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Cuntz-Krieger-Pimsner Algebras Associated with Amalgamated Free Product Groups

By

Rui Okayasu*

Abstract

We give a construction of a nuclear C^*-algebra associated with an amalgamated free product of groups, generalizing Spielberg's construction of a certain Cuntz-Krieger algebra associated with a finitely generated free product of cyclic groups. Our nuclear C^*-algebras can be identified with certain Cuntz-Krieger-Pimsner algebras. We will also show that our algebras can be obtained by the crossed product construction of the canonical actions on the hyperbolic boundaries, which proves a special case of Adams' result about amenability of the boundary action for hyperbolic groups. We will also give an explicit formula of the K-groups of our algebras. Finally we will investigate a relationship between the KMS states of the generalized gauge actions on our C^* algebras and random walks on the groups.

§1. Introduction

In [5], Choi proved that the reduced group C^*-algebra $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ of the free product of cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ is embedded in $\mathcal{O}_2$. Consequently, this shows that $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ is a non-nuclear exact C^*-algebra, (see S. Wassermann [31] for a good introduction to exact C^*-algebras). Spielberg generalized it to finitely generated free products of cyclic groups in [28]. Namely, he constructed a certain action on a compact space and proved that some Cuntz-Krieger algebras (see [8]) can be obtained by the crossed product construction for the action. For a related topic, see W. Szymański and S. Zhang's work [30].

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*Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan.
e-mail: rui@kusm.kyoto-u.ac.jp

More generally, the above mentioned compact space coincides with Gromov's notion of the boundaries of hyperbolic groups (e.g. see [18]). In [1], Adams proved that the action of any discrete hyperbolic group Γ on the hyperbolic boundary $\partial\Gamma$ is amenable in the sense of Anantharaman-Delaroche [2]. It follows from [2] that the corresponding crossed product $C(\partial\Gamma) \rtimes_r \Gamma$ is nuclear, and this implies that $C_r^*(\Gamma)$ is an exact C^*-algebra.

Although we know that $C(\partial\Gamma) \rtimes_r \Gamma$ is nuclear for a general discrete hyperbolic group Γ as mentioned above, there are only few things known about this C^*-algebra. So one of our purposes is to generalize Spielberg's construction to some finitely generated amalgamated free product Γ and to give detailed description of the algebra $C(\partial\Gamma) \rtimes_r \Gamma$. More precisely, let I be a finite index set and G_i be a group containing a copy of a finite group H as a subgroup for $i \in I$. We always assume that each G_i is either a finite group or $\mathbb{Z} \times H$. Let $\Gamma = *_H G_i$ be the amalgamated free product group. We will construct a nuclear C^*-algebra $\mathcal{O}_\Gamma$ associated with Γ by mimicking the construction for Cuntz-Krieger algebras with respect to the full Fock space in M. Enomoto, M. Fujii and Y. Watatani [12] and D. E. Evans [14]. This generalizes Spielberg's construction.

First we show that $\mathcal{O}_\Gamma$ has a certain universal property as in the case of the Cuntz-Krieger algebras, which allows several descriptions of $\mathcal{O}_\Gamma$. For example, it turns out that $\mathcal{O}_\Gamma$ is a Cuntz-Krieger-Pimsner algebra, introduced by Pimsner in [23] and studied by several authors, e.g. T. Kajiwara, C. Pinzari and Y. Watatani [19]. We will also show that $\mathcal{O}_\Gamma$ can be obtained by the crossed product construction. Namely, we will introduce a boundary space Ω with a natural Γ-action, which coincides with the boundary of the associated tree (see [27], [32]). Then we will prove that $C(\Omega) \rtimes_r \Gamma$ is isomorphic to $\mathcal{O}_\Gamma$. Since the hyperbolic boundary $\partial\Gamma$ coincides with Ω and the two actions of Γ on $\partial\Gamma$ and Ω are conjugate, $\mathcal{O}_\Gamma$ is also isomorphic to $C(\partial\Gamma) \rtimes_r \Gamma$, and depends only on the group structure of Γ. As a consequence, we give a proof to Adams' theorem in this special case.

Next, we will consider the K-groups of $\mathcal{O}_\Gamma$. In [22], Pimsner gave a certain exact sequence of KK-groups of the crossed product by groups acting on trees. However, it is not a trivial task to apply Pimsner's exact sequence to $C(\partial\Gamma) \rtimes_r \Gamma$ and obtain its K-groups. We will give explicit formulae of the K-groups of $\mathcal{O}_\Gamma$ following the method used for the Cuntz-Krieger algebras instead of using $C(\partial\Gamma) \rtimes_r \Gamma$. We can compute the K-groups of $C(\partial\Gamma) \rtimes_r \Gamma$ for concrete examples. They are completely determined by the representation theory of H and the actions of H on G_i/H (the space of right cosets) by left multiplication.

Finally we will prove that KMS states on $\mathcal{O}_\Gamma$ for generalized gauge actions arise from harmonic measures on the Poisson boundary with respect to random walks on the discrete group Γ. Consequently, for special cases, we can determine easily the type of factor $\mathcal{O}_\Gamma''$ for the corresponding unique KMS state of the gauge action by essentially the same arguments in M. Enomoto, M. Fujii and Y. Watatani [13], which generalized J. Ramagge and G. Robertson's result [25].

§2. Preliminaries

In this section, we collect basic facts used in the present article. We begin by reviewing the Cuntz-Krieger-Pimsner algebras in [23]. Let A be a C^*-algebra and X be a Hilbert bimodule over A, which means that X is a right Hilbert A-module with an injective $*$-homomorphism of A to $\mathcal{L}(X)$, where $\mathcal{L}(X)$ is the C^*-algebra of all adjointable A-linear operators on X. We assume that X is full, that is, $\{\langle x, y\rangle_A \mid x, y \in X\}$ generates A as a C^*-algebra, where $\langle \cdot, \cdot \rangle_A$ is the A-valued inner product on X. We further assume that X has a finite basis $\{u_1, \dots, u_n\}$, which means that $x = \sum_{i=1}^n u_i \langle u_i, x\rangle_A$ for any $x \in X$. We fix a basis $\{u_1, \dots, u_n\}$ of X. Let $\mathcal{F}(X) = A \oplus \bigoplus_{n\geq 1} X^{(n)}$ be the full Fock space over X, where $X^{(n)}$ is the n-fold tensor product $X \otimes_A X \otimes_A \cdots \otimes_A X$. Note that $\mathcal{F}(X)$ is naturally equipped with Hilbert A-bimodule structure. For each $x \in X$, the operator $T_x : \mathcal{F}(X) \to \mathcal{F}(X)$ is defined by

$$T_x(x_1 \otimes \cdots \otimes x_n) = x \otimes x_1 \otimes \cdots \otimes x_n,$$
$$T_x(a) = xa,$$

for $x, x_1, \dots, x_n \in X$ and $a \in A$. Note that $T_x \in \mathcal{L}(\mathcal{F}(X))$ satisfies the following relations

$$T_x^* T_y = \langle x, y\rangle_A, \qquad x, y \in X,$$
$$aT_x b = T_{axb}, \qquad x \in X, a, b \in A.$$

Let π be the quotient map of $\mathcal{L}(\mathcal{F}(X))$ onto $\mathcal{L}(\mathcal{F}(X))/\mathcal{K}(\mathcal{F}(X))$ where $\mathcal{K}(\mathcal{F}(X))$ is the C^*-algebra of all compact operators of $\mathcal{L}(\mathcal{F}(X))$. We denote $S_x = \pi(T_x)$ for $x \in X$. Then we define the Cuntz-Krieger-Pimsner algebra $\mathcal{O}_X$ to be

$$\mathcal{O}_X = C^*(S_x \mid x \in X).$$

Since X is full, a copy of A acting by left multiplication on $\mathcal{F}(X)$ is contained in $\mathcal{O}_X$. Furthermore we have the relation

$$(\dagger) \qquad \sum_{i=1}^n S_{u_i} S_{u_i}^* = 1.$$

On the other hand, $\mathcal{O}_X$ is characterized as the universal C^*-algebra generated by A and S_x, satisfying the above relations [23, Theorem 3.12]. More precisely, we have

Theorem 2.1 ([23, Theorem 3.12]). *Let X be a full Hilbert A-bimodule and $\mathcal{O}_X$ be the corresponding Cuntz-Krieger-Pimsner algebra. Suppose that $\{u_1, \dots, u_n\}$ is a finite basis for X. If B is a C^*-algebra generated by $\{s_x\}_{x\in X}$ satisfying*

$$\begin{aligned} s_x + s_y &= s_{x+y}, && x \in X, \\ a s_x b &= s_{axb}, && x \in X, a, b \in A, \\ s_x^* s_y &= \langle x, y\rangle_A, && x, y \in X, \\ \sum_{i=1}^n s_{u_i} s_{u_i}^* &= 1. \end{aligned}$$

Then there exists a unique surjective $*$-homomorphism from $\mathcal{O}_X$ onto $C^*(s_x)$ that maps S_x to s_x.

Next we recall the notion of amenability for discrete C^*-dynamical systems introduced by C. Anantharaman-Delaroche in [2]. Let (A, G, α) be a C^*-dynamical system, where A is a C^*-algebra, G is a group and α is an action of G on A. An A-valued function h on G is said to be of *positive type* if the matrix $[\alpha_{s_i}(h(s_i^{-1}s_j))] \in M_n(A)$ is positive for any $s_1, \dots, s_n \in G$. We assume that G is discrete. Then α is said to be *amenable* if there exists a net $(h_i)_{i\in I} \subset C_c(G, Z(A''))$ of functions of positive type such that

$$\begin{cases} h_i(e) \le 1 & \text{for } i \in I, \\ \lim_i h_i(s) = 1 & \text{for } s \in G, \end{cases}$$

where the limit is taken in the σ-weak topology in the enveloping von Neumann algebra A'' of A. We remark that this is one of several equivalent conditions given in [2, Théorème 3.3]. We will use the following theorems without a proof.

Theorem 2.2 ([2, Théorème 4.5]). *Let (A, G, α) be a C^*-dynamical system such that A is nuclear and G is discrete. Then the following are equivalent*:

1) *The full C^*-crossed product $A \rtimes_\alpha G$ is nuclear*;
2) *The reduced C^*-crossed product $A \rtimes_{\alpha r} G$ is nuclear*;
3) *The W^*-crossed product $A'' \rtimes_{\alpha w} G$ is injective*;
4) *The action α of G on A is amenable.*

Theorem 2.3 ([2, Théorème 4.8]). *Let (A, G, α) be an amenable C^*-dynamical system such that G is discrete. Then the natural quotient map from $A \rtimes_\alpha G$ onto $A \rtimes_{\alpha r} G$ is an isomorphism.*

Finally, we review the notion of the strong boundary actions in [21]. Let Γ be a discrete group acting by homeomorphisms on a compact Hausdorff space Ω. Suppose that Ω has at least three points. The action of Γ on Ω is said to be a *strong boundary action* if for every pair U, V of non-empty open subsets of Ω there exists $\gamma \in \Gamma$ such that $\gamma U^c \subset V$. The action of Γ on Ω is said to be *topologically free* in the sense of [3] if the fixed point set of each non-trivial element of Γ has empty interior.

Theorem 2.4 ([21, Theorem 5]). *Let (Ω, Γ) be a strong boundary action where Ω is compact. We further assume that the action is topologically free. Then $C(\Omega) \rtimes_r \Gamma$ is purely infinite and simple.*

§3. A Motivating Example

Before introducing our algebras, we present a simple case of Spielberg's construction for $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z}$ with generators a and b as a motivating example. See also [26]. The Cayley graph of $\mathbb{F}_2$ is a homogeneous tree of degree 4. The boundary Ω of the tree in the sense of [16] (see also [17]) can be thought of as the set of all infinite reduced words $\omega = x_1 x_2 x_3 \cdots$, where $x_i \in S = \{a, b, a^{-1}, b^{-1}\}$. Note that Ω is compact in the relative topology of the product topology of $\prod_{\mathbb{N}} S$. In an appendix, several facts about trees are collected for the convenience of the reader, (see also [15]). Left multiplication of $\mathbb{F}_2$ on Ω induces an action of $\mathbb{F}_2$ on $C(\Omega)$. For $x \in \mathbb{F}_2$, let $\Omega(x)$ be the set of infinite words beginning with x. We identify the implementing unitaries in the full crossed product $C(\Omega) \rtimes \mathbb{F}_2$ with elements of $\mathbb{F}_2$. Let p_x denote the projection defined by the characteristic function $\chi_{\Omega(x)} \in C(\Omega)$. Note that for each $x \in S$,

$$p_x + x p_{x^{-1}} x^{-1} = 1,$$
$$p_a + p_{a^{-1}} + p_b + p_{b^{-1}} = 1,$$

hold. For $x \in S$, let $S_x \in C(\Omega) \rtimes \mathbb{F}_2$ be a partial isometry

$$S_x = x(1 - p_{x^{-1}}).$$

Then we have

$$S_x^* S_y = x^{-1} p_x p_y y = \delta_{x,y} S_x^* S_x = \delta_{x,y}(1 - p_{x^{-1}}),$$

$$S_x S_x^* = x(1 - p_{x^{-1}})x^{-1} = p_x,$$
$$S_x^* S_x = 1 - p_{x^{-1}} = \sum_{y \neq x^{-1}} S_y S_y^*.$$

These relations show that the partial isometries S_x generate the Cuntz-Krieger algebra $\mathcal{O}_A$ [8], where

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

On the other hand, we can recover the generators of $C(\Omega) \rtimes \mathbb{F}_2$ by setting

$$x = S_x + S_{x^{-1}}^* \quad \text{and} \quad p_x = S_x S_x^*.$$

Hence we have $C(\Omega) \rtimes \mathbb{F}_2 \simeq \mathcal{O}_A$.

Next we recall the Fock space realization of the Cuntz-Krieger algebras, (e.g. see [14], [12]). Let $\{e_a, e_b, e_{a^{-1}}, e_{b^{-1}}\}$ be a basis of $\mathbb{C}^4$. We define the Fock space associated with the matrix A by

$$\mathcal{F}_A = \mathbb{C}e_0 \oplus \bigoplus_{n \geq 1} \left(\overline{\text{span}}\{e_{x_1} \otimes \cdots \otimes e_{x_n} \mid A(x_i, x_{i+1}) = 1\} \right),$$

where e_0 is the vacuum vector. For any $x \in S$, let T_x be the creation operator on $\mathcal{F}$, given by

$$T_x e_0 = e_x,$$
$$T_x(e_{x_1} \otimes \cdots \otimes e_{x_n}) = \begin{cases} e_x \otimes e_{x_1} \otimes \cdots \otimes e_{x_n} & \text{if } A(x, x_1) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let p_0 be the rank one projection on the vacuum vector e_0. Note that we have

$$T_a T_a^* + T_b T_b^* + T_{a^{-1}} T_{a^{-1}}^* + T_{b^{-1}} T_{b^{-1}}^* + p_0 = 1.$$

If π is the quotient map of $\mathcal{B}(\mathcal{F})$ onto the Calkin algebra $\mathcal{Q}(\mathcal{F})$, then the C^*-algebra generated by the partial isometries $\{\pi(T_a), \pi(T_b), \pi(T_{a^{-1}}), \pi(T_{b^{-1}})\}$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_A$.

Now we look at this construction from another point of view. We can perform the following natural identification:

$$\mathcal{F} \ni \begin{matrix} e_0 & \longleftrightarrow & \delta_e \\ e_{x_1} \otimes \cdots \otimes e_{x_n} & \longleftrightarrow & \delta_{x_1 \cdots x_n} \end{matrix} \in l^2(\mathbb{F}_2).$$

Under this identification, the creation operator T_x on $l^2(\mathbb{F}_2)$ can be expressed as

$$T_x\delta_e = \lambda_x\delta_e,$$
$$T_x\delta_{x_1\cdots x_n} = \begin{cases} \lambda_x\delta_{x_1\cdots x_n} & \text{if } x \neq x_1^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

where λ is the left regular representation of $\mathbb{F}_2$.

For a reduced word $x_1 \cdots x_n \in \mathbb{F}_2$, we define the length function $|\cdot|$ on $\mathbb{F}_2$ by $|x_1 \cdots x_n| = n$. Let p_n be the projection onto the closed linear span of $\{\delta_\gamma \in l^2(\mathbb{F}_2) \mid |\gamma| = n\}$. Then we can express T_x for $x \in S$ by

$$T_x = \sum_{n\geq 0} p_{n+1}\lambda_x p_n.$$

Note that this expression makes sense for every finitely generated group. In the next section, we generalize this construction to amalgamated free product groups.

