On Tameness and Growth Conditions

Jörg Winkelmann

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Abstract. We study discrete subsets of $\mathbb{C}^d$, relating "tameness" with growth conditions.

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1. Results

A discrete subset D in $\mathbb{C}^n$ ($n \geq 2$) is called "tame" if there exists a holomorphic automorphism ϕ of $\mathbb{C}^n$ such that $\phi(D) = \mathbb{Z} \times \{0\}^{n-1}$ (see [3]). If there exists a linear projection π of $\mathbb{C}^n$ onto some $\mathbb{C}^k$ $(0 < k < n)$ for which the image $\pi(D)$ is discrete, then D is tame ([3]). If D is a discrete subgroup (e.g. a lattice) of the additive group $(\mathbb{C}^n +)$, then D must be tame ([1], lemma 4.4 in combination with corollary 2.6). On the other hand there do exist discrete subsets which are not tame (see [3], theorem 3.9).
Here we will investigate how "tameness" is related to growth conditions for D. Slow growth implies tameness as we well see. On the other hand, rapid growth can not imply non-tameness, since every discrete subset of $\mathbb{C}^{n-1}$ is tame regarded as subset of $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$.
The key method is to show that sufficiently slow growth implies that a generic linear projection will have discrete image for D.
The main result is:

Theorem 1. *Let n be a natural number and let v_k be a sequence of elements in $V = \mathbb{C}^n$.*
Assume that

$$\sum_k \frac{1}{||v_k||^{2n-2}} < \infty$$

Then $D = \{v_k : k \in \mathbb{N}\}$ is tame*, i.e., there exists a biholomorphic map $\phi : \mathbb{C}^n \to \mathbb{C}^n$ such that*

$$\phi(D) = \mathbb{Z} \times \{0\}^{n-1}.$$

This growth condition is fulfilled for discrete subgroups of rank at most $2n-3$, implying the following well-known fact:

Corollary 1. *Let Γ be a discrete subgroup of $\mathbb{Z}$-rank at most $2n-3$ of the additive group $(\mathbb{C}^n,+)$.*
Then Γ is a tame discrete subset of $\mathbb{C}^n$.

While this is well-known (even with no condition on the $\mathbb{Z}$-rank of Γ), our approach yields the additional information that these discrete subsets remain tame after a small deformation:

Corollary 2. *Let Γ be a discrete subgroup of $\mathbb{Z}$-rank at most $2n-3$ of the additive group $(\mathbb{C}^n,+)$, $0<\lambda<1$ and $K>0$. Let D be a subset of $\mathbb{C}^n$ for which there exists a bijective map $\zeta:\Gamma\to D$ with*

$$||\zeta(v)-v||\le\lambda||v||+K$$

for all $v\in\Gamma$.
Then D is a tame discrete subset of $\mathbb{C}^n$.

This confirms the idea that tame sets should be stable under deformation. Similarily one would hope that the category of non-tame sets is also stable under deformation. Here, however, one has to be careful not to be too optimistic, because in fact the following is true:

Proposition. *For every non-tame discrete subset $D\subset\mathbb{C}^n$ $(n>1)$ there is a tame discrete subset D' and a bijection $\alpha:D\to D'$ such that*

$$||\alpha(v)-v||\le\frac{1}{\sqrt{2}}||v|| \quad \forall v\in D$$

and

$$||w-\alpha^{-1}(w)||\le||w|| \quad \forall w\in D'.$$

In particular, if D is a tame discrete subset and $\zeta:D\to\mathbb{C}^n$ is a bijective map with $||\zeta(v)-v||\le||v||$ for all $v\in D$, it is possible that $\zeta(D)$ is not tame.
Still, one might hope for a positive answer to the following question:

