Homology of the Steinberg Variety and Weyl Group Coinvariants

J. Matthew Douglass and Gerhard Röhrle [1]

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Abstract. Let G be a complex, connected, reductive algebraic group with Weyl group W and Steinberg variety Z. We show that the graded Borel-Moore homology of Z is isomorphic to the smash product of the coinvariant algebra of W and the group algebra of W.

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1. Introduction

Suppose G is a complex, reductive algebraic group, $\mathcal{B}$ is the variety of Borel subgroups of G. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathcal{N}$ the cone of nilpotent elements in $\mathfrak{g}$. Let $T^*\mathcal{B}$ denote the cotangent bundle of $\mathcal{B}$. Then there is a *moment map*, $\mu_0\colon T^*\mathcal{B} \to \mathcal{N}$. The *Steinberg variety* of G is the fibered product $T^*\mathcal{B} \times_{\mathcal{N}} T^*\mathcal{B}$ which we will identify with the closed subvariety

$$Z = \{\, (x, B', B'') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \operatorname{Lie}(B') \cap \operatorname{Lie}(B'') \,\}$$

of $\mathcal{N} \times \mathcal{B} \times \mathcal{B}$. Set $n = \dim \mathcal{B}$. Then Z is a $2n$-dimensional, complex algebraic variety.

If $V = \oplus_{i\geq 0} V_i$ is a graded vector space, we will frequently denote V by $V_\bullet$. Similarly, if X is a topological space, then $H_i(X)$ denotes the i^{th} rational Borel-Moore homology of X and $H_\bullet(X) = \oplus_{i\geq 0} H_i(X)$ denotes the total Borel-Moore homology of X.

Fix a maximal torus, T, of G, with Lie algebra $\mathfrak{t}$, and let $W = N_G(T)/T$ be the Weyl group of (G, T). In [6] Kazhdan and Lusztig defined an action of $W \times W$ on $H_\bullet(Z)$ and they showed that the representation of $W \times W$ on the

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top-dimensional homology of Z, $H_{4n}(Z)$, is equivalent to the two-sided regular representation of W. Tanisaki [11] and, more recently, Chriss and Ginzburg [3] have strengthened the connection between $H_\bullet(Z)$ and W by defining a $\mathbb{Q}$-algebra structure on $H_\bullet(Z)$ so that $H_i(Z) * H_j(Z) \subseteq H_{i+j-4n}(Z)$. Chriss and Ginzburg [3, §3.4] have also given an elementary construction of an isomorphism between $H_{4n}(Z)$ and the group algebra $\mathbb{Q}W$.

Let Z_1 denote the "diagonal" in Z:

$$Z_1 = \{ (x, B', B') \in \mathcal{N} \times \mathcal{B} \times \mathcal{B} \mid x \in \mathrm{Lie}(B') \}.$$

In this paper we extend the results of Chriss and Ginzburg [3, §3.4] and show in Theorem 2.3 that for any i, the convolution product defines an isomorphism $H_i(Z_1) \otimes H_{4n}(Z) \xrightarrow{\sim} H_i(Z)$. It then follows easily that with the convolution product, $H_\bullet(Z)$ is isomorphic to the smash product of the coinvariant algebra of W and the group algebra of W.

Precisely, for $0 \leq i \leq n$ let $\mathrm{Coinv}_{2i}(W)$ denote the degree i subspace of the rational coinvariant algebra of W, so $\mathrm{Coinv}_{2i}(W)$ may be identified with the space of degree i, W-harmonic polynomials on $\mathfrak{t}$. If j is odd, define $\mathrm{Coinv}_j(W) = 0$. Recall that the smash product, $\mathrm{Coinv}(W)\#\mathbb{Q}W$, is the $\mathbb{Q}$-algebra whose underlying vector space is $\mathrm{Coinv}(W) \otimes_{\mathbb{Q}} \mathbb{Q}W$ with multiplication satisfying $(f_1 \otimes \phi_1) \cdot (f_2 \otimes \phi_2) = f_1 \phi_1(f_2) \otimes \phi_1 \phi_2$ where f_1 and f_2 are in $\mathrm{Coinv}(W)$, ϕ_1 and ϕ_2 are in $\mathbb{Q}W$, and $\mathbb{Q}W$ acts on $\mathrm{Coinv}(W)$ in the usual way. The algebra $\mathrm{Coinv}(W)\#\mathbb{Q}W$ is graded by $(\mathrm{Coinv}(W)\#\mathbb{Q}W)_i = \mathrm{Coinv}_i(W)\#\mathbb{Q}W$ and we will denote this graded algebra by $\mathrm{Coinv}_\bullet(W)\#\mathbb{Q}W$. In Theorem 2.5 we construct an explicit isomorphism of graded algebras $H_{4n-\bullet}(Z) \cong \mathrm{Coinv}_\bullet(W)\#\mathbb{Q}W$.

This paper was motivated by the observation, pointed out to the first author by Catharina Stroppel, that the argument in [3, 8.1.5] can be used to show that $H_\bullet(Z)$ is isomorphic to the smash product of $\mathbb{Q}W$ and $\mathrm{Coinv}_\bullet(W)$. The details of such an argument have been carried out in a recent preprint of Namhee Kwon [8]. This argument relies on some deep and technical results: the localization theorem in K-theory proved by Thomason [12], the bivariant Riemann-Roch Theorem [3, 5.11.11], and the Kazhdan-Lusztig isomorphism between the equivariant K-theory of Z and the extended, affine, Hecke algebra [7]. In contrast, and also in the spirit of Kazhdan and Lusztig's original analysis of $H_{4n}(Z)$, and the analysis of $H_{4n}(Z)$ in [3, 3.4], our argument uses more elementary notions and is accessible to readers who are not experts in equivariant K-theory and to readers who are not experts in the representation theory of reductive, algebraic groups.

Another approach to the Borel-Moore homology of the Steinberg variety uses intersection homology. Let $\mu\colon Z \to \mathcal{N}$ be projection on the first factor. Then, as in [3, §8.6], $H_\bullet(Z) \cong \mathrm{Ext}^{4n-\bullet}_{D(\mathcal{N})}(R\mu_* \mathbb{Q}_{\mathcal{N}}, R\mu_* \mathbb{Q}_{\mathcal{N}})$. The Decomposition Theorem of Beilinson, Bernstein, and Deligne can be used to decompose $R\mu_*\mathbb{Q}_{\mathcal{N}}$ into a direct sum of simple perverse sheaves $R\mu_*\mathbb{Q}_{\mathcal{N}} \cong \oplus_{x,\phi} \mathrm{IC}_{x,\phi}^{n_{x,\phi}}$ where x runs over a set of orbit representatives in $\mathcal{N}$, for each x, ϕ runs over a set of irreducible representations of the component group of $Z_G(x)$, and $\mathrm{IC}_{x,\phi}$ denotes

an intersection complex (see [2] or [9, §4,5]). Chriss and Ginzburg have used this construction to describe an isomorphism $H_{4n}(Z) \cong \mathbb{Q}W$ and to in addition give a description of the projective, indecomposable $H_\bullet(Z)$-modules.
It follows from Theorem 2.3 that $H_i(Z) \cong \mathrm{Coinv}_{4n-i}(W) \otimes H_{4n}(Z)$ and so

$$\begin{aligned}\mathrm{Coinv}_i(W) \otimes H_{4n}(Z) &\cong \mathrm{Ext}^{4n-i}_{D(\mathcal{N})}\left(R\mu_*\mathbb{Q}_{\mathcal{N}}, R\mu_*\mathbb{Q}_{\mathcal{N}}\right)\\ &\cong \bigoplus_{x,\phi}\bigoplus_{y,\psi} \mathrm{Ext}^{4n-i}_{D(\mathcal{N})}\left(\mathrm{IC}^{n_{x,\phi}}_{x,\phi}, \mathrm{IC}^{n_{y,\psi}}_{y,\psi}\right).\end{aligned} \tag{1.1}$$

In the special case when $i = 0$ we have that

$$\mathrm{Coinv}_0(W) \otimes H_{4n}(Z) \cong \mathrm{End}_{D(\mathcal{N})}\left(R\mu_*\mathbb{Q}_{\mathcal{N}}\right) \cong \bigoplus_{x,\phi} \mathrm{End}_{D(\mathcal{N})}\left(\mathrm{IC}^{n_{x,\phi}}_{x,\phi}\right).$$

The image of the one-dimensional vector space $\mathrm{Coinv}_0(W)$ in $\mathrm{End}_{D(\mathcal{N})}\left(R\mu_*\mathbb{Q}_{\mathcal{N}}\right)$ is the line through the identity endomorphism and $\mathbb{Q}W \cong H_{4n}(Z) \cong \oplus_{x,\phi} \mathrm{End}_{D(\mathcal{N})}\left(\mathrm{IC}^{n_{x,\phi}}_{x,\phi}\right)$ is the Wedderburn decomposition of $\mathbb{Q}W$ as a direct sum of minimal two-sided ideals. For $i > 0$ we have not been able to find a nice description of the image of $\mathrm{Coinv}_i(W)$ in the right-hand side of (1.1).
The rest of this paper is organized as follows: in §2 we set up our notation and state the main results; in §3 we construct an isomorphism of graded vector spaces between $\mathrm{Coinv}_\bullet(W) \otimes \mathbb{Q}W$ and $H_{4n-\bullet}(Z)$; and in §4 we complete the proof that this isomorphism is in fact an algebra isomorphism when $\mathrm{Coinv}_\bullet(W) \otimes \mathbb{Q}W$ is given the smash product multiplication. Some very general results about graphs and convolution that we need for the proofs of the main theorems are proved in an appendix.
In this paper $\otimes = \otimes_{\mathbb{Q}}$, if X is a set, then δ_X, or just δ, will denote the diagonal embedding of X in $X \times X$, and for g in G and x in $\mathfrak{g}$, $g \cdot x$ denotes the adjoint action of g on x.

