Local Classes and Pairwise Mutually Permutable Products of Finite Groups

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Abstract. The main aim of the paper is to present some results about products of pairwise mutually permutable subgroups and local classes.

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1 Introduction

If A and B are subgroups of a group G, the product AB of A and B is defined to be the subset of all elements of G with the form ab, where $a \in A, b \in B$. It is well known that AB is a subgroup of G if and only if $AB = BA$, that is, if the subgroups A and B permute. Should it happen that AB coincides with the group G, with the result that $G = AB = BA$, then G is said to be factorized by its subgroups A and B. More generally, a group G is said to be the product of its pairwise permutable subgroups $G_1, G_2, \ldots, G_n$ if $G = G_1G_2 \ldots G_n$ and $G_iG_j = G_jG_i$ for all integers i and j with $i, j \in \{1, 2, \ldots, n\}$. This implies that for every choice of indices $1 \leq i_1 \leq i_2 \leq \ldots \leq i_k \leq n$, the product $G_{i_1}G_{i_2} \ldots G_{i_k}$ is a subgroup of G. Groups which are product of two of its subgroups have played a significant part in the theory of groups over the past sixty years. Among the central problems considered the following ones are of interest to us:

Let the group $G = G_1G_2 \ldots G_n$ be the product of its pairwise permutable subgroups $G_1, G_2, \ldots, G_n$ and suppose that the factors G_i, $1 \leq i \leq n$, belong to a class of groups $\mathcal{X}$. When does the group G belong to $\mathcal{X}$?. How does the *structure of the factors G_i, $1 \leq i \leq n$ affect the structure of the group G?.*

Obviously, if G_i, $1 \leq i \leq n$, are finite, then the group G is finite. However not many properties carry over from the factors of a factorized group to the group itself. Indeed if one thinks about properties such as solubility, supersolubility, or nilpotency, one soon realizes the difficulty of using factorization to obtain information about the structure of the whole group. Two well known examples support the above claim: there exist non abelian groups which are products of two abelian subgroups and every finite soluble group is the product of pairwise permutable nilpotent subgroups. However a prominent result by Itô shows that every product of two abelian groups is metabelian, and an important result of Kegel and Wielandt shows the solubility of every finite group $G = G_1G_2 \dots G_n$ which is the product of pairwise permutable nilpotent subgroups G_i, $1 \leq i \leq n$. In the much more special case when G_i, $1 \leq i \leq n$, are normal nilpotent subgroups of G, the product $G_1G_2 \dots G_n$ is nilpotent. This is a well known result of Fitting. However, if $G_1, G_2, \dots, G_n$ are normal supersoluble subgroups of G, the product $G_1G_2 \dots G_n$ is not supersoluble in general even in the finite case (see [1]). Consequently it seems reasonable to look into these problems under additional assumptions. In this context, assumptions on permutability connections between the factors turn out to be very useful. One of the most important ones is the mutual permutability introduced by Asaad and Shaalan in [1]. We say that two subgroups A and B of a group G are mutually permutable if A permutes with every subgroup of B and B permutes with every subgroup of A. If $G = AB$ and A and B are mutually permutable, then G is called a mutually permutable product of A and B. More generally, a group $G = G_1G_2 \dots G_n$ is said to be the product of the pairwise mutually permutable subgroups $G_1, G_2, \dots, G_n$ if G_i and G_j are mutually permutable subgroups of G for all $i, j \in \{1, 2, \dots, n\}$. Asaad and Shaalan ([1]) proved that if G is a mutually permutable product of the subgroups A and B and A and B are finite and supersoluble, then G is supersoluble provided that either G', the derived subgroup of G, is nilpotent or A or B is nilpotent. This result was the beginning of an intensive study of such factorized groups (see, for instance, [2, 4, 6, 9] and the papers cited therein).

