument yields the results on $H_0(K)$. $\square$

The openness results in the preceding lemma can also be proved (without applying the transversality openness theorem) as follows. For instance, by compactness of L the mapping $\mathbb{H}_0(U) \ni h \mapsto F^0(h,.)) \in C^r(T_0U,\mathbb{R}^m)$ is continuous in the Whitney topology. This fact and openness of $\{F \in C^r(T_0U,\mathbb{R}^m) :$ zero is a regular value of $F\}$ (this fact follows, for instance, from the lower semicontinuity of the mapping $C^r(L,\mathbb{R}) \times T_0U \ni (h,x,x') \mapsto rank(d_{(h,x,x')}F^0)$ and compactness of L) in the Whitney topology implies that $\mathcal{H}_0(L;U)$ is open in the Whitney topology and by compactness of L also open in the compact-open topology of $C^r(L,\mathbb{R})$. Other parts follow likewise.

In light of the preceding lemmas we can now prove the genericity results on output functions just by comparing dimensions.

Theorem 2.1 Suppose that K is contained in an s-dimensional C^r submanifold of M and $f \in \mathbb{X}^r_{0,1}(M)$ is complete. Then the following assertions hold.
a) Assume that $r \geq 2$ and $m \geq n+s$. Then functions h belonging to $C^r(M,\mathbb{R})$ such that $\Theta_{f,h}$ is immersive at each point of K, constitute a residual set. This set is also open, if K is closed.
b) Assume that $m \geq 2s+1$. Then the set of functions h belonging to $C^r(M,\mathbb{R})$ such that $\Theta_{f,h}$ is injective (respectively an injective immersion) on K, is residual (respectively, if $r \geq 2$). It contains an open set, if K is closed.
Supposing $m \leq 2s$ and $r > 2s-m$ the same results hold for the set of functions h belonging to $C^r(M,\mathbb{R})$ such that the Θ-unobservable points of $K \times K$ belong to a submanifold of dimension $2s-m$.
Suppose moreover that M is compact and $f \in \mathbb{X}^r_2(M)$. Then the assertions also hold for $\tau = \infty$.

Proof: Ad a) Since $m \geq n+s$, the set of functions h belonging to $C^r(M,\mathbb{R})$ such that $\Theta_{f,h}$ is an immersion at each point of K, coincides with $\mathcal{H}_1(K)$, i.e., $d\Theta_{f,h}|_{T_1K}$ is transversal to $\{0\} \in \mathbb{R}^m$. Therefore the assertion follows immediately from the previous lemma.
Ad b) According to the previous lemma $\mathcal{H}_0(K)$ is residual and contains an open set, if K is closed. If $m \geq 2s+1$, then the set of functions h belonging to $C^r(M,\mathbb{R})$ such that the restriction of $\Theta_{f,h}$ to K is injective (respectively an injective immersion), is precisely $\mathcal{H}_0(K)$ (respectively $\mathcal{H}_0(K) \cap \mathcal{H}_1(K)$). The statement on the indistinguishable set follows from the preimage theorem. $\square$

Remark 2.2 As it can easily be seen from the proof of lemma 2.2, the dimension condition $m \geq n+s$ in part a) of the preceding theorem can be weakend to $m \geq 2s$, if immersivity is replaced by immersivity on TS.

If the state space is compact, we get sharper results. In particular the following theorem is important, since embedding of the state space gives information on limit behavior of the system.

THEOREM 2.2 Suppose that $m \geq 2n+1$, $f \in \mathbb{X}^r_{0,1}(M)$ and M is compact. Then output functions $h \in C^r(M, \mathbb{R})$ such that $\Theta_{f,h}$ is an embedding constitute an open and dense set.
If we further assume that $f \in \mathbb{X}^r_2(M)$, then the assertion remains true in the case $\tau = \infty$.

PROOF: Recall that by compactness of M an injective immersion is also an embedding (case $2 \leq r \leq \infty$ in Theorem 2.1) and an injective mapping is also a topological embedding (case $r = 1$). Hence density follows from Theorem 2.1. Openness is a consequence of the fact that by compactness of M the mapping $C^r(M, \mathbb{R}) \times T_0M \ni h \mapsto \Delta\Theta_{f,h} \in C^r(M, \mathbb{R}^m)$ is continuous and the set of embeddings $Emb^r(M, \mathbb{R}^m) := \{F \in C^r(M, \mathbb{R}^m) : F \text{ is embedding}\}$ is open. □

Next we shall prove residuality of $\mathbb{X}^r_{0,1}(M)$ by using the characterization of simplicity and Θ-simplicity of a singularity, in terms of transversal nonintersection.

LEMMA 2.4 Assume that $m \geq n$. Then $\mathbb{X}^r_{0,1}(M)$ is open and dense in $\mathbb{X}^r(M)$. Moreover, the assertion holds also for $\tau = \infty$, if we restrict the vector fields to $X^r(M, a)$.