2. Preliminaries and Statement of Results

Fix a Borel subgroup, B, of G with $T \subseteq B$ and define U to be the unipotent radical of B. We will denote the Lie algebras of B and U by $\mathfrak{b}$ and $\mathfrak{u}$ respectively.
Our proof that $H_\bullet(Z)$ is isomorphic to $\mathrm{Coinv}_\bullet(W)\#\mathbb{Q}W$ makes use of the specialization construction used by Chriss and Ginzburg in [3, §3.4] to establish the isomorphism between $H_{4n}(Z)$ and $\mathbb{Q}W$. We begin by reviewing their construction.
The group G acts diagonally on $\mathcal{B} \times \mathcal{B}$. Let $\mathcal{O}_w$ denote the orbit containing (B, wBw^{-1}). Then the rule $w \mapsto \mathcal{O}_w$ defines a bijection between W and the set of G-orbits in $\mathcal{B} \times \mathcal{B}$.
Let $\pi_Z\colon Z \to \mathcal{B} \times \mathcal{B}$ denote the projection on the second and third factors and for w in W define $Z_w = \pi_Z^{-1}(\mathcal{O}_w)$. For w in W we also set $\mathfrak{u}_w = \mathfrak{u} \cap w \cdot \mathfrak{u}$. The following facts are well-known (see [10] and [9, §1.1]):

- $Z_w \cong G \times^{B \cap {}^wB} \mathfrak{u}_w$.

- $\dim Z_w = 2n$.
- The set $\{\, \overline{Z_w} \mid w \in W \,\}$ is the set of irreducible components of Z.

Define

$$\begin{aligned}
\widetilde{\mathfrak{g}} &= \{\, (x, B') \in \mathfrak{g} \times \mathcal{B} \mid x \in \operatorname{Lie}(B') \,\}, \\
\widetilde{\mathcal{N}} &= \{\, (x, B') \in \mathcal{N} \times \mathcal{B} \mid x \in \operatorname{Lie}(B') \,\}, \text{ and} \\
\widehat{Z} &= \{\, (x, B', B'') \in \mathfrak{g} \times \mathcal{B} \times \mathcal{B} \mid x \in \operatorname{Lie}(B') \cap \operatorname{Lie}(B'') \,\},
\end{aligned}$$

and let $\mu \colon \widetilde{\mathfrak{g}} \to \mathfrak{g}$ denote the projection on the first factor. Then $\widetilde{\mathcal{N}} \cong T^*\mathcal{B}$, $\mu(\widetilde{\mathcal{N}}) = \mathcal{N}$, $Z \cong \widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}$, and $\widehat{Z} \cong \widetilde{\mathfrak{g}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$.

Let $\hat{\pi} \colon \widehat{Z} \to \mathcal{B} \times \mathcal{B}$ denote the projection on the second and third factors and for w in W define $\widehat{Z}_w = \hat{\pi}^{-1}(\mathcal{O}_w)$. Then it is well-known that $\dim \widehat{Z}_w = \dim \mathfrak{g}$ and that the closures of the $\widehat{Z}_w$'s for w in W are the irreducible components of $\widehat{Z}$ (see [9, §1.1]).

Next, for (x, gBg^{-1}) in $\widetilde{\mathfrak{g}}$, define $\nu(x, gBg^{-1})$ to be the projection of $g^{-1} \cdot x$ in $\mathfrak{t}$. Then μ and ν are two of the maps in Grothendieck's simultaneous resolution:

$$\begin{tikzcd}
\widetilde{\mathfrak{g}} \arrow[r, "\mu"] \arrow[d, "\nu"'] & \mathfrak{g} \arrow[d] \\
\mathfrak{t} \arrow[r] & \mathfrak{t}/W
\end{tikzcd}$$

It is easily seen that if $\widehat{\mu} \colon \widehat{Z} \to \mathfrak{g}$ is the projection on the first factor, then the square

$$\begin{tikzcd}
\widehat{Z} \arrow[r, "\widehat{\mu}"] \arrow[d] & \mathfrak{g} \arrow[d, "\delta_{\mathfrak{g}}"] \\
\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}} \arrow[r, "\mu \times \mu"'] & \mathfrak{g} \times \mathfrak{g}
\end{tikzcd}$$

is cartesian, where the vertical map on the left is given by $(x, B', B'') \mapsto ((x, B'), (x, B''))$. We will frequently identify $\widehat{Z}$ with the subvariety of $\widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}}$ consisting of all pairs $((x, B'), (x, B''))$ with x in $\operatorname{Lie}(B') \cap \operatorname{Lie}(B'')$.

For w in W, let $\Gamma_{w^{-1}} = \{\, (h, w^{-1} \cdot h) \mid h \in \mathfrak{t} \,\} \subseteq \mathfrak{t} \times \mathfrak{t}$ denote the graph of the action of w^{-1} on $\mathfrak{t}$ and define

$$\Lambda_w = \widehat{Z} \cap (\nu \times \nu)^{-1} (\Gamma_{w^{-1}}) = \{\, (x, B', B'') \in \widehat{Z} \mid \nu(x, B'') = w^{-1}\nu(x, B') \,\}.$$

In the special case when w is the identity element in W, we will denote Λ_w by Λ_1.

The spaces we have defined so far fit into a commutative diagram with cartesian squares:

(2.1)
$$\begin{tikzcd}
\Lambda_w \arrow[r] \arrow[d] & \widehat{Z} \arrow[r, "\widehat{\mu}"] \arrow[d] & \mathfrak{g} \arrow[d, "\delta_{\mathfrak{g}}"] \\
(\nu\times\nu)^{-1}\left(\Gamma_{w^{-1}}\right) \arrow[r] \arrow[d] & \widetilde{\mathfrak{g}}\times\widetilde{\mathfrak{g}} \arrow[r, "\mu\times\mu"'] \arrow[d, "\nu\times\nu"'] & \mathfrak{g}\times\mathfrak{g} \\
\Gamma_{w^{-1}} \arrow[r] & \mathfrak{t}\times\mathfrak{t} &
\end{tikzcd}$$

Let $\nu_w\colon \Lambda_w \to \Gamma_{w^{-1}}$ denote the composition of the leftmost vertical maps in (2.1), so ν_w is the restriction of $\nu\times\nu$ to Λ_w.
For the specialization construction, we consider subsets of $\widehat{Z}$ of the form $\nu_w^{-1}(S')$ for $S' \subseteq \Gamma_{w^{-1}}$. Thus, for h in $\mathfrak{t}$ we define $\Lambda_w^h = \nu_w^{-1}(h, w^{-1}h)$. Notice in particular that $\Lambda_w^0 = Z$. More generally, for a subset S of $\mathfrak{t}$ we define $\Lambda_w^S = \coprod_{h\in S}\Lambda_w^h$. Then, $\Lambda_w^S = \nu_w^{-1}(S')$, where S' is the graph of w^{-1} restricted to S. Let $\mathfrak{t}_{\mathrm{reg}}$ denote the set of regular elements in $\mathfrak{t}$.
Fix a one-dimensional subspace, ℓ, of $\mathfrak{t}$ so that $\ell\cap\mathfrak{t}_{\mathrm{reg}} = \ell\setminus\{0\}$ and set $\ell^* = \ell\setminus\{0\}$. Then $\Lambda_w^\ell = \Lambda_w^{\ell^*}\coprod\Lambda_w^0 = \Lambda_w^{\ell^*}\coprod Z$. We will see in Corollary 3.6 that the restriction of ν_w to $\Lambda_w^{\ell^*}$ is a locally trivial fibration with fibre G/T. Thus, using a construction due to Fulton and MacPherson ([4, §3.4], [3, §2.6.30]), there is a specialization map

$$\lim\colon H_{\bullet+2}(\Lambda_w^{\ell^*}) \longrightarrow H_\bullet(Z).$$

Since $\Lambda_w^{\ell^*}$ is an irreducible, $(2n+1)$-dimensional variety, if $[\Lambda_w^{\ell^*}]$ denotes the fundamental class of $\Lambda_w^{\ell^*}$, then $H_{4n+2}(\Lambda_w^{\ell^*})$ is one-dimensional with basis $\{[\Lambda_w^{\ell^*}]\}$. Define $\lambda_w = \lim([\Lambda_w^{\ell^*}])$ in $H_{4n}(Z)$. Chriss and Ginzburg [3, §3.4] have proved the following theorem.

Theorem 2.2. *Consider $H_\bullet(Z)$ endowed with the convolution product.*

(a) *For $0\le i,j\le 4n$, $H_i(Z)*H_j(Z)\subseteq H_{i+j-4n}(Z)$. In particular, $H_{4n}(Z)$ is a subalgebra of $H_\bullet(Z)$.*

(b) *The element λ_w in $H_{4n}(Z)$ does not depend on the choice of ℓ.*

(c) *The assignment $w\mapsto\lambda_w$ extends to an algebra isomorphism $\alpha\colon \mathbb{Q}W \xrightarrow{\cong} H_{4n}(Z)$.*

Now consider

$$Z_1 = \{\,(x, B', B')\in\mathcal{N}\times\mathcal{B}\times\mathcal{B}\mid x\in\mathrm{Lie}(B')\,\}.$$

Then Z_1 may be identified with the diagonal in $\widetilde{\mathcal{N}}\times\widetilde{\mathcal{N}}$. It follows that Z_1 is closed in Z and isomorphic to $\widetilde{\mathcal{N}}$.

Since $\widetilde{\mathcal{N}} \cong T^*\mathcal{B}$, it follows from the Thom isomorphism in Borel-Moore homology [3, §2.6] that $H_{i+2n}(Z_1) \cong H_i(\mathcal{B})$ for all i. Since $\mathcal{B}$ is smooth and compact, $H_i(\mathcal{B}) \cong H^{2n-i}(\mathcal{B})$ by Poincaré duality. Therefore, $H_{4n-i}(Z_1) \cong H^i(\mathcal{B})$ for all i.

The cohomology of $\mathcal{B}$ is well-understood: there is an isomorphism of graded algebras, $H^\bullet(\mathcal{B}) \cong \mathrm{Coinv}_\bullet(W)$. It follows that $H_j(Z_1) = 0$ if j is odd and $H_{4n-2i}(Z_1) \cong \mathrm{Coinv}_{2i}(W)$ for $0 \leq i \leq n$.