The extension of the above results on mutually permutable products of two subgroups to general pairwise mutually permutable products turns out to be difficult in many cases. Carocca proved (see [10]) that if the derived subgroup of a pairwise mutually permutable product of supersoluble subgroups is nilpotent, then the group G is supersoluble. However a pairwise mutually permutable product of supersoluble groups in which one of them is nilpotent is not supersoluble in general (see [4, Example]). Nevertheless in [4] we obtained that if G is the pairwise mutually permutable product of supersoluble subgroups with all factors but one nilpotent, then the group is supersoluble.

Some interesting results on pairwise mutually permutable products arise when the factors belong to some classes of finite groups which are defined in terms of permutability. They are the class of PST-groups, or finite groups G in which every subnormal subgroup of G permutes with every Sylow subgroup of G, the class of PT-groups, or finite groups in which every subnormal subgroup is a permutable subgroup of the group, the class of T-groups, or groups in which every subnormal subgroup is normal, and the class of $\mathcal{Y}$-groups, or finite groups G for which for every subgroup H and for all primes q dividing the index $|G : H|$ there exists a subgroup K of G such that H is contained in K and $|K : H| = q$, and their corresponding local versions (see [2, 3]).

The main purpose of this article is to take this program of research a step further by analyzing the structure of the pairwise mutually permutable products whose factors belong to some local classes of finite groups closely related to the classes of all T-groups and $\mathcal{Y}$-groups.

Therefore in the sequel all groups considered are finite.

2 The class $\bar{\mathcal{C}_p}$ and pairwise mutually permutable products

Throughout this section, p will be a prime.

Recall that a group G satisfies property $\mathcal{C}_p$, or G is a $\mathcal{C}_p$-group, if each subgroup of a Sylow p-subgroup P of G is normal in the normalizer $N_G(P)$. This class of groups was introduced by Robinson in his seminal paper [14] as a local version of the class of all soluble T-groups. In fact, he proved there that a group G is a soluble T-group if and only if G is a $\mathcal{C}_p$-group for all primes p.

In [7] the second and third authors introduce and analyze an interesting class of groups closely related to the class of all T-groups. A group G is a T_1-group if $G/Z_\infty(G)$ is a T-group. Here $Z_\infty(G)$ denotes the hypercenter of G, that is, the largest normal subgroup of G having a G-invariant series with central G-chief factors. The local version of the class T_1 in the soluble universe is the class $\bar{\mathcal{C}_p}$ introduced and studied in [8]:

Definition 1. *Let G be a group and let $Z_p(G)$ be the Sylow p-subgroup of $Z_\infty(G)$. A group satisfies $\bar{\mathcal{C}_p}$ if and only if $G/Z_p(G)$ is a $\mathcal{C}_p$-group.*

Theorem A ([8]) *A group G is a soluble T_1-group if and only if G is a $\bar{\mathcal{C}_p}$-group for all primes p.*

The objective of this section is to analyze the behaviour of pairwise mutually permutable products with respect to the class $\bar{\mathcal{C}_p}$.

We begin with some results concerning the classes $\mathcal{C}_p$ and $\bar{\mathcal{C}_p}$.

Lemma 1. *[8, Lemma 2] Let p be a prime. Then:*

(i) $\mathcal{C}_p$ is a subgroup-closed class.

(ii) Let M be a normal p'-subgroup of a group G. If G/M is a $\mathcal{C}_p$-group, then so is G.

(iii) If G is a $\mathcal{C}_p$-group and N is a normal subgroup of G, then G/N is a $\mathcal{C}_p$-group.

Lemma 2. *Let G be a $\bar{\mathcal{C}}_p$-group and let N be a normal subgroup of G. Then G/N is a $\bar{\mathcal{C}}_p$-group.*

Proof Let $Z_p(G)$ be the Sylow p-subgroup of $Z_\infty(G)$. Since $G/Z_p(G)$ is a $\mathcal{C}_p$-group, it follows that $G/Z_p(G)N$ is a $\mathcal{C}_p$-group by Lemma 1. Let H/N denote the Sylow p-subgroup of $Z_\infty(G/N)$. Since $Z_p(G)N/N$ is contained in H/N, we have that $(G/N)/(H/N)$ is isomorphic to a quotient of $G/Z_p(G)N$. By Lemma 1, $(G/N)/(H/N)$ is a $\mathcal{C}_p$-group. Therefore G/N is a $\bar{\mathcal{C}}_p$-group.