PROOF: We give a proof for $\tau < \infty$. The arguments for $\tau = \infty$ are similar. It sufficies to show that $\mathbb{X}^r_1(M)$ is open and dense. Let $\mathcal{O}$ resp. $\mathcal{U}$ denote the set of n-dimensional Θ-observable resp. unobservable linear systems. Let $f \in \mathbb{X}^r_1(M)$, $x_0 \in Sing(f)$ and $d_{x_0}f$ denote the principial part of the linearization of f.
Note that $f \in \mathbb{X}^r_1(M)$ if and only if there exists a $v \in T_{x_0}M$ such that the system $(d_{x_0}f, v)$ avoids the set of Θ-unobservable linear systems on $T_{x_0}M$.
We now give a local characterization. Let $\psi : U \to \mathbb{R}^n$ be a local chart such that U has compact closure and $\psi(x_0) = 0$. The tangent mapping $d\psi : TU \to \mathbb{R}^n \times \mathbb{R}^n$ defined by $d\psi(v) = (\psi(x), d_x\psi v)$ with $x = \pi(v)$ is a local chart for TM. Let Pr_2 denote the projection $\mathbb{R}^n \times \mathbb{R}^n \ni (x, w) \mapsto w$. Consider the mapping $\xi_f := Pr_2 d\psi f \circ \psi^{-1}$ on $\psi(U)$. Hence $d_0\xi_f = d_{x_0}\psi d_{x_0} f d_0\psi^{-1}$. The singularity x_0 is Θ-simple if and only if $(d_0\xi_f, d_{x_0}v) \notin \mathcal{U}$ for some $v \in T_{x_0}M$. Obviously $\mathcal{U}$ is closed, analytic and $\neq End(\mathbb{R}^n) \times \mathbb{R}^n$, hence finite union of closed positive codimensional real analytic submanifold of $End(\mathbb{R}^n) \times \mathbb{R}^n$. Given a pair $(G, w) \in \mathcal{O}$, obviously there exists a vector field $g \in \mathbb{X}^r(M)$ such that g and f coincide on $M \setminus U$, $x_0 \in Sing(g)$ and $d_{x_0}\xi_g = G$. An immediate application (details are similar to those in the proof of lemma 2.3) of the transversality density and openness theorems completes the proof. □

We can now prove the main result on generic observability with respect to the mapping Θ.

Theorem 2.3 Suppose that $m \geq 2n+1$ and M is compact. Then pairs (f,h) in $\mathbb{X}^r(M) \times C^r(M,\mathbb{R})$ such that $\Theta_{f,h}$ is an embedding constitute an open and dense subset of $\mathbb{X}^r(M) \times C^r(M,\mathbb{R})$. Restricing the vector fields to $\mathbb{X}_2^r(M)$, the assertion remains valid for $\tau = \infty$ as well.

Proof: Recall that $\mathbb{X}_{0,1}^r(M)$ is open and dense. Furthermore $\mathbb{X}_2^r(M)$ is open in $\mathbb{X}^r(M,a)$. The assertions follow immediately from this facts and theorem 2.2. $\square$

Similarily, genericity results for noncompact state spaces can be deriven, if we restrict ourself to the set of complete vector fields, replace open and dense by residual and embedding by injective immersion. We remark also that considering the observation mapping

$$\Theta_{g,h}(x) := \sum_{k=-N}^{0} e^{-kA} b h(g^k(x))$$

for $x \in M$ and $(g,h) \in Diff^r(M) \times C^r(M,\mathbb{R})$ with $g^k := g \circ g^{k-1}$, corresponding results for discrete dynamical systems can be proved likewise.

3 Concluding Remarks

Remark 3.1 Since the set of m-dimensional controllable linear filters is open and dense in $End(\mathbb{R}^m) \times \mathbb{R}^m$, all genericity results of the last section hold also generically with respect to the linear filters.

The following examples show that the conditions on the observation dimension are also necessary, thus $m \geq 2n$ for generic local observability and $m \geq 2n+1$ for generic observability can not be weakend.

Example 3.1 Let $M = S^1$ and $f(\varphi) = 1$, where φ denotes the standard angular coordinate of the circle. Furthermore consider the pair (λ, b) with $\lambda < 0$ and $b \neq 0$. Taking $\tau = 2\pi$ and the output function $h(\varphi) = cos\,\varphi$ leads to $\Theta_{f,h}(\varphi) = \frac{1-e^{2\lambda\pi}}{1+\lambda^2}(\lambda cos\,\varphi - sin\,\varphi)$, which is not an immersion. Moreover the zero of $d\Theta_{f,h}$ at $\varphi_0 = -arctan\frac{1}{\lambda}$ is transversal. Hence the nonimmersivity of $\Theta_{f,h}$ is preserved under small perturbations of the output function, the vector field and the linear filter.