In §3 below we will prove the following theorem.

Theorem 2.3. *Consider the Borel-Moore homology of the variety Z_1.*

(a) *There is a convolution product on $H_\bullet(Z_1)$. With this product, $H_\bullet(Z_1)$ is a commutative $\mathbb{Q}$-algebra and there is an isomorphism of graded $\mathbb{Q}$-algebras*

$$\beta\colon \mathrm{Coinv}_\bullet(W) \xrightarrow{\cong} H_{4n-\bullet}(Z_1).$$

(b) *If $r\colon Z_1 \to Z$ denotes the inclusion, then the direct image map in Borel-Moore homology, $r_*\colon H_\bullet(Z_1) \longrightarrow H_\bullet(Z)$, is an injective ring homomorphism.*

(c) *If we identify $H_\bullet(Z_1)$ with its image in $H_\bullet(Z)$ as in (b), then the linear transformation given by the convolution product*

$$H_i(Z_1) \otimes H_{4n}(Z) \xrightarrow{*} H_i(Z)$$

is an isomorphism of vector spaces for $0 \leq i \leq 4n$.

The algebra $\mathrm{Coinv}_\bullet(W)$ has a natural action of W by algebra automorphisms, and the isomorphism β in Theorem 2.3(a) is in fact an isomorphism of W-algebras. The W-algebra structure on $H_\bullet(Z_1)$ is described in the next theorem, which will be proved in §4.

Theorem 2.4. *If w is in W and $H_\bullet(Z_1)$ is identified with its image in $H_\bullet(Z)$, then*

$$\lambda_w * H_i(Z_1) * \lambda_{w^{-1}} = H_i(Z_1).$$

Thus, conjugation by λ_w defines a W-algebra structure on $H_\bullet(Z_1)$. With this W-algebra structure, the isomorphism $\beta\colon \mathrm{Coinv}_\bullet(W) \xrightarrow{\cong} H_{4n-\bullet}(Z_1)$ in Theorem 2.3(a) is an isomorphism of W-algebras.

Recall that the algebra $\mathrm{Coinv}(W)\#\mathbb{Q}W$ is graded by $(\mathrm{Coinv}(W)\#\mathbb{Q}W)_i = \mathrm{Coinv}_i(W) \otimes \mathbb{Q}W$. Then combining Theorem 2.2(c), Theorem 2.3(c), and Theorem 2.4 we get our main result.

Theorem 2.5. *The composition*

$$\mathrm{Coinv}_\bullet(W)\#\mathbb{Q}W \xrightarrow{\beta\otimes\alpha} H_{4n-\bullet}(Z_1) \otimes H_{4n}(Z) \xrightarrow{*} H_{4n-\bullet}(Z)$$

is an isomorphism of graded $\mathbb{Q}$-algebras.

3. Factorization of $H_\bullet(Z)$

Proof of Theorem 2.3(a). We need to prove that $H_\bullet(Z_1)$ is a commutative $\mathbb{Q}$-algebra and that $\mathrm{Coinv}_\bullet(W) \cong H_{4n-\bullet}(Z_1)$.

Let $\pi\colon \widetilde{\mathcal{N}} \to \mathcal{B}$ by $\pi(x, B') = B'$. Then π may be identified with the vector bundle projection $T^*\mathcal{B} \to \mathcal{B}$ and so the induced map in cohomology $\pi^*\colon H^i(\mathcal{B}) \to H^i(\widetilde{\mathcal{N}})$ is an isomorphism. The projection π determines an isomorphism in Borel-Moore homology that we will also denote by π^* (see [3, §2.6.42]). We have $\pi^*\colon H_i(\mathcal{B}) \xrightarrow{\cong} H_{i+2n}(\widetilde{\mathcal{N}})$.

For a smooth m-dimensional variety X, let $\mathrm{pd}\colon H^i(X) \longrightarrow H_{2m-i}(X)$ denote the Poincaré duality isomorphism. Then the composition

$$H_{2n-i}(\mathcal{B}) \xrightarrow{\mathrm{pd}^{-1}} H^i(\mathcal{B}) \xrightarrow{\pi^*} H^i(\widetilde{\mathcal{N}}) \xrightarrow{\mathrm{pd}} H_{4n-i}(\widetilde{\mathcal{N}})$$

is an isomorphism. It follows from the uniqueness construction in [3, §2.6.26] that

$$\mathrm{pd}\circ\pi^*\circ\mathrm{pd}^{-1} = \pi^*\colon H_{2n-i}(\mathcal{B}) \longrightarrow H_{4n-i}(\widetilde{\mathcal{N}})$$

and so $\pi^*\circ\mathrm{pd} = \mathrm{pd}\circ\pi^*\colon H^i(\mathcal{B}) \longrightarrow H_{4n-i}(\widetilde{\mathcal{N}})$.

Recall that $\mathrm{Coinv}_j(W) = 0$ if j is odd and $\mathrm{Coinv}_{2i}(W)$ is the degree i subspace of the coinvariant algebra of W. Let $\mathrm{bi}\colon \mathrm{Coinv}_\bullet(W) \longrightarrow H^\bullet(\mathcal{B})$ be the Borel isomorphism (see [1, §1.5] or [5]). Then with the cup product, $H^\bullet(\mathcal{B})$ is a graded algebra and bi is an isomorphism of graded algebras.

Define $\beta\colon \mathrm{Coinv}_i(W) \to H_{4n-i}(Z_1)$ to be the composition

$$\mathrm{Coinv}_i(W) \xrightarrow{\mathrm{bi}} H^i(\mathcal{B}) \xrightarrow{\pi^*} H^i(\widetilde{\mathcal{N}}) \xrightarrow{\mathrm{pd}} H_{4n-i}(\widetilde{\mathcal{N}}) \xrightarrow{\delta_*} H_{4n-i}(Z_1)$$

where $\delta = \delta_{\widetilde{\mathcal{N}}}$. Then β is an isomorphism of graded vector spaces and

$$\beta = \delta_*\circ\mathrm{pd}\circ\pi^*\circ\mathrm{bi} = \delta_*\circ\pi^*\circ\mathrm{pd}\circ\mathrm{bi}.$$

The algebra structure of $H^\bullet(\mathcal{B})$ and $H^\bullet(\widetilde{\mathcal{N}})$ is given by the cup product, and $\pi^*\colon H^\bullet(\mathcal{B}) \to H^\bullet(\widetilde{\mathcal{N}})$ is an isomorphism of graded algebras. Since $\widetilde{\mathcal{N}}$ is smooth, as in [3, §2.6.15], there is an intersection product defined on $H_\bullet(\widetilde{\mathcal{N}})$ using Poincaré duality and the cup product on $H^\bullet(\widetilde{\mathcal{N}})$. Thus, $\mathrm{pd}\colon H^\bullet(\widetilde{\mathcal{N}}) \to H_{4n-\bullet}(\widetilde{\mathcal{N}})$ is an algebra isomorphism. Finally, it is observed in [3, §2.7.10] that $\delta_*\colon H_\bullet(\widetilde{\mathcal{N}}) \to H_\bullet(Z_1)$ is a ring homomorphism and hence an algebra isomorphism. This shows that β is an isomorphism of graded algebras and proves Theorem 2.3(a).

Proof of Theorem 2.3(b). To prove the remaining parts of Theorem 2.3, we need a linear order on W. Suppose $|W| = N$. Fix a linear order on W that extends the Bruhat order. Say $W = \{w_1, \ldots, w_N\}$, where $w_1 = 1$ and w_N is the longest element in W.

For $1 \le j \le N$, define $Z_j = \coprod_{i=1}^{j} Z_{w_i}$. Then, for each j, Z_j is closed in Z, Z_{w_j} is open in Z_j, and $Z_j = Z_{j-1} \coprod Z_{w_j}$. Notice that $Z_N = Z$ and $Z_1 = Z_{w_1}$.

Similarly, define $\widehat{Z}_j = \coprod_{i=1}^{j} \widehat{Z}_{w_i}$. Then each $\widehat{Z}_j$ is closed in $\widehat{Z}$, $\widehat{Z}_{w_j}$ is open in $\widehat{Z}_j$, and $\widehat{Z}_j = \widehat{Z}_{j-1} \coprod \widehat{Z}_{w_j}$.

We need to show that $r_*\colon H_\bullet(Z_1) \longrightarrow H_\bullet(Z)$ is an injective ring homomorphism.
Let $\mathrm{res}_j\colon H_i(Z_j) \to H_i(Z_{w_j})$ denote the restriction map in Borel-Moore homology induced by the open embedding $Z_{w_j} \subseteq Z_j$ and let $r_j\colon H_i(Z_{j-1}) \longrightarrow H_i(Z_j)$ denote the direct image map in Borel-Moore homology induced by the closed embedding $Z_{j-1} \subseteq Z_j$. Then there is a long exact sequence in homology

$$\cdots \longrightarrow H_i(Z_{j-1}) \xrightarrow{r_j} H_i(Z_j) \xrightarrow{\mathrm{res}_j} H_i(Z_{w_j}) \xrightarrow{\partial} H_{i-1}(Z_{j-1}) \longrightarrow \cdots$$

It is shown in [3, §6.2] that $\partial = 0$ and so the sequence

$$0 \longrightarrow H_i(Z_{j-1}) \xrightarrow{r_j} H_i(Z_j) \xrightarrow{\mathrm{res}_j} H_i(Z_{w_j}) \longrightarrow 0 \tag{3.1}$$

is exact for every i and j. Therefore, if $r\colon Z_j \to Z$ denotes the inclusion, then the direct image $r_*\colon H_i(Z_j) \to H_i(Z)$ is an injection for all i. (The fact that r depends on j should not lead to any confusion.)
We will frequently identify $H_i(Z_j)$ with its image in $H_i(Z)$ and consider $H_i(Z_j)$ as a subset of $H_i(Z)$. Thus, we have a flag of subspaces $0 \subseteq H_i(Z_1) \subseteq \cdots \subseteq H_i(Z_{N-1}) \subseteq H_i(Z)$.
In particular, $r_*\colon H_i(Z_1) \to H_i(Z)$ is an injection for all i. It follows from [3, Lemma 5.2.23] that r_* is a ring homomorphism. This proves part (b) of Theorem 2.3.

