On the Solutions of Quadratic Diophantine Equations

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Abstract. We determine a finite set of representatives of the set of local solutions in a maximal lattice modulo the stabilizer of the lattice in question for a quadratic Diophantine equation. Our study is based on the works of Shimura on quadratic forms, especially [AQC] and [IQD]. Indeed, as an application of the result, we present a criterion (in both global and local cases) of the maximality of the lattice of (11.6a) in [AQC]. This gives an answer to the question (11.6a). As one more global application, we investigate primitive solutions contained in a maximal lattice for the sums of squares on each vector space of dimension 4, 6, 8, or 10 over the field of rational numbers.

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1 Introduction

In this paper we study quadratic forms over global and local fields of characteristic zero, i.e. over number fields and their p-adic completions. Let F be a field of one of these two types. We let $\mathfrak{g}$ denote the ring of all integers in F (in both global and local cases). We denote by $\mathfrak{p}$ the maximal ideal of $\mathfrak{g}$ in the local case. Throughout the paper we mainly follow the notion and the notation in Shimura's book [AQC] and the paper [IQD]. We denote by V an n-dimensional vector space over F. Let $\varphi : V \times V \to F$ be a nondegenerate symmetric F-bilinear form. We denote by $\varphi[x]$ the quadratic form $\varphi(x, x)$ on V. By a maximal lattice L in V with respect to φ, we understand a $\mathfrak{g}$-lattice L in V, which is maximal among $\mathfrak{g}$-lattices on which the values $\varphi[x]$ are contained in $\mathfrak{g}$. For simplicity, when φ is fixed on V, we will often refer to a maximal lattice in V, omitting reference to the φ needed to define it. All results in the paper concern only maximal lattices in V. Let SO^{φ} be the special orthogonal group of φ. In this paper we consider the set of the solutions of the quadratic Diophantine equation $\varphi[x] = q$ in L, that is

$$L[q] = \{x \in L \mid \varphi[x] = q\},$$

and

$$L[q, \mathfrak{b}] = \{x \in V \mid \varphi[x] = q, \varphi(x, L) = \mathfrak{b}\},$$

where $q \in \mathfrak{g} \cap F^{\times}$ and a fractional ideal $\mathfrak{b}$ of F.

Assume now that F is local, put $C(L) = \{\gamma \in SO^{\varphi} \mid L\gamma = L\}$, and take $h \in L$ such that $\varphi[h] \neq 0$. It was shown by Shimura that there exists a finite subset A of SO^{φ} such that

$$L[\varphi[h]] = \bigsqcup_{\alpha \in A} h\alpha C(L)$$

([AQC, Theorem 10.3]) and

$$\#\{L[q, \mathfrak{b}]/C(L)\} \leq 1 \text{ if } n > 2$$

([IQD, Theorem 1.3]). Note that [AQC, Theorem 10.3] is true even when L is not maximal. In Theorem 3.5 we shall obtain, using the proof of [AQC, Theorem 10.3], an explicit complete set $\{h\alpha\}_{\alpha \in A}$ of representatives for $L[\varphi[h]]/C(L)$. Also, we show that

$$L[\varphi[h]] = \begin{cases} L[\varphi[h], 2^{-1}\mathfrak{p}^{\tau(\varphi[h])}] & \text{if } \varphi \text{ is anisotropic,} \\ \bigsqcup_{i=0}^{\tau(\varphi[h])} L[\varphi[h], 2^{-1}\mathfrak{p}^{i}] & \text{if } \varphi \text{ is isotropic,} \end{cases}$$

with the value $\tau(\varphi[h])$; see Theorem 3.5.

As a result of this theorem we prove Theorem 5.3: Suppose F is local and $n \geq 2$. Then

$$L \cap (Fh)^{\perp} \text{ is maximal in } (Fh)^{\perp} \text{ if and only if } h \in L[\varphi[h], 2^{-1}\mathfrak{p}^{\tau(\varphi[h])}]$$

for $h \in L$ such that $\varphi[h] \neq 0$. Here $(Fh)^{\perp} = \{x \in V \mid \varphi(x, h) = 0\}$. We also obtain the global version of the maximality of the lattice $L \cap (Fh)^{\perp}$ in $(Fh)^{\perp}$ in Theorem 6.3. This theorem answers the question raised in [AQC, (11.6a)].

As a global application of Theorem 3.5, in Theorem 7.5 we give the criterion of the existence of solutions contained in $L[q, \mathbf{Z}]$ and $L[q, 2^{-1}\mathbf{Z}]$ in both cases when q is a squarefree positive integer, by taking $V = \mathbf{Q}_n^1$ ($4 \leq n \leq 10$, n even), the sums of squares as φ, and a maximal lattice L in V. It is known that $L[q] = L[q, 2^{-1}\mathbf{Z}] \sqcup L[q, \mathbf{Z}]$; see [AQC, (12.17)]. For example, when $n = 6$, the set $L[q, \mathbf{Z}] = \emptyset$ if and only if $q - 1 \in 4\mathbf{Z}$. When $n = 10$, the genus of L consists of two SO^{φ}-classes $L_{10}SO^{\varphi}$ and ΛSO^{φ} (cf. [CGQ, §3.2]). In this case,

$$L[q, \mathbf{Z}] = \emptyset \text{ if and only if } \begin{cases} L \in L_{10}SO^{\varphi}, q = 1 \text{ or } q - 3 \in 4\mathbf{Z}; \text{ or} \\ L \in \Lambda SO^{\varphi}, q - 3 \in 4\mathbf{Z}. \end{cases}$$

We summarize the contents of the paper. In Section 2 we recall the notion of Shimura [AQC] and [IQD] and introduce the basic facts of a local Witt decomposition with respect to φ. In Sections 3 through 5 we treat local cases. In Section 3 we introduce the result obtained from the proof of [AQC, Theorem 10.3] and state the first result. In Section 4 we prove Theorem 3.5. In Section 5 we shall give a criterion of the maximality of the lattice $L \cap (Fh)^{\perp}$ in $(Fh)^{\perp}$ in the local case. In Section 6 we prove the global version of Theorem 5.3. In Section 7 we prove Theorem 7.5.

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Notations and Conventions As usual, $\mathbf{Z}$ (resp. $\mathbf{Z}_p$) is the ring of rational (resp. p-adic) integers, $\mathbf{Q}$ (resp. $\mathbf{Q}_p$) the field of rational (resp. p-adic) numbers. In this paper we consider the base field F in two cases. One is a global field and the other is a local field. When we do not need to specify the case of F, we call it only "a field".

If R is an associative ring with identity element, then $R^{\times}$ is the group of units of R. If K is a finite algebraic extension of a field F, then $D_{K/F}$ denotes the relative discriminant of K over F. Let $\mathfrak{d}_{K/F}$ be the different of K relative to F.

If F is a local field, then for $x \in F^{\times}$, put

$$\xi(x) = \begin{cases} 1 & \text{if } \sqrt{x} \in F, \\ -1 & \text{if } F(\sqrt{x}) \text{ is an unramified quadratic extension of } F, \\ 0 & \text{if } F(\sqrt{x}) \text{ is a ramified quadratic extension of } F \end{cases}$$

as in [NRQ, (3.3.1)].

If F is the field of quotients of a Dedekind domain $\mathfrak{g}$ and V an n-dimensional vector space over F, then by a $\mathfrak{g}$-lattice in V, we understand a finitely generated $\mathfrak{g}$-module in V that spans V over F. In particular, if $\mathfrak{a}$ is a $\mathfrak{g}$-lattice in F, we call $\mathfrak{a}$ a $\mathfrak{g}$-ideal of F. We write $\dim_F(V)$ for the dimension of V over F. We let $GL(V, F)$ denote the group of all F-linear automorphisms of V. If $R = F$ or $\mathfrak{g}$, then we write R_n^m for the ring of all $m \times n$-matrices with entries in R and let $GL_n(R) = (R_n^n)^{\times}$.

If X is a set, then $\#X$ denotes the cardinality of X. If X is a disjoint union of its subsets $Y_1, \cdots, Y_m$, we write $X = \bigsqcup_{i=1}^m Y_i$ or $X = Y_1 \sqcup \cdots \sqcup Y_m$. For a subgroup H of a group G, we let $[G : H] = \#(H \setminus G)$.

We denote by δ_{ij} Kronecker's delta. For a real number a, we let $[a]$ denote the greatest integer not greater than a.

2 Preliminaries

2.1. Let F be a field and we consider the pair (V, φ) as in the introduction. Define

$$SO^{\varphi}(V) = \{\alpha \in GL(V, F) \mid \det(\alpha) = 1,\ \varphi[x\alpha] = \varphi[x] \text{ for all } x \in V\}.$$

We understand that $GL(V, F)$ acts on V on the right. Let $\varphi_0 = [\varphi(x_i, x_j)]_{i,j=1}^{n}$ for an F-basis $\{x_i\}_{i=1}^{n}$ of V, then $\varphi_0 \in GL_n(F)$ such that $\varphi_0 = {}^t\varphi_0$. Define the discriminant of (V, φ) by

$$\delta(\varphi) = \delta(V, \varphi) = (-1)^{n(n-1)/2} \det(\varphi_0) F^{\times 2}. \tag{2.1}$$

Let $A(V) = A(V, \varphi)$ be the Clifford algebra of φ (cf. [AQC, Chap. I Section 2]). We say that (V_1, φ_1) is isomorphic to (V_2, φ_2) if there is an F-linear isomorphism f of V_1 onto V_2 such that $\varphi_1[x] = \varphi_2[xf]$ for any $x \in V_1$. If W is a subspace of V, then we always consider (W, ψ), where ψ is the restriction of φ to W ($\psi[x] = \varphi[x]$ for $x \in W$).
For a $\mathfrak{g}$-lattice Λ in V, put

$$\tilde{\Lambda} = \Lambda^{\sim} = \{x \in V \mid \varphi(x, \Lambda) \subset 2^{-1}\mathfrak{g}\}, \tag{2.2}$$

$$C(\Lambda) = \{\gamma \in SO^{\varphi}(V) \mid \Lambda\gamma = \Lambda\}. \tag{2.3}$$

By an integral lattice L in V (with respect to φ), we understand a $\mathfrak{g}$-lattice L in V such that $\varphi[x] \in \mathfrak{g}$ for every $x \in L$. We call L maximal (with respect to φ) if it is maximal among integral lattices in V. We note that $L \subset \tilde{L}$ when L is an integral lattice in V.

2.2. Here we assume that F is a local field and L is a maximal lattice in V with respect to φ. Considering the maximality of L, we have a Witt decomposition by [AQC, Lemma 6.5];

$$V = Z + \sum_{i=1}^{r} (Ff_i + Fe_i), \quad L = N + \sum_{i=1}^{r} (\mathfrak{g}f_i + \mathfrak{g}e_i), \tag{2.4}$$

where

$$\varphi(e_i, e_j) = \varphi(f_i, f_j) = 0, \quad \varphi(e_i, f_j) = 2^{-1}\delta_{ij}, \tag{2.5}$$

$$Z = \{z \in V \mid \varphi(e_i, z) = \varphi(f_i, z) = 0 \text{ for all } i\}, \tag{2.6}$$

$$N = \{z \in Z \mid \varphi[z] \in \mathfrak{g}\}. \tag{2.7}$$

Here the restriction of φ to Z is anisotropic and N is a unique maximal lattice in Z by [AQC, Lemma 6.4]. We say that Z is a core subspace of V with respect to φ. Until the end of Section 5, we fix these decompositions. Put $t = \dim_F(Z)$ then $n = 2r + t$. We have $t \leq 4$ by [AQC, Theorem 7.6(ii)]. We call t the core dimension of (V, φ).

2.3. We introduce here the basic notions of (Z, φ) and of N, which play an important role in this paper. Note that we use the same letters c and δ, for simplification, in the following different cases (I) (2.10), (II) (2.13), and (III) (2.15).
(I) Assume $t = 1$ (cf. [AQC, §7.1 and §7.7(I)] and [IQD, §1.5(A)]). Take $g \in Z$ such that

$$N = \mathfrak{g}g \tag{2.8}$$

and put

$$c = \varphi[g]. \tag{2.9}$$

Then $Z = Fg$ and $\varphi[xg] = cx^2$ for $x \in F$. Furthermore we obtain $c \in \mathfrak{g}^\times$ (resp. $c\mathfrak{g} = \mathfrak{p}$) if $\delta(\varphi) \cap \mathfrak{g} \neq \emptyset$ (resp. $\delta(\varphi) \cap \mathfrak{g} = \emptyset$) by (2.7). Put

$$c\mathfrak{g} = \mathfrak{p}^\delta \text{ with } \delta \in \mathbf{Z}. \tag{2.10}$$

By (2.2) and (2.8), we easily see that

$$\tilde{N} = 2^{-1}\mathfrak{p}^{-\delta}g. \tag{2.11}$$

(II) Next suppose $t = 2$ (cf. [AQC, §7.2 and §7.7(II)]). We can take $g_1, g_2 \in Z$ such that $Z = Fg_1 + Fg_2$ and $\varphi(g_1, g_2) = 0$ by [EPE, Lemma 1.8]. Put

$$b = \varphi[g_1] \text{ and } c = \varphi[g_2]. \tag{2.12}$$

Put $K = F + Fg_1g_2$ in $A(Z)$. Then K is a quadratic extension of F, which is isomorphic to $F(\sqrt{-bc})$, $Z = Kg_2$, and $\varphi[xg_2] = cN_{K/F}(x)$ for $x \in K$. We may assume $c \in \mathfrak{g}^\times$ or $c\mathfrak{g} = \mathfrak{p}$. Moreover when K is a ramified extension of F, we can take $c \in \mathfrak{g}^\times$. Then by (2.7) we have $N = \mathfrak{r}g_2$ if K is either unramified or ramified, where $\mathfrak{r}$ is the valuation ring of K. We put

$$c\mathfrak{g} = \mathfrak{p}^\delta \text{ with } \delta \in \mathbf{Z}. \tag{2.13}$$

(III) Suppose $t = 3$ (cf. [AQC, §7.3 and §7.7(III)] and [IQD, §1.5(B)]). There exist $g_i \in Z$ such that $Z = Fg_1 + Fg_2 + Fg_3$ and $\varphi(g_i, g_j) = 0$ if $i \neq j$ by [EPE, Lemma 1.8]. Put

$$c = \varphi[g_1]\varphi[g_2]\varphi[g_3]. \tag{2.14}$$

Then we can take $c \in \mathfrak{g}^\times$ (resp. $c\mathfrak{g} = \mathfrak{p}$) if $\delta(\varphi) \cap \mathfrak{g} \neq \emptyset$ (resp. $\delta(\varphi) \cap \mathfrak{g} = \emptyset$). We put

$$c\mathfrak{g} = \mathfrak{p}^\delta \text{ with } \delta \in \mathbf{Z}. \tag{2.15}$$

Put $\zeta = g_1g_2g_3$, $T = Fg_1g_2 + Fg_2g_3 + Fg_3g_1$, and $B = F + T$ in $A(Z)$. Then B is a division quaternion algebra over F and $T = \{x \in B \mid x + x^\iota = 0\}$, where

ι is the main involution of B. Moreover we have $Z = T\zeta$ and $\varphi[x\zeta] = cxx^\iota$ for $x \in T$. Then by (2.7),

$$N = (T \cap \mathfrak{P}^{-\delta})\zeta, \tag{2.16}$$

where $\mathfrak{P} = \{x \in B \mid xx^\iota \in \mathfrak{p}\}$. By [AQC, Theorem 5.13], there exist an unramified quadratic extension K over F and an element $\omega \in B$ such that $B = K + K\omega$, $a\omega = \omega a^\iota$ for each $a \in K$, and $\omega^2 \in \pi\mathfrak{g}^\times$. Here π is a prime element of F. Let $\mathfrak{r}$ be the valuation ring of K. There exists $u \in \mathfrak{r}$ such that $\mathfrak{r} = \mathfrak{g}[u]$ and $u - u^\iota \in \mathfrak{r}^\times$ by [AQC, Lemma 5.7]. Put $v = u - u^\iota$. Then $T = Fv + K\omega$. For $a, \alpha \in \mathfrak{g}$ and $b, \beta \in \mathfrak{r}$,

$$\varphi[(av + b\omega^{1-2\delta})\zeta] = -c(a^2v^2 + \omega^{2(1-2\delta)}N_{K/F}(b)), \tag{2.17}$$

(2.18)
$$\varphi((av + b\omega^{1-2\delta})\zeta, (\alpha v + \beta\omega^{1-2\delta})\zeta) = -2^{-1}c(2a\alpha v^2 + \omega^{2(1-2\delta)}Tr_{K/F}(b\beta^\iota)).$$

From (2.16) and (2.17),

$$N = (\mathfrak{g}v + \mathfrak{r}\omega^{1-2\delta})\zeta = (\mathfrak{g}v + \mathfrak{g}\omega^{1-2\delta} + \mathfrak{g}u\omega^{1-2\delta})\zeta. \tag{2.19}$$

From (2.2) and (2.19),

$$\tilde{N} = (2^{-1}\mathfrak{p}^{-\delta}v + \mathfrak{r}\omega^{-1})\zeta. \tag{2.20}$$

Put $Tr_{B/F}(x) = x + x^\iota$ and $N_{B/F}(x) = xx^\iota$ for $x \in B$.
(IV) Finally assume $t = 4$ (cf. [AQC, Theorem 7.5 and §7.7(IV)]). There exist a division quaternion algebra B over F and an F-linear isomorphism $\gamma : B \to Z$ such that $\varphi[x\gamma] = xx^\iota$ for $x \in B$, where ι is the main involution of B. Then $N = \mathfrak{O}\gamma$. Here $\mathfrak{O}$ is the unique maximal order of B.

3 A complete set of representatives for $L[q]/C(L)$

Until the end of Section 5, we assume that F is a local field and L is a maximal lattice in V with respect to φ. In this section, we first introduce the facts obtained from the proof of [AQC, Theorem 10.3]. After that, we state our first main theorem.

3.1. We suppose that V and L are represented as in (2.4). If $r \geq 1$, put $M = N + \sum_{i=2}^{r}(\mathfrak{g}f_i + \mathfrak{g}e_i)$. We consider $M = N$ if $r = 1$. Then

$$L = \mathfrak{g}f_1 + M + \mathfrak{g}e_1 \tag{3.1}$$

for every $r \geq 1$. For $0 \leq i \in \mathbf{Z}$ and $q \in \mathfrak{g} \cap F^\times$, put

$$X_i(q) = \{x \in M \mid \varphi[x] - q \in \mathfrak{p}^i\}. \tag{3.2}$$

Note that $X_i(q) \supset X_{i+1}(q)$. Hereafter we take a prime element π of F and fix it.
We obtain the following theorem from the proof of [AQC, Theorem 10.3].

