Algebraic Groups of Type D_4, Triality, and Composition Algebras

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Abstract. Conjugacy classes of outer automorphisms of order 3 of simple algebraic groups of classical type D_4 are classified over arbitrary fields. There are two main types of conjugacy classes. For one type the fixed algebraic groups are simple of type G_2; for the other type they are simple of type A_2 when the characteristic is different from 3 and are not smooth when the characteristic is 3. A large part of the paper is dedicated to the exceptional case of characteristic 3. A key ingredient of the classification of conjugacy classes of trialitarian automorphisms is the fact that the fixed groups are automorphism groups of certain composition algebras.

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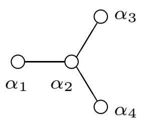
1. Introduction

The projective linear algebraic group $\mathbf{PGL_n}$ admits two types of conjugacy classes of outer automorphisms of order two. For one type the fixed subgroups are symplectic groups; for the other type the fixed groups are orthogonal groups, which are not smooth when the base field has characteristic 2.

The picture is similar for trialitarian automorphisms (i.e., outer automorphisms of order three) of the algebraic groups $G = \mathbf{PGO_8^+}$ or $G = \mathbf{Spin_8}$. There are two types of conjugacy classes. For one type the fixed groups are simple of type G_2; for the other type they are simple of type A_2 when the characteristic is different from 3, and not smooth when the characteristic is 3. The first case is well known: there is a split exact sequence

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathfrak{S}_3 \to 1 \tag{1.1}$$

where the permutation group of three elements $\mathfrak{S}_3$ is viewed as the group of automorphisms of the Dynkin diagram of type D_4



(the simple roots α_i are numbered after [Bou81]). Viewing G as a Chevalley group, there is a canonical section of the exact sequence (1.1) which leads to a distinguished trialitarian automorphism of G, known as a *graph automorphism*. The permutation $\rho\colon \alpha_1 \mapsto \alpha_3 \mapsto \alpha_4 \mapsto \alpha_1$, $\alpha_2 \mapsto \alpha_2$ of the Dynkin diagram above induces a permutation of all simple roots α of G, also denoted by ρ. The graph automorphism is such that $\rho\big(x_\alpha(u)\big) = x_{\alpha^\rho}(u)$ for each simple root group $x_\alpha(u)$ of G (see [Ste68b] or Section 2). The subgroup of G fixed by the graph automorphism is of type G_2.

The aim of this paper is to describe all conjugacy classes of trialitarian automorphisms of simple algebraic groups of classical inner type D_4 over arbitrary fields, from the (functorial) point of view of algebraic groups[1]. In particular the case of characteristic 3 has not been considered before in this setting, and a large part of this paper is dedicated to it. In a previous paper [CKT12], three of the authors gave a description based on the (known) classification of symmetric composition algebras of dimension 8 and a correspondence between symmetric compositions and trialitarian automorphisms. Here we take a different approach: we classify trialitarian automorphisms directly, and then use the correspondence to derive a new proof of the classification of symmetric composition algebras of dimension 8.

In the first part of the paper, consisting of §§2–7, we consider algebraic groups over algebraically closed fields. The first step, achieved in §§2–3, is to prove

[1] Trialitarian automorphisms (or more generally automorphisms of finite order) of simple Lie algebras have been extensively studied, see i.a., [Hel78], [Kač69], [Knu09], [Ree10] and [WG68].

that every trialitarian automorphism admits an invariant maximal torus. This is clear in characteristic different from 3 since trialitarian automorphisms are semisimple and semisimple automorphisms admit invariant tori (see [BM55], [Ste68a] or [Pia85]). However we did not find references for the result over fields of characteristic 3. In §§4–5, we recall the correspondence between symmetric compositions and trialitarian automorphisms set up in [CKT12], and use it to define two standard trialitarian automorphisms $\rho_\diamond$ and $\rho_\triangle$ corresponding respectively to the para-Zorn composition $\diamond$ and the split Okubo composition $\triangle$ on the 8-dimensional quadratic space (C, n) of Zorn matrices. We then define in §6 a split maximal torus T of $\mathbf{PGO}^+(n)$ invariant under $\rho_\diamond$ and $\rho_\triangle$, and use it in §7 to show that over an algebraically closed field, every trialitarian automorphism of $\mathbf{PGO}^+(n)$ is conjugate to $\rho_\diamond$ or to $\rho_\triangle$. In view of the results of §3, and since over a separably closed field all the maximal tori are conjugate, it suffices to consider trialitarian automorphisms that preserve the given torus T. In the second part of the paper, we describe the subgroups of $\mathbf{PGO}_8^+$ fixed under trialitarian automorphisms, over an arbitrary field F. These fixed subgroups are the automorphism groups of the corresponding symmetric composition algebras. They are divided into two classes according to their isomorphism class over an algebraic closure. In §8 we show that for one of the classes the symmetric composition algebras are para-octonion algebras and the automorphism groups are of type G_2; for the other the symmetric composition algebras are Okubo algebras and the automorphism groups are of type A_2 when the characteristic is different from 3. The following §§9–11 deal with the case of Okubo algebras in characteristic 3. A particular type of idempotents, which we call *quaternionic idempotents*, is singled out in §9, where we show that the existence of a quaternionic idempotent characterizes split Okubo algebras, and that split Okubo algebras over a field of characteristic 3 contain a unique quaternionic idempotent. This idempotent is used to describe in §§10–11 the group scheme $\mathbf{Aut}(\triangle)$ of automorphisms of the split Okubo algebra in characteristic 3. (The corresponding groups of rational points already occur in various settings, see [Tit59], [GL83] and [Eld99].) Using the description of $\mathbf{Aut}(\triangle)$, we recover in §12 the classification of Okubo algebras over arbitrary fields of characteristic 3 by a cohomological approach. Finally, in §13 we give a cohomological version of the correspondence between symmetric compositions and trialitarian automorphisms, and show that the only groups of classical type ${}^{1,2}\mathsf{D}_4$ that admit trialitarian automorphisms are $\mathbf{PGO}^+(n)$ and $\mathbf{Spin}(n)$ for n a 3-Pfister quadratic form. Note that the cohomology we use is faithfully flat finitely presented cohomology, or fppf-cohomology, since we need to deal with inseparable base field extensions. (Galois cohomology would be sufficient for fields of characteristic different from 3.) We refer to [DG70], [Wat79] and [KO74] for details on fppf-cohomology and descent theory.

The fact that the composition algebras of octonions could be used to define trialitarian automorphisms was already known to Élie Cartan (see [Car25]). We refer to [KMRT98] and [SV00] for historical comments on triality.

If not explicitly mentioned F denotes throughout the paper an arbitrary field and the algebras considered are defined over F.

We gratefully thank Philippe Gille for his help during the preparation of this paper.

2. Generators and relations in Chevalley groups

We recall in this section results about Chevalley groups which will be used later. Let G be a split simple simply connected group over F. Since G is a Chevalley group over F, its F-structure is well known. For details and proofs of all standard facts about $G(F)$ used in this section we refer to [Ste68b]. Let $\mathcal{G}$ be the Lie algebra of G. Choose a split maximal torus $T \subset G$ and a Borel subgroup $T \subset B \subset G$. Let $\Sigma = \Sigma(G,T)$ be the root system of G relative to T. The Borel subgroup B determines an ordering of Σ, hence the system of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$. We pick a Chevalley basis [Ste68b]

$$\{H_{\alpha_1}, \dots, H_{\alpha_n},\ X_\alpha,\ \alpha \in \Sigma\}$$

in $\mathcal{G}$ corresponding to the pair (T,B). This basis is unique up to signs and automorphisms of $\mathcal{G}$ which preserve B and T (see [Ste68b], §1, Remark 1). The group $G(F)$ is generated by the so-called root subgroups $U_\alpha = \langle x_\alpha(u) \mid u \in F\rangle$, where $\alpha \in \Sigma$ and

$$x_\alpha(u) = \sum_{n=0}^{\infty} u^n X_\alpha^n / n! \tag{2.1}$$

(we refer to [Ste68b] for the definition of the operators $X_\alpha^n/n!$ in the case $\mathrm{char}(F) = p > 0$, see also [SGA3]).

If $\alpha \in \Sigma$ and $t \in \mathbf{G}_m(F)$, the following elements are also of great importance:

$$w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t) \quad \text{and} \quad h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1}.$$

To ease notation we set $w_\alpha = w_\alpha(1)$. The elements $h_\alpha(t)$ give rise to a cocharacter $h_\alpha : \mathbf{G}_m \to T$ whose image is $T_\alpha = T \cap G_\alpha$ where G_α is the subgroup generated by $U_{\pm\alpha}$.

Example 2.2. The group G_α is isomorphic in a natural way to $\mathbf{SL}_2$. This isomorphism is given by

$$x_\alpha(u) \longrightarrow \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad x_{-\alpha}(u) \longrightarrow \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix},$$

$$h_\alpha(t) \longrightarrow \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad w_\alpha \longrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The following relations hold in G (cf. [Ste68b], Lemma 19 a), Lemma 28 b), Lemma 20 a)):

$$T = T_{\alpha_1} \times \cdots \times T_{\alpha_n}; \tag{2.3}$$

for any two roots $\alpha, \beta \in \Sigma$ we have

$$w_\alpha h_\beta(t) w_\alpha^{-1} = h_{w_\alpha(\beta)}(t); \tag{2.4}$$

$$w_\alpha X_\beta w_\alpha^{-1} = \varepsilon X_{w_\alpha(\beta)} \tag{2.5}$$

where $\varepsilon = \pm 1$.
We finally recall a commutator formula due to Chevalley, see [Ste68b, Lemma 32']. For arbitrary roots α, β, one has

$$\big(x_\alpha(u), x_\beta(v)\big) = \prod x_{i\alpha+j\beta}(c_{ij}u^i v^j) \tag{2.6}$$

where the product is taken over all integers $i, j > 0$ such that $i\alpha + j\beta$ is a root and the c_{ij} are integers depending on α, β and a chosen ordering of roots $i\alpha + j\beta$, but not on u, v.

3. Invariant tori

The main result in this section is the existence of invariant tori under trialitarian automorphisms. As already mentioned in the introduction the claim is known over fields of characteristic different from 3. We give here a proof for any field F.

We first reduce to the case of simply connected groups. Let G be a split simple simply connected group of type D_4 over F, and let G^{ad} be the isogenous adjoint group. We have a canonical exact sequence

$$1 \to \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \to G \to G^{\mathrm{ad}} \to 1, \tag{3.1}$$

where $\boldsymbol{\mu}_2$ is the group scheme of square roots of unity. Every automorphism of G induces an automorphism of G^{ad}.

Lemma 3.2. *Every trialitarian automorphism ϕ of G^{ad} lifts to a trialitarian automorphism $\widetilde{\phi}$ of G. If T is a torus in G invariant under $\widetilde{\phi}$, the image of T in G^{ad} is invariant under ϕ.*

Proof. Viewing G as a Chevalley group, we have a group of graph automorphisms of G isomorphic to $\mathfrak{S}_3$ defined over F, and we may consider the semidirect products $G \rtimes \mathfrak{S}_3$ and $G^{\mathrm{ad}} \rtimes \mathfrak{S}_3$. Note that

$$\mathbf{Aut}(G^{\mathrm{ad}}) = \mathbf{Int}(G^{\mathrm{ad}}) \rtimes \mathfrak{S}_3 = G^{\mathrm{ad}} \rtimes \mathfrak{S}_3,$$

and the exact sequence (3.1) yields an exact sequence

$$1 \to \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \to G \rtimes \mathfrak{S}_3 \to G^{\mathrm{ad}} \rtimes \mathfrak{S}_3 \to 1.$$

Consider the associated exact sequence in faithfully flat finitely presented cohomology (see [DG70, p. 272–273]):

$$(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)(F) \to (G \rtimes \mathfrak{S}_3)(F) \to (G^{\mathrm{ad}} \rtimes \mathfrak{S}_3)(F) \to H^1_{\mathrm{fppf}}(F, \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2).$$

Viewing ϕ as an element in $(G^{\mathrm{ad}} \rtimes \mathfrak{S}_3)(F)$, we may consider its image ϕ' in $H^1_{\mathrm{fppf}}(F, \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$. We have ${\phi'}^2 = 1$ because $H^1_{\mathrm{fppf}}(F, \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$ has exponent 2, and ${\phi'}^3 = 1$ because $\phi^3 = 1$, hence $\phi' = 1$. It follows that ϕ has a preimage ϕ'' in $(G \rtimes \mathfrak{S}_3)(F)$. We have ${\phi''}^3 \in (\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)(F)$ since $\phi^3 = 1$, hence conjugation by ϕ'' in $G \rtimes \mathfrak{S}_3$ restricts to an automorphism $\widetilde{\phi}$ of order 3 of G. The induced automorphism on G^{ad} is ϕ, so $\widetilde{\phi}$ is a lift of ϕ and is an outer automorphism.

A torus T in G is invariant under $\widetilde{\phi}$ if and only if it centralizes ϕ''. It is then clear that the image of T in G^{ad} is invariant under ϕ. □

The crucial tool to prove the existence of invariant tori is the following result of G. Harder[2]:

PROPOSITION 3.3 ([Har75, Lemma 3.2.4]). *Let F be an infinite field and let G be a split simple simply connected algebraic group or a split simple adjoint algebraic group of type D_4 over F. Let $\phi \in \mathbf{Aut}(G)(F)$ be an outer automorphism of order 3 of G over F. There exists a parabolic subgroup $P \subset G$ such that $H = P \cap P^{\phi} \cap P^{\phi^2}$ is a reductive subgroup of G of dimension 10 defined over F whose semi-simple part $H' = [H, H]$ is a simple group of type A_2.*

We first make some comments about properties of H. Assume that F is separably closed. Being a reductive group, H is an almost direct product of its central 2-dimensional torus S and its derived subgroup H'. Let T be an arbitrary maximal torus in H. Since $\dim T = 4$, T is also maximal in G. It then follows that H' is generated by some root subgroups $U_{\pm\gamma}$ and $U_{\pm\delta}$ where γ, δ are roots in G with respect to T. Note that every such subgroup in G is simply connected, hence $H' \simeq \mathbf{SL}_3$. Let $\alpha_1, \dots, \alpha_4$ be the basis of the root system Σ of type D_4 as defined in [Bou81], see also (1), and let $\beta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \in \Sigma$. For an arbitrary pair of roots γ, δ with $(\gamma, \delta) = -1$ in Σ there exists an element w in the corresponding Weyl group such that $w(\gamma) = \alpha_2$, $w(\delta) = \beta$. In view of (2.5) we may then assume without loss of generality that H' is the subgroup $G_{\{\alpha_2, \beta\}}$ generated by the root subgroups $U_{\pm\alpha_2}$ and $U_{\pm\beta}$.

Another ingredient in the proof of existence of invariant tori is the following lemma. Let G be a split simple simply connected algebraic group of type D_4 over a field F. Let $T \subset B$ be a maximal split torus and a Borel subgroup in G. As usual, the subgroup B determines the ordering in the root system $\Sigma = \Sigma(G, T)$ of G with respect to T and hence a basis of Σ. As above we set $\beta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$.

LEMMA 3.4. *Assume that F is a field of characteristic 3. Let ρ be the graph automorphism of G. If $x = x_{\alpha_2}(u)$, where $u \in F^\times$, there exists $g \in B(F) \rtimes \langle \rho \rangle$ such that $g(\rho x)g^{-1} = \rho x_{\alpha_2}(u_1) x_\beta(u_2) x_{\alpha_2+\beta}(u_3) \in B(F) \rtimes \langle \rho \rangle$, where $u_1, u_2 \in F^\times$ and $u_3 \in F$.*

Proof. The proof is based on the commutator formula (2.6). Note that for a group of type G_2 in the commutator formula applied to two simple roots γ_1, γ_2 of G_2 (numbered as in [Bou81]), all integers c_{ij} are non-zero modulo 3. Consider the subgroup $\overline{G}$ of G of elements invariant under ρ, which is of type G_2. It contains the subgroup of G generated by G_{α_2}, G_β of type A_2. Also, the root subgroups in $\overline{G}$ corresponding to its two simple roots γ_1 and γ_2 are generated by elements $x_{\alpha_2}(u)$ and $x_{\alpha_1}(v)x_{\alpha_3}(v)x_{\alpha_4}(v)$. Taking into consideration (2.6) and the fact that ρ commutes with $x_{\alpha_1}(v)x_{\alpha_3}(v)x_{\alpha_4}(v)$ we can write the element

$$X = \big(x_{\alpha_1}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)\big)\big(\rho x_{\alpha_2}(u)\big)\big(x_{\alpha_1}(1)x_{\alpha_3}(1)x_{\alpha_4}(1)\big)^{-1}$$

[2]We are indebted to Philippe Gille for pointing out this result to us.

in the form

$$X = \rho\big(x_{\alpha_2}(v_1)x_\beta(v_2)x_{\alpha_2+\beta}(v_3)\big)YZ \tag{3.5}$$

where $v_1, v_2 \neq 0$ (see the above remark about the integers c_{ij}) and

$$Y = x_{\alpha_1+\alpha_2}(v_4)x_{\alpha_3+\alpha_2}(v_5)x_{\alpha_4+\alpha_2}(v_6)$$

$$Z = x_{\alpha_1+\alpha_3+\alpha_2}(v_7)x_{\alpha_1+\alpha_4+\alpha_2}(v_8)x_{\alpha_3+\alpha_4+\alpha_2}(v_9)$$

where $v_i \in F$. We now note that in the expression (3.5) for X the root elements $x_{\alpha_1}(\cdot), x_{\alpha_3}(\cdot)$ and $x_{\alpha_4}(\cdot)$ are missing and the commutator of the other root subgroups which appear in (3.5) is either 1 or of the form $x_{\alpha_2+\beta}(\cdot)$ which is in the center of $B(F) \rtimes \langle\, \rho \,\rangle$. So there is no harm if we ignore factors $x_{\alpha_2+\beta}(\cdot)$ in the computations below and thus we shall assume that all root subgroups x_γ which appear commute with each other. Since ρx has order 3 (because ρ and x commute and $x^3 = x_{\alpha_2}(3u) = 1$) it follows almost immediately that Z satisfies the "cocycle condition" $Z\rho(Z)\rho^2(Z) = 1$. Since ρ acts freely on the subgroup

$$A = \langle\, x_{\alpha_1+\alpha_2+\alpha_3}(\cdot)\,\rangle \times \langle\, x_{\alpha_1+\alpha_2+\alpha_4}(\cdot)\,\rangle \times \langle\, x_{\alpha_3+\alpha_2+\alpha_4}(\cdot)\,\rangle$$

one can see that Z can be written in the form $Z = \rho^{-1}Z_1\rho Z_1^{-1}$ where $Z_1 \in A$. We then have

$$\begin{array}{rcl} Z_1^{-1}XZ_1 &=& Z_1^{-1}\rho\, x_{\alpha_2}(v_1)\, x_\beta(v_2)\, Y(\rho^{-1}Z_1\rho) \\ &=& \rho(\rho^{-1}\, Z_1^{-1}\, \rho)\, x_{\alpha_2}(v_1)\, x_\beta(v_2)\, Y(\rho^{-1}\, Z_1\, \rho) \\ &=& \rho x_{\alpha_2}(v_1)x_\beta(v_2)Y. \end{array}$$

Arguing similarly we can now up to conjugacy eliminate Y and this completes the proof. We emphasize once more that all the above equalities are considered modulo the central subgroup $x_{\alpha_2+\beta}(\cdot)$ of $B(F) \rtimes \langle\, \rho \,\rangle$. □

Theorem 3.6. *Let F be a separably closed field and let G be a simple simply connected algebraic group or a simple adjoint algebraic group of type D_4. Let $\phi \in \mathbf{Aut}(G)(F)$ be an outer automorphism of order 3 of G. There exists a maximal F-torus T which is invariant under ϕ. Moreover, if τ is another outer automorphism of order 3, there exists a conjugate of τ in $\mathbf{Aut}(G)(F)$ which leaves the F-torus T invariant.*

Proof. The last claim follows from the fact that over a separably closed field two maximal tori in G are always conjugate ([Bor56]). To show the existence of an invariant torus we treat the cases $\operatorname{char} F \neq 3$ and $\operatorname{char} F = 3$ separately. We may assume that G is simply connected by Lemma 3.2. We keep the above notation. Let H be the subgroup in G provided by Proposition 3.3. By construction it is ϕ-stable. Since H' has no outer automorphisms of order 3 the restriction of ϕ to H' is an inner automorphism of H' given by an element $x \in H'$.

A) Characteristic $F \neq 3$.

Note that ϕ is a semi-simple automorphism of G in characteristic different from 3, hence x is also semi-simple and therefore x is contained in a maximal torus, say S', in H'. By our construction we have $\dim S' = 2$ and $\dim S = 2$

and this implies that the torus $T = S \cdot S'$ is maximal in G and stable with respect to ϕ (because so are S and S').

B) Characteristic $F = 3$.
We keep the notation used above. Since x has order 3 it is unipotent. Therefore there are two possibilities for its Jordan normal form.
We first assume that

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.7}$$

Let $\{e_1, e_2, e_3\}$ be the basis of the vector space over F in which x has form (3.7). Let $S' \subset H'$ be a maximal torus in H' whose three weight subspaces are spanned by $e_1 + e_2,\ 2e_1 + e_2,\ 2e_1 + 2e_2 + e_3$. One easily checks that S' is stable with respect to conjugation given by x. As above we then get that the maximal torus $T = S \cdot S'$ in G is invariant with respect to ϕ.

We next assume that x is of the form

$$x = x_{\alpha_2}(1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let S' be the corresponding diagonal torus in $H' = \mathbf{SL}_3$ and let $T = S \cdot S'$. Choose a graph automorphism σ of G such that it acts trivially on $H' = G_{\{\alpha_2, \beta\}}$ and the images of ϕ and σ with respect to $G \rtimes \mathfrak{S}_3 \to \mathfrak{S}_3$ are equal. Let $\psi = \phi \circ (\sigma x)^{-1}$. It is an inner automorphism of G acting trivially on H' and hence on S'. It follows that ψ stabilizes $C_G(S') = T$. Note that the last equality follows from the fact that in general case $C_G(S')$ is a reductive group generated by T and root subgroups U_γ such that γ is orthogonal to α_2 and β and in case D_4 there are no roots orthogonal to α_2, β. Thus we showed that ψ is an inner conjugation in G given by an element say y which belongs to $N_G(T)(F)$. By our construction y commutes with S' and hence with its generators $h_{\alpha_2}(u)$ and $h_\beta(u)$. In view of (2.4) we conclude that the image $\dot{y}$ of y in the Weyl group $W = N_G(T)/T$ acts trivially on the roots α_2, β. One checks that in the Weyl group of type D_4 none of the elements acts trivially on two roots whose scalar product is -1. It follows $\dot{y} = 1$ implying $y \in T$. Thus we showed that $\phi = y\sigma x$ where $y \in T(F)$. By (2.3) we can write y in the form $y = h_{\alpha_2}(u_2)h_{\alpha_1}(u_1)h_{\alpha_3}(u_3)h_{\alpha_4}(u_4)$ where $u_i \in F^\times$. Since x commutes with σ it is easy to see that the condition that ϕ has order 3 implies $h_{\alpha_1}(u_1)h_{\alpha_3}(u_3)h_{\alpha_4}(u_4)\sigma$ also has order 3 and $h_{\alpha_2}(u_2)^3 = 1$. The latter of course implies that $u_2 = 1$. Then arguing as in Lemma 3.4 we may write $h_{\alpha_1}(u_1)h_{\alpha_3}(u_3)h_{\alpha_4}(u_4)\sigma$ in the form $t\sigma t^{-1}$ for some element $t \in T(F)$. It follows $\phi = t\sigma t^{-1}x$ or $t^{-1}\phi t = \sigma(t^{-1}xt)$. Note that $\tilde{x} = t^{-1}xt \in U_{\alpha_2}$. Let $B \subset G$ be the corresponding Borel subgroup in G. By Lemma 3.4 the element $\sigma\tilde{x}$ is conjugate in $B(F) \rtimes \mathbb{Z}/3\mathbb{Z}$ to an element of the form $\sigma x'$ where $x' = x_{\alpha_2}(u_1)x_\beta(u_2)x_{\alpha_2+\beta}(u_3)$ and u_1, u_2 are non-zero elements in F. The Jordan normal form of such unipotent element is of the shape (3.7) and this reduces the proof to the previous case. $\square$

COROLLARY 3.8. *Let G be a simple simply connected algebraic group or a simple adjoint algebraic group of type D_4 over an arbitrary field F and let ϕ be a trialitarian automorphism of G over F. There exists a finite separable field extension $\widetilde{F}$ of F such that over $\widetilde{F}$ there is a split maximal torus T which is invariant under ϕ. Moreover, if τ is another trialitarian automorphism of G over F, there exists a conjugate of τ in the automorphism group of G over a finite separable field extension of $\widetilde{F}$ which leaves the torus T invariant.*

Let G be split over F and $T \subset G$ be a split maximal torus over F. We denote the group of automorphisms of G which leave the torus T invariant by $\mathbf{Aut}(G{\supset}T)$. Let $\rho \in \mathbf{Aut}(G{\supset}T)(F)$ be a fixed trialitarian automorphism of G, for example the graph automorphism of G described in the introduction. The normalizer N of T in G is also invariant under ρ and ρ induces a well defined action $\overline{\rho}$ on N/T, which is the Weyl group W of G. This action is not inner. The group W also admits a group of outer automorphisms isomorphic to $\mathfrak{S}_3$ and we still call *trialitarian* the outer automorphisms of W of order 3 (see [Ban69], [Fra01] or [FH03] for automorphism groups of Weyl groups).