Proof of Theorem 2.3(c). We need to show that the linear transformation given by the convolution product $H_i(Z_1) \otimes H_{4n}(Z) \to H_i(Z)$ is an isomorphism of vector spaces for $0 \leq i \leq 4n$.
The proof is a consequence of the following lemma.

Lemma 3.2. *The image of the convolution map* $*: H_i(Z_1) \otimes H_{4n}(Z_j) \longrightarrow H_i(Z)$ *is precisely* $H_i(Z_j)$ *for* $0 \leq i \leq 4n$ *and* $1 \leq j \leq N$.

Assuming that the lemma has been proved, taking $j = N$, we conclude that the convolution product in $H_\bullet(Z)$ induces a surjection $H_i(Z_1) \otimes H_{4n}(Z) \longrightarrow H_i(Z)$. It is shown in [3, §6.2] that $\dim H_\bullet(Z) = |W|^2$ and so $\dim H_\bullet(Z_1) \otimes H_{4n}(Z) = |W|^2 = \dim H_\bullet(Z)$. Thus, the convolution product induces an isomorphism $H_i(Z_1) \otimes H_{4n}(Z) \cong H_i(Z)$.
The rest of this section is devoted to the proof of Lemma 3.2.
To prove Lemma 3.2 we need to analyze the specialization map, $\lim\colon H_{\bullet+2}(\Lambda_w^{\ell^*}) \to H_\bullet(Z)$, beginning with the subvarieties Λ_w^ℓ and $\Lambda_w^{\ell^*}$ of Λ_w.

Subvarieties of Λ_w. Suppose that ℓ is a one-dimensional subspace of $\mathfrak{t}$ with $\ell^* = \ell \setminus \{0\} = \ell \cap \mathfrak{t}_{\mathrm{reg}}$. Recall that $\mathfrak{u}_w = \mathfrak{u} \cap w \cdot \mathfrak{u}$ for w in W.

Lemma 3.3. *The variety* $\Lambda_w^\ell \cap \widehat{Z}_w$ *is the* G*-saturation in* $\widehat{Z}$ *of the subset*

$$\{\, (h+n, B, wBw^{-1}) \mid h \in \ell,\, n \in \mathfrak{u}_w \,\}.$$

Proof. By definition,

$$\Lambda_w^\ell = \Lambda_w^{\ell^*} \coprod \Lambda_w^0 = \{\, (x, B', B'') \in \widehat{Z} \mid \nu(x, B'') = w^{-1}\nu(x, B') \in w^{-1}(\ell) \,\}.$$

Suppose that h is in $\mathfrak{t}_{\mathrm{reg}}$ and $(x, g_1Bg_1^{-1}, g_2Bg_2^{-1})$ is in Λ_w^h. Then $g_1^{-1}\cdot x = h+n_1$ and $g_2^{-1} \cdot x = w^{-1}h + n_2$ for some n_1 and n_2 in $\mathfrak{u}$. Since h is regular, there are elements u_1 and u_2 in U so that $u_1^{-1}g_1^{-1} \cdot h = h$ and $u_2^{-1}g_2^{-1} \cdot h = w^{-1}h$. Then $x = g_1u_1 \cdot h = g_2u_2w^{-1} \cdot h$ and so $g_1u_1 = g_2u_2w^{-1}t$ for some t in T. Therefore, $(x, g_1Bg_1^{-1}, g_2Bg_2^{-1}) = g_1u_1 \cdot (h, B, wBw^{-1})$. Thus, Λ_w^h is contained in the G-orbit of (h, B, wBw^{-1}). Since ν is G-equivariant, it follows that Λ_w^h is G-stable and so Λ_w^h is the full G-orbit of (h, B, wBw^{-1}). Therefore, $\Lambda_w^{\ell^*}$ is the G-saturation of $\{\, (h+n, B, wBw^{-1} \mid h \in \ell^*,\, n \in \mathfrak{u}_w \,\}$ and $\Lambda_w^h \subseteq \widehat{Z}_w$ for h in ℓ^*.

We have already observed that $\Lambda_w^0 = Z$ and so

$$\Lambda_w^\ell \cap \widehat{Z}_w = \left(\Lambda_w^{\ell^*} \cap \widehat{Z}_w\right) \coprod \left(\Lambda_w^0 \cap \widehat{Z}_w\right) = \Lambda_w^{\ell^*} \coprod Z_w.$$

It is easy to see that Z_w is the G-saturation of $\{\, (n, B, wBw^{-1}) \mid n \in \mathfrak{u}_w \,\}$ in Z. This proves the lemma. □

Corollary 3.4. *The variety $\Lambda_w^\ell \cap \widehat{Z}_w$ is a locally trivial, affine space bundle over $\mathcal{O}_w$ with fibre isomorphic to $\ell + \mathfrak{u}_w$, and hence there is an isomorphism $\Lambda_w^\ell \cap \widehat{Z}_w \cong G \times^{B\cap {}^wB} (\ell + \mathfrak{u}_w)$.*

Proof. It follows from Lemma 3.3 that the map given by projection on the second and third factors is a G-equivariant morphism from Λ_w^ℓ onto $\mathcal{O}_w$ and that the fibre over (B, wBw^{-1}) is $\{\, (h+n, B, wBw^{-1}) \mid h \in \ell,\, n \in \mathfrak{u}_w \,\}$. Therefore, $\Lambda_w^\ell \cong G \times^{B\cap {}^wB} (\ell + \mathfrak{u}_w)$. □

Let $\mathfrak{g}_{\mathrm{rs}}$ denote the set of regular semisimple elements in $\mathfrak{g}$ and define $\widetilde{\mathfrak{g}}_{\mathrm{rs}} = \{\, (x, B') \in \widetilde{\mathfrak{g}} \mid x \in \mathfrak{g}_{\mathrm{rs}} \,\}$. For an arbitrary subset S of $\mathfrak{t}$, define

$$\widetilde{\mathfrak{g}}^S = \nu^{-1}(S) = \{\, (x, B') \in \widetilde{\mathfrak{g}} \mid \nu(x, B') \in S \,\}.$$

For w in W, define $\widetilde{w}\colon G/T \times \mathfrak{t}_{\mathrm{reg}} \longrightarrow G/T \times \mathfrak{t}_{\mathrm{reg}}$ by $\widetilde{w}(gT, h) = (gwT, w^{-1}h)$. The rule $(gT, h) \mapsto (g\cdot h, gB)$ defines an isomorphism of varieties

$$f\colon G/T \times \mathfrak{t}_{\mathrm{reg}} \xrightarrow{\cong} \widetilde{\mathfrak{g}}_{\mathrm{rs}}$$

and we will denote the automorphism $f \circ \widetilde{w} \circ f^{-1}$ of $\widetilde{\mathfrak{g}}_{\mathrm{rs}}$ also by $\widetilde{w}$. Notice that if h is in $\mathfrak{t}_{\mathrm{reg}}$ and g is in G, then $\widetilde{w}(g\cdot h, gB) = (g\cdot h, gwBw^{-1}g^{-1})$.

Lemma 3.5. *The variety $\Lambda_w^{\ell^*}$ is the graph of $\widetilde{w}|_{\widetilde{\mathfrak{g}}^{\ell^*}} \colon \widetilde{\mathfrak{g}}^{\ell^*} \to \widetilde{\mathfrak{g}}^{w^{-1}(\ell^*)}$.*

Proof. It follows from Lemma 3.3 that

$$\begin{aligned}\Lambda_w^{\ell^*} &= \{\, (g\cdot h, gBg^{-1}, gwBw^{-1}g^{-1}) \in \mathfrak{g}_{\mathrm{rs}} \times \mathcal{B} \times \mathcal{B} \mid h \in \ell^*,\, g \in G \,\} \\ &= \{\, ((g\cdot h, gBg^{-1}), (g\cdot h, gwBw^{-1}g^{-1})) \in \widetilde{\mathfrak{g}} \times \widetilde{\mathfrak{g}} \mid h \in \ell^*,\, g \in G \,\}.\end{aligned}$$

The argument in the proof of Lemma 3.3 shows that

$$\widetilde{\mathfrak{g}}^{\ell^*} = \{\, (g\cdot h, gBg^{-1}) \mid h \in \ell^*,\, g \in G \,\}$$

and by definition $\widetilde{w}(g\cdot h, gB) = (g\cdot h, gwBw^{-1}g^{-1})$. Therefore, $\Lambda_w^{\ell^*}$ is the graph of $\widetilde{w}|_{\widetilde{\mathfrak{g}}^{\ell^*}}$. □

Corollary 3.6. *The map $\nu_w\colon \Lambda_w^{\ell^*}\to \ell^*$ is a locally trivial fibration with fibre isomorphic to G/T.*

Proof. This follows from the lemma and the fact that $\widetilde{\mathfrak{g}}^{\ell^*}\cong G/T\times\ell^*$. □

The specialization map. Suppose that w is in W and that ℓ is a one-dimensional subspace of $\mathfrak{t}$ with $\ell^* = \ell\setminus\{0\} = \ell\cap\mathfrak{t}_{\mathrm{reg}}$. As in [4] and [3, §2.6.30], $\lim\colon H_{i+2}(\Lambda_w^{\ell^*})\to H_i(Z)$ is the composition of three maps, defined as follows.

As a vector space over $\mathbb{R}$, ℓ is two-dimensional. Fix an $\mathbb{R}$-basis of ℓ, say $\{v_1, v_2\}$. Define P to be the open half plane $\mathbb{R}_{>0}v_1\oplus\mathbb{R}v_2$, define $I_{>0}$ to be the ray $\mathbb{R}_{>0}v_1$, and define I to be the closure of $I_{>0}$, so $I=\mathbb{R}_{\geq 0}v_1$.

Since P is an open subset of ℓ^*, Λ_w^P is an open subset of $\Lambda_w^{\ell^*}$ and so there is a restriction map in Borel-Moore homology $\mathrm{res}\colon H_{i+2}(\Lambda_w^{\ell^*})\to H_{i+2}(\Lambda_w^P)$.