3.2 Theorem. (Shimura) *Let the notation be as above. Let* $h \in L$ *such that* $\varphi[h] \neq 0$ *and* $\nu \in \mathbf{Z}$ *such that* $\varphi[h]\mathfrak{g} = \mathfrak{p}^\nu$. *Put* $C = C(L)$ *in the notation of* (2.3). *Let* t, e_1, *and* f_1 *be as in* §2.2.
(1) *Suppose* $r = 0$. *Then*

$$L[\varphi[h]] = \begin{cases} hC \sqcup (-h)C, \quad C = \{1\} & \text{if } t = 1, \\ hC & \text{if } t > 1. \end{cases}$$

(2) *Suppose* $n = 2r = 2$. *Then* $L[\varphi[h]] = \bigsqcup_{i=0}^{\nu}(\pi^i f_1 + \varphi[h]\pi^{-i}e_1)C$.
(3) *Suppose* $n > 2$, $r > 0$, *and* $M[\varphi[h]] = \emptyset$. *Put*

$$\kappa_0 = \min(\{k \in \mathbf{Z} \mid X_k(\varphi[h]) = \emptyset\}). \tag{3.3}$$

Then

$$L[\varphi[h]] = \bigcup_{i=0}^{\kappa_0-1} \bigcup_{b \in X_i(\varphi[h])/\mathfrak{p}^i M} [\pi^i f_1 + b + \pi^{-i}(\varphi[h] - \varphi[b])e_1]C.$$

Here b *runs over all elements of* $X_i(\varphi[h])/\mathfrak{p}^i M$.
(4) *Suppose that* $n > 2$, $r > 0$, $M[\varphi[h]] \neq \emptyset$, *and that there exists a finite subset* B *of* $M[\varphi[h]]$ *such that* $M[\varphi[h]] = \sqcup_{b \in B} bC(M)$. *Then*

$$L[\varphi[h]] = \bigcup_{b \in B} \bigcup_{y \in \mathfrak{g}/2\varphi(b,M)} (b + ye_1)C.$$

3.3 Lemma. *Let the notation be the same as in Theorem* 3.2. *We let* $q \in \mathfrak{g} \cap F^\times$. *Assume* $r \geq 2$. *If there are a finite number of elements* $x_0, \cdots, x_\tau$ *of* M *such that*

$$M[q] = \sqcup_{i=0}^{\tau} x_i C(M) \text{ and } \varphi(x_i, M) = 2^{-1}\mathfrak{p}^i, \tag{3.4}$$

then we have $L[q] = \sqcup_{i=0}^{\tau} x_i C$ *and* $\varphi(x_i, L) = 2^{-1}\mathfrak{p}^i$.

Proof. From (3.4) and Theorem 3.2(4),

$$L[q] = \bigcup_{i=0}^{\tau} \bigcup_{y \in \mathfrak{g}/\mathfrak{p}^i} (x_i + ye_1)C. \tag{3.5}$$

We fix $0 \leq i \leq \tau$. By (2.5), (2.6), (3.1), and (3.4),

$$\varphi(x_i + ye_1, L) = \varphi(x_i, M) + 2^{-1}y\mathfrak{g} = \begin{cases} 2^{-1}\mathfrak{p}^i & \text{if } y \in \mathfrak{p}^i, \\ 2^{-1}y\mathfrak{g} & \text{if } y \notin \mathfrak{p}^i. \end{cases}$$

From this and [IQD, Theorem 1.3],

$$(x_i + ye_1)C = L[q, \varphi(x_i + ye_1, L)] = \begin{cases} L[q, 2^{-1}\mathfrak{p}^i] & \text{if } y \in \mathfrak{p}^i, \\ L[q, 2^{-1}y\mathfrak{g}] & \text{if } y \notin \mathfrak{p}^i. \end{cases} \tag{3.6}$$

For $y \in \mathfrak{g}$ such that $y \notin \mathfrak{p}^i$, if $y\mathfrak{g} = \mathfrak{p}^j$ then $0 \leq j \leq i-1$. Thus we see that $\cup_{y \in \mathfrak{g}/\mathfrak{p}^i}(x_i + ye_1)C = \sqcup_{j=0}^{i} L[q, 2^{-1}\mathfrak{p}^j]$ and $L[q, 2^{-1}\mathfrak{p}^i] = x_i C$ by (3.6). From this and (3.5) we obtain

$$L[q] = \bigcup_{i=0}^{\tau}\left[\bigsqcup_{j=0}^{i} L[q, 2^{-1}\mathfrak{p}^j]\right] = \bigsqcup_{i=0}^{\tau} L[q, 2^{-1}\mathfrak{p}^i] = \bigsqcup_{i=0}^{\tau} x_i C.$$

Clearly $\varphi(x_i, L) = 2^{-1}\mathfrak{p}^i$ by (2.5), (2.6), (3.1), and (3.4). This completes the proof. □

3.4 LEMMA. *In the Witt decomposition of V of* (2.4)*, let N be as in* (2.7)*. Let q be an element of $\mathfrak{g} \cap F^\times$ and ξ as in Notation. Let t and c be as in* §2.2 *and* §2.3*, respectively. Then we obtain the following assertions:*

(1) *If $t = 1$, then $N[q] \neq \emptyset$ if and only if $\xi(cq) = 1$.*
(2) *Assume $t = 2$. Let K, $\mathfrak{r}$, and δ be as in* §2.3(II)*. Let $\nu \in \mathbf{Z}$ such that $q\mathfrak{g} = \mathfrak{p}^\nu$. Then $N[q] \neq \emptyset$ if and only if $c^{-1}q \in N_{K/F}(\mathfrak{r})$. Moreover if K is unramified over F, then this is the case if and only if $\nu \equiv \delta \pmod 2$.*
(3) *If $t = 3$, then $N[q] \neq \emptyset$ if and only if $\xi(-cq) \neq 1$.*
(4) *If $t = 4$, then we have $N[q] \neq \emptyset$ for all $q \in \mathfrak{g} \cap F^\times$.*
(5) *Let L be a maximal lattice in V and r as in* (2.4)*. If $r > 0$, then we have $L[q] \neq \emptyset$ for all $q \in \mathfrak{g} \cap F^\times$.*

Proof. We may assume that:

if $t = 1$, then $Z = F$, $N = \mathfrak{g}$, and $\varphi[x] = cx^2$ for $x \in F$;
if $t = 2$, then $Z = K$, $N = \mathfrak{r}$, and $\varphi[x] = cN_{K/F}(x)$ for $x \in K$;
if $t = 3$, then $Z = T$, $N = T \cap \mathfrak{P}^{-\delta}$, and $\varphi[x] = cN_{B/F}(x) = -cx^2$ for $x \in T$;
if $t = 4$, then $Z = B$, $N = \mathfrak{O}$, and $\varphi[x] = N_{B/F}(x)$ for $x \in B$

in (2.4); see §2.3. Then (1) and the first statement of (2) are trivial. We prove the second assertion of (2). Assume that $t = 2$ and K is unramified over F, then $\pi\mathfrak{r} = \mathfrak{q}$ and $N_{K/F}(\mathfrak{r}^\times) = \mathfrak{g}^\times$ by [BNT, Chapter VIII, Proposition 3]. Here $\mathfrak{q}$ is the maximal ideal of $\mathfrak{r}$. From these, we obtain the second assertion of (2). Assume $t = 3$. Noticing that B is division, the "only if"-part of (3) is immediate. If $\xi(-cq) \neq 1$, then $F(\sqrt{-c^{-1}q})$ is a quadratic extension of F, and hence there exists $z \in B$ such that $z \notin F$ and $z^2 = -c^{-1}q$ by [AQC, Proposition 5.15(ii)]. We easily see that $z \in T \cap \mathfrak{P}^{-\delta}$, and hence $N[q] \neq \emptyset$. Assume $t = 4$. Then we see that $N_{B/F}(\mathfrak{O}) = \mathfrak{g}$ from [AQC, Theorem 5.13] and [AQC, Proposition 5.15(i)]. This implies (4). Finally we prove (5). Assume $r > 0$. Since L can be represented as in (2.4), we have $f_1 + qe_1 \in L[q]$ for every $q \in \mathfrak{g} \cap F^\times$ with e_1 and f_1 in (2.4). This completes the proof. □

Now, our first main result in this paper can be stated as follows:

3.5 THEOREM. *Let L be a maximal lattice in V and put $C = C(L)$ as in Theorem* 3.2*. Let ξ be as in Notation. Let $q \in \mathfrak{g} \cap F^\times$ and $\nu \in \mathbf{Z}$ such that*

$q\mathfrak{g} = \mathfrak{p}^\nu$. *Let* $\kappa \in \mathbf{Z}$ *such that* $2\mathfrak{g} = \mathfrak{p}^\kappa$. *Let* r, t, e_r, f_r, *and* N *be as in* §2.2. *For* $0 \le i \in \mathbf{Z}$ *and* $x \in N$, *put*

$$h_{i,x} = x + \pi^i e_r,\ k_{i,x} = \pi^i f_r + x + \frac{q - \varphi[x]}{\pi^i} e_r,\ \ell_i = \pi^i f_r + q\pi^{-i} e_r. \tag{3.7}$$

Then we have

$$L[q] = \bigsqcup_{u \in R} uC = \begin{cases} L[q, 2^{-1}\mathfrak{p}^{\tau(q)}] & \text{if } r = 0, \\ \bigsqcup_{i=0}^{\tau(q)} L[q, 2^{-1}\mathfrak{p}^i] & \text{if } r \ge 1. \end{cases} \tag{3.8}$$

Here the set R *and the index* $\tau(q)$ *are defined as follows:*
(i) *Suppose* $t = 0$ *and* $r \ge 1$. *Then*

$$R = \begin{cases} \{\ell_i\}_{i=0}^{\nu} & \text{if } r = 1, \\ \{\ell_i\}_{i=0}^{\tau(q)} & \text{if } r \ge 2, \end{cases}$$

$$\tau(q) = [\nu/2]. \tag{3.9}$$

Moreover

$$L[q, 2^{-1}\mathfrak{p}^i] = \begin{cases} \ell_i C \sqcup \ell_{\nu-i} C & \text{if } r = 1 \text{ and } 0 \le i < \nu/2, \\ \ell_{\nu/2} C & \text{if } r = 1,\ \nu \in 2\mathbf{Z}, \text{ and } i = \nu/2, \\ \ell_i C & \text{if } r \ge 2. \end{cases} \tag{3.10}$$

(ii) *Suppose* $t = 1$. *Let* c *be as in* (2.9) *and* δ *as in* (2.10). *Let us define an integer* $d \in \mathbf{Z}$ *as follows:* $D_{F(\sqrt{cq})/F} = \mathfrak{p}^d$ *when* $\xi(cq) = 0$ *(in the ordinary sense) and* $d = 1$ *when* $\xi(cq) = -1$ *(This is only for a simplification of the following statements* (3.11) *and* (3.12)*). When* $\xi(cq) = 1$, *we take any element* y *of* $N[q]$ *and fix it (By Lemma* 3.4(1), $N[q] \ne \emptyset$*). When* $2 \in \mathfrak{p}$, $\xi(cq) \ne 1$, *and* $\nu \equiv \delta \pmod 2$, *take any element* z *of* $N[sq]$ *and fix it, with*

$$s \in 1 + \pi^{2\kappa+1-d}\mathfrak{g}^\times \text{ such that } c^{-1}q\pi^{\delta-\nu} \in s^{-1}\mathfrak{g}^{\times 2}. \tag{3.11}$$

(As for the existence of s *and* z, *see* (4.30) *and* (4.31), *respectively.) Then* R *and* $\tau(q)$ *are given as follows:*

$$R = \begin{cases} \{\pm y\} & \text{if } r = 0 \text{ and } \xi(cq) = 1, \\ \{y\} \sqcup \{h_{i,y}\}_{i=0}^{\tau(q)-1} & \text{if } r \ge 1 \text{ and } \xi(cq) = 1, \\ \{k_{i,z}\}_{i=0}^{\tau(q)} & \text{if } r \ge 1,\ \xi(cq) \ne 1, \\ & \quad \nu \equiv \delta \pmod 2, \text{ and } 2 \in \mathfrak{p}, \\ \{\ell_i\}_{i=0}^{\tau(q)} & \text{otherwise,} \end{cases}$$

$$\tau(q) = \begin{cases} \kappa + \frac{\nu+\delta}{2} & \text{if } \xi(cq) = 1, \\ \kappa + \left[\frac{\nu+1-d}{2}\right] & \text{if } \xi(cq) \ne 1,\ \nu \equiv \delta \pmod 2, \text{ and } 2 \in \mathfrak{p}, \\ \left[\frac{\nu}{2}\right] & \text{otherwise.} \end{cases} \tag{3.12}$$

Moreover

$$L[q,2^{-1}\mathfrak{p}^i]=\begin{cases} yC\sqcup(-y)C & \text{if } r=0,\ \xi(cq)=1,\ \text{and } i=\tau(q),\\ yC & \text{if } r\geq 1,\ \xi(cq)=1,\ \text{and } i=\tau(q),\\ h_{i,y}C & \text{if } r\geq 1,\ \xi(cq)=1,\ \text{and } i<\tau(q),\\ k_{i,z}C & \text{if } r\geq 1,\ \xi(cq)\neq 1,\\ & \qquad \nu\equiv\delta \pmod 2,\ \text{and } 2\in\mathfrak{p},\\ \ell_i C & \text{otherwise.}\end{cases} \tag{3.13}$$

(iii) *Suppose* $t=2$. *Let* c *and* δ *be as in* (2.12) *and* (2.13), *respectively. Let* K *and* $\mathfrak{r}$ *be as in* §2.3(II). *Let* $\mathfrak{d}$ *be the different of* K *relative to* F. *Let* $d\in\mathbf{Z}$ *such that* $D_{K/F}=\mathfrak{p}^d$ *when* $\mathfrak{d}\neq\mathfrak{r}$. *Put* $d=1$ *when* $\mathfrak{d}=\mathfrak{r}$ *(This is the same simplification as in* (ii)*). When* $c^{-1}q\in N_{K/F}(\mathfrak{r})$, *we take any element* y *of* $N[q]$ *and fix it (By Lemma* 3.4(2), $N[q]\neq\emptyset$*). When* $c^{-1}q\notin N_{K/F}(\mathfrak{r})$ *and* $d>1$, *we take any element* z *of* $N[sq]$ *and fix it, with*

$$s\in 1+\pi^{d-1}\mathfrak{g}^\times \text{ such that } c^{-1}q\pi^{-\nu}\in s^{-1}N_{K/F}(\mathfrak{r}^\times). \tag{3.14}$$

(As for the existence of s *and* z, *see* (4.32) *and* (4.33), *respectively.) Then* R *and* $\tau(q)$ *are given as follows:*

$$R=\begin{cases} \{y\} & \text{if } r=0 \text{ and } c^{-1}q\in N_{K/F}(\mathfrak{r}),\\ \{y\}\sqcup\{h_{i,y}\}_{i=0}^{\tau(q)-1} & \text{if } r\geq 1 \text{ and } c^{-1}q\in N_{K/F}(\mathfrak{r}),\\ \{k_{i,z}\}_{i=0}^{\tau(q)} & \text{if } r\geq 1,\ c^{-1}q\notin N_{K/F}(\mathfrak{r}),\ \text{and } d>1,\\ \{\ell_i\}_{i=0}^{\tau(q)} & \text{otherwise,}\end{cases}$$

$$\tau(q)=\begin{cases} \frac{\nu+\delta}{2} & \text{if } c^{-1}q\in N_{K/F}(\mathfrak{r}) \text{ and } \mathfrak{d}=\mathfrak{r},\\ \left[\frac{\nu+d}{2}\right] & \text{if } c^{-1}q\in N_{K/F}(\mathfrak{r}) \text{ and } \mathfrak{d}\neq\mathfrak{r},\\ \left[\frac{\nu+d-1}{2}\right] & \text{otherwise.}\end{cases} \tag{3.15}$$

Moreover

$$L[q,2^{-1}\mathfrak{p}^i]=\begin{cases} yC & \text{if } c^{-1}q\in N_{K/F}(\mathfrak{r}) \text{ and } i=\tau(q),\\ h_{i,y}C & \text{if } c^{-1}q\in N_{K/F}(\mathfrak{r}) \text{ and } i<\tau(q),\\ k_{i,z}C & \text{if } c^{-1}q\notin N_{K/F}(\mathfrak{r}) \text{ and } d>1,\\ \ell_i C & \text{otherwise.}\end{cases} \tag{3.16}$$

(iv) *Suppose* $t=3$. *Let* c *and* δ *be as in* (2.14) *and* (2.15), *respectively. When* $\xi(-cq)\neq 1$, *we take any element* y *of* $N[q]$ *and fix it (By Lemma* 3.4(3), $N[q]\neq\emptyset$*). When* $\xi(-cq)=1$ *and* $2\in\mathfrak{p}$, *we take any element* z *of* $N[sq]$ *and fix it, with*

$$s\in 1+4\mathfrak{g} \text{ such that } s\notin\mathfrak{g}^{\times 2}. \tag{3.17}$$

(As for the existence of s and z, see (4.34) *and* (4.35)*, respectively.) Then R and $\tau(q)$ are given as follows:*

$$R = \begin{cases} \{y\} & \text{if } r = 0 \text{ and } \xi(-cq) \neq 1, \\ \{y\} \sqcup \{h_{i,y}\}_{i=0}^{\tau(q)-1} & \text{if } r \geq 1 \text{ and } \xi(-cq) \neq 1, \\ \{k_{i,z}\}_{i=0}^{\tau(q)} & \text{if } r \geq 1,\ \xi(-cq) = 1, \text{ and } 2 \in \mathfrak{p}, \\ \{\ell_i\}_{i=0}^{\tau(q)} & \text{if } r \geq 1,\ \xi(-cq) = 1, \text{ and } 2 \in \mathfrak{g}^\times, \end{cases}$$

$$\tau(q) = \begin{cases} \kappa + \left[\frac{\nu}{2}\right] & \text{if } \xi(-cq) = 1, \\ \frac{\nu-\delta+1}{2} & \text{if } \xi(-cq) \neq 1 \text{ and } \nu \not\equiv \delta \pmod 2, \\ \kappa + 1 + \frac{\nu-\delta-d}{2} & \text{if } \xi(-cq) = 0,\ \nu \equiv \delta \pmod 2, \text{ and } 2 \in \mathfrak{p}, \\ \kappa + \frac{\nu+\delta}{2} & \text{otherwise,} \end{cases} \tag{3.18}$$

where $d \in \mathbf{Z}$ such that $D_{F(\sqrt{-cq})/F} = \mathfrak{p}^d$. Moreover

$$L[q, 2^{-1}\mathfrak{p}^i] = \begin{cases} yC & \text{if } \xi(-cq) \neq 1 \text{ and } i = \tau(q), \\ h_{i,y}C & \text{if } \xi(-cq) \neq 1 \text{ and } i < \tau(q), \\ k_{i,z}C & \text{if } \xi(-cq) = 1 \text{ and } 2 \in \mathfrak{p}, \\ \ell_i C & \text{if } \xi(-cq) = 1 \text{ and } 2 \in \mathfrak{g}^\times. \end{cases} \tag{3.19}$$

(v) *Suppose $t = 4$. Take any element y of $N[q]$ and fix it. Then R and $\tau(q)$ are given as follows:*

$$R = \begin{cases} \{y\} & \text{if } r = 0, \\ \{y\} \sqcup \{h_{i,y}\}_{i=0}^{\tau(q)-1} & \text{if } r \geq 1, \end{cases}$$

$$\tau(q) = [(\nu+1)/2]. \tag{3.20}$$

Moreover

$$L[q, 2^{-1}\mathfrak{p}^i] = \begin{cases} yC & \text{if } i = \tau(q), \\ h_{i,y}C & \text{if } i < \tau(q). \end{cases} \tag{3.21}$$

The proof of this theorem will be given in the following Section 4. Here we insert one elementary lemma:

3.6 Lemma. *Let F be a local field and L a maximal lattice in V. Let $\kappa \in \mathbf{Z}$ such that $2\mathfrak{g} = \mathfrak{p}^\kappa$. Then for $q \in \mathfrak{g}$ and $i \in \mathbf{Z}$, we have $L[q, \mathfrak{p}^i] \subset L[q]$ if and only if $i \geq -\kappa$.*

Proof. From (2.2), clearly $i \geq -\kappa$ if and only if $L[q, \mathfrak{p}^i] \subset \tilde{L}[q]$. Here [AQC, Lemma 6.2(3)] implies $\tilde{L}[q] = L[q]$. This proves the lemma. □