Question. *Let $n\in\mathbb{N}$, $n\ge 2$, let $1>\lambda>0$, $K>0$, let D be a tame discrete subset of $\mathbb{C}^n$ and let $\zeta:D\to\mathbb{C}^n$ be a map such that*

$$||\zeta(v)-v||\le\lambda||v||+K$$

for all $v\in D$. Does this imply that $\zeta(D)$ is a tame *discrete subset of $\mathbb{C}^n$?*

Technically, the following is the key point for the proof of our main result (theorem 1):

Theorem 2. *Let $n>d>0$. Let V be a complex vector space of dimension n and let v_k be a sequence of elements in V.*
Assume that

$$\sum_k\frac{1}{||v_k||^{2d}}<\infty$$

Then there exists a complex linear map $\pi:V\to\mathbb{C}^d$ such that the set of all $\pi(v_k)$ is discrete in $\mathbb{C}^d$.

In a similar way on can prove such a result for real vector spaces:

THEOREM 3. *Let $n > d > 0$. Let V be a real vector space of dimension n and let v_k be a sequence of elements in V.*
Assume that

$$\sum_k \frac{1}{||v_k||^d} < \infty$$

Then there exists a real linear map $\pi : V \to \mathbb{R}^d$ such that the set of all $\pi(v_k)$ is discrete in $\mathbb{R}^d$.

For the proof of the existence of a linear projection π with $\pi(D)$ discrete we proceed by regarding randomly chosen linear projections and verifying that the image of D under a random projection has discrete image with probability 1 if the above stated series converges.

2. PROOFS

First we deduce an auxiliary lemma.

LEMMA 1. *Let $k, m > 0$, $n = k + m$ and let S denote the unit sphere in $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^m$. Furthermore let*

$$M_\epsilon = \{(v, w) \in \mathbb{R}^k \times \mathbb{R}^m : ||v|| \leq \epsilon, (v, w) \in S\}.$$

Then there are constants $\delta > 0$, $C_1 > C_2 > 0$ such that for all $\epsilon < \delta$ we have

$$C_1 \epsilon^k \geq \lambda(M_\epsilon) \geq C_2 \epsilon^k$$

where λ denotes the rotationally invariant probability measure on S.

Proof. For each $\epsilon \in]0, 1[$ there is a bijection

$$\phi_\epsilon : B \times S' \to M_\epsilon$$

where

$$B = \{v \in \mathbb{R}^k : ||v|| \leq 1\}, \quad S' = \{w \in \mathbb{R}^m : ||w|| = 1\}$$

and

$$\phi_\epsilon(v, w) = \left(\epsilon v; \sqrt{1 - ||\epsilon v||^2} w\right).$$

The functional determinant for ϕ_ϵ equals

$$\epsilon^k \left(\sqrt{1 - ||\epsilon v||^2}\right)^m.$$

It follows that

$$\epsilon^k \left(\sqrt{1 - \epsilon^2}\right)^m volume(S' \times B) \leq volume(M_\epsilon) \leq \epsilon^k volume(S' \times B),$$

which in turn implies

$$\lim_{\epsilon \to 0} \epsilon^{-k} \frac{volume(M_\epsilon)}{volume(S' \times B)} = 1.$$

Hence the assertion. □

Lemma 2. *Let Γ be a discrete subgroup of $\mathbb{Z}$-rank d in $V = \mathbb{R}^n$. Then*

$$\sum_{\gamma \in \Gamma} ||\gamma||^{-d-\epsilon} < \infty$$

for all $\epsilon > 0$.