Recall that a group G is said to be p-supersoluble if it is p-soluble and every p-chief factor of G is cyclic. It is rather clear that the derived subgroup of a p-supersoluble group is p-nilpotent and, if $p = 2$, the group itself is 2-nilpotent.

Lemma 3. *[8, Lemma 3] Let G be a p-soluble group. If G is a $\bar{\mathcal{C}}_p$-group, then G is p-supersoluble.*

The main aim of this section is to show that pairwise mutually permutable products of p-soluble $\bar{\mathcal{C}}_p$-groups are p-supersoluble.

Theorem 1. *Let $G = G_1G_2 \dots G_k$ be the pairwise mutually permutable product of the subgroups $G_1, G_2, \dots, G_k$. If G_i is a p-soluble $\bar{\mathcal{C}}_p$-group for every $i \in \{1, 2, \dots, k\}$, then G is p-supersoluble.*

Proof Assume that the theorem is false, and let G be a counterexample with minimal order. By [4, Theorem 1], G is p-soluble. If $p = 2$, then G_i is 2-nilpotent for all $i = 1, 2, \dots, k$ and so G is 2-supersoluble by [4, Theorem 3]. This contradiction implies that p is odd. Note, that the hypotheses of the theorem are inherited by all proper quotients of G. Therefore the minimal choice of G yields G/N p-supersoluble for every minimal normal subgroup N of G. Since the class of p-supersoluble groups is a saturated formation, it follows that G has a unique minimal normal subgroup, say N, G/N is p-supersoluble, the Frattini subgroup of G is trivial and then $N = C_G(N) = F(G) = O_p(G)$. Moreover, N is an elementary abelian p-group of rank greater than 1.
By Lemma 3, G_i is p-supersoluble, for all $i = 1, 2, \dots, k$. Consequently $(G_i)'$ is p-nilpotent. Furthermore, by [4, Lemma 1(iii)], we have that $(G_i)'$ is a subnormal subgroup of G for all i. Since $O_{p'}(G) = 1$, it follows that $(G_i)'$ is a p-group and then G_i is supersoluble for all i. Then G_i is a Sylow tower group with respect to the reverse natural ordering of the prime numbers for all i. Applying [4, Corollary 1], G is a Sylow tower group with respect to the reverse natural ordering of the prime numbers. Therefore p is the largest prime dividing the order of G and $F(G) = N$ is the Sylow p-subgroup of G.
Now we observe the following facts:

(i) For each $i \in \{1, 2, \dots, k\}$, either $N \leq G_i$ or $N \cap G_i = 1$.
Put $R := N \cap G_i$, and assume that $R \neq 1$. Let H_j be a Hall p'-subgroup of

G_j (such Hall subgroups exist since G_j is soluble). Then $R = G_iH_j \cap N$, so $G_j \leq NH_j \leq N_G(R)$ for every j. Hence R is a normal subgroup of G and $R = N$.

(ii) Let $N \leq G_i$, with $i \in \{1, 2, \ldots, k\}$. Then $Z_\infty(G_i) = Z(G_i)$, $N = Z(G_i) \times [N, G_i]$, and every subgroup of $[N, G_i]$ is G_i-invariant.
Clearly $N \leq F(G_i)$ and $O_{p'}(F(G_i)) \leq C_G(N) = N$. Thus $F(G_i) = N$. Therefore $Z_\infty(G_i) \leq N$, and $Z_\infty(G_i) = Z(G_i)$, since $G_i/C_{G_i}(Z_\infty(G_i))$ is a p-group (G_i stabilizes a series of subgroups of $Z_\infty(G_i)$, see [11, A, 12.4]) and N is a Sylow p-subgroup of G_i. Moreover, $N = Z(G_i) \times [N, G_i]$ since G_i/N is a p'-group. As $G_i \in \bar{\mathcal{C}}_p$ and $Z(G_i) = Z_\infty(G_i)$, G_i normalizes every subgroup of $[N, G_i]$.