The projection map from P to $I_{>0}$ determines an isomorphism in Borel-Moore homology $\psi\colon H_{i+2}(\Lambda_w^P)\to H_{i+1}(\Lambda_w^{I_{>0}})$.

Since $I = I_{>0}\coprod\{0\}$, we have $\Lambda_w^I = \Lambda_w^{I_{>0}}\coprod\Lambda_w^0 = \Lambda_w^{I_{>0}}\coprod Z$, where Z is closed in Λ_w^I. The connecting homomorphism of the long exact sequence in Borel-Moore homology arising from the partition $\Lambda_w^I = \Lambda_w^{I_{>0}}\coprod Z$ is a map

$$\partial\colon H_{i+1}(\Lambda_w^{I_{>0}})\to H_i(Z).$$

By definition, $\lim = \partial\circ\psi\circ\mathrm{res}$.

Now fix j with $1\leq j\leq N$ and set $w=w_j$.

Consider the intersection $\Lambda_w^I\cap\widehat{Z}_j = (\Lambda_w^{I_{>0}}\cap\widehat{Z}_j)\coprod(Z\cap\widehat{Z}_j)$. Then $Z\cap\widehat{Z}_j$ is closed in $\Lambda_w^I\cap\widehat{Z}_j$ and by construction, $\Lambda_w^{I_{>0}}\subseteq\widehat{Z}_j$ and $Z\cap\widehat{Z}_j = Z_j$. Thus, $\Lambda_w^I\cap\widehat{Z}_j = \Lambda_w^{I_{>0}}\coprod Z_j$. Let $\partial_j\colon H_{i+1}(\Lambda_w^{I_{>0}})\to H_i(Z_j)$ be the connecting homomorphism of the long exact sequence in Borel-Moore homology arising from this partition. Because the long exact sequence in Borel-Moore homology is natural, we have a commutative square:

$$\begin{array}{ccc} H_{i+1}(\Lambda_w^{I_{>0}}) & \xrightarrow{\partial} & H_i(Z) \\ \Vert & & \uparrow{\scriptstyle r_*} \\ H_{i+1}(\Lambda_w^{I_{>0}}) & \xrightarrow[\partial_j]{} & H_i(Z_j) \end{array}$$

This proves the following lemma.

Lemma 3.7. *Fix j with $1\leq j\leq N$ and set $w=w_j$. Then the map $\partial\colon H_{i+1}(\Lambda_w^{I_{>0}})\longrightarrow H_i(Z)$ factors as $r_*\circ\partial_j$ where $\partial_j\colon H_{i+1}(\Lambda_w^{I_{>0}})\longrightarrow H_i(Z_j)$ is the connecting homomorphism of the long exact sequence arising from the partition $\Lambda_w^I\cap\widehat{Z} = \Lambda_w^{I_{>0}}\coprod Z_j$.*

It follows from the lemma that $\lim\colon H_{i+2}(\Lambda_w^{\ell^*}) \longrightarrow H_i(Z)$ factors as

$$H_{i+2}(\Lambda_w^{\ell^*}) \xrightarrow{\text{res}} H_{i+2}(\Lambda_w^P) \xrightarrow{\psi} H_{i+1}(\Lambda_w^{I>0}) \xrightarrow{\partial_j} H_i(Z_j) \xrightarrow{r_*} H_i(Z). \tag{3.8}$$

Define $\lim_j\colon H_{i+2}(\Lambda_w^{\ell^*}) \longrightarrow H_i(Z_j)$ by $\lim_j = \partial_j \circ \psi \circ \text{res}$.

Specialization and restriction. As above, fix j with $1 \leq j \leq N$ and a one-dimensional subspace ℓ of $\mathfrak{t}$ with $\ell^* = \ell \setminus \{0\} = \ell \cap \mathfrak{t}_{\text{reg}}$. Set $w = w_j$. Recall the restriction map $\text{res}_j\colon H_i(Z_j) \to H_i(Z_w)$ from (3.1).

Lemma 3.9. *The composition* $\text{res}_j \circ \lim_j\colon H_{i+2}(\Lambda_w^{\ell^*}) \longrightarrow H_i(Z_w)$ *is surjective for* $0 \leq i \leq 4n$.

Proof. Using (3.8), $\text{res}_j \circ \lim_j$ factors as

$$H_{i+2}(\Lambda_w^{\ell^*}) \xrightarrow{\text{res}} H_{i+2}(\Lambda_w^P) \xrightarrow{\psi} H_{i+1}(\Lambda_w^{I>0}) \xrightarrow{\partial_j} H_i(Z_j) \xrightarrow{\text{res}_j} H_i(Z_w).$$

Lemma 3.11 below shows that res is always surjective and the map ψ is an isomorphism, so we need to show that the composition $\text{res}_j \circ \partial_j$ is surjective. Consider $\Lambda_w^I \cap \widehat{Z}_w = (\Lambda_w^I \cap \widehat{Z}_j) \cap \widehat{Z}_w = \Lambda_w^{I>0} \coprod Z_w$. Then $\Lambda_w^{I>0}$ is open in $\Lambda_w^I \cap \widehat{Z}_w$ and we have a commutative diagram of long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & H_{i+1}(\Lambda_w^I \cap \widehat{Z}_w) \arrow[r] & H_{i+1}(\Lambda_w^{I>0}) \arrow[r, "\partial_w"] & H_i(Z_w) \arrow[r] & \cdots \\
\cdots \arrow[r] & H_{i+1}(\Lambda_w^I \cap \widehat{Z}_j) \arrow[r] \arrow[u] & H_{i+1}(\Lambda_w^{I>0}) \arrow[r, "\partial_j"] \arrow[u, equal] & H_i(Z_j) \arrow[r] \arrow[u, "\text{res}_j"'] & \cdots
\end{tikzcd}$$

where ∂_w is the connecting homomorphism of the long exact sequence arising from the partition $\Lambda_w^I \cap \widehat{Z}_w = \Lambda_w^{I>0} \coprod Z_w$. We have seen at the beginning of this section that res_j is surjective and so it is enough to show that ∂_w is surjective. Recall that $\{v_1, v_2\}$ is an $\mathbb{R}$-basis of ℓ and $I = \mathbb{R}_{\geq 0} v_1$. Define

$$\begin{aligned}
E_I &= G \times^{B\cap {}^wB} (\mathbb{R}_{\geq 0} v_1 + \mathfrak{u}_w), \\
E_{I_{>0}} &= G \times^{B\cap {}^wB} (\mathbb{R}_{>0} v_1 + \mathfrak{u}_w), \text{ and} \\
E_0 &= G \times^{B\cap {}^wB} \mathfrak{u}_w.
\end{aligned}$$

It follows from Corollary 3.4 that $E_I \cong \Lambda_w^I$, $E_{I_{>0}} \cong \Lambda_w^{I>0}$, and $E_0 \cong Z_w$, so the long exact sequence arising from the partition $\Lambda_w^I \cap \widehat{Z}_w = \Lambda_w^{I>0} \coprod Z_w$ may be identified with the long exact sequence arising from the partition $E_I = E_{I_{>0}} \coprod E_0$:

$$\cdots \longrightarrow H_{i+1}(E_I) \longrightarrow H_{i+1}(E_{I_{>0}}) \xrightarrow{\partial_E} H_i(E_0) \longrightarrow \cdots$$

Therefore, it is enough to show that ∂_E is surjective. In fact, we show that $H_\bullet(E_I) = 0$ and so ∂_E is an isomorphism.
Define $E_{\mathbb{R}} = G \times^{B\cap {}^wB} (\mathbb{R}v_1 + \mathfrak{u}_w)$. Then $E_{\mathbb{R}}$ is a smooth, real vector bundle over $G/B\cap {}^wB$ and so $E_{\mathbb{R}}$ is a smooth manifold containing E_I as a closed subset. We may apply [3, 2.6.1] and conclude that $H_i(E_I) \cong H^{4n+1-i}(E_{\mathbb{R}}, E_{\mathbb{R}} \setminus E_I)$.

Consider the cohomology long exact sequence of the pair $(E_{\mathbb{R}}, E_{\mathbb{R}} \setminus E_I)$. Since $E_{\mathbb{R}}$ is a vector bundle over $G/B \cap {}^wB$, it is homotopy equivalent to $G/B \cap {}^wB$. Similarly, $E_{\mathbb{R}} \setminus E_I \cong G \times^{B\cap {}^wB} (\mathbb{R}_{<0}v_1 + \mathfrak{u}_w)$ and so is also homotopy equivalent to $G/B \cap {}^wB$. Therefore, $H^i(E_{\mathbb{R}}) \cong H^i(E_{\mathbb{R}} \setminus E_I)$ and it follows that the relative cohomology group $H^i(E_{\mathbb{R}}, E_{\mathbb{R}} \setminus E_I)$ is trivial for every i. Therefore, $H_\bullet(E_I) = 0$, as claimed.

This completes the proof of the lemma. □

Corollary 3.10. *The specialization map* $\lim_1 \colon H_{i+2}(\Lambda_1^{\ell^*}) \longrightarrow H_i(Z_1)$ *is surjective for* $0 \leq i \leq 4n$.

Proof. This follows from Lemma 3.9, because $Z_1 = Z_{w_1}$ and so res_1 is the identity map. □

The next lemma is true for any specialization map.

Lemma 3.11. *The restriction map* $\mathrm{res} \colon H_{i+2}(\Lambda_w^{\ell^*}) \longrightarrow H_{i+2}(\Lambda_w^P)$ *is surjective for every* w *in* W *and every* $i \geq 0$.

Proof. There are homeomorphisms $\Lambda_w^{\ell^*} \cong G/T \times \ell^*$ and $\Lambda_w^P \cong G/T \times P$. By definition, P is an open subset of ℓ^* and so there is a restriction map $\mathrm{res} \colon H_2(\ell^*) \to H_2(P)$. This map is a non-zero linear transformation between one-dimensional $\mathbb{Q}$-vector spaces so it is an isomorphism.