PROPOSITION 3.9.
(i) *Any trialitarian automorphism $\alpha \in \mathbf{Aut}(G)(F)$ leaving T invariant induces a trialitarian automorphism $\overline{\alpha}$ of W.*
(ii) *If two trialitarian automorphisms α and β of G over F leaving the same torus T invariant satisfy $\overline{\alpha} = \overline{\beta}$ in $\mathrm{Aut}(W)$, then $\alpha = \mathrm{Int}(t) \circ \beta$ for some element $t \in T(F)$.*

Proof. We may assume that G is adjoint and we may view trialitarian automorphisms as elements of $G \rtimes \mathfrak{S}_3$. In particular trialitarian automorphisms of G which leave T invariant are represented by elements of $N \rtimes \mathfrak{S}_3$. Let now α be represented by an element $\alpha_0 \in (N \rtimes \mathfrak{S}_3)(F)$. Let α_0' be the class of α_0 in $(W \rtimes \mathfrak{S}_3)(F) = W \rtimes \mathfrak{S}_3$ according to the last map in the exact sequence

$$1 \to T \to N \rtimes \mathfrak{S}_3 \to W \rtimes \mathfrak{S}_3 \to 1. \tag{3.10}$$

Let $Z(W) = \mathbb{Z}/2\mathbb{Z}$ be the center of W. The class of the automorphism $\overline{\alpha}$ of W induced by α is the class $\overline{\alpha} = \overline{\alpha_0'}$ of α_0' in $\mathrm{Aut}(W) \subset W/Z(W) \rtimes \mathfrak{S}_3$, hence Claim (i). For Claim (ii), the image of an element $\alpha \circ \beta^{-1}$ in W induces the trivial automorphism of W. Since the center of W has order 2 and α, β have order 3 this image is trivial and this implies $\alpha = \mathrm{Int}(t) \circ \beta$ for some $t \in T(F)$. $\square$

4. TRIALITARIAN AUTOMORPHISMS AND SYMMETRIC COMPOSITIONS

In the following sections, we recall a construction of trialitarian automorphisms given in [KMRT98, §35.B], [CKT12, Th. 4.6], and we construct explicitly invariant tori for some of these automorphisms.

Let (V, q) be a *quadratic space* over F, i.e., V is a finite-dimensional vector space over F and $q\colon V \to F$ is a quadratic form. We always assume that q is nonsingular, in the sense that the *polar bilinear form* b_q defined by

$$b_q(x, y) = q(x + y) - q(x) - q(y) \qquad \text{for } x,\, y \in V$$

has radical $\{0\}$. We also assume throughout that $\dim V$ is even.
Let $\mathbf{GO}(q)$ be the F-algebraic group of similarities of (V, q), whose group of rational points $\mathbf{GO}(q)(F)$ consists of linear maps $f\colon V \to V$ for which there exists a scalar $\mu(f) \in F^\times$, called the *multiplier* of f, such that

$$q\big(f(x)\big) = \mu(f)q(x) \qquad \text{for all } x \in V.$$

Let also $\mathbf{O}(q)$ be the F-algebraic group of isometries of (V, q), i.e., the kernel of the multiplier map $\mu\colon \mathbf{GO}(q) \to \mathbf{G}_m$. The center of $\mathbf{GO}(q)$ is the multiplicative group $\mathbf{G}_m$, whose rational points are viewed as homotheties. For $f \in \mathbf{GO}(q)(F)$, we let $[f]$ be the image of f in $\mathbf{PGO}(q)(F) = \mathbf{GO}(q)(F)/\,\mathbf{G}_m(F)$. For simplicity, we write

$$\mathrm{GO}(q) = \mathbf{GO}(q)(F) \qquad \text{and} \qquad \mathrm{PGO}(q) = \mathbf{PGO}(q)(F) = \mathrm{GO}(q)/F^\times.$$

Let $C(V, q)$ be the Clifford algebra of the quadratic space (V, q) and let $C_0(V, q)$ be the even Clifford algebra. We let σ be the canonical involution of $C(V, q)$, such that $\sigma(x) = x$ for $x \in V$, and use the same notation for its restriction to $C_0(V, q)$. Every similarity $f \in \mathrm{GO}(q)$ induces an automorphism $C_0(f)$ of $(C_0(V, q), \sigma)$ such that

$$C_0(f)(xy) = \mu(f)^{-1} f(x)f(y) \qquad \text{for } x,\, y \in V, \tag{4.1}$$

see [KMRT98, (13.1)]. This automorphism depends only on the image $[f] = fF^\times$ of f in $\mathrm{PGO}(q)$, and we shall use the notation $C_0[f]$ for $C_0(f)$. The similarity f is *proper* if $C_0[f]$ fixes the center of $C_0(V, q)$ and *improper* if it induces a nontrivial automorphism of the center of $C_0(V, q)$ (see [KMRT98, (13.2)]). Proper similarities define an algebraic subgroup $\mathbf{GO}^+(q)$ in $\mathbf{GO}(q)$, and we let $\mathbf{PGO}^+(q) = \mathbf{GO}^+(q)/\,\mathbf{G}_m$, a subgroup of $\mathbf{PGO}(q)$. The groups $\mathbf{GO}^+(q)$ and $\mathbf{PGO}^+(q)$ are the connected components of the identity in $\mathbf{GO}(q)$ and $\mathbf{PGO}(q)$ respectively, see [KMRT98, §23.B]. We write $\mathbf{O}^+(q)$ for the algebraic group of proper isometries of (V, q).

The F-rational points of the F-algebraic group $\mathbf{Spin}(q)$ are given by

$$\mathbf{Spin}(q)(F) = \mathrm{Spin}(q) = \{c \in C_0(V, q)^\times \mid cVc^{-1} \subset V \quad \text{and} \quad c\sigma(c) = 1\},$$

see for example [KMRT98, §35.C]. For any element $c \in \mathrm{Spin}(q)$ the linear map $x \mapsto cxc^{-1}$, $x \in V$, is a proper similitude of (V, q). Thus there exists a morphism of algebraic groups $\pi\colon \mathbf{Spin}(q) \to \mathbf{PGO}^+(q)$. In the next proposition we collect known results about the algebraic groups $\mathbf{Spin}(q)$ and $\mathbf{PGO}^+(q)$, see for example [KMRT98, 26.A].

Proposition 4.2. *Let (V, q) be a quadratic space and let* $\dim V = 2n$ *with* $n \equiv 0 \pmod 2$.

(i) *The algebraic group* $\mathbf{Spin}(q)$ *is simple, simply connected of type* D_n *and its center is isomorphic to* $\boldsymbol{\mu}_2^2 = \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$*, with* $\boldsymbol{\mu}_2(F) = \{\pm 1\}$.

(ii) *The algebraic group* $\mathbf{PGO}^+(q)$ *is simple, adjoint of type* D_n.

(iii) *The sequence of algebraic groups*

$$1 \to \boldsymbol{\mu}_2^2 \to \mathbf{Spin}(q) \xrightarrow{\pi} \mathbf{PGO}^+(q) \to 1$$

is exact.

Following [KMRT98, §35.B] or [CKT12, §4] we next recall how trialitarian automorphisms can be generated with the help of *symmetric compositions*, which we define next, following [KMRT98, §34] or[3] [CKT12, §3].

DEFINITION 4.3. A *symmetric composition* on a nonsingular quadratic space (S, n) is a bilinear map $\star\colon S \times S \to S$ such that for all x, y, $z \in S$

(i) $n(x \star y) = n(x)n(y)$,

(ii) $b_n(x \star y, z) = b_n(x, y \star z)$,

(iii) $x \star (y \star x) = n(x)y = (x \star y) \star x$.

If $\star$ is a symmetric composition on (S, n), the triple $(S, \star, n)$ is called a *symmetric composition algebra*. The composition $\star$ is its *multiplication*.

Symmetric composition algebras exist in dimension 2, 4, and 8. Their norms are Pfister forms. From now on, we restrict to dimension 8.

THEOREM 4.4. *Let* $(S, \star, n)$ *be an* 8*-dimensional symmetric composition algebra. Let* f *be a similitude of* n *with multiplier* $\mu(f)$.

(i) *If* f *is proper there exist proper similitudes* f_1, f_2 *of* n *such that for all* x, $y \in S$

$$\begin{aligned}
\mu(f)^{-1} f(x \star y) &= f_2(x) \star f_1(y)\\
\mu(f_1)^{-1} f_1(x \star y) &= f(x) \star f_2(y)\\
\mu(f_2)^{-1} f_2(x \star y) &= f_1(x) \star f(y).
\end{aligned}$$

(ii) *If* f *is improper there exist improper similitudes* f_1, f_2 *of* n *such that for all* x, $y \in S$

$$\begin{aligned}
\mu(f)^{-1} f(x \star y) &= f_2(y) \star f_1(x)\\
\mu(f_1)^{-1} f_1(x \star y) &= f(y) \star f_2(x)\\
\mu(f_2)^{-1} f_2(x \star y) &= f_1(y) \star f(x).
\end{aligned}$$

The pair (f_1, f_2) *is determined by* f *up to a factor* (μ, μ^{-1}), $\mu \in F^\times$*, and we have* $\mu(f)\mu(f_1)\mu(f_2) = 1$*. Furthermore any of the three formulas in* (i) *(resp.* (ii)*) implies the two others.*

For a proof, see [KMRT98, (35.4)] or [CKT12, Th. 4.5].

[3]The symmetric compositions considered in [CKT12] are slightly more general; the ones we use here are referred to as *normalized* in [CKT12].

In view of Theorem 4.4, the elements $[f_1]$, $[f_2]$ in $\mathbf{PGO}^+(n)(F)$ are uniquely determined by $[f]$. Thus they give rise to well-defined maps

$$\begin{aligned} \rho_\star \colon \mathbf{PGO}^+(n)(F) &\to \mathbf{PGO}^+(n)(F), & [f] &\mapsto [f_2], \\ \hat{\rho}_\star \colon \mathbf{PGO}^+(n)(F) &\to \mathbf{PGO}^+(n)(F), & [f] &\mapsto [f_1] \end{aligned}$$

THEOREM 4.5. *The mappings $\rho_\star$ and $\hat{\rho}_\star$ are outer automorphisms over F of order 3 of $\mathbf{PGO}^+(n)$, and $\hat{\rho}_\star = \rho_\star^2$.*

Proof. See [KMRT98, (35.6)]. □

The algebraic group $\mathbf{Spin}(n)$ can be described with the help of a symmetric composition $\star$ on (S, n):

PROPOSITION 4.6. *Let $(S, \star, n)$ be a symmetric composition algebra with norm n. Then*

$$\mathbf{Spin}(n)(F) = \{(f, f_1, f_2) \mid f_i \in \mathbf{O}^+(n)(F),\ f(x \star y) = f_2(x) \star f_1(y),\ x, y \in S\}.$$

Moreover any of the three relations

$$\begin{aligned} f(x \star y) &= f_2(x) \star f_1(y) \\ f_1(x \star y) &= f(x) \star f_2(y) \\ f_2(x \star y) &= f_1(x) \star f(y) \end{aligned}$$

implies the two others.

Proof. See [KMRT98, (35.2)] for a proof in characteristic different from 2 and [KT13] for a proof in any characteristic. □

We have an obvious trialitarian automorphism $\rho_\star$ of $\mathbf{Spin}(n)$:

$$\rho_\star \colon (f, f_1, f_2) \mapsto (f_2, f, f_1)$$

and properties similar to those for $\mathbf{PGO}^+(n)$ hold.

Viewing $\boldsymbol{\mu}_2^2$ as kernel of the multiplication map

$$p \colon \boldsymbol{\mu}_2^3 \to \boldsymbol{\mu}_2,\ p(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \varepsilon_1\varepsilon_2\varepsilon_3,$$

we get a natural $\mathfrak{A}_3$-action on the center $\boldsymbol{\mu}_2^2$ of $\mathbf{Spin}(n)$, compatible with the trialitarian action on $\mathbf{Spin}(n)$ and the sequence

$$1 \to \boldsymbol{\mu}_2^2 \to \mathbf{Spin}(n) \to \mathbf{PGO}^+(n) \to 1$$

is $\mathfrak{A}_3$-equivariant (see [KMRT98, (35.13)]). Here $\mathfrak{A}_3 \subset \mathfrak{S}_3$ is the subgroup of even permutations. In particular the trialitarian action on $\mathbf{Spin}(n)$ is a lift of the trialitarian action on $\mathbf{PGO}^+(n)$.

Isomorphisms of symmetric compositions are defined in the obvious way. The trialitarian automorphisms associated to isomorphic symmetric compositions are conjugate; more precisely, we have:

LEMMA 4.7. *Let $\phi \colon (S, \diamond, n) \xrightarrow{\sim} (S, \star, n)$ be an isomorphism of symmetric composition algebras. Then*

$$\rho_\star = \mathrm{Int}(\phi) \circ \rho_\diamond \circ \mathrm{Int}(\phi)^{-1}.$$

Proof. See [CKT12, Proposition 6.1]. □

The main result of [CKT12] is the following:

THEOREM 4.8. *Let (S, n) be a 3-Pfister quadratic space over a field F. The assignment $\star \mapsto \rho_\star$ defines a one-to-one correspondence between isomorphism classes of symmetric compositions on (S, n) and conjugacy classes over F of trialitarian automorphisms of $\mathbf{PGO}^+(n)$ defined over F.*

Proof. This follows from Theorems 5.8 and 6.4 of [CKT12]. □

Automorphisms of order 3 of a given symmetric composition lead to new symmetric compositions by "twisting":

PROPOSITION 4.9. *Let $(S, \star, n)$ be a symmetric composition algebra and let ϕ be an automorphism of order 3 of $(S, \star, n)$.*

(i) *The multiplication*

$$(\xi, \eta) \mapsto \xi \star_\phi \eta = \phi(\xi) \star \phi^2(\eta), \quad \text{for } \xi,\ \eta \in S$$

defines a symmetric composition on the quadratic space (S, n).

(ii) $\rho_{\star_\phi} = \operatorname{Int}(\phi^{-1}) \circ \rho_\star$.

Proof. See [Pet69] for the first claim and [CKT12, Lemma 5.2] for the second. □

The symmetric composition $\star_\phi$ is called the *(Petersson) twist* of $\star$.

5. ZORN MATRICES

In this section we use Zorn's nice description of the split octonion algebra in [Zor30] to give two examples of 8-dimensional symmetric compositions. The associated trialitarian automorphisms play a fundamental rôle in this work. The *Zorn matrix algebra* is defined as follows. Let $\bullet$ be the usual scalar product on the 3-dimensional space F^3 and let $\times$ the vector product: for vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3) \in F^3$, we have $\vec{a} \bullet \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$ and $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$. The *Zorn algebra* is the set of matrices

$$C = \left\{ \begin{pmatrix} \alpha & \vec{a} \\ \vec{b} & \beta \end{pmatrix} \middle| \, \alpha, \beta \in F,\ \vec{a}, \vec{b} \in F^3 \right\}$$

with the product

$$\begin{pmatrix} \alpha & \vec{a} \\ \vec{b} & \beta \end{pmatrix} \cdot \begin{pmatrix} \gamma & \vec{c} \\ \vec{d} & \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma + \vec{a} \bullet \vec{d} & \alpha\vec{c} + \delta\vec{a} - \vec{b} \times \vec{d} \\ \gamma\vec{b} + \beta\vec{d} + \vec{a} \times \vec{c} & \beta\delta + \vec{b} \bullet \vec{c} \end{pmatrix}, \tag{5.1}$$

the norm

$$n \begin{pmatrix} \alpha & \vec{a} \\ \vec{b} & \beta \end{pmatrix} = \alpha\beta - \vec{a} \bullet \vec{b},$$

and the conjugation

$$\overline{\begin{pmatrix} \alpha & \vec{a} \\ \vec{b} & \beta \end{pmatrix}} = \begin{pmatrix} \beta & -\vec{a} \\ -\vec{b} & \alpha \end{pmatrix},$$

which is such that $\xi \cdot \overline{\xi} = \overline{\xi} \cdot \xi = n(\xi)$ for all $\xi \in C$ (see[4] [Zor30, p. 144]). The element $1 = \begin{pmatrix} 1 & \vec{0} \\ \vec{0} & 1 \end{pmatrix}$ is an identity element for the product. One checks that the norm is *multiplicative*:

$$n(\xi \cdot \eta) = n(\xi)n(\eta) \text{ for all } \xi,\ \eta \text{ in } C. \tag{5.2}$$

Thus $(C, \cdot, n)$ is an 8-dimensional composition algebra with identity. Since the norm n is obviously a hyperbolic form, $(C, \cdot, n)$ is a split octonion algebra.

Let $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ be the standard basis of F^3. The elements

(5.3)
$$e_1 = \begin{pmatrix} 1 & \vec{0} \\ \vec{0} & 0 \end{pmatrix},\ f_1 = \begin{pmatrix} 0 & \vec{0} \\ \vec{0} & 1 \end{pmatrix},\ u_i = \begin{pmatrix} 0 & -\vec{a}_i \\ \vec{0} & 0 \end{pmatrix},\ v_i = \begin{pmatrix} 0 & \vec{0} \\ \vec{a}_i & 0 \end{pmatrix},\ i = 1, 2, 3,$$

form a hyperbolic basis, called the *canonical basis* of the Zorn algebra $(C, \cdot, n)$. We have

$$\overline{e_1} = f_1, \quad \text{and} \quad \overline{u_i} = -u_i,\ \overline{v_i} = -v_i \quad \text{for } i = 1,\ 2,\ 3.$$

Next we associate to $(C, \cdot, n)$ a symmetric composition. More generally, let $(H, \cdot, n)$ be a Hurwitz algebra (i.e., a quadratic algebra, a quaternion algebra or an octonion algebra) with conjugation $x \mapsto \overline{x}$, $x \in H$.

LEMMA 5.4. *Setting $x \diamond y = \overline{x} \cdot \overline{y}$ we get a symmetric composition $(H, \diamond, n)$.*

Proof. See for example [KMRT98, §34A]. □

We call $(H, \diamond, n)$ the *para-Hurwitz algebra* (resp. *para-quadratic, para-quaternion* or *para-octonion algebra*) associated with H. Applying the same construction to the Zorn algebra, we obtain the *para-Zorn algebra* with multiplication

$$\begin{pmatrix} \alpha & \vec{a} \\ \vec{b} & \beta \end{pmatrix} \diamond \begin{pmatrix} \gamma & \vec{c} \\ \vec{d} & \delta \end{pmatrix} = \begin{pmatrix} \beta\delta + \vec{a} \bullet \vec{d} & -\beta\vec{c} - \gamma\vec{a} - \vec{b} \times \vec{d} \\ -\delta\vec{b} - \alpha\vec{d} + \vec{a} \times \vec{c} & \alpha\gamma + \vec{b} \bullet \vec{c} \end{pmatrix}. \tag{5.5}$$

As we shall see in Remark 8.11 the trialitarian automorphism associated to the para-Zorn algebra is conjugate to the graph automorphism described in the introduction.

The group $\mathbf{GL}_3$ acts on the vector space C of Zorn matrices by

$$s_g \begin{pmatrix} \alpha & \vec{a} \\ \vec{b} & \beta \end{pmatrix} = \begin{pmatrix} \alpha & g(\vec{a}) \\ g^\sharp(\vec{b}) & \beta \end{pmatrix}, \tag{5.6}$$

where $g \in \mathbf{GL}_3(F)$ and $g^\sharp = (g^{-1})^t$.

PROPOSITION 5.7. (i) *The mappings s_g for $g \in \mathbf{SL}_3(F)$ are automorphisms of the para-Zorn algebra $(C, \diamond, n)$.*

(ii) *The mappings s_g for $g \in \mathbf{GL}_3(F)$ are isometries of (C, n).*

[4] Zorn's definition is different, but isomorphic under the map $\left(\begin{smallmatrix} \alpha & \vec{a} \\ \vec{b} & \beta \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} \alpha & -\vec{a} \\ \vec{b} & \beta \end{smallmatrix}\right)$.

Proof. Let $g \in \mathbf{GL}_3(F)$. For $\vec{x}$, $\vec{y} \in F^3$ we have $g(\vec{x}) \bullet \vec{y} = \vec{x} \bullet g^t(\vec{y})$, hence

$$g(\vec{x}) \bullet g^\sharp(\vec{y}) = \vec{x} \bullet \vec{y}. \tag{5.8}$$

This readily implies (ii). To prove (i), observe that $\vec{x} \times \vec{y}$ is characterized by the condition that for all $\vec{z} \in F^3$

$$[(\vec{x} \times \vec{y}) \bullet \vec{z}]\, \vec{a}_1 \wedge \vec{a}_2 \wedge \vec{a}_3 = \vec{x} \wedge \vec{y} \wedge \vec{z} \quad \text{in } \textstyle\bigwedge^3 F^3.$$

For all $\vec{x}$, $\vec{y}$, $\vec{z} \in F^3$ we have

$$(\det g)\, \vec{x} \wedge \vec{y} \wedge \vec{z} = g(\vec{x}) \wedge g(\vec{y}) \wedge g(\vec{z}),$$

hence

$$(\det g)\, (\vec{x} \times \vec{y}) \bullet \vec{z} = \big(g(\vec{x}) \times g(\vec{y})\big) \bullet g(\vec{z}) = g^t\big(g(\vec{x}) \times g(\vec{y})\big) \bullet \vec{z}.$$

Since these equations hold for all $\vec{z} \in F^3$ it follows that

$$(\det g)\, \vec{x} \times \vec{y} = g^t\big(g(\vec{x}) \times g(\vec{y})\big) \quad \text{for all } \vec{x},\, \vec{y} \in F^3,$$

hence

$$(\det g)\, g^\sharp(\vec{x} \times \vec{y}) = g(\vec{x}) \times g(\vec{y}) \quad \text{for all } \vec{x},\, \vec{y} \in F^3. \tag{5.9}$$

A straightforward computation then yields (i). □

Let π be the cyclic permutation of the standard basis vectors $\pi\colon \vec{a}_1 \mapsto \vec{a}_2 \mapsto \vec{a}_3 \mapsto \vec{a}_1$, extended by linearity to F^3, i.e.,

$$\pi = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \mathbf{SL}_3(F). \tag{5.10}$$

The symmetric composition algebra $(C, \diamond_{s_\pi}, n)$ obtained by twisting the para-Zorn algebra $(C, \diamond, n)$ with the automorphism s_π is called the *split Okubo algebra*[5]. We let $\triangle$ denote its multiplication. Thus

$$\begin{pmatrix} \alpha & \vec{a} \\ \vec{b} & \beta \end{pmatrix} \triangle \begin{pmatrix} \gamma & \vec{c} \\ \vec{d} & \delta \end{pmatrix} = \\ \begin{pmatrix} \beta\delta + \pi(\vec{a}) \bullet \pi^2(\vec{d}) & -\beta\pi^2(\vec{c}) - \gamma\pi(\vec{a}) - \pi(\vec{b}) \times \pi^2(\vec{d}) \\ -\delta\pi(\vec{b}) - \alpha\pi^2(\vec{d}) + \pi(\vec{a}) \times \pi^2(\vec{c}) & \alpha\gamma + \pi(\vec{b}) \bullet \pi^2(\vec{c}) \end{pmatrix}.$$

The multiplication table of $\triangle$ with respect to the canonical basis (5.3) is

[5]Okubo ([Oku78]) gave a different description of this algebra, valid over fields of characteristic not 3, see also Proposition 8.5 in this paper.

(5.11)

$\triangle$	e_1	f_1	u_1	v_1	u_2	v_2	u_3	v_3
e_1	f_1	0	0	$-v_3$	0	$-v_1$	0	$-v_2$
f_1	0	e_1	$-u_3$	0	$-u_1$	0	$-u_2$	0
u_1	$-u_2$	0	v_1	0	$-v_3$	0	0	$-e_1$
v_1	0	$-v_2$	0	u_1	0	$-u_3$	$-f_1$	0
u_2	$-u_3$	0	0	$-e_1$	v_2	0	$-v_1$	0
v_2	0	$-v_3$	$-f_1$	0	0	u_2	0	$-u_1$
u_3	$-u_1$	0	$-v_2$	0	0	$-e_1$	v_3	0
v_3	0	$-v_1$	0	$-u_2$	$-f_1$	0	0	u_3

If the field F contains a primitive cube root ω of 1 (so that in particular F has characteristic different from 3), there is another way to twist the multiplication of the para-Zorn algebra to get the split Okubo algebra: let $\overline{\omega} = \mathrm{Diag}(1, \omega, \omega^2) \in \mathbf{SL}_3(F)$ and let $s_{\overline{\omega}}$ be the corresponding automorphism of $(C, \diamond, n)$, see Proposition 5.7. We may then consider the Petersson twist $\blacktriangledown = \diamond_{s_{\overline{\omega}}}$. The next lemma shows that the symmetric composition $\blacktriangledown$ is isomorphic to $\triangle$.

LEMMA 5.12. *Let*

$$g = \frac{\omega - \omega^2}{3} \begin{pmatrix} 1 & 1 & 1 \\ -1 & -\omega & -\omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \in \mathbf{SL}_3(F).$$

The isometry s_g of (C, n) is an isomorphism of symmetric compositions $\triangle \xrightarrow{\sim} \blacktriangledown$.

Proof. The matrix g satisfies $\overline{\omega} g = g\pi$, hence also $\overline{\omega}^\sharp g^\sharp = g^\sharp \pi^\sharp = g^\sharp \pi$. The lemma is then verified by straightforward computations, using the identities (5.8), (5.9). $\square$

By contrast, the para-Zorn multiplication $\diamond$ and the split Okubo multiplication $\triangle$ are not isomorphic, even if the base field F is algebraically closed. This can be seen directly because the para-Zorn algebra contains a para-unit (see Definition 9.1 below), whereas the split Okubo algebra does not.