(iii) Let $N \leq G_i$ with $i \in \{1, 2, \ldots, k\}$, then every $y \in G_i \setminus N$ induces a non-trivial $GF(p)$-scalar multiplication on $[N, G_i]$; in particular $C_N(y) = Z(G_i)$ and G_i/N is cyclic.
Note that G/N acts faithfully on N. So y induces a non-trivial linear mapping on the $GF(p)$-space $[N, G_i]$ that leaves invariant every subspace. It is well-known that these mappings come from multiplication with an element of $GF(p)$.

(iv) Let $N \leq G_i$ and $N \leq G_j$ with $i \in \{1, 2, \ldots, k\}$. Suppose that $N_{G_i}(Z(G_j)) \nleq N$. Then $G_i \leq N_G(Z(G_j))$.
Put $R := N_{G_i}(Z(G_j))$. By (iii), $Z(G_j) = (Z(G_j) \cap Z(G_i)) \times [Z(G_j), R]$, and $[Z(G_j), R] \leq [N, G_i]$. Thus by (ii), $Z(G_j)$ is G_i-invariant.

(v) Suppose that $N \leq G_i$ and $G_j \leq N_G(Z(G_i))$. Then $[G_i, G_j] \leq N$; in particular, if $N \leq G_j$, $G_i \leq N_G(Z(G_j))$.
Put $H := G_iG_j$. Then H/N is a p'-group. By Maschke's Theorem there exists an H-invariant complement N_0 for $Z(G_i)$ in N. By (iii) $N_0 = [N, G_i]$ and $[G_i, G_j] \leq C_H(N_0)$. Since also $[G_i, G_j] \leq C_H(Z(G_i))$, it follows that $[G_i, G_j] \leq C_H(N) \leq N$. Moreover if $N \leq G_j$ we have that $G_j^H = G_j^{G_i} = G_j[G_i, G_j] = G_j$, that is, G_j is a normal subgroup of H and then G_i normalizes $Z(G_j)$.

(vi) Let $R \leq N \cap G_i$ and $G_j \cap N = 1$. Then G_j normalizes R.
Since RG_j is a subgroup of G, $RG_j \cap N = R$ is a normal subgroup of RG_j.

(vii) Suppose that $N \leq G_j$. Then $Z(G_j)$ is a normal subgroup of G.
We may assume that there exists $i \in \{1, 2, \ldots, k\}$ such that $G_i \nleq N_G(Z(G_j))$. In particular $Z(G_j) \neq 1$, and $Z(G_j) \leq N$ by (ii). Now the application of (vi) yields $G_i \cap N \neq 1$ and so by (i) also $N \leq G_i$. Moreover, $G_i \notin \mathcal{C}_p$ and so $Z_\infty(G_i) \neq 1$. Applying (ii) $Z_\infty(G_i) = Z(G_i) \neq 1$, and by (v) $G_j \nleq N_G(Z(G_i))$. Hence the situation is completely symmetric in i and j.
Put $H := G_iG_j$. We first show:

(*) $G_i \cap G_j = N$, and $N_H(RN) = G_k$ for every $k \in \{i,j\}$ and $R \leq G_k$ with $R \not\leq N$.
Since G_k/N is cyclic by (iii), RN is a normal subgroup of G_k and so $N_H(RN) = N_{G_t}(RN)G_k$, where $\{t,k\} = \{i,j\}$. Now (iv) yields $N_{G_t}(RN) \leq N$. This shows that $G_i \cap G_j = N$ and $N_H(RN) = G_k$.
As a consequence of (*), G_i/N and G_j/N have trivial intersection, therefore $H/N = (G_i/N)(G_j/N)$ is the totally permutable product of G_i/N and G_j/N (see [6, Lemma 1]), that is, every subgroup of G_i/N permutes with every subgroup of G_j/N. Thus there exists RN/N a minimal normal subgroup of H/N contained in G_i/N or in G_j/N (see [10]), suppose $RN/N \leq G_i/N$ without loss of generality. Then $N_H(RN) = H$. On the other hand, by (*) $N_H(RN) = G_i$. But then $G_j \leq G_i$, a contradiction since $G_j \not\leq N_G(Z(G_i))$.