Using the Künneth formula we get a commutative square where the horizontal maps are isomorphisms and the right-hand vertical map is surjective:

$$\begin{tikzcd}
H_{i+2}(\Lambda_w^{\ell^*}) \arrow[r, "\cong"] \arrow[d, "\mathrm{res}"'] & H_i(G/T) \otimes H_2(\ell^*) + H_{i+1}(G/T) \otimes H_1(\ell^*) \arrow[d, "id\otimes\mathrm{res}+0"] \\
H_{i+2}(\Lambda_w^P) \arrow[r, "\cong"'] & H_i(G/T) \otimes H_2(P)
\end{tikzcd}$$

It follows that $\mathrm{res} \colon H_{i+2}(\Lambda_w^{\ell^*}) \to H_{i+2}(\Lambda_w^P)$ is surjective. □

Proof of Lemma 3.2. Fix i with $0 \leq i \leq 4n$. We show that the image of the convolution map $* : H_i(Z_1) \otimes H_{4n}(Z_j) \longrightarrow H_i(Z)$ is precisely $H_i(Z_j)$ for $1 \leq j \leq N$ using induction on j.

For $j = 1$, $H_{4n}(Z_1)$ is one-dimensional with basis $\{\lambda_1\}$. It follows from Theorem 2.2(c) that λ_1 is the identity in $H_\bullet(Z)$ and so clearly the image of the convolution map $H_i(Z_1) \otimes H_{4n}(Z_1) \longrightarrow H_i(Z)$ is precisely $H_i(Z_1)$.

Assume that $j > 1$ and set $w = w_j$. We will complete the proof using the commutative diagram with exact rows

$$\begin{tikzcd}
H_i \otimes H_{4n}(Z_{j-1}) \arrow[r, hook, "id\otimes(r_j)_*"] \arrow[d, "*"'] & H_i \otimes H_{4n}(Z_j) \arrow[r, two heads, "id\otimes\mathrm{res}_j"] \arrow[d, "*"'] & H_i \otimes H_{4n}(Z_w) \arrow[d, "*"'] \\
H_i(Z_{j-1}) \arrow[r, hook, "(r_j)_*"] & H_i(Z_j) \arrow[r, two heads, "\mathrm{res}_j"] & H_i(Z_w)
\end{tikzcd} \tag{3.12}$$

and the Five Lemma, where in the first line H_i means $H_i(Z_1)$. We saw in (3.1) that the bottom row is exact and it follows that the top row is also exact. By induction, the convolution product in $H_\bullet(Z)$ determines a surjective map $*: H_i(Z_1)\otimes H_{4n}(Z_{j-1}) \longrightarrow H_i(Z_{j-1})$. To conclude from the Five Lemma that the middle vertical map is a surjection, it remains to define the other vertical maps so that the diagram commutes and to show that the right-hand vertical map is a surjection.

First we show that the image of the map $H_i(Z_1)\otimes H_{4n}(Z_j) \longrightarrow H_i(Z_j)$ determined by the convolution product in $H_\bullet(Z)$ is contained in $H_i(Z_j)$. It then follows that the middle vertical map in (3.12) is defined and so by exactness there is an induced map from $H_i(Z_1)\otimes H_{4n}(Z_w)$ to $H_i(Z_w)$ so that the diagram (3.12) commutes. Second we show that the right-hand vertical map is a surjection.

By Lemma 3.5, $\Lambda_1^{\ell^*}$ is the graph of the identity map of $\widetilde{\mathfrak{g}}^{\ell^*}$, and $\Lambda_w^{\ell^*}$ is the graph of $\widetilde{w}|_{\widetilde{\mathfrak{g}}^{\ell^*}}$. Therefore, $\Lambda_1^{\ell^*}\circ\Lambda_w^{\ell^*}=\Lambda_w^{\ell^*}$ and there is a convolution product

$$H_{i+2}(\Lambda_1^{\ell^*})\otimes H_{4n+2}(\Lambda_w^{\ell^*}) \xrightarrow{\ *\ } H_{i+2}(\Lambda_w^{\ell^*}).$$

Suppose a is in $H_i(Z_1)$. Then by Corollary 3.10, $a=\lim_1(a_1)$ for some a_1 in $H_{i+2}(\Lambda_1^{\ell^*})$. It is shown in [3, Proposition 2.7.23] that specialization commutes with convolution, so $\lim(a_1*[\Lambda_w^{\ell^*}])=\lim(a_1)*\lim([\Lambda_w^{\ell^*}])=a*\lambda_w$. Also, $a_1*[\Lambda_w^{\ell^*}]$ is in $H_{i+2}(\Lambda_w^{\ell^*})$ and $\lim=r_*\circ\lim_j$ and so $a*\lambda_w=r_*\circ\lim_j(a_1*[\Lambda_w^{\ell^*}])$ is in $H_i(Z_j)$. By induction, if $k<j$, then $a*\lambda_{w_k}$ is in $H_i(Z_k)$ and so $a*\lambda_{w_k}$ is in $H_i(Z_k)$. Since the set $\{\,\lambda_{w_k}\mid 1\le k\le j\,\}$ is a basis of $H_{4n}(Z_j)$, it follows that $a*H_{4n}(Z_j)\subseteq H_i(Z_j)$. Therefore, the image of the convolution map $H_i(Z_1)\otimes H_{4n}(Z_j)\longrightarrow H_i(Z)$ is contained in $H_i(Z_j)$.

To complete the proof of Lemma 3.2, we need to show that the induced map from $H_i(Z_1)\otimes H_{4n}(Z_w)$ to $H_i(Z_w)$ is surjective.

Consider the following diagram:

$$\begin{tikzcd}
H_{i+2}(\Lambda_1^{\ell^*})\otimes H_{4n+2}(\Lambda_w^{\ell^*}) \arrow[r, "*"] \arrow[d, "\lim_1\otimes\lim_j"'] & H_{i+2}(\Lambda_w^{\ell^*}) \arrow[d, "\lim_j"] \\
H_i(Z_1)\otimes H_{4n}(Z_j) \arrow[r, "*"] \arrow[d, "id\otimes\mathrm{res}_j"'] & H_i(Z_j) \arrow[d, "\mathrm{res}_j"] \\
H_i(Z_1)\otimes H_{4n}(Z_w) \arrow[r, "*"] & H_i(Z_w)
\end{tikzcd}$$

We have seen that the bottom square is commutative. It follows from the fact that specialization commutes with convolution that the top square is also commutative. It is shown in Proposition A.2 that the convolution product $H_{i+2}(\Lambda_1^{\ell^*})\otimes H_{4n+2}(\Lambda_w^{\ell^*})\to H_{i+2}(\Lambda_w^{\ell^*})$ is an injection. Since $H_{i+2}(\Lambda_1^{\ell^*})$ is finite-dimensional and $H_{4n+2}(\Lambda_w^{\ell^*})$ is one-dimensional, it follows that this convolution mapping is an isomorphism. Also, we saw in Lemma 3.9 that $\mathrm{res}_j\circ\lim_j$ is surjective. Therefore, the composition $\mathrm{res}_j\circ\lim_j\circ *$ is surjective and it follows that the bottom convolution map $H_i(Z_1)\otimes H_{4n}(Z_w)\to H_i(Z_w)$ is also surjective. This completes the proof of Lemma 3.2.

4. Smash Product Structure

In this section we prove Theorem 2.4. We need to show that $\lambda_w * H_i(Z_1) * \lambda_{w^{-1}} = H_i(Z_1)$ and that $\beta\colon \operatorname{Coinv}_\bullet(W) \xrightarrow{\cong} H_{4n-\bullet}(Z_1)$ is an isomorphism of W-algebras.

Suppose that ℓ is a one-dimensional subspace of $\mathfrak{t}$ so that $\ell^* = \ell \setminus \{0\} = \ell \cap \mathfrak{t}_{\mathrm{reg}}$. Recall that for $S \subseteq \mathfrak{t}$, $\widetilde{\mathfrak{g}}^S = \nu^{-1}(S)$. By Lemma 3.5, if w is in W, then $\Lambda_w^{\ell^*}$ is the graph of the restriction of $\widetilde{w}$ to $\widetilde{\mathfrak{g}}^{\ell^*}$. It follows that there is a convolution product

$$H_{4n+2}(\Lambda_w^{\ell^*}) \otimes H_{i+2}(\Lambda_1^{w^{-1}(\ell^*)}) \otimes H_{4n+2}(\Lambda_w^{w^{-1}(\ell^*)}) \xrightarrow{\ *\ } H_{i+2}(\Lambda_1^{\ell^*}).$$

Because specialization commutes with convolution, the diagram

$$\begin{CD}
H_{4n+2}(\Lambda_w^{\ell^*}) \otimes H_{i+2}(\Lambda_1^{w^{-1}(\ell^*)}) \otimes H_{4n+2}(\Lambda_w^{w^{-1}(\ell^*)}) @>{*}>> H_{i+2}(\Lambda_1^{\ell^*}) \\
@V{\lim\otimes\lim\otimes\lim}VV @VV{\lim}V \\
H_{4n}(Z) \otimes H_i(Z_1) \otimes H_{4n}(Z) @>>{*}> H_i(Z)
\end{CD}$$

commutes.

We saw in Corollary 3.10 that $\lim_1\colon H_{i+2}(\Lambda_1^{\ell^*}) \to H_i(Z_1)$ is surjective. Thus, if c is in $H_i(Z_1)$, then $c = \lim(c_1)$ for some c_1 in $H_{i+2}(\Lambda_{w_1}^{w^{-1}(\ell^*)})$. Therefore,

$$\lambda_w * c * \lambda_{w^{-1}} = \lim([\Lambda_w^{\ell^*}]) * \lim(c_1) * \lim([\Lambda_{w^{-1}}^{w^{-1}(\ell^*)}]) = \lim\left([\Lambda_w^{\ell^*}] * c_1 * [\Lambda_{w^{-1}}^{w^{-1}(\ell^*)}]\right).$$

Since $\Lambda_w^{\ell^*}$ and $\Lambda_{w^{-1}}^{w^{-1}(\ell^*)}$ are the graphs of $\widetilde{w}$ and $\widetilde{w}^{-1}$ respectively, and $\Lambda_1^{w^{-1}(\ell^*)}$ is the graph of the identity function, it follows that $[\Lambda_w^{\ell^*}] * c_1 * [\Lambda_{w^{-1}}^{w^{-1}(\ell^*)}]$ is in $H_{i+2}(\Lambda_1^{w^{-1}(\ell^*)})$ and so by (3.8), $\lambda_w * c * \lambda_{w^{-1}}$ is in $H_i(Z_1)$. This shows that $\lambda_w * H_i(Z_1) * \lambda_{w^{-1}} = H_i(Z_1)$ for all i.