6. A MAXIMAL TORUS FOR $\mathbf{PGO}_8^+$ AND $\mathbf{Spin}_8$

Let G denote one of the algebraic groups $\mathbf{PGO}_8^+$ or $\mathbf{Spin}_8$. In view of Theorem 3.6, to classify trialitarian automorphisms over a separably closed field, we may fix a maximal torus T and consider only trialitarian automorphisms of G which leave T invariant. We now describe a special maximal split torus which is invariant under the trialitarian automorphisms associated to the para-Zorn algebra $(C, \diamond, n)$ and the split Okubo algebra $(C, \triangle, n)$.
Consider the following torus

$$T_0 = \mathbf{G}_m^5 = \mathbf{G}_m^2 \times \mathbf{Diag}_3 \tag{6.1}$$

where $\mathbf{Diag}_3$ is the torus of diagonal matrices in $\mathbf{GL}_3$. The torus T_0 acts by similitudes on the quadratic space (C, n) of Zorn matrices as follows:

$$(\lambda, \mu, \mathbf{t}) \circledast \begin{pmatrix} \alpha & \vec{a} \\ \vec{b} & \beta \end{pmatrix} = \begin{pmatrix} \lambda\alpha & \lambda\mu\mathbf{t}^{-1}\vec{a} \\ \mathbf{t}\vec{b} & \mu\beta \end{pmatrix}. \tag{6.2}$$

The group T_0 is a maximal torus of $\mathbf{GO}_8^+$ $(= \mathbf{GO}^+(n))$, and $\mathbf{G}_m$ diagonally embedded into T_0 is the center of $\mathbf{GO}_8^+$. Thus $T_0/\,\mathbf{G}_m$ is a maximal torus of $\mathbf{PGO}_8^+$, which we denote by T. For $(\lambda, \mu, \mathbf{t}) \in T_0(F)$, we write $[\lambda, \mu, \mathbf{t}]$ for $[(\lambda, \mu, \mathbf{t})] \in T(F)$. We compute the action on T of the trialitarian automorphism $\rho_\diamond$ associated to the para-Zorn algebra $(C, \diamond, n)$:

LEMMA 6.3. *Let $(\lambda, \mu, \mathbf{t}) \in T_0(F)$. If λ', μ', λ'', $\mu'' \in F^\times$ are such that*

$$\mu'\mu'' = \lambda, \quad \lambda'\lambda'' = \mu, \quad \text{and} \quad \lambda'\mu'' = (\det \mathbf{t})(\lambda\mu)^{-1}, \tag{6.4}$$

then for ξ, $\eta \in C$, we have

$$(\lambda, \mu, \mathbf{t}) \circledast (\xi \diamond \eta) = \big((\lambda', \mu', \mu''^{-1}\mathbf{t}) \circledast \xi\big) \diamond \big((\lambda'', \mu'', \lambda'^{-1}\mathbf{t}) \circledast \eta\big);$$

hence

$$\rho_\diamond[\lambda, \mu, \mathbf{t}] = [\lambda', \mu', \mu''^{-1}\mathbf{t}] \quad \text{and} \quad \rho_\diamond^2[\lambda, \mu, \mathbf{t}] = [\lambda'', \mu'', \lambda'^{-1}\mathbf{t}].$$

The lemma is verified by a straightforward computation. It shows that the torus T is invariant under $\rho_\diamond$, provided a solution to (6.4) exists for all $(\lambda, \mu, \mathbf{t}) \in T_0(F)$. This is indeed the case:

PROPOSITION 6.5. *For $(\lambda, \mu, \mathbf{t}) \in T_0(F)$,*

$$\rho_\diamond[\lambda, \mu, \mathbf{t}] = [\det \mathbf{t}, \lambda^2\mu, \lambda\mu\mathbf{t}] = [(\lambda\mu)^{-1}\det \mathbf{t}, \lambda, \mathbf{t}]$$

and

$$\rho_\diamond^2[\lambda, \mu, \mathbf{t}] = [\mu(\det \mathbf{t})^{-1}, (\lambda\mu)^{-1}, (\det \mathbf{t})^{-1}\mathbf{t}] = [\mu, (\lambda\mu)^{-1}\det \mathbf{t}, \mathbf{t}].$$

Proof. Check that the equations (6.4) have the following solution:

$$\lambda' = \det \mathbf{t}, \quad \mu' = \lambda^2\mu, \quad \lambda'' = \mu(\det \mathbf{t})^{-1}, \quad \mu'' = (\lambda\mu)^{-1}.$$

□

We next show that the torus T is also stable under the trialitarian automorphism $\rho_\triangle$ associated to the split Okubo algebra $(C, \triangle)$. We recall that the multiplication $\triangle$ is the Petersson twist $\diamond_{s_\pi}$ of the multiplication $\diamond$ by the automorphism s_π of the para-Zorn algebra.

PROPOSITION 6.6. *For $(\lambda, \mu, \mathbf{t}) \in T_0(F)$ we have*

$$s_\pi^{-1} \circ (\lambda, \mu, \mathbf{t}) \circ s_\pi = (\lambda, \mu, \pi^{-1}\mathbf{t}\pi),$$

and

$$\rho_\triangle[\lambda, \mu, \mathbf{t}] = [(\lambda\mu)^{-1}\det \mathbf{t}, \lambda, \pi^{-1}\mathbf{t}\pi], \quad \rho_\triangle^2[\lambda, \mu, \mathbf{t}] = [\mu, (\lambda\mu)^{-1}\det \mathbf{t}, \pi^{-2}\mathbf{t}\pi^2].$$

Proof. The first equation is easily checked by direct computation. The others follow from Proposition 4.9 and Proposition 6.5. □

For the algebraic group $\mathbf{Spin}_8$ we consider the torus

$$T_0' = \mathbf{G}_m^4 = \mathbf{G}_m \times \mathbf{Diag}_3$$

as a subtorus of the torus T_0 defined in (6.1) through the embedding $(\lambda, \mathbf{t}) \mapsto (\lambda, \lambda^{-1}, \mathbf{t})$. The torus T_0' is a maximal torus of $\mathbf{O}_8^+$ through the action $\circledast$ defined in (6.2). The spinor norm of the element $(\lambda, \mathbf{t})$ is the class of $\lambda \det \mathbf{t}$ in $F^\times/F^{\times 2}$. Thus $(\lambda, \mathbf{t})$ lifts to an element in $\mathbf{Spin}_8(F)$ if and only if the product $\lambda \det \mathbf{t}$ is a square in F. By Lemma 6.3, the equation

$$(\lambda, \mathbf{t}) \circledast (\xi \diamond \eta) = \big((\lambda', \mathbf{t}') \circledast \xi\big) \diamond \big((\lambda'', \mathbf{t}'') \circledast \eta\big)$$

holds for all $\xi, \eta \in C$ if

$$\lambda\lambda'\lambda'' = 1, \quad \mathbf{t}' = \lambda''\mathbf{t}, \quad \mathbf{t}'' = \lambda'^{-1}\mathbf{t}, \quad \text{and} \quad \lambda'\lambda''^{-1} = \det \mathbf{t}. \tag{6.7}$$

These equations imply

$$\lambda'^2 = \lambda^{-1} \det \mathbf{t} \quad \text{and} \quad \lambda''^2 = (\lambda \det \mathbf{t})^{-1}. \tag{6.8}$$

Thus the set of triples $\big((\lambda, \mathbf{t}), (\lambda', \mathbf{t}'), (\lambda'', \mathbf{t}'')\big)$ satisfying the above conditions is a maximal torus $\widetilde{T}$ in $\mathbf{Spin}_8$.

For the twisted composition $\triangle$ we have

$$\lambda\lambda'\lambda'' = 1, \quad \mathbf{t}' = \lambda''\pi^{-1}\mathbf{t}\pi, \quad \mathbf{t}'' = \lambda'^{-1}\pi^{-2}\mathbf{t}\pi^2, \quad \text{and} \quad \lambda'\lambda''^{-1} = \det \mathbf{t} \tag{6.9}$$

instead of (6.7).

The natural coverings $\widetilde{T} \to T_0'$ and $T_0' \to T$ give natural embeddings of the character groups $X(T) \hookrightarrow X(T_0') \hookrightarrow X(\widetilde{T})$. Letting $\mathbf{t} = \mathrm{Diag}(t_1, t_2, t_3)$, we define characters $\varepsilon_1, \ldots, \varepsilon_4$ of T_0' generating $X(T_0')$ by

$$\varepsilon_1 \colon (\lambda, \mathbf{t}) \mapsto t_1, \quad \varepsilon_2 \colon (\lambda, \mathbf{t}) \mapsto t_2^{-1}, \quad \varepsilon_3 \colon (\lambda, \mathbf{t}) \mapsto t_3^{-1}, \quad \text{and} \quad \varepsilon_4 \colon (\lambda, \mathbf{t}) \mapsto \lambda.$$

View each ε_i as a character of $\widetilde{T}$ through the embedding $X(T_0') \hookrightarrow X(\widetilde{T})$. Using additive notation, it follows from (6.8) that

$$\begin{array}{rcl} \rho_\diamond(\varepsilon_1) & = & \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4) \\ \rho_\diamond(\varepsilon_2) & = & \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \varepsilon_4) \\ \rho_\diamond(\varepsilon_3) & = & \frac{1}{2}(\varepsilon_1 - \varepsilon_2 + \varepsilon_3 + \varepsilon_4) \\ \rho_\diamond(\varepsilon_4) & = & \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4). \end{array} \tag{6.10}$$

Formulas (6.10) imply that the action of $\rho_\diamond$ on the vector space $\mathbb{R}^4 = X(\widetilde{T}) \otimes_\mathbb{Z} \mathbb{R} = X(T_0') \otimes_\mathbb{Z} \mathbb{R} = X(T) \otimes_\mathbb{Z} \mathbb{R}$ is given by the matrix

$$R_\diamond = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

In view of (6.9) the action of $\rho_\triangle$ is given by

$$R_\triangle = \frac{1}{2}\begin{pmatrix} -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

Recall from [Bou81] that the simple roots of $\mathbf{Spin}_8$ are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \quad \alpha_3 = \varepsilon_3 - \varepsilon_4, \quad \text{and} \quad \alpha_4 = \varepsilon_3 + \varepsilon_4.$$

Let also $\widetilde{\alpha} = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ be the highest root.

Corollary 6.11. *The action of $\rho_\diamond$ on $X(\widetilde{T})$ permutes the simple roots as follows:*

$$\rho_\diamond\colon \alpha_1 \mapsto \alpha_3, \quad \alpha_3 \mapsto \alpha_4, \quad \alpha_4 \mapsto \alpha_1, \quad \alpha_2 \mapsto \alpha_2.$$

The action of $\rho_\triangle$ is given by

$$\alpha_1 \mapsto \alpha_3 + \alpha_2, \quad \alpha_3 \mapsto \alpha_4 + \alpha_2, \quad \alpha_4 \mapsto \alpha_1 + \alpha_2, \quad \alpha_2 \mapsto -\widetilde{\alpha}.$$

Proof. This is readily verified by computation using the matrices $R_\diamond$ and $R_\triangle$. □

Remark 6.12. Viewing the Weyl group W of $\mathbf{Spin}_8$ as the subgroup of the real orthogonal group $\mathbf{O}_4(\mathbb{R})$ generated by the reflections with respect to the roots α_1, α_2, α_3, α_4, we conclude from the above discussion that the trialitarian automorphism $\overline{\rho_\diamond}$ (resp. $\overline{\rho_\triangle}$) of W induced by $\rho_\diamond$ (resp. $\rho_\triangle$) is given by conjugation in $\mathbf{O}_4(\mathbb{R})$ by the matrix $R_\diamond$ (resp. $R_\triangle$). In particular the action $\overline{\rho}$ induced by the graph automorphism ρ (see the introduction) is the same as $\overline{\rho_\diamond}$.

7. Trialitarian automorphisms of $\mathbf{PGO}_8^+$

In this section we show that every trialitarian automorphism of[6] $\mathbf{PGO}_8^+$ over an algebraically closed field is conjugate to either $\rho_\diamond$ or $\rho_\triangle$, which we call the *standard trialitarian automorphisms.*

We denote by $\mathbf{Aut}(\mathbf{PGO}_8^+ \supset T)$ the subgroup of $\mathbf{Aut}(\mathbf{PGO}_8^+)$ consisting of automorphisms which map the maximal split torus T introduced in Section 6 to itself. It follows from Theorem 3.6 that every trialitarian automorphism of $\mathbf{PGO}_8^+$ is conjugate over a separable closure to an automorphism which belongs to $\mathbf{Aut}(\mathbf{PGO}_8^+ \supset T)$. Let $N_0 \subset \mathbf{GO}_8^+$ be the normalizer of the torus T_0 (see (6.1)) and let $N = N_0/\mathbf{G}_m \subset \mathbf{PGO}_8^+$. Let further

$$W = N_0/T_0 = N/T$$

be the Weyl group of $\mathbf{PGO}_8^+$. For $\alpha \in \mathbf{Aut}(\mathbf{PGO}_8^+ \supset T)(F)$, we have obviously $\alpha(N) = N$, hence α induces an automorphism $\overline{\alpha} \in \mathrm{Aut}(W)$ defined by

$$\overline{\alpha}\big([g]\cdot T\big) = \alpha\big([g]\big)\cdot T \qquad \text{for } g \in N_0.$$

In particular, for $g \in N_0(F)$, we have $\mathrm{Int}\,\big([g]\big) \in \mathbf{Aut}(\mathbf{PGO}_8^+ \supset T)(F)$, and

$$\overline{\mathrm{Int}\,\big([g]\big)} = \mathrm{Int}(gT_0).$$

[6]We could as well have discussed the case of $\mathbf{Spin}_8$.

By Proposition 3.9, $\overline{\rho_\diamond}$ and $\overline{\rho_\Delta}$ are outer automorphisms of order 3 of W.

Proposition 7.1. *Every outer automorphism of order* 3 *of* W *is conjugate in* $\mathrm{Aut}(W)$ *to exactly one of* $\overline{\rho_\diamond}$ *or* $\overline{\rho_\Delta} \in \mathrm{Aut}(W)$*.*

Proof. This is well-known; see for instance [KT10, Proposition 5.8] or [KT12, Theorem 5.11]. □

Proposition 7.2. *Let F be an arbitrary field and $\phi \in \mathbf{Aut}(\mathbf{PGO}_8^+)(F)$ be a trialitarian automorphism which admits an invariant maximal split torus over F. Then ϕ is conjugate in $\mathbf{Aut}(\mathbf{PGO}_8^+)(F)$ to an automorphism $\beta \in \mathbf{Aut}(\mathbf{PGO}_8^+ \supset T)(F)$ such that $\overline{\beta} = \overline{\rho_\diamond}$ or $\overline{\rho_\Delta}$.*

Proof. Two maximal split tori in $\mathbf{PGO}_8^+$ are conjugate over F. Thus ϕ is conjugate to a trialitarian automorphism $\beta \in \mathbf{Aut}(\mathbf{PGO}_8^+ \supset T)(F)$. Its image $\overline{\beta} \in \mathrm{Aut}(W)$ is well defined and is also trialitarian. Note that $N(F) \to W$ is surjective (because T is split) and therefore so is $N(F) \rtimes \mathfrak{S}_3 \to \mathrm{Aut}(W)$. By Proposition 7.1, one can find $g \in N(F) \rtimes \mathfrak{S}_3 \subset \mathbf{Aut}(\mathbf{PGO}_8^+)(F)$ such that

$$\overline{\beta} = \mathrm{Int}(gT) \circ \overline{\rho_\diamond} \circ \mathrm{Int}(gT)^{-1} \quad \text{or} \quad \overline{\beta} = \mathrm{Int}(gT) \circ \overline{\rho_\Delta} \circ \mathrm{Int}(gT)^{-1}.$$

Let $\beta_0 = \mathrm{Int}\big([g]\big)^{-1} \circ \beta \circ \mathrm{Int}\big([g]\big)$. Then β_0 is conjugate to β, and $\overline{\beta_0} = \overline{\rho_\diamond}$ or $\overline{\rho_\Delta}$. □

From now on we only consider automorphisms $\beta \in \mathbf{Aut}(\mathbf{PGO}_8^+ \supset T)(F)$ such that $\overline{\beta} = \overline{\rho_\diamond}$ or $\overline{\rho_\Delta}$.

Proposition 7.3. *Let $\beta \in \mathbf{Aut}(\mathbf{PGO}_8^+ \supset T)(F)$ be a trialitarian automorphism such that $\overline{\beta} = \overline{\rho_\Delta}$. There exists an extension $\widetilde{F} = F(\xi, \eta)$ of F, with $\xi^3 \in F^\times$ and $\eta^3 \in F^\times$, such that β and ρ_Δ are conjugate in $\mathbf{Aut}(\mathbf{PGO}_8^+)(\widetilde{F})$.*

Proof. Since $\overline{\beta} = \overline{\rho_\Delta}$, we have $\overline{\beta\rho_\Delta^{-1}} = 1_W$, thus by Proposition 3.9 we have $\beta = \mathrm{Int}\big([\lambda, \mu, \mathbf{t}]\big) \circ \rho_\Delta$ for some $(\lambda, \mu, \mathbf{t}) \in T_0(F)$. Let $\mathbf{t} = \mathrm{Diag}(t_1, t_2, t_3)$. In an algebraic closure of F, choose ξ such that $\xi^3 = \det \mathbf{t}$ and η such that $\eta^3 = \lambda^{-1}\mu t_1 t_3^2$, and let $\widetilde{F} = F(\xi, \eta)$. Let also

$$\mathbf{u} = \mathrm{Diag}(t_3, \xi^2 t_2^{-1}, \xi) \in \mathbf{Diag}_3(\widetilde{F}).$$

Consider $[\eta, \mu^{-1}\xi\eta, \mathbf{u}] \in T(\widetilde{F})$. By Proposition 6.6 we have

$$\rho_\Delta[\eta, \mu^{-1}\xi\eta, \mathbf{u}] = [(\mu^{-1}\xi\eta^2)^{-1} \det \mathbf{u}, \eta, \pi^{-1}\mathbf{u}\pi].$$

On the other hand,

$$[\eta, \mu^{-1}\xi\eta, \mathbf{u}] \cdot [\lambda, \mu, \mathbf{t}] = [\lambda\eta, \xi\eta, \mathbf{ut}],$$

and computation shows that

$$\xi(\mu^{-1}\xi\eta^2)^{-1} \det \mathbf{u} = \lambda\eta \quad \text{and} \quad \xi(\pi^{-1}\mathbf{u}\pi) = \mathbf{ut}.$$

Therefore, we have

$$[\eta, \mu^{-1}\xi\eta, \mathbf{u}] \cdot [\lambda, \mu, \mathbf{t}] \cdot \rho_\Delta[\eta, \mu^{-1}\xi\eta, \mathbf{u}]^{-1} = 1 \quad \text{in } \mathbf{PGO}_8^+(\widetilde{F}),$$

hence, letting $\tau = \mathrm{Int}\big([\eta, \mu^{-1}\xi\eta, \mathbf{u}]\big)$,

$$\tau \circ \big(\mathrm{Int}\big([\lambda, \mu, \mathbf{t}]\big) \circ \rho_\triangle\big) \circ \tau^{-1} = \rho_\triangle.$$

Since $\beta = \mathrm{Int}\big([\lambda, \mu, \mathbf{t}]\big) \circ \rho_\triangle$, this shows that β and $\rho_\triangle$ are conjugate. $\square$

We next consider the case where $\overline{\beta} = \overline{\rho_\diamond}$.

Proposition 7.4. *Let $\beta \in \mathbf{Aut}(\mathbf{PGO}_8^+ \supset T)(F)$ be such that $\overline{\beta} = \overline{\rho_\diamond}$ and $\beta^3 = 1$. If F does not contain any $\omega \neq 1$ such that $\omega^3 = 1$ (in particular if F has characteristic 3), then β is conjugate in $\mathbf{Aut}(\mathbf{PGO}_8^+)(F)$ to $\rho_\diamond$. If F contains an element $\omega \neq 1$ such that $\omega^3 = 1$, then β is conjugate to $\rho_\diamond$ or to $\rho_\triangle$.*

Proof. As in Proposition 7.3, we may assume that $\beta = \mathrm{Int}\big([\lambda, \mu, \mathbf{t}]\big) \circ \rho_\diamond$ for some $(\lambda, \mu, \mathbf{t}) \in T(F)$. Since $\beta^3 = 1$, we must have

$$\mathrm{Int}\big([\lambda, \mu, \mathbf{t}] \cdot \rho_\diamond[\lambda, \mu, \mathbf{t}] \cdot \rho_\diamond^2[\lambda, \mu, \mathbf{t}]\big) = 1,$$

hence

$$[\lambda, \mu, \mathbf{t}] \cdot \rho_\diamond[\lambda, \mu, \mathbf{t}] \cdot \rho_\diamond^2[\lambda, \mu, \mathbf{t}] = 1 \qquad \text{in} \qquad \mathbf{PGO}_8^+(F).$$

By Proposition 6.5, we have

$$[\lambda, \mu, \mathbf{t}] \cdot \rho_\diamond[\lambda, \mu, \mathbf{t}] \cdot \rho_\diamond^2[\lambda, \mu, \mathbf{t}] = [\det \mathbf{t}, \det \mathbf{t}, \mathbf{t}^3].$$

Therefore, letting $\mathbf{t} = \mathrm{Diag}(t_1, t_2, t_3)$, we see that the condition $\beta^3 = 1$ implies

$$t_1^3 = t_2^3 = t_3^3 = t_1 t_2 t_3.$$

Let $\omega = t_2 t_1^{-1}$. We then have $\omega^3 = 1$ and $t_3 = \omega^2 t_1$, hence $\mathbf{t} = t_1 \mathbf{w}$ where $\mathbf{w} = \mathrm{Diag}(1, \omega, \omega^2)$. For $\mathbf{u} = \mathrm{Diag}(\lambda, \mu, \mu)$, computations using Proposition 6.5 show that

$$[\mu, t_1, \mathbf{u}] \cdot [\lambda, \mu, t_1\mathbf{w}] \cdot \rho_\diamond[\mu, t_1, \mathbf{u}]^{-1} = [t_1, t_1, t_1\mathbf{w}] = [1, 1, \mathbf{w}].$$

Therefore, letting $\sigma = \mathrm{Int}\big([\mu, t_1, \mathbf{u}]\big)$ we have

$$\sigma \circ \big(\mathrm{Int}\big([\lambda, \mu, t_1\mathbf{w}]\big) \circ \rho_\diamond\big) \circ \sigma^{-1} = \mathrm{Int}\big([1, 1, \mathbf{w}]\big) \circ \rho_\diamond. \tag{7.5}$$

If $\omega = 1$, then $\mathbf{w} = 1$, and (7.5) shows that β is conjugate to $\rho_\diamond$. If $\omega \neq 1$, observe that $(1, 1, \mathbf{w}) \in T_0(F)$ acts on the quadratic space of Zorn matrices as the automorphism $s_{\mathbf{w}}^{-1}$ of (5.6): we have

$$[1, 1, \mathbf{w}] \circledast \xi = s_{\mathbf{w}}^{-1}(\xi) \qquad \text{for all } \xi \in C.$$

Therefore, for the Petersson twist $\blacktriangledown = \diamond_{s_{\mathbf{w}}}$ we have $\mathrm{Int}\big([1, 1, \mathbf{w}]\big) \circ \rho_\diamond = \rho_\blacktriangledown$ by Proposition 4.9(ii), and (7.5) shows that β is conjugate to $\rho_\blacktriangledown$. But we saw in Lemma 5.12 that $\blacktriangledown$ is isomorphic to $\triangle$, hence by Lemma 4.7 the trialitarian automorphisms $\rho_\blacktriangledown$ and $\rho_\triangle$ are conjugate. Therefore, β is conjugate to $\rho_\triangle$. $\square$

We get to the main result:

Theorem 7.6. *Let n be a 3-Pfister form over a field F, let G be either $\mathbf{PGO}^+(n)$ or $\mathbf{Spin}(n)$ and let $\phi \in \mathbf{Aut}(G)(F)$ be any trialitarian automorphism. There is a finite field extension $\widetilde{F}$ of F splitting G and such that ϕ is conjugate over $\widetilde{F}$ to one of the two standard trialitarian automorphisms of $\mathbf{PGO}_8^+$ or $\mathbf{Spin}_8$.*

Proof. After enlarging F to split G, the claim follows from Propositions 7.2, 7.3, 7.4 and Theorem 3.6. $\square$

Thus there are two types of trialitarian automorphisms of $\mathbf{PGO}^+(n)$ or $\mathbf{Spin}(n)$. We call those conjugate over a field extension of F to $\rho_\diamond$ *of octonion type* and the others *of Okubo type*. Using the correspondence between trialitarian automorphisms and symmetric compositions in Theorem 4.8, we readily derive the classification of 8-dimensional symmetric composition algebras, first established by Petersson [Pet69, Satz 2.7] over fields of characteristic different from 2 and 3, and by Elduque-Pérez [EP96] over arbitrary fields:

Theorem 7.7. (i) *Let F be an algebraically closed field. The para-Zorn algebra $(C, \diamond, n)$ and the split Okubo algebra $(C, \triangle, n)$ are non-isomorphic symmetric composition algebras.*

(ii) *For every 8-dimensional symmetric composition algebra $(S, \star, n)$ over an arbitrary field F there is a finite field extension $\widetilde{F}$ of F such that $(S, \star, n) \otimes_F \widetilde{F}$ is either isomorphic to the para-Zorn algebra or to the split Okubo algebra over $\widetilde{F}$.*

It follows from Theorem 7.7 that symmetric composition algebras of dimension 8 over an arbitrary base field can be divided into two types, according to their isomorphism class over an algebraic closure. Those that after scalar extension are isomorphic to the split Okubo algebra are called *Okubo algebras*. As we shall see in Theorem 8.2, those that after scalar extension are isomorphic to the para-Zorn algebra are the para-octonion algebras defined after Lemma 5.4. Thus:

Corollary 7.8. *Symmetric composition algebras of dimension 8 are either para-octonion algebras or Okubo algebras.*

8. Automorphisms of symmetric compositions

Let $(S, \star, n)$ be a symmetric composition algebra of dimension 8 with associated trialitarian automorphism $\rho_\star$. We let $\mathbf{Aut}(\star)$ or $\mathbf{Aut}(S, \star, n)$ denote the F-algebraic group of automorphisms of $\star$, whose group of F-rational points $\mathrm{Aut}(\star)$ consists of the automorphisms $f\colon \star \to \star$. This group is related to $\rho_\star$ as follows:

Proposition 8.1. *The canonical map $\mathbf{GO}^+(n) \to \mathbf{PGO}^+(n)$ induces an isomorphism from $\mathbf{Aut}(\star)$ to the subgroup $\mathbf{PGO}^+(n)^{\rho_\star}$ of $\mathbf{PGO}^+(n)$ fixed under the trialitarian automorphism $\rho_\star$.*

Proof. See [CKT12, Theorem 6.6]. $\square$

In view of Theorem 7.6 we have two types of subgroups fixed under trialitarian automorphisms, those which are automorphism groups of para-octonions algebras and those which are automorphism groups of Okubo algebras.

We next determine the type of the algebraic group $\mathbf{Aut}(\star)$, and show that this group determines the composition $\star$ up to isomorphism or anti-isomorphism.

Type I: Para-octonion algebras. Let $(C,\cdot,n)$ be an octonion algebra over an arbitrary field F, and let $\diamond$ be the corresponding para-octonion multiplication on (C,n) (see Lemma 5.4).

Theorem 8.2. (i) *The octonion algebra $(C,\cdot,n)$ and the para-octonion algebra $(C,\diamond,n)$ have the same groups of automorphisms.*

(ii) *Forms of para-octonion algebras are para-octonion algebras.*

(iii) *The group of automorphisms of any para-octonion algebra is an algebraic group of type G_2.*

Proof. The equality $\mathbf{Aut}(\diamond)=\mathbf{Aut}(\cdot)$ is proved in [KMRT98, (34.4)]. Claim (i) and (ii) now follow from the fact that algebraic groups of type G_2 are the automorphism groups of octonion algebras (see i.a. [SV00, §2.3]). □

We collect more related known results in the next proposition. Sketches of proofs are given for completeness.