§4. Construction of a Nuclear C^*-algebra $\mathcal{O}_\Gamma$

In what follows, we always assume that I is a finite index set and G_i is a group containing a copy of a finite group H as a subgroup for $i \in I$. Moreover, we assume that each G_i is either a finite group or $\mathbb{Z} \times H$. We set $I_0 = \{i \in I \mid |G_i| < \infty\}$. Let $\Gamma = *_H G_i$ be the amalgamated free product.

First we introduce a "length function" $|\cdot|$ on each G_i. If $i \in I_0$, we set $|g| = 1$ for any $g \in G_i \setminus H$ and $|h| = 0$ for any $h \in H$. If $i \in I \setminus I_0$ we set $|(a_i^n, h)| = |n|$ for any $(a_i^n, h) \in G_i = \mathbb{Z} \times H$ where a_i is a generator of $\mathbb{Z}$. Now we extend the length function to Γ. Let Ω_i be a set of left representatives of G_i/H with $e \in \Omega_i$. If $\gamma \in \Gamma$ is written uniquely as $g_1 \cdots g_n h$, where $g_1 \in \Omega_{i_1}, \ldots, g_n \in \Omega_{i_n}$ with $i_1 \neq i_2, \ldots, i_{n-1} \neq i_n$(we write simply $i_1 \neq \cdots \neq i_n$), then we define

$$|\gamma| = \sum_{k=1}^{n} |g_k|.$$

Let p_n be the projection of $l^2(\Gamma)$ onto $l^2(\Gamma_n)$ for each n, where $\Gamma_n = \{\gamma \in \Gamma \mid |\gamma| = n\}$. We define partial isometries and unitary operators on $l^2(\Gamma)$ by

$$\begin{cases} T_g = \sum_{n\geq 0} p_{n+1}\lambda_g p_n & \text{if } g \in \bigcup_{i\in I} G_i \setminus H, \\ V_h = \lambda_h & \text{if } h \in H, \end{cases}$$

where λ is the left regular representation of Γ. Let π be the quotient map of $\mathcal{B}(l^2(\Gamma))$ onto $\mathcal{B}(l^2(\Gamma))/\mathcal{K}(l^2(\Gamma))$, where $\mathcal{B}(l^2(\Gamma))$ is the C^*-algebra of all bounded linear operators on $l^2(\Gamma)$ and $\mathcal{K}(l^2(\Gamma))$ is the C^*-subalgebra of all compact operators of $\mathcal{B}(l^2(\Gamma))$. We set $\pi(T_g) = S_g$ and $\pi(V_h) = U_h$. For $\gamma \in \Gamma$, we define S_γ by

$$S_\gamma = S_{g_1} \cdots S_{g_n},$$

where $\gamma = g_1 \cdots g_n$ for some $g_1 \in G_{i_1} \setminus H, \dots, g_n \in G_{i_n} \setminus H$ with $i_1 \neq \cdots \neq i_n$. Note that S_γ does not depend on the expression $\gamma = g_1 \cdots g_n$. We denote the initial projections of S_γ by $Q_\gamma = S_\gamma^* \cdot S_\gamma$ and the range projections by $P_\gamma = S_\gamma \cdot S_\gamma^*$ for $\gamma \in \Gamma$.

We collect several relations, which the family $\{ S_g, U_h \mid g \in \bigcup_{i \in I} G_i \setminus H, h \in H \}$ satisfies.

For $g, g' \in \bigcup_i G_i \setminus H$ with $|g| = |g'| = 1$ and $h \in H$,

$$S_{gh} = S_g \cdot U_h, \qquad S_{hg} = U_h \cdot S_g, \tag{1}$$

$$P_g \cdot P_{g'} = \begin{cases} P_g = P_{g'} & \text{if } gH = g'H, \\ 0 & \text{if } gH \neq g'H. \end{cases} \tag{2}$$

Moreover, if $g \in G_i \setminus H$ and $i \in I_0$, then

$$Q_g = \sum_{\substack{j \in I_0 \\ j \neq i}} \sum_{g' \in \Omega_j \setminus \{e\}} P_{g'} + \sum_{j \in I \setminus I_0} P_{a_j} + P_{a_j^{-1}}, \tag{3}$$

and if $g = a_i^{\pm 1}$ and $i \in I \setminus I_0$, then

$$Q_{a_i^{\pm 1}} = \sum_{j \in I_0} \sum_{g' \in \Omega_j \setminus \{e\}} P_{g'} + \sum_{\substack{j \in I \setminus I_0 \\ j \neq i}} \left(P_{a_j} + P_{a_j^{-1}} \right) + P_{a_i^{\pm 1}}. \tag{3$'$}$$

Finally,

$$1 = \sum_{i \in I_0} \sum_{g \in \Omega_i \setminus \{e\}} P_g + \sum_{i \in I \setminus I_0} \left(P_{a_i} + P_{a_i^{-1}} \right). \tag{4}$$

Indeed, (1) follows from the relations $T_{gh} = T_g V_h$ and $T_{hg} = V_h T_g$. From the definition, we have $T_{g'}^* T_g = \sum_{n \geq 0} p_n \lambda_{g'}^* p_{n+1} \lambda_g p_n$. This can be non-zero if and only if $|{g'}^{-1} g| = 0$, i.e. ${g'}^{-1} g \in H$. We have (2) immediately. The relation

$$1 = \sum_{i \in I_0} \sum_{g \in \Omega_i} T_g T_g^* + \sum_{i \in I \setminus I_0} \left(T_{a_i} T_{a_i}^* + T_{a_i^{-1}} T_{a_i^{-1}}^* \right) + p_0,$$

implies (4). By multiplying S_g^* on the left and S_g on the right of equation (4) respectively, we obtain (3).

Moreover, the following condition holds: Let $P_i = \sum_{g\in\Omega_i} P_g$ for $i \in I_0$, and $P_i = P_{a_i} + P_{a_i^{-1}}$ for $i \in I \setminus I_0$. For every $i \in I$, we have

$$C^*(H) \simeq C^*\left(P_i U_h P_i \mid h \in H\right). \tag{5}$$

Indeed, since the unitary representation $P_i' V_h P_i'$ contains the left regular representation of H with infinite multiplicity, where P_i' is some projection with $\pi(P_i') = P_i$. we have relation (5).

Now we consider the universal C^*-algebra generated by the family $\{S_g, U_h \mid g \in \bigcup_{i\in I} G_i \setminus H, h \in H\}$ satisfying (1), (2), (3) and (4). We denote it by $\mathcal{O}_\Gamma$. Here, the universality means that if another family $\{s_g, u_h\}$ satisfies (1), (2), (3) and (4), then there exists a surjective $*$-homomorphism ϕ of $\mathcal{O}_\Gamma$ onto $C^*(s_g, u_h)$ such that $\phi(S_g) = s_g$ and $\phi(U_h) = u_h$. Summing up the above, we employ the following definitions and notation:

Definition 4.1. Let I be a finite index set and G_i be a group containing a copy of a finite group H as a subgroup for $i \in I$. Suppose that each G_i is either a finite group or $\mathbb{Z} \times H$. Let I_0 be the subset of I such that G_i is finite for all $i \in I_0$. We denote the amalgamated free product $*_H G_i$ by Γ.

We fix a set Ω_i of left representatives of G_i/H with $e \in \Omega_i$ and a set X_i of representatives of $H\backslash G_i/H$ which is contained in Ω_i. Let (a_i, e) be a generator of G_i for $i \in I \setminus I_0$. We write a_i, for short. Here we choose $\Omega_i = X_i = \{a_i^n \mid n \in \mathbb{N}\}$. We exclude the case where $\bigcup_i \Omega_i \setminus \{e\}$ has only one or two points.

We define the corresponding universal C^*-algebra $\mathcal{O}_\Gamma$ generated by partial isometries S_g for $g \in \bigcup_{i\in I} G_i \setminus H$ and unitaries U_h for $h \in H$ satisfying (1), (2), (3) and (4).

We set for $\gamma \in \Gamma$,

$$Q_\gamma = S_\gamma^* \cdot S_\gamma, \quad P_\gamma = S_\gamma \cdot S_\gamma^*,$$

$$P_i = \sum_{g\in\Omega_i} P_g \quad \text{if } i \in I_0,$$

$$P_i = P_{a_i} + P_{a_i^{-1}} \quad \text{if } i \in I \setminus I_0.$$

For convenience, we set for any integer n,

$$\Gamma_n = \{\gamma \in \Gamma \mid |\gamma| = n\},$$
$$\Delta_n = \{\gamma \in \Gamma_n \mid \gamma = \gamma_1 \cdots \gamma_n, \gamma_k \in \Omega_{i_k}, i_1 \neq \cdots \neq i_n\}.$$

We also set $\Delta = \bigcup_{n\geq 1} \Delta_n$.

Lemma 4.2. *For $i \in I$ and $h \in H$,*

$$U_h P_i = P_i U_h.$$

Proof. Use the above relations (2). □

Lemma 4.3. *Let $\gamma_1, \gamma_2 \in \Gamma$. Suppose that $S_{\gamma_1}^* S_{\gamma_2} \neq 0$.*
If $|\gamma_1| = |\gamma_2|$, then $S_{\gamma_1}^* S_{\gamma_2} = Q_g U_h$ for some $g \in \bigcup_{i \in I} G_i, h \in H$.
If $|\gamma_1| > |\gamma_2|$, then $S_{\gamma_1}^* S_{\gamma_2} = S_\gamma^*$ for some $\gamma \in \Gamma$ with $|\gamma| = |\gamma_1| - |\gamma_2|$.
If $|\gamma_1| < |\gamma_2|$, then $S_{\gamma_1}^* S_{\gamma_2} = S_\gamma$ for some $\gamma \in \Gamma$ with $|\gamma| = |\gamma_2| - |\gamma_1|$.

Proof. By (2), we obtain the lemma. □

Corollary 4.4.

$$\mathcal{O}_\Gamma = \overline{\operatorname{span}}\{S_\mu P_i S_\nu^* \mid \mu, \nu \in \Gamma, i \in I\}.$$

Proof. This follows from the previous lemma. □

Next we consider the gauge action of $\mathcal{O}_\Gamma$. Namely, if $z \in \mathbb{T}$ then the family $\{zS_g, U_h\}$ also satisfies (1), (2), (3), (4) and generates $\mathcal{O}_\Gamma$. The universality gives an automorphism α_z on $\mathcal{O}_\Gamma$ such that $\alpha_z(S_g) = zS_g$ and $\alpha_z(U_h) = U_h$. In fact, α is a continuous action of $\mathbb{T}$ on $\mathcal{O}_\Gamma$, which is called *the gauge action.* Let dz be the normalized Haar measure on $\mathbb{T}$ and we define a conditional expectation Φ of $\mathcal{O}_\Gamma$ onto the fixed-point algebra $\mathcal{O}_\Gamma^\mathbb{T} = \{a \in \mathcal{O}_\Gamma \mid \alpha_z(a) = a, \text{ for } z \in \mathbb{T}\}$ by

$$\Phi(a) = \int_\mathbb{T} \alpha_z(a)\, dz, \qquad \text{for } a \in \mathcal{O}_\Gamma.$$

Lemma 4.5. *The fixed-point algebra $\mathcal{O}_\Gamma^\mathbb{T}$ is an AF-algebra.*

Proof. For each $i \in I$, set

$$\mathcal{F}_n^i = \overline{\operatorname{span}}\{S_\mu P_i S_\nu^* \mid \mu, \nu \in \Gamma_n\}.$$

We can find systems of matrix units in $\mathcal{F}_n^i$, parameterized by $\mu, \nu \in \Delta_n$, as follows:

$$e_{\mu,\nu}^i = S_\mu P_i S_\nu^*.$$

Indeed, using the previous lemma, we compute

$$e_{\mu_1,\nu_1}^i e_{\mu_2,\nu_2}^i = \delta_{\nu_1,\mu_2} S_{\mu_1} P_i Q_{\nu_1} P_i S_{\nu_2}^* = \delta_{\nu_1,\mu_2} e_{\mu_1,\nu_2}^i.$$

Thus we obtain the identifications

$$\mathcal{F}_n^i \simeq M_{N(n,i)}(\mathbb{C}) \otimes e_{\mu,\mu}^i \mathcal{F}_n^i e_{\mu,\mu}^i,$$

for some integer $N(n,i)$ and some $\mu \in \Delta_n$. Moreover, for ξ, η,

$$e_{\mu,\mu}^i \left(S_\xi P_i S_\eta^*\right) e_{\mu,\mu}^i = \begin{cases} S_\mu P_i U_h P_i S_\mu^* & \text{if} \quad \xi, \eta \in \mu H, \\ 0 & \text{otherwise.} \end{cases}$$

for some $h \in H$. Note that $C^*(S_\mu P_i U_h P_i S_\mu^* \mid h \in H)$ is isomorphic to $C^*(P_i U_h P_i \mid h \in H)$ via the map $x \mapsto S_\mu^* x S_\mu$. Therefore the relation (5) gives

$$\mathcal{F}_n^i \simeq M_k(\mathbb{C}) \otimes \overline{\operatorname{span}}\{S_\mu P_i U_h P_i S_\mu^* \mid h \in H\} \simeq M_k(\mathbb{C}) \otimes C^*(H).$$

Note that $\{\mathcal{F}_n^i \mid i \in I\}$ are mutually orthogonal and

$$\mathcal{F}_n = \bigoplus_{i \in I} \mathcal{F}_n^i$$

is a finite-dimensional C^*-algebra.

The relation (2) gives $\mathcal{F}_n \hookrightarrow \mathcal{F}_{n+1}$. Hence,

$$\mathcal{F} = \overline{\bigcup_{n \geq 0} \mathcal{F}_n}$$

is an AF-algebra. Therefore it suffices to show that $\mathcal{F} = \mathcal{O}_\Gamma^\mathbb{T}$. It is trivial that $\mathcal{F} \subseteq \mathcal{O}_\Gamma^\mathbb{T}$. On the other hand, we can approximate any $a \in \mathcal{O}_\Gamma^\mathbb{T}$ by a linear combination of elements of the form $S_\mu P_i S_\nu^*$. Since $\Phi(a) = a$, a can be approximated by a linear combination of elements of the form $S_\mu P_i S_\nu^*$ with $|\mu| = |\nu|$. Thus $a \in \mathcal{F}$. ☐

We need another lemma to prove the uniqueness of $\mathcal{O}_\Gamma$.

Lemma 4.6. *Suppose that $i_0 \in I$ and W consists of finitely many elements $(\mu, h) \in \Delta \times H$ such that the last word of μ is not contained in Ω_{i_0} and $W \cap \{e\} \times H = \emptyset$. Then there exists $\gamma = g_0 \cdots g_n$ with $g_k \in \Omega_{i_k}$ and $i_0 \neq \cdots \neq i_n \neq i_0$ such that for any $(\mu, h) \in W$, $\mu h \gamma$ never have the form $\gamma\gamma'$ for some $\gamma' \in \Gamma$.*

Proof. Let $i_0 \in I$ and W be a finite subset of $\Delta \times H$ as above. We first assume that $|I| \geq 3$. Then we can choose $x \in \Omega_{i_0}, y \in \Omega_j$ and $z \in \Omega_{j'}$ such that $j \neq i_0 \neq j'$ and $j \neq j'$. For sufficiently long word

$$\gamma = (xy)(xz)(xyxy)(xzxz)(xyxyxy)(xzxzxz) \cdots (\cdots z),$$

we are done. We next assume that $|I| = 2$. Since we exclude the case where $\Omega_1 \cup \Omega_2 \setminus \{e\}$ has only one or two elements, we can choose at least three distinct points $x \in \Omega_{i_0}, y \in \Omega_j$ and $z \in \Omega_{j'}$. If $i_0 \neq j = j'$ we set

$$\gamma = (xy)(xz)(xyxy)(xzxz)(xyxyxy)(xzxzxz)\cdots(\cdots z),$$

as well. If $i_0 = j \neq j'$ we set

$$\gamma = (xz)(yz)(xzxz)(yzyz)(xzxzxz)(yzyzyz)\cdots(\cdots z).$$

Then if γ has the desired properties, we are done. Now assume that there exist some $(\mu, h) \in W$ such that $\mu h \gamma = \gamma\gamma'$ for some γ'. Fix such an element $(\mu, h) \in W$. By hypothesis, we can choose $\delta \in \Delta$ with $|\gamma'| \leq |\delta|$ such that the last word of δ does not belong to Ω_{i_0} and δ does not have the form $\gamma'\delta'$ for some δ'. Set $\tilde{\gamma} = \gamma\delta$. Then $\mu h \tilde{\gamma}$ does not have the form $\gamma\gamma''$ for any γ''. Indeed,

$$\mu h \tilde{\gamma} = \mu h \gamma \delta = \gamma\gamma'\delta \neq \tilde{\gamma}\gamma'',$$

for some γ''. Since W is finite, we can obtain a desired element γ by replacing $\tilde{\gamma}$, inductively. □

We now obtain the uniqueness theorem for $\mathcal{O}_\Gamma$.