To complete the proof of Theorem 2.4 we need to show that if w is in W and f is in $\operatorname{Coinv}_i(W)$, then $\beta(w \cdot f) = \lambda_w * \beta(f) * \lambda_{w^{-1}}$ where $w \cdot f$ denotes the natural action of w on f. To do this, we need some preliminary results.

First, since $\Lambda_1^{\ell^*}$ is the diagonal in $\widetilde{\mathfrak{g}}^{\ell^*} \times \widetilde{\mathfrak{g}}^{\ell^*}$, it is obvious that

$$\delta \circ \widetilde{w}^{-1} = (\widetilde{w}^{-1} \times \widetilde{w}^{-1}) \circ \delta\colon \widetilde{\mathfrak{g}}^{w^{-1}(\ell^*)} \longrightarrow \Lambda_1^{\ell^*}.$$

Therefore,

$$\delta_* \circ \widetilde{w}_*^{-1} = (\widetilde{w}^{-1} \times \widetilde{w}^{-1})_* \circ \delta_*\colon H_i(\widetilde{\mathfrak{g}}^{w^{-1}(\ell^*)}) \longrightarrow H_i(\Lambda_1^{\ell^*}) \tag{4.1}$$

for all i. (The first δ in (4.1) is the diagonal embedding $\widetilde{\mathfrak{g}}^{\ell^*} \cong \Lambda_1^{\ell^*}$ and the second δ is the diagonal embedding $\widetilde{\mathfrak{g}}^{w^{-1}(\ell^*)} \cong \Lambda_1^{w^{-1}(\ell^*)}$.)

Next, with $\ell \subseteq \mathfrak{t}$ as above, $\widetilde{\mathfrak{g}}^\ell = \widetilde{\mathfrak{g}}^{\ell^*} \coprod \nu^{-1}(0) = \widetilde{\mathfrak{g}}^{\ell^*} \coprod \widetilde{\mathcal{N}}$ and the restriction of $\nu\colon \widetilde{\mathfrak{g}}^\ell \to \ell$ to $\widetilde{\mathfrak{g}}^{\ell^*}$ is a locally trivial fibration. Therefore, there is a specialization map $\lim_0\colon H_{i+2}(\widetilde{\mathfrak{g}}^{\ell^*}) \to H_i(\widetilde{\mathcal{N}})$. Since $\delta_*\colon H_{i+2}(\widetilde{\mathfrak{g}}^{\ell^*}) \to H_{i+2}(\Lambda_1^{\ell^*})$ and $\delta_*\colon H_i(Z) \to H_i(Z_1)$ are isomorphisms, the next lemma is obvious.

Lemma 4.2. *Suppose that ℓ is a one-dimensional subspace of $\mathfrak{t}$ so that $\ell^* = \ell \setminus \{0\} \subseteq \mathfrak{t}_{\mathrm{reg}}$. Then the diagram*

$$\begin{array}{ccc} H_{i+2}(\widetilde{\mathfrak{g}}^{\ell^*}) & \xrightarrow{\delta_*} & H_{i+2}(\Lambda_1^{\ell^*}) \\ \downarrow{\scriptstyle \lim_0} & & \downarrow{\scriptstyle \lim_1} \\ H_i(Z) & \xrightarrow[\delta_*]{} & H_i(Z_1) \end{array}$$

commutes.

Finally, $\widetilde{\mathcal{N}} \times_{\mathcal{N}} \mathcal{N} = \widetilde{\mathcal{N}}$ and so $Z \circ \widetilde{\mathcal{N}} = (\widetilde{\mathcal{N}} \times_{\mathcal{N}} \widetilde{\mathcal{N}}) \circ (\widetilde{\mathcal{N}} \times_{\mathcal{N}} \mathcal{N}) = \widetilde{\mathcal{N}} \times_{\mathcal{N}} \mathcal{N}$. Thus, there is a convolution action, $H_{4n}(Z) \otimes H_i(\widetilde{\mathcal{N}}) \longrightarrow H_i(\widetilde{\mathcal{N}})$, of $H_{4n}(Z)$ on $H_i(\widetilde{\mathcal{N}})$.

Suppose that w is in W and z is in $H^i(\mathcal{B})$. Then $\pi^* \circ \mathrm{pd}(z)$ is in $H_{4n-i}(\widetilde{\mathcal{N}})$ and so $\lambda_w * (\pi^* \circ \mathrm{pd}(z))$ is in $H_{4n-i}(\widetilde{\mathcal{N}})$. It is shown in [3, Proposition 7.3.31] that for y in $H_\bullet(\mathcal{B})$, $\lambda_w * \pi^*(y) = \epsilon_w \pi^*(w \cdot y)$ where ϵ_w is the sign of w and $w \cdot y$ denotes the action of W on $H_\bullet(\mathcal{B})$ coming from the action of W on G/T and the homotopy equivalence $G/T \simeq \mathcal{B}$. It is also shown in [3, Proposition 7.3.31] that $\mathrm{pd}(w \cdot z) = \epsilon_w w \cdot \mathrm{pd}(z)$. Therefore,

$$\lambda_w * (\pi^* \circ \mathrm{pd}(z)) = \epsilon_w \pi^*(w \cdot \mathrm{pd}(z)) = \epsilon_w \epsilon_w \pi^* \circ \mathrm{pd}(w \cdot z) = \pi^* \circ \mathrm{pd}(w \cdot z).$$

This proves the next lemma.

Lemma 4.3. *If w is in W and z is in $H_i(\mathcal{B})$, then*

$$\lambda_w * (\pi^* \circ \mathrm{pd}(z)) = \pi^* \circ \mathrm{pd}(w \cdot z).$$

Proof of Theorem 2.4. Fix w in W and f in $\mathrm{Coinv}_i(W)$. We need to show that $\lambda_w * \beta(f) * \lambda_{w^{-1}} = \beta(w \cdot f)$. Set $C = \lambda_w * \beta(f) * \lambda_{w^{-1}}$. Using the fact that $\beta = \delta_* \circ \pi^* \circ \mathrm{pd} \circ \mathrm{bi}$ we compute

$$\begin{aligned} C &= \lim_1 \left([\Lambda_w^{\ell^*}] * \lim_1^{-1}(\beta(f)) * [\Lambda_{w^{-1}}^{w^{-1}(\ell^*)}] \right) && [3, 2.7.23] \\ &= \lim_1 \circ (\widetilde{w}^{-1} \times \widetilde{w}^{-1})_* \circ \lim_1^{-1} \circ \beta(f) && \text{Proposition A.3} \\ &= \lim_1 \circ \delta_* \circ \widetilde{w}_*^{-1} \circ \delta_*^{-1} \circ \lim_1^{-1} \circ \beta(f) && (4.1) \\ &= \delta_* \circ \lim_0 \circ \widetilde{w}_*^{-1} \circ \delta_*^{-1} \circ \lim_1^{-1} \circ \delta_* \circ \delta_*^{-1} \circ \beta(f) && \text{Lemma 4.2} \\ &= \delta_* \circ \lim_0 \circ \widetilde{w}_*^{-1} \circ \lim_0^{-1} \circ \delta_*^{-1} \circ \beta(f) && \text{Lemma 4.2} \\ &= \delta_* \circ \lim_0 \circ \widetilde{w}_*^{-1} \circ \lim_0^{-1} \circ \pi^* \circ \mathrm{pd} \circ \mathrm{bi}(f) && \\ &= \delta_* \circ \lim_0 \left((\lim_0^{-1} \circ \pi^* \circ \mathrm{pd} \circ \mathrm{bi}(f)) * [\Lambda_{w^{-1}}^{w^{-1}(\ell^*)}] \right) && [3, 2.7.11] \\ &= \delta_* \left((\pi^* \circ \mathrm{pd} \circ \mathrm{bi}(f)) * \lambda_{w^{-1}} \right) && [3, 2.7.23] \\ &= \delta_* \left(\lambda_w * (\pi^* \circ \mathrm{pd} \circ \mathrm{bi}(f)) \right) && \text{Lemma A.1 and } [3, 3.6.11] \\ &= \delta_* \circ \pi^* \circ \mathrm{pd}(w \cdot \mathrm{bi}(f)) && \text{Lemma 4.3} \\ &= \delta_* \circ \pi^* \circ \mathrm{pd} \circ \mathrm{bi}(w \cdot f) && \mathrm{bi} \text{ is } W\text{-equivariant} \\ &= \beta(w \cdot f). \end{aligned}$$

This completes the proof of Theorem 2.4.

Appendix A. Convolution and Graphs

In this appendix we prove some general properties of convolution and graphs. Suppose M_1, M_2, and M_3 are smooth varieties, $\dim M_2 = d$, and that $Z_{1,2} \subseteq M_1 \times M_2$ and $Z_{2,3} \subseteq M_2 \times M_3$ are two closed subvarieties so that the convolution product,

$$H_i(Z_{1,2}) \otimes H_j(Z_{2,3}) \xrightarrow{\ *\ } H_{i+j-2d}(Z_{1,2} \circ Z_{2,3}),$$

in [3, §2.7.5] is defined. For $1 \leq i, j \leq 3$, let $\tau_{i,j} \colon M_i \times M_j \to M_j \times M_i$ be the map that switches the factors. Define $Z_{2,1} = \tau_{1,2}(Z_{1,2}) \subseteq M_2 \times M_1$ and $Z_{3,2} = \tau_{2,3}(Z_{2,3}) \subseteq M_3 \times M_2$. Then the convolution product

$$H_j(Z_{3,2}) \otimes H_i(Z_{2,1}) \xrightarrow{\ *'\ } H_{i+j-2d}(Z_{3,2} \circ Z_{2,1})$$

is defined. We omit the easy proof of the following lemma.