Proposition 8.3. *Let $(C,\cdot,n)$ and $(C',\cdot',n')$ be octonion algebras, and let $\diamond$ resp. $\diamond'$ be the corresponding para-octonion multiplications. The following properties are equivalent:*

(i) *$(C,\cdot,n)$ and $(C',\cdot',n')$ are isomorphic.*

(ii) *$(C,\diamond,n)$ and $(C',\diamond',n')$ are isomorphic.*

(iii) *The norms n and n' are isometric.*

(iv) *The algebraic groups $\mathbf{Aut}(\cdot)$ and $\mathbf{Aut}(\cdot')$ are isomorphic.*

(v) *The algebraic groups $\mathbf{Aut}(\diamond)$ and $\mathbf{Aut}(\diamond')$ are isomorphic.*

(vi) *The algebraic groups $\mathbf{PGO}^+(n)$ and $\mathbf{PGO}^+(n')$ are isomorphic.*

Proof. The equivalence of (i) and (ii) is proved in [KMRT98, (34.4)] and the equivalence of (i) and (iii) is for example in [KMRT98, (33.19)] or [SV00, §1.7]. Claims (i) $\Longrightarrow$ (iv), (ii) $\Longrightarrow$ (v), and (iii) $\Longrightarrow$ (vi) are clear, and the equivalence of (iv) and (v) is Theorem 8.2. If $\mathbf{Aut}(\cdot)\xrightarrow{\sim}\mathbf{Aut}(\cdot')$, then $\mathbf{Aut}(\cdot)(F)\xrightarrow{\sim}\mathbf{Aut}(\cdot')(F)$ and (i) follows by [vdBS59, (2.3)]. The implication (iv) $\Longrightarrow$ (i) can also be obtained by a cohomological argument. Let $G=\mathbf{Aut}(\cdot)$ and let $G'=\mathbf{Aut}(\cdot')$. Forms of $(C,\cdot,n)$ are classified by $H^1(F,G)$ and forms of G by $H^1\big(F,\mathbf{Aut}(G)\big)$, for étale or fppf-cohomology. The algebra $(C',\cdot',n')$, being a form of $(C,\cdot,n)$, gives a cohomology class $[\xi]\in H^1(F,G)$ and G' is the twisted group ${}^{\xi}G$. If $G'\xrightarrow{\sim}G$, then the image of ξ in $H^1\big(F,\mathbf{Aut}(G)\big)$ under the map $H^1(F,G)\to H^1\big(F,\mathbf{Aut}(G)\big)$ induced by inner conjugation $\iota\colon G\to\mathbf{Aut}(G)$ is trivial. However the map ι is an isomorphism for groups of type G_2. Thus $[\xi]$ is trivial and $(C,\cdot,n)\xrightarrow{\sim}(C',\cdot',n')$. The implication (vi) $\Longrightarrow$ (iii) can also be obtained by a similar cohomological argument. Namely, let $[\xi]\in H^1\big(F,\mathbf{Aut}(\cdot)\big)$ correspond to $(C',\cdot',n')$ and assume that its image in $H^1(F,\mathbf{Aut}\left(\mathbf{PGO}^+(n)\right))$ is trivial. First we recall that $\mathbf{Aut}(\cdot)$ is a subgroup of $\mathbf{GO}^+(n)$ and that similar Pfister forms are isomorphic. Thus it suffices to consider the norm forms n

and n' up to similarity and we are reduced to show that the image of $[\xi]$ in $H^1\big(F, \mathbf{GO}^+(n)\big)$ is trivial. We next remark that the map $H^1\big(F, \mathbf{GO}^+(n)\big) \to H^1\big(F, \mathbf{PGO}^+(n)\big)$ induced by the projection has trivial kernel. This follows from the fact that in the exact sequence

$$H^1(F, \mathbf{G}_{\mathrm{m}}) \to H^1\big(F, \mathbf{GO}^+(n)\big) \to H^1\big(F, \mathbf{PGO}^+(n)\big)$$

induced by

$$1 \to \mathbf{G}_{\mathrm{m}} \to \mathbf{GO}^+(n) \to \mathbf{PGO}^+(n) \to 1,$$

the set $H^1(F, \mathbf{G}_{\mathrm{m}})$ is trivial by Hilbert 90. Let $H = \mathbf{PGO}^+(n)$. It remains to verify that $H^1(F, H) \to H^1\big(F, \mathbf{Aut}(H)\big)$ has trivial kernel. Since $\mathbf{Aut}(H)(F) = (H \rtimes \mathfrak{S}_3)(F) \to \mathfrak{S}_3(F) = \mathfrak{S}_3$ is surjective this follows from the cohomology sequence

$$\big(H \rtimes \mathfrak{S}_3\big)(F) \to \mathfrak{S}_3(F) \to H^1(F, H) \to H^1\big(F, \mathbf{Aut}(H)\big)$$

induced by the exact sequence

$$1 \to H \to H \rtimes \mathfrak{S}_3 \to \mathfrak{S}_3 \to 1.$$

Thus, the composition

$$H^1\big(F, \mathbf{Aut}(\cdot)\big) \to H^1\big(F, \mathbf{GO}^+(n)\big) \to H^1(F, H) \to H^1\big(F, \mathbf{Aut}(H)\big)$$

has trivial kernel and we are done. □

COROLLARY 8.4. *Forms of para-Zorn algebras are para-octonion algebras. In particular there is up to isomorphism a unique para-octonion algebra with given 3-Pfister form n.*

Proof. In view of Proposition 8.3 the classification of para-octonion algebras is equivalent to the classification of octonion algebras. Thus the claim follows from the known fact that forms of octonion algebras are octonion algebras. □

For Okubo algebras we treat separately fields of characteristic not 3 and fields of characteristic 3.

TYPE II: OKUBO ALGEBRAS IN CHARACTERISTIC DIFFERENT FROM 3. We distinguish two subtypes IIa and IIb, depending on whether the base field contains a primitive cube root of unity or not. Suppose first F contains a primitive cube root of unity ω. Let A be a central simple F-algebra of degree 3. For the reduced characteristic polynomial of $a \in A$, we use the notation

$$X^3 - \operatorname{Trd}(a)X^2 + \operatorname{Srd}(a)X - \operatorname{Nrd}(a)\,1,$$

so Trd is the reduced trace map on A, Srd is the reduced quadratic trace map, and Nrd is the reduced norm. Let $A^0 \subset A$ be the kernel of Trd.

PROPOSITION 8.5. *Assume that F contains a primitive cube root of unity.*

(i) *The multiplication $\star$ on A^0 given by*

$$x \star y = \frac{yx - \omega xy}{1-\omega} - \tfrac{1}{3}\operatorname{Trd}(xy)\,1 \tag{8.6}$$

together with the quadratic form

$$n(x) = -\tfrac{1}{3}\operatorname{Srd}(x). \tag{8.7}$$

define an Okubo algebra on the quadratic space (A^0, n), which is hyperbolic.

(ii) *The symmetric composition $(A^0, \star, n)$ and the algebra A have the same groups of automorphisms.*

Proof. See for instance [KMRT98, (34.19), (34.25)] and [KMRT98, (34.35)]. □

If F does not contain a primitive cube root of unity ω, we denote $K = F(\omega)$ the separable quadratic extension generated by ω. Let B be a central simple K-algebra of degree 3 with a unitary involution τ leaving F fixed and let $\operatorname{Sym}(B,\tau)^0$ be the F-vector space of τ-symmetric elements of reduced trace zero.

Proposition 8.8. *Assume that F does not contain a primitive cube root of unity ω.*

(i) *The quadratic space $(\operatorname{Sym}(B,\tau)^0, n)$ is a 3-fold Pfister quadratic space that becomes hyperbolic over K, and the restriction of the product $\star$ and of the norm n of B to $\operatorname{Sym}(B,\tau)^0$ define a symmetric composition on this space.*

(ii) *The symmetric composition $\big((\operatorname{Sym}(B,\tau)^0, \star, n\big)$ and the algebra with involution (B,τ) have the same groups of automorphisms.*

Proof. See [KMRT98, (34.35)]. □

Corollary 8.9. (i) *If F contains a primitive cube root of unity ω, Okubo algebras over F are in functorial bijective correspondence with central simple algebras over F of degree 3.*

(ii) *Let $K = F(\omega)$ if F does not contain ω. Okubo algebras over F are in functorial bijective correspondence with central simple algebras of degree 3 over K, with unitary K/F-involution.*

(iii) *The group of automorphisms of an Okubo algebra over a field F of characteristic not 3 is an algebraic group of inner type ${}^1\mathsf{A}_2$ if F contains a primitive cube root of unity ω and of outer type ${}^2\mathsf{A}_2$ if F does not contain ω. In the latter case it becomes a group of inner type over $F(\omega)$.*

Proof. It follows from Proposition 8.5 and Proposition 8.8 that the classification of Okubo algebras over F is equivalent to the classification of central simple algebras of degree 3 over F, resp. of central simple algebras of degree 3 over $F(\omega)$. In particular they have isomorphic groups of automorphisms. □

Proposition 8.10. *Two Okubo algebras $(S,\star,n)$ and $(S',\star',n')$ over a field F of characteristic not 3 are isomorphic or anti-isomorphic if and only if their groups of automorphisms $\mathbf{Aut}(\star)$ and $\mathbf{Aut}(\star')$ are isomorphic.*

Proof. In view of Corollary 8.9, the claim follows from the corresponding fact for central simple algebras, which is certainly known. However, since the only reference we could find was at the level of Lie algebras ([Jac79, Chap. X, §4]), we give here a cohomological proof in the spirit of the proof of Proposition 8.3. Assume that F contains a cube root of unity. Let A and A' be central simple F-algebras of degree 3 and let $G = \mathbf{Aut}(A)$, $G' = \mathbf{Aut}(A')$. First assume that A is split, so that $G = \mathbf{PGL}_3$ and $\mathbf{Aut}(G) = G \rtimes \mathfrak{S}_2$. Let $[\xi] \in H^1(F,G)$ be a cocycle defining $(S',\star',n')$ or equivalently A'. If $G' \xrightarrow{\sim} G$, the image of $[\xi]$ in $H^1\big(F,\mathbf{Aut}(G)\big)$ is trivial. From the exact sequence

$$G(F) \to \mathbf{Aut}(G)(F) \xrightarrow{\lambda} \mathfrak{S}_2 \xrightarrow{\nu} H^1(F,G) \to H^1\big(F,\mathbf{Aut}(G)\big)$$

we conclude that $[\xi]$ lies in the image of ν. But λ is surjective because we are in the split case, hence $[\xi] = 1$ and $A' \xrightarrow{\sim} A$. If A is not split, λ is not surjective and the kernel of $H^1(F,G) \to H^1\big(F,\mathbf{Aut}(G)\big)$ consists of two elements. Thus we have exactly two choices for $[\xi]$, resp. for the algebra A'. On the other hand we have two candidates, the algebra A and the opposite algebra A^{op}. Obviously A and A^{op} have the same group of automorphisms and they are not isomorphic: if $A \xrightarrow{\sim} A^{\mathrm{op}}$, the class of A in the Brauer group of F would have order smaller than or equal to 2. But since A is of degree 3 and is not split, its Brauer class has order 3. A similar argument can be given if F does not contain a cube root of unity. □

Type III: Okubo algebras in characteristic 3. The algebraic group of automorphisms of the split Okubo algebra in characteristic 3 is more intricate than in characteristic different from 3. In particular the group is not smooth. Its set of F-points was computed in [Eld99, Theorem 7], see also [Tit59, §10] and [GL83, (9.1)] for the computation of fixed F-points under triality. The next sections are devoted to the computation of this algebraic group of automorphisms.

Remark 8.11. We conclude this section by verifying that the graph automorphism ρ and the trialitarian automorphism $\rho_\diamond$ defined through the para-Zorn algebra $(C,\diamond,n)$ are conjugate in $\mathbf{Aut}(\mathbf{PGO}_8^+)(F)$. We know already that the induced actions on the Weyl group are conjugate (see Remark 6.12). Thus, by Proposition 7.4 ρ is conjugate to $\rho_\diamond$ or to $\rho_\triangle$, the automorphism associated to the split Okubo algebra. Since the subgroup fixed by ρ is of type G_2, while the fixed subgroup of $\rho_\triangle$ is a simple group of type A_2 if the characteristic is different from 3 (Proposition 8.10), or not smooth if the characteristic is 3 (Section 10), we obtain that ρ is conjugate to $\rho_\diamond$.

9. IDEMPOTENTS OF COMPOSITION ALGEBRAS IN CHARACTERISTIC 3

For simplicity we assume throughout this section that the base field F has characteristic 3, even if some of the results hold for arbitrary fields. Let $(H, \cdot, n)$ be a Hurwitz algebra and let $(H, \diamond, n)$ be the associated para-Hurwitz symmetric composition (see Lemma 5.4). The identity element of $(H, \cdot)$ also plays a special role for the associated para-Hurwitz algebra. It is an idempotent (i.e., $1 \diamond 1 = 1$) and satisfies $1 \diamond x = x \diamond 1 = -x$ for $x \in H$ such that $b_n(1, x) = 0$.

DEFINITION 9.1. An idempotent e of a symmetric composition $(S, \star, n)$ is called a *para-unit* if $e \star x = x \star e = -x$ for $x \in S$ such that $b_n(e, x) = 0$.

We recall that a symmetric composition is para-Hurwitz if and only if it admits a para-unit (see [KMRT98, (34.8)]).

Let now $(S, \star, n)$ be an Okubo algebra.

DEFINITION 9.2. We say that an idempotent e of $(S, \star, n)$ is *quaternionic* if e is the para-unit of a para-quaternion subalgebra of $(S, \star, n)$. We say that an idempotent e of $(S, \star, n)$ is *quadratic* if e is not quaternionic, and e is the para-unit of a para-quadratic subalgebra of $(S, \star, n)$.

We give examples of such elements in the split Okubo algebra $(C, \triangle, n)$, as described in (5.5) using Zorn matrices:

LEMMA 9.3. (i) *The Zorn matrix* $e = \begin{pmatrix} 1 & \vec{0} \\ \vec{0} & 1 \end{pmatrix}$ *is a quadratic idempotent.*

(ii) *The Zorn matrix* $\hat{e} = \begin{pmatrix} 1 & -(1,1,1) \\ (1,1,1) & 1 \end{pmatrix}$ *is a quaternionic idempotent.*

Proof. (i) The element e is the para-unit of the para-quadratic subalgebra

$$Q = \left\{ \begin{pmatrix} \alpha & \vec{0} \\ \vec{0} & \beta \end{pmatrix} \mid \alpha,\, \beta \in F \right\}.$$

Moreover e cannot be the para-unit of any para-quaternion subalgebra, as this would imply that this subalgebra (with nondegenerate norm!) would be contained in the centralizer of e. But the centralizer $\mathrm{Cent}_{(C,\triangle,n)}(e) = \{x \in C \mid x \triangle e = e \triangle x\}$ of e is the set of Zorn matrices

$$Z = \left\{ \begin{pmatrix} \alpha & (x,x,x) \\ (y,y,y) & \beta \end{pmatrix} \mid \alpha, \beta, x, y \in F \right\}$$

and the restriction of the norm to this centralizer is degenerate.
(ii) In the same vein, $\hat{e}$ is an idempotent and the set of Zorn matrices

$$H = \left\{ \begin{pmatrix} \alpha & -(x,y,z) \\ (x,y,z) & \alpha \end{pmatrix} \mid \alpha, x, y, z \in F \right\}$$

is a para-quaternion subalgebra with para-unit $\hat{e}$. □

We next give another construction of an Okubo algebra with a quaternionic idempotent:

EXAMPLE 9.4. Let $Q = \mathrm{Mat}_2(F)$ be the split quaternion algebra (multiplication denoted by juxtaposition) with conjugation:

$$\overline{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The split octonion algebra can be constructed through the Cayley-Dickson doubling process as $C = Q \oplus Qv$, with multiplication, conjugation and norm given by

$$\begin{aligned} (a+bv)\cdot(c+dv) &= (ac+\overline{d}b)+(da+b\overline{c})v, \\ \overline{a+bv} &= \overline{a}-bv, \\ n(a+bv) &= \det(a)-\det(b). \end{aligned} \tag{9.5}$$

The element $w = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ satisfies $(w-1)^2 = 0$, $w\overline{w} = 1$, $\overline{w} = w^2 = -1-w$, and the map

$$\vartheta\colon a+bv \mapsto a+(wb)v, \quad a,\, b \in Q \tag{9.6}$$

is an order 3 automorphism of $(C,\cdot)$. Let $\star = \cdot_\vartheta$ be the associated Petersson twist, i.e.,

$$x \star y = \vartheta(\overline{x})\cdot\vartheta^2(\overline{y}) \qquad \text{for } x,\, y \in C. \tag{9.7}$$

Obviously, $e = 1$ is an idempotent of the symmetric composition $(C,\star,n)$, and $(Q,\star,n)$ is a para-quaternion subalgebra, since the restriction of ϑ to Q is the identity, with para-unit e. Thus e is also quaternionic.

LEMMA 9.8. *The symmetric composition algebra $(C,\star,n)$ is an Okubo algebra.*

Proof. In view of Corollary 7.8 it suffices to show that $(C,\star,n)$ is not a para-octonion algebra. The decomposition $C = Q\oplus Qv$ is a $\mathbb{Z}_2$-grading of both $(C,\cdot)$ and $(C,\star)$. Hence the commutative center

$$K(C,\star) = \{x \in C \mid x\star y = y\star x,\ \forall y \in C\}$$

is graded: $K(C,\star) = (K(C,\star)\cap Q)\oplus(K(C,\star)\cap Qv)$. If $(C,\star,n)$ were a para-octonion algebra, its commutative center would be one-dimensional and spanned by a nonzero idempotent. Hence we would have $K(C,\star) = K(C,\star)\cap Q$, and this is contained in the commutative center of the para-quaternion algebra $(Q,\star)$, which is Fe. But $e\star v = -w^2v$ while $v\star e = -wv$. □

PROPOSITION 9.9. *If an Okubo algebra $(S,*,n_S)$ admits a quaternionic idempotent f, there is an isomorphism $\varphi\colon (S,*,n_S)\to(C,\star,n)$ such that $\varphi(f) = e$. In particular $(C,\star,n)$ is isomorphic to the split Okubo algebra, and the unique Okubo algebra that contains a quaternionic idempotent is, up to isomorphism, the split Okubo algebra.*

Proof. For the new multiplication

$$x \cdot y = (f * x) * (y * f),$$

$(S, \cdot, n_S)$ is a Cayley algebra with unity f, the map

$$\vartheta_S(x) = \big(f * (f * x)\big) = n_S(f, x)f - x * f$$

is an automorphism of $(S, \cdot)$ and of $(S, *)$, of order 3, such that $* = \cdot_{\vartheta_S}$ is the twist with respect to ϑ_S (see [KMRT98, (34.9)]). Let P be a para-quaternion subalgebra of $(S, *)$ with para-unit f. Then the restriction of ϑ_S to P is the identity. Let $z \in P^\perp$ with $n_S(z) \neq 0$, so $P^\perp = P \cdot z$. Then $S = P \oplus P \cdot z$ and $\vartheta_S(z) = u \cdot z$ for some $u \in P$. But then $u^{\cdot 3} = 1$, so $(u-1)^{\cdot 3} = 0 \neq u - 1$. We conclude that $(P, \cdot)$ is the split quaternion algebra. Then z can be taken with $n_S(z) = -1$. Moreover, there is an isomorphism $(P, \cdot) \to \mathrm{Mat}_2(F)$ that takes u to the element w in Example 9.4, and this isomorphism extends to a unique isomorphism $\varphi\colon (S, \cdot) \to (C, \cdot)$ such that $\varphi(z) = v$. Obviously $\varphi(f) = e$ and $\varphi\vartheta_S = \vartheta\varphi$. This last condition implies that φ is also an isomorphism $(S, *, n_S) \to (C, \star, n)$. Moreover, since the split Okubo algebra contains a quaternionic idempotent (see Lemma 9.3), there is a unique Okubo algebra up to isomorphism containing quaternionic idempotents. □

Our aim now is to prove that the quaternionic idempotent in the split Okubo algebra is unique. The Okubo algebra $(C, \star, n)$ in Example 9.4 is isomorphic to the split Okubo algebra, since it contains a quaternionic idempotent. Thus we may use it as a model of the split Okubo algebra. We keep the notation used in Example 9.4. The subalgebra of $(C, \star)$ of elements fixed under the order 3 automorphism ϑ is $Q \oplus Iv$, where $I = \{a \in Q \mid wa = a\} = \{a \in Q \mid (w-1)a = 0\} = (w-1)Q$, because $(w-1)^2 = 0$. (Note that it is clear that $(w-1)Q$ is contained in the subspace of elements of Q annihilated by $(w-1)$, the equality holds since both subspaces are proper right ideals of Q, and hence both of them have dimension 2.) Therefore, the centralizer of e in $(C, \star, n)$ is

$$\begin{aligned} \mathrm{Cent}_{(C,\star,n)}(e) &= \{x \in C \mid e \star x = x \star e\} \\ &= \{x \in C \mid \vartheta^2(\overline{x}) = \vartheta(\overline{x})\} \\ &= \{x \in C \mid \vartheta(x) = x\} \\ &= Q \oplus Iv, \end{aligned} \tag{9.10}$$

and it is a six-dimensional subalgebra of both $(C, \cdot)$ and $(C, \star)$. Note also that $Iv = \{x \in \mathrm{Cent}_{(C,\star,n)}(e) \mid b_n\big(x, \mathrm{Cent}_{(C,\star,n)}(e)\big) = 0\}$ is precisely the radical of $\mathrm{Cent}_{(C,\star,n)}(e)$ relative to the polar form of the norm, and $(Iv) \star (Iv) = \overline{I}I = Q(w-1)^2Q = 0$, so Iv is a nilpotent ideal of $\big(\mathrm{Cent}_{(C,\star,n)}(e), \star\big)$. Since $(Q, \star)$ is simple, Iv is the largest nilpotent ideal of $(\mathrm{Cent}_{(C,\star,n)}(e), \star)$.

Let $Z = \mathrm{Cent}_{(C,\star,n)}(e)$ and let $K(Z, \star) = \{x \in Z \mid x \star y = y \star x,\ \forall y \in Z\}$.

Lemma 9.11. (i) $K(Z, \star) = Fe$.

(ii) *Any subalgebra of* $(Z, \star)$ *of dimension* ≥ 4 *contains* e.

Proof. Clearly e belongs to $K(Z,\star)$. Also $(Z,\star)$ is $\mathbb{Z}_2$-graded, so

$$K(Z,\star) = \big(K(Z,\star)\cap Q\big)\oplus\big(K(Z,\star)\cap Iv\big).$$

Since $(Q,\star)$ is a para-quaternion algebra, its commutative center is spanned by e, which is its para-unit. On the other hand, for any $a\in Q$ and $b\in I$, $a\star(bv) = -\overline{a}\cdot(bv) = -(b\overline{a})v$, while $(bv)\star a = -(bv)\cdot\overline{a} = -(ba)v$. Hence, if bv is in $K(Z,\star)$, then $ba = b\overline{a}$ for any $a\in Q$, so $b(a-\overline{a}) = 0$ for any $a\in Q$, and $b=0$ since we may find elements $a\in Q$ with $n(a-\overline{a})\neq 0$.
For the second part, let $Z_0 = \{x\in Z \mid b_n(e,x) = 0\}$, so $Z = Fe\oplus Z_0$. If T is a subalgebra of $(Z,\star)$ and $e\notin T$, then the projection $\pi\colon T\to Z_0$ is one-to-one, and hence for any $x\in\pi(T)$ there is a unique $\alpha\in F$ such that $\alpha e + x\in T$. But $(\alpha e+x)^{\star 2} = (\alpha e - x)^{\cdot 2} = \alpha^2 e - 2\alpha x + x^{\cdot 2} = \alpha(\alpha e + x) - n(x)e$, so $n(x)e\in T$ for any $x\in\pi(T)$. Since we are assuming $e\notin T$, $\pi(T)$ consists of isotropic vectors and hence $\dim T = \dim\pi(T)\le 3$, because $\dim Z_0 = 5$ and the rank of the restriction of n to Z_0 is 3. $\square$

COROLLARY 9.12. *If f is a quaternionic idempotent of the split Okubo algebra $(C,\star,n)$, then $f\in Z$.*

Proof. We know by Proposition 9.9 and by formulas (9.10) that $Z = \mathrm{Cent}_{(C,\star,n)}(e)$ and $\mathrm{Cent}_{(C,\star,n)}(f)$ are six-dimensional subalgebras. Hence $\mathrm{Cent}_{(C,\star,n)}(f)\cap Z$ has dimension at least 4, so it contains e by Lemma 9.11. Hence e lies in $\mathrm{Cent}_{(C,\star,n)}(f)$, $e\star f = f\star e$, and this is equivalent to $f\in Z$. $\square$

THEOREM 9.13. *The split Okubo algebra contains a unique quaternionic idempotent.*

Proof. By Corollary 9.12 it suffices to study the nonzero idempotents in the subalgebra Z of the split Okubo algebra $(C,\star,n)$. For $a\in Q$ and $b\in I = (w-1)Q$, we have

$$(a+bv)^{\star 2} = (\overline{a}-bv)^{\cdot 2} = b_n(\overline{a},1)(\overline{a}-bv) - n(\overline{a}-bv)1 = b_n(a,1)(\overline{a}-bv) - n(a)1,$$

and this is an idempotent if and only if

$$a = b_n(a,1)\overline{a} - n(a)1,\quad b = -b_n(a,1)b. \tag{9.14}$$

Hence $a = b_n(a,1)\big(b_n(a,1)1 - a\big) - n(a)1$, or

$$\big(1+b_n(a,1)\big)a = \big(b_n(a,1)^2 - n(a)\big)1. \tag{9.15}$$

Now, if $b_n(a,1)\neq -1$ we conclude from (9.14) that $a\in F1$ and $b=0$, so $a = 1 = e$.
On the other hand, if $b_n(a,1) = -1$, then we conclude from (9.15) that $n(a) = 1$ and hence $(a-1)^2 = 0$. Then we assume from now on that $(a-1)^2 = 0$.
If $b=0$, then either $a = 1 = e$, or a is conjugate in Q to $w = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. In this last case, for $x,y\in Q$, $x+yv$ is in the centralizer of a if and only if so are x and yv, but $x\star a = a\star x$ if and only if $\overline{x}\,\overline{a} = \overline{a}\,\overline{x}$, if and only if $ax = xa$, if and only if $x\in F1 + Fa$. And $(yv)\cdot a = a\cdot(yv)$ if and only if $-((wy)v)\overline{a} = -\overline{a}((w^2y)v)$, if and only if $wya = w^2y\overline{a}$. Since $\overline{a} = a^2 = a^{-1}$, this is equivalent to $ya^2 = wy$.