Theorem 4.7. *Let $\{s_g, u_h\}$ be another family of partial isometries and unitaries satisfying* (1), (2), (3) *and* (4). *Assume that*

$$C^*(H) \simeq C^*(p_i u_h p_i \mid h \in H),$$

where $p_i = \sum_{g \in \Omega_i \setminus \{e\}} s_g s_g^*$ for $i \in I_0$ and $p_i = s_{a_i} s_{a_i}^* + s_{a_i^{-1}} s_{a_i^{-1}}^*$ for $i \in I \setminus I_0$. Then the canonical surjective $*$-homomorphism π of $\mathcal{O}_\Gamma$ onto $C^*(s_g, u_h)$ is faithful.

Proof. To prove the theorem, it is enough to show that (a) π is faithful on the fixed-point algebra $\mathcal{O}_\Gamma^{\mathbb{T}}$, and (b) $\|\pi(\Phi(a))\| \leq \|\pi(a)\|$ for all $a \in \mathcal{O}_\Gamma$ thanks to [4, Lemma 2.2].

To establish (a), it suffices to show that π is faithful on $\mathcal{F}_n$ for all $n \geq 0$. By the proof of Lemma 4.5, we have

$$\mathcal{F}_n^i = M_{N(n,i)}(\mathbb{C}) \otimes C^*(H),$$

for some integer $N(n, i)$. Note that $s_g s_g^*$ is non-zero. Hence π is injective on $M_{N(n,i)}(\mathbb{C})$. By the other hypothesis, π is injective on $C^*(H)$.

Next we will show (b). It is enough to check (b) for

$$a = \sum_{\mu,\nu\in F}\sum_{j\in J} C^j_{\mu,\nu} S_\mu P_j S^*_\nu,$$

where F is a finite subset of Γ and J is a subset of I. For $n = \max\{|\mu| \mid \mu \in F\}$, we have

$$\Phi(a) = \sum_{\{\mu,\nu\in F \mid |\mu|=|\nu|\}}\sum_{j\in J} C^j_{\mu,\nu} S_\mu P_j S^*_\nu \in \mathcal{F}_n.$$

Now by changing F if necessary, we may assume that $\min\{|\mu|,|\nu|\} = n$ for every pair $\mu,\nu \in F$ with $C^j_{\mu,\nu} \neq 0$. Since $\mathcal{F}_n = \oplus_i \mathcal{F}^i_n$, there exists some $i_0 \in J$ such that

$$\|\pi(\Phi(a))\| = \left\| \sum_{|\mu|=|\nu|} C^{i_0}_{\mu,\nu} s_\mu p_{i_0} s^*_\nu \right\|.$$

By changing F such that $F \subset \Delta$ again, we may further assume that

$$\|\pi(\Phi(a))\| = \left\| \sum_{\substack{\mu,\nu\in F\\ |\mu|=|\nu|}} \sum_{h\in F'} C^{i_0}_{\mu,\nu,h} s_\mu p_{i_0} u_h p_{i_0} s^*_\nu \right\|$$

where F' consists of elements of H, (perhaps with multiplicity). By applying the preceding lemma to

$$W = \{(\mu',h) \in \Delta\times H \mid \mu' \text{ is subword of } \mu \in F, h^{-1} \in F'\},$$

we have $\gamma \in \Delta$ satisfying the property in the previous lemma. Then we define a projection

$$Q = \sum_{\tau\in\Delta_n} s_\tau s_\gamma p_{i_0} s^*_\gamma s^*_\tau.$$

By hypothesis, Q is non-zero.

If $\mu,\nu \in \Delta_n$ then

$$Q\left(s_\mu p_{i_0} s^*_\nu\right) Q = s_\mu s_\gamma p_{i_0} s^*_\gamma p_{i_0} s_\gamma p_{i_0} s^*_\gamma s^*_\nu = s_\mu s_\gamma p_{i_0} s^*_\gamma s^*_\nu$$

is non-zero. Therefore $s_\mu(s_\gamma p_{i_0} s^*_\gamma)s^*_\nu$ is also a family of matrix units parameterized by $\mu,\nu \in \Delta_n$. Hence the same arguments as in the proof of Lemma 4.5 give

$$\pi(\mathcal{F}^{i_0}_n) \simeq M_{N(n,i_0)}(\mathbb{C}) \otimes C^*\left(s_\mu s_\gamma p_{i_0} u_h p_{i_0} s^*_\gamma s^*_\mu \mid h \in H\right).$$

By hypothesis, we deduce that $b \mapsto Q\pi(b)Q$ is faithful on $\mathcal{F}^{i_0}_n$. In particular, we conclude that $\|\pi(\Phi(a))\| = \|Q\pi(\Phi(a))Q\|$.

We next claim that $Q\pi(\Phi(a))Q = Q\pi(a)Q$. We fix $\mu, \nu \in F$. If $|\mu| \neq |\nu|$ then one of μ, ν has length n and the other is longer; say $|\mu| = n$ and $|\nu| > n$. Then

$$Q\left(s_\mu p_{i_0} u_h p_{i_0} s_\nu^*\right) Q = s_\mu s_\gamma p_{i_0} s_\gamma^* p_{i_0} u_h p_{i_0} s_\nu^* \left(\sum_{\tau \in \Delta_n} s_\tau s_\gamma p_{i_0} s_\gamma^* s_\tau^* \right).$$

Since $|\nu| > |\tau|$, this can have a non-zero summand only if $\nu = \tau\nu'$ for some ν'. However $s_\gamma^* u_h s_\nu^* s_\tau s_\gamma = s_\gamma^* u_h s_{\nu'}^* s_\gamma$, and $s_{\nu' h^{-1}\gamma}^* s_\gamma$ is non-zero only if $\nu' h^{-1}\gamma$ has the form $\gamma\gamma'$. This is impossible by the choice of γ. Therefore we have $Q\left(s_\mu p_{i_0} s_\nu\right) Q = 0$ if $|\mu| \neq |\nu|$, namely $Q\pi(\Phi(a))Q = Q\pi(a)Q$. Hence we can finish proving (b):

$$\|\pi(\Phi(a))\| = \|Q\pi(\Phi(a))Q\| = \|Q\pi(a)Q\| \leq \|\pi(a)\|.$$

Therefore [4, Lemma 2.2] gives the theorem. □

By essentially the same arguments, we can prove the following.

Corollary 4.8. *Let $\{t_g, v_h\}$ and $\{s_g, u_h\}$ be two families of partial isometries and unitaries satisfying* (1), (2), (3) *and* (4). *Suppose that the map $p_i v_h p_i \mapsto q_i u_h q_i$ gives an isomorphism*:

$$C^*(p_i v_h p_i \mid h \in H) \simeq C^*(q_i v_h q_i \mid h \in H),$$

where $p_i = \sum_{g \in \Omega_i \setminus \{e\}} t_g t_g^*, q_i = \sum_{g \in \Omega_i \setminus \{e\}} s_g s_g^*$ and so on. Then the canonical map gives the isomorphism between $C^*(t_g, v_h)$ and $C^*(s_g, u_h)$.

Before closing this section, we will show that our algebra $\mathcal{O}_\Gamma$ is isomorphic to a certain Cuntz-Krieger-Pimsner algebra. Let $A = C^*\left(P_i U_h P_i \,|\, h \in H, i \in I\right) \simeq \bigoplus_{i \in I} C_r^*(H)$. We define a Hilbert A-bimodule X as follows:

$$X = \overline{\operatorname{span}} \left\{ S_g P_i \mid g \in \bigcup_{j \neq i} G_j,\, |g| = 1,\, i \in I \right\}$$

with respect to the inner product $\langle S_g P_i, S_{g'} P_j \rangle = P_i S_g^* S_{g'} P_j \in A$. In terms of the groups, the A-A bimodule structure can be described as follows: we set

$$A = \bigoplus_{i \in I} A_i = \bigoplus_{i \in I} \mathbb{C}[H],$$

and define an A-bimodule $\mathcal{H}_i$ by

$$\mathcal{H}_i = \mathbb{C}\left[\left\{ g \in \bigcup_{j \neq i} G_j \mid |g| = 1 \right\} \right]$$

with left and right A-multiplications such that for $a = (h_i)_{i \in I} \in A$ and $g \in G_j \setminus H \subset \mathcal{H}_i$,

$$a \cdot g = h_j g \quad \text{and} \quad g \cdot a = g h_i,$$

and with respect to the inner product

$$\langle g, g' \rangle_{\mathcal{H}_i} = \begin{cases} g^{-1} g' \in A_i & \text{if } g^{-1} g' \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Then we define the A-bimodule X by

$$X = \bigoplus_{i \in I} \mathcal{H}_i,$$

and we obtain the CKP-algebra $\mathcal{O}_X$.

Proposition 4.9. *Assume that A and X are as above. Then*

$$\mathcal{O}_\Gamma \simeq \mathcal{O}_X.$$

Proof. We fix a finite basis $u(g, i) = g \in \mathcal{H}_i$ for $g \in \Omega_j, i \in I$ with $j \neq i, |g| = 1$. Then we have $\mathcal{O}_X = C^*(S_{u(g,i)})$. Let $s_{u(g,i)} = S_g P_i$ in $\mathcal{O}_\Gamma$. Note that we have $\mathcal{O}_\Gamma = C^*(s_{u(g,i)})$. The relation (4) corresponds to the relations (†) of the CKP-algebras. The family $\{s_{u(g,i)}\}$ therefore satisfies the relations of the CKP-algebras. Since the CKP-algebra has universal properties, there exists a canonical surjective $*$-homomorphism of $\mathcal{O}_X$ onto $\mathcal{O}_\Gamma$. Conversely, let $s_g = \sum_{i \in I} S_{u(g,i)}$ and $u_h = \oplus_{i \in I} h$ for $h \in H$ in $\mathcal{O}_X$, and then we have $\mathcal{O}_X = C^*(s_g, u_h)$. By the universality of $\mathcal{O}_\Gamma$, we can also obtain a canonical surjective $*$-homomorphism of $\mathcal{O}_\Gamma$ onto $\mathcal{O}_X$. These maps are mutual inverses. Indeed,

$$\begin{aligned} S_g &\mapsto \textstyle\sum_{i \in I} S_{u(g,i)} \mapsto \sum_{i \in I} S_g P_i = S_g, \\ U_h &\mapsto \textstyle\bigoplus_{i \in I} h \mapsto \sum_{i \in I} P_i U_h P_i = U_h. \end{aligned}$$

□

§5. Crossed Product Algebras Associated with $\mathcal{O}_\Gamma$

In this section, we will show that $\mathcal{O}_\Gamma$ is isomorphic to a crossed product algebra. We first define a "boundary space". We set

$$\tilde{\Lambda} = \{(\gamma_n) | \gamma_n \in \Gamma, |\gamma_n| + 1 = |\gamma_{n+1}|, |\gamma_n^{-1} \gamma_{n+1}| = 1 \text{ for a sufficiently large } n \geq 0\}.$$

We introduce the following equivalence relation $\sim$; $(\gamma_n)_{n \geq 0}, (\gamma'_n)_{n \geq 0} \in \tilde{\Lambda}$ are equivalent if there exists some $k \in \mathbb{Z}$ such that $\gamma_n H = \gamma'_{n+k} H$ for a sufficiently large n. Then we define $\Lambda = \tilde{\Lambda}/\sim$. We denote the equivalent class of $(\gamma_n)_{n\geq 0}$ by $[\gamma_n]_{n\geq 0}$.

Before we define an action of Γ on Λ, we construct another space Ω to introduce a compact space structure, on which Γ acts continuously. Let Ω denote the set of sequences $x : \mathbb{N} \to \Gamma$ such that

$$\begin{cases} x(n) \in \Omega_{i_n} \setminus \{e\} & \text{for } n \geq 1, \\ x(n) \in \{a_{i_n}^{\pm 1}\} & \text{if } i_n \in I \setminus I_0, \\ i_n \neq i_{n+1} & \text{if } i_n \in I_0, \\ x(n) = x(n+1) & \text{if } i_n \in I \setminus I_0, i_n = i_{n+1}. \end{cases}$$

Note that Ω is a compact Hausdorff subspace of $\prod_{\mathbb{N}} (\bigcup_i \Omega_i \setminus \{e\})$. We introduce a map ϕ between Λ and Ω; for $x = (x(n))_{n\geq 1} \in \Omega$, we define a map $\phi(x) = [\gamma_n] \in \Lambda$ by

$$\gamma_0 = e \qquad \text{if } n = 0,$$
$$\gamma_n = x(1) \cdots x(n), \qquad \text{if } n \geq 1.$$

Lemma 5.1. *The above map ϕ is a bijection from Λ onto Ω and hence Λ inherits a compact space structure via ϕ.*

Proof. For $x = (x(n)) \neq x' = (x'(n))$, there exists an integer k such that $x(k) \neq x'(k)$. If $\phi(x) = [\gamma_n]$ and $\phi(x') = [\gamma'_n]$, then $\gamma_k H \neq \gamma'_k H$. Hence we have injectivity of ϕ. Next we will show surjectivity. Let $[\gamma_n] \in \Sigma$. We may take a representative (γ_n) satisfying $|\gamma_n| = n$. Now we assume that γ_n is uniquely expressed as $\gamma_n = g_1 \cdots g_n h, \gamma_{n+1} = g'_1 \cdots g'_{n+1} h'$ for $g_k \in \Omega_{i_k}, g'_k \in \Omega_{j_k}, h, h' \in H$. Since $|\gamma_n^{-1}\gamma_{n+1}| = 1$, we have

$$h^{-1} g_n^{-1} \cdots g_1^{-1} g'_1 \cdots g'_{n+1} h' = g,$$

for some $g \notin H$ with $|g| = 1$. Inductively, we have $g_1 = g'_1, \ldots, g_n = g'_n$. Hence we can assume that $\gamma_n = g_1 \cdots g_n$. We set $x(n) = g_n$ and get $\phi((x(n))) = [\gamma_n]$. □

Next we define an action of Γ on Λ. Let $[\gamma_n]_{n\geq 0} \in \Lambda$. For $\gamma \in \Gamma$, define

$$\gamma \cdot [\gamma_n]_{n\geq 0} = [\gamma\gamma_n]_{n\geq 0}.$$

We will show that this is a continuous action of Γ on Λ. Let $[\gamma_n]$, $[\gamma'_n] \in \Lambda$ such that $(\gamma_n) \sim (\gamma'_n)$ and $\gamma \in \Gamma$. Since there exists some integer k such that $\gamma_n H = \gamma'_{n+k} H$ for sufficiently large integers n, we have $\gamma\gamma_n H = \gamma\gamma'_{n+k} H$.

Hence this is well-defined. To show that γ is continuous, we consider how γ acts on Ω via the map ϕ. For $g \in \Omega_i$ with $|g| = 1$ and $x = (x(n))_{n\geq 1} \in \Omega$,

$$(g \cdot x)(1) = \begin{cases} g & \text{if } i \neq i_1, \\ g_1 & \text{if } i = i_1,\ gx(1) \notin H,\ i \in I_0, \\ & \text{and } gx(1) = g_1 h_1\, (g_1 \in \Omega_{i_1}, h_1 \in H), \\ g & \text{if } i = i_1,\ gx(1) \notin H,\ i \in I \setminus I_0, \\ g_2 & \text{if } i = i_1,\ gx(1) \in H,\ i \in I_0, \\ & \text{and } gx(1) = h_1,\ h_1 x(2) = g_2 h_2 (g_2 \in \Omega_{i_2}, h_1, h_2 \in H), \\ x(2) & \text{if } i = i_1,\ gx(1) \in H,\ i \in I \setminus I_0, \end{cases}$$

and for $n > 1$,

$$(g \cdot x)(n) = \begin{cases} x(n-1) & \text{if } i \neq i_1, \\ g_n & \text{if } i = i_1,\ gx(1) \notin H, \\ & \text{and } h_{n-1} x(n) = g_n h_n\, (g_n \in \Omega_{i_n}, h_n \in H), \\ x(n-1) & \text{if } i = i_1,\ gx(1) \notin H,\ i \in I \setminus I_0, \\ g_{n+1} & \text{if } i = i_1,\ gx(1) \in H, \\ & \text{and } h_n x(n+1) = g_{n+1} h_{n+1}, (g_{n+1} \in \Omega_{i_{n+1}}, h_{n+1} \in H), \\ x(n+1) & \text{if } i = i_1,\ gx(1) \in H,\ i \in I \setminus I_0. \end{cases}$$

For $h \in H$,

$$(h \cdot x)(n) = \begin{cases} g_1 & \text{if } n = 1, \\ & \text{and } hx(1) = g_1 h_1,\ (g_1 \in \Omega_{i_1}, h_n \in H), \\ g_n & \text{if } n > 1, \\ & \text{and } h_{n-1} x(n) = g_n h_n,\ (g_n \in \Omega_{i_n}, h_n \in H). \end{cases}$$

Then one can check easily that the pull-back of any open set of Ω by γ is also an open set of Ω. Thus we have proved that γ is a homeomorphism on Λ. The equations

$$(\gamma\gamma')[\gamma_n] = [\gamma\gamma'\gamma_n] = \gamma([\gamma'\gamma_n]) = \gamma \circ \gamma'[\gamma_n],$$

imply associativity.