3.7 Corollary. *Let the notation be the same as in Theorem* 3.5*. Assume $L[q] \neq \emptyset$ for $q \in \mathfrak{g} \cap F^\times$. Then for every $i \in \mathbf{Z}$,*

$$L[q, 2^{-1}\mathfrak{p}^i] \neq \emptyset \iff \begin{cases} i = \tau(q) & \text{if } n = t, \\ i \leq \tau(q) & \text{otherwise.} \end{cases}$$

Proof. For $0 \leq i \in \mathbf{Z}$, the result follows from Theorem 3.5 and Lemma 3.6. Assume $i < 0$. Clearly $L[q, 2^{-1}\mathfrak{p}^i] = \pi^i \cdot L[\pi^{-2i}q, 2^{-1}\mathfrak{g}]$. Here Lemma 3.6 implies $L[\pi^{-2i}q, 2^{-1}\mathfrak{g}] \subset L[\pi^{-2i}q]$. Since $\pi^i \cdot L[\pi^{-2i}q] = (\pi^i L)[q] \supset L[q]$, we obtain $L[\pi^{-2i}q] \neq \emptyset$. Applying Theorem 3.5 to (V, φ), L, and $\pi^{-2i}q$, we find that if $n = t$ then $L[\pi^{-2i}q] = L[\pi^{-2i}q, 2^{-1}\mathfrak{p}^{\tau(\pi^{-2i}q)}]$ and $\tau(\pi^{-2i}q) = \tau(q) - i > 0$. Therefore $L[\pi^{-2i}q, 2^{-1}\mathfrak{g}] = \emptyset$, and hence $L[q, 2^{-1}\mathfrak{p}^i] = \emptyset$. If $n \neq t$, then $L[\pi^{-2i}q, 2^{-1}\mathfrak{g}] \neq \emptyset$ by Theorem 3.5, and hence $L[q, 2^{-1}\mathfrak{p}^i] \neq \emptyset$. This completes the proof. □

4 Proof of Theorem 3.5

4.1. We first prove Theorem 3.5(i). Assume $t = 0$. Then $L[q] \neq \emptyset$ for all $q \in \mathfrak{g} \cap F^\times$ by Lemma 3.4(5).
First suppose $r = 1$. Then $L = \mathfrak{g}f_1 + \mathfrak{g}e_1$ by (2.4). We obtain

$$L[q] = \bigsqcup_{i=0}^{\nu} \ell_i C \tag{4.1}$$

by Theorem 3.2(2) with ℓ_i in (3.7). We have clearly

$$\varphi(\ell_i, L) = 2^{-1}\pi^i\mathfrak{g} + 2^{-1}q\pi^{-i}\mathfrak{g} = \begin{cases} 2^{-1}\mathfrak{p}^i & \text{if } 0 \leq i \leq [\nu/2], \\ 2^{-1}\mathfrak{p}^{\nu-i} & \text{if } i > [\nu/2] \end{cases} \tag{4.2}$$

from (2.5). Assume $\nu \in 2\mathbf{Z}$. Then

$$\varphi(\ell_i, L) = \varphi(\ell_{\nu-i}, L) = 2^{-1}\mathfrak{p}^i \tag{4.3}$$

for $0 \leq i \leq (\nu-2)/2$ and $\varphi(\ell_{\nu/2}, L) = 2^{-1}\mathfrak{p}^{\nu/2}$ by (4.2). Thus $\ell_i C \sqcup \ell_{\nu-i}C \subset L[q, 2^{-1}\mathfrak{p}^i]$ for $0 \leq i \leq (\nu-2)/2$ and $\ell_{\nu/2}C \subset L[q, 2^{-1}\mathfrak{p}^{\nu/2}]$. On the other hand, we have $L[q, 2^{-1}\mathfrak{p}^i] \subset L[q]$ for $0 \leq i \leq \nu/2$ by Lemma 3.6. Hence $\ell_i C \sqcup \ell_{\nu-i}C = L[q, 2^{-1}\mathfrak{p}^i]$ for $0 \leq i \leq (\nu-2)/2$ and $\ell_{\nu/2}C = L[q, 2^{-1}\mathfrak{p}^{\nu/2}]$. From this and (4.1) we obtain the assertion in the case $r = 1$, $t = 0$, and $\nu \in 2\mathbf{Z}$. Next assume $\nu \notin 2\mathbf{Z}$. Then

$$\varphi(\ell_i, L) = \varphi(\ell_{\nu-i}, L) = 2^{-1}\mathfrak{p}^i \tag{4.4}$$

for $0 \leq i \leq (\nu-1)/2$. Thus $\ell_i C \sqcup \ell_{\nu-i}C = L[q, 2^{-1}\mathfrak{p}^i]$ for $0 \leq i \leq (\nu-1)/2$, in the same manner as in the case $\nu \in 2\mathbf{Z}$. This proves the assertion when $r = 1$ and $t = 0$.
Next suppose $r = 2$. Then $L = \mathfrak{g}f_1 + M + \mathfrak{g}e_1$ by (3.1). We obtain

$$\emptyset \neq M[q] = \sqcup_{i=0}^{\nu} \ell_i C(M) \tag{4.5}$$

by Theorem 3.2(2) and Lemma 3.4(5). In this case we can not apply Lemma 3.3 since we have (4.3) and (4.4) in the notation of (4.1). By (4.5) and Theorem 3.2(4),

$$L[q] = \bigcup_{i=0}^{\nu} \bigcup_{a \in \mathfrak{g}/2\varphi(\ell_i, M)} (\ell_i + ae_1)C.$$

Since

$$2\varphi(\ell_i, M) = \begin{cases} \mathfrak{p}^i & \text{if } 0 \le i \le [\nu/2], \\ \mathfrak{p}^{\nu-i} & \text{if } i > [\nu/2] \end{cases}$$

by (4.2), we have

$$L[q] = \Big[\bigcup_{i=0}^{[\nu/2]} \bigcup_{a\in\mathfrak{g}/\mathfrak{p}^i} (\ell_i + ae_1)C\Big] \bigcup \Big[\bigcup_{j>[\nu/2]}^{\nu} \bigcup_{b\in\mathfrak{g}/\mathfrak{p}^{\nu-j}} (\ell_j + be_1)C\Big]. \tag{4.6}$$

For $0 \le i \le [\nu/2]$ and $a \in \mathfrak{g}/\mathfrak{p}^i$,

$$\varphi(\ell_i + ae_1, L) = \varphi(\ell_i, M) + 2^{-1}a\mathfrak{g} = \begin{cases} 2^{-1}\mathfrak{p}^i & \text{if } a \in \mathfrak{p}^i, \\ 2^{-1}a\mathfrak{g} & \text{if } a \notin \mathfrak{p}^i \end{cases}$$

by (2.5) and (3.1). Therefore by [IQD, Theorem 1.3],

$$(\ell_i + ae_1)C = L[q, \varphi(\ell_i + ae_1, L)] = \begin{cases} L[q, 2^{-1}\mathfrak{p}^i] & \text{if } a \in \mathfrak{p}^i, \\ L[q, 2^{-1}a\mathfrak{g}] & \text{if } a \notin \mathfrak{p}^i. \end{cases} \tag{4.7}$$

Similarly we have

$$(\ell_j + be_1)C = \begin{cases} L[q, 2^{-1}\mathfrak{p}^{\nu-j}] & \text{if } b \in \mathfrak{p}^{\nu-j}, \\ L[q, 2^{-1}b\mathfrak{g}] & \text{if } b \notin \mathfrak{p}^{\nu-j} \end{cases} \tag{4.8}$$

for $j > [\nu/2]$ and $b \in \mathfrak{g}/\mathfrak{p}^{\nu-j}$. From (4.7) and (4.8), the argument in the proof of Lemma 3.3 shows that

$$\bigcup_{i=0}^{[\nu/2]} \bigcup_{a\in\mathfrak{g}/\mathfrak{p}^i} (\ell_i + ae_1)C = \bigsqcup_{i=0}^{[\nu/2]} \ell_i C, \quad \ell_i C = L[q, 2^{-1}\mathfrak{p}^i], \tag{4.9}$$

$$\bigcup_{j>[\nu/2]}^{\nu} \bigcup_{b\in\mathfrak{g}/\mathfrak{p}^{\nu-j}} (\ell_j + be_1)C = \bigsqcup_{j>[\nu/2]}^{\nu} \ell_j C, \quad \ell_j C = L[q, 2^{-1}\mathfrak{p}^{\nu-j}]. \tag{4.10}$$

Combining (4.6), (4.9), and (4.10), we have

$$L[q] = \Big(\bigsqcup_{i=0}^{[\nu/2]} \ell_i C\Big) \cup \Big(\bigsqcup_{j>[\nu/2]}^{\nu} \ell_j C\Big) = \bigsqcup_{i=0}^{[\nu/2]} L[q, 2^{-1}\mathfrak{p}^i] = \bigsqcup_{i=0}^{[\nu/2]} \ell_i C$$

and $L[q, 2^{-1}\mathfrak{p}^i] = \ell_i C$. This proves our theorem in the case $r = 2$ and $t = 0$. As for the case $r \ge 3$, we apply (repeatedly, if necessary) Lemma 3.3 and [IQD, Theorem 1.3] to this case, we can reduce the proof to the case $r = 2$. This completes the proof of (i).

4.2 Lemma. *Let F be a local field. Assume $2 \in \mathfrak{p}$ and let $\kappa \in \mathbf{Z}$ such that $2\mathfrak{g} = \mathfrak{p}^{\kappa}$. Let ξ be as in Notation. Put*

$$\varepsilon(a) = \max(\{e \in \mathbf{Z} \mid e \leq 2\kappa + 1 \text{ and } a \in (1 + \mathfrak{p}^e)\mathfrak{g}^{\times 2}\}) \tag{4.11}$$

for $a \in \mathfrak{g}^{\times}$. Then we obtain the following assertions:

(1) *For $a \in \mathfrak{g}^{\times}$,*

$$\varepsilon(a) = \begin{cases} 2\kappa + 1 - d & \text{if } \xi(a) = 0, \\ 2\kappa & \text{if } \xi(a) = -1, \\ 2\kappa + 1 & \text{if } \xi(a) = 1, \end{cases} \tag{4.12}$$

where $d \in \mathbf{Z}$ such that $D_{F(\sqrt{a})/F} = \mathfrak{p}^d$.
(2) *If $\xi(a) = 0$, then we have $2\kappa > \varepsilon(a) \notin 2\mathbf{Z}$.*
(3) *If $0 < \ell < 2\kappa$ and $\ell \in \mathbf{Z}$, $\notin 2\mathbf{Z}$, then $\mathfrak{g}^{\times 2} \cap (1 + \pi^{\ell}\mathfrak{g}^{\times}) = \emptyset$.*
(4) *If $a \in (1 + \pi^{\ell}\mathfrak{g}^{\times})\mathfrak{g}^{\times 2}$ with $0 < \ell \in \mathbf{Z}$, $\notin 2\mathbf{Z}$ and $\varepsilon(a) < 2\kappa$, then $\varepsilon(a) = \ell$.*
(5) *If $a \in (1 + \pi^{\ell}\mathfrak{g}^{\times})\mathfrak{g}^{\times 2}$ with $0 < \ell \in \mathbf{Z}$, $\notin 2\mathbf{Z}$ and $\varepsilon(a) = 2\kappa + 1$, then $\varepsilon(a) \leq \ell$.*

Proof. Assertions (1) and (2) are in [NRQ, Lemma 3.5]. (3): If there exists an element $b \in \mathfrak{g}^{\times}$ such that $b^2 \in 1 + \pi^{\ell}\mathfrak{g}^{\times}$, then $b \in Z_{\ell}$. Here $Z_{\ell} = \{x \in \mathfrak{g}^{\times} \mid x^2 - 1 \in \mathfrak{p}^{\ell}\}$ as in [NRQ, §3.4]. By [NRQ, (3.5.1)], $Z_{\ell} = 1 + \mathfrak{p}^{(\ell+1)/2}$, and hence we can take $y \in \mathfrak{g}$ such that $b = 1 + \pi^{(\ell+1)/2}y$. Then $b^2 = 1 + \pi^{\ell+1}y(2\pi^{-(\ell+1)/2} + y) \in 1 + \mathfrak{p}^{\ell+1}$ since $2^{-1}(\ell+1) \leq \kappa$. This gives a contradiction. Thus we obtain (3). (4): We find $\ell \leq \varepsilon(a)$ from (4.11), (4.12), and [NRQ, Lemma 3.2(1)]. Clearly

$$(1 + \pi^{\ell}\mathfrak{g}^{\times})\mathfrak{g}^{\times 2} \cap (1 + \mathfrak{p}^{\varepsilon(a)})\mathfrak{g}^{\times 2} \neq \emptyset. \tag{4.13}$$

If $\ell < \varepsilon(a)$, then (4.13) contradicts (3) settled above, and hence $\ell = \varepsilon(a)$. (5): By (4.12), we have $\xi(a) = 1$. Thus

$$\mathfrak{g}^{\times 2} \cap (1 + \pi^{\ell}\mathfrak{g}^{\times}) \neq \emptyset. \tag{4.14}$$

If $\ell < \varepsilon(a) = 2\kappa + 1$, then (4.14) contradicts (3). This completes the proof. □

4.3. Now we prove (ii), (iii), (iv), and (v) of Theorem 3.5. We may assume that:

if $t = 1$, then $Z = F$, $N = \mathfrak{g}$, and $\varphi[x] = cx^2$ for $x \in F$;
if $t = 2$, then $Z = K$, $N = \mathfrak{r}$, and $\varphi[x] = cN_{K/F}(x)$ for $x \in K$;
if $t = 3$, then $Z = T$, $N = \mathfrak{g}v + \mathfrak{r}\omega^{1-2\delta}$, and $\varphi[x] = cN_{B/F}(x)$ for $x \in T$;
if $t = 4$, then $Z = B$, $N = \mathfrak{O}$, and $\varphi[x] = N_{B/F}(x)$ for $x \in B$.

Then for $x, w \in Z$,

$$\varphi(x, w) = \begin{cases} cxw & \text{if } t = 1, \\ 2^{-1}cTr_{K/F}(xw^{\rho}) & \text{if } t = 2, \\ 2^{-1}cTr_{B/F}(xw^{\iota}) & \text{if } t = 3, \\ 2^{-1}Tr_{B/F}(xw^{\iota}) & \text{if } t = 4. \end{cases} \tag{4.15}$$

Here $\rho \in Gal(K/F)$ such that $\rho \neq 1$. In this §4.3 we prove the theorem in the case $r = 0$ and $t > 0$. Note that $L = N$ in this case. If $L[q] \neq \emptyset$, then

$$L[q] = \begin{cases} yC \sqcup (-y)C & \text{if } t = 1, \\ yC & \text{otherwise} \end{cases} \tag{4.16}$$

by Theorem 3.2(1). Here, y is any element of $L[q]$ and fix it until the end of §4.3. This proves the first equality of (3.8) in this case. From Lemma 3.6 and (4.16),

$$L[q] = L[q, \varphi(y, L)] \tag{4.17}$$

for $1 \leq t \leq 4$. Note that $\varphi(y, L) \subset 2^{-1}\mathfrak{g}$ since L is an integral lattice in V. To prove the second equality of (3.8) we determine the ideal $\varphi(y, L)$ as the next step.

(4.18) We let μ denote the normalized order function of F.

First suppose $t = 1$, then $C = \{1\}$. Here Lemma 3.4(1) implies that $L[q] \neq \emptyset$ if and only if $\xi(cq) = 1$. Since $y^2 = c^{-1}q$, we have $\varphi(y, L) = cy\mathfrak{g} = \mathfrak{p}^{(\nu+\delta)/2}$ by (4.15).
Next suppose $t = 2$. We have $L[q] \neq \emptyset$ if and only if $c^{-1}q \in N_{K/F}(\mathfrak{r})$ by Lemma 3.4(2). From [BNT, Chapter VIII, Propositon 4] and (4.15), we see that

$$\varphi(y, L) = 2^{-1}cTr_{K/F}(y\mathfrak{r}) = \begin{cases} 2^{-1}\mathfrak{p}^{(\nu+\delta)/2} & \text{if } \mathfrak{d} = \mathfrak{r}, \\ 2^{-1}\mathfrak{p}^{[(\nu+d)/2]} & \text{if } \mathfrak{d} \neq \mathfrak{r}. \end{cases}$$

Note that we take $c \in \mathfrak{g}^\times$ if K is ramified over F; see §2.3.
Suppose $t = 3$. By Lemma 3.4(3),

$$L[q] \neq \emptyset \text{ if and only if } \xi(-cq) \neq 1. \tag{4.19}$$

Take $m \in \mathbf{Z}$ such that $\varphi(y, L) = 2^{-1}\mathfrak{p}^m$. Let us determine m. Let μ_K be the normalized order function of K. Since $L = \mathfrak{g}v + \mathfrak{r}\omega^{1-2\delta} = v(\mathfrak{g} + \mathfrak{r}\omega^{1-2\delta})$, we can put $y = v(a + b\omega^{1-2\delta})$ with $a \in \mathfrak{g}$ and $b \in \mathfrak{r}$. Then by (2.18), $\varphi(y, L) = 2^{-1}c(2a\mathfrak{g} + \omega^{2(1-2\delta)}Tr_{K/F}(b\mathfrak{r}))$, and hence

$$m = \min(\kappa + \delta + \mu(a), 1 + \mu_K(b) - \delta). \tag{4.20}$$

We have also

$$q = \varphi[y] = -cv^2(a^2 - \omega^{2(1-2\delta)}N_{K/F}(b)) \tag{4.21}$$

by (2.17), and hence

$$\nu - \delta = \min(2\mu(a), 1 + 2(\mu_K(b) - \delta)). \tag{4.22}$$

Assume $\nu \not\equiv \delta \pmod 2$. Then $\nu - \delta = 1 + 2(\mu_K(b) - \delta) < 2\mu(a)$ by (4.22), and hence $2^{-1}(\nu-\delta+1) = 1+\mu_K(b)-\delta \le \kappa+\delta+\mu(a)$. Thus we find $m = 2^{-1}(\nu-\delta+1)$ by (4.20). Next assume $\nu \equiv \delta \pmod 2$. Then $\nu-\delta = 2\mu(a) < 1+2(\mu_K(b)-\delta)$ by (4.22). If $2 \in \mathfrak{g}^\times$, then $\kappa + 2^{-1}(\nu + \delta) = \kappa + \delta + \mu(a) \le 1 + \mu_K(b) - \delta$, and hence $m = \kappa + 2^{-1}(\nu + \delta)$ from (4.20). If $2 \in \mathfrak{p}$, then Lemma 4.2 implies $\varepsilon(-c^{-1}q\pi^{\delta-\nu}) \le 2\kappa$ since $\xi(-c^{-1}q\pi^{\delta-\nu}) \ne 1$ by (4.19). Put $\beta = -c^{-1}q\pi^{\delta-\nu}$, then

$$\varepsilon(v^{-2}\beta) = \begin{cases} \varepsilon(\beta) & \text{if } \varepsilon(\beta) < 2\kappa, \\ 2\kappa+1 & \text{if } \varepsilon(\beta) = 2\kappa; \end{cases} \tag{4.23}$$

see the proof of [NRQ, Lemma 4.5]. By (4.21),

$$v^{-2}\beta = (\pi^{(\delta-\nu)/2}a)^2(1-\omega^{2(1-2\delta)}a^{-2}N_{K/F}(b)) \in (1-\omega^{2(1-2\delta)}a^{-2}N_{K/F}(b))\mathfrak{g}^{\times 2}. \tag{4.24}$$

Hence if $\varepsilon(\beta) < 2\kappa$, that is $\xi(-cq) = 0$ from Lemma 4.2, then Lemma 4.2(4) and (4.23) imply $\varepsilon(\beta) = 1 - 2\delta - 2\mu(a) + 2\mu_K(b)$, and hence $1 + \mu_K(b) - \delta = 2^{-1}(\varepsilon(\beta)+1) + 2^{-1}(\nu-\delta) \le \kappa + \mu(a) + \delta$. Thus $m = 2^{-1}(\varepsilon(\beta)+1) + 2^{-1}(\nu-\delta)$ from (4.20). Here Lemma 4.2(1) implies $\varepsilon(\beta) = 2\kappa + 1 - d$, where $d \in \mathbf{Z}$ such that $D_{F(\sqrt{-cq})/F} = \mathfrak{p}^d$. Therefore $m = \kappa + 1 + (\nu - \delta - d)/2$. If $\varepsilon(\beta) = 2\kappa$, then Lemma 4.2(5), (4.23), and (4.24) imply $2\kappa + 1 \le 1 - 2\delta - 2\mu(a) + 2\mu_K(b)$, and hence $1 + \mu_K(b) - \delta \ge \kappa + \mu(a) + \delta$. Thus $m = \kappa + \mu(a) + \delta = \kappa + (\nu + \delta)/2$ from (4.20). Consequently we obtain

$$\varphi(y, L) = \begin{cases} 2^{-1}\mathfrak{p}^{(\nu-\delta+1)/2} & \text{if } \nu \not\equiv \delta \pmod 2, \\ \mathfrak{p}^{1+(\nu-\delta-d)/2} & \text{if } \nu \equiv \delta \pmod 2,\ 2 \in \mathfrak{p},\ \xi(-cq) = 0, \\ \mathfrak{p}^{(\nu+\delta)/2} & \text{otherwise} \end{cases} \tag{4.25}$$

under the assumption $t = 3$, $r = 0$, and $\xi(-cq) \ne 1$. Here $d \in \mathbf{Z}$ such that $D_{F(\sqrt{-cq})/F} = \mathfrak{p}^d$. We see the second equality of (3.8) by (4.17) and (4.25). Moreover combining this with (4.16), we obtain the theorem in the case $r = 0$ and $t = 3$.