But this equation does not hold for all $y \in Q$, and hence the dimension of $\mathrm{Cent}_{(C,\star,n)}(a)$ is not 6, and a is not a quaternionic idempotent.
If $b \neq 0$, we compute again $\mathrm{Cent}_{(C,\star,n)}(a+bv)$. For x, $y \in Q$,

$$\begin{aligned}(x+yv)\star(a+bv) &= (\overline{x}-(wy)v)\cdot(\overline{a}-bv)\\ &= ((\overline{x}\,\overline{a}+\overline{b}wy)-(b\overline{x}+wya)v,\\ (a+bv)\star(x+yv) &= (\overline{a}-bv)\cdot(\overline{x}-(w^2y)v)\\ &= (\overline{a}\,\overline{x}+\overline{w^2y}b)-(bx+w^2y\overline{a})v.\end{aligned}$$

Hence, $x+yv$ is in $\mathrm{Cent}_{(C,\star,n)}(a+bv)$ if and only if

$$\overline{x}\,\overline{a}+\overline{b}wy=\overline{a}\,\overline{x}+\overline{y}wb \qquad \text{and} \qquad b\overline{x}+wya=bx+w^2y\overline{a}.$$

But $wb=b$, because $a+bv\in Z$, so $\overline{b}w=\overline{w^2b}=\overline{b}$. Thus, this is equivalent to

$$\overline{x}\,\overline{a}+\overline{b}y=\overline{a}\,\overline{x}+\overline{y}b \qquad \text{and} \qquad b\overline{x}+wya=bx+w^2y\overline{a}. \tag{9.16}$$

If $a=1$, this gives $\overline{b}y=\overline{y}b$, so $\overline{b}y\in F1$. But $n(b)=0$, so $\overline{b}y=0$, or $y\in bQ$. Note that I is a minimal right ideal of Q, so $bQ=I$ and $y\in I$. Hence the centralizer of $a+bv$ is contained in $Q+Iv$. But then $wy=y$ and we obtain $b\overline{x}=bx$, or $b(x-\overline{x})=0$. We conclude that Q is not contained in the centralizer of $a+bv$, so $\dim\mathrm{Cent}_{(C,\star,n)}(a+bv)$ is not 6 and $a+bv$ is not a quaternionic idempotent.
Finally, if $a\neq 1$ and we take $y\in bQ=I$, we have as before $\overline{b}y=0=\overline{y}b$, so $\overline{x}\,\overline{a}=\overline{a}\,\overline{x}$, and hence $x\in F1+Fa$. If $a+bv$ were a quaternionic element, the dimension of $\mathrm{Cent}_{(C,\star,n)}(a+bv)$ would be 6, and hence the dimension of $\mathrm{Cent}_{(Z,\star)}(a+bv)$ would be at least 4. Therefore we would have $\mathrm{Cent}_{(Z,\star)}(a+bv)=(F1+Fa)+Iv$. But for $y\in I$, $yv\in\mathrm{Cent}_{(Z,\star)}(a+bv)$ if and only if $ya=y\overline{a}$ because of (9.16), if and only if $y(a-1)=0$. We conclude that $I(a-1)=0$, or $(w-1)Q(a-1)=0$, and this is impossible since Q is simple (and hence prime). □

For any Okubo algebra $(S,\star,n)$, the cube form $g(x)=b_n(x,x\star x)$ plays a key role (see [EP96, §5], [KMRT98, (36.11)]).

Lemma 9.17 (see [Eld97]). *Let $(S,\star,n)$ be an Okubo algebra. Then:*

(i) *The cubic form g is semilinear, i.e. $g(\alpha x+y)=\alpha^3g(x)+g(y)$ for any $\alpha\in F$ and $x,y\in S$.*
(ii) *For any $x\in S$, we have $g(x^{\star 2})=g(x)^2+n(x)^3$.*
(iii) *The set $F^3+g(S)$ is a subfield of F (a purely inseparable field extension of F^3 of exponent 1).*
(iv) *The quadratic form n is isotropic (and hence its Witt index is 4).*

Theorem 9.18. *Let $(C,\triangle,n)$ be the split Okubo algebra represented by Zorn matrices (see §5), over an algebraically closed field F. Consider the subset*

$$O=\{x\in C\mid n(x)=0,\ g(x)=1\}.$$

Then:

(i) *The set O is an irreducible closed subset of C in the Zariski topology.*

(ii) *For any $x \in O$, $e = x + x \vartriangle x$ is an idempotent of $(C, \vartriangle, n)$ which is either quadratic or quaternionic.*

(iii) *The subset $O_0 = \{x \in O \mid x + x \vartriangle x \text{ is a quadratic idempotent}\}$ coincides with the orbit of e_1 under the automorphism group of $(C, \vartriangle, n)$.*

(iv) *For any $x \in O_0$ and any $y \in C$ such that $x \vartriangle y = 0$, $b_n(x, y) = 0$ and $g(y) = 1$, there is an automorphism φ of $(C, \vartriangle, n)$ such that $\varphi(e_1) = x$ and $\varphi(u_1) = y$.*

(v) *The subset O_0 is a nonempty open subset of O, and the subset $O_1 = \{x \in O \mid x + x \vartriangle x \text{ is a quaternionic idempotent}\}$ is a closed subset of O.*

Proof. (i) Let e_1 and f_1 be as in (5.3). Note first that e_1 belongs to O, as $e_1 \vartriangle e_1 = f_1$ and $g(e_1) = b_n(e_1, f_1) = 1$. Since the cubic form g is semilinear (Lemma 9.17), the subset $\{x \in C \mid g(x) = 1\}$ coincides with the set $e_1 + \{x \in C \mid g(x) = 0\}$, which is isomorphic to an affine space of dimension 7. It follows that O is a quadric in this affine space, and one easily checks that this quadric is irreducible.
(ii) For any $x \in O$, $x \vartriangle (x \vartriangle x) = n(x)x = 0$, while

$$(x \vartriangle x) \vartriangle (x \vartriangle x) + ((x \vartriangle x) \vartriangle x) \vartriangle x = b_n(x, x \vartriangle x)x$$

(see 4.3), so $(x \vartriangle x) \vartriangle (x \vartriangle x) = x$. Hence $e = x + x \vartriangle x$ is an idempotent, the para-unit of the para-quadratic subalgebra spanned by x and $x \vartriangle x$. Thus e is either quadratic or quaternionic.
(iii) Note that $e_1 \in O_0$ as $e = e_1 + e_1 \vartriangle e_1 = e_1 + f_1$ is a quadratic idempotent (Lemma 9.3). Recall that the Hurwitz product $\cdot$ is recovered as

$$x \cdot y = (e \vartriangle x) \vartriangle (y \vartriangle e).$$

For any $z \in O_0$, let $f = z + z \vartriangle z$, and consider the new multiplication on C:

$$x \circ y = (f \vartriangle x) \vartriangle (y \vartriangle f).$$

Then $(C, \circ, n)$ is a Cayley algebra with identity element f and the map $\vartheta(x) = f \vartriangle (f \vartriangle x) = b_n(f, x)f - x \vartriangle f$ is an automorphism of $(C, \circ, n)$ of order 3. Besides, $x \vartriangle y = \vartheta(\widehat{x}) \circ \vartheta^2(\widehat{y})$, where $\widehat{x} = b_n(f, x) - x$ for any x, so the Okubo algebra $(C, \vartriangle, n)$ is the twist $(C, \circ, n)_\vartheta$ of $(C, \circ, n)$ by ϑ (see [KMRT98, (34.9)]). The automorphism ϑ fixes the elements z and $z \vartriangle z$ of $(C, \circ, n)$, and hence the subalgebra $K = \operatorname{span}\{z, z \vartriangle z\}$ of $(C, \circ, n)$. Moreover, $\vartheta(x) = x$ if and only if $b_n(f, x)f - x \vartriangle f = x$, and this implies $f \vartriangle x = b_n(f, x) - f \vartriangle (x \vartriangle f) = b_n(f, x)f - x = x \vartriangle f$. Conversely, if $f \vartriangle x = x \vartriangle f$, then $\vartheta(x) = f \vartriangle (f \vartriangle x) = f \vartriangle (x \vartriangle f) = n(f)x = x$, so the subalgebra of $(C, \circ, n)$ of the elements fixed by ϑ coincides with the centralizer of f in $(C, \vartriangle, n)$. Besides, $\vartheta^3 = \mathrm{Id}$, so $(\vartheta - \mathrm{Id})^3 = 0$. If $(\vartheta - \mathrm{Id})^2$ were 0, then there would exist an element x orthogonal to K such that $\vartheta(x) = x$ and $n(x) \neq 0$, but then the subalgebra $K \oplus K \circ x$ would be a para-quaternion subalgebra with para-unit f, a contradiction with f being quadratic. Hence the restriction of ϑ to $K^\perp$ has

minimal polynomial $(X-1)^3$ and the arguments in [EP96, proof of Proposition 1.1] show that there exists an isomorphism $\varphi\colon (C,\cdot,n) \to (C,\circ,n)$ such that $\varphi(e_1) = z$, $\varphi(f_1) = z \vartriangle z$ and such that $\varphi s_\pi = \vartheta\varphi$ (recall that $(C,\vartriangle,n)$ is the twist $(C,\diamond,n)_{s_\pi}$). Therefore φ is an automorphism of $(C,\vartriangle,n)$ and $\varphi(e_1) = z$.
(iv) Note that $0 = y \vartriangle (x \vartriangle y) = n(y)x$, so $n(y) = 0$. Also,

$$b_n(x \vartriangle x, y) = -b_n(x, x \vartriangle y) = 0, \quad b_n(x, y \vartriangle y) = b_n(x \vartriangle y, y) = 0$$

and

$$b_n(x \vartriangle x, y \vartriangle y) = b_n\big(y, y \vartriangle (x \vartriangle x)\big) = b_n\big(y, -b_n(x,y)x + x \vartriangle (x \vartriangle y)\big) = 0.$$

Therefore the subalgebras span $\{x, x \vartriangle x\}$ and span $\{y, y \vartriangle y\}$ are orthogonal and the result follows now from [Eld09, Theorem 3.12].
(v) As $\mathrm{Cent}_{(C,\vartriangle,n)}(e_1 + e_1 \vartriangle e_1) = \mathrm{span}\,\{e_1, f_1, u_1 + u_2 + u_3, v_1 + v_2 + v_3\}$, we have that $\mathrm{Cent}_{(C,\vartriangle,n)}(x + x \vartriangle x)$ has dimension 4 for any $x \in O_0$. On the other hand, for any $x \in O_1$, $x + x \vartriangle x$ is the unique quaternionic idempotent, whose centralizer has dimension 6. Since for any x, $\mathrm{Cent}_{(C,\vartriangle,n)}(x) = \ker(l_x - r_x)$, where l_x and r_x denote the left and right multiplication by x in $(C,\vartriangle,n)$, we obtain:

$$\begin{aligned} O_0 &= \{x \in O \mid \mathrm{rank}(l_{x+x\vartriangle x} - r_{x+x\vartriangle x}) = 4\} \\ &= \{x \in O \mid \mathrm{rank}(l_{x+x\vartriangle x} - r_{x+x\vartriangle x}) > 2\}, \end{aligned}$$

and this is open in O. $\square$

Corollary 9.19. *Let $(C,\vartriangle,n)$ be the Okubo algebra with the multiplication table* (5.11) *over an algebraically closed field F, and let $\lambda \in F$ be nonzero. The subset*

$$O^\lambda = \{x \in C \mid n(x) = 0,\ g(x) = \lambda\}$$

is an irreducible closed subset of C, and its subset $O_0^\lambda = \{x \in O^\lambda \mid \lambda^{1/3}x \in O_0\}$ is a nonempty open subset of O^λ.

Given an Okubo algebra $(S,\star,n)$ over a field F and a field extension K/F, denote by $O(K)$ the subset $\{x \in S \otimes_F K \mid n(x) = 0,\ g(x) = 1\}$ (the extensions of n and g to $S \otimes_F K$ are denoted by the same letters). The same applies to $O^\lambda(K)$. For any field F, let $\overline{F}$ denote an algebraic closure of F.

Corollary 9.20. *Let $(S,\star,n)$ be an Okubo algebra over an infinite field F and let $\lambda \in F$ be nonzero. If the set $O^\lambda(F)$ is not empty, then it is a closed irreducible subset of S and its subset $O_0^\lambda(F) := O^\lambda(F) \cap O_0^\lambda(\overline{F})$ is a nonempty open subset.*

10. Automorphisms of the split Okubo algebra

The rational points of the groups of automorphisms of Okubo algebras over fields of characteristic 3 have been computed in [Eld99], by relating the Okubo algebras with some noncommutative Jordan algebras. In this section the existence of a unique quaternionic idempotent in the split Okubo algebra will be used to compute its group of automorphisms inside the group of automorphisms of the split Cayley algebra. We systematically assume that the base field F has characteristic 3. As in Example 9.4, consider the split Cayley algebra $(C,\cdot,n)$, with $C = Q \oplus Qv$, and the order 3 automorphism ϑ in (9.6) such that its restriction to Q is the identity and $\vartheta(v) = wv$ with $1 \neq w \in Q$ such that $w^3 = 1$. The Petersson twist $\star = \cdot_\vartheta$ is the split Okubo algebra multiplication.

Proposition 10.1. $\mathbf{Aut}(C,\star,n)(F) = \{\psi \in \mathbf{Aut}(C,\cdot,n)(F) \mid \psi\vartheta = \vartheta\psi\}$.

Proof. Obviously, any automorphism of $(C,\cdot,n)$ commuting with ϑ is an automorphism of $(C,\star,n)$. Conversely, if ψ is an automorphism of $(C,\star,n)$, then $\psi(e) = e$, because e is the unique quaternionic idempotent, and since $\vartheta(x) = e\star(e\star x)$ for any $x \in C$, we conclude that ψ commutes with ϑ. □

It is no longer true that the quaternionic idempotent e of $(C,\star,n)$ is fixed by all automorphisms of the extension $(C\otimes_F R,\star,n)$, for any unital commutative associative algebra R. However, the automorphisms that fix $e\otimes 1_R$ coincide with the automorphisms of $(C\otimes_F R,\cdot,n)$ that commute with ϑ (more precisely with the automorphism $\vartheta\otimes\mathrm{Id}$). Denote by $\mathbf{H}$ the centralizer of the automorphism ϑ in the group scheme of automorphisms $\mathbf{Aut}(C,\cdot,n)$. Alternatively, $\mathbf{H}$ is the stabilizer of e (or of Fe). By (9.10) any $\varphi \in \mathbf{H}(R)$ leaves $Iv\otimes_F R$ invariant, since Iv is the radical of the restriction of the norm to the subalgebra of fixed elements by ϑ. Therefore there is a morphism

$$\Phi\colon \mathbf{H} \to \mathbf{GL}(I), \tag{10.2}$$

that takes any automorphism $\varphi \in \mathrm{Aut}\big((C\otimes_F R,\star,n)\big)$ commuting with $\vartheta\otimes\mathrm{Id}$ to the linear automorphism $\Phi(\varphi) = f$ of $I\otimes_F R$ defined by $\varphi(xv) = f(x)v$, for any $x \in I\otimes_F R$. Given any element $a \in \mathbf{SL}_2(R)$ there is a unique automorphism ψ_a of $(C\otimes_F R,\cdot)$ such that $\psi_a(x) = axa^{-1}$ and $\psi_a(xv) = (xa^{-1})v$ for $x \in Q\otimes_F R = \mathrm{Mat}_2(R)$. This automorphism commutes with ϑ. Thus we get a closed embedding of $\mathbf{SL}_2$ into $\mathbf{H}$. Denote by $\mathbf{S}$ its image. Moreover, $\mathbf{GL}(I)$ will be identified with $\mathbf{GL}_2$, with $a \in \mathbf{GL}_2(R) = (Q\otimes_F R)^\times$ being identified with the map $xv \mapsto (xa^{-1})v$. With these identifications we get the following result:

Theorem 10.3. (i) $\Phi(\mathbf{H}) = \mathbf{SL}_2$,

(ii) $\mathbf{H}$ *is the semidirect product of* $\mathbf{ker}\,\Phi$ *and* $\mathbf{S}$.

Proof. As in the proof of Theorem 9.13, we may assume $w = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Hence $I = (w-1)Q$ is the right ideal of 2×2 matrices with trivial second row. Let J be the right ideal of 2×2 matrices with trivial first row, so $Q = I\oplus J$. For simplicity, given a unital commutative associative algebra R, we write $Q_R = Q\otimes_F R$ and similarly I_R, J_R. Take $\varphi \in \mathbf{H}(R)$ and let $a \in \mathbf{GL}_2(R)$ be its image under Φ, so $\varphi(xv) = (xa^{-1})v$ for any $x \in I$. For $x \in Q_R$, $\varphi(x) = \varphi_0(x) + \sigma(x)v$ for R-linear maps $\varphi_0\colon Q_R \to Q_R$ and $\sigma\colon Q_R \to I_R$. Then, for $x \in Q_R$ and $y \in I_R$,

$$\varphi\big(x\cdot(yv)\big) = \begin{cases} \varphi\big((yx)v\big) = (yxa^{-1})v, \\ \varphi(x)\cdot\varphi(yv) = \big(\varphi_0(x)+\sigma(x)v\big)\cdot\big((ya^{-1})v\big) = \big(ya^{-1}\varphi_0(x)\big)v, \end{cases}$$

as $(Iv)^2 = 0$. Thus $y\big(xa^{-1} - a^{-1}\varphi_0(x)\big) = 0$ for any $x \in Q_R$, $y \in I_R$, so $\varphi_0(x) = axa^{-1}$. Now, for $x \in J_R$, $\varphi(xv) = \mu(x)v + \nu(x) + \delta(x)v$ for some R-linear maps $\mu\colon J_R \to J_R$, $\nu\colon J_R \to Q_R$ and $\delta\colon J_R \to I_R$. Hence, with ϑ as in (9.6), we have

$$\begin{aligned}\varphi\vartheta(xv) &= \varphi\big((wx)v\big) = \varphi(xv) + \varphi\big(((w-1)x)v\big)\\ &= \mu(x)v + \nu(x) + \big(\delta(x) + (w-1)xa^{-1}\big)v,\\ \vartheta\varphi(xv) &= \vartheta\big(\mu(x)v + \nu(x) + \delta(x)v\big)\\ &= (w\mu(x))v + \nu(x) + \delta(x)v\\ &= \mu(x)v + \nu(x) + \big((w-1)\mu(x) + \delta(x)\big)v.\end{aligned}$$

We get $(w-1)xa^{-1} = (w-1)\mu(x)$, so $\mu(x) - xa^{-1}$ lies in J_R and it is annihilated by $(w-1)$, and hence lies in I_R. Thus $\mu(x) = xa^{-1}$ for any $x \in J_R$.
But φ is an isometry of the norm too, so for any $x \in J_R$ and $y \in I_R$,

$$\begin{aligned}b_n(x,y) &= -b_n(xv,yv) = -b_n\big(\varphi(xv),\varphi(yv)\big)\\ &= -b_n\big((xa^{-1})v,(ya^{-1})v\big) = b_n(xa^{-1},ya^{-1}) = n(a^{-1})b_n(x,y),\end{aligned}$$

and we conclude that $\det(a) = n(a) = 1$. This proves (i). Finally, the assignment $a \mapsto \psi_a$ gives a section of the epimorphism $\Phi\colon \mathbf{H} \to \mathbf{SL}_2$. $\square$

We next compute the group $\mathbf{ker}\,\Phi$.

Lemma 10.4. *Let $(C,\cdot,n)$ be the Zorn algebra as in § 5. Let R be a unital commutative associative algebra over F and let W be a free R-submodule of $C \otimes_F R$ of rank 3 with $n(W) = 0 = b_n(W,1)$ and $1 \in b_n(W,W^{\cdot 2})$. Let $\{w_1,w_2,w_3\}$ be a basis of W with $b_n(w_1,w_2\cdot w_3) = 1$. Let $\{e_1,f_1,u_1,u_2,u_3,v_1,v_2,v_3\}$ be the canonical basis of C. Then there is a unique automorphism $\varphi \in \operatorname{Aut}(C\otimes_F R,\cdot,n)$ such that $\varphi(u_i) = w_i$, for $i = 1$, 2, 3.*

Proof. The uniqueness follows from the fact that u_1,u_2,u_3 generate $(C,\cdot)$. Consider the trilinear form given by $\Lambda(x,y,z) = b_n(x,y\cdot z)$. For any $x \in W$, $n(x) = 0 = b_n(x,1)$, so $x^{\cdot 2} = 0$, and hence $x\cdot y = -y\cdot x$ for any x, $y \in W$. Moreover, for all x, $y \in W$,

$$\begin{aligned}\Lambda(x,x,y) &= b_n(x,x\cdot y) = n(x)b_n(1,y) = 0,\\ \Lambda(x,y,y) &= b_n(x,y^{\cdot 2}) = 0,\end{aligned}$$

so Λ is alternating on W. In particular, we have $\Lambda(x,y,z) = \Lambda(y,z,x) = \Lambda(z,x,y)$ for any $x,y,z \in W$.
Let $\hat W$ be the linear span (over R) of $w_2\cdot w_3$, $w_3\cdot w_1$ and $w_1\cdot w_2$. Taking indices modulo 3 we get $n(w_i\cdot w_{i+1}) = 0$ and

$$\begin{aligned}b_n(w_i\cdot w_{i+1},w_{i+1}\cdot w_{i+2}) &= -b_n(w_{i+1}\cdot w_i,w_{i+1}\cdot w_{i+2})\\ &= -n(w_{i+1})b_n(w_i,w_{i+2}) = 0,\end{aligned}$$

so $\hat W$ is isotropic. Also, $b_n(w_i\cdot w_{i+1},1) = -b_n(w_i,w_{i+1}) = 0$. Moreover

$$b_n(w_i,w_{i+1}\cdot w_{i+2}) = \Lambda(w_i,w_{i+1},w_{i+2}) = 1,$$

while

$$b_n(w_i, w_i \cdot w_{i+1}) = n(w_i)b_n(1, w_{i+1}) = 0 = b_n(w_i, w_{i+2} \cdot w_i).$$

Therefore, W and $\hat{W}$ are paired by the polar form of the norm.
Now, since we have $x \cdot (y \cdot z) + y \cdot (x \cdot z) = (x \cdot y + y \cdot x) \cdot z$, we get

$$w_1 \cdot (w_2 \cdot w_3) = -w_2 \cdot (w_1 \cdot w_3) = w_2 \cdot (w_3 \cdot w_1) = w_3 \cdot (w_1 \cdot w_3),$$

and also

$$(w_1 \cdot w_2) \cdot w_3 = (w_2 \cdot w_3) \cdot w_1 = (w_3 \cdot w_1) \cdot w_2.$$

But

$$w_1 \cdot (w_2 \cdot w_3) + (w_2 \cdot w_3) \cdot w_1 = -b_n(w_1, w_2 \cdot w_3)1 = -1.$$

As $n\big(w_1 \cdot (w_2 \cdot w_3)\big) = 0$ and $b_n\big(w_1 \cdot (w_2 \cdot w_3), 1\big) = -b_n(w_2 \cdot w_3, w_1) = -1$, it turns out that $\hat{e}_1 = -w_1 \cdot (w_2 \cdot w_3)$ is an idempotent, and so is $\hat{f}_1 = 1 - \hat{e}_1 = -(w_2 \cdot w_3) \cdot w_1$. Then

$$w_i \cdot \hat{e}_1 = w_i \cdot \big(w_i \cdot (w_{i+1} \cdot w_{i+1})\big) = w_i^{\cdot 2} \cdot (w_{i+1} \cdot w_{i+2}) = 0,$$

and we get $\hat{f}_1 \cdot w_i = 0$ in a similar vein. Also $\hat{e}_1 \cdot (w_i \cdot w_{i+1}) = 0 = (w_i \cdot w_{i+1}) \cdot \hat{f}_1$, and it follows easily that $\{\hat{e}_1, \hat{f}_1, w_1, w_2, w_3, w_2 \cdot w_3, w_3 \cdot w_1, w_1 \cdot w_2\}$ is a basis of $C \otimes_F R$ with the same multiplication table as for the canonical basis over F. $\square$

Consider now an automorphism $\varphi \in \mathrm{Aut}(C \otimes_F R, \cdot, n)$ that commutes with ϑ and which belongs to $\mathbf{ker}\,\Phi$, that is, φ fixes the elements in $I_R v$. Then for any $x \in Q_R$, x is fixed by ϑ, and hence so is $\varphi(x)$. Thus, as in the proof of Theorem 10.3 (with $a = 1$), we obtain $\varphi(x) = x + \sigma(x)v$ for any $x \in Q$, for an R-linear map $\sigma\colon Q_R \to I_R$.
We may take a canonical basis $\{e_1, f_1, u_1, u_2, u_3, v_1, v_2, v_3\}$ of C such that $Q = \mathrm{span}\{e_1, f_1, u_1, v_1\}$, $w = 1 - u_1$ and $v = u_3 - v_3$. Then $I = (w-1)Q = \mathrm{span}\{e_1, u_1\}$, $Iv = \mathrm{span}\{u_3, v_2\}$. Also, $\vartheta(u_2) = \vartheta(v_1 v) = (wv_1)v = \big((1 - u_1)v_1\big)v = (v_1 - e_1) \cdot (u_3 - v_3) = u_2 + u_3$. In particular, $\varphi(u_3) = u_3$, since we are assuming that φ leaves fixed the elements in $I_R v$, and $\varphi(u_1) = u_1 + \alpha u_3 + \beta v_2$ for some $\alpha, \beta \in R$, since $\varphi(u_1) - u_1 \in I_R v$. Besides, for any $x \in I_R v$, $b_n\big(\varphi(u_2), x\big) = b_n\big(\varphi(u_2), \varphi(x)\big) = b_n(u_2, x)$, so $\varphi(u_2) - u_2 \in (I_R v)^\perp = Q_R \oplus I_R v$. As $b_n\big(\varphi(u_2), 1\big) = b_n(u_2, 1) = 0$, we conclude that $\varphi(u_2) = u_2 + \gamma(e_1 - f_1) + \delta u_1 + \mu v_1 + \nu u_3 + \rho v_2$ for some $\gamma, \delta, \mu, \nu, \rho \in R$. Hence

$$\begin{aligned} \varphi(u_1) &= u_1 + \alpha u_3 + \beta v_2, \\ \varphi(u_2) &= u_2 + \gamma(e_1 - f_1) + \delta u_1 + \mu v_1 + \nu u_3 + \rho v_2, \\ \varphi(u_3) &= u_3, \end{aligned} \tag{10.5}$$

and this determines φ completely because the elements u_1, u_2, u_3 generate C. Conversely, Lemma 10.4 shows that for any α, β, γ, δ, μ, ν, $\rho \in R$, there is a unique automorphism φ satisfying (10.5) if and only if the elements $u_1 + \alpha u_3 + \beta v_2$, $u_2 + \gamma(e_1 - f_1) + \delta u_1 + \mu v_1 + \nu u_3 + \rho v_2$ and u_3 span an isotropic space and

$$b_n\big(u_1 + \alpha u_2 + \beta v_2, (u_2 + \gamma(e_1 - f_1) + \delta u_1 + \mu v_1 + \nu u_3 + \rho v_2) \cdot u_3\big) = 1.$$

But

$$n(u_1+\alpha u_3+\beta v_2)=0=n(u_3),$$
$$n(u_2+\gamma(e_1-f_1)+\delta u_1+\mu v_1+\nu u_3+\rho v_2)=-\gamma^2+\rho+\delta\mu,$$

while

$$b_n(u_1+\alpha u_3+\beta u_2,u_3)=0=b_n(u_2+\gamma(e_1-f_1)+\delta u_1+\mu v_1+\nu u_3+\rho v_2,u_3)$$
$$b_n(u_1+\alpha u_3+\beta v_2,u_2+\gamma(e_1-f_1)+\delta u_1+\mu v_1+\nu u_3+\rho v_2)=\mu+\beta,$$

and

$$b_n\big(u_1+\alpha u_3+\beta u_2,(u_2+\gamma(e_1-f_1)+\delta u_1+\mu v_1+\nu u_3+\rho v_2)\cdot u_3\big)= \\ b_n(u_1+\alpha u_3+\beta v_2,v_1+\gamma u_3-\delta v_2)=1.$$

Hence the only restrictions on the scalars $\alpha,\dots,\rho$ so that equation (10.5) determines an automorphism are

$$\beta=-\mu,\qquad \rho=\gamma^2-\delta\mu.$$

If these conditions are satisfied, then $\varphi\vartheta(u_1)=\vartheta\varphi(u_1)$ and $\varphi\vartheta(u_3)=\vartheta\varphi(u_3)$ since $u_1,u_3,\varphi(u_1),\varphi(u_3)$ are fixed by ϑ. Also, $\varphi\vartheta(u_2)=\varphi(u_2+u_3)=\varphi(u_2)+u_3$, while $\vartheta\big(\varphi(u_2)-u_2\big)=\varphi(u_2)-u_2$, so $\vartheta\varphi(u_2)=\varphi(u_2)-u_2+\vartheta(u_2)=\varphi(u_2)+u_3=\varphi\vartheta(u_2)$. Hence $\varphi\vartheta=\vartheta\varphi$.