Therefore we have obtained the following:

Lemma 5.2. *The above space Ω is a compact Hausdorff space and Γ acts on Ω continuously.*

The following result is the main theorem of this section.

Theorem 5.3. *Assume that Ω and the action of Γ on Ω are as above. Then we have the identifications*

$$\mathcal{O}_\Gamma \simeq C(\Omega) \rtimes \Gamma \simeq C(\Omega) \rtimes_r \Gamma.$$

Proof. We first consider the full crossed product $C(\Omega) \rtimes \Gamma$. Let $Y_i = \{(x(n)) \mid x(1) \in \Omega_i\} \subset \Omega$ be clopen sets for $i \in I$. Note that if $i \in I_0$, then Y_i is the disjoint union of the clopen sets $\{g(\Omega \setminus Y_i) \mid g \in \Omega_i \setminus \{e\}\}$, and if $i \in I \setminus I_0$, then $Y_i = Y_i^+ \cup Y_i^-$ where $Y_i^\pm = \{(x(n)) \mid x(1) = a_i^\pm\}$. Let $p_i = \chi_{\Omega\setminus Y_i}$ and $p_i^\pm = \chi_{Y_i^\pm}$. We define $T_g = gp_i$ for $g \in G_i \setminus H$ and $i \in I_0$ and $T_{a_i^{\pm1}} = a_i^{\pm1}\left(p_i + p_i^\pm\right)$ for $i \in I \setminus I_0$. Let $V_h = h$ for $h \in H$. Then the family $\{T_g, V_h\}$ satisfies the relations (1), (2), (3) and (4). Indeed, we can first check that $h \in H$ commutes with p_i and $p_i^{\pm1}$. So the relation (1) holds. Let $g \in G_i \setminus H$ and $g' \in G_j \setminus H$ with $i, j \in I_0$. Then

$$T_g^* T_{g'} = p_i g^{-1} g' p_j = g^{-1} \chi_{g(\Omega\setminus Y_i)} \chi_{g'(\Omega\setminus Y_j)} g' = \delta_{i,j} \delta_{gH,g'H} p_i g^{-1} g'.$$

Moreover it follows from $\Omega \setminus Y_i = \bigcup_{j\neq i} Y_j$ that

$$\begin{aligned} T_g^* T_g = \chi_{\Omega\setminus Y_i} &= \sum_{j\neq i} \chi_{Y_j} \\ &= \sum_{j\in I_0, j\neq i} \sum_{g\in\Omega_j\setminus\{e\}} \chi_{g(\Omega\setminus Y_j)} + \sum_{j\in I\setminus I_0} \chi_{a_j(\Omega\setminus Y_j)} + \chi_{a_j^{-1}(\Omega\setminus Y_j)} \\ &= \sum_{j\in I_0, j\neq i} \sum_{g\in\Omega_j\setminus\{e\}} g p_j g^{-1} + \sum_{j\in I\setminus I_0} p_j^+ + p_j^- \\ &= \sum_{j\in I_0, j\neq i} \sum_{g\in\Omega_j\setminus\{e\}} T_g T_g^* + \sum_{j\in I\setminus I_0} T_{a_j} T_{a_j}^* + T_{a_j^{-1}} T_{a_j^{-1}}^*. \end{aligned}$$

For all other cases, we can also check the relations (2) and (3) by similar calculations. Since Ω is the disjoint union of Y_i, we have (4). Note that $g, p_i, p_i^\pm \in C^*(T_g, V_h)$. Moreover, since the family $\{\gamma(\Omega \setminus Y_i) \mid \gamma \in \Gamma, i \in I\} \cup \{\gamma Y_i^\pm \mid \gamma \in \Gamma, i \in I \setminus I_0\}$ generates the topology of Ω, we have $C(\Omega) \rtimes \Gamma = C^*(T_g, V_h)$. By the universality of $\mathcal{O}_\Gamma$, there exists a canonical surjective $*$-homomorphism of $\mathcal{O}_\Gamma$ onto $C(\Omega) \rtimes \Gamma$, sending S_g to T_g and U_h to V_h.

Conversely, let $q_i = \sum_{j\neq i} P_j$ and $q_i^\pm = S_{a_i^{\pm1}} S_{a_i^{\pm1}}^*$. Let

$$\begin{cases} w_g = S_g + \sum_{g'\in\Omega_i\setminus H\cup g^{-1}H} S_{gg'} S_{g'}^* + S_g^* & \text{for } g \in G_i \setminus H, i \in I_0, \\ w_{a_i} = S_{a_i} + S_{a_i^{-1}}^* & \text{for } i \in I \setminus I_0, \\ w_h = U_h & \text{for } h \in H. \end{cases}$$

We will check that w_g are unitaries for $g \in G_i \setminus H$ with $i \in I_0$. If $g' \in \Omega_i \setminus H \cup g^{-1}H$, then $gg'H = \gamma H$ for some $\gamma \in \Omega_i \setminus \{e, g\}$. Hence

$$\begin{aligned} & w_g w_g^* \\ &= \left(S_g + \sum_{g' \in \Omega_i \setminus H \cup g^{-1}H} S_{gg'} S_{g'}^* + S_{g^{-1}}^*\right)\left(S_g + \sum_{g' \in \Omega_i \setminus H \cup g^{-1}H} S_{gg'} S_{g'}^* + S_{g^{-1}}^*\right)^* \\ &= S_g S_g^* + \sum_{g' \in \Omega_i \setminus H \cup g^{-1}H} S_{gg'} S_{g'}^* S_{g'} S_{gg'}^* + S_{g^{-1}}^* S_{g^{-1}} \\ &= P_g + \sum_{g' \in \Omega_i \setminus \{e, g\}} P_{g'} + Q_g = 1. \end{aligned}$$

Similarly, we have $w_g^* w_g = 1$. For the other case, we can check in the same way.

If $i \in I_0, \tau \in \Omega_i \setminus \{e\}$ then

$$\begin{aligned} \sum_{g \in \Omega_i} w_g q_i w_g^* &= \sum_{g \in \Omega_i} \left(S_g + \sum_{g' \in \Omega_i \setminus H \cup g^{-1}H} S_{gg'} S_{g'}^* + S_{g^{-1}}^*\right) S_\tau^* S_\tau w_g^* \\ &= \sum_{g \in \Omega_i} S_g S_\tau^* S_\tau \left(S_g^* + \sum_{g' \in \Omega_i \setminus H \cup g^{-1}H} S_g S_{gg'}^* + S_{g^{-1}}\right) \\ &= \sum_{g \in \Omega_i} S_g S_\tau^* S_\tau S_g^* = 1. \end{aligned}$$

For $i \in I \setminus I_0$, we have $q_i^+ + w_{a_i} q_i^- w_{a_i}^* = 1$ and $q_i^+ + q_i^- + q_i = 1$ as well. Therefore the conjugates of the family $\{q_i, q_i^\pm\}$ by the elements of Γ generate a commutative C^*-algebra. This is the image of a representation of $C(\Omega)$. Therefore (q_i, w) gives a covariant representation of the C^*-dynamical system $(C(\Omega), \Gamma)$. Note that (q_i, w_g) generates $\mathcal{O}_\Gamma$. Hence by the universality of the full crossed product $C(\Omega) \rtimes \Gamma$, there exists a canonical surjective $*$-homomorphism of $C(\Omega) \rtimes \Gamma$ onto $\mathcal{O}_\Gamma$. It is easy to show that the above two $*$-homomorphisms are the inverses of each other.

$$\begin{array}{ccccc} S_g & \mapsto & gp_i & \mapsto & w_g Q_g = S_g, \\ S_{a_i^{\pm 1}} & \mapsto & a_i^{\pm 1}(p_i + p_i^\pm) & \mapsto & w_{a_i^{\pm 1}}(Q_{a_i^{\pm 1}} + P_{a_i^{\pm 1}}) = S_{a_i^{\pm 1}}, \\ U_h & \mapsto & h & \mapsto & U_h. \end{array}$$

We have shown the identification $\mathcal{O}_\Gamma \simeq C(\Omega) \rtimes \Gamma$. Since there exists a canonical surjective map of $C(\Omega) \rtimes \Gamma$ onto $C(\Omega) \rtimes_r \Gamma$, we have a surjective $*$-homomorphism of $\mathcal{O}_\Gamma$ onto $C(\Omega) \rtimes_r \Gamma$. Let $C(\Omega) \rtimes_r \Gamma = C^*(\tilde{\pi}(p_i), \lambda)$ where

$\tilde{\pi}$ is the induced representation on the Hilbert space $l^2(\Gamma, \mathcal{H})$ by the universal representation π of $C(\Omega)$ on a Hilbert space $\mathcal{H}$ and λ is the unitary representation of Γ on $l^2(\Gamma, \mathcal{H})$ such that $(\lambda_s x)(t) = x(s^{-1}t)$ for $x \in l^2(\Gamma, \mathcal{H})$. By the uniqueness theorem for $\mathcal{O}_\Gamma$, it suffices to check

$$C^*\left(\tilde{\pi}(\chi_{Y_i})\lambda_h\tilde{\pi}(\chi_{Y_i})\right) \simeq C^*(H).$$

But the unitary representation $\tilde{\pi}(\chi_{Y_i})\lambda_h\tilde{\pi}(\chi_{Y_i})$ is quasi-equivalent to the left regular representation of H. This completes the proof of the theorem. □

In [27], Serre defined the tree G_T, on which Γ acts. In an appendix, we will give the definition of the tree $G_T = (V, E)$ where V is the set of vertices and E is the set of edges. We denote the corresponding natural boundary by ∂G_T. We also show how to construct boundaries of trees in the appendix. (See Furstenberg [17] and Freudenthal [16] for details.)

Proposition 5.4. *The space ∂G_T is homeomorphic to Ω and the above two actions of Γ on ∂G_T and Ω are conjugate.*

Proof. We define a map ψ from ∂G_T to Ω. First we assume that $I = \{1, 2\}$. The corresponding tree G_T consists of the vertex set $V = \Gamma/G_1 \coprod \Gamma/G_2$ and the edge set $E = \Gamma/H$. For $\omega \in \partial G_T$, we can identify ω with an infinite chain $\{G_{i_1}, g_1G_{i_2}, g_1g_2G_{i_3}, \dots\}$ with $g_k \in \Omega_{i_k} \setminus \{e\}$ and $i_1 \neq i_2 \neq \cdots$. Then we define $\psi(\omega) = [x(n) = g_{i_n}]$. We will recall the definition of the corresponding tree G_T, in general, on the appendix, (see [27]). Similarly, we can identify $\omega \in \partial G_T$ with an infinite chain $\{G_0, G_{i_1}, g_1G_0, g_1G_{i_2}, g_1g_2G_0, \dots\}$. Moreover we may ignore vertices γG_0 for an infinite chain ω,

$$\{G_0, G_{i_1}, (g_1G_0 \to \text{ignoring}), g_1G_{i_2}, (g_1g_2G_0 \to \text{ignoring}), g_1g_2G_{i_3}, \dots\}.$$

Therefore, we define a map ψ of ∂G_T to Ω by

$$\psi(\omega) = [x(n) = g_n].$$

The pull-back by ψ of any open set of ∂G_T is an open set on Ω. It follows that ψ is a homeomorphism. The two actions on ∂G_T and Ω are defined by left multiplication. So it immediately follows that these actions are conjugate. □

It is known that Γ is a hyperbolic group (see a proof in the appendix, where we recall the notion of hyperbolicity for finitely generated groups as introduced by Gromov e.g. see [18]). Let $S = \{\bigcup_{i \in I} G_i\}$ and $G(\Gamma, S)$ be the Cayley graph of Γ with the word metric d. Let $\partial\Gamma$ be the hyperbolic boundary.

Proposition 5.5. *The hyperbolic boundary* $\partial\Gamma$ *is homeomorphic to* Ω *and the actions of* Γ *are conjugate.*

Proof. We can define a map ψ from Ω to $\partial\Gamma$ by $(x(n)) \mapsto [x_n = x(1)\cdots x(n)]$. Indeed, since $\langle x_n \,|\, x_m\rangle = \min\{n,m\} \to \infty\,(n,m\to\infty)$, it is well-defined. For $x \neq y$ in Ω, there exists k such that $x(k) \neq y(k)$. Then $\langle \psi(x)|\psi(y)\rangle \leq k+1$, which shows injectivity. Let $(x_n) \in \partial\Gamma$. Suppose that $x_n = g_{n(1)}\cdots g_{n(k_n)}h_n$ for some $g_l \in \bigcup_i \Omega_i \setminus \{e\}$ with $n(1) \neq \cdots \neq n(k_n)$. If $g_{n(1)} = g_{m(1)}, \dots, g_{n(l)} = g_{m(l)}$ and $g_{n(l+1)} \neq g_{m(l+1)}$, then we set $a_{n,m} = g_{n(1)}\cdots g_{n(l)} = g_{m(1)}\cdots g_{m(l)}$. So we have

$$\langle x_n \,|\, x_m\rangle \leq d(e, a_{n,m}) + 1 \to \infty \;(n,m\to\infty).$$

Therefore we can choose sequences $n_1 < n_2 < \cdots$, and $m_1 < m_2 < \cdots$, such that a_{n_k,m_k} is a sub-word of $a_{n_{k+1},m_{k+1}}$. Then a sequence $\{g_{n_k(1)},\dots,g_{n_k(l)}, g_{n_{k+1}(l+1)},\dots\}$ is mapped to (x_n) by ψ. We have proved that ψ is surjective. The pull-back of any open set in $\partial\Gamma$ is an open set in Ω. So ψ is continuous. Since $\Omega, \partial\Gamma$ are compact Hausdorff spaces, ψ is a homeomorphism. Again, the two actions on Ω and $\partial\Gamma$ are defined by left multiplication and hence are conjugate. □

Remark. Since the action of Γ on $\partial\Gamma$ depends only on the group structure of Γ in [18], the above proposition shows that $\mathcal{O}_\Gamma$ is, up to isomorophism, independent of the choice of generators of Γ.

§6. Nuclearity, Simplicity and Pure Infiniteness of $\mathcal{O}_\Gamma$

We first begin by reviewing the crossed product $B \rtimes \mathbb{N}$ of a C^*-algebra B by a $*$-endomorphism; this construction was first introduced by Cuntz [6] to describe the Cuntz algebra $\mathcal{O}_n$ as the crossed product of UHF algebras by $*$-endomorphisms. See Stacey's paper [29] for a more detailed discussion. Suppose that ρ is an injective $*$-endomorphism on a unital C^*-algebra B. Let $\overline{B}$ be the inductive limit $\varinjlim(B \xrightarrow{\rho} B)$ with the corresponding injective homomorphisms $\sigma_n : B \to \overline{B}$ $(n \in \mathbb{N})$. Let p be the projection $\sigma_0(1)$. There exists an automorphism $\bar{\rho}$ given by $\bar{\rho}\circ\sigma_n = \sigma_n \circ \rho$ with inverse $\sigma_n(b) \mapsto \sigma_{n+1}(b)$. Then the crossed product $B \rtimes_\rho \mathbb{N}$ is defined to be the hereditary C^*-algebra $p(\overline{B} \rtimes_{\bar{\rho}} \mathbb{Z})p$. The map σ_0 induces an embedding of B into $\overline{B}$. Therefore the canonical embedding of $\overline{B}$ into $\overline{B} \rtimes_{\bar{\rho}} \mathbb{Z}$ gives an embedding $\pi : B \to B \rtimes_\rho \mathbb{N}$. Moreover the compression by p of the implementing unitary is an isometry V belonging to $B \rtimes_\rho \mathbb{N}$ satisfying

$$V\pi(b)V^* = \pi(\rho(b)).$$

In fact, $B \rtimes_\rho \mathbb{N}$ is also the universal C^*-algebra generated by a copy $\pi(B)$ of B and an isometry V satisfying the above relation. If B is nuclear, then so is $B \rtimes_\rho \mathbb{N}$.