Finally suppose $t = 4$. Then Lemma 3.4(4) implies $L[q] \ne \emptyset$ for every $q \in \mathfrak{g} \cap F^\times$. From [AQC, Theorem 5.9(2), (6), (7)] and (4.15), we see that

$$\varphi(y, L) = 2^{-1}Tr_{B/F}(\mathfrak{P}^{\mu(N_{B/F}(y))}) = 2^{-1}\mathfrak{p}^{[(\nu+1)/2]}, \tag{4.26}$$

where $\mathfrak{P} = \{x \in \mathfrak{O} \mid N_{B/F}(x) \in \mathfrak{p}\}$, with μ of (4.18). This completes the proof of our theorem in the case $r = 0$ and $t > 0$.

4.4. Here we prove the theorem in the case $r \ge 1$ and $t > 0$. First we note that when $r \ge 2$ we apply (repeatedly, if necessary) Lemma 3.3 and [IQD, Theorem 1.3] to this case, we can reduce the proof to the case $r = 1$.

Thus hereafter until the end of §4.7, we assume $r = 1$. Then $L = \mathfrak{g}f_1 + N + \mathfrak{g}e_1$ by (2.4). Assume $1 \le t \le 4$. We recall here Lemma 3.4. We know that $L[q] \ne \emptyset$

for all $q \in \mathfrak{g} \cap F^\times$ and

$$N[q] \neq \emptyset \iff \begin{cases} \xi(cq) = 1 & \text{if } t = 1, \\ c^{-1}q \in N_{K/F}(\mathfrak{r}) & \text{if } t = 2, \\ \xi(-cq) \neq 1 & \text{if } t = 3. \end{cases} \tag{4.27}$$

When $t = 4$, we have $N[q] \neq \emptyset$ for every $q \in \mathfrak{g} \cap F^\times$.

4.5. In this § we prove the theorem in the case $r = 1$, $1 \leq t \leq 4$, and $N[q] \neq \emptyset$. Applying Theorem 3.2(1) and (4), we find

$$L[q] = \begin{cases} \left[\bigcup_{a \in \mathfrak{g}/2\varphi(y,N)} (y + ae_1)C\right] \cup \left[\bigcup_{a \in \mathfrak{g}/2\varphi(-y,N)} (-y + ae_1)C\right] & \text{if } t = 1, \\ \bigcup_{a \in \mathfrak{g}/2\varphi(y,N)} (y + ae_1)C & \text{otherwise,} \end{cases}$$

where y is any element of $N[q]$ and fix it. Since $\varphi(y + ae_1, L) = \varphi(-y + ae_1, L)$, we obtain $(y + ae_1)C = (-y + ae_1)C$ by [IQD, Theorem 1.3]. Thus $L[q] = \bigcup_{a \in \mathfrak{g}/2\varphi(y,N)} (y + ae_1)C$ for $1 \leq t \leq 4$. We have already obtained our theorem in the case $r = 0$ and $t > 0$. Thus $\varphi(y, N) = 2^{-1}\mathfrak{p}^{\tau(q)}$. Put simply $\tau = \tau(q)$. The same argument as in the proof of Lemma 3.3 shows that

$$(y + ae_1)C = \begin{cases} L[q, 2^{-1}\mathfrak{p}^\tau] & \text{if } a \in \mathfrak{p}^\tau, \\ L[q, 2^{-1}\mathfrak{p}^{\mu(a)}] & \text{if } a \notin \mathfrak{p}^\tau \end{cases}$$

with μ of (4.18). If $\tau \geq 1$, then $0 \leq \mu(a) \leq \tau - 1$ for $a \in \mathfrak{g}$ such that $a \notin \mathfrak{p}^\tau$. Thus

$$L[q] = \bigcup_{a \in \mathfrak{g}/\mathfrak{p}^\tau} (y + ae_1)C = \bigsqcup_{i=0}^{\tau} L[q, 2^{-1}\mathfrak{p}^i],$$

$L[q, 2^{-1}\mathfrak{p}^\tau] = yC$, and $L[q, 2^{-1}\mathfrak{p}^i] = (y + \pi^i e_1)C$ for $0 \leq i \leq \tau - 1$. If $\tau = 0$, then it is clear that $L[q] = yC = L[q, 2^{-1}\mathfrak{g}]$. This proves the theorem in the case $r = 1$, $t > 0$, and $N[q] \neq \emptyset$.

4.6. In §§4.6 and 4.7 we assume $r = 1$, $t > 0$, and $N[q] = \emptyset$. Then $1 \leq t \leq 3$ since $N[q] \neq \emptyset$ for all $q \in \mathfrak{g} \cap F^\times$ if $t = 4$. By Theorem 3.2(3),

$$L[q] = \bigcup_{i=0}^{\kappa_0 - 1} \bigcup_{b \in X_i(q)/\mathfrak{p}^i N} k_{i,b}C \tag{4.28}$$

with $k_{i,b}$ in (3.7). Let us determine κ_0 in this §4.6. With the notation of (3.2)

$$0 \in X_\nu(q) \tag{4.29}$$

since $q \in \mathfrak{p}^\nu$, and hence $\kappa_0 > \nu$ with the notation of (3.3). Suppose $t = 1$, then $\xi(cq) \neq 1$ by (4.27) and the assumption $N[q] = \emptyset$. First assume $2 \notin \mathfrak{p}$ or $\nu \not\equiv \delta \pmod 2$. If there exists $x \in X_{\nu+1}(q)$, then $x^2 \in c^{-1}q(1 + \mathfrak{p})$ by (3.2). This implies $\nu \equiv \delta \pmod 2$, and hence $2 \notin \mathfrak{p}$. Then by [NRQ, Lemma 3.2(1)], we have $1+\mathfrak{p} \subset \mathfrak{g}^{\times 2}$, which contradicts $\xi(cq) \neq 1$. Thus $\kappa_0 = \nu+1$ by (4.29). Next assume $2 \in \mathfrak{p}$ and $\nu \equiv \delta \pmod 2$. Then $\varepsilon(c^{-1}q\pi^{\delta-\nu}) \leq 2\kappa$ from $\xi(cq) \neq 1$ and Lemma 4.2. Hereafter we put $\varepsilon = \varepsilon(c^{-1}q\pi^{\delta-\nu})$. There exist

$$s \in 1+\pi^{\varepsilon}\mathfrak{g}^{\times} \text{ and } \alpha \in \mathfrak{g}^{\times} \text{ such that } c^{-1}q\pi^{\delta-\nu} = s^{-1}\alpha^2 \tag{4.30}$$

by (4.11). From this we have $c(\pi^{(\nu-\delta)/2}\alpha)^2 = sq$, and hence

$$N[sq] \neq \emptyset. \tag{4.31}$$

Since $N[sq] \subset X_{\nu+\varepsilon}(q)$, we obtain $X_{\nu+\varepsilon}(q) \neq \emptyset$, and hence $\kappa_0 > \nu+\varepsilon$ in the notation of (3.3). If there exists $x \in X_{\nu+\varepsilon+1}(q)$, then we can take $a \in \mathfrak{p}^{\nu+\varepsilon+1}$ such that $cx^2 + a = q$. Thus $c^{-1}q\pi^{\delta-\nu} = (1+c^{-1}x^{-2}a)(\pi^{(\delta-\nu)/2}x)^2 \in (1+\mathfrak{p}^{\varepsilon+1})\mathfrak{g}^{\times 2}$, which contradicts (4.11). Hence $\kappa_0 = \nu+\varepsilon+1$. Moreover Lemma 4.2(1) implies that: if $\xi(cq) = -1$, then $\varepsilon = 2\kappa$, and hence $\kappa_0 = 2\kappa+\nu+1$; if $\xi(cq) = 0$ and $D_{F(\sqrt{cq})/F} = \mathfrak{p}^d$, then $\varepsilon = 2\kappa+1-d$, and hence $\kappa_0 = 2\kappa+\nu+2-d$.
Next suppose $t = 2$, then $c^{-1}q \notin N_{K/F}(\mathfrak{r})$ by (4.27). Let d be as in Theorem 3.5(iii). If $X_{\nu+d}(q) \neq \emptyset$, then $c^{-1}q \in N_{K/F}(\mathfrak{r})(1+\mathfrak{p}^d) \subset N_{K/F}(\mathfrak{r})$ by [BNT, Chapter VIII, Proposition 3] or the conductor-discriminant theorem according as $\mathfrak{d} = \mathfrak{r}$ or $\mathfrak{d} \neq \mathfrak{r}$. This contradicts $c^{-1}q \notin N_{K/F}(\mathfrak{r})$. Thus $\kappa_0 \leq \nu+d$. In particular if $\mathfrak{d} = \mathfrak{r}$ or $\mathfrak{q}$, that is $d = 1$, then $\kappa_0 = \nu+1$ from (4.29). Here $\mathfrak{q}$ is the maximal ideal of $\mathfrak{r}$. Assume $\mathfrak{d} = \mathfrak{q}^d$ and $d > 1$. Take a prime element π_K of K such that $N_{K/F}(\pi_K) = \pi$. We see that $2 \in \mathfrak{p}$ by [BNT, Chapter VIII, Corollary 3 of Proposition 7]. By local class field theory, we have $(1+\mathfrak{p}^{d-1})N_{K/F}(\mathfrak{r}^{\times}) = \mathfrak{g}^{\times}$. Thus there exist

$$s \in 1+\pi^{d-1}\mathfrak{g}^{\times} \text{ and } \alpha \in \mathfrak{r}^{\times} \text{ such that } c^{-1}q\pi^{-\nu} = s^{-1}N_{K/F}(\alpha). \tag{4.32}$$

Note that $c \in \mathfrak{g}^{\times}$ since K is ramified over F; see §2.3. Then $cN_{K/F}(\pi_K^{\nu}\alpha) = sq$, and hence

$$N[sq] \neq \emptyset. \tag{4.33}$$

We obtain $N[sq] \subset X_{\nu+d-1}(q)$ by the definition of s. Thus $\kappa_0 = \nu+d$.
Finally suppose $t = 3$. Then $\xi(-cq) = 1$ by (4.27), and hence $\nu \equiv \delta \pmod 2$. For $b \in \mathfrak{g}$ and $0 \leq m \in \mathbf{Z}$, put

$$Y_m(b) = \{y \in N \mid y^2 - b \in \mathfrak{p}^m\}$$

as in [NRQ, (4.1.3)]. Then $X_i(q) = \pi^{(\nu-\delta)/2}Y_{i-\nu}(-c^{-1}q\pi^{\delta-\nu})$ if $i > \nu$. The proof of [NRQ, Lemma 4.2(2)] shows that $Y_m(-c^{-1}q\pi^{\delta-\nu}) = \emptyset$ for $m \geq 2\kappa+1$ even for $c\mathfrak{g} = \mathfrak{p}$. Thus $\kappa_0 \leq \nu+2\kappa+1$. If $2 \in \mathfrak{g}^{\times}$, then $\kappa_0 = \nu+1$ by (4.29). Assume $2 \in \mathfrak{p}$. There exists

$$s \in 1+4\mathfrak{g} \text{ such that } s \notin \mathfrak{g}^{\times 2} \tag{4.34}$$

by [NRQ, Lemma 3.2(1)], then $\xi(s) = -1$ from Lemma 4.2(1). Thus $\xi(-csq) = -1$, and hence, by Lemma 3.4(3),

$$N[sq] \neq \emptyset. \tag{4.35}$$

We find $N[sq] \subset X_{\nu+2\kappa}(q)$, and hence $\kappa_0 = \nu + 2\kappa + 1$. Consequently we have

$$\kappa_0 = \begin{cases} \nu + 2\kappa + 2 - d & \text{if } t = 1,\ \xi(cq) \neq 1,\ \nu \equiv \delta \pmod 2, \text{ and } 2 \in \mathfrak{p}, \\ \nu + d & \text{if } t = 2,\ c^{-1}q \notin N_{K/F}(\mathfrak{r}), \text{ and } d > 1, \\ \nu + 2\kappa + 1 & \text{if } t = 3,\ \xi(-cq) = 1, \text{ and } 2 \in \mathfrak{p}, \\ \nu + 1 & \text{otherwise.} \end{cases}$$

This completes the determination of the number κ_0.

4.7. Now, in (4.28) we have $\varphi(k_{i,b}, L) = 2^{-1}\mathfrak{p}^i + \varphi(b, N) + 2^{-1}(q - \varphi[b])\mathfrak{p}^{-i}$ by (2.5), (2.6), and (2.7) for $0 \leq i \leq \kappa_0 - 1$ and $b \in X_i(q)$. Let $m(i, b) \in \mathbf{Z}$ such that $\varphi(k_{i,b}, L) = 2^{-1}\mathfrak{p}^{m(i,b)}$. Here [IQD, Theorem 1.3] implies $k_{i,b}C = L[q, 2^{-1}\mathfrak{p}^{m(i,b)}]$. We have $0 \leq m(i, b) \leq \mu(q - \varphi[b]) - i \leq (\kappa_0 - 1) - i$ in the notation of (3.3), with μ of (4.18). From this and $m(i, b) \leq i$, we see that

$$0 \leq m(i, b) \leq [(\kappa_0 - 1)/2] \tag{4.36}$$

for $0 \leq i \leq \kappa_0 - 1$ and $b \in X_i(q)$. On the other hand, when $\kappa_0 = \nu + 1$, put $z = 0$; when $\kappa_0 > \nu + 1$, take any element $z \in N[sq]$ and fix it. Here s is of (4.30), (4.32), or (4.34) according as $t = 1$, 2, or 3. Then $z \in X_{\kappa_0 - 1}(q)$. We assert that

$$m(i, z) = i \tag{4.37}$$

for $0 \leq i \leq [(\kappa_0 - 1)/2]$. Indeed, if $z = 0$, then it is obvious. Suppose $z \in N[sq]$. Then $\mu(q - \varphi[z]) = \kappa_0 - 1$. From the theorem in the case $r = 0$ and $t > 0$, we find that

$$\varphi(z, N) = \begin{cases} \mathfrak{p}^{(\nu+\delta)/2} & \text{if } t = 1, 3, \\ 2^{-1}\mathfrak{p}^{[(\nu+d)/2]} & \text{if } t = 2. \end{cases}$$

Therefore

$$m(i, z) = \begin{cases} \min(i, \kappa + (\nu + \delta)/2, (\kappa_0 - 1) - i) & \text{if } t = 1, 3, \\ \min(i, [(\nu + d)/2], (\kappa_0 - 1) - i) & \text{if } t = 2. \end{cases}$$

From this, we obtain (4.37). As a consequence, from (4.36) and (4.37),

$$L[q] = \bigcup_{i=0}^{\kappa_0 - 1} \bigcup_{b \in X_i(q)/\mathfrak{p}^i N} k_{i,b}C = \bigsqcup_{i=0}^{[(\kappa_0 - 1)/2]} L[q, 2^{-1}\mathfrak{p}^i]$$

and $L[q, 2^{-1}\mathfrak{p}^i] = k_{i,z}C$. This completes the proof.

5 THE MAXIMALITY OF $L \cap (Fh)^\perp$

In this section still the field F is local and we prove the second main Theorem 5.3 as an application of Theorem 3.5. We first prepare two lemmas.

5.1 LEMMA. *Let ξ be as in Notation. Let $\alpha \in F^\times$ such that $\xi(\alpha) \neq 1$. Let $\mathfrak{o}$ be the valuation ring of $F(\sqrt{\alpha})$, $\mathfrak{q}_\mathfrak{o}$ the maximal ideal of $\mathfrak{o}$, $\mathfrak{d}_{F(\sqrt{\alpha})/F}$ the different of $F(\sqrt{\alpha})$ relative to F, and μ the normalized order function of F. Then we obtain the following assertions:*

(1) *If $2 \in \mathfrak{g}^\times$ and $\mu(\alpha) \in 2\mathbf{Z}$, then $\mathfrak{d}_{F(\sqrt{\alpha})/F} = \mathfrak{o}$.*
(2) *If $\mu(\alpha) \notin 2\mathbf{Z}$, then $\mathfrak{d}_{F(\sqrt{\alpha})/F} = 2\mathfrak{q}_\mathfrak{o}$.*
(3) *If $2 \in \mathfrak{p}$, $\mu(\alpha) \in 2\mathbf{Z}$, $\xi(\alpha) = 0$, and $\mathfrak{d}_{F(\sqrt{\alpha})/F} = \mathfrak{q}_\mathfrak{o}^d$, then $d \in 2\mathbf{Z}$.*

Proof. The first two assertions are well known. Assertion (3) follows from Lemma 4.2(1), (2). □

5.2 LEMMA. *Let H be an integral lattice in V. Let t be the core dimension of V. Let $\delta(\varphi)$ be defined as in* (2.1). *Assume $n \notin 2\mathbf{Z}$ and $\delta(\varphi) \cap \mathfrak{g}^\times = \emptyset$. Then we have H is maximal in V if and only if $[\tilde{H} : H] = [\mathfrak{g} : 2\mathfrak{p}]$. Here $\tilde{H}$ is defined as in* (2.2).