Since u_1, u_2, u_3 generate the Cayley algebra $(C,\cdot)$, we can easily compute the images of any basic element. It turns out that the coordinate matrix in the basis $\{u_2,v_3,e_1,f_1,u_1,v_1,u_3,v_2\}$ of an arbitrary element in $\mathbf{ker}\,\Phi(R)$ is of the form

$$\left(\begin{array}{cc|cccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
\gamma & -\mu & 1 & 0 & 0 & 0 & 0 & 0\\
-\gamma & \mu & 0 & 1 & 0 & 0 & 0 & 0\\
\delta & -\gamma & 0 & 0 & 1 & 0 & 0 & 0\\
\mu & -\alpha & 0 & 0 & 0 & 1 & 0 & 0\\
\hline
\nu & \mu^2-\alpha\gamma & -\mu & \mu & \alpha & \gamma & 1 & 0\\
\gamma^2-\delta\mu & \alpha\delta-\mu\gamma-\nu & \gamma & -\gamma & -\mu & -\delta & 0 & 1
\end{array}\right) \tag{10.6}$$

PROPOSITION 10.7. *The algebraic group* $\mathbf{ker}\,\Phi$ *is smooth unipotent of dimension* 5. *It is represented by the Hopf algebra which, as an algebra, is the polynomial ring in* 5 *variables* $F[\alpha,\gamma,\delta,\mu,\nu]$, *with comultiplication given by* α,γ,δ,μ *being primitive and* $\Delta(\nu)=\nu\otimes1+1\otimes\nu+\mu\otimes\gamma+\gamma\otimes\mu+\alpha\otimes\delta$.

Proof. Everything follows from the coordinate expression in equation (10.6). □

COROLLARY 10.8. $\mathbf{H}$ *is a smooth group scheme.*

Proof. Both $\mathbf{ker}\,\Phi$ and $\mathbf{SL}_2$ are smooth group schemes, so $\mathbf{H}$ is smooth ([KMRT98, (22.12)].) □

PROPOSITION 10.9. *The group* $\mathbf{ker}\,\Phi$ *is split unipotent.*

Proof. Consider the ideal in the ring of regular functions $F[\alpha,\gamma,\delta,\mu,\nu]$ of $\mathbf{ker}\,\Phi$ generated by α,γ,δ,μ. It is a Hopf ideal because α,γ,δ,μ are primitive elements. Hence it corresponds to a closed group subscheme, say Z, isomorphic to $\mathbf{G}_a$ and consisting of all matrices of the form

$$\left(\begin{array}{cc|cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline \nu & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\nu & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

It is straightforward to check that Z is in the center of $\mathbf{ker}\,\Phi$, hence normal. The quotient map $\mathbf{ker}\,\Phi \to (\mathbf{ker}\,\Phi)/Z$ corresponds to the embedding $F[\alpha,\gamma,\delta,\mu] \hookrightarrow F[\alpha,\gamma,\delta,\mu,\nu]$ of Hopf algebras. Since α,γ,δ,μ are primitive elements it follows that $(\mathbf{ker}\,\Phi)/Z \simeq \mathbf{G_a} \times \mathbf{G}_a \times \mathbf{G}_a \times \mathbf{G}_a$ and the claim follows. □

PROPOSITION 10.10. *The scheme* $\mathbf{H}$ *is a closed subgroup of the group scheme of automorphisms* $\mathbf{Aut}(C,\star,n)$ *of the split Okubo algebra and* $\mathbf{H} = \mathbf{Aut}(C,\star,n)_{\mathrm{red}}$.

Proof. The scheme $\mathbf{H}$ is the stabilizer in $\mathbf{Aut}(C,\star,n)$ of the quaternionic idempotent e. Let $\mathcal{A}$ be the Hopf algebra representing $\mathbf{Aut}(C,\star,n)$, and let $\mathcal{I}$ be the Hopf ideal such that $\mathcal{A}/\mathcal{I}$ represent $\mathbf{H}$. Since $\mathbf{H}$ is smooth, $\mathcal{I}$ is a radical ideal. But $\mathbf{H}(\overline{F}) = \mathrm{Aut}(C \otimes_F \overline{F},\star,n) = \mathbf{Aut}(C,\star,n)(\overline{F})$ by Proposition 10.1. By Hilbert's Nullstellensatz $\mathcal{I}$ coincides with the radical of $\mathcal{A}$. Hence the radical of $\mathcal{A}$ is a Hopf ideal, and $\mathbf{H} = \mathbf{Aut}(C,\star,n)_{\mathrm{red}}$. □

11. The group scheme of automorphisms of the split Okubo algebra

We continue assuming that the characteristic of the ground field F is 3.
Given any Okubo algebra $(S,\star,n)$, the arguments in [Eld97] and [Eld99] show that the vector space $A = F\mathbf{1} \oplus S$, with the multiplication given by

$$\begin{aligned} &\mathbf{1}\cdot x = x = x\cdot\mathbf{1}, \\ &a\cdot b = \frac{1}{2}\big(a\star b + b\star a + 2b_n(a,b)\mathbf{1}\big) = -(a\star b + b\star a) + b_n(a,b)\mathbf{1}, \end{aligned} \tag{11.1}$$

for $x \in A$ and $a,b \in S$, is a 9-dimensional commutative and associative algebra. We give a proof for completeness. We start with

LEMMA 11.2. *Every Okubo algebra* $(S,\star,n)$ *is Lie admissible, i.e.,* S *becomes a Lie algebra with bracket* $[x,y]^\star = x\star y - y\star x$.

Proof. This can be proved using the multiplication table in (5.11), but we can proceed in a different way, which has its own independent interest. The result is clear for the unique (split) Okubo algebra over the complex numbers, since it can be defined as the space of zero trace 3×3 matrices with the product

$$x\star y=\frac{yx-\omega xy}{1-\omega}-\tfrac{1}{3}\,\mathrm{trace}(xy)\,1$$

where ω is a primitive cube root of unity and xy denotes the usual matrix multiplication (see (8.6)). In this case $[x,y]^\star=\frac{\omega^2}{1-\omega}(xy-yx)$, so the Jacobi identity holds, and the Okubo algebra is Lie admissible. It has a basis $\{e_1,f_1,u_1,v_1,u_2,v_2,u_3,v_3\}$ with the same multiplication table as in (5.11). Consider the subalgebra E over $\mathbb{Z}$ spanned by the elements of this basis. Then $(E,\star)$ is a Lie algebra over $\mathbb{Z}$ with the bracket $[x,y]^\star=x\star y-y\star x$, and so is $(E\otimes_{\mathbb{Z}}F,\star)$ over F. But $(E\otimes_{\mathbb{Z}}F,\star)$ is the split Okubo algebra over F. Hence the split Okubo algebra is Lie-admissible, and the result follows since any Okubo algebra splits after a scalar extension. □

Proposition 11.3. *The algebra $(A,\cdot)$, with multiplication* (11.1)*, is a commutative and associative algebra. Moreover, if $(S,\star,n)$ is split, then $(A,\cdot)$ is isomorphic to the algebra of truncated polynomials $F[X,Y]/(X^3,Y^3)$.*

Proof. By its own definition $(A,\cdot)$ is commutative. To prove the associativity, it is enough to check that any associator $(a,b,c)^\cdot=(a\cdot b)\cdot c-a\cdot(b\cdot c)$ is trivial. This is clear if one of the elements is $\mathbf{1}$, and hence we may assume that a,b,c are in S. Then we get, using Lemma 11.2,

$$\begin{aligned}
0 &= \big[[a,b]^\star,c\big]^\star+\big[[b,c]^\star,aig\big]^\star+\big[[c,a]^\star,b\big]^\star\\
&= (a\star c-c\star a)\star b-b\star(a\star c-c\star a)+(c\star b-b\star c)\star a\\
&\qquad -a\star(b\star c-c\star b)+(b\star a-a\star b)\star c-c\star(b\star a-a\star b)\\
&= -(b\star c)\star a+b_n(a,b)c+(b\star a)\star c-b_n(b,c)a+c\star(a\star b)\\
&\qquad -b_n(b,c)a-a\star(c\star b)+b_n(a,b)c+(c\star b-b\star c)\star a\\
&\qquad\quad -a\star(b\star c-c\star b)+(b\star a-a\star b)\star c-c\star(b\star a-a\star b)\\
&= (c\star b+b\star c)\star a+a\star(c\star b+b\star c)+b_n(b,c)a\\
&\qquad -(b\star a+a\star b)\star c-c\star(b\star a+a\star b)-b_n(a,b)c\\
&= a\cdot(b\cdot c)-(a\cdot b)\cdot c=-(a,b,c)^\cdot,
\end{aligned}$$

where we have used that $b_n(a,b\star c)=b_n(a\star b,c)$ and $b_n(a,c\star b)=b_n(b\star a,c)$. Therefore $(A,\cdot)$ is associative. Take now a canonical basis of the split Okubo algebra $(C,\triangle,n)$ as in (5.11). The elements

$$x=e_1-\mathbf{1}\quad\text{and}\quad y=u_1-\mathbf{1}\tag{11.4}$$

satisfy $x^{\cdot 3}=0=y^{\cdot 3}$, and the algebra $(A,\cdot)$ is isomorphic to the truncated polynomial algebra $F[X,Y]/(X^3,Y^3)$, with x and y corresponding to the classes of X and Y modulo (X^3,Y^3). □

Remark 11.5. For any $a\in S$, the third power $a^{\cdot 3}$ in A is

$$\begin{aligned}
a^{\cdot 3}&=(a\cdot a)\cdot a=(a\star a-n(a)\mathbf{1})\cdot a\\
&=(a\star a)\star a+b_n(a\star a,a)\mathbf{1}-n(a)a=g(a)\mathbf{1},
\end{aligned}$$

where $g(x)=b_n(x,x\star x)$ is the cubic form considered in Lemma 9.17.

Consider now the split Okubo algebra $(C, \triangle, n)$ and the associated commutative and associative algebra $(A, \cdot)$, $A = F\mathbf{1} \oplus C$, as above. With the notation used in the previous proof, the ideal N generated by the elements x and y in $(A, \cdot)$ is the radical of $(A, \cdot)$ and $N^{\cdot 4} = F(x^{\cdot 2} \cdot y^{\cdot 2})$. A straightforward computation gives

$$x^{\cdot 2} \cdot y^{\cdot 2} = \mathbf{1} + (e_1 + f_1 + u_1 + v_1 + u_2 + v_2 + u_3 + v_3) = \mathbf{1} + e, \tag{11.6}$$

where e is the quaternionic idempotent of $(C, \triangle, n)$ (Lemma 9.3).

Any automorphism φ of $(C, \triangle, n)$ extends to a unique automorphism of $(A, \cdot)$ by means of $\varphi(\mathbf{1}) = \mathbf{1}$. This is also valid if we extend scalars to a unital commutative associative algebra over F. As usual, given a unital commutative associative algebra R over F, N_R will denote $N \otimes_F R$. Recall that the algebraic group $\mathbf{H}$ is the stabilizer in $\mathbf{Aut}(C, \triangle, n)$ of the quaternionic idempotent e. (Initially, $\mathbf{H}$ was defined in terms of the model $(C, \star, n)$ in Example 9.4 of the split Okubo algebra, but this should cause no confusion.)

LEMMA 11.7. *Let $(C, \triangle, n)$ be the split Okubo algebra. Given any unital commutative associative algebra R over F, an element φ in $\mathbf{Aut}(C, \triangle, n)(R)$ lies in $\mathbf{H}(R)$ if and only if, when extended to an automorphism of $(A_R, \cdot)$, it satisfies $\varphi((N_R)^{\cdot 4}) = (N_R)^{\cdot 4}$.*

Proof. If an automorphism φ of $(C_R, \triangle, n)$ fixes e, then its extension to $(A_R, \cdot)$ fixes $\mathbf{1} + e$ and hence φ preserves $(N_R)^{\cdot 4}$. Conversely, if φ preserves $(N_R)^{\cdot 4}$, then there is an element $\alpha \in R$ such that $\varphi(\mathbf{1} + e) = \alpha(\mathbf{1} + e)$. But $\varphi(\mathbf{1}) = \mathbf{1}$ and $\varphi(e) \in C_R$. Hence $\alpha = 1$ and φ fixes e. □

The elements e_1 and u_1 generate the algebra $(C, \triangle)$, and the assignment $\deg(e_1) = (\bar{1}, \bar{0})$, $\deg(u_1) = (\bar{0}, \bar{1})$, endows $(C, \triangle)$ with a grading by $\mathbb{Z}_3 \times \mathbb{Z}_3$. This gives a morphism of group schemes

$$\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3 \to \mathbf{Aut}(C, \triangle, n),$$

such that for any $\alpha, \beta \in \boldsymbol{\mu}_3(R) \times \boldsymbol{\mu}_3(R)$, i.e., $\alpha^3 = \beta^3 = 1$, the image of (α, β) is the automorphism $\psi_{\alpha,\beta}$ of $(C \otimes_F R, \triangle)$ with the property that $\psi_{\alpha,\beta}(e_1) = \alpha e_1$ and $\psi_{\alpha,\beta}(u_1) = \beta u_1$. The image $\mathbf{D}$ of this morphism is isomorphic to $\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3$. Moreover, since $b_n(e_1, e_1 \triangle e_1) = 1$ and $b_n(u_1, u_1 \triangle u_1) = 1$, it turns out that $\mathbf{D}$ is the group scheme of diagonal automorphisms relative to the basis of $(C, \triangle, n)$ in Table (5.11).

THEOREM 11.8. *Let $(C, \triangle, n)$ be the split Okubo algebra. For any unital commutative associative algebra R over F, $\mathbf{Aut}(C, \triangle, n)(R) = \mathbf{H}(R)\mathbf{D}(R)$. Moreover $\mathbf{H}(R) \cap \mathbf{D}(R) = 1$.*

Proof. Since the quaternionic idempotent is $e = e_1 + f_1 + u_1 + v_1 + u_2 + v_2 + u_3 + v_3$ (the sum of the elements in the basis, see Lemma 9.3), an automorphism fixes e and is diagonal relative to this basis if and only if it is the identity. Hence we have $\mathbf{H}(R) \cap \mathbf{D}(R) = 1$.
Let φ be an element in $\mathbf{Aut}(C, \triangle, n)(R)$. Extend it to an automorphism of $(A_R, \cdot)$. Then, with x and y as in (11.4), there are elements $\mu, \nu \in R$ such that

$\varphi(x)-\mu\mathbf{1} \in N_R$ and $\varphi(y)-\nu\mathbf{1} \in N_R$. Since $x^{\cdot 3} = 0 = y^{\cdot 3}$ and N_R is an ideal, we have $\mu^3 = \nu^3 = 0$. Consider the element $(\alpha, \beta) := (\mu+1, \nu+1) \in \boldsymbol{\mu}_3(R)\times\boldsymbol{\mu}_3(R)$ and the automorphism $\psi_{\alpha,\beta} \in \mathbf{D}(R)$. Then we have

$$\begin{aligned}
\varphi\psi_{\alpha,\beta}^{-1}(x) &= \varphi\psi_{\alpha^2,\beta^2}(x) \\
&= \varphi\psi_{\alpha^2,\beta^2}(e_1 - \mathbf{1}) \\
&= \varphi(\alpha^2 e_1 - \mathbf{1}) = \varphi\bigl(\alpha^2 x + (\alpha^2 - 1)\mathbf{1}\bigr) \\
&\equiv (\alpha^2 - 1)\mathbf{1} + \alpha^2\mu\mathbf{1} \quad \text{modulo } N_R \\
&\equiv \bigl(\alpha^2 - 1 + \alpha^2(\alpha - 1)\bigr)\mathbf{1} \equiv 0 \quad \text{modulo } N_R, \text{ as } \alpha^3 = 1,
\end{aligned}$$

so $\varphi\psi_{\alpha,\beta}^{-1}(x) \in N_R$. In a similar vein we get $\varphi\psi_{\alpha,\beta}^{-1}(y) \in N_R$, and this proves $\varphi\psi_{\alpha,\beta}^{-1}(N_R) = N_R$, and hence $\varphi\psi_{\alpha,\beta}^{-1}\bigl((N_R)^{\cdot 4}\bigr) = (N_R)^{\cdot 4}$. By Lemma 11.7 we conclude that $\varphi\psi_{\alpha,\beta}^{-1}$ lies in $\mathbf{H}(R)$, and hence $\varphi = \bigl(\varphi\psi_{\alpha,\beta}^{-1}\bigr)\psi_{\alpha,\beta} \in \mathbf{H}(R)\mathbf{D}(R)$. □

Proposition 11.9. *Neither of the subgroups* $\mathbf{H}$ *and* $\mathbf{D}$ *of* $\mathbf{Aut}(C, \triangle, n)$ *is normal.*

Proof. Consider the automorphism φ of the split Cayley algebra $(C, \cdot, n)$ in (5.1) defined on the canonical basis as

$$\varphi(e_1) = e_1,\ \varphi(f_1) = f_1 \quad \text{and} \quad \varphi(u_i) = -(u_i + u_{i+1}),\ \varphi(v_i) = v_i + v_{i+1} - v_{i+2},$$

for $i = 1, 2, 3$ (indices modulo 3). It is straightforward to check that φ is indeed an automorphism, and that it commutes with the automorphism s_π that permutes cyclically the u_i's and the v_i's. Therefore φ is an automorphism of the split Okubo algebra $(C, \triangle, n)$, where $\triangle = \cdot_{s_\pi}$ is the Petersson twist. And φ preserves the quaternionic idempotent, so φ is in $\mathbf{H}(F)$. Its inverse fixes e_1 and f_1 and satisfies

$$\varphi^{-1}(u_i) = u_i - u_{i+1} + u_{i+2}, \quad \varphi^{-1}(v_i) = -v_i - v_{i+2}$$

for $i = 1, 2, 3$. For any unital commutative and associative algebra R over F containing an element $1 \neq \alpha \in \boldsymbol{\mu}_3(R)$, consider the automorphism $\psi_{\alpha,1} \in \mathbf{D}(R)$. Then the commutator $[\varphi, \psi_{\alpha,1}] = \varphi^{-1}\psi_{\alpha,1}^{-1}\varphi\psi_{\alpha,1}$ fixes e_1 and takes u_1 to

$$\begin{aligned}
\varphi^{-1}\psi_{\alpha,1}^{-1}\varphi\psi_{\alpha,1}(u_1) &= \varphi^{-1}\psi_{\alpha,1}^{-1}\varphi(u_1) \\
&= \varphi^{-1}\psi_{\alpha,1}^{-1}(-u_1 - u_2) \\
&= \varphi^{-1}(-u_1 - \alpha u_2) \\
&= -(u_1 - u_2 + u_3) - \alpha(u_2 - u_3 + u_1) \\
&= -(1 + \alpha)u_1 + (1 - \alpha)u_2 - (1 - \alpha)u_3.
\end{aligned}$$

In particular $\Phi = [\varphi, \psi_{\alpha,1}]$ is not in $\mathbf{D}(R)$, as it does not act diagonally in our basis. But it does not belong to $\mathbf{H}(R)$ either because it does not fix the quaternionic idempotent $e = e_1 + f_1 + u_1 + v_1 + u_2 + v_2 + u_3 + v_3$. To check this, note that $\Phi(u_2) = -\Phi(u_1 \triangle e_1) = -(1+\alpha)u_2 + (1-\alpha)u_3 - (1-\alpha)u_1$ and

$\Phi(u_3) = -\Phi(u_2 \vartriangle e_1) = -(1+\alpha)u_3 + (1-\alpha)u_1 - (1-\alpha)u_2$. In the same vein $\Phi(v_i)$ belongs to the linear span of v_1, v_2, v_3. Then

$$\begin{aligned}\Phi(e) &= \Phi(e_1 + f_1) + \Phi(u_1 + u_2 + u_3) + \Phi(v_1 + v_2 + v_3) \\ &= e_1 + f_1 - (1+\alpha)(u_1 + u_2 + u_3) + \text{a linear combination of } v_1 v_2, v_3 \\ &\neq e.\end{aligned}$$

□

REMARK 11.10. The subscheme $\mathbf{H}$ embeds in the simple group scheme $\mathbf{Aut}(C, \cdot, n)$ of type G_2 (Proposition 10.1). However, *there is no such embedding for* $\mathbf{G} = \mathbf{Aut}(C, \vartriangle, n)$. Actually, an embedding $\mathbf{G} \to \mathbf{Aut}(C, \cdot, n)$ would give a monomorphism of Lie algebras

$$\iota\colon \operatorname{Lie}(\mathbf{G}) = \operatorname{Der}(C, \vartriangle, n) \to \operatorname{Der}(C, \cdot, n).$$

But $\operatorname{Der}(C, \vartriangle, n)$ contains a simple nonclassical ideal $\mathfrak{i}$ of dimension 8 [Eld99], while $\mathfrak{g}$ is not simple (the characteristic is 3!) but contains a simple ideal $\mathfrak{j}$ of dimension 7 such that the quotient $\mathfrak{g}/\mathfrak{j}$ is again simple and isomorphic, as a Lie algebra, to $\mathfrak{j}$ (see [AEMN02]). Then ι would induce a Lie algebra homomorphism $\mathfrak{i} \to \mathfrak{g}/\mathfrak{j}$, which must be 0 by dimension count, and this would give that $\mathfrak{i}$ embeds in $\mathfrak{j}$, a contradiction, again by dimension count.

12. THE CLASSIFICATION OF OKUBO ALGEBRAS REVISITED

The classification of the Okubo algebras in characteristic 3 was obtained in [Eld97, §5], using previous classification results in [EP96, Theorems B and 5.1]. Here this classification will be revisited in light of the results in the preceding sections.
Still the characteristic of F is assumed to be 3 in this section.

PROPOSITION 12.1. *The cohomology set* $H^1_{\mathrm{fppf}}(F, \mathbf{H})$ *is trivial.*

Proof. The group $\mathbf{H}$ is the semidirect product of the split unipotent group $\mathbf{ker}\,\Phi$ (see Theorem 10.3) and of $\mathbf{SL}_2$. Hence we have an exact sequence

$$H^1_{\mathrm{fppf}}(F, \mathbf{ker}\,\Phi) \to H^1_{\mathrm{fppf}}(F, \mathbf{H}) \to H^1_{\mathrm{fppf}}(F, \mathbf{SL}_2).$$

The left and right terms are trivial, so is the central term. □

REMARK 12.2. Alternatively, one may proceed as follows. Let $(C, \vartriangle, n)$ be the split Okubo algebra and let e be its (unique) quaternionic idempotent. The group scheme $\mathbf{H}$ is the subscheme of $\mathbf{Aut}(C, \vartriangle, n)$ consisting of those elements fixing e:

$$\mathbf{H} = \mathbf{Aut}(C, \vartriangle, n, e).$$

The twisted forms of $(C, \vartriangle, n, e)$ are the Okubo algebras with a quaternionic idempotent. But Proposition 9.9 asserts that the only Okubo algebra containing a quaternionic idempotent is the split Okubo algebra. This shows that $H^1_{\mathrm{fppf}}(F, \mathbf{H})$ is trivial.