Since not all the factors G_i are p'-groups, there exists G_i with $N \leq G_i$. It suffices to show that every subgroup R of N is normal in G. By (i) and (vi) every G_j with $N \not\leq G_j$ normalizes R. On the other hand, by (vii) for every G_j with $N \leq G_j$ either $N = Z(G_j) = G_j$ or $Z(G_j) = 1$. In the first case obviously $G_j \leq N_G(R)$. In the second case $G_j \in \mathcal{C}_p$ and again $G_j \leq N_G(R)$. Consequently $\mid N \mid= p$, the final contradiction.

Combining Theorems A and 1 we have:

Corollary 1. *Let $G = G_1G_2\ldots G_k$ be a product of the pairwise mutually permutable soluble T_1-groups $G_1, G_2, \ldots, G_k$. Then G is supersoluble.*

3 The class $\hat{\mathcal{Z}}_p$ and pairwise mutually permutable products

Another interesting class of groups closely related to T-groups is the class T_0 of all groups G whose Frattini quotient $G/\Phi(G)$ is a T-group. This class was introduced in [15] and studied in [12, 13, 15].
The procedure of defining local versions in order to simplify the study of global properties has also been successfully applied to the study of the classes T_0 ([12]) and $\mathcal{Y}$ ([3]).

Definition 2. *Let p be a prime and let G be a group.*

(i) ([12]) Let $\Phi(G)_p$ be the Sylow p-subgroup of the Frattini subgroup of G. G is said to be a $\hat{\mathcal{C}}_p$-group if $G/\Phi(G)_p$ is a $\mathcal{C}_p$-group.

(ii) ([3, Definition 11]) We say that G satisfies $\mathcal{Z}_p$ or G is a $\mathcal{Z}_p$-group when for every p-subgroup X of G and for every power of a prime q, q^m, dividing $\mid G : XO_{p'}(G) \mid$, there exists a subgroup K of G containing $XO_{p'}(G)$ such that $\mid K : XO_{p'}(G) \mid= q^m$.

It is rather clear that the class of all $\hat{\mathcal{C}}_p$-groups is closed under taking epimorphic images and all p-soluble groups belonging to $\hat{\mathcal{C}}_p$ are p-supersoluble. Moreover:

Theorem B ([12]) *A group G is a soluble T_0-group if and only if G is a $\hat{\mathcal{C}}_p$-group for all primes p.*

In the following two results we gather some useful properties of the $\mathcal{Z}_p$-groups.

Lemma 4. *Let G be a group.*

(i) If G is a p-soluble $\mathcal{Z}_p$-group, then G is p-supersoluble. [3, Lemma 20]

(ii) If G is a p-soluble $\mathcal{Z}_p$-group and N is a normal subgroup of G, then G/N is a $\mathcal{Z}_p$-group. [3, Lemma 18]

(iiii) Let G be a soluble group. G is a $\mathcal{Y}$-group if and only if G is a $\mathcal{Z}_p$-group for every prime p. [3, Theorem 15]

Theorem C [3, Theorem 13] *Let p be a prime and G a p-soluble group. Then G satisfies $\mathcal{Z}_p$ if and only if G satisfies one of the following conditions:*

(1) G is p-nilpotent.

(2) $G(p)/O_{p'}(G(p))$ is a Sylow p-subgroup of $G/O_{p'}(G(p))$ and for every p-subgroup H of $G(p)$, we have that $G = N_G(H)G(p)$.

Here $G(p)$ denotes the p-nilpotent residual of G, that is, the smallest normal subgroup of G with p-nilpotent quotient.

The results of [5] show that the class $\mathcal{C}_p$ is a proper subclass of the class $\mathcal{Z}_p$.