Proof. Assume that H is maximal in V. Then [AQC, Lemma 6.9] implies $[\tilde{H} : H] = [\tilde{L} : L]$ with L of (2.4). Since $\tilde{L} = \tilde{N} + \sum_{i=1}^r (\mathfrak{g}f_i + \mathfrak{g}e_i)$, we have $[\tilde{L} : L] = [\tilde{N} : N]$. By (2.8), (2.11), (2.19), and (2.20), we obtain $[\tilde{N} : N] = [\mathfrak{g} : 2\mathfrak{p}]$ since $\delta = 1$. Thus we obtain the "only if"-part of the assertion. Conversely, we assume that H is an integral lattice in V such that

$$[\tilde{H} : H] = [\mathfrak{g} : 2\mathfrak{p}]. \tag{5.1}$$

By [AQC, Lemma 6.2(1)], there exists a maximal lattice H_0 in V such that $H \subset H_0$. Then

$$H \subset H_0 \subset \tilde{H}_0 \subset \tilde{H}. \tag{5.2}$$

From the "only if"-part of the lemma, which is settled above, we obtain $[\tilde{H}_0 : H_0] = [\mathfrak{g} : 2\mathfrak{p}]$. Combining this with (5.1) and (5.2), we obtain $H = H_0$, and hence H is maximal in V. □

We remark that the index $[\tilde{H} : H]$ of a maximal lattice H in V is given in [AQC, Lemma 8.4(iv)] when $n \in 2\mathbf{Z}$ or $\delta(\varphi) \cap \mathfrak{g}^\times \neq \emptyset$.
Now, for $h \in L$ such that $\varphi[h] \neq 0$, put

$$(Fh)^\perp = \{x \in V \mid \varphi(x, h) = 0\}. \tag{5.3}$$

5.3 THEOREM. *Let L be a maximal lattice in V and $\tau(q)$ as in* (3.8) *for a given $q \in \mathfrak{g} \cap F^\times$. Assume $n \geq 2$. Then for $h \in L$ such that $\varphi[h] \neq 0$, we have*

$$L \cap (Fh)^\perp \text{ is maximal in } (Fh)^\perp \iff h \in L\big[\varphi[h], 2^{-1}\mathfrak{p}^{\tau(\varphi[h])}\big].$$

Hereafter we prove this theorem until the end of §5.14.

5.4. Before stating the proof let us recall the basic notion and terminology in the previous subsections, which will be needed in the next arguments. Put $q = \varphi[h]$, then $h \in L[q]$. Put simply $\tau = \tau(q)$. We have

$$L[q] = \begin{cases} L[q, 2^{-1}\mathfrak{p}^{\tau}] & \text{if } r = 0, \\ \bigsqcup_{i=0}^{\tau} L[q, 2^{-1}\mathfrak{p}^{i}] & \text{if } r \geq 1 \end{cases} \tag{5.4}$$

by (3.8). Hence $h \in L[q, 2^{-1}\mathfrak{p}^i]$ for some $0 \leq i \leq \tau$. Put $W = (Fh)^{\perp}$. Our aim is to show that $L \cap W$ is maximal in W if and only if $i = \tau$.
We have a Witt decomposition of V with respect to φ

$$V = Z + \sum_{j=1}^{r}(Ff_j + Fe_j), \quad L = N + \sum_{j=1}^{r}(\mathfrak{g}f_j + \mathfrak{g}e_j) \tag{5.5}$$

as in (2.4). Let t be the core dimension of (V, φ). Let ξ be as in Notation.
Assume $t = 1$. Let c and δ be as in (2.9) and (2.10), respectively. For q, let d and s be as in Theorem 3.5(ii).
Assume $t = 2$. Let b and c be as in (2.12) and δ as in (2.13). Let K and $\mathfrak{r}$ be as in §2.3(II). Let $\mathfrak{q}$ be the maximal ideal of $\mathfrak{r}$ and $\rho \in Gal(K/F)$ such that $\rho \neq 1$. Let $\mathfrak{d}$ and d be as in Theorem 3.5(iii). Then

$$K \text{ is isomorphic to } F(\sqrt{-bc}), \tag{5.6}$$

$$\mathfrak{d} = \mathfrak{d}_{K/F} = \mathfrak{q}^d \text{ when } \mathfrak{d} \neq \mathfrak{r}. \tag{5.7}$$

For q, let s be as in Theorem 3.5(iii).
Assume $t = 3$. Let c and δ be as in (2.14) and (2.15), respectively. For q, let s be as in Theorem 3.5(iv).
Let $\delta(\varphi)$ be as in (2.1). We may assume that: if $t = 1$ or 3 and $\delta(\varphi) \cap \mathfrak{g}^{\times} \neq \emptyset$, then $c \in \mathfrak{g}^{\times}$; if $t = 1$ or 3 and $\delta(\varphi) \cap \mathfrak{g}^{\times} = \emptyset$, then $c\mathfrak{g} = \mathfrak{p}$; if $t = 2$, then $b, c \in \mathfrak{g}^{\times} \sqcup \pi\mathfrak{g}^{\times}$; if $t = 2$ and K is ramified over F, then $c \in \mathfrak{g}^{\times}$; see §2.3.

5.5. First suppose that:

$t = 1$, $r \geq 1$, and $\xi(cq) = 1$; or
$t = 2$ and $c^{-1}q \in N_{K/F}(\mathfrak{r})$; or
$t = 3$ and $\xi(-cq) \neq 1$; or
$t = 4$.

This assumption is equivalent to $N[q] \neq \emptyset$ by Lemma 3.4. Here $0 \neq q = \varphi[h]$, $h \in L$. Then $h \in L[q, 2^{-1}\mathfrak{p}^i]$ for some $0 \leq i \leq \tau$ by (5.4). We obtain

$$L[q, 2^{-1}\mathfrak{p}^i] = \begin{cases} yC & \text{if } i = \tau, \\ h_{i,y}C & \text{if } i < \tau \end{cases} \tag{5.8}$$

by (3.13), (3.16), (3.19), and (3.21). Here, y is any element of $N[q]$ and fix it.

We first prove that $L \cap W$ is maximal in W when $i = \tau$. In this case, $h \in L[q, 2^{-1}\mathfrak{p}^\tau] = yC$ by (5.8). Therefore there exists $\gamma \in C$ such that $y = h\gamma$. We have $L \cap (Fy)^\perp = (L \cap W)\gamma$ since $L\gamma = L$ and $(Fy)^\perp = W\gamma$. Therefore $L \cap (Fy)^\perp$ is maximal in $(Fy)^\perp$ if and only if $L \cap W$ is maximal in W. Hence we assume $h = y$ and $W = (Fy)^\perp$. Now, we have

$$W = (Z \cap W) + \sum_{j=1}^{r} (Ff_j + Fe_j)$$

since $h \in N$, (5.3), and (5.5). This is a Witt decomposition of W with respect to the restriction of φ to W. Moreover we obtain

$$(5.9)\quad L \cap W = (N \cap W) + \sum_{j=1}^{r} (\mathfrak{g}f_j + \mathfrak{g}e_j), \quad N \cap W = \{x \in Z \cap W \mid \varphi[x] \in \mathfrak{g}\}$$

from (2.7) and (5.5). Thus by [AQC, Lemma 6.5], $L \cap W$ is maximal in W when $i = \tau$.

Next suppose $i < \tau$. We shall show that $L \cap W$ is not a maximal lattice in W in this case. We obtain $h \in h_{i,y}C$ by (5.8). Thus we may assume $h = h_{i,y}$. By (3.8), $N[q] = N[q, 2^{-1}\mathfrak{p}^\tau]$. From this,

$$(5.10)\qquad \varphi(y, N) = 2^{-1}\mathfrak{p}^\tau.$$

We see that

(5.11)

$$W = X + \sum_{j=1}^{r-1} (Ff_j + Fe_j), X = \{ae_r + x - 2\pi^{-i}\varphi(y, x)f_r \mid a \in F, x \in Z\},$$

$$L \cap W = H + \sum_{j=1}^{r-1} (\mathfrak{g}f_j + \mathfrak{g}e_j), H = \{ae_r + x - 2\pi^{-i}\varphi(y, x)f_r \mid a \in \mathfrak{g}, x \in N\}$$

by the definition of $h_{i,y}$ (in (3.7)), (5.3), (5.5), and (5.10). Take

$$(5.12)\qquad w \in N \text{ such that } \varphi(y, w) = -2^{-1}\pi^\tau$$

and fix it. Put $u = \pi^{\tau-i}f_r + w - \pi^{i-\tau}\varphi[w]e_r$, $v = \pi^{i-\tau}e_r$, and

$$Y = \{x - 2\pi^{i-\tau}\varphi(x, w)e_r \mid x \in Z \text{ such that } \varphi(x, y) = 0\}.$$

Then we find that $X = Y + Fu + Fv$ is a Witt decomposition of X by a straightforward calculation. Here X is defined as in (5.11). Put $\Lambda = \{k \in Y \mid \varphi[k] \in \mathfrak{g}\}$, then $\Lambda + \mathfrak{g}u + \mathfrak{g}v$ is maximal in X by [AQC, Lemma 6.5]. We assert that $H \subsetneq \Lambda + \mathfrak{g}u + \mathfrak{g}v$. Indeed, it is clear that $v \notin H$ since $i - \tau < 0$. For any $\ell = ae_r + x - 2\pi^{-i}\varphi(y, x)f_r \in H$, put $\xi = -2\pi^{-\tau}\varphi(y, x)$ and $\eta = a\pi^{\tau-i} + 2\pi^{-\tau}\varphi(y, x)\varphi[w] + 2\varphi(x, w)$ with w in (5.12). Then a straightforward computation shows that $\xi, \eta \in \mathfrak{g}$ and $\ell - \xi u - \eta v \in \Lambda$ by (5.10). Therefore $H \subsetneq \Lambda + \mathfrak{g}u + \mathfrak{g}v$. Thus H is not maximal in X, and hence $L \cap W$ is not maximal in W by [AQC, Lemma 6.3]. This completes the proof in the case $N[q] \neq \emptyset$.

5.6. Let us now suppose that:

$t = 0$ and $r \geq 1$; or
$t = 1$, $r \geq 1$, $\xi(cq) \neq 1$, and $2 \in \mathfrak{g}^\times$; or
$t = 1$, $r \geq 1$, $\xi(cq) \neq 1$, and $\nu \not\equiv \delta \pmod 2$; or
$t = 2$, $r \geq 1$, $c^{-1}q \notin N_{K/F}(\mathfrak{r})$, and $\mathfrak{d}_{K/F} = \mathfrak{r}$; or
$t = 2$, $r \geq 1$, $c^{-1}q \notin N_{K/F}(\mathfrak{r})$, and $\mathfrak{d}_{K/F} = \mathfrak{q}$; or
$t = 3$, $r \geq 1$, $\xi(-cq) = 1$, and $2 \in \mathfrak{g}^\times$.

Here $\nu \in \mathbf{Z}$ such that $q\mathfrak{g} = \mathfrak{p}^\nu$ with $0 \neq q = \varphi[h]$, $h \in L$. In this case we obtain

$$L[q] = \begin{cases} \sqcup_{j=0}^{\tau} \ell_j C & \text{if } t \geq 1 \text{ or } r \geq 2, \\ \sqcup_{j=0}^{\nu} \ell_j C & \text{if } t = 0 \text{ and } r = 1 \end{cases}$$

with ℓ_j in (3.7) and $\tau = [\nu/2]$ of Theorem 3.5. Moreover

$$N[q] = \emptyset \tag{5.13}$$

by Lemma 3.4. We have $h \in L[q, 2^{-1}\mathfrak{p}^i]$ for some $0 \leq i \leq \tau$ by (5.4). Hereafter until the end of §5.7 we prove the theorem in the case $t \geq 1$ or $r \geq 2$. Then

$$L[q, 2^{-1}\mathfrak{p}^i] = \ell_i C \text{ for } 0 \leq i \leq \tau$$

by (3.10), (3.13), (3.16), and (3.19). Thus we may assume $h = \ell_i$ and $W = (F\ell_i)^\perp$ since $h \in \ell_i C$.
In this §5.6 we determine $[(L \cap W)\tilde{} : L \cap W]$. Put

$$w = f_r - q\pi^{-2i} e_r \tag{5.14}$$

with e_r and f_r in (5.5). Then

$$W = (Fw + Z) + \sum_{j=1}^{r-1} (Ff_j + Fe_j) \tag{5.15}$$

from the definition of ℓ_i (in (3.7)), (5.3), and (5.5). We understand that: $\sum_{j=1}^{r-1}(Ff_j + Fe_j) = \{0\}$ when $t > 0$ and $r = 1$; $Z = \{0\}$ when $t = 0$ and $r \geq 2$. We assert that (5.15) is a Witt decomposition. Indeed, it is clear when $t = 0$ and $r \geq 2$. Assume $t \geq 1$. If $\varphi[aw + x] = 0$ for $a \in F$ and $x \in Z$, then $\varphi[x] = q\pi^{-2i}a^2$. If $a \neq 0$, then this is the case if and only if $\varphi[a^{-1}\pi^i x] = q$, and hence $N[q] \neq \emptyset$. This contradicts (5.13). Thus $a = 0$, and hence $x = 0$. Therefore the restriction of φ to $Fw + Z$ is anisotropic. Combining this with

(2.5), (2.6), and (5.14), we see that (5.15) is a Witt decomposition. Now, we have

$$L \cap W = (\mathfrak{g}w + N) + \sum_{j=1}^{r-1} (\mathfrak{g}f_j + \mathfrak{g}e_j) \tag{5.16}$$

by (5.5) and (5.15). Therefore a straightforward computation shows that

$$(L \cap W)\tilde{} = 2^{-1}\mathfrak{p}^{2i-\nu}w + \tilde{N} + \sum_{j=1}^{r-1} (\mathfrak{g}f_j + \mathfrak{g}e_j) \tag{5.17}$$

from (2.2) with (5.16). Combining (5.16) with (5.17), we have

$$[(L \cap W)\tilde{} : L \cap W] = [\mathfrak{g} : 2\mathfrak{p}^{\nu-2i}] \cdot [\tilde{N} : N]. \tag{5.18}$$

Here we obtain the index $[\tilde{N} : N]$ by [AQC, Lemma 8.4(iv)] and Lemma 5.2. Combining this with (5.18), we have

(5.19)

$$[(L \cap W)\tilde{} : L \cap W] = \begin{cases} [\mathfrak{g} : 2\mathfrak{p}^{\nu-2i}] & \text{if } t = 0 \text{ and } r \geq 2, \\ [\mathfrak{g} : 4\mathfrak{p}^{\nu-2i+\delta}] & \text{if } t = 1, r \geq 1, \xi(cq) \neq 1, \text{ and } 2 \in \mathfrak{g}^\times, \\ [\mathfrak{g} : 4\mathfrak{p}^{\nu-2i+\delta}] & \text{if } t = 1, r \geq 1, \xi(cq) \neq 1, \text{ and } \nu \not\equiv \delta \pmod 2, \\ [\mathfrak{g} : 2\mathfrak{p}^{\nu-2i+2\delta}] & \text{if } t = 2, r \geq 1, c^{-1}q \notin N_{K/F}(\mathfrak{r}), \text{ and } \mathfrak{d} = \mathfrak{r}, \\ [\mathfrak{g} : 2\mathfrak{p}^{\nu-2i+1}] & \text{if } t = 2, r \geq 1, c^{-1}q \notin N_{K/F}(\mathfrak{r}), \text{ and } \mathfrak{d} = \mathfrak{q}, \\ [\mathfrak{g} : \mathfrak{p}^{\nu-2i+2-\delta}] & \text{if } t = 3, r \geq 1, \xi(-cq) = 1, \text{ and } 2 \in \mathfrak{g}^\times. \end{cases}$$

Note that: if $t = 2$, $r \geq 1$, and $\mathfrak{d} = \mathfrak{r}$, then $c^{-1}q \notin N_{K/F}(\mathfrak{r})$ if and only if $\nu \not\equiv \delta \pmod 2$ by Lemma 3.4(2); if $t = 3$, $r \geq 1$, $\xi(-cq) = 1$, and $2 \in \mathfrak{g}^\times$, then $\nu \equiv \delta \pmod 2$.

5.7. In this § we still assume $t \geq 1$ or $r \geq 2$. For an integral lattice R in W, the following assertion holds:

(5.20)

$$R \text{ is maximal in } W$$
$$\iff [\tilde{R}:R] = \begin{cases} [\mathfrak{g}:2\mathfrak{p}^{\nu-2\tau}] & \text{if } t=0 \text{ and } r \geq 2, \\ [\mathfrak{g}:\mathfrak{p}^{2\delta}] & \text{if } t=1, r\geq 1, \xi(cq)\neq 1, \\ & \quad\text{and } \nu \equiv \delta \pmod 2, \\ [\mathfrak{g}:4\mathfrak{p}] & \text{if } t=1, r\geq 1, \xi(cq)\neq 1, \\ & \quad\text{and } \nu \not\equiv \delta \pmod 2, \\ [\mathfrak{g}:2\mathfrak{p}^{1+\delta}] & \text{if } t=2, r\geq 1, c^{-1}q\notin N_{K/F}(\mathfrak{r}), \text{ and } \mathfrak{d}=\mathfrak{r}, \\ [\mathfrak{g}:2\mathfrak{p}^{\nu-2\tau+1}] & \text{if } t=2, r\geq 1, c^{-1}q\notin N_{K/F}(\mathfrak{r}), \text{ and } \mathfrak{d}=\mathfrak{q}, \\ [\mathfrak{g}:\mathfrak{p}^{2}] & \text{if } t=3, r\geq 1, \xi(-cq)=1, \text{ and } 2\in\mathfrak{g}^{\times}. \end{cases}$$

Note that if $t=1$, $r\geq 1$, $\xi(cq)\neq 1$, and $\nu\equiv\delta \pmod 2$, then $2\in\mathfrak{g}^{\times}$. If (5.20) holds, then combining (5.19) with (5.20), we obtain our theorem in the case $t\geq 1$ or $r\geq 2$ since $L\cap W$ is an integral lattice in W. Let us prove (5.20). We observe a core subspace $Fw+Z$ of W with w of (5.14). If $t=0$ and $r\geq 2$, then $\dim_F(Fw+Z)=1$ and $\delta(Fw+Z,\varphi)=\varphi[w]F^{\times 2}=-qF^{\times 2}$. Here $\delta(Fw+Z,\varphi)$ is defined as in (2.1). This implies $\delta(Fw+Z,\varphi)\cap\mathfrak{g}^{\times}\neq\emptyset$ if $\nu\in 2\mathbf{Z}$ and $\delta(Fw+Z,\varphi)\cap\mathfrak{g}^{\times}=\emptyset$ if $\nu\notin 2\mathbf{Z}$. Therefore we have (5.20) by [AQC, Lemma 8.4(iv)] and Lemma 5.2 when $t=0$ and $r\geq 2$. Assume $t=1$, $r\geq 1$, and $\xi(cq)\neq 1$. Since $\varphi[w]=-q\pi^{-2i}$ and $Z=Fg$ with g of (2.8), $(Fw+Z,\varphi)$ is isomorphic to $(F(\sqrt{cq}),\psi)$, where $\psi[x]=cN_{F(\sqrt{cq})/F}(x)$ for $x\in F(\sqrt{cq})$, as explained in §2.3(II). Suppose $\nu\equiv\delta \pmod 2$, then $2\in\mathfrak{g}^{\times}$. Thus Lemma 5.1(1) implies that $F(\sqrt{cq})$ is unramified. Therefore we have (5.20) in this case by [AQC, Lemma 8.4(iv)]. Next suppose $\nu\not\equiv\delta \pmod 2$, then $\mathfrak{d}_{F(\sqrt{cq})/F}=2\mathfrak{q}_{F(\sqrt{cq})}$ by Lemma 5.1(2), where $\mathfrak{q}_{F(\sqrt{cq})}$ is the maximal ideal of the valuation ring of $F(\sqrt{cq})$. Therefore [AQC, Lemma 8.4(iv)] implies (5.20) when $t=1$, $r\geq 1$, $\xi(cq)\neq 1$, and $2\in\mathfrak{g}^{\times}$ or $\nu\not\equiv\delta \pmod 2$. Assume $t=2$, $r\geq 1$, and $c^{-1}q\notin N_{K/F}(\mathfrak{r})$, then $\dim_F(Fw+Z)=3$. Since $Z=Fg_1+Fg_2$ with g_1 and g_2 in (2.12) and $\varphi[w]=-q\pi^{-2i}$, we have

$$\delta(Fw+Z,\varphi)=bcqF^{\times 2}. \tag{5.21}$$

Suppose $\mathfrak{d}=\mathfrak{r}$. Then $\nu\not\equiv\delta \pmod 2$ by Lemma 3.4(2). Since K is unramified over F, we have $b\mathfrak{g}=c\mathfrak{g}(=\mathfrak{p}^{\delta})$ by Lemma 5.1(2). Thus (5.21) implies $\delta(Fw+Z,\varphi)\cap\mathfrak{g}^{\times}=\emptyset$ if $\delta=0$ and $\delta(Fw+Z,\varphi)\cap\mathfrak{g}^{\times}\neq\emptyset$ if $\delta=1$. From this we obtain (5.20) by [AQC, Lemma 8.4(iv)] and Lemma 5.2 in this case. Next suppose $\mathfrak{d}=\mathfrak{q}$. Here Lemma 5.1 implies $b\mathfrak{g}=\mathfrak{p}$ since $c\in\mathfrak{g}^{\times}$. Therefore (5.21) implies $\delta(Fw+Z,\varphi)\cap\mathfrak{g}^{\times}=\emptyset$ if $\nu\in 2\mathbf{Z}$ and $\delta(Fw+Z,\varphi)\cap\mathfrak{g}^{\times}\neq\emptyset$ if $\nu\notin 2\mathbf{Z}$. Hence we have (5.20) by [AQC, Lemma 8.4(iv)] and Lemma 5.2 when $t = 2$, $r \geq 1$, $c^{-1}q \notin N_{K/F}(\mathfrak{r})$, and $\mathfrak{d}_{K/F} = \mathfrak{r}$ or $\mathfrak{q}$. Finally if $t = 3$, $r \geq 1$, $\xi(-cq) = 1$, and $2 \in \mathfrak{g}^\times$, then $\dim_F(Fw + Z) = 4$, and hence [AQC, Lemma 8.4(iv)] implies (5.20). This proves the theorem in the case when $N[q] = \emptyset$, $L[q] = \sqcup_j \ell_j C$, and $t \geq 1$ or $r \geq 2$.