Also, $\mathbf{H}$ is smooth and hence we have ([Wat79, (18.5) and (17.8)]):

$$\begin{aligned} H^1_{\mathrm{fppf}}(F,\mathbf{H}) &= H^1_{\text{ét}}(F,\mathbf{H}) = H^1\big(\Gamma,\mathbf{H}(F_{\mathrm{sep}})\big) \\ &= H^1\big(\Gamma,\mathrm{Aut}(C\otimes_F F_{\mathrm{sep}},\triangle,n)\big) = H^1_{\text{ét}}\big(F,\mathbf{Aut}(C,\triangle,n)\big), \end{aligned}$$

where F_{sep} denotes the separable closure of F in $\overline{F}$ and Γ the Galois group of the extension F_{sep}/F. (Note that over any field extension K of F, $\mathbf{H}(K)$ exhausts $\mathrm{Aut}(C\otimes_F K,\triangle,n)$, since $\mathbf{D}(K)=1$.)
Hence we conclude, without making any appeal to the classification in [Eld97], that the only twisted F_{sep}/F-form of $(C,\triangle,n)$ is, up to isomorphism, $(C,\triangle,n)$ itself:

Corollary 12.3. *The cohomology set $H^1_{\text{ét}}\big(F,\mathbf{Aut}(C,\triangle,n)\big)$ is trivial. In particular, if F is perfect, there is up to isomorphism a unique Okubo algebra over F.*

Take now nonzero elements α, $\beta\in F$, and let $(C_{\alpha,\beta},\triangle,n)$ be the F-subalgebra of $(C\otimes_F\overline{F},\triangle,n)$ generated by the elements $e_1\otimes\alpha^{\frac13}$ and $u_1\otimes\beta^{\frac13}$. This is a twisted form of $(C,\triangle,n)$. Denote by $g_{\alpha,\beta}$ the cubic form on $(C_{\alpha,\beta},\triangle,n)$ given by $g_{\alpha,\beta}(x)=b_n(x,x\triangle x)$. The image $g_{\alpha,\beta}(C_{\alpha,\beta})$ is a F^3-subspace of F spanned by the elements α, α^2, β, β^2, $\alpha\beta$, $\alpha^2\beta^2$, $\alpha\beta^2$, $\alpha^2\beta$. Moreover, under the bijection

$$H^1_{\mathrm{fppf}}\big(F,\mathbf{Aut}(C,\triangle,n)\big)\cong\{\text{Isomorphism classes of Okubo algebras}\},$$

the isomorphism class of $(C_{\alpha,\beta},\triangle,n)$ corresponds to the class of the cocycle

$$\Psi^{\alpha,\beta}\in\mathbf{Aut}(C,\triangle,n)(\overline{F}\otimes_F\overline{F})=\mathrm{Aut}(C\otimes_F\overline{F}\otimes_F\overline{F}) \tag{12.4}$$

determined by

$$\Psi^{\alpha,\beta}(e_1\otimes1\otimes1)=e_1\otimes\alpha^{-\frac13}\otimes\alpha^{\frac13},\qquad \Psi^{\alpha,\beta}(u_1\otimes1\otimes1)=u_1\otimes\beta^{-\frac13}\otimes\beta^{\frac13}.$$

Note that $\alpha^{-\frac13}\otimes\alpha^{\frac13}$ and $\beta^{-\frac13}\otimes\beta^{\frac13}$ are cube roots of unity, so $\Psi^{\alpha,\beta}$ belongs to the subscheme $\mathbf{D}$.

Remark 12.5. The scheme $\mathbf{D}$ is isomorphic to $\boldsymbol{\mu}_3\times\boldsymbol{\mu}_3$, so

$$H^1_{\mathrm{fppf}}(F,\mathbf{D})=\{[\Psi^{\alpha,\beta}]\mid\alpha,\beta\in F^\times\},$$

where $[\Psi^{\alpha,\beta}]=[\Psi^{\alpha',\beta'}]$ if and only if $\alpha'\alpha^{-1},\beta'\beta^{-1}\in F^3$, see [Wat79, 18.2(a)]. In particular, over perfect fields, $H^1_{\mathrm{fppf}}(F,\mathbf{D})$ is trivial.

Theorem 12.6. *The mapping $H^1_{\mathrm{fppf}}(F,\boldsymbol{\mu}_3\times\boldsymbol{\mu}_3)\to H^1_{\mathrm{fppf}}\big(F,\mathbf{Aut}(C,\triangle,n)\big)$, induced by the inclusion $\mathbf{D}\hookrightarrow\mathbf{Aut}(C,\triangle,n)$, is surjective.*

Proof. This is trivial if F is perfect, in particular if F is finite. Hence we will assume that F is infinite. Let $\zeta\in Z^1_{\mathrm{fppf}}\big(F,\mathbf{Aut}(C,\triangle,n)\big)$ be a cocycle and let $(C^\zeta,\triangle,n)$ be the Okubo algebra $C^\zeta=\{x\in C\otimes_F\overline{F}\mid\zeta(x\otimes1)=\theta(x\otimes1)\}$, where $\theta\colon C\otimes_F\overline{F}\otimes_F\overline{F}\to C\otimes_F\overline{F}\otimes_F\overline{F}$, $x\otimes\alpha\otimes\beta\mapsto x\otimes\beta\otimes\alpha$.

Since the norm of an Okubo algebra is isotropic (Lemma 9.17), C^ζ has a basis of isotropic elements and hence there are elements $x \in C^\zeta$ with $n(x) = 0$ and $g(x) = \alpha \neq 0$. Then, by Corollary 9.20, we may take

$$x \in O_0^\alpha(F) \subset C^\zeta \subset C^\zeta \otimes_F \overline{F} = C \otimes_F \overline{F}.$$

Then Theorem 9.18 shows that there is an automorphism $\varphi \in \mathrm{Aut}(C \otimes_F \overline{F}, \triangle, n)$ such that $\varphi(e_1 \otimes 1) = \alpha^{-\frac{1}{3}} x$. Replacing ζ by an equivalent cocycle we may assume $x = e_1 \otimes \alpha^{\frac{1}{3}}$. The subspace $\{z \in C \mid e_1 \triangle z = 0,\, b_n(e_1, z) = 0\}$ is spanned by u_1, u_2, u_3, thus the subspace $\{y \in C^\zeta \mid x \triangle y = 0,\, b_n(x, y) = 0\}$ has dimension 3 and it is not contained in the kernel of the semilinear map g. Take an element $y \in C^\zeta$ with $x \triangle y = 0$, $b_n(x, y) = 0$ and $g(y) = \beta \neq 0$. Theorem 9.18 shows that there is an automorphism $\psi \in \mathrm{Aut}(C \otimes_F \overline{F}, \triangle, n)$ such that $\psi(e_1 \otimes 1) = e_1 \otimes 1$, $\psi(u_1 \otimes 1) = \beta^{-\frac{1}{3}} y$. Replacing ζ by an equivalent cocycle we may assume that $x = e_1 \otimes \alpha^{\frac{1}{3}}$ and $y = u_1 \otimes \beta^{\frac{1}{3}}$ belong to C^ζ. Therefore $\zeta(e_1 \otimes \alpha^{\frac{1}{3}} \otimes 1) = e_1 \otimes 1 \otimes \alpha^{\frac{1}{3}}$, so $\zeta(e_1 \otimes 1 \otimes 1) = e_1 \otimes \alpha^{-\frac{1}{3}} \otimes \alpha^{\frac{1}{3}}$, and similarly $\zeta(u_1 \otimes 1 \otimes 1) = u_1 \otimes \beta^{-\frac{1}{3}} \otimes \beta^{\frac{1}{3}}$. Hence we get $\zeta = \Psi^{\alpha,\beta}$ (see (12.4)). □

This shows that any Okubo algebra is isomorphic to $(C_{\alpha,\beta}, \triangle, n)$ for some nonzero α, $\beta \in F$. Moreover, the set of isomorphism classes of Okubo algebras is a quotient set of $H^1_{\mathrm{fppf}}(F, \boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3) \cong F^\times/(F^\times)^3 \times F^\times/(F^\times)^3$. It remains to consider the isomorphism conditions:

Theorem 12.7 (see [Eld97]). *For $\alpha, \beta, \alpha', \beta' \in F^\times$, the algebras $(C_{\alpha,\beta}, \triangle, n)$ and $(C_{\alpha',\beta'}, \triangle, n)$ are isomorphic if and only if*

$$g_{\alpha,\beta}(C_{\alpha,\beta}) = g_{\alpha',\beta'}(C_{\alpha',\beta'})$$

and if this common F^3-subspace has dimension 8 (i.e., $g_{\alpha,\beta}$ and $g_{\alpha',\beta'}$ are bijective), then the following restriction is required too:

$$g^{-1}_{\alpha,\beta}(\alpha') \triangle g^{-1}_{\alpha,\beta}(\beta') = 0$$

(this is a product of two elements in $(C_{\alpha,\beta}, \triangle, n)$).

Proof. The result is trivial for perfect fields, and hence for finite fields. Hence we will assume that the ground field F is infinite. If the compositions $(C_{\alpha,\beta}, \triangle, n)$ and $(C_{\alpha',\beta'}, \triangle, n)$ are isomorphic, any isomorphism φ satisfies $n\big(\varphi(x)\big) = n(x)$ and $g_{\alpha',\beta'}\big(\varphi(x)\big) = g_{\alpha,\beta}(x)$ for any x, so $g_{\alpha,\beta}(C_{\alpha,\beta}) = g_{\alpha',\beta'}(C_{\alpha',\beta'})$. Moreover, Lemma 9.17 shows that $F^3 + g_{\alpha,\beta}(C_{\alpha,\beta})$ is a subfield of F. If $g_{\alpha,\beta}(C_{\alpha,\beta}) = F^3$, then the elements x and y in the proof of Theorem 12.6 can be taken with $g(x) = 1 = g(y)$, and hence $C_{\alpha,\beta}$ is isomorphic to $C_{1,1}$, which is the split Okubo algebra. If the semilinear map $g_{\alpha,\beta}$ is not bijective, then $\dim_{F^3}\big(F^3 + g_{\alpha,\beta}(C_{\alpha,\beta})\big)$ equals 1 or 3, and the dimension of $\ker g_{\alpha,\beta}$ is at least 5. Hence the restriction of the norm to $\ker g_{\alpha,\beta}$ is isotropic, so there exists a hyperbolic pair a, b in $\ker g_{\alpha,\beta}$: $n(a) = n(b) = 0$, $b_n(a, b) = 1$. Take $x \in \ker g_{\alpha,\beta}$ with $n(x) \neq 0$. Then $g_{\alpha,\beta}(x \triangle x) = n(x)^3$ (Lemma 9.17), so $F^3 \subseteq g_{\alpha,\beta}(C_{\alpha,\beta})$. If $\gamma \in g_{\alpha,\beta}(C_{\alpha,\beta}) \setminus F^3$ and $x \in C_{\alpha,\beta}$ satisfies $g(x) = \gamma$, then we may find μ, $\nu \in F$

such that the element $x' = x+\mu a+\nu b$ is isotropic (and satisfies $g(x') = \gamma$). The three-dimensional subspace $\{z \in C_{\alpha,\beta} \mid x' \vartriangle z = 0,\, b_n(x', z) = 0\}$ intersects nontrivially the six-dimensional subspace $g_{\alpha,\beta}^{-1}(F^3)$, so we may take an element y' satisfying $x' \vartriangle y' = 0$, $b_n(x', y') = 0$ and $g_{\alpha,\beta}(y') = 1$. The proof of Theorem 12.6 shows that $C_{\alpha,\beta}$ is isomorphic to $C_{\gamma,1}$ for any $0 \neq \gamma \in g_{\alpha,\beta}(C_{\alpha,\beta}) \setminus F^3$. We are left with the case in which $g_{\alpha,\beta}$ is one-to-one, so $F^3 \cap g_{\alpha,\beta}(C_{\alpha,\beta}) = 0$ and $F^3 + g_{\alpha,\beta}(C_{\alpha,\beta})$ is a field extension of degree 9 of F^3. Then, if $g_{\alpha,\beta}(C_{\alpha,\beta}) = g_{\alpha',\beta'}(C_{\alpha',\beta'})$, then both $g_{\alpha,\beta}$ and $g_{\alpha',\beta'}$ are one-to-one, so there is a unique linear map $\phi\colon C_{\alpha',\beta'} \to C_{\alpha,\beta}$ such that $g_{\alpha',\beta'}(x) = g_{\alpha,\beta}\big(\phi(x)\big)$ for any $x \in C_{\alpha',\beta'}$ (so $\phi = g_{\alpha,\beta}^{-1} g_{\alpha',\beta'}$). Then, by the uniqueness of ϕ, $C_{\alpha,\beta}$ and $C_{\alpha',\beta'}$ are isomorphic if and only if ϕ is an isomorphism. But Lemma 9.17 shows that

$$\begin{aligned} g_{\alpha,\beta}\big(\phi(x)^{\vartriangle 2}\big) &= g_{\alpha,\beta}\big(\phi(x)\big)^2 + n\big(\phi(x)\big)^3 \\ &= g_{\alpha',\beta'}(x)^2 + n\big(\phi(x)\big)^3 \\ &= g_{\alpha',\beta'}(x^{\vartriangle 2}) - n(x)^3 + n\big(\phi(x)\big)^3, \end{aligned}$$

thus $g_{\alpha,\beta}(\phi(x)^{\vartriangle 2}) - g_{\alpha',\beta'}(x^{\vartriangle 2}) \in g_{\alpha,\beta}(C_{\alpha,\beta}) \cap F^3 = 0$ and $\phi(x \vartriangle x) = \phi(x) \vartriangle \phi(x)$ for any $x \in C_{\alpha',\beta'}$. This implies, using $(x \vartriangle x) \vartriangle x = x \vartriangle (x \vartriangle x) = n(x)x$, that $n\big(\phi(x)\big) = n(x)$ for any $x \in C_{\alpha',\beta'}$. If ϕ is an isomorphism, then $\phi(e_1 \otimes (\alpha')^{\frac{1}{3}}) \vartriangle \phi(u_1 \otimes (\beta')^{\frac{1}{3}}) = 0$, which is equivalent to $g_{\alpha,\beta}^{-1}(\alpha') \vartriangle g_{\alpha,\beta}^{-1}(\beta') = 0$. Conversely, if $\phi(e_1 \otimes (\alpha')^{\frac{1}{3}}) \vartriangle \phi(u_1 \otimes (\beta')^{\frac{1}{3}}) = 0$, then since the subspaces

$$\{\phi(e_1 \otimes (\alpha')^{\frac{1}{3}}), \phi(e_1 \otimes (\alpha')^{\frac{1}{3}})^{\vartriangle 2}\} \quad \text{and} \quad \{\phi(u_1 \otimes (\beta')^{\frac{1}{3}}), \phi(u_1 \otimes (\beta')^{\frac{1}{3}})^{\vartriangle 2}\}$$

are orthogonal subspaces of $C_{\alpha,\beta}$, [Eld09, Theorem 3.12] shows that $C_{\alpha,\beta}$ and $C_{\alpha',\beta'}$ are isomorphic. $\square$

In the last part of this section we show that Okubo algebras with isomorphic automorphisms groups (as algebraic groups) are isomorphic or anti-isomorphic (compare with Proposition 8.10). We start with a definition:

Definition 12.8. Let $(S, *, n_S)$ be an Okubo algebra and let K be a 2-dimensional composition subalgebra (i.e., the restriction of n_S to the subalgebra K is nondegenerate). Then K is said to be a *regular subalgebra* if the para unit of $K \otimes_F \overline{F}$ is a quadratic idempotent.

Lemma 12.9. *Let $(C, \vartriangle, n)$ be the (split) Okubo algebra over an algebraically closed field F (as in Table* (5.11)*), and let K be a two-dimensional composition subalgebra. Then the following conditions are equivalent:*

(i) *K is regular.*

(ii) *There is an automorphism φ of $(C, \vartriangle, n)$ such that $\varphi(K) = Fe_1 + Ff_1$.*

(iii) *There is another two-dimensional composition subalgebra K' of $(C, \vartriangle, n)$ orthogonal to K: $b_n(K, K') = 0$.*

Proof. (i)$\Rightarrow$(ii) K is, up to isomorphism, the unique para-quadratic algebra over the algebraically closed field F, so there is a basis $\{a, b\}$ of K such that

$a \vartriangle a = b$, $b \vartriangle b = a$, $a \vartriangle b = 0 = b \vartriangle a$, and $n(a) = 0 = n(b)$, $b_n(a, b) = 1$. Since K is regular $a + a \vartriangle a = a + b$ is a quadratic idempotent, and by Theorem 9.18 there is an automorphism φ of $(C, \vartriangle, n)$ such that $\varphi(a) = e_1$; then $\varphi(K) = Fe_1 + Ff_1$.
(ii)⇒(iii) This is clear since $Fu_1 + Fv_1$ is a two-dimensional composition subalgebra of $(C, \vartriangle, n)$ orthogonal to $Fe_1 + Ff_1$.
(iii)⇒(i) Both K and K' are para-quadratic algebras. Take bases $\{x, x \vartriangle x\}$ and $\{y, y \vartriangle y\}$ of K and K' respectively, with $n(x) = 0 = n(y)$ and $b_n(x, x \vartriangle x) = 1 = b_n(y, y \vartriangle y)$. Then [Eld09, Theorem 3.12] shows that either $x \vartriangle y = 0$ or $y \vartriangle x = 0$. But $y \vartriangle x = 0$ implies $x \vartriangle (y \vartriangle y) = -y \vartriangle (y \vartriangle x) = 0$, so replacing y by $y \vartriangle y$ we may assume $x \vartriangle y = 0$. Then by Theorem 9.18 there is an automorphism of $(C, \vartriangle, n)$ such that $\varphi(x) = e_1$ and $\varphi(y) = u_1$. The para-unit $x + x \vartriangle x$ of K corresponds to $e_1 + f_1$, which is a quadratic idempotent. □

Lemma 12.10. *Let K be a regular two-dimensional composition subalgebra of an Okubo algebra $(S, *, n_S)$. Let x, $y \in K$ be two nonzero elements such that the endomorphism $\mathrm{ad}^*_x \mathrm{ad}^*_y$ is diagonalizable (where $\mathrm{ad}^*_x \colon z \mapsto [x, z]^* = x*z - z*x$). Then $n(x) = 0$, y is a scalar multiple of $x * x$, and the eigenvalues of $(\mathrm{ad}^*_x)^3$ are 0, with multiplicity 2, and $\pm g_S(x)$, each with multiplicity 3. (Recall that $g_S(x) = b_{n_S}(x, x * x)$.)*

Proof. Extending scalars to $\overline{F}$ we may assume by Lemma 12.9 that $(S, *, n_S)$ is the algebra $(C, \vartriangle, n)$ in Table (5.11) and that $K = Fe_1 + Ff_1$. Hence $x = \alpha e_1 + \beta f_1$ and $y = \alpha' e_1 + \beta' f_1$. Without loss of generality we may assume $\alpha \neq 0$. The subspace $U = Fu_1 + Fu_2 + Fu_3$ is invariant under $\mathrm{ad}^{\vartriangle}_x \mathrm{ad}^{\vartriangle}_y$, and the coordinate matrix of the restriction of $\mathrm{ad}^{\vartriangle}_x$ and $\mathrm{ad}^{\vartriangle}_y$ to U in the basis $\{u_1, u_2, u_3\}$ are $\alpha C - \beta C^2$ and $\alpha' C - \beta' C^2$ where $C = \left(\begin{smallmatrix} 0&0&1\\ 1&0&0\\ 0&1&0 \end{smallmatrix}\right)$. Hence the coordinate matrix of the restriction of $\mathrm{ad}^{\vartriangle}_x \mathrm{ad}^{\vartriangle}_y$ to U is

$$(\alpha C - \beta C^2)(\alpha' C - \beta' C^2) = -(\alpha\beta' + \beta\alpha')I + \beta\beta' C + \alpha\alpha' C^2,$$

because $C^3 = I$ (the identity matrix). Therefore we have

$$\left(\mathrm{ad}^{\vartriangle}_x \mathrm{ad}^{\vartriangle}_y |_U\right)^3 = (-\alpha\beta' - \beta\alpha' + \beta\beta' + \alpha\alpha')^3 I = \left((\alpha - \beta)(\alpha' - \beta')\right)^3 I,$$

and hence $\mathrm{ad}^{\vartriangle}_x \mathrm{ad}^{\vartriangle}_y |_U - (\alpha - \beta)(\alpha' - \beta')I$ is simultaneously diagonalizable and nilpotent. It follows that $\mathrm{ad}^{\vartriangle}_x \mathrm{ad}^{\vartriangle}_y |_U$ is a scalar multiple of the identity, so $\beta\beta' = \alpha\alpha' = 0$. But $\alpha \neq 0$, so $\alpha' = 0$ and hence (as $y \neq 0$) $\beta' \neq 0$ and $\beta = 0$. Thus $x = \alpha e_1$, $y = \beta' f_1$, and the result follows, since $(\mathrm{ad}^{\vartriangle}_x)^3$ acts trivially on K, by multiplication by $\alpha^3 = g_S(x)$ on U, and by multiplication by $-\alpha^3$ on $V = Fv_1 + Fv_2 + Fv_3$. □

Even if the group $\mathbf{Aut}(S, *, n_S)$ is not smooth for an Okubo algebra over a field of characteristic 3, a result corresponding to Proposition 8.10 holds:

Theorem 12.11. *Let $(S, *, n_S)$ and $(S', \star, n_{S'})$ be two Okubo algebras. The following conditions are equivalent:*

(i) *The algebraic groups* $\mathbf{Aut}(S,*,n)$ *and* $\mathbf{Aut}(S',\star,n_{S'})$ *are isomorphic.*

(ii) *The Lie algebras (see 11.2)* $(S,*)^-$ *and* $(S',\star)^-$ *are isomorphic. (Here* $(S,*)^-$ *denotes the Lie algebra defined over* S *with bracket* $[x,y]^* = x*y-y*x$*.)*

(iii) *The Okubo algebras* $(S,*,n_S)$ *and* $(S',\star,n_{S'})$ *are either isomorphic or anti-isomorphic.*

Proof. (i)$\Rightarrow$(ii) If the algebraic groups $\mathbf{Aut}(S,*,n)$ and $\mathbf{Aut}(S',\star,n_{S'})$ are isomorphic, so are their Lie algebras $\mathrm{Der}(S,*,n_S)$ and $\mathrm{Der}(S',\star,n_{S'})$. But the simple Lie algebra $(S,*)^-$ is isomorphic to the only minimal ideal of $\mathrm{Der}(S,*,n_S)$ [Eld99, Theorem 4], and the same holds for $(S',\star)^-$, whence the result.

(iii)$\Rightarrow$(i) This is straightforward.

(ii)$\Rightarrow$(iii) There are nonzero elements α, $\beta\in F$ such that the Okubo algebra $(S,*,n_S)$ is, up to isomorphism, the Okubo algebra $(C_{\alpha,\beta},\triangle,n)$ (see Theorem 12.6), which is the F-subalgebra of $(C\otimes_F\overline{F},\triangle,n)$ generated by $x=e_1\otimes\alpha^{\frac13}$ and $y=u_1\otimes\beta^{\frac13}$. Note that $x\triangle y=0$. Then, $(S,*,n_S)$ is graded by $\mathbb{Z}_3^2$, with $\deg(x)=(\bar1,\bar0)$ and $\deg(y)=(\bar0,\bar1)$. Therefore $S=\bigoplus_{\mu\in\mathbb{Z}_3^2}S_\mu$, with $\dim S_\mu=1$ for any nonzero $\mu\in\mathbb{Z}_3^2$, $S_0=0$, and $S_\mu*S_\nu\subseteq S_{\mu+\nu}$. Moreover, for any nonzero $\mu\in\mathbb{Z}_3^2$ and any nonzero $z\in S_\mu$, we have

- $n_S(z)=0$,
- $S_{-\mu}=Fz*z$,
- $S_\mu\oplus S_{-\mu}=\ker\mathrm{ad}^*_z\bigl(=\{t\in S\mid t*z=z*t\}\bigr)$, and this coincides with the subalgebra $\mathrm{alg}\langle z\rangle$ generated by z, and
- $\bigoplus_{\nu\neq0,\pm\mu}S_\nu$ equals the subspace orthogonal to $\mathrm{alg}\langle z\rangle$ relative to n_S, and coincides with the image of ad^*_z.

Let $\varphi\colon(S,*)^-\to(S',\star)^-$ be an isomorphism of Lie algebras. Then the Lie algebra $(S',\star)^-$ inherits a grading by $\mathbb{Z}_3^2$: $S'=\bigoplus_{\nu\in\mathbb{Z}_3^2}S'_\mu$, with $S'_\mu=\varphi(S_\mu)$ for any $\mu\in\mathbb{Z}_3^2$. Since φ is an isomorphism of Lie algebras, for any nonzero $\mu\in\mathbb{Z}_3^2$ and any nonzero $u\in S'_\mu$, we have

- $S'_\mu\oplus S'_{-\mu}=\ker\mathrm{ad}^\star_u$,
- $\bigoplus_{\nu\neq0,\pm\mu}S'_\nu=\mathrm{ad}^\star_u(S')$.

In particular we have $(\ker\mathrm{ad}^\star_u)\cap\mathrm{ad}^\star_u(S')=0$. Since $n_{S'}$ is associative, we get $n_{S'}\bigl(\ker\mathrm{ad}^\star_u,\mathrm{ad}^\star_u(S')\bigr)=\ 0$, and hence

- $S'_\mu\oplus S'_{-\mu}$ is orthogonal to $\bigoplus_{\nu\neq0,\pm\mu}S'_\nu$.