In [2, Theorem 16] it is proved that a pairwise mutually permutable product of $\mathcal{Y}$-groups is supersoluble. There it is asked whether a pairwise mutually permutable product of $\mathcal{Z}_p$-groups is p-supersoluble. In this section, we answer to this question affirmatively. In fact, the main purpose here is to study pairwise mutually permutable products whose factors belong to some class of groups closely related to $\mathcal{Z}_p$-groups.

Definition 3. *Let p be a prime, let G be a group and let $\Phi(G)_p$ be the Sylow p-subgroup of the Frattini subgroup of G. G is said to be a $\hat{\mathcal{Z}}_p$-group if $G/\Phi(G)_p$ is a $\mathcal{Z}_p$-group.*

Lemma 5. *Let p be a prime and M a normal subgroup of G. If G is a $\hat{\mathcal{Z}}_p$-group, then G/M is a $\hat{\mathcal{Z}}_p$-group.*

Proof Assume that G is a $\hat{\mathcal{Z}}_p$-group. Then $G/\Phi(G)_p$ is a $\mathcal{Z}_p$-group. Since $\Phi(G)_pM/M$ is contained in $\Phi(G/M)_p = L/M$ and the class of all $\mathcal{Z}_p$-groups is closed under taking epimorphic images, we have that G/L belongs to $\mathcal{Z}_p$. This is to say that G/M is a $\hat{\mathcal{Z}}_p$-group.

Since the class of all p-supersoluble groups is a saturated formation and, by Lemma 4(i), every p-soluble $\mathcal{Z}_p$-group is p-supersoluble, we have:

Lemma 6. *Let p be a prime and let G be a p-soluble group. If G is a $\hat{\mathcal{Z}}_p$-group, then G is p-supersoluble.*

The main result of this section shows that pairwise mutually permutable products of $\hat{\mathcal{Z}}_p$-groups are p-supersoluble.

Theorem 2. *Let $G = AB$ be the mutually permutable product of the p-supersoluble group A and the p-soluble $\hat{\mathcal{Z}}_p$-group B. Then G is p-supersoluble.*

Proof Assume that the result is false, and let G be a counterexample of minimal order. Applying [4, Theorem 1], G is p-soluble. Let N be a minimal normal subgroup of G. Then G/N is the mutually permutable product of the subgroups AN/N and BN/N. Moreover, AN/N is p-supersoluble and BN/N is a p-soluble $\hat{\mathcal{Z}}_p$-group by Lemma 5. The minimality of G implies that G/N is p-supersoluble. Since p-supersoluble groups is a saturated formation, it follows that G has a unique minimal normal subgroup, N say. Moreover N is an elementary abelian p-group of rank greater than 1 and $N = C_G(N) = F(G) = O_p(G)$. Note further that, by Lemma 6, A and B are p-supersoluble.
Applying [6, Lemma 1(vii)], we have that A and B either cover or avoid N. If A and B both avoid N, then $|N| = p$ by [6, Lemma 2] and G is p-supersoluble. This contradiction allows us to assume that $N \leq A$. Suppose that $B \cap N = 1$ and let X be a minimal normal subgroup of A such that $X \leq N$. Then $|X| = p$ and $XB \cap N = X$ is a normal subgroup of XB. It means that B normalizes X and so X is a normal subgroup of G. This would imply that G is p-supersoluble, contrary to our supposition. We obtain also a contradiction if we assume $N \leq B$ and $A \cap N = 1$. Therefore we may suppose that $N \leq A \cap B$. Note that, by [4, Theorem 3], neither A nor B is p-nilpotent.
On the other hand, by [6, Theorem 1], we have that A' and B' are subnormal subgroups of G. Since they are p-nilpotent and $O_{p'p}(G) = N$, it follows that $\langle A', B' \rangle \leq N$. Let $1 \neq B(p)$ be the p-nilpotent residual of B. Then $B(p) \leq B' \leq N$. Now observe that $O_{p'}(B) = 1$ and B is p-closed. Then it is an elementary fact that $\Phi(B) = \Phi(O_p(B)) = \Phi(B)_p$. Since B is not p-nilpotent, Theorem C gives $B(p) \in Syl_p(B)$, so $N = B(p)$ and $\Phi(B) = \Phi(N) = 1$. In particular $B \in \mathcal{Z}_p$ and by Theorem C every subgroup of N is normal in B. Therefore, if X is a minimal normal subgroup of A contained in N, we have that X is a normal subgroup of G of order p. Consequently, G is p-supersoluble, the final contradiction.