5.8. In this § we prove the theorem in the case $t = 0$ and $r = 1$. Then

$$L[q, 2^{-1}\mathfrak{p}^i] = \begin{cases} \ell_i C \sqcup \ell_{\nu-i} C & \text{if } t = 0,\ r = 1, \text{ and } i \neq \nu/2, \\ \ell_i C & \text{if } t = 0,\ r = 1, \text{ and } i = \nu/2 \end{cases}$$

for $0 \leq i \leq \tau$, by (3.10). Since $h \in L[q, 2^{-1}\mathfrak{p}^i]$ for some $0 \leq i \leq \tau$ by (5.4), we may assume that $h = \ell_i$ or $h = \ell_{\nu-i}$ when $i \neq \nu/2$ and $h = \ell_i$ when $i = \nu/2$. If $h = \ell_i$ for $0 \leq i \leq \tau$ (including the case $h = \ell_{\nu/2}$), we can obtain the assertion in the same way as §§5.6 and 5.7. Assume $h = \ell_{\nu-i}$ for $0 \leq i < \nu/2$. Put here $w = f_1 - q\pi^{-2(\nu-i)}e_1$. Then we see that

$$W = (F\ell_{\nu-i})^\perp = Fw, \quad L \cap W = \mathfrak{p}^{\nu-2i}w, \text{ and } (L \cap W)\tilde{} = 2^{-1}\mathfrak{g}w$$

in a similar way as §5.6. Thus $[(L \cap W)\tilde{} : L \cap W] = [\mathfrak{g} : 2\mathfrak{p}^{\nu-2i}]$. Therefore we obtain the theorem in the same way as in the case when $t = 0$ and $r \geq 2$ since $\varphi[w]F^{\times 2} = -qF^{\times 2}$. This completes the proof in the case when $N[q] = \emptyset$ and $L[q] = \sqcup_j \ell_j C$.

5.9. Finally we suppose that:

$t = 1$, $r \geq 1$, $\nu \equiv \delta \pmod 2$, $\xi(cq) \neq 1$, and $2 \in \mathfrak{p}$; or
$t = 2$, $r \geq 1$, $c^{-1}q \notin N_{K/F}(\mathfrak{r})$, and $d > 1$; or
$t = 3$, $r \geq 1$, $\xi(-cq) = 1$, and $2 \in \mathfrak{p}$.

Note that $c \in \mathfrak{g}^\times$ when $t = 2$, $r \geq 1$, $c^{-1}q \notin N_{K/F}(\mathfrak{r})$, and $d > 1$. Then we have

$$L[q] = \bigsqcup_{j=0}^{\tau} k_{j,z}C \text{ and } k_{j,z}C = L[q, 2^{-1}\mathfrak{p}^j] \tag{5.22}$$

with $k_{j,z}$ in (3.7), from (3.8), (3.13), (3.16), and (3.19). Here z is any element of $N[sq]$ with s of (3.11), (3.14), or (3.17) of Theorem 3.5 according as $t = 1, 2,$ or 3. We fix z. We obtain

(5.23)
$$\tau = \begin{cases} \kappa + \left[\frac{\nu+1-d}{2}\right] & \text{if } t = 1, r \geq 1, \xi(cq) \neq 1, \nu \equiv \delta \pmod 2, \text{ and } 2 \in \mathfrak{p}, \\ \left[\frac{\nu+d-1}{2}\right] & \text{if } t = 2, r \geq 1, c^{-1}q \notin N_{K/F}(\mathfrak{r}), \text{ and } d > 1, \\ \kappa + \left[\frac{\nu}{2}\right] & \text{if } t = 3, r \geq 1, \xi(-cq) = 1, \text{ and } 2 \in \mathfrak{p} \end{cases}$$

from (3.12), (3.15), and (3.18), where $\kappa \in \mathbf{Z}$ such that $2\mathfrak{g} = \mathfrak{p}^\kappa$. Moreover, by Lemma 3.4,

$$N[q] = \emptyset. \tag{5.24}$$

Applying Theorem 3.5 to (Z, φ), N, and sq, we have $N[sq] = N[sq, 2^{-1}\mathfrak{p}^{\tau(sq)}]$ by (3.8). Here

$$(5.25)\tau(sq) = \begin{cases} \kappa + \frac{\nu+\delta}{2} & \text{if } t = 1, r \geq 1, \xi(cq) \neq 1, \\ & \quad \nu \equiv \delta \pmod 2, \text{ and } 2 \in \mathfrak{p}, \\ \left[\frac{\nu+d}{2}\right] & \text{if } t = 2, r \geq 1, c^{-1}q \notin N_{K/F}(\mathfrak{r}), \text{ and } d > 1, \\ \kappa + \frac{\nu+\delta}{2} & \text{if } t = 3, r \geq 1, \xi(-cq) = 1, \text{ and } 2 \in \mathfrak{p} \end{cases}$$

from (3.12), (3.15), and (3.18). Thus

$$\varphi(z, N) = 2^{-1}\mathfrak{p}^{\tau(sq)}. \tag{5.26}$$

Assume $h \in L[q, 2^{-1}\mathfrak{p}^i]$ for some $0 \leq i \leq \tau$. Then by (5.22), $h \in k_{i,z}C$, and hence we may assume $h = k_{i,z}$. Hereafter we show that $L \cap W$ is maximal in W if and only if $i = \tau$. Put

$$x_1 = \pi^{i-\tau}[f_r - \pi^{-2i}(q - \varphi[z])e_r] \tag{5.27}$$

with e_r and f_r in (5.5). Then

$$\varphi[x_1] = \pi^{-2\tau}(s-1)q \text{ and } \pi^{\tau-i}x_1 \in L. \tag{5.28}$$

By a straightforward computation, we obtain
(5.29)

$$W = X + \sum_{j=1}^{r-1}(Ff_j + Fe_j), \quad X = \{ax_1 + x - 2\pi^{-i}\varphi(x, z)e_r \mid a \in F, x \in Z\}$$

with x_1 of (5.27). Then (5.29) is a Witt decomposition. Indeed, if $\varphi[ax_1 + x - 2\pi^{-i}\varphi(x, z)e_r] = 0$ for $a \in F$ and $x \in Z$, then $\varphi[x - \pi^{-\tau}az] = (\pi^{-\tau}a)^2q$. Assuming $a \neq 0$, we have $\varphi[\pi^{\tau}a^{-1}x - z] = q$, and hence $N[q] \neq \emptyset$. This contradicts (5.24). Thus $a = 0$, and hence $x = 0$. Therefore the restriction of φ to X is anisotropic. Combining this with (2.5), (2.6), and (5.27), we see that (5.29) is a Witt decomposition. Now, we easily see that $2\pi^{-i}\varphi(x, z) \in \mathfrak{g}$ for $x \in N$ from (5.23), (5.25), (5.26), and $0 \leq i \leq \tau$. From this and (5.28),
(5.30)

$$L \cap W = H + \sum_{j=1}^{r-1}(\mathfrak{g}f_j + \mathfrak{g}e_j), \quad H = \{ax_1 + x - 2\pi^{-i}\varphi(x, z)e_r \mid a \in \mathfrak{p}^{\tau-i}, x \in N\}.$$

By [AQC, Lemma 6.3(1)], $L \cap W$ is maximal in W if and only if H is maximal in X. Thus we consider the lattice H in X instead of $L \cap W$ in W.

5.10. In this § we first determine the structure of N under the assumption of §5.9. Now we put

$$Y = \{k \in Z \mid \varphi(k, z) = 0\} \text{ and } z_1 = \pi^{-[\nu/2]}z. \tag{5.31}$$

Then $Z = Fz_1 + Y$.
First assume $t = 1$, $r \geq 1$, $\nu \equiv \delta \pmod 2$, $\xi(cq) \neq 1$, and $2 \in \mathfrak{p}$. Since $\varphi[z_1]\mathfrak{g} = \mathfrak{p}^\delta$,

$$N = \mathfrak{g}z_1. \tag{5.32}$$

Next assume $t = 2$, $r \geq 1$, $c^{-1}q \notin N_{K/F}(\mathfrak{r})$, and $d > 1$. Then Lemma 5.1(1) and (2) imply $2 \in \mathfrak{p}$. Take $y \in Y$ such that $Y = Fy$, then

$$Z = Fz_1 + Fy. \tag{5.33}$$

Thus from (2.1) and (2.12), we have $\varphi[z_1]\varphi[y]F^{\times 2} = -\delta(Z, \varphi) = bcF^{\times 2}$, and hence we may assume

$$\varphi[z_1]\varphi[y](bc)^{-1} = \begin{cases} 1 & \text{if } b\mathfrak{g} = \mathfrak{p}, \\ \pi^{2\lambda} & \text{if } b \in \mathfrak{g}^\times, \end{cases} \tag{5.34}$$

where

$$\lambda = \nu - 2[\nu/2]. \tag{5.35}$$

From (5.6) and (5.34), we see that $F + Fz_1y \subset A(Z, \varphi)$ is isomorphic to K. Here we identify $F + Fz_1y$ with K. Suppose $b\mathfrak{g} = \mathfrak{p}$. Then

$$d = 2\kappa + 1 \tag{5.36}$$

by $c \in \mathfrak{g}^\times$, Lemma 5.1(2), and (5.7). Put

$$z_\nu = \begin{cases} z_1 & \text{if } \nu \in 2\mathbf{Z}, \\ y & \text{if } \nu \notin 2\mathbf{Z}. \end{cases}$$

Then $\varphi[z_\nu] \in \mathfrak{g}^\times$ from (5.31) and (5.34), and hence $Z = Kz_\nu$ and $N = \mathfrak{r}z_\nu$ by (5.33). We find $z_1y \in \mathfrak{r}$ and $(z_1y - (z_1y)^\rho)\mathfrak{r} = 2\mathfrak{q} = \mathfrak{d}$, and hence $\mathfrak{r} = \mathfrak{g}[z_1y]$ by [AQC, Lemma 5.6(ii)]. Thus

$$N = \mathfrak{g}z_1 + \mathfrak{g}y. \tag{5.37}$$

Next suppose $b \in \mathfrak{g}^\times$. Then $bc \in \mathfrak{g}^\times$ and $2 \in \mathfrak{p}$ from $c \in \mathfrak{g}^\times$, $d > 1$, and Lemma 5.1(1). Thus there exist

$$\alpha, \beta \in \mathfrak{g}^\times \text{ such that } -bc = \alpha^{-2}(1 + \pi^{2\kappa+1-d}\beta) \tag{5.38}$$

by $d > 1$ and Lemma 4.2(1). Put $\eta = \pi^{(d-2\kappa)/2}(1 + \alpha\pi^{-\lambda}z_1y)$ in K with λ of (5.35). Then η is a root of an Eisenstein equation $\mathbf{x}^2 - 2\pi^{(d-2\kappa)/2}\mathbf{x} - \pi\beta = 0$, and hence η is a prime element of K and $(\eta - \eta^\rho)\mathfrak{r} = (2\eta - 2\pi^{(d-2\kappa)/2})\mathfrak{r} = \mathfrak{d}$. Here $\mathbf{x}$ is an indeterminate. Thus $\mathfrak{r} = \mathfrak{g}[\eta]$ by [AQC, Lemma 5.6(ii)]. By (5.31) and (5.34), we have $\varphi[y]\mathfrak{g} = \mathfrak{p}^{\lambda}$. From this and $\eta^{-1} = -\pi^{-1+(d-2\kappa)/2}\beta^{-1}(1 - \alpha\pi^{-\lambda}z_1 y)$, we obtain

$$N = \mathfrak{r}\eta^{-\lambda}y = \mathfrak{g}y + \mathfrak{g}\pi^{-\lambda+(d-2\kappa)/2}[y + (-1)^{\nu}\alpha\pi^{-\lambda}\varphi[y]z_1]. \tag{5.39}$$

Finally assume $t = 3$, $r \geq 1$, $\xi(-cq) = 1$, and $2 \in \mathfrak{p}$. Then $\nu \equiv \delta \pmod 2$, and hence $\varphi[z_1]\mathfrak{g} = \mathfrak{p}^{\delta}$. We can take

$$y_1, y_2 \in Y \text{ so that } Y = Fy_1 + Fy_2 \text{ and } \varphi(y_1, y_2) = 0 \tag{5.40}$$

by [EPE, Lemma 1.8]. Then we may assume

$$\varphi[y_1]\varphi[y_2] \in \mathfrak{g}^{\times} \tag{5.41}$$

since

$$-\varphi[z_1]\varphi[y_1]\varphi[y_2]F^{\times 2} = \delta(Z, \varphi) = -cF^{\times 2} \tag{5.42}$$

by (2.14), (5.31), and (5.40). Put

$$T = Fy_1y_2 + Fz_1y_1 + Fz_1y_2,\ K_Y = F + Fy_1y_2,\ B = F + T,\ \zeta = z_1y_1y_2 \tag{5.43}$$

in $A(Z)$. Moreover put $c_1 = \varphi[z_1]\varphi[y_1]\varphi[y_2]$. Then $Z = T\zeta$, $Y = K_Y y_2$, B is a division quaternion algebra over F, $c_1\mathfrak{g} = c\mathfrak{g}$, and $\varphi[x\zeta] = c_1 N_{B/F}(x)$ for $x \in T$. From $\xi(-cq) = 1$ and (5.42), we have $(y_1y_2)^2F^{\times 2} = sF^{\times 2}$. Thus K_Y is an unramified quadratic extension of F by (3.17). We may assume

$$\varphi[y_1]\mathfrak{g} = \varphi[y_2]\mathfrak{g} = \mathfrak{g} \text{ or } \varphi[y_1]^{-1}\mathfrak{g} = \varphi[y_2]\mathfrak{g} = \mathfrak{p} \tag{5.44}$$

by (5.41). Then we see that

$$\varphi[y_2]\mathfrak{g} = \mathfrak{p}^{1-\delta} = \begin{cases} \mathfrak{p} & \text{if } \nu \equiv 0 \pmod 2, \\ \mathfrak{g} & \text{if } \nu \equiv 1 \pmod 2 \end{cases} \tag{5.45}$$

as shown below. Put

$$\omega = z_1y_1 \text{ and } v = y_1y_2.$$

Then $B = K_Y + K_Y\omega$, $\omega v = -v\omega$, $\omega^2\mathfrak{g} = \mathfrak{p}^{2\delta-1}$, $v \in K_Y \cap T$, and $v^2 \in \mathfrak{g}^{\times}$ from (5.31), (5.40), (5.41), (5.42), (5.43), and (5.45). Therefore

$$N = (\mathfrak{g}v + \mathfrak{r}_Y\omega^{-1})\zeta = \mathfrak{g}z_1 + \mathfrak{r}_Y y_2; \tag{5.46}$$

see §2.3. Here $\mathfrak{r}_Y$ is the valuation ring of K_Y. Now we assert (5.45). Indeed, if this is not the case, then

$$\varphi[y_2]\mathfrak{g} = \mathfrak{p}^{\delta} = \begin{cases} \mathfrak{g} & \text{if } \nu \equiv 0 \pmod 2, \\ \mathfrak{p} & \text{if } \nu \equiv 1 \pmod 2 \end{cases}$$

by (5.44). Since K_Y is unramified over F, there exists $\theta \in B$ such that $B = K_Y + K_Y\theta$, $\theta^2\mathfrak{g} = \mathfrak{p}$, and $\theta y_1 y_2 = -y_1 y_2 \theta$ by the proof of [AQC, Theorem 5.13]. Then we easily see that $\theta \in Fz_1y_1 + Fz_1y_2$. Put $J = Fz_1y_1 + Fz_1y_2$ and $\psi[x] = N_{B/F}(x)$ for $x \in J$. We consider the Clifford algebra $A(J, \psi)$ of ψ. Put $K_J = F + Fz_1y_1 \cdot z_1y_2$, then $J = K_J z_1 y_2$, $\psi[z_1y_2]\mathfrak{g} = \mathfrak{p}^{2\delta}$, and K_J is isomorphic to $F(\sqrt{s})$ which is an unramified quadratic extension of F. Thus Lemma 3.4(2) implies $\Lambda_J[\psi[\theta]] = \emptyset$, where Λ_J is a maximal lattice in J. Since $\theta^2\mathfrak{g} = \mathfrak{p}$, this gives a contradiction, and hence $\varphi[y_2]\mathfrak{g} = \mathfrak{p}^{1-\delta}$.

5.11. Put

$$x_2 = \pi^{-[\nu/2]}(z - 2\pi^{-i}\varphi[z]e_r). \tag{5.47}$$

Then $\varphi[x_2] = \varphi[z_1]$ with z_1 in (5.31). From (5.30), (5.32), (5.37), (5.39), and (5.46),

$$H = \begin{cases} \mathfrak{p}^{\tau-i}x_1 + \mathfrak{g}x_2 & \text{if } t = 1, r \geq 1, \xi(cq) \neq 1, \\ & \quad \nu \equiv \delta \pmod 2, \text{ and } 2 \in \mathfrak{p}, \\ \mathfrak{p}^{\tau-i}x_1 + \mathfrak{g}x_2 + \mathfrak{g}y & \text{if } t = 2, r \geq 1, c^{-1}q \notin N_{K/F}(\mathfrak{r}), d > 1, \\ & \quad \text{and } b\mathfrak{g} = \mathfrak{p}, \\ \mathfrak{p}^{\tau-i}x_1 + \mathfrak{g}x_3 + \mathfrak{g}y & \text{if } t = 2, r \geq 1, c^{-1}q \notin N_{K/F}(\mathfrak{r}), d > 1, \\ & \quad \text{and } b \in \mathfrak{g}^\times, \\ \mathfrak{p}^{\tau-i}x_1 + \mathfrak{g}x_2 + \mathfrak{r}_Y y_2 & \text{if } t = 3, r \geq 1, \xi(-cq) = 1, \text{ and } 2 \in \mathfrak{p} \end{cases} \tag{5.48}$$

with x_1 of (5.27), $y \in Y$ satisfying (5.34), y_2 of (5.40), and $\mathfrak{r}_Y$ in (5.46). Moreover x_3 is given as

$$x_3 = \pi^{-\lambda+(d-2\kappa)/2}[y + (-1)^\nu \alpha\pi^{-\lambda}\varphi[y]x_2], \tag{5.49}$$

with λ of (5.35).