Proposition 6.1.

$$\mathcal{O}_\Gamma \simeq \mathcal{O}_\Gamma^{\mathbb{T}} \rtimes_\rho \mathbb{N}$$

In particular, $\mathcal{O}_\Gamma$ is nuclear.

Proof. We fix $g_i \in G_i \setminus H$ for all $i \in I$. We can choose projections e_i which are sums of projections P_g such that $e_i \le Q_{g_i}$ and $\sum_{i\in I} e_i = 1$. Then $V = \sum_{i\in I} S_{g_i} e_i$ is an isometry in $\mathcal{O}_\Gamma$.

We claim that $V\mathcal{O}_\Gamma^{\mathbb{T}}V^* \subseteq \mathcal{O}_\Gamma^{\mathbb{T}}$ and $\mathcal{O}_\Gamma = C^*\left(\mathcal{O}_\Gamma^{\mathbb{T}}, V\right)$. Let $a \in \mathcal{O}_\Gamma^{\mathbb{T}}$. It is obvious that $VaV^* \in \mathcal{O}_\Gamma^{\mathbb{T}}$ and $C^*\left(\mathcal{O}_\Gamma^{\mathbb{T}}, V\right) \subseteq \mathcal{O}_\Gamma$. To show the second claim, it suffices to check that $S_\mu P_i S_\nu^* \in \mathcal{O}_\Gamma$ for all μ, ν and i. If $|\mu| = |\nu|$, we have $S_\mu P_i S_\nu^* \in \mathcal{O}_\Gamma^{\mathbb{T}}$. If $|\mu| \ne |\nu|$, then we may assume $|\mu| < |\nu|$. Let $|\nu| - |\mu| = k$. Thus $S_\mu P_i S_\nu^* = (V^*)^k V^k S_\mu P_i S_\nu^*$ and $V^k S_\mu P_i S_\nu^* \in \mathcal{O}_\Gamma^{\mathbb{T}}$. This proves our claim.

We define a $*$-endomorphism ρ of $\mathcal{O}_\Gamma^{\mathbb{T}}$ by $\rho(a) = VaV^*$ for $a \in \mathcal{O}_\Gamma^{\mathbb{T}}$. Thanks to the universality of the crossed product $\mathcal{O}_\Gamma^{\mathbb{T}} \rtimes_\rho \mathbb{N}$, we obtain a canonical surjective $*$-homomorphism σ of $\mathcal{O}_\Gamma^{\mathbb{T}} \rtimes_\rho \mathbb{N}$ onto $C^*(\mathcal{O}_\Gamma^{\mathbb{T}}, V)$. Since $\mathcal{O}_\Gamma^{\mathbb{T}} \rtimes_\rho \mathbb{N}$ has the universal property, there also exists a gauge action β on $\mathcal{O}_\Gamma^{\mathbb{T}} \rtimes_\rho \mathbb{N}$. Let Ψ be the corresponding canonical conditional expectation of $\mathcal{O}_\Gamma^{\mathbb{T}} \rtimes_\rho \mathbb{N}$ onto $\mathcal{O}_\Gamma^{\mathbb{T}}$. Suppose that $a \in \ker\sigma$. Then $\sigma(a^*a) = 0$. Since $\alpha \circ \sigma = \sigma \circ \beta$, we have $\sigma \circ \Psi(a^*a) = 0$. The injectivity of σ on $\mathcal{O}_\Gamma^{\mathbb{T}}$ implies $\Psi(a^*a) = 0$ and hence $a^*a = 0$ and $a = 0$. It follows that $\mathcal{O}_\Gamma \simeq \mathcal{O}_\Gamma^{\mathbb{T}} \rtimes_\rho \mathbb{N}$. □

In Section 2, we reviewed the notion of amenability for discrete group actions. The following is a special case of [1].

Corollary 6.2. *The action of Γ on $\partial\Gamma$ is amenable.*

Proof. This follows from Theorem 2.2 and the above proposition. □

We also have a partial result of [20], [9], [10] and [11].

Corollary 6.3. *The reduced group C^*-algebra $C_r^*(\Gamma)$ is exact.*

Proof. It is well-known that every C^*-subalgebra of an exact C^*-algebra is exact; see Wassermann's monograph [31]. Therefore the inclusion $C_r^*(\Gamma) \subset \mathcal{O}_\Gamma$ implies exactness. □

Finally we give a sufficient condition for the simplicity and pure infiniteness of $\mathcal{O}_\Gamma$.

Corollary 6.4. *Suppose that* $\Gamma = *_H G_i$ *satisfies the following condition*:

There exists at least one element $j \in I$ *such that*

$$\bigcap_{i \neq j} N_i = \{e\},$$

where $N_i = \bigcap_{g \in G_i} gHg^{-1}$.

Then $\mathcal{O}_\Gamma$ *is simple and purely infinite.*

Proof. We first claim that for any $\mu \in \Delta$ and $|g| = 1$ with $|\mu g| = |\mu| + 1$,

$$\mu H \mu^{-1} \cap H \supseteq \mu g H g^{-1} \mu^{-1} \cap H.$$

Suppose that $\mu = \mu_1 \cdots \mu_n$ such that $\mu_k \in \Omega_{i_k}$ with $\mu_1 \neq \cdots \neq \mu_n$ and $g \in G_i$ with $i \neq i_n$. We first assume that $\mu = \mu_1$. If $\mu g h g^{-1} \mu^{-1} \in \mu g H g^{-1} \mu^{-1} \cap H$, then $ghg^{-1} \in \mu^{-1} H \mu \subseteq G_{i_1}$. Thus $ghg^{-1} \in G_i \cap G_{i_1}$ implies $ghg^{-1} \in H$. Next we assume that $|\mu| > 1$. If $\mu g h g^{-1} \mu^{-1} \in \mu g H g^{-1} \mu^{-1} \cap H$, then

$$\mu_2 \cdots \mu_n g h g^{-1} \mu_k^{-1} \cdots \mu_2^{-1} \in \mu_1^{-1} H \mu_1 \subseteq G_{i_1}.$$

Thus $|\mu_2 \cdots \mu_n g h g^{-1} \mu_k^{-1} \cdots \mu_2^{-1}| \leq 1$ implies $ghg^{-1} \in H$. This proves the claim.

Let $\{S_g, U_h\}$ be any family satisfying the relations (1), (2), (3) and (4). By the uniqueness theorem, it is enough to show that $C^*(P_i U_h P_i \mid h \in H) \simeq C^*(H)$ for any $i \in I$. We next claim that there exists $\nu \in \Gamma$ such that the initial letter of ν belongs to Ω_i and $\{U_h S_\nu\}_{h \in H}$ have mutually orthogonal ranges.

Let $g \in \Omega_i$. If $gHg^{-1} \cap H = \{e\}$, then it is enough to set $\nu = g$. Now suppose that there exists some $h \in gHg^{-1} \cap H$ with $h \neq e$. We first assume that $i = j$. By the hypothesis, there exists some $i_1 \in I$ such that $g^{-1} h g \notin N_{i_1}$ and $i \neq i_1$. Hence there exists $g_1 \in \Omega_{i_1}$ such that $g^{-1} h g \notin g_1 H g_1^{-1}$ and so $h \notin g g_1 H g_1^{-1} g^{-1}$. If $g g_1 H g_1^{-1} g^{-1} \cap H = \{e\}$, then it is enough to put $\nu = g g_1$. If not, we set $\gamma_1 = g_1 g_1'$ for some $g_1' \in \Omega_j$. By the first part of the proof, we have

$$gHg^{-1} \cap H \supsetneqq \mu \gamma_1 H \gamma_1^{-1} \mu^{-1} \cap H.$$

Since H is finite, we can inductively obtain $\gamma_1, \gamma_2, \ldots \gamma_n$ satisfying

$$gHg^{-1} \cap H \supsetneqq g\gamma_1 H \gamma_1^{-1} g^{-1} \cap H \supsetneqq \cdots \supsetneqq g\gamma_1 \cdots \gamma_n H \gamma_n^{-1} \cdots \gamma_1^{-1} g^{-1} \cap H = \{e\}.$$

Then we set $\nu = g\gamma_1 \cdots \gamma_n$. If $i \neq j$, we can carry out the same arguments by replacing g by $\gamma = g g_j$ for some $g_j \in \Omega_j$. Hence from the identification

$U_hS_\nu \leftrightarrow \delta_h \in l^2(H)$, it follows that the unitary representation $P_iU_hP_i$ is quasi-equivalent to the left regular representation of H. Thus $\mathcal{O}_\Gamma$ is simple.

In Section 5, we have proved that $\mathcal{O}_\Gamma \simeq C(\Omega) \rtimes_r \Gamma$. We show that the action of Γ on Ω is the strong boundary action (see Preliminaries). Let U, V be any non-empty open sets in Ω. There exists some open set $O = \{(x(n)) \in \Omega \mid x(1) = g_1, \dots, x(k) = g_k\}$ which is contained in V. We may also assume that U^c is an open of the form $\{(x(n)) \in \Omega \mid x(1) = \gamma_1, \dots, x(m) = \gamma_m\}$. Let $\gamma = g_1 \cdots g_k \gamma_m^{-1} \cdots \gamma_1^{-1}$. Then we have $\gamma U^c \subset O \subset V$. Since $C(\Omega) \rtimes_r \Gamma$ is simple, it follows from [3] that the action of Γ is topological free. Therefore it follows from Theorem 2.4 that $C(\Omega) \rtimes_r \Gamma$, namely $\mathcal{O}_\Gamma$, is purely infinite. □

Remark. We gave a sufficient condition for $\mathcal{O}_\Gamma$ to be simple. However, we can completely determine the ideal structure of $\mathcal{O}_\Gamma$ with further effort. Indeed, we will obtain a matrix A_Γ to compute K-groups of $\mathcal{O}_\Gamma$ in the next section. The same argument as in [7] also works for the ideal structure of $\mathcal{O}_\Gamma$. For Cuntz-Krieger algebras, we need to assume that corresponding matrices have the condition (II) of [7] to apply the uniqueness theorem. Since we have another uniqueness theorem for our algebras, we can always apply the ideal structure theorem.

Let $\Sigma = I \times \{1, \dots, r\}$ be a finite set, where r is the number of all irreducible unitary representations of H. For $x, y \in \Sigma$, we define $x \geq y$ if there exists a sequence $x_1, \dots, x_m$ of elements in Σ such that $x_1 = x, x_m = y$ and $A_\Gamma(x_a, x_{a+1}) \neq 0 (a = 1, \dots, m-1)$. We call x and y equivalent if $x \geq y \geq x$ and write $\mathit{\Gamma}_{A_\Gamma}$ for the partially ordered set of equivalence classes of elements x in Σ for which $x \geq x$. A subset K of $\mathit{\Gamma}_{A_\Gamma}$ is called hereditary if $\gamma_1 \geq \gamma_2$ and $\gamma_1 \in K$ implies $\gamma_2 \in K$. Let

$$\Sigma(K) = \left\{ x \in \Sigma \mid x_1 \geq x \geq x_2 \quad \text{for some} \quad x_1, x_2 \in \bigcup_{\gamma \in K} \gamma \right\}.$$

We denote by I_K the closed ideal of $\mathcal{O}_\Gamma$ generated by projections $P(i, k)$, which is defined in the next section, for all $(i, k) \in \Sigma(K)$.

Theorem 6.5 ([7, Theorem 2.5]). *The map $K \mapsto I_K$ is an inclusion preserving bijection of the set of hereditary subsets of $\mathit{\Gamma}_{A_\Gamma}$ onto the set of closed ideals of $\mathcal{O}_\Gamma$.*

§7. K-theory for $\mathcal{O}_\Gamma$

In this section we give explicit formulae of the K-groups of $\mathcal{O}_\Gamma$. We have described $\mathcal{O}_\Gamma$ as the crossed product $\mathcal{O}_\Gamma^{\mathbb{T}} \rtimes \mathbb{N}$ in Section 6. So to apply the

Pimsner-Voiculescu exact sequence [24], we need to compute the K-groups of the AF-algebra $\mathcal{O}_\Gamma^{\mathbb{T}}$. We assume that each G_i is finite for simplicity throughout this section. We can also compute the K-groups for general cases by essentially the same arguments. Recall that the fixed-point algebra is described as follows:

$$\mathcal{O}_\Gamma^{\mathbb{T}} = \overline{\bigcup_{n \geqq 0} \mathcal{F}_n},$$
$$\mathcal{F}_n = \oplus_{i \in I} \mathcal{F}_n^i.$$

For each n, we consider a direct summand of $\mathcal{F}_n$, which is

$$\mathcal{F}_n^i = C^*(S_\mu P_i U_h P_i S_\nu^* \mid h \in H, |\mu| = |\nu| = n),$$

and the embedding $\mathcal{F}_n^i \hookrightarrow \mathcal{F}_{n+1}$ is given by

$$\begin{aligned} S_\mu P_i U_h P_i S_\nu^* &= \sum_{g \in \Omega_i \setminus \{e\}} S_\mu U_h (S_g Q_g S_g^*) S_\nu^* \\ &= \sum_g \sum_{i' \neq i} S_\mu S_{hg} P_{i'} S_{\nu g}^*. \end{aligned}$$

Let $\{\chi_1, \dots, \chi_r\}$ be the set of characters corresponding with all irreducible unitary representations of the finite group H with degrees $n_1, \dots, n_r$. Then we have the identification $C^*(H) \simeq M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$. We can write a unit p_k of the k-th component $M_{n_k}(\mathbb{C})$ of $C^*(H)$ as follows:

$$p_k = \frac{n_k}{|H|} \sum_{h \in H} \overline{\chi_k(h)} U_h.$$

Suppose that for $i \neq j$,

$$\mathcal{F}_n^i \simeq M_{N(n,i)}(\mathbb{C}) \otimes C^*(H),$$

$$\mathcal{F}_{n+1}^j \simeq M_{N(n+1,j)}(\mathbb{C}) \otimes C^*(H).$$

Now we compute each embedding of $\mathcal{F}_n^i \hookrightarrow \mathcal{F}_{n+1}^j$,

$$M_{N(n,i)}(\mathbb{C}) \otimes M_{n_i}(\mathbb{C}) \hookrightarrow M_{N(n+1,j)}(\mathbb{C}) \otimes M_{n_j}(\mathbb{C})$$

at the K-theory level. $P(i,k)$ denotes $P_i p_k P_i$. Let P be the projection $e \otimes 1$ in $M_{N(n,i)}(\mathbb{C}) \otimes M_{n_k}(\mathbb{C})$ given by

$$P = S_\mu P(i,k) S_\mu^* \qquad \text{for some } \ \mu \in \Delta_n,$$

where e is a minimal projection in the matrix algebras, and Q be the unit of $M_{N(n+1,j)}(\mathbb{C}) \otimes M_{n_l}(\mathbb{C})$ given by

$$Q = \sum_{\nu \in \Delta_{n+1}} S_\nu P(j,l) S_\nu^*.$$

At the K-theory level, we have $[P] = n_k[e]$. Hence it suffices to compute $\mathrm{tr}(PQ)/n_k$, where tr is the canonical trace in the matrix algebras.

$$\begin{aligned}
&\frac{\mathrm{tr}(PQ)}{n_k}\\
&= \mathrm{tr}\left(\frac{1}{n_k}(S_\mu P(i,k) S_\mu^*)\left(\sum_{\nu \in \Delta_{n+1}} S_\nu P(j,l) S_\nu^*\right)\right)\\
&= \mathrm{tr}\left(\frac{1}{|H|}\left(\sum_{h \in H} \overline{\chi_k(h)} S_\mu U_h P_i S_\mu^*\right)\left(\sum_{\nu \in \Delta_{n+1}} S_\nu P(j,l) S_\nu^*\right)\right)\\
&= \frac{1}{|H|}\mathrm{tr}\left(\sum_{h \in H} \overline{\chi_k(h)}\left(\sum_{g \in \Omega_i \setminus \{e\}} \sum_{i' \neq i} S_\mu S_{hg} P_{i'} S_{\mu g}^*\right)\left(\sum_{\nu \in \Delta_{n+1}} S_\nu P(j,l) S_\nu^*\right)\right)\\
&= \frac{1}{|H|}\mathrm{tr}\left(\sum_{h \in H} \overline{\chi_k(h)}\left(\sum_{g \in \Omega_i \setminus \{e\}} S_\mu S_{hg} P(j,l) S_{\mu g}^*\right)\right)\\
&= \frac{1}{|H|}\sum_{g \in \Omega_i \setminus \{e\}} \sum_{h \in H(g)} \overline{\chi_k(h)} \mathrm{tr}\left(S_{\mu g} U_{g^{-1}hg} P(j,l) S_{\mu g}^*\right)\\
&= \frac{1}{|H|}\sum_{g \in \Omega_i \setminus \{e\}} \sum_{h \in H(g)} \overline{\chi_k(h)} \chi_l(g^{-1}hg),
\end{aligned}$$

where $H(g)$ is the stabilizer of gH by the left multiplication of H.