Theorem 3. *Let $G = G_1G_2 \dots G_n$ be the pairwise mutually permutable product of the subgroups $G_1, G_2, \dots, G_n$. If G_i is a p-soluble $\hat{\mathcal{Z}}_p$-group for every i, then G is p-supersoluble.*

Proof Assume that the theorem is false, and among the counterexamples with minimal order choose one $G = G_1G_2 \dots G_n$ such that the sum $|G_1| + |G_2| + \dots + |G_n|$ is minimal. By Theorem 2, we have $n > 2$. Moreover, by [4, Theorem 1], G is p-soluble. It is rather clear that the hypotheses of the theorem are inherited by all proper quotients of G. Hence G contains a unique minimal normal subgroup, N say, N is not cyclic, G/N is p-supersoluble and $N = C_G(N) = O_p(G)$. Hence $O_{p'}(G) = 1$. By Lemma 6, G_i is p-supersoluble and hence G_i' is p-nilpotent for every i. Applying Lemma 1(iii) of [4], we have that G_i' is a subnormal subgroup of G for each $i \in \{1, 2, \dots, n\}$. Hence G_i' is contained in N and so G_i is supersoluble for each $i \in \{1, 2, \dots, n\}$. By [4, Corollary 1], G is a Sylow tower group with respect to the reverse natural ordering of the prime numbers, p is the largest prime divisor of $|G|$ and N is the Sylow p-subgroup of G.

Let $i \in \{1, 2, \dots, n\}$ such that p divides $|G_i|$. Then $N \cap G_i$ is the non-trivial Sylow p-subgroup of G_i. Let $j \in \{1, 2, \dots, n\}$ such that $j \neq i$. Then $G_i(G_j)_{p'}$ is a subgroup of G and $N \cap G_i$ is a Sylow p-subgroup of $G_i(G_j)_{p'}$. Since $G_i(G_j)_{p'}$ is a Sylow tower group with respect to the reverse natural ordering of the prime numbers, it follows that $N \cap G_i$ is normal in $G_i(G_j)_{p'}$. This implies that $N \cap G_i$ is a normal subgroup of G and so $N = N \cap G_i$ is contained in G_i.

Assume that there exists $j \in \{1, 2, \dots, n\}$ such that p does not divide $|G_j|$. We may assume without loss of generality $j = 1$. Then $G_1' = 1$, that is, G_1 is an abelian p'-group, and $T = G_2G_3 \dots G_n$ is p-supersoluble by the choice of G. Let R be a minimal normal subgroup of T contained in N. Then $|R| = p$. Moreover, G_1R is a subgroup of G because N is contained in some of the factors G_l, $l > 1$. Hence $G_1R \cap N = R$ is a normal subgroup of G_1R. Hence R is a normal subgroup of G and so $N = R$. This is a contradiction. Therefore p divides the order of G_i for every $i \in \{1, 2, \dots, n\}$. Consequently, N is contained in G_i for every $i \in \{1, 2, \dots, n\}$.

Consider now $W = G_2G_3 \dots G_n$. Then W is p-supersoluble. Let X be a minimal normal subgroup of W contained in N. Then $|X| = p$. Recall that G_1 is a $\hat{\mathcal{Z}}_p$-group. Assume that $G_1/\Phi(G_1)_p$ is p-nilpotent. Then G_1 is p-nilpotent. Since N is self-centralizing in G, it follows that $G_1 = N$. Suppose that $G_1/\Phi(G_1)_p$ satisfies condition (2) of Theorem C. Then we can argue as in the proof of Theorem 2 to obtain that $N = G_1(p)$, the p-nilpotent residual of G_1, and $\Phi(G_1)_p = 1$. Consequently every subgroup of N is normal in G_1. In both cases, we have that G_1 normalizes X. It means that $N = X$, the final contradiction.