5.12. On the other hand for the space X in (5.29) we put

$$\Lambda = \{x \in X \mid \varphi[x] \in \mathfrak{g}\}. \tag{5.50}$$

Then [AQC, Lemma 6.4] implies that Λ is a unique maximal lattice in X. Here we put

$$w = -\pi^i(q - \varphi[z])^{-1}\varphi[z]f_r + z - \pi^{-i}\varphi[z]e_r \tag{5.51}$$

with e_r and f_r in (5.5). Then we find that

$$x_2 = \pi^{-[\nu/2]}[w + \pi^\tau(q - \varphi[z])^{-1}\varphi[z]x_1], \tag{5.52}$$

$$\varphi[w] = (1 - s)^{-1}sq, \tag{5.53}$$

$$X = Fx_1 + Fw + Y,\ \varphi(x_1, w) = 0,\ Y = \{k \in X \mid \varphi(k, x_1) = \varphi(k, w) = 0\}. \tag{5.54}$$

Here Y is given in (5.31) and x_2 is of (5.47).

5.13. In §§5.13 and 5.14 we determine the structure of Λ in the above (5.50). In §5.13 we suppose that: $t = 1$, $r \geq 1$, $\nu \equiv \delta \pmod 2$, $2 \in \mathfrak{p}$, and $\xi(cq) \neq 1$; or $t = 2$, $r \geq 1$, $c^{-1}q \notin N_{K/F}(\mathfrak{r})$, $d > 1$, and $b\mathfrak{g} = \mathfrak{p}$; or $t = 3$, $r \geq 1$, $\xi(-cq) = 1$, and $2 \in \mathfrak{p}$. (The case when $t = 2$, $r \geq 1$, $c^{-1}q \notin N_{K/F}(\mathfrak{r})$, $d > 1$, and $b \in \mathfrak{g}^\times$ will be treated in §5.14.) To prove the theorem in this case, it suffices to show that

(5.55)

$$\Lambda = \begin{cases} \mathfrak{g}x_1 + \mathfrak{g}x_2 & \text{if } t = 1, r \geq 1, \xi(cq) \neq 1, \nu \equiv \delta \pmod 2, \text{ and } 2 \in \mathfrak{p}, \\ \mathfrak{g}x_1 + \mathfrak{g}x_2 + \mathfrak{g}y & \text{if } t = 2, r \geq 1, c^{-1}q \notin N_{K/F}(\mathfrak{r}), d > 1, \text{ and } b\mathfrak{g} = \mathfrak{p}, \\ \mathfrak{g}x_1 + \mathfrak{g}x_2 + \mathfrak{r}_Y y_2 & \text{if } t = 3, r \geq 1, \xi(-cq) = 1, \text{ and } 2 \in \mathfrak{p} \end{cases}$$

by (5.48).
First we shall show that

$$\mathfrak{g}x_1 + \mathfrak{g}x_2 \text{ is maximal in } Fx_1 + Fw. \tag{5.56}$$

For the purpose, we consider the Clifford algebra $A(Fx_1+Fw)$ of the restriction of φ to $Fx_1 + Fw$. Put $E = F + Fwx_1$ in $A(Fx_1 + Fw)$. Then we obtain that

$$Fx_1 + Fw = Ex_1, \ \varphi[xx_1] = \varphi[x_1]N_{E/F}(x) \text{ for } x \in E, \tag{5.57}$$

and E is isomorphic to $F(\sqrt{s})$ since $(wx_1)^2F^{\times 2} = sF^{\times 2}$ by (5.28) and (5.53). First we suppose that:

$t = 1$, $r \geq 1$, $\xi(cq) = 0$, $\nu \notin 2\mathbf{Z}$, $2 \in \mathfrak{p}$, and $\delta = 1$; or
$t = 1$, $r \geq 1$, $\xi(cq) = -1$, $\nu \equiv \delta \pmod 2$, and $2 \in \mathfrak{p}$; or
$t = 2$, $r \geq 1$, $c^{-1}q \notin N_{K/F}(\mathfrak{r})$, $d > 1$, and $b\mathfrak{g} = \mathfrak{p}$; or
$t = 3$, $r \geq 1$, $\xi(-cq) = 1$, and $2 \in \mathfrak{p}$.

Then

$$\mathfrak{D}_{E/F} = \begin{cases} \mathfrak{p}^d & \text{if } t = 1, r \geq 1, \xi(cq) = 0, \nu \notin 2\mathbf{Z}, 2 \in \mathfrak{p}, \text{ and } \delta = 1, \\ \mathfrak{g} & \text{otherwise.} \end{cases} \tag{5.58}$$

Indeed, if $t = 1$, $r \geq 1$, $\xi(cq) = 0$, $\nu \notin 2\mathbf{Z}$, $2 \in \mathfrak{p}$, and $\delta = 1$, then $cqF^{\times 2} = sF^{\times 2}$. Thus we have (5.58) by the definition of d; see Theorem 3.5(ii). If $t = 1$, $r \geq 1$, $\xi(cq) = -1$, $\nu \equiv \delta \pmod 2$, and $2 \in \mathfrak{p}$, then from (3.11), Lemma 4.2(1), and $cq \notin F^{\times 2}$, we have $\varepsilon(s) = 2\kappa$ with ε of (4.11). Thus we obtain (5.58). If $t = 2$, $r \geq 1$, $c^{-1}q \notin N_{K/F}(\mathfrak{r})$, $d > 1$, and $b\mathfrak{g} = \mathfrak{p}$, then $\varepsilon(s) = 2\kappa$ from (3.14), (5.36), and $c^{-1}q \notin N_{K/F}(\mathfrak{r})$. If $t = 3$, $r \geq 1$, $\xi(-cq) = 1$, and $2 \in \mathfrak{p}$, then clearly we have (5.58) by (4.34) and Lemma 4.2(1). Therefore we have (5.58) for all cases. Now, by (5.23), (5.36), and Lemma 5.1(3),

$$\tau = \begin{cases} \kappa + (\nu + 1 - d)/2 & \text{if } t = 1, r \geq 1, \xi(cq) = 0, \nu \notin 2\mathbf{Z}, 2 \in \mathfrak{p}, \text{ and } \delta = 1, \\ \kappa + [\nu/2] & \text{otherwise.} \end{cases}$$

Thus by (3.11), (3.14), (3.17), and (5.28), we have
(5.59)

$$\varphi[x_1]\mathfrak{g} = \begin{cases} \mathfrak{g} & \text{if } t = 1, r \geq 1, \xi(cq) = 0, \nu \notin 2\mathbf{Z}, 2 \in \mathfrak{p}, \text{ and } \delta = 1, \\ \mathfrak{p}^{\delta} & \text{if } t = 1, r \geq 1, \xi(cq) = -1, \nu \equiv \delta \pmod 2, \text{ and } 2 \in \mathfrak{p}, \\ \mathfrak{p}^{\lambda} & \text{if } t = 2, r \geq 1, c^{-1}q \notin N_{K/F}(\mathfrak{r}), d > 1, \text{ and } b\mathfrak{g} = \mathfrak{p}, \\ \mathfrak{p}^{\delta} & \text{if } t = 3, r \geq 1, \xi(-cq) = 1, \text{ and } 2 \in \mathfrak{p}. \end{cases}$$

Therefore $\mathfrak{r}_E x_1$ is a unique maximal lattice in Ex_1, where $\mathfrak{r}_E$ is the valuation ring of E. Since $N_{E/F}(x_2 x_1^{-1}) = \varphi[x_2]\varphi[x_1]^{-1} \in \mathfrak{g}$, we have $x_2 x_1^{-1} \in \mathfrak{r}_E$. By (5.52) and (5.58), $N_{E/F}(x_2 x_1^{-1} - (x_2 x_1^{-1})^{\rho_E})\mathfrak{g} = D_{E/F}$, where $1 \neq \rho_E \in Gal(E/F)$. Thus [AQC, Lemma 5.6(ii)] implies $\mathfrak{r}_E = \mathfrak{g}[x_2 x_1^{-1}]$, and hence $\mathfrak{g}x_1 + \mathfrak{g}x_2$ is a maximal lattice in Ex_1 in this case.
Next suppose that $t = 1$, $r \geq 1$, $\xi(cq) = 0$, $\nu \in 2\mathbf{Z}$, $2 \in \mathfrak{p}$, and $\delta = 0$. Then $d \in 2\mathbf{Z}$ by Lemma 5.1(3). From this and (5.23), $\tau = \kappa + 2^{-1}(\nu - d)$. Thus $\varphi[x_1]\mathfrak{g} = \mathfrak{p}$ from (3.11) and (5.28). Put $\eta = \pi^{2^{-1}d-\kappa}(1 + \pi^{\tau}\varphi[z]^{-1}wx_1) \in E$, then η is a root of an Eisenstein equation $\mathbf{x}^2 - 2\pi^{2^{-1}d-\kappa}\mathbf{x} - \pi^{d-2\kappa}\varphi[z]^{-1}(q - \varphi[z]) = 0$. Here $\mathbf{x}$ is an indeterminate. Therefore η is a prime element of E and $(\eta - \eta^{\rho_E})\mathfrak{r}_E = (2\eta - 2\pi^{2^{-1}d-\kappa})\mathfrak{r}_E = \mathfrak{d}_{E/F}$. Here $\mathfrak{r}_E$ and ρ_E are the same symbols as above. Thus $\mathfrak{r}_E = \mathfrak{g}[\eta]$ by [AQC, Lemma 5.6(ii)]. From $\varphi[x_1]\mathfrak{g} = \mathfrak{p}$, we have $\Lambda = \mathfrak{r}_E\eta^{-1}x_1 = \mathfrak{g}x_1 + \mathfrak{g}\eta^{-1}x_1$ with Λ of (5.50). Since $\eta^{-1} = -\pi^{\kappa-2^{-1}d}\varphi[z](q - \varphi[z])^{-1}(1 - \pi^{\tau}\varphi[z]^{-1}wx_1)$, we obtain $\Lambda = \mathfrak{g}x_1 + \mathfrak{g}x_2$. This proves (5.56).
Now, we obtain (5.55) when $t = 1$, $r \geq 1$, $\nu \equiv \delta \pmod 2$, $\xi(cq) \neq 1$, and $2 \in \mathfrak{p}$ by (5.56). When $t > 1$, we have

(5.60)

$$[(\mathfrak{g}x_1 + \mathfrak{g}x_2)\tilde{} : \mathfrak{g}x_1 + \mathfrak{g}x_2] = \begin{cases} [\mathfrak{g} : \mathfrak{p}^{2\lambda}] & \text{if } t = 2, r \geq 1, c^{-1}q \notin N_{K/F}(\mathfrak{r}), d > 1, \text{ and } b\mathfrak{g} = \mathfrak{p}, \\ [\mathfrak{g} : \mathfrak{p}^{2\delta}] & \text{if } t = 3, r \geq 1, \xi(-cq) = 1, \text{ and } 2 \in \mathfrak{p}. \end{cases}$$

by (5.56), (5.57), (5.58), (5.59), and [AQC, Lemma 8.4(iv)]. Put

$$\Lambda_Y = \begin{cases} \mathfrak{g}y & \text{if } t = 2, r \geq 1, c^{-1}q \notin N_{K/F}(\mathfrak{r}), d > 1, \text{ and } b\mathfrak{g} = \mathfrak{p}, \\ \mathfrak{r}_Y y_2 & \text{if } t = 3, r \geq 1, \xi(-cq) = 1, \text{ and } 2 \in \mathfrak{p}. \end{cases}$$

Then Λ_Y is maximal in Y. Indeed, if $t = 2$, $r \geq 1$, $c^{-1}q \notin N_{K/F}(\mathfrak{r})$, $d > 1$, and $b\mathfrak{g} = \mathfrak{p}$, then $\dim_F(Fy) = 1$. We have $\varphi[y]\mathfrak{g} = \mathfrak{p}^{1-\lambda}$ from (5.31) and

(5.34), and hence $\mathfrak{g}y$ is maximal in Y by [AQC, Lemma 6.4]. If $t = 3$, $r \geq 1$, $\xi(-cq) = 1$, and $2 \in \mathfrak{p}$, then K_Y in (5.43) is unramified over F and $Y = K_Y y_2$. Also we have $\varphi[ky_2] = \varphi[y_2]N_{K_Y/F}(k)$ for $k \in K_Y$. Thus $\mathfrak{r}_Y y_2$ is maximal in Y by (5.45) and [AQC, Lemma 6.4]. Since Λ_Y is maximal in Y, [AQC, Lemma 8.4(iv)] and Lemma 5.2 imply

$$[\widetilde{\Lambda_Y} : \Lambda_Y] = \begin{cases} [\mathfrak{g} : 2\mathfrak{p}^{1-\lambda}] & \text{if } t = 2, r \geq 1, c^{-1}q \notin N_{K/F}(\mathfrak{r}), d > 1, \text{ and } b\mathfrak{g} = \mathfrak{p}, \\ [\mathfrak{g} : \mathfrak{p}^{2(1-\delta)}] & \text{if } t = 3, r \geq 1, \xi(-cq) = 1, \text{ and } 2 \in \mathfrak{p}. \end{cases}$$

Combining this with (5.60), we obtain

$$\begin{aligned} [(\mathfrak{g}x_1 + \mathfrak{g}x_2 + \Lambda_Y)\tilde{} : \mathfrak{g}x_1 + \mathfrak{g}x_2 + \Lambda_Y] &= [(\mathfrak{g}x_1 + \mathfrak{g}x_2)\tilde{} : \mathfrak{g}x_1 + \mathfrak{g}x_2] \cdot [\widetilde{\Lambda_Y} : \Lambda_Y] \\ &= \begin{cases} [\mathfrak{g} : 2\mathfrak{p}^{1+\lambda}] & \text{if } t = 2, r \geq 1, c^{-1}q \notin N_{K/F}(\mathfrak{r}), d > 1, \\ & \qquad \text{and } b\mathfrak{g} = \mathfrak{p}, \\ [\mathfrak{g} : \mathfrak{p}^2] & \text{if } t = 3, r \geq 1, \xi(-cq) = 1, \text{ and } 2 \in \mathfrak{p}. \end{cases} \end{aligned}$$

Therefore $\mathfrak{g}x_1 + \mathfrak{g}x_2 + \Lambda_Y$ is maximal in X by [AQC, Lemma 8.4(iv)] and Lemma 5.2. Note that $\varphi[x_1]\varphi[w]\varphi[y]\mathfrak{g} = \mathfrak{p}^{\nu-2\kappa+1}$ by (5.34), (5.53), and (5.59), when $t = 2$, $r \geq 1$, $c^{-1}q \notin N_{K/F}(\mathfrak{r})$, $d > 1$, and $b\mathfrak{g} = \mathfrak{p}$. Therefore we obtain (5.55).

5.14. Finally, we suppose that $t = 2$, $r \geq 1$, $c^{-1}q \notin N_{K/F}(\mathfrak{r})$, $d > 1$, and $b \in \mathfrak{g}^\times$. We have already obtained

$$H = \mathfrak{g}\pi^{\tau-i}x_1 + \mathfrak{g}x_3 + \mathfrak{g}y \tag{5.61}$$

by (5.48), with x_1 of (5.27), y satisfying (5.34), and x_3 of (5.49). Our aim is to show that H coincides with the unique maximal lattice Λ in X if and only if $i = \tau$, with Λ of (5.50). For the purpose, we shall find a $\mathfrak{g}$-basis of Λ in a similar way as §2.3(III) (cf. (2.19)).
We consider the Clifford algebra $A(X)$ of the restriction of φ to X. Put

$$x_0 = \pi^{\lambda-1}x_1 \text{ and } w_0 = \pi^{-(\nu-d+\lambda)/2}w$$

with λ of (5.35) and w of (5.51). Then $\{x_0, w_0, y\}$ is an F-basis of X satisfying $\varphi(x_0, w_0) = \varphi(w_0, y) = \varphi(y, x_0) = 0$ by (5.54) and $Y = Fy$ with Y in (5.31). Thus we obtain

$$X = T_X\zeta_X \text{ and } \varphi[x\zeta_X] = c_X N_{B/F}(x) \text{ for } x \in T_X.$$

Here

$$T_X = Fx_0w_0 + Fx_0y + Fw_0y, B = F + T_X, c_X = \varphi[x_0]\varphi[w_0]\varphi[y], \zeta_X = x_0w_0y.$$

Note that B is a division quaternion algebra over F. Moreover we have $c_X\mathfrak{g} = \mathfrak{p}^\lambda$ by (5.28), (5.34), and (5.53). Therefore

$$\Lambda = (T_X \cap \mathfrak{P}^{-\lambda})\zeta_X, \tag{5.62}$$

where $\mathfrak{P} = \{x \in B \mid N_{B/F}(x) \in \mathfrak{p}\}$. Put

$$\omega_X = -x_1\zeta_X^{-1}, u_X = x_3\zeta_X^{-1}\omega_X^{-1}, \text{ and } v_X = u_X - u_X^\iota \tag{5.63}$$

in B, where ι is the main involution of B. Then we assert that

$$\Lambda = [\mathfrak{g}v_X + (\mathfrak{g} + \mathfrak{g}u_X)\omega_X]\zeta_X = \mathfrak{g}x_1 + \mathfrak{g}x_3 + \mathfrak{g}v_X\zeta_X. \tag{5.64}$$

Indeed, we have $\omega_X^2\mathfrak{g} = \mathfrak{p}^{1-2\lambda}$ by (5.34) and (5.53). Thus we obtain

$$\begin{aligned} N_{B/F}(u_X) &= c_X^{-1}N_{B/F}(\omega_X)^{-1}\pi^{d-2\kappa-2\lambda}\varphi[y](-\pi^{2\kappa+1-d}\beta) \in \mathfrak{g}^\times, \\ N_{B/F}(v_X) &= N_{B/F}(u_X - u_X^\iota) \\ &= 4\pi^{-2\lambda+d-2\kappa}\varphi[x_1]^{-1}\varphi[y](s + \pi^{2\kappa+1-d}\beta)(s-1)^{-1} \in \mathfrak{g}^\times \end{aligned}$$

by (5.28), (5.34), (5.38), (5.49), and (5.53). Note that $2\kappa+1 > d$ by $d > 1$ and Lemma 4.2. Thus [AQC, Lemma 5.6(ii)] implies that $F + Fu_X$ is an unramified quadratic extension of F and $\mathfrak{g} + \mathfrak{g}u_X$ is the valuation ring of $F + Fu_X$. Also we have $v_X \in \mathfrak{g}[u_X]^\times$. We see that $B = (F + Fu_X) + (F + Fu_X)\omega_X$ and $v_X\omega_X = -\omega_X v_X$ by a straightforward caluculation. Combining these with (5.62), we obtain the first equality of (5.64) in the same way as §2.3(III). The second equality of (5.64) is trivial from (5.63). This proves (5.64).