Now fix $x \in X_i \setminus \{e\}$. Let $\{g \in \Omega_i \mid HgH = HxH\} = \{g_0 = x, g_1, \ldots, g_{m-1}\}$. Then there exists $h_1, h_1', \ldots, h_{m-1}, h_{m-1}' \in H$ such that $h_1 x = g_1 h_1', \ldots, h_{m-1} x = g_{m-1} h_{m-1}'$. Note that $h_s H(x) h_s^{-1} = H(g_s)$ for $s = 1, \ldots, m-1$. Since χ_k, χ_l are class functions, we have

$$\begin{aligned}
\frac{\mathrm{tr}(PQ)}{n_k} &= \frac{1}{|H|}\sum_{x \in X_i}\left(\sum_{s=1}^{m-1} \sum_{h \in H(x)} \overline{\chi_k(h_s h h_s^{-1})} \chi_l(h_s' x^{-1} h_s^{-1} \cdot h_s h h_s^{-1} \cdot h_s x {h'}_s^{-1})\right)\\
&= \frac{1}{|H|}\sum_{x \in X_i}\left(\sum_{s=1}^{m-1} \sum_{h \in H(x)} \overline{\chi_k(h_s h h_s^{-1})} \chi_l(h_s' x^{-1} h x {h'}_s^{-1})\right)
\end{aligned}$$

$$= \frac{1}{|H|} \sum_{x \in X_i} \left(\sum_{s=1}^{m-1} \sum_{h \in H(x)} \overline{\chi_k(h)} \chi_l(x^{-1}hx) \right)$$
$$= \frac{1}{|H|} \sum_{x \in X_i} \left(\sum_{s=1}^{m-1} \sum_{h \in H(x)} \overline{\chi_k(h)} \chi_l^x(h) \right)$$
$$= \sum_{x \in X_i} \left(\frac{|H(x)|}{|H|} \sum_{s=1}^{m-1} \langle \chi_k, \chi_l^x \rangle_{H(x)} \right)$$
$$= \sum_{x \in X_i} \langle \chi_k, \chi_l^x \rangle_{H(x)},$$

where

$$\chi_l^x(h) = \chi_l\left(x^{-1}hx\right)$$
$$\langle \chi_k, \chi_l^x \rangle_{H(x)} = \frac{1}{|H(x)|} \sum_{h \in H(x)} \overline{\chi_k(h)} \chi_l^x(h).$$

Let $A_\Gamma((j,l),(i,k)) = \sum_{x \in X_i \setminus \{e\}} \langle \chi_k, \chi_l^x \rangle_{H(x)}$ for $i \neq j$ and $A_\Gamma((i,k),(i,l)) = 0$ for $1 \leq k,l \leq r$. Then we describe the embedding $\mathcal{F}_n^i \hookrightarrow \mathcal{F}_{n+1}^j$ at the K-theory level by the matrix $[A_\Gamma((i,k),(j,l))]_{1 \leq k,l \leq r}$. Let $A_\Gamma = [A_\Gamma((i,k),(j,l))]$. We have the following lemma.

Lemma 7.1.
$$K_0\left(\mathcal{O}_\Gamma^{\mathbb{T}}\right) = \varinjlim \left(\mathbb{Z}^N \xrightarrow{A_\Gamma} \mathbb{Z}^N\right)$$
$$K_1\left(\mathcal{O}_\Gamma^{\mathbb{T}}\right) = 0$$

where $N = |I|r$.

We can compute the K-groups of $\mathcal{O}_\Gamma$ by using the Pimsner-Voiculescu sequence with essentially the same argument as in the Cuntz-Krieger algebra case (see [7]).

Theorem 7.2.

$$K_0(\mathcal{O}_\Gamma) = \mathbb{Z}^N/(1 - A_\Gamma)\mathbb{Z}^N.$$
$$K_1(\mathcal{O}_\Gamma) = \mathrm{Ker}\{1 - A_\Gamma : \mathbb{Z}^N \to \mathbb{Z}^N\} \quad \textit{on} \quad \mathbb{Z}^N.$$

Proof. It suffices to compute the K-groups of $\overline{\mathcal{O}}_\gamma = \overline{\mathcal{O}}_\Gamma^{\mathbb{T}} \rtimes_{\bar{\rho}} \mathbb{Z}$. We represent the inductive limit
$$\varinjlim \left(\mathbb{Z}^N \xrightarrow{A_\Gamma} \mathbb{Z}^N\right)$$

as the set of equivalence classes of $x = (x_1, x_2, \dots)$ such that $x_k \in \mathbb{Z}^N$ with $x_{k+1} = A(x_k)$. If S is a partial isometry in $\mathcal{O}_\Gamma$ such that $\alpha_z(S) = zS$ and P is a projection in $\mathcal{O}_\Gamma^{\mathbb{T}}$ with $P \leq S^*S$, then $[\rho(P)] = [VPV^*] = [(VS^*S)P(VS^*S)^*]$ $= [SPS^*]$ in $K_0(\mathcal{O}_\Gamma^{\mathbb{T}})$. Recall that

$$p_k = \frac{n_k}{|H|} \sum_{h \in H} \overline{\chi_k(h)} U_h.$$

Let $P = S_\mu P(i,k) S_\mu^*$ for some $\mu \in \Delta_n$. If $\mu = \mu_1 \cdots \mu_n$, then

$$\begin{aligned}
[\bar{\rho}^{-1}(P)] &= [S_{\mu_1}^* P S_{\mu_1}] \\
&= \left[\frac{n_k}{|H|} \sum_{h \in H} \overline{\chi_k(h)} \left(S_{\mu_2} \cdots S_{\mu_n} P_i U_h P_i S_{\mu_n} \cdots S_{\mu_2}^* \right) \right] \\
&= \cdots \\
&= \sum_{j \neq i} \sum_{l=1}^{r} n_i \left(\sum_{x \in X_i \setminus \{e\}} \langle \chi_k, \chi_l^x \rangle [e_l] \right),
\end{aligned}$$

where the e_l are non-zero minimal projections for $1 \leq l \leq r$. Thus it follows that $\bar{\rho}_*^{-1}$ is the shift on $K_0(\overline{\mathcal{O}}_\Gamma^{\mathbb{T}})$. We denote the shift by σ. If $x = (x_1, x_2, x_3, \dots) \in K_0(\overline{\mathcal{O}}_\Gamma^{\mathbb{T}})$, then $\sigma(x) = (x_2, x_3, \dots)$. By the Pimsner-Voiculescu exact sequence, there exists an exact sequence

$$0 \to K_1(\overline{\mathcal{O}}_\Gamma) \to K_0(\overline{\mathcal{O}}_\Gamma^{\mathbb{T}}) \to K_0(\overline{\mathcal{O}}_\Gamma^{\mathbb{T}}) \to K_0(\overline{\mathcal{O}}_\Gamma) \to 0.$$

It therefore follows that $K_0(\overline{\mathcal{O}}_\Gamma) = K_0(\overline{\mathcal{O}}_\Gamma^{\mathbb{T}})/(1-\sigma)K_0(\overline{\mathcal{O}}_\Gamma^{\mathbb{T}})$ and $K_1(\overline{\mathcal{O}}_\Gamma) = \ker(1-\sigma)$ on $K_0(\overline{\mathcal{O}}_\Gamma^{\mathbb{T}})$. □

Finally we consider some simple examples. First let $\Gamma = SL(2,\mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$. Let χ_1 be the unit character of $\mathbb{Z}_2$ and let χ_2 be the character such that $\chi_2(a) = -1$ where a is a generator of $\mathbb{Z}_2$. These are one-dimensional and exhaust all the irreducible characters. Then we have the corresponding matrix

$$A_\Gamma = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}.$$

Hence the corresponding K-groups are $K_0(\mathcal{O}_\Gamma) = 0$ and $K_1(\mathcal{O}_\Gamma) = 0$. In fact, $\mathcal{O}_{\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6} \simeq \mathcal{O}_{\mathbb{Z}_2 * \mathbb{Z}_3} \oplus \mathcal{O}_{\mathbb{Z}_2 * \mathbb{Z}_3} \simeq \mathcal{O}_2 \oplus \mathcal{O}_2$.

Next let $\Gamma = \mathfrak{S}_4 *_{\mathfrak{S}_3} \mathfrak{S}_4$, $\tau = (1\,2)$ and $\sigma = (1\,2\,3)$. Note that $\mathfrak{S}_3 = \langle 1, \tau, \sigma \rangle$. $\mathfrak{S}_3$ has three irreducible characters:

	1	τ	σ
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Moreover, $\mathfrak{S}_3 \backslash \mathfrak{S}_4 / \mathfrak{S}_3$ has only two points; say $\mathfrak{S}_3$ and $\mathfrak{S}_3 x \mathfrak{S}_3$ with $x = (1\,2)(3\,4)$. Then we obtain the corresponding matrix

$$A_\Gamma = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

Hence this gives $K_0(\mathcal{O}_\Gamma) = \mathbb{Z} \oplus \mathbb{Z}_4$ and $K_1(\mathcal{O}_\Gamma) = \mathbb{Z}$. In this case, Γ satisfies the condition of Theorem 6.3. So $\mathcal{O}_\Gamma$ is a simple, nuclear, purely infinite C^*-algebra.

§8. KMS States on $\mathcal{O}_\Gamma$

In this section, we investigate the relationship between KMS states on $\mathcal{O}_\Gamma$ for generalized gauge actions and random walks on Γ. Throughout this section, we assume that all groups G_i are finite though we can carry out the same arguments if $G_i = \mathbb{Z} \times H$ for some $i \in I$. Let $\omega = (\omega_i)_{i \in I} \in \mathbb{R}_+^{|I|}$. By the universality of $\mathcal{O}_\Gamma$, we can define an automorphism α_t^ω for any $t \in \mathbb{R}$ on $\mathcal{O}_\Gamma$ by $\alpha_t^\omega(S_g) = e^{\sqrt{-1}\,\omega_i t} S_g$ for $g \in G_i \setminus H$ and $\alpha_t^\omega(U_h) = U_h$ for $h \in H$. Hence we obtain the $\mathbb{R}$-action α^ω on $\mathcal{O}_\Gamma$. We call it *the generalized gauge action* with respect to ω. We will only consider actions of these types and determine KMS states on $\mathcal{O}_\Gamma$ for these actions.

In [32], Woess showed that our boundary Ω can be identified with the Poisson boundary of random walks satisfying certain conditions. The reader is referred to [33] for a good book of random walks.

Let μ be a probability measure on Γ and consider a random walk governed by μ, i.e. the transition probability from x to y given by

$$p(x, y) = \mu(x^{-1} y).$$

A random walk is said to be *irreducible* if for any $x, y \in \Gamma$, $p^{(n)}(x,y) \neq 0$ for some integer n, where

$$p^{(n)}(x,y) = \sum_{x_1, x_2, \ldots, x_{n-1} \in \Gamma} p(x, x_1)p(x_1, x_2) \cdots p(x_{n-1}, y).$$

A probability measure ν on Ω is said to be *stationary* with respect to μ if $\nu = \mu * \nu$, where $\mu * \nu$ is defined by

$$\int_\Omega f(\omega) d\mu * \nu(\omega) = \int_\Omega \int_{\mathrm{supp}\mu} f(g\omega) d\mu(g) d\nu(\omega), \quad \text{for} \quad f \in C(\Omega, \nu).$$

By [32, Theorem 9.1], if a random walk governed by a probability measure μ on Γ is irreducible, then there exists a unique stationary probability measure ν on Ω with respect to μ. Moreover if μ has finite support, then the Poisson boundary coincides with (Ω, ν).

If ν is a probability measure on the compact space Ω, then we can define a state ϕ_ν by

$$\phi_\nu(X) = \int_\Omega E(X) d\nu \quad \text{for} \quad X \in \mathcal{O}_\Gamma,$$

where E is the canonical conditional expectation of $C(\Omega) \rtimes_r \Gamma$ onto $C(\Omega)$.

One of our purposes in this section is to prove that there exists a random walk governed by a probability measure μ that induces the stationary measure ν on Ω such that the corresponding state ϕ_ν is the unique KMS state for α^ω. Namely,

Theorem 8.1. *Assume that the matrix A_Γ obtained in the preceding section is irreducible. For any $\omega = (\omega_i)_{i \in I} \in \mathbb{R}_+^{|I|}$, there exists a unique probability measure μ with the following properties*:

(i) $\mathrm{supp}(\mu) = \bigcup_{i \in I} G_i \setminus H$.

(ii) $\mu(gh) = \mu(g)$ *for any* $g \in \bigcup_{i \in I} G_i \setminus H$ *and* $h \in H$.

(iii) *The corresponding unique stationary measure ν on Ω induces the unique KMS state ϕ_ν for α^ω and the corresponding inverse temperature β is also unique.*

We need the hypothesis of the irreducibility of the matrix A_Γ for the uniqueness of the KMS state. Though it is, in general, difficult to check the irreducibility of A_Γ, by Theorem 6.5, the condition of simplicity of $\mathcal{O}_\Gamma$ in Corollary 6.4 is also a sufficient condition for irreducibility of A_Γ. To obtain the theorem, we first present two lemmas.

Lemma 8.2. *Assume that ν is a probability measure on Ω. Then the corresponding state ϕ_ν is the KMS state for α^ω if and only if ν satisfies the following conditions*:

$$\nu(\Omega(x_1 \cdots x_m)) = \frac{e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_{m-1}}}}{[G_{i_m} : H] - 1 + e^{\beta\omega_{i_m}}},$$

for $x_k \in \Omega_{i_k}$ with $i_1 \neq \cdots \neq i_m$, where $\Omega(x_1 \cdots x_m)$ is the cylinder subset of Ω defined by

$$\Omega(x_1 \cdots x_m) = \{(x(n))_{n\geq 1} \in \Omega \mid x(1) = x_1, \ldots, x(m) = x_m\}.$$

Proof. ϕ_ν is the KMS state for α^ω if and only if

$$\phi_\nu(S_\xi P_i U_h S_\eta^* \cdot S_\sigma P_j U_k S_\tau^*) = \phi(S_\sigma P_j U_k S_\tau^* \cdot \alpha^\omega_{\sqrt{-1}\beta}(S_\xi P_i U_h S_\eta^*)),$$

for any $\xi, \eta, \sigma, \tau \in \Delta, h, k \in H$ and $i, j \in I$.