Applying Theorems C and 3 we have:

Corollary 2. *Let $G = G_1G_2 \dots G_n$ be a group such that $G_1, G_2, \dots, G_n$ are pairwise mutually permutable subgroups of G. If all G_i are p-nilpotent, then G is p-supersoluble.*

Since every $\mathcal{Z}_p$-group is a $\hat{\mathcal{Z}}_p$-group, we can apply Lemma 4(iii) and Theorem 3 to obtain the following:

Corollary 3. *[2, Theorem 16] Let $G = G_1G_2 \dots G_n$ be a group such that $G_1, G_2, \dots, G_n$ are pairwise mutually permutable subgroups of G. If all G_i are $\mathcal{Y}$-groups, then G is supersoluble.*

Finally, applying Theorems B and 3, we have:

Corollary 4. *Let $G = G_1G_2 \dots G_n$ be a group such that $G_1, G_2, \dots, G_n$ are pairwise mutually permutable subgroups of G. If all G_i are soluble T_0-groups, then G is supersoluble.*

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References

[1] M. Asaad and A. Shaalan. *On the supersolvability of finite groups,* Arch. Math., 53(1989), 318-326.

[2] M. Asaad, A. Ballester-Bolinches, J.C. Beidleman and R. Esteban-Romero. *Some classes of finite groups and mutually permutable products,* J. Algebra, 319 (2008), 3343-3351.

[3] A. Ballester-Bolinches, J.C. Beidleman, R. Esteban-Romero. *On some classes of supersoluble groups,* J. Algebra, 312 (2007), 445-454.

[4] A. Ballester-Bolinches, J.C. Beidleman, H. Heineken and M.C. Pedraza-Aguilera. *On pairwise mutually permutable products,* Forum Math., 21 (2009), 1081-1090.

[5] A. Ballester-Bolinches and R. Esteban-Romero. *Sylow Permutable Subnormal Subgroups of Finite Groups II,* Bull. Austral. Math. Soc. 64 (2001), 479-486.

[6] J. C. Beidleman and H. Heineken. *Mutually permutable subgroups and group classes,* Arch. Math., 85 (2005), 18-30.

[7] J. C. Beidleman and H. Heineken. *Groups in which the hypercentral factor group is a T-group,* Ric. Mat. 55 (2006), no. 2, 219–225.

[8] J. C. Beidleman, H. Heineken and M. F. Ragland. *On hypercentral factor groups from certain classes,* J. Group Theory, 11 (2008), 525-535.

[9] J. Bochtler and P. Hauck. *Mutually permutable subgroups and Fitting classes,* Arch. Math. (Basel) 88 (2007), no. 5, 385–388.

[10] A. Carocca and R. Maier. *Theorems of Kegel-Wielandt type,* Groups St. Andrews 1997 in Bath I, 195-201. London Math. Soc. Lecture Note Ser. 260. Cambridge University Press, Cambridge, 1999.

[11] K. Doerk and T. Hawkes. *Finite Soluble Groups,* Walter De Gruyter, Berlin/New York, 1992.

[12] M. F. Ragland. *On generalizations of groups in which normality is a transitive relation,* Ph. D. thesis. University of Kentucky (2005).

[13] M. F. Ragland. *Generalizations of groups in which normality is transitive,* Comm. Algebra 35 (2007), 3242-3252.

[14] D.J.S. Robinson. *A note on finite groups in which normality is transitive,* Proc. Amer. Math. Soc. 19 (1968), 933-937.

[15] R. W. van der Wall and A. Frassman. *On products of groups for which normality is a transitive relation on their Frattini factor groups,* Quaestiones Math. 19 (1996), 59-82.

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