Now, we consider the $\mathfrak{g}$-base of both H and Λ, that is, $\{\pi^{\tau-i}x_1, x_3, y\}$ and $\{x_1, x_3, v_X\zeta_X\}$; see (5.61) and (5.64) above. We see that

$$\begin{aligned} v_X\zeta_X &= 2\pi^{d-\kappa-1-(\nu+\lambda)/2}\varphi[y][w + (-1)^{\nu+1}\pi^{-(\nu+\lambda)/2}\alpha\varphi[w]y] \\ &= A(\pi^{\tau-i}x_1) + Bx_3 + Cy \end{aligned}$$

with

$$\begin{aligned} A &= -2\varphi[y]\varphi[z](q - \varphi[z])^{-1}\pi^{i+d-\kappa-1-(\nu+\lambda)/2} \in \mathfrak{p}^{i-\tau}, \\ B &= (-1)^\nu 2\alpha^{-1}\pi^{\lambda-1+2^{-1}d} \in \mathfrak{g}, \\ C &= (-1)^{\nu+1}2\pi^{d-\kappa-1}(\alpha^{-1} + \alpha\varphi[y]\varphi[w]\pi^{-\nu-\lambda}) \in \mathfrak{g}^\times \end{aligned}$$

by (5.23), (5.34), (5.38), (5.53), and $d > 1$. Thus

$$(\pi^{\tau-i}x_1, x_3, y) = (x_1, x_3, v_X\zeta_X)\gamma, \quad \gamma = \begin{bmatrix} \pi^{\tau-i} & 0 & -\pi^{\tau-i}AC^{-1} \\ 0 & 1 & -BC^{-1} \\ 0 & 0 & C^{-1} \end{bmatrix}.$$

Since $\gamma \in \mathfrak{g}_3^3$ and $\det(\gamma)\mathfrak{g} = \mathfrak{p}^{\tau-i}$, we obtain $H = \Lambda$ if and only if $i = \tau$, also in this case. This completes the proof.

6 Global results

In this section we assume that F is a global field and L is a maximal lattice in V with respect to φ. We state two global results which are derived from the local cases.

6.1. Let $\mathbf{h}$ be the set of nonarchmedean primes of F and fix $v \in \mathbf{h}$. We let F_v denote the completion of F at v. Then F_v is a local field. Let $\mathfrak{g}_v$ be the valuation ring of F_v and $\mathfrak{p}_v$ the maximal ideal of $\mathfrak{g}_v$. We also write $\mathfrak{p}_v$ for the prime ideal of $\mathfrak{g}$ corresponding to v. Put $X_v = X \otimes_F F_v$ for a subspace X of V and $\Lambda_v = \Lambda \otimes_{\mathfrak{g}} \mathfrak{g}_v$ for a $\mathfrak{g}$-lattice Λ in V. Let φ_v be the F_v-bilinear extension of φ to $V_v \times V_v$. We consider (V_v, φ_v). By [AQC, Lemma 9.4(iii)], L_v is a maximal lattice in V_v. For $q \in \mathfrak{g} \cap F^\times$ such that $L_v[q] \neq \emptyset$, put

$$\tau_v(q) = \max(\{i \in \mathbf{Z} \mid L_v[q] \supset L_v[q, 2^{-1}\mathfrak{p}_v^i] \neq \emptyset\}). \tag{6.1}$$

This is given by (3.9), (3.12), (3.15), (3.18), and (3.20) for every $v \in \mathbf{h}$.

6.2 Proposition. *Let the notation be as above. Let L be a maximal lattice in V and $q \in \mathfrak{g} \cap F^\times$. Let t_v be the core dimension of (V_v, φ_v) for $v \in \mathbf{h}$. Put $n = \dim_F(V)$. Then for a $\mathfrak{g}$-ideal $\mathfrak{a} = \prod_{v \in \mathbf{h}} \mathfrak{p}_v^{i_v}$ such that $\mathfrak{a} \subset \mathfrak{g}$, we have*

$$L[q, 2^{-1}\mathfrak{a}] \neq \emptyset \Longrightarrow \begin{cases} i_v = \tau_v(q) & \text{if } n = t_v, \\ i_v \leq \tau_v(q) & \text{otherwise} \end{cases}$$

for all $v \in \mathbf{h}$.

Proof. Assume $L[q, 2^{-1}\mathfrak{a}] \neq \emptyset$. For every $v \in \mathbf{h}$, we have $L[q, 2^{-1}\mathfrak{a}] \subset L_v[q, 2^{-1}\mathfrak{p}_v^{i_v}]$ since $\varphi(x, L)_v = \varphi_v(x, L_v)$ for any $x \in V$. Thus we obtain $\emptyset \neq L_v[q, 2^{-1}\mathfrak{p}_v^{i_v}] \subset L_v[q]$ by Lemma 3.6, and hence Corollary 3.7 implies the assertion. ☐

6.3 Theorem. *Let the notation be the same as in Proposition 6.2. Let L be a maximal lattice in V. Assume $n \geq 2$. Then for $h \in L$ such that $\varphi[h] \neq 0$, we have*

$$L \cap (Fh)^\perp \text{ is maximal in } (Fh)^\perp \Longleftrightarrow h \in L\big[\varphi[h], 2^{-1} \prod_{v \in \mathbf{h}} \mathfrak{p}_v^{\tau_v(\varphi[h])}\big].$$

Here $(Fh)^\perp = \{x \in V \mid \varphi(x, h) = 0\}$.

Proof. Put $W = (Fh)^\perp$. Then we see that $W_v = (F_v h)^\perp$ in V_v for all $v \in \mathbf{h}$. By [AQC, Lemma 9.4(iii)], $L \cap W$ is maximal in W if and only if $L_v \cap W_v = (L \cap W)_v$ is maximal in W_v for every $v \in \mathbf{h}$; Moreover Theorem 5.3 shows that this is the case if and only if $h \in L_v\big[\varphi[h], 2^{-1}\mathfrak{p}_v^{\tau_v(\varphi[h])}\big]$ for all $v \in \mathbf{h}$. Since $\varphi(h, L)_v = \varphi_v(h, L_v)$, the assertion holds. ☐

This theorem answers the question raised in [AQC, (11.6a)].

7 Sums of squares

7.1. Put $V = \mathbf{Q}_n^1$ and $\varphi(x, y) = x \cdot {}^t y$ for $x, y \in V$. Let L be a maximal lattice in V and $\{e_i\}_{i=1}^n$ the standard $\mathbf{Q}$-basis of V in this section. Then $\varphi[x] = \sum_{i=1}^n x_i^2$

for $x = \sum_{i=1}^{n} x_i e_i \in V$. Hereafter we assume that q is a squarefree positive integer. By [AQC, (12.17)],

$$L[q] = L[q, 2^{-1}\mathbf{Z}] \sqcup L[q, \mathbf{Z}].$$

Here we apply our results on $L[q]$ in this case and investigate the sets $L[q, 2^{-1}\mathbf{Z}]$ and $L[q, \mathbf{Z}]$ when $4 \leq n \leq 10$ and $n \in 2\mathbf{Z}$. As for the case $n \notin 2\mathbf{Z}$, we can refer to [AQC, Section 12].

7.2 Lemma. *Assume $n \geq 4$. Let L be a maximal lattice in V and q a squarefree positive integer. Then*

$$L[q, \mathbf{Z}] = \emptyset \ \textit{if} \ \begin{cases} n \equiv 0 \pmod 8; \ \textit{or} \\ n \equiv \pm 1 \pmod 8 \ \textit{and} \ (-1)^{(n-1)/2} q \not\equiv 1 \pmod 4; \ \textit{or} \\ n \equiv \pm 2 \pmod 8 \ \textit{and} \ (-1)^{(n-2)/4} q \equiv 3 \pmod 4; \ \textit{or} \\ n \equiv 4 \pmod 8 \ \textit{and} \ q \equiv 1 \pmod 2 \end{cases}$$

and $L[q, 2^{-1}\mathbf{Z}] = \emptyset$ *if* $n = 4$ *and* $q \equiv 0 \pmod 2$.

Proof. Let p be a rational prime number. The core dimension t_p of (V_p, φ_p) is given by [AQC, (7.12a) and (7.12b)]. Let c_p be as in §2.3 when $1 \leq t_p \leq 3$. By a Witt decomposition of V_p as in (2.4), we have $(-1)^{(n-t_p)/2} c_p \mathbf{Q}_p^{\times 2} = \delta(V_p, \varphi_p) = \mathbf{Q}_p^{\times 2}$ for $t_p = 1, 3$. From this and [AQC, §7.15], we can take c_p so that $c_p \in \mathbf{Z}_p^{\times}$ when $p \neq 2$ and

$$c_2 = \begin{cases} (-1)^{(n-t_2)/2} & \text{if } t_2 = 1, 3, \\ (-1)^{(n-2)/4} & \text{if } t_2 = 2 \end{cases}$$

when $p = 2$. Let $\tau_p(q)$ be as in (6.1). Then we see that

$$\tau_p(q) = \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod 8, p = 2, \text{ and } (-1)^{(n-1)/2} q \equiv 1 \pmod 4; \\ & \text{or } n \equiv \pm 2 \pmod 8, p = 2, \text{ and } (-1)^{(n-2)/4} q \not\equiv 3 \pmod 4; \\ & \text{or } n \equiv \pm 3 \pmod 8 \text{ and } p = 2; \\ & \text{or } n \equiv 4 \pmod 8, p = 2, \text{ and } q \equiv 0 \pmod 2, \\ 0 & \text{otherwise} \end{cases} \tag{7.1}$$

by Theorem 3.5 and Lemma 4.2(1). Note that $N_{\mathbf{Q}(\sqrt{-1})_2/\mathbf{Q}_2}(\mathbf{Z}[\sqrt{-1}]_2^{\times}) = (1 + 4\mathbf{Z}_2)\mathbf{Z}_2^{\times 2}$. Combining (7.1) with Proposition 6.2, the assertion holds. □

If $n \equiv \pm 1 \pmod 8$, then this lemma is a restatement of [AQC, Lemma 12.13(ii)].

7.3. Put

$$L_n = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \sum_{i=2}^{n/2} (\mathbf{Z}e_{2i-1} + \mathbf{Z}g_{2i}) \tag{7.2}$$

for $4 \leq n \equiv 2, 4, 6 \pmod 8$, where $g_{2i} = 2^{-1}(e_{2i-3} + e_{2i-2} + e_{2i-1} + e_{2i})$. Then L_n is maximal in V by [CGQ, Lemma 3.1]. When $n = 10$, we put

$$\Lambda = H + M \tag{7.3}$$

with a maximal lattice H (resp. M) in $\sum_{i=1}^{8} \mathbf{Q}e_i$ (resp. $\mathbf{Q}e_9 + \mathbf{Q}e_{10}$). Then Λ is a maximal lattice in V by [NRQ, §6.8]. Hereafter we suppose $4 \leq n \leq 10$ and $n \in 2\mathbf{Z}$. By [AQC, §12.12], if $n < 10$, then the genus $LSO^{\varphi}_{\mathbf{A}}$ of L (cf. [AQC, §§9.3 and 9.7]) equals to the SO^{φ}-class LSO^{φ}. Here $SO^{\varphi}_{\mathbf{A}}$ is the adelization of SO^{φ}. If $n = 10$, then [CGQ, §3.2] and [AQC, Lemma 9.23(i)] imply $LSO^{\varphi}_{\mathbf{A}} = L_{10}SO^{\varphi} \sqcup \Lambda SO^{\varphi}$.

7.4 Lemma. *Let L_n be as in (7.2) and q a squarefree positive integer. Assume $n = 4, 6$, or 10. Then we obtain the following assertions:*

(1)*Assume $n = 4$. Then we have $L_4[q, 2^{-1}\mathbf{Z}] = \emptyset$ if and only if $q \equiv 0 \pmod 2$ and $L_4[q, \mathbf{Z}] = \emptyset$ if and only if $q \equiv 1 \pmod 2$.*
(2) *If $n > 4$, then $L_n[q, 2^{-1}\mathbf{Z}] \neq \emptyset$.*
(3) *If $n > 4$ and $q \equiv 0 \pmod 2$, then $L_n[q, \mathbf{Z}] \neq \emptyset$.*
(4) *If $n = 6$ and $q \equiv 3 \pmod 4$, then $L_6[q, \mathbf{Z}] \neq \emptyset$.*
(5) *Assume $n = 10$ and $q \equiv 1 \pmod 4$. Then we have $L_{10}[q, \mathbf{Z}] = \emptyset$ if and only if $q = 1$.*

Proof. (1): Assume $n = 4$. Since L_4 is maximal in V,

$$L_4[q] = \begin{cases} L_4[q, 2^{-1}\mathbf{Z}] & \text{if } q \equiv 1 \pmod 2, \\ L_4[q, \mathbf{Z}] & \text{if } q \equiv 0 \pmod 2 \end{cases}$$

by Lemma 7.2. We have $\sum_{i=1}^{4} \mathbf{Z}e_i \subset L_4$, and hence

$$L_4[q] \neq \emptyset \text{ for any squarefree positive integer } q. \tag{7.4}$$

This proves (1). (2): Assume $n > 4$. We can take $x \in L_4$ so that $\varphi[x] = q$ or $q - 1$ according as $q \equiv 1 \pmod 2$ or $q \equiv 0 \pmod 2$ by (7.4). If $q \equiv 1 \pmod 2$, then put $h = x$; if $q \equiv 0 \pmod 2$, then put $h = x + e_5$. By (1) settled above, $h \in L_n[q, 2^{-1}\mathbf{Z}]$ in both cases. This proves (2). In the proof of (3) and (4) we take

$$y = \sum_{i=1}^{4} y_i e_i \in \sum_{i=1}^{4} \mathbf{Z}e_i \text{ such that } \varphi[y] = q \tag{7.5}$$

for a given q. (3): Suppose $n > 4$ and $q \equiv 0 \pmod 2$. Since q is even and squarefree, at least two of y_1, y_2, y_3, and y_4 are even. We may assume $y_3, y_4 \in 2\mathbf{Z}$. Then $y \in L_n[q, \mathbf{Z}]$ from (1). This proves (3). Now, for $h = \sum_{i=1}^{n} h_i e_i \in V$ such that $\varphi[h] = q$, we have

$$h \in L_n[q, \mathbf{Z}] \Longleftrightarrow h \in \sum_{i=1}^{n} \mathbf{Z}e_i \text{ and } \sum_{k=0}^{3} h_{2j-k} \in 2\mathbf{Z} \text{ for } 2 \leq j \leq n/2. \tag{7.6}$$

(4): Suppose $n = 6$ and $q \equiv 3 \pmod 4$. Then one and only one of y_1, y_2, y_3, and y_4 in (7.5) is even. We may assume $y_1 \in 2\mathbf{Z}$. Put $h = \sum_{i=1}^{3} y_i e_i + y_4 e_5$, then $h \in L_6[q, \mathbf{Z}]$ by (7.6). This proves (4). (5): Assume $n = 10$ and $q \equiv 1 \pmod 4$. Then $L_{10}[1, \mathbf{Z}] = \emptyset$ by (7.6). If $q > 1$, then there exists $z = \sum_{i=1}^{4} z_i e_i \in \sum_{i=1}^{4} \mathbf{Z} e_i$ such that $\sum_{i=1}^{4} z_i^2 = 4^{-1}(q-5)$. Put $h = \sum_{i=1}^{4} 2z_i e_{2i} + \sum_{j=1}^{5} e_{2j-1}$. Then $h \in L_{10}[q, \mathbf{Z}]$ by (7.6). This completes the proof. $\square$

7.5 Theorem. *Let L be a maximal lattice in V and q a squarefree positive integer. We assume $4 \leq n \leq 10$ and $n \in 2\mathbf{Z}$. Then*

$$L[q, \mathbf{Z}] = \emptyset \textit{ if and only if } \begin{cases} n = 4 \textit{ and } q \equiv 1 \pmod 2; \textit{ or} \\ n = 6 \textit{ and } q \equiv 1 \pmod 4; \textit{ or} \\ n = 8; \textit{ or} \\ n = 10, L \in L_{10} SO^{\varphi}, q = 1 \textit{ or } q \equiv 3 \pmod 4; \textit{ or} \\ n = 10, L \in \Lambda SO^{\varphi}, q \equiv 3 \pmod 4 \end{cases}$$

and $L[q, 2^{-1}\mathbf{Z}] = \emptyset$ if and only if $n = 4$ and $q \equiv 0 \pmod 2$. Here L_{10} (resp. Λ) is of (7.2) *(resp.* (7.3)*).*

Proof. If $n = 4, 6$, or, $n = 10$ and $LSO^{\varphi} = L_{10} SO^{\varphi}$, then we have $LSO^{\varphi} = L_n SO^{\varphi}$. Therefore we obtain the assertion by Lemma 7.2 and Lemma 7.4. Assume $n = 8$. Then Lemma 7.2 implies $L[q] = L[q, 2^{-1}\mathbf{Z}]$. By [AQC, Lemma 6.2(1)], we may assume $\sum_{i=1}^{4} \mathbf{Z} e_i \subset L$, and hence $L[q] \neq \emptyset$. This proves our theorem in the case $n = 8$. Next assume $n = 10$ and $LSO^{\varphi} = \Lambda SO^{\varphi}$. Then we may put $L = \Lambda$. For $x = h + m \in H + M = \Lambda$, we have $\varphi[x] = \varphi[h] + \varphi[m]$ and $\varphi(x, \Lambda) = \varphi(h, H) + \varphi(m, M)$. Thus we obtain $H[q, 2^{-1}\mathbf{Z}] \subset \Lambda[q, 2^{-1}\mathbf{Z}]$. From this and the result of the case $n = 8$, we have $\Lambda[q, 2^{-1}\mathbf{Z}] \neq \emptyset$. Next we consider $\Lambda[q, \mathbf{Z}]$. We see that $L_6 + \mathbf{Z} f_7 + \mathbf{Z} g_8$ (resp. $\mathbf{Z} e_9 + \mathbf{Z} e_{10}$) is maximal in $\sum_{i=1}^{8} \mathbf{Q} e_i$ (resp. $\mathbf{Q} e_9 + \mathbf{Q} e_{10}$) by [CGQ, Lemma 3.1]. Here L_6 and g_8 are given in (7.2) and $f_7 = 2^{-1}(e_1 + e_3 + e_5 + e_7)$. Thus we can put

$$\Lambda = H + M = L_6 + \mathbf{Z} f_7 + \mathbf{Z} g_8 + \mathbf{Z} e_9 + \mathbf{Z} e_{10}.$$

Then, for $x = \sum_{i=1}^{10} x_i e_i \in V$ such that $\varphi[h] = q$, we have
(7.7)

$$x \in \Lambda[q, \mathbf{Z}] \Longleftrightarrow x \in \sum_{i=1}^{10} \mathbf{Z} e_i, \ \sum_{k=0}^{3} x_{2k+1} \in 2\mathbf{Z}, \ \sum_{k=0}^{3} x_{2j-k} \in 2\mathbf{Z} \text{ for } 2 \leq j \leq 4.$$

Assuming $q \not\equiv 3 \pmod 4$, we take y so that (7.5). Then at least two of y_1, y_2, y_3, and y_4 are even. We may assume $y_1, y_2 \in 2\mathbf{Z}$. Put $h = y_1 e_1 + y_2 e_2 + y_3 e_9 + y_4 e_{10}$, then $h \in \Lambda[q, \mathbf{Z}]$ by (7.7). Therefore if $\Lambda[q, \mathbf{Z}] = \emptyset$, then $q \equiv 3 \pmod 4$. Combining this with Lemma 7.2, we obtain our theorem. $\square$

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