We may assume that $|\xi| + |\sigma| = |\eta| + |\tau|$ and $|\eta| \geq |\sigma|$. Set $|\xi| = p, |\eta| = q, |\sigma| = s, |\tau| = t$ and let $\xi = \xi_1 \cdots \xi_p$, $\eta = \eta_1 \cdots \eta_q$ with $\xi_k \in \Omega_{i_k} \setminus \{e\}, \eta_l \in \Omega_{j_l} \setminus \{e\}$ and $i_1 \neq \cdots \neq i_p, j_1 \neq \cdots \neq j_q$. Then

$$\begin{aligned}\phi_\nu(S_\xi P_i U_h S_\eta^* \cdot S_\sigma P_j U_k S_\tau^*) &= \delta_{\eta_1\cdots\eta_s,\sigma}\delta_{\eta_{s+1},j}\phi_\nu(S_\xi P_i U_h S^*_{\eta_{s+1}\cdots\eta_q} U_k S_\tau^*) \\ &= \delta_{\eta_1\cdots\eta_s,\sigma}\delta_{\eta_{s+1},j}\phi_\nu(S_{\xi h} P_i S_{\tau k^{-1}\eta_{s+1}\cdots\eta_q}) \\ &= \delta_{\eta_1\cdots\eta_s,\sigma}\delta_{\eta_{s+1},j}\delta_{\xi h,\tau k^{-1}\eta_{s+1}\cdots\eta_q} \sum_{x\in\Omega_i\setminus\{e\}} \nu(\Omega(\xi x)),\end{aligned}$$

and

$$\begin{aligned}&\phi_\nu(S_\sigma P_j U_k S_\tau^* \cdot \alpha^\omega_{\sqrt{-1}\beta}(S_\xi P_i U_h S_\eta^*)) \\ &= e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_p}} e^{\beta\omega_{j_1}} \cdots e^{\beta\omega_{j_q}} \phi_\nu(S_\sigma P_j U_k S_\tau^* \cdot S_\xi P_i U_h S_\eta^*) \\ &= e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_p}} e^{\beta\omega_{j_1}} \cdots e^{\beta\omega_{j_q}} \delta_{\tau,\xi_1\cdots\xi_t}\delta_{\xi_{t+1},j}\phi_\nu(S_{\sigma k\xi_{t+1}\cdots\xi_p h} P_i S_\eta^*) \\ &= e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_p}} e^{\beta\omega_{j_1}} \cdots e^{\beta\omega_{j_q}} \delta_{\tau,\xi_1\cdots\xi_t}\delta_{\xi_{t+1},j}\delta_{\sigma k\xi_{t+1}\cdots\xi_p h,\eta} \sum_{x\in\Omega_i\setminus\{e\}} \nu(\Omega(\eta x)),\end{aligned}$$

where $\delta_{g,i} = 1$ only if $g \in G_i \setminus H$. Therefore the corresponding state ϕ_ν is the KMS state for α^ω if and only if ν satisfies the following conditions:

$$\nu(\Omega(\xi_1 \ldots \xi_p x)) = e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_p}} \nu(\Omega(x)),$$

for $x \in \Omega_i \setminus \{e\}$ with $i \neq i_p$.

Now we assume that ϕ_ν is the KMS state for α^ω. Then for $i \in I$,

$$\begin{aligned}
\nu(Y_i) &= \phi_\nu(P_i) \\
&= \sum_{g \in \Omega_i \setminus \{e\}} \phi_\nu(S_g S_g^*) \\
&= \sum_{g \in \Omega_i \setminus \{e\}} \phi_\nu(S_g^* \alpha^\omega_{\sqrt{-1}\beta}(S_g)) \\
&= e^{-\beta\omega_i} \sum_{g \in \Omega_i \setminus \{e\}} \phi_\nu(Q_g) \\
&= e^{-\beta\omega_i} \sum_{g \in \Omega_i \setminus \{e\}} \phi_\nu(1 - P_i) \\
&= e^{-\beta\omega_i}([G_i : H] - 1)(1 - \nu(Y_i)).
\end{aligned}$$

Hence,

$$\nu(Y_i) = \frac{[G_i : H] - 1}{[G_i : H] - 1 + e^{\beta\omega_i}}.$$

Moreover,

$$\begin{aligned}
\nu(\Omega(x_1 \dots x_m)) &= \phi_\nu(S_{x_1} \cdots S_{x_m} S_{x_m}^* \cdots S_{x_1}^*) \\
&= \phi_\nu(S_{x_m}^* \cdots S_{x_1}^* \alpha^\omega_{\sqrt{-1}\beta}(S_{x_1} \cdots S_{x_m})) \\
&= e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_m}} \phi_\nu(Q_{x_m}) \\
&= e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_m}} (1 - \nu(\Omega(Y_{i_m}))) \\
&= \frac{e^{-\beta\omega_{i_1}} \cdots e^{-\beta\omega_{i_{m-1}}}}{[G_{i_m} : H] - 1 + e^{\beta\omega_{i_m}}}.
\end{aligned}$$

Conversely, suppose that a probability measure ν satisfies the condition of this lemma. By the first part of this proof, ϕ_ν is the KMS state for α^ω. □

Lemma 8.3. *Assume that ν is the unique stationary measure on Ω with respect to a random walk on Γ, governed by a probability measure μ with the conditions* (i), (ii) *in Theorem* 8.1. *Then ϕ_ν is a β-KMS state for α^ω if and only if μ satisfies the following conditions*:

$$\mu(g) = \frac{\prod_{j \neq i} C_j}{\sum_{k \in I} (g_k \prod_{l \neq k} C_l)} \qquad \text{for} \quad g \in G_i \setminus H \quad \text{and} \quad i \in I,$$

where $g_i = |G_i \setminus H|$ and $C_i = (1 - e^{-\beta\omega_i})g_i - (1 - e^{\beta\omega_i})|H|$ for $i \in I$.

Proof. Assume that ϕ_ν is a β-KMS state for α^ω. For any $f \in C(\Omega)$,

$$\begin{aligned}\iint f(\omega)d\nu(\omega) &= \iint f(\omega)d\mu * \nu(\omega) \\ &= \iint f(g\omega)d\nu(\omega)d\mu(g) \\ &= \iint (\lambda_g^* f \lambda_g)(\omega)d\nu(\omega)d\mu(g) \\ &= \sum_{g\in \mathrm{supp}(\mu)} \mu(g)\phi_\nu(\lambda_g^* f\lambda_g) \\ &= \sum_{g\in \mathrm{supp}(\mu)} \mu(g)\phi_\nu(f\lambda_g \alpha^\omega_{\sqrt{-1}\beta}(\lambda_g^*)),\end{aligned}$$

where $\mathcal{O}_\Gamma \simeq C(\Omega) \rtimes_r \Gamma = C^*(f, \lambda_\gamma \mid f \in C(\Omega), \gamma \in \Gamma)$.

Put $f = \chi_{\Omega(x)} = P_x$ for $i \in I$ and $x \in \Omega_i \setminus \{e\}$. Since $\lambda_g = S_g + \sum_{g' \in \Omega_{i'} \setminus H \cup g^{-1}H} S_{gg'}S_{g'}^* + S_{g^{-1}}^*$ for $g \in G_{i'} \setminus H$ and $i' \in I$, we have

$$1 = \sum_{gH=xH} \mu(g)e^{\beta\omega_i} + \sum_{g\in G_i\setminus H, gH\neq xH} \mu(g) + \sum_{g\in G_j\setminus H, j\neq i} \mu(g)e^{-\beta\omega_j}$$

for any $i \in I$ and $x \in \Omega_i \setminus \{e\}$. Let $x, y \in \Omega_i \setminus \{e\}$ with $xH \neq yH$. Then

$$1 = \sum_{gH=xH} \mu(g)e^{\beta\omega_i} + \sum_{gH\neq xH} \mu(g) + \sum_{g\in G_j\setminus H, j\neq i} \mu(g)e^{-\beta\omega_j},$$

$$1 = \sum_{gH=yH} \mu(g)e^{\beta\omega_i} + \sum_{gH\neq yH} \mu(g) + \sum_{g\in G_j\setminus H, j\neq i} \mu(g)e^{-\beta\omega_j}.$$

By the above equations, we have $\mu(x) = \mu(y)$, and then it follows from hypothesis (ii) in Theorem 8.1 that $\mu(g) = \mu_i$ for any $g \in G_i \setminus H$. Therefore we have

$$1 = |H|e^{\beta\omega_i}\mu_i + (g_i - |H|)\mu_i + \sum_{j\neq i} g_j e^{-\beta\omega_j}\mu_j,$$

for any $i \in I$, where $g_i = |G_i \setminus H|$. Thus by considering the above equations for i and $j \in I$,

$$|H|e^{\beta\omega_i}\mu_i - |H|e^{\beta\omega_j}\mu_j + (g_i - |H|)\mu_i - (g_j - |H|)\mu_j + g_j e^{-\beta\omega_j}\mu_j - g_i e^{-\beta\omega_i}\mu_i = 0.$$

Hence we obtain the equation,

$$(|H|e^{\beta\omega_i} + g_i - |H| - g_i e^{-\beta\omega_i})\mu_i = (|H|e^{\beta\omega_j} + g_j - |H| - g_j e^{-\beta\omega_j})\mu_j.$$

Since $\mu(\bigcup_{i\in I} G_i \setminus H) = 1$, we have

$$g_i\mu_i + \sum_{j\neq i} g_j \frac{(1-e^{-\beta\omega_i})g_i - (1-e^{-\beta\omega_i})|H|}{(1-e^{-\beta\omega_j})g_j - (1-e^{-\beta\omega_j})|H|}\mu_i = 1.$$

We put $C_i = (1-e^{-\beta\omega_i})g_i - (1-e^{-\beta\omega_i})|H|$ and then

$$\left(g_i + C_i \sum_{j\neq i} \frac{g_j}{C_j}\right)\mu_i = 1.$$

Therefore

$$\begin{aligned}\mu_i &= \frac{1}{g_i + C_i\sum_{j\neq i} g_j/C_j} \\ &= \frac{\prod_{j\neq i} C_j}{g_i\prod_{j\neq i} C_j + \sum_{j\neq i}(g_jC_i\prod_{k\neq i,j} C_k)} \\ &= \frac{\prod_{j\neq i} C_j}{\sum_{k\in I} g_k \prod_{l\neq k} C_l}.\end{aligned}$$

On the other hand, let ν be the probability measure on Ω satisfying the condition in Lemma 8.2. Then the corresponding state ϕ_ν is the KMS state. It is enough to check that $\mu * \nu = \nu$ by [32]. Since

$$\nu(\Omega(x_1\cdots x_n)) = e^{-\beta\omega_{i_1}}\cdots e^{-\beta\omega_{i_{n-1}}}\nu(\Omega(x_n)),$$

for $x_k \in \Omega_{i_k}\setminus\{e\}$ with $i_1 \neq \cdots \neq i_n$, we have

$$\begin{aligned}&\mu * \nu(\Omega(x_1\cdots x_n)) \\ &\quad= \iint \chi_{\Omega(x_1\cdots x_n)}(\omega)d\mu*\nu(\omega) \\ &\quad= \sum_{g\in\mathrm{supp}\mu} \mu(g)\int(\lambda_g^*\chi_{\Omega(x_1\cdots x_n)}\lambda_g)(\omega)d\nu(\omega) \\ &\quad= \sum_{g\in G_{i_1}\setminus H, x_1H=gH} \mu_{i_1}\phi_\nu(S_{x_2}\cdots S_{x_n}S_{x_n}^*\cdots S_{x_2}^*) \\ &\qquad+ \sum_{g\in G_{i_1}\setminus H, x_1H\neq gH} \mu_{i_1}\phi_\nu(S_{g^{-1}x_1}S_{x_2}\cdots S_{x_n}S_{x_n}^*\cdots S_{x_2}^*S_{g^{-1}x_1}^*) \\ &\qquad+ \sum_{g\in G_i\setminus H, i\neq i_1} \mu_i\phi_\nu(S_{g^{-1}}S_{x_1}S_{x_2}\cdots S_{x_n}S_{x_n}^*\cdots S_{x_2}^*S_{x_1}^*S_{g^1}^*) \\ &\quad= \left(|H|e^{\beta\omega_{i_1}}\mu_{i_1} + (g_{i_1}-|H|)\mu_{i_1} + \sum_{i\neq i_1} g_ie^{-\beta\omega_i}\mu_i\right)\nu(\Omega(x_1\cdots x_n))\end{aligned}$$

$$= \nu(\Omega(x_1 \dots x_n)).$$

□

To prove the uniqueness of KMS states of $\mathcal{O}_\Gamma$, we need the irreducibility of the matrix A_Γ. (See [13] for KMS states on Cuntz-Krieger algebras.) Set an irreducible matrix $B = [B((i,k),(j,l))] = [e^{-\beta\omega_i} A_\Gamma^t((i,k),(j,l))]$. Let K_β be the set of all β-KMS states for the action α^ω. We put

$$L_\beta = \left\{ y = [y(i,k)] \in \mathbb{R}^N \mid By = y, \quad y(i,k) \geq 0, \quad \sum_{i\in I}\sum_{k=1}^{r} n_k y(i,k) = 1 \right\}.$$

We now have the necessary ingredients for the proof of Theorem 8.1.

Proof of Theorem 8.1. We first prove the uniqueness of the corresponding inverse temperature. Let ϕ be a β-KMS state for α^ω. For $i \in I$,

$$\begin{aligned} \phi(P_i) &= \sum_{g\in\Omega_i\setminus\{e\}} \phi(S_g S_g^*) \\ &= \sum_{g\in\Omega_i\setminus\{e\}} \phi(S_g^* \alpha^\omega_{\sqrt{-1}\beta}(S_g)) \\ &= e^{-\beta\omega_i} \sum_{g\in\Omega_i\setminus\{e\}} \phi(Q_g) \\ &= e^{-\beta\omega_i}([G_i : H] - 1)(1 - \phi(P_i)). \end{aligned}$$

Thus $\phi(P_i) = \lambda_i(\beta)/(1+\lambda_i(\beta))$, where $\lambda_i(\beta) = e^{-\beta\omega_i}([G_i : H] - 1)$. Since $\sum_{i\in I} P_i = 1$,

$$|I| - 1 = \sum_{i\in I} \frac{1}{1+\lambda_i(\beta)}.$$

The function $\sum_{i\in I} 1/(1+\lambda_i(\beta))$ is a monotone increasing continuous function such that

$$\sum_{i\in I} \frac{1}{1+\lambda_i(\beta)} = \begin{cases} \sum_{i\in I} 1/[G_i : H] & \text{if } \beta = 0, \\ |I| & \text{if } \beta \to \infty. \end{cases}$$

Since $\sum_{i\in I} 1/[G_i : H] \leq |I|/2 \leq |I| - 1$, there exists a unique β satisfying

$$|I| - 1 = \sum_{i\in I} \frac{1}{([G_i : H] - 1)e^{-\beta\omega_i} + 1}.$$

Therefore we obtain the uniqueness of the inverse temperature β.

We will next show the uniqueness of the KMS state ϕ_ν. We claim that K_β is in one-to-one correspondence with $Ł_\beta$. In fact, we define a map f from K_β to $Ł_\beta$ by

$$f(\phi) = [\phi(P(i,k))/n_k].$$

Indeed,

$$\begin{aligned}
e^{\beta\omega_i}\phi(P(i,k)) &= \sum_{g\in\Omega_i\setminus\{e\}} \phi(p_k S_g \alpha^{\omega}_{\sqrt{-1}\beta}(S_g^*)) \\
&= \sum_{g\in\Omega_i\setminus\{e\}} \phi(S_g^* p_k S_g) \\
&= \frac{n_k}{|H|} \sum_{g\in\Omega_i\setminus\{e\}} \sum_{h\in H} \overline{\chi_k(h)}\phi(S_g^* U_h S_g) \\
&= \frac{n_k}{|H|} \sum_{g\in\Omega_i\setminus\{e\}} \sum_{h\in H(g)} \overline{\chi_k(h)}\phi(Q_g U_{g^{-1}hg}) \\
&= \frac{n_k}{|H|} \sum_{g\in\Omega_i\setminus\{e\}} \sum_{h\in H(g)} \overline{\chi_k(h)} \sum_{j\neq i} \phi(P_j U_{g^{-1}hg} P_j) \\
&= \frac{n_k}{|H|} \sum_{g\in\Omega_i\setminus\{e\}} \sum_{h\in H(g)} \overline{\chi_k(h)} \sum_{j\neq i} \sum_{l=1}^{r} \phi(P(j,l) U_{g^{-1}hg} P(j,l)).
\end{aligned}$$

Since ϕ is a trace on $C^*(P(j,l)U_h P(j,l) \mid h\in H) \simeq M_{n_l}(\mathbb{C})$ and $M_{n_l}(\mathbb{C})$ has a unique tracial state, we have

$$\phi(P(j,l)U_{g^{-1}hg}P(j,l)) = \chi_l(g^{-1}hg)\frac{\phi(P(j,l))}{n_l}.$$

Therefore, by the same arguments as in the previous section, we obtain

$$\begin{aligned}
e^{\beta\omega_i}\phi(P(i,k)) &= \frac{n_k}{|H|} \sum_{g\in\Omega_i\setminus\{e\}} \sum_{h\in H(g)} \overline{\chi_k(h)} \sum_{j\neq i} \sum_{l=1}^{r} \phi(P(j,l) U_{g^{-1}hg} P(j,l)) \\
&= n_k \sum_{x\in X_i\setminus\{e\}} \sum_{j\neq i} \sum_{l=1}^{r} \langle \chi_k, \chi_l^x \rangle_{H(x)} \phi(P(j,l))/n_l \\
&= n_k \sum_{(j,l)} A_\Gamma((j,l),(i,k))\phi(P(j,l))/n_l.
\end{aligned}$$

Hence this is well-defined.