Lemma A.1. *If c is in $H_i(Z_{1,2})$ and d is in $H_j(Z_{2,3})$, then $(\tau_{1,3})_*(c * d) = (\tau_{2,3})_*(d) *' (\tau_{1,2})_*(c)$.*

Now suppose X is an irreducible, smooth, m-dimensional variety, Y is a smooth variety, and $f \colon X \to Y$ is a morphism. Then if Γ_X and Γ_f denote the graphs of id_X and f respectively, using the notation in [3, §2.7], we have $\Gamma_X \circ \Gamma_f = \Gamma_f$ and there is a convolution product $* \colon H_i(\Gamma_X) \otimes H_{2m}(\Gamma_f) \longrightarrow H_i(\Gamma_f)$.

Proposition A.2. *The convolution product $* \colon H_i(\Gamma_X) \otimes H_{2m}(\Gamma_f) \longrightarrow H_i(\Gamma_f)$ is an injection.*

Proof. For $i, j = 1, 2, 3$, let $p_{i,j}$ denote the projection of $X \times X \times Y$ on the i^{th} and j^{th} factors. Then the restriction of $p_{1,3}$ to $(\Gamma_X \times Y) \cap (X \times \Gamma_f)$ is the map that sends $(x, x, f(x))$ to $(x, f(x))$. Thus, the restriction of $p_{1,3}$ to $(\Gamma_X \times Y) \cap (X \times \Gamma_f)$ is an isomorphism onto Γ_f and hence is proper. Therefore, the convolution product in homology is defined.

Since X is irreducible, so is Γ_f and so $H_{2m}(\Gamma_f)$ is one-dimensional with basis $[\Gamma_f]$. Suppose that c is in $H_i(\Gamma_X)$. We need to show that if $c * [\Gamma_f] = 0$, then $c = 0$.

Fix c in $H_i(\Gamma_X)$. Notice that the restriction of $p_{1,3}$ to $(\Gamma_X \times Y) \cap (X \times \Gamma_f)$ is the same as the restriction of $p_{2,3}$ to $(\Gamma_X \times Y) \cap (X \times \Gamma_f)$. Thus, using the projection formula, we have

$$\begin{aligned} c * [\Gamma_f] &= (p_{1,3})_* \left(p_{1,2}^* c \cap p_{2,3}^* [\Gamma_f] \right) \\ &= (p_{2,3})_* \left(p_{1,2}^* c \cap p_{2,3}^* [\Gamma_f] \right) \\ &= \left((p_{2,3})_* p_{1,2}^* c \right) \cap [\Gamma_f], \end{aligned}$$

where the intersection product in the last line is from the cartesian square:

$$\begin{tikzcd} \Gamma_f \arrow[r, "="] \arrow[d] & \Gamma_f \arrow[d] \\ X \times Y \arrow[r, "="'] & X \times Y \end{tikzcd}$$

Let $p\colon X \times Y \to X$ and $q\colon \Gamma_X \to X$ be the first and second projections, respectively. Then the square

$$\begin{tikzcd} \Gamma_X \times Y \arrow[r, "p_{2,3}"] \arrow[d, "p_{1,2}"'] & X \times Y \arrow[d, "p"] \\ \Gamma_X \arrow[r, "q"'] & X \end{tikzcd}$$

is cartesian. Thus,

$$\begin{aligned} p_*\left(c * [\Gamma_f]\right) &= p_*\left(\left((p_{2,3})_* p_{1,2}^* c\right) \cap [\Gamma_f]\right) \\ &= p_*\left((p^* q_* c) \cap [\Gamma_f]\right) \\ &= q_* c \cap (p|_{\Gamma_f})_* [\Gamma_f] \\ &= q_* c \cap [X] \\ &= q_* c, \end{aligned}$$

where we have used the projection formula and the fact that $(p|_{\Gamma_f})_*[\Gamma_f] = [X]$. Now if $c * [\Gamma_f] = 0$, then $q_* c = 0$ and so $c = 0$, because q is an isomorphism. $\square$

Let Γ_Y denote the graph of the identity functions id_Y. Then the following compositions and convolution products in Borel-Moore homology are defined:

- $\Gamma_f \circ \Gamma_X = \Gamma_f$ and so there is a convolution product
$$H_i(\Gamma_f) \otimes H_j(\Gamma_X) \longrightarrow H_{i+j-m}(\Gamma_f).$$
- $\Gamma_Y \circ \Gamma_{f^{-1}} = \Gamma_{f^{-1}}$ and so there is a convolution product
$$H_i(\Gamma_X) \otimes H_j(\Gamma_{f^{-1}}) \longrightarrow H_{i+j-m}(\Gamma_{f^{-1}}).$$
- $\Gamma_f \circ \Gamma_{f^{-1}} = \Gamma_X$ and so there is a convolution product
$$H_i(\Gamma_f) \otimes H_j(\Gamma_{f^{-1}}) \longrightarrow H_{i+j-m}(\Gamma_X).$$

Thus, if c is in $H_i(\Gamma_Y)$, then $[\Gamma_f] * c * [\Gamma_{f^{-1}}]$ is in $H_i(\Gamma_X)$. Note that the map $f^{-1} \times f^{-1}\colon \Gamma_Y \to \Gamma_X$ is an isomorphism, so in particular it is proper.

Proposition A.3. *If c is in $H_i(\Gamma_Y)$, then $[\Gamma_f] * c * [\Gamma_{f^{-1}}] = (f^{-1} \times f^{-1})_*(c)$.*

Proof. We compute $([\Gamma_f] * c) * [\Gamma_{f^{-1}}]$, starting with $[\Gamma_f] * c$.
For $1 \le i, j \le 3$ let $q_{i,j}$ be the projection of the subset

$$\Gamma_f \times Y \cap X \times \Gamma_Y = \{\, (x, f(x), f(x)) \mid x \in X \,\}$$

of $X \times Y \times Y$ onto the i,j-factors. Then $q_{1,3} = q_{1,2}$. Therefore, using the projection formula, we see that

$$\begin{aligned} [\Gamma_f] * c &= (q_{1,3})_* \left(q_{1,2}^*[\Gamma_f] \cap q_{2,3}^* c\right) \\ &= (q_{1,2})_* \left(q_{1,2}^*[\Gamma_f] \cap q_{2,3}^* c\right) \\ &= [\Gamma_f] \cap (q_{1,2})_* q_{2,3}^* c \\ &= (q_{1,2})_* q_{2,3}^* c. \end{aligned}$$

Next, for $1 \leq i, j \leq 3$ let $p_{i,j}$ be the projection of the subset

$$\Gamma_f \times X \cap X \times \Gamma_{f^{-1}} = \{\, (x, f(x), x) \mid x \in X \,\}$$

of $X \times Y \times X$ onto the i, j-factors. Then $p_{1,3} = (f^{-1} \times id) \circ p_{2,3}$. Therefore, using the fact that $[\Gamma_f] * c = (q_{1,2})_* q_{2,3}^* c$ and the projection formula, we have

$$\begin{aligned}
([\Gamma_f] * c) * [\Gamma_{f^{-1}}] &= (p_{1,3})_* \left(p_{1,2}^*((q_{1,2})_* q_{2,3}^* c) \cap p_{2,3}^*[\Gamma_{f^{-1}}]\right) \\
&= (f^{-1} \times id)_* (p_{2,3})_* \left(p_{1,2}^*((q_{1,2})_* q_{2,3}^* c) \cap p_{2,3}^*[\Gamma_{f^{-1}}]\right) \\
&= (f^{-1} \times id)_* \left((p_{2,3})_* p_{1,2}^* (q_{1,2})_* q_{2,3}^* c \cap [\Gamma_{f^{-1}}]\right) \\
&= (f^{-1} \times id)_* (p_{2,3})_* p_{1,2}^* (q_{1,2})_* q_{2,3}^* c.
\end{aligned}$$

The commutative square

$$\begin{tikzcd}
\Gamma_f \times X \cap X \times \Gamma_{f^{-1}} \arrow[r, "id \times id \times f"] \arrow[d, "id"'] & \Gamma_f \times Y \cap X \times \Gamma_Y \arrow[d, "q_{1,2}"] \\
\Gamma_f \times X \cap X \times \Gamma_{f^{-1}} \arrow[r, "p_{1,2}"'] & \Gamma_f
\end{tikzcd}$$

is cartesian, so $p_{1,2}^*(q_{1,2})_* = (id \times id \times f)^*$.
Also, the commutative square

$$\begin{tikzcd}
\Gamma_f \times X \cap X \times \Gamma_{f^{-1}} \arrow[r, "q_{2,3} \circ (id \times id \times f)"] \arrow[d, "(f^{-1} \times id) \circ p_{2,3}"'] & \Gamma_Y \arrow[d, "f^{-1} \times f^{-1}"] \\
\Gamma_X \arrow[r, "id"'] & \Gamma_X
\end{tikzcd}$$

is cartesian, so $(f^{-1} \times id)_* (p_{2,3})_* (id \times id \times f)^* q_{2,3}^* = (f^{-1} \times f^{-1})_*$.
Therefore,

$$\begin{aligned}
([\Gamma_f] * c) * [\Gamma_{f^{-1}}] &= (f^{-1} \times id)_* (p_{2,3})_* p_{1,2}^* (q_{1,2})_* q_{2,3}^* c \\
&= (f^{-1} \times id)_* (p_{2,3})_* (id \times id \times f)^* q_{2,3}^* c \\
&= (f^{-1} \times f^{-1})_* c.
\end{aligned}$$

This completes the proof of the proposition. □

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J. Matthew Douglass	Gerhard Röhrle
Department of Mathematics	Fakultät für Mathematik
University of North Texas	Ruhr-Universität Bochum
Denton TX	D-44780 Bochum
USA 76203	Germany
douglass@unt.edu	gerhard.roehrle@rub.de