Example 3.2 Let M, f, φ and τ be as in the preceding example. Furthermore let $A = diag(-1,-2)$, $b = (1-e^{-2\pi})^{-1}(1,1)^T$ and $h(\varphi) = 2cos\,\varphi + 5cos\,2\varphi$.

Then a straightforward computation yields

$$\Theta_{f,h}(\varphi) = \begin{bmatrix} cos\,\varphi + sin\,\varphi + cos\,2\varphi + 2sin\,2\varphi \\ (e^{-2\pi}+1)(\frac{2}{5}(2cos\,\varphi + sin\,\varphi) + \frac{5}{4}(cos\,2\varphi + sin\,2\varphi)) \end{bmatrix}.$$

The following figure shows the image of S^1 by $\Theta_{f,h}$.

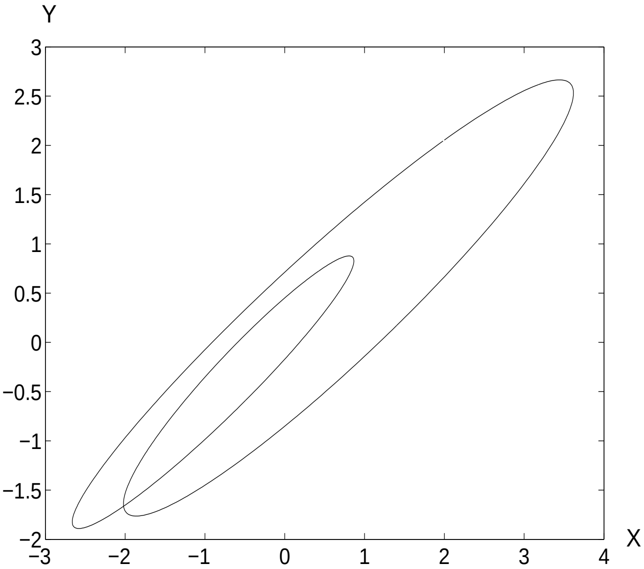

Image of S^1 by the continuous linear filter mapping given by
$\varphi \mapsto (cos\varphi + sin\varphi + cos2\varphi + 2sin2\varphi, (e^{-2\pi}+1)(\frac{2}{5}(2cos\varphi + sin\varphi) + \frac{5}{4}(cos2\varphi + sin2\varphi))^T$
in the XY-plane

The selfintersection of the image is transversal. Hence the noninjectiveness of $\Theta_{f,h}$ is persistent under small perturbations of the output function, the vector field and the linear filter.
For instance, small perturbations of h do not result in injectivity, i.e., there is an $\epsilon > 0$ such that for each output function $\tilde{h}$, which is C^r near to h within ϵ, the mapping $\Theta_{f,\tilde{h}}$ is not injective.

Note that the considered system is also unobservable with respect to the high-gain mapping given by $\varphi \mapsto (2cos\,\varphi + 5cos\,2\varphi, 2sin\,\varphi - 10sin\,2\varphi)^T$ as well as sampling mapping $\varphi \mapsto (2cos(\varphi + t_1) + 5cos(2\varphi + 2t_1, 2cos(\varphi + t_2) + 5cos(2\varphi + 2t_2))^T$ with sampling times t_1, t_2. The following figure shows the image of the state space by the sampling mapping with sampling times 0 and $\frac{\pi}{2}$.

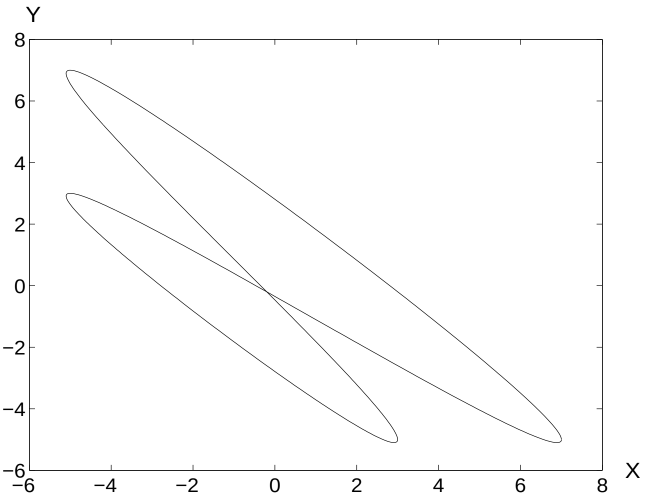

Image of S^1 by the sampling mapping given by
$\varphi \mapsto (2cos\, \varphi + 5cos\, 2\varphi, -2sin\, \varphi - 5cos\, 2\varphi)^T$
in the XY-plane

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