Proof. Since all norms on a finite-dimensional vector space are equivalent, there is no loss in generality if we assume that the norm is the maximum norm and $\Gamma = \mathbb{Z}^d \times \{0\}^{n-d}$. Then the assertion is an easy consequence of the fact that $\sum_{n \in \mathbb{N}} n^{-s} < \infty$ if and only if $s > 1$. □

Now we proceed with the proof of theorem 2:

Proof. We fix a surjective linear map $L : V \to W = \mathbb{C}^d$. Let K denote $U(n)$ (the group of unitary complex linear transformations of V). For each $g \in K$ we define a linear map $\pi_g : V \to W$ as follows:

$$\pi_g : v \mapsto L(g \cdot v).$$

For $k \in \mathbb{N}$ and $r \in \mathbb{R}^+$ define

$$S_{k,r} = \{g \in K : ||\pi_g(v_k)|| \leq r\},$$

$$M_{N,r} = \{g \in K : \#\{k \in \mathbb{N} : g \in S_{k,r}\} \geq N\}$$

and

$$M_r = \cap_N M_{N,r}.$$

Now for each $g \in K$ the set $\{\pi_g(v_k) : k \in \mathbb{N}\}$ is discrete unless there is a number $r > 0$ such that infinitely many distinct image points are contained in a ball of radius r. By the definition of the sets M_r it follows that $\{\pi_g(v_k) : k \in \mathbb{N}\}$ is discrete unless $g \in M = \cup M_r$.
Let us now assume that there is no linear map $L' : V \to W$ with $L'(D)$ discrete. Then $K = M$. In particular $\mu(M) > 0$, where μ denotes the Haar measure on the compact topological group K. Since the sets M_r are increasing in r, we have

$$M = \cup_{r \in \mathbb{R}^+} M_r = \cup_{r \in \mathbb{N}} M_r$$

and may thus deduce that $\mu(M_r) > 0$ for some number r. Fix such a number $r > 0$ and define $c = \mu(M_r) > 0$. Then $\mu(M_{N,r}) \geq c$ for all N, since $M_r = \cap M_{N,r}$. However, for fixed N and r we have

$$N\mu(M_{N,r}) \leq \sum_k \mu(S_{k,r}).$$

Hence

$$\sum_{k \in \mathbb{N}} \mu(S_{k,r}) \geq N\mu(M_{N,r}) \geq Nc$$

for all $N \in \mathbb{N}$. Since $c > 0$, it follows that $\sum_k \mu(S_{k,r}) = +\infty$.

Let us now embedd $\mathbb{C}^d$ into $\mathbb{C}^n$ as the orthogonal complement of $\ker L$. In this way we may assume that L is simply the map which projects a vector onto its first d coordinates, i.e.,

$$L(w_1, \ldots, w_n) = (w_1, \ldots, w_d; 0, \ldots, 0).$$

Now $g \in S_{k,r}$ is equivalent to the condition that $g(v_k)$ is a real multiple of an element in M_ϵ where M_ϵ is defined as in lemma 1 with $\epsilon = r/||v_k||$. Using lemma 1 we may deduce that $\sum_k \mu(S_{k,r})$ converges if and only if $\sum_k ||v_k||^{-2d}$ converges. □

Proof of theorem 1. The growth condition allows us to employ theorem 2 in order to deduce that there is a linear projection onto a space of complex dimension $d-1$ which maps D onto a discrete image. By the results of Rosay and Rudin it follows that D is tame. □

Proof of the proposition. We fix a decomposition $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$ and write D as the union of all $(a_k, b_k) \in \mathbb{C} \times \mathbb{C}^{n-1}$ $(k \in \mathbb{N})$. We define

$$\alpha(a_k, b_k) = \begin{cases} (a_k, 0) & \text{if } ||a_k|| > ||b_k|| \\ (0, b_k) & \text{if } ||a_k|| \leq ||b_k|| \end{cases}$$

Then $D' = \alpha(D)$ is tame because each of the projections to one of the two factors $\mathbb{C}$ and $\mathbb{C}^{n-1}$ maps D' onto a discrete subset.
The other assertions follow from the triangle inequality. □

The proof of thm. 3 works in the same way as the proof of thm. 2, simply using the group of all orthogonal transformations instead of the group of unitary transformations.

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Jörg Winkelmann
Mathematisches Institut
Universität Bayreuth
Universitätsstraße 30
D-95447 Bayreuth
Germany
jwinkel@member.ams.org