Now, $\mathrm{alg}\langle u\rangle=Fu+Fu\star u$ is contained in $\ker\mathrm{ad}^\star_u=S'_\mu\oplus S'_{-\mu}$. If $u\star u$ were a scalar multiple of u, we would have $u\star u=\lambda u$ for some $0\neq\lambda\in F$. But

$$\begin{aligned}(\mathrm{ad}^\star_u)^3&=(l^\star_u-r^\star_u)^3 &&\text{(where } l^\star_u(v)=u\star v=r^\star_v(u))\\ &=(l^\star_u)^3-(r^\star_u)^3 &&\text{as } l^\star_ur^\star_u=r^\star_ul^\star_u=n_{S'}(u)\,\mathrm{Id}.\end{aligned}$$

But for any v orthogonal to u,

$$(l^\star_u)^3(v)=u\star\bigl(u\star(u\star v)\bigr)=-u\star\bigl(v\star(u\star u)\bigr)=-\lambda u\star(v\star u)=-\lambda n_{S'}(u)v,$$

and also $(r_u^\star)^3(v) = -\lambda n_{S'}(u)v$. Thus $(\mathrm{ad}_u^\star)^3(v) = 0$ for any v orthogonal to u, but this contradicts the fact that $\ker \mathrm{ad}_u^\star \cap \mathrm{ad}_u^\star(S') = 0$. We conclude that $u \star u$ is not a scalar multiple of u and hence $\mathrm{alg}\langle u\rangle = Fu + Fu \star u = S'_\mu \oplus S'_{-\mu}$. This shows that $\mathrm{alg}\langle u\rangle$ is a regular subalgebra, as we may consider the subalgebra $S'_\nu \oplus S'_{-\nu}$ for $\nu \neq 0, \pm\mu$ and use the characterization of regular subalgebras in item (iii) of Lemma 12.9. Finally, for $0 \neq u \in S'_\mu$ and $0 \neq v \in S'_{-\mu}$, $\mathrm{ad}_u^\star \mathrm{ad}_v^\star(S'_\nu) \subseteq S'_{\mu-\mu+\nu} = S'_\nu$ for any $\nu \in \mathbb{Z}_3^2$, and this implies that $\mathrm{ad}_u^\star \mathrm{ad}_v^\star$ is diagonalizable. By Lemma 12.10 we conclude that $n_{S'}(u) = 0$, $v \in Fu \star u$, and the eigenvalues of $(\mathrm{ad}_u^\star)^3$ are 0, with multiplicity 2, and $\pm g_{S'}(u)$, each with multiplicity 3. Take now $u = \varphi(x) \in S'_{(\bar{1},\bar{0})}$ and $v = \varphi(y) \in S'_{(\bar{0},\bar{1})}$. Then $n_{S'}(u) = 0 = n_{S'}(v)$, and

$$n_{S'}\big(\mathrm{alg}\langle u\rangle, \mathrm{alg}\langle v\rangle\big) = n_{S'}\big(S'_{(\bar{1},\bar{0})} \oplus S'_{-(\bar{1},\bar{0})}, S'_{(\bar{0},\bar{1})} \oplus S'_{-(\bar{0},\bar{1})}\big) = 0.$$

Then $0 \neq g_{S'}(u) = n_{S'}(u, u \star u)$ is an eigenvector of $(\mathrm{ad}_u^\star)^3$, and hence also of $(\mathrm{ad}_x^*)^3$, so $g_{S'}(u)$ is either $g_S(x)$ or $-g_S(x)$. Also $g_{S'}(v) = \pm g_S(y)$. Changing u by $-u$ and v by $-v$ if necessary, we get elements $u \in S'_{(\bar{1},\bar{0})}$, $v \in S'_{(\bar{0},\bar{1})}$, with $g_{S'}(u) = g_S(x)$ and $g_{S'}(v) = g_S(y)$. Now, [Eld09, Theorem 3.12] shows that either $u \star v = 0$ or $v \star u = 0$, and that $(S', \star, n_{S'})$ and $(S, *, n_S)$ are isomorphic if $u \star v = 0$, and anti-isomorphic if $v \star u = 0$. $\square$

Remark 12.12. With the notation in Section 12, if $\dim_{F^3} g_{\alpha,\beta}(C_{\alpha,\beta})$ equals 1 or 3, then $(C_{\alpha,\beta}, \triangle, n)$ and its opposite algebra $(C_{\beta,\alpha}, \triangle, n)$ are isomorphic (see Theorem 12.7). However, if $\dim_{F^3} g_{\alpha,\beta}(C_{\alpha,\beta})$ is 8, then $(C_{\alpha,\beta}, \triangle, n)$ is not isomorphic to its opposite algebra.

13. Groups admitting triality over arbitrary fields

In this section we classify all simple adjoint groups G of classical type ${}^{1,2}\mathsf{D}_4$ which admit trialitarian automorphisms over an arbitrary field F. The first reduction is to groups of type $\mathbf{PGO}^+(n)$ and $\mathbf{Spin}(n)$, where n is a 3-Pfister form. More precisely:

Theorem 13.1. *Let F be an arbitrary field. Let G be an adjoint (resp. simply connected) simple group of type ${}^{1,2}\mathsf{D}_4$ which admits a trialitarian automorphism ϕ. There exists a symmetric composition $(S, \star, n)$ such that the pair (G, ϕ) is isomorphic to a pair $(\mathbf{PGO}^+(n), \rho_\star)$ (resp. $(\mathbf{Spin}\,(n), \rho_\star)$).*

Proof. It suffices to consider the adjoint case. The group G is a twisted form of $G_0 = \mathbf{PGO}_8^+$, i.e., there exists a finite field extension L/F and a cocycle $\xi \in (G_0 \rtimes \mathbb{Z}/2)(L \otimes L)$ such that

$$G(F) = \{x \in G_0(L) \mid \xi x_1 = x_2 \xi\}$$

where $\mathbb{Z}/2 \hookrightarrow \mathfrak{S}_3$ is a fixed embedding and x_1, x_2 are images of x under two natural mappings $\pi_i\colon G_0(L) \to G_0(L \otimes L)$. The isomorphism class of G is given by the image of the cohomology class $[\xi] \in H^1_{\mathrm{fppf}}(F, G_0 \rtimes \mathbb{Z}/2)$ in $H^1_{\mathrm{fppf}}(F, G_0 \rtimes \mathfrak{S}_3)$. We view ϕ as an element of $(G_0 \rtimes \mathfrak{S}_3)(L)$ (as in the proof of Lemma 3.2). The fact that ϕ is F-defined implies that $\xi\phi_1 = \phi_2\xi$.

Replacing if necessary L by a bigger field extension of F, we may assume in view of Theorem 7.6 that ϕ is conjugate in $(G_0 \rtimes \mathfrak{S}_3)(L)$ to one of the standard trialitarian automorphisms $\beta = \rho_\diamond$ or $\beta = \rho_\Delta$. Let $\beta = \gamma\phi\gamma^{-1}$. Take the cocycle $\xi' = \gamma_2\xi\gamma_1^{-1}$, which is cohomologous to ξ. We have

$$\xi'\beta_1 = \gamma_2\xi\gamma_1^{-1}\gamma_1\phi_1\gamma_1^{-1} = \gamma_2\xi\phi_1\gamma_1 = \beta_2\xi'.$$

Thus replacing ξ by ξ' we may assume that $\phi = \beta$. We now note that by construction $\beta \in (G_0 \rtimes \mathfrak{S}_3)(F)$, hence $\beta_1 = \beta_2 = \beta$ and this implies that ξ takes values in $C_{G_0 \rtimes \mathfrak{S}_3}(\beta) = C_{G_0}(\beta) \rtimes \langle \beta \rangle$. Furthermore, since G has type ${}^{1,2}\mathsf{D}_4$ the image of the class $[\xi]$ under the projection

$$H^1_{\mathrm{fppf}}(F, G_0 \rtimes \mathfrak{S}_3) \to H^1_{\mathrm{fppf}}(F, \mathfrak{S}_3)$$

takes values in a subgroup of order 2 and on the other hand in the subgroup $\langle \beta \rangle$ of order 3. It follows that $[\xi] \in H^1_{\mathrm{fppf}}(F, H_0)$ where $H_0 = C_{G_0}(\beta)$ is the subgroup of G_0 fixed under β. In view of Lemma 4.7 and Proposition 8.1 the cohomology set $H^1_{\mathrm{fppf}}(F, H_0)$ classifies isomorphism classes of symmetric compositions which over an algebraic closure of F induce trialitarian automorphisms conjugate to β. The map $H^1_{\mathrm{fppf}}(F, H_0) \to H^1_{\mathrm{fppf}}(F, G_0 \rtimes \mathfrak{S}_3)$ induced by the embedding $H_0 \to G_0 \rtimes \mathfrak{S}_3$ maps the class of the symmetric composition $(S, \star, n)$ to the isomorphism class of the group $\mathbf{PGO}^+(n)$. It follows that $G \simeq \mathbf{PGO}^+(n)$ and that our automorphism ϕ is of the form $\rho_\star$ for some symmetric composition $\star$ on S. □

Let n be a 3-Pfister form over F and let G be either $\mathbf{PGO}^+(n)$ or $\mathbf{Spin}(n)$. The next aim is to describe the conjugacy classes of trialitarian automorphisms of G. Let $\sigma \in \widetilde{G}(F) = (G \rtimes \mathfrak{S}_3)(F)$ be a trialitarian automorphism of G order 3. We proved above that σ is of the form $\sigma = \rho_\star$ for a proper symmetric composition algebra $(S, \star, n)$ which is either a para-octonion algebra or an Okubo algebra. The fixed subgroup $H = C_G(\langle \sigma \rangle)$ in G is isomorphic to the automorphism group $\mathbf{Aut}(S, \star, n)$, by Proposition 8.1. The group $\widetilde{G}$ acts on itself by conjugation and we denote by $X = Cl_{\widetilde{G}}(\sigma) \subset \widetilde{G}$ the orbit of σ. This is a quasi-projective variety defined over F. As we proved before an arbitrary F-defined outer automorphism ϕ of G whose centralizer in G has the same type as that of H is conjugate to σ over an algebraic closure $\overline{F}$ of F and hence ϕ can be viewed as an F-point of X. We denote by $X(F)/{\sim}$ the set of conjugacy classes of F-defined outer automorphisms of G of order 3 whose centralizers have the same type as that of H (equivalently, the set of $\widetilde{G}(F)$-orbits in $X(F)$). Let now $\widetilde{H} = H \times \langle \sigma \rangle = C_{\widetilde{G}}(\sigma)$. The group $\widetilde{G}$ acts in a natural way on the quotient space $\widetilde{G}/\widetilde{H}$ and by the general formalism of cohomology (see [DG70, p. 372–373]) we have a natural bijection between the set of $\widetilde{G}(F)$-orbits in $(\widetilde{G}/\widetilde{H})(F)$ and the set $\operatorname{Ker}[H^1_{\mathrm{fppf}}(F, \widetilde{H}) \to H^1_{\mathrm{fppf}}(F, \widetilde{G})]$. Also, by the universal property of the quotient $\widetilde{G}/\widetilde{H}$ we have a natural $\widetilde{G}$-equivariant morphism $\widetilde{G}/\widetilde{H} \to X$ defined over F which induces a bijection $(\widetilde{G}/\widetilde{H})(\overline{F}) \to X(\overline{F})$.

Consider the commutative diagram

$$\begin{tikzcd}
(\widetilde{G}/\widetilde{H})(F) \arrow[r, hook] \arrow[d, "\lambda"'] & (\widetilde{G}/\widetilde{H})(\overline{F}) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, "\phi"'] & (\widetilde{G}/\widetilde{H})(\overline{F}\otimes\overline{F}) \arrow[d, "\psi"] \\
X(F) \arrow[r, hook] & X(\overline{F}) \arrow[r, shift left, "\pi'_1"] \arrow[r, shift right, "\pi'_2"'] & X(\overline{F}\otimes\overline{F})
\end{tikzcd}$$

Here π_1, π_2 (resp. π'_1, π'_2) are maps induced by $a \to a\otimes 1$ and $a \to 1\otimes a$ respectively and the top and bottom lines are the diagrams appearing in descent theory. We want to show that λ is a bijection. Let $x \in X(F)$. Since ϕ is a bijection there exists $g \in G(\overline{F})$ such that $\phi(g\widetilde{H}) = x$. Let $\theta\colon (\widetilde{G}/\widetilde{H})(\overline{F}\otimes\overline{F}) \to (\widetilde{G}/\widetilde{H})(\overline{F}\otimes\overline{F})$ (resp. $X(\overline{F}\otimes\overline{F}) \to X(\overline{F}\otimes\overline{F})$ and $G(\overline{F}\otimes\overline{F}) \to G(\overline{F}\otimes\overline{F})$) be the bijection corresponding to $a\otimes b \to b\otimes a$. This is the descent data for the variety $\widetilde{G}/\widetilde{H}$ (resp. X and $\widetilde{G}$). Since $x \in X(F)$ we get $\theta\big(\pi'_1(x)\big) = \pi'_2(x)$. Then $\psi\big(\theta(\pi_1(g\widetilde{H}))\big) = \psi\big(\pi_2(g\widetilde{H})\big)$ and hence $\psi\big(\pi_2(g)^{-1}\theta\big(\pi_1(g)\big)\widetilde{H}\big) = \sigma$. This implies that $\pi_2(g)^{-1}\theta\big(\pi_1(g)\big) \in \widetilde{H}(\overline{F}\otimes\overline{F})$ or $\theta(\pi_1(g)\widetilde{H}) = \pi_2(g\widetilde{H})$. By descent theory it follows that $g\widetilde{H} \in (\widetilde{G}/\widetilde{H})(F)$. Thus we have a natural bijection between the set of $\widetilde{G}(F)$-orbits in $(\widetilde{G}/\widetilde{H})(F)$ and the set $X(F)/{\sim}$. Combining all these facts we obtain

THEOREM 13.2. *There exists a natural bijection between the set of conjugacy classes of F-defined outer automorphisms of G of order 3 whose centralizers have the same type as that of H and the set* $\operatorname{Ker}[H^1_{\mathrm{fppf}}(F,\widetilde{H}) \to H^1_{\mathrm{fppf}}(F,\widetilde{G})]$.

Before proceeding with consequences of the theorem we compute the kernel of the last map in terms of the connected groups H and G.

PROPOSITION 13.3. *The natural mapping*

$$\lambda\colon \operatorname{Ker}\big[\,H^1_{\mathrm{fppf}}(F,H) \to H^1_{\mathrm{fppf}}(F,G)\,\big] \to \operatorname{Ker}\big[\,H^1_{\mathrm{fppf}}(F,\widetilde{H}) \to H^1_{\mathrm{fppf}}(F,\widetilde{G})\,\big]$$

induced by the embedding $H \to \widetilde{H}$ is a bijection.

Proof. The mapping λ is induced by the embedding $H \hookrightarrow \widetilde{H} = H \times \langle\,\sigma\,\rangle$. Since

$$H^1_{\mathrm{fppf}}(F,H) \to H^1_{\mathrm{fppf}}\big(F, H\times\langle\,\sigma\,\rangle\big)$$

is clearly injective so is λ. As for surjectivity of λ, let

$$\xi \in \operatorname{Ker}\big[H^1_{\mathrm{fppf}}(F,\widetilde{H}) \to H^1_{\mathrm{fppf}}(F,\widetilde{G})\big]. \tag{13.4}$$

Since $\widetilde{H} = H\times\langle\,\sigma\,\rangle$ we can write ξ in the form $\xi = \xi_1\cdot\xi_2$ where $\xi_1 \in H^1_{\mathrm{fppf}}(F,H)$ and $\xi_2 \in H^1_{\mathrm{fppf}}\big(F,\langle\,\sigma\,\rangle\big)$. The composition $\widetilde{H} \to \widetilde{G} \to \mathfrak{S}_3$ induces the map

$$H^1_{\mathrm{fppf}}(F, H\times\langle\,\sigma\,\rangle) \to H^1_{\mathrm{fppf}}(F,\mathfrak{S}_3)$$

which can be factored through

$$H^1_{\mathrm{fppf}}(F,\langle\,\sigma\,\rangle) \to H^1_{\mathrm{fppf}}(F,\mathfrak{S}_3)$$

and the last map has trivial kernel. But it follows from (13.4) that ξ_2 is in its kernel. Then $\xi_2 = 1$ implies $\xi = \xi_1 \in H^1_{\mathrm{fppf}}(F, H)$. It remains to show that ξ viewed as an element in $H^1_{\mathrm{fppf}}(F, G)$ is trivial. To do this, we look at the exact sequence

$$1 \longrightarrow G \longrightarrow \widetilde{G} \longrightarrow \mathfrak{S}_3 \longrightarrow 1$$

which induces

$$\widetilde{G}(F) \xrightarrow{\mu_1} \mathfrak{S}_3(F) \longrightarrow H^1_{\mathrm{fppf}}(F, G) \xrightarrow{\mu_2} H^1_{\mathrm{fppf}}(F, \widetilde{G}).$$

Surjectivity of μ_1 implies that the kernel of μ_2 is trivial. Since by our construction the cocycle ξ viewed as an element in $H^1_{\mathrm{fppf}}(F, \widetilde{G})$ is trivial, it is also trivial as an element in $H^1_{\mathrm{fppf}}(F, G)$. Thus we proved that ξ is in $\operatorname{Ker}[\, H^1_{\mathrm{fppf}}(F, H) \to H^1_{\mathrm{fppf}}(F, G)\,]$ and we are done. $\square$

In the next corollaries we describe explicitly the set

$$\operatorname{Ker}\left[H^1_{\mathrm{fppf}}(F, H) \to H^1_{\mathrm{fppf}}(F, G)\,\right]$$

where F is an arbitrary field. In view of 13.2 and 13.3 this leads to a complete list of conjugacy classes of trialitarian automorphisms of G. Clearly, the description depends on the structure of the algebraic group H in question. We recall that this group is the automorphism group of a symmetric composition algebra of dimension 8. Each corollary corresponds to one of the types of symmetric algebras described in Section 8. We recall that we have para-octonion algebras (Type I), Okubo algebras in characteristic different from 3 (Type IIa and Type IIb), which are obtained from central simple algebras of degree 3, and Okubo algebras in characteristic 3 (Type III). In each case the trialitarian automorphisms are those linked to symmetric compositions, as described in Theorem 4.8.

Corollary 13.5. *Let F be an arbitrary field. Let n be a 3-Pfister form over F and let $G = \mathbf{PGO}^+(n)$ or $G = \mathbf{Spin}(n)$. Then there is a unique conjugacy class of trialitarian F-automorphisms of G whose centralizers in G have type G_2. The class is represented by the trialitarian automorphism $\rho_\diamond$ associated to the isomorphism class of the para-octonion algebra with norm n.*

Proof. Let σ be an F-defined trialitarian automorphism of G with the property $H = C_G(\sigma) = \mathbf{G}_2(n)$ where $\mathbf{G}_2(n)$ is the automorphism group of an octonion algebra with norm n. By Proposition 8.3 the map $H^1_{\mathrm{fppf}}(F, H) \to H^1_{\mathrm{fppf}}(F, G)$ is injective. The result now follows from Theorem 13.2 and Proposition 13.3. $\square$

The next corollary corresponds to the case where the algebraic group H is the group of automorphisms of the split Okubo algebra over a field of characteristic not 3 containing a primitive cube root of unity, hence H is the split group $\mathbf{PGL}_3$.

Corollary 13.6. *Let F be an arbitrary field of characteristic different from 3 and assume that a primitive cube root of unity ω is contained in F. Let $G = \mathbf{PGO}_8^+$. Then the set of conjugacy classes of trialitarian F-automorphisms of*

G whose centralizers in G have type A_2 is in one-to-one correspondence with the set of isomorphism classes of central simple algebras over F of degree 3.

Proof. We apply Theorem 13.2 and Proposition 13.3 with σ the standard trialitarian automorphism $\rho_\vartriangle$ associated to the split Okubo multiplication. Its centralizer H in G is $\mathbf{PGL}_3$, hence $H^1_{\mathrm{fppf}}(F,H)$ classifies central simple algebras over F of degree 3. Also, it is clear that $H^1_{\mathrm{fppf}}(F,H) \to H^1_{\mathrm{fppf}}(F,G)$ is a trivial mapping because on one side every element in $H^1_{\mathrm{fppf}}(F,H)$ is split by a cubic extension of F and, on the other side, no element in $H^1_{\mathrm{fppf}}(F,G)$ is split by a cubic extension, in view of Springer's Theorem (see for instance [Sch85, Theorem 5.4]). The result now follows from Theorem 13.2 and Proposition 13.3. $\square$

Remark 13.7. Tracing through our constructions, we can make the correspondence of Corollary 13.6 explicit as follows, using Galois cohomology since the characteristic of F is not 3. Let $\rho_\vartriangle$ be the standard trialitarian automorphism associated to the split Okubo multiplication over F. Its centralizer in $G = \mathbf{PGO}_8^+$ is a split group $\mathbf{PGL}_3$. Let ρ be another trialitarian automorphism over F whose centralizer is a group of type A_2. We may view $\rho_\vartriangle$ and ρ as elements of the group $(\mathbf{PGO}_8^+ \rtimes \mathfrak{S}_3)(F)$. We know that they are conjugate over a separable closure F_{sep} of F by an element of $\mathbf{PGO}_8^+$. Let $g \in \mathbf{PGO}_8^+(F_{\mathrm{sep}})$ be such that $\rho = g\rho_\vartriangle g^{-1}$. For $\gamma \in \mathrm{Gal}(F_{\mathrm{sep}}/F)$ we have $\rho^\gamma = \rho$ and $\rho_\vartriangle^\gamma = \rho_\vartriangle$ since ρ and $\rho_\vartriangle$ are defined over F, hence $(g^\gamma)^{-1}g$ lies in the centralizer of $\rho_\vartriangle$ in $\mathbf{PGO}_8^+$. Thus we get a cocycle $a_\gamma = (g^\gamma)^{-1}g \in Z^1(F,\mathbf{PGL}_3)$. This cocycle gives rise to a central simple algebra, say A, of degree 3 over F. Note that it follows from the construction that the group $\mathbf{PGL}(A)$ is the centralizer of ρ in $\mathbf{PGO}_8^+$.

Conversely, assume that we have a central simple algebra A of degree 3 over F. It is determined by a cocycle a_γ with coefficients in $\mathbf{PGL}_3$. Since the map $H^1(F,\mathbf{PGL}_3) \to H^1(F,\mathbf{PGO}_8^+)$ is trivial (as observed in the proof of Corollary 13.6), there exists $g \in \mathbf{PGO}_8^+(F_s)$ such that $a_\gamma = (g^\gamma)^{-1}g$. Consider $\rho = g\rho_\vartriangle g^{-1}$. Since $\rho_\vartriangle$ is defined over F and $(g^\gamma)^{-1}g$ centralizes $\rho_\vartriangle$ for all γ, it follows that ρ is defined over F.

Similar considerations apply to corollaries 13.8 and 13.9 below.

Now, suppose the field F does not contain ω, and $\operatorname{char} F \neq 3$. Let $K = F(\omega)$, which is a quadratic field extension of F. Before stating the next corollary we recall that on a given central simple K-algebra B of degree 3, unitary involutions τ fixing F are classified by a quadratic form invariant $\pi(\tau)$, which is a 3-Pfister form over F split by K, see [KMRT98, (19.6)]. This result is proved in [KMRT98] under the hypothesis that $\operatorname{char} F \neq 2$, but the following observations show that it also holds when $\operatorname{char} F = 2$.

If $\operatorname{char} F \neq 2$, the 3-fold Pfister form $\pi(\tau)$ is obtained by modifying the form $Q_\tau(x) = \mathrm{Trd}(x^2)$ defined on the F-vector space $\mathrm{Sym}(\tau)$ of τ-symmetric elements: see [KMRT98, (19.4)]. The arguments in the proof of the classification theorem [KMRT98, (19.6)] can be used when $\operatorname{char} F = 2$, substituting for Q_τ the restriction to $\mathrm{Sym}(\tau)$ of the quadratic form $\mathrm{Srd}(x)$, which is the coefficient of the indeterminate t in the reduced characteristic polynomial of x (as in Proposition 8.5). Let φ_τ be the restriction of Srd to the F-vector space $\mathrm{Sym}(\tau)^0$ of τ-symmetric elements of trace zero, and let $[1,1]$ denote the quadratic form $X^2 + XY + Y^2$ over F. If B is split and τ is adjoint to a hermitian form h with diagonalization $\langle\delta_1, \delta_2, \delta_3\rangle_K$, computation shows that $\varphi_\tau \cong [1,1] \perp \langle\delta_1\delta_2, \delta_2\delta_3, \delta_3\delta_1\rangle \cdot n_K$, where n_K is the norm form of K/F. Therefore, if τ' is also a unitary involution on the split algebra B, and τ' is adjoint to a hermitian form h', the arguments on [KMRT98, p. 305] show that $\varphi_{\tau'} \cong \varphi_\tau$ implies h' is similar to h, hence τ' and τ are isomorphic. Therefore, the 3-fold Pfister form $\pi(\tau) = \langle 1, \delta_1\delta_2, \delta_2\delta_3, \delta_3\delta_1\rangle \cdot n_K$ determines the involution τ up to isomorphism. The case where B is not split reduces to the split case by an odd-degree scalar extension. The arguments on p. 305 of [KMRT98] apply in characteristic 2, since [KMRT98, (6.17)] relies on a result of Bayer-Lenstra that also holds in characteristic 2 (see [BFT07, Theorem 1.13] for a discussion of the Bayer-Lenstra result in characteristic 2).

Corollary 13.8. *Let F be an arbitrary field of characteristic not 3 and assume that F does not contain a primitive cube root of unity ω. Let n be a 3-Pfister form split by $K = F(\omega)$ and let G be $\mathbf{PGO}^+(n)$ or $\mathbf{Spin}(n)$. Then the set of conjugacy classes of trialitarian F-automorphisms of G of order 3 whose centralizers in G have type A_2 is in one-to-one correspondence with the set of F-isomorphism classes of pairs (B, τ) where B is a central simple K-algebra of degree 3 and τ is a unitary involution on B fixing F such that $\pi(\tau) = n$.*

Proof. Let σ be an outer F-automorphism of G of order 3 whose centralizer H in G is an outer form of type ${}^2\mathsf{A}_2$, of inner type over K. Its existence follows from our previous results. Indeed, take the standard trialitarian automorphism $\rho_\star$ of G over F corresponding to the para-octonion algebra $(C, \star, n)$ with norm n. Its automorphism group has type G_2 and splits over $K = F(\omega)$. Hence it contains a subtorus of the form $R^{(1)}_{K/F}(\mathbf{G}_m)$. Such a torus contains an element of order 3 over F, namely ω. Twisting the multiplication $\star$ by ω we get an Okubo algebra $(C, \star_\omega, n)$. The corresponding automorphism $\rho_{\star_\omega}$ of G is as required.
Let H_0 be an adjoint quasi-split F-group of type ${}^2\mathsf{A}_2$, of inner type over K. It is known that H is a twisted form of H_0, i.e. $H = {}^\xi H_0$ where $\xi \in Z^1(F, H_0)$. Also, we know that the pointed set $H^1_{\mathrm{fppf}}(F, H_0)$ classifies pairs (B, τ) where B is a central simple algebra over K of degree 3 and τ is a unitary involution on B (see [KMRT98, (30.21)]) and that there exists a natural bijection $H^1(F, H_0) \to H^1(F, H)$ which takes the class $[\xi]$ into the trivial class. Thus the pointed set $H^1_{\mathrm{fppf}}(F, H)$ also classifies the same pairs (B, τ). The mapping $H^1_{\mathrm{fppf}}(F, H) \to H^1_{\mathrm{fppf}}(F, G)$ takes a pair (B, τ) to the class in $H^1_{\mathrm{fppf}}(F, G)$ corresponding to the 3-Pfister form $\pi(\tau)$ (see [KMRT98, §30.C]). So the result follows from Theorem 13.2 and Proposition 13.3. $\square$

Finally, suppose char $F = 3$. If the centralizer H of a trialitarian automorphism ϕ is not a form of type G_2, this centralizer is the automorphism group of an Okubo algebra, as computed in Section 10.

COROLLARY 13.9. *Let $G = \mathbf{PGO}_8^+$ be defined over an arbitrary field F of characteristic 3. The set of conjugacy classes of trialitarian F-automorphisms of G of Okubo type in G is in one-to-one correspondence with the set of isomorphism classes of Okubo algebras over F.*

Proof. Taking Theorem 12.6 into account, the proof is along the same lines as the proof of Corollary 13.6. □

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