Suppose that ν is the probability measure in Lemma 8.2 and ϕ_ν is the induced β-KMS state for α^ω. Set a vector $y = [y(i,k) = \phi_\nu(P(i,k))/n_k]$. Since y is strictly positive and B is irreducible, 1 is the eigenvalue which dominates the absolute value of all eigenvalue of B by the Perron-Frobenius theorem. It also follows from the Perron-Frobenius theorem that L_β has only one element. Hence f is surjective.

Let $\phi \in K_\beta$. For $\xi = \xi_{i_1} \cdots \xi_{i_n}, \eta = \eta_{j_1} \cdots \eta_{j_n}$ with $i_1 \neq \cdots \neq i_n, j_1 \neq \cdots \neq j_n$, $h \in H$ and $i \in I$,

$$\begin{aligned} e^{\beta\omega_{j_1}} \cdots e^{\beta\omega_{j_n}} \phi(S_\xi U_h P_i S_\eta^*) &= \phi(S_\xi U_h P_i \alpha^\omega_{\sqrt{-1}\beta}(S_\eta^*)) \\ &= \phi(S_\eta^* S_\xi U_h P_i) \\ &= \delta_{\xi,\eta}\phi(U_h P_i) \\ &= \delta_{\xi,\eta} \sum_{k=1}^{r} \phi(U_h P(i,k)) \\ &= \delta_{\xi,\eta} \sum_{k=1}^{r} \chi_k(h)\phi(P(i,k))/n_k, \end{aligned}$$

because ϕ is a trace on $C^*(U_h P(i,k) \mid h \in H) \simeq M_{n_k}(\mathbb{C})$. If $f(\phi) = f(\psi)$, then the above calculations imply $\phi = \psi$ on $\mathcal{O}_\Gamma^{\mathbb{T}}$. By the KMS condition, $\phi(b) = 0 = \psi(b)$ for $b \notin \mathcal{O}_\Gamma^{\mathbb{T}}$. Thus $\phi = \psi$ and f is injective. Therefore ϕ_ν is the unique β-KMS state for α^ω. □

Remark. Let ν be the corresponding probability measure with the gauge action α. Under the identification $L^\infty(\Omega, \nu) \rtimes_w \Gamma \simeq \pi_\nu(\mathcal{O}_\Gamma)''$, we can determine the type of the factor by essentially the same arguments as in [13]. If H is trivial, then $\mathcal{O}_\Gamma$ is a Cuntz-Krieger algebra for some irreducible matrix with 0-1 entries. In this case, we can always apply the result in [13]. This fact generalizes [25]. If H is not trivial, then by using the condition of simplicity of $\mathcal{O}_\Gamma$ in Corollary 6.4 to check the irreducibility of the matrix A_Γ, we can apply Theorem 8.1. In the special case where $G_i = G$ for all $i \in I$, we can easily determine the type of the factor $\pi_\nu(\mathcal{O}_\Gamma)''$ for the gauge action. The factor $\pi_\nu(\mathcal{O}_\Gamma)''$ is of type III_λ where $\lambda = 1/([G:H]-1)^2$ if $|I| = 2$ and $\lambda = 1/(|I|-1)([G:H]-1)$ if $|I| > 2$. For instance, let $\Gamma = \mathfrak{S}_4 *_{\mathfrak{S}_3} \mathfrak{S}_4$. We have already obtained the matrix A_Γ in Section 7, but we can determine that the factor $L^\infty(\Omega, \nu) \rtimes_w \Gamma$ is of type $\mathrm{III}_{1/9}$ without using A_Γ.

We next discuss the converse. Namely any $\mathbb{R}$-actions that have KMS states induced by a probability measure μ on Γ with some conditions is, in fact, a generalized gauge action.

Let μ be a given probability measure on Γ with $\mathrm{supp}(\mu) = \bigcup_{i \in I} G_i \setminus H$. By [32], there exists an unique probability measure ν on Ω such that $\mu * \nu = \nu$. Let (π_ν, H_ν, x_ν) be the GNS-representation of $\mathcal{O}_\Gamma$ with respect to the state ϕ_ν.

We also denote a vector state of x_ν by ϕ_ν.

$$\phi_\nu(a) = \langle a x_\nu, x_\nu \rangle \qquad \text{for} \quad a \in \pi_\nu(\mathcal{O}_\Gamma)''.$$

Let σ_t^ν be the modular automorphism group of ϕ_ν.

Theorem 8.4. *Suppose that μ is a probability measure on Γ such that $\mathrm{supp}(\mu) = \bigcup_{i \in I} G_i \setminus H$ and $\mu(g) = \mu(hg)$ for any $g \in \bigcup_{i\in I} G_i \setminus H$, $h \in H$. If ν is the corresponding stationary measure with respect to μ, then there exists $\omega_g \in \mathbb{R}_+$ such that*

$$\sigma_t^\nu(\pi_\nu(S_g)) = e^{\sqrt{-1}\omega_g t}\pi_\nu(S_g) \quad \textit{for} \quad g \in G_i \setminus H, i \in I,$$

and

$$\sigma_t^\nu(\pi_\nu(U_h)) = \pi_\nu(U_h) \quad \textit{for} \quad h \in H.$$

Proof. To prove that $\sigma_t^\nu(\pi_\nu(S_g)) = e^{\sqrt{-1}\omega_g t}\pi_\nu(S_g)$, it suffices to show that there exists $\zeta_g \in \mathbb{R}_+$ such that

$$(*) \qquad \phi_\nu(\pi_\nu(S_g)a) = \zeta_g \phi_\nu(a\pi_\nu(S_g)) \quad \text{for} \quad g \in G_i \setminus H, a \in \pi_\nu(\mathcal{O}_\Gamma)''.$$

In fact, Let Δ_ν be the modular operator and J_ν be the modular conjugate of ϕ_ν.

$$\begin{aligned}
(\text{left hand side of } (*)) &= \langle \pi_\nu(S_g) a x_\nu, x_\nu\rangle \\
&= \langle a x_\nu, \pi_\nu(S_g)^* x_\nu\rangle \\
&= \langle a x_\nu, J_\nu \Delta_\nu^{1/2}\pi_\nu(S_g)x_\nu\rangle \\
&= \langle \Delta_\nu^{1/2}\pi_\nu(S_g)x_\nu, J_\nu a x_\nu\rangle \\
&= \langle \Delta_\nu^{1/2}\pi_\nu(S_g)x_\nu, \Delta_\nu^{1/2} a^* x_\nu\rangle.
\end{aligned}$$

and

$$\begin{aligned}
(\text{right hand side of } (*)) &= \zeta_g \langle a\pi_\nu(S_g)x_\nu, x_\nu\rangle \\
&= \zeta_g \langle \pi_\nu(S_g)x_\nu, a^* x_\nu\rangle.
\end{aligned}$$

Therefore for $a \in \pi_\nu(\mathcal{O}_\Gamma)''$,

$$\langle \Delta_\nu^{1/2}\pi_\nu(S_g)x_\nu, \Delta_\nu^{1/2} a^* x_\nu\rangle = \zeta_g \langle \pi_\nu(S_g)x_\nu, a^* x_\nu\rangle.$$

and hence for $y \in \mathrm{dom}(\Delta_\nu^{1/2})$, we have

$$\langle \Delta_\nu^{1/2}\pi_\nu(S_g)x_\nu, \Delta_\nu^{1/2} y\rangle = \zeta_g \langle \pi_\nu(S_g)x_\nu, y\rangle.$$

Thus $\Delta_\nu^{1/2}\pi_\nu(S_g)x_\nu \in \mathrm{dom}(\Delta_\nu^{1/2})$ and we obtain

$$\Delta_\nu \pi_\nu(S_g)x_\nu = \zeta_g \pi_\nu(S_g)x_\nu.$$

Therefore

$$\Delta_\nu^{\sqrt{-1}t}\pi_\nu(S_g)x_\nu = \zeta_g^{\sqrt{-1}t}\pi_\nu(S_g)x_\nu,$$

and then

$$(\sigma_t^\nu(\pi_\nu(S_g)) - \zeta_g^{\sqrt{-1}t}\pi_\nu(S_g))x_\nu = 0,$$

where σ_t^ν is the modular automorphism group of ϕ_ν. Since x_ν is a separating vector,

$$\sigma_t^\nu(\pi_\nu(S_g)) = \zeta_g^{\sqrt{-1}t}\pi_\nu(S_g).$$

Now we will show that

$$\phi_\nu(\pi_\nu(S_g)a) = \zeta_g \phi_\nu(a\pi_\nu(S_g)) \quad \text{for} \quad g \in G_i \setminus H, a \in \pi_\nu(\mathcal{O}_\Gamma)''.$$

We may assume that $a = f\lambda_{g^{-1}}$ for $f \in C(\Omega)$. Recall that $S_g = \lambda_g \chi_{\Omega\setminus Y_i} \in C(\Omega) \rtimes_r \Gamma$. Since

$$\phi_\nu(\pi_\nu(S_g a)) = \int_{\Omega\setminus Y_i} f(g^{-1}\omega)d\nu(\omega) = \int_{\Omega\setminus Y_i} f(\omega)\frac{dg^{-1}\nu}{d\nu}(\omega)d\nu(\omega),$$

we claim that

$$\frac{dg^{-1}\nu}{d\nu}(\omega) = \zeta_g \quad \text{on} \quad \Omega \setminus Y_i.$$

This is the Martin kernel $K(g^{-1},\omega)$, (See [32]). Hence it suffices to show that $K(g^{-1},x)$ is constant for any $x = x_1 \cdots x_n \in \Gamma$ such that $x_1 \notin G_i$. By [32], we have

$$K(g^{-1},x) = \frac{G(g^{-1},x)}{G(e,x)},$$

where $G(y,z) = \sum_{k=1}^\infty p^{(k)}(y,z)$ is the Green kernel. Since any probability from g^{-1} to x must be through elements of H at least once, we have

$$G(g^{-1},x) = \sum_{h\in H} F(g^{-1},h)G(h,x),$$

where $s^x = \inf\{n \geq 0 \mid Z_n = x\}$ and $F(g,x) = \sum_{n=0}^\infty \mathrm{Pr}_g[s^x = n]$ in [33]. By hypothesis $\mu(g) = \mu(hg)$ for any $g \in \bigcup_{i\in I} G_i \setminus H$ and $h \in H$, we have

$$G(h,x) = G(e,x) \quad \text{for any} \quad h \in H.$$

Therefore we have $\omega_g = \log(\sum_{h\in H} F(g^{-1},h))$. $\sigma_t^\nu(\pi_\nu(U_h)) = \pi_\nu(U_h)$ can be proved in the same way. Hence we are done. ☐

§9. Appendix

Trees. We first review trees based on [15]. A *graph* is a pair (V, E) consisting of a set of vertices V and a family E of two-element subsets of V, called edges. A *path* is a finite sequence $\{x_1, \ldots, x_n\} \subseteq V$ such that $\{x_i, x_{i+1}\} \in E$. (V, E) is said to be *connected* if for $x, y \in V$ there exists a path $\{x_1, \ldots, x_n\}$ with $x_1 = x, x_n = y$. If (V, E) is a tree, then for $x, y \in V$ there exists a unique path $\{x_1, \ldots, x_n\}$ joining x to y such that $x_i \neq x_{i+2}$. We denote this path by $[x, y]$. A tree is said to be *locally finite* if every vertex belongs to finitely many edges. The number of edges to which a vertex of a locally finite tree belongs is called a *degree*. If the degree is independent of the choice of vertices, then the tree is called *homogeneous*.

We introduce trees for amalgamated free product groups based on [27]. Let $(G_i)_{i\in I}$ be a family of groups with an index set I. When H is a group and every G_i contains H as a subgroup, then we denote $*_H G_i$ by Γ, which is the amalgamated free product of the groups. If we choose sets Ω_i of left representatives of G_i/H with $e \in \Omega_i$ for any $i \in I$, then each $\gamma \in \Gamma$ can be written uniquely as

$$\gamma = g_1 g_2 \cdots g_n h,$$

where $h \in H, g_1 \in \Omega_{i_1} \setminus \{e\}, \ldots, g_n \in \Omega_{i_n} \setminus \{e\}$ and $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{n-1} \neq i_n$.

Now we construct the corresponding tree. At first, we assume that $I = \{1, 2\}$. Let

$$V = \Gamma/G_1 \coprod \Gamma/G_2 \quad \text{and} \quad E = \Gamma/H,$$

and the original and terminal maps $o : \Gamma/H \to \Gamma/G_1$ and $t : \Gamma/H \to \Gamma/G_2$ are natural surjections. It is easy to see that $G_T = (V, E)$ is a tree. In general, we assume that the element 0 does not belong to I. Let $G_0 = H$ and $H_i = H$ for $i \in I$. Then we define

$$V = \coprod_{i\in I\cup\{0\}} \Gamma/G_i \quad \text{and} \quad E = \coprod_{i\in I} \Gamma/H_i.$$

Now we define two maps $o, t : E \to V$. For $H_i \in E$, let

$$o(H_i) = G_0 \quad \text{and} \quad t(H_i) = G_i.$$

For any $\gamma H_i \in E$, we may assume that $\gamma H = g_1 \cdots g_n H_i$ such that $g_k \in \Omega_{i_k}$ with $i_1 \neq \cdots \neq i_n$. If $i = i_n$ we define

$$o(\gamma H_i) = \gamma G_{i_n} \quad \text{and} \quad t(\gamma H_i) = \gamma G_0.$$

If $i \neq i_n$ we define

$$o(\gamma H_i) = \gamma G_0 \quad \text{and} \quad t(\gamma H_i) = \gamma G_i.$$

Then we have a tree $G_T = (V, E)$.

For a tree (V, E), the set V is naturally a metric space. The distance $d(x, y)$ is defined by the number of edges in the unique path $[x, y]$. An *infinite chain* is an infinite path $\{x_1, x_2, \dots\}$ such that $x_i \neq x_{i+2}$. We define an equivalence relation on the set of infinite chains. Two infinite chains $\{x_1, x_2, \dots\}, \{y_1, y_2, \dots\}$ are equivalent if there exists an integer k such that $x_n = y_{n+k}$ for a sufficiently large n. The boundary Ω of a tree is the set of the equivalence classes of infinite chains. The boundary may be thought of as a point at infinity. Next we introduce the topology into the space $V \cup \Omega$ such that $V \cup \Omega$ is compact, the points of V are open and V is dense in $V \cup \Omega$. It suffices to define a basis of neighborhoods for each $\omega \in \Omega$. Let x be a vertex. Let $\{x, x_1, x_2, \dots\}$ be an infinite chain representing ω. For each $y = x_n$, the neighborhood of ω is defined to consist of all vertices and all boundary points of the infinite chains which include $[x, y]$.

Hyperbolic groups. We introduce hyperbolic groups defined by Gromov. See [18] for details. Suppose that (X, d) is a metric space. We define a product by

$$\langle x|y\rangle_z = \frac{1}{2}\{d(x, z) + d(y, z) - d(x, y)\},$$

for $x, y, z \in X$. This is called the Gromov product. Let $\delta \geq 0$ and $w \in X$. A metric space X is said to be *δ-hyperbolic* with respect to w if for $x, y, z \in X$,

$$\langle x|y\rangle_w \geq \min\{\langle x|z\rangle_w, \langle y|z\rangle_w\} - \delta. \tag{‡}$$

Note that if X is δ-hyperbolic with respect to w, then X is δ-hyperbolic with respect to any $w' \in X$.

Definition 9.1. The space X is said to be *hyperbolic* if X is δ-hyperbolic with respect to some $w \in X$ and some $\delta \geq 0$.

Suppose that Γ is a group generated by a finite subset S such that $S^{-1} = S$. Let $G(\Gamma, S)$ be the Cayley graph. The graph $G(\Gamma, S)$ has a natural word metric. Hence $G(\Gamma, S)$ is a metric space.

Definition 9.2. A finitely generated group Γ is said to be *hyperbolic* with respect to a finite generator system S if the corresponding Cayley graph $G(\Gamma, S)$ is hyperbolic with respect to the word metric.

In fact, hyperbolicity is independent of the choice of S. Therefore we say that Γ is a hyperbolic group, for short.

We define the hyperbolic boundary of a hyperbolic space X. Let $w \in X$ be a point. A sequence (x_n) in X is said to *converge to infinity* if $\langle x_n|x_m\rangle_w \to \infty$, $(n, m \to \infty)$. Note that this is independent of the choice of w. The set X_∞ is the set of all sequences converging to infinity in X. Then we define an equivalence relation in X_∞. Two sequences $(x_n), (y_n)$ are equivalent if $\langle x_n|y_n\rangle_w \to \infty$, $(n \to \infty)$. Although this is not an equivalence relation in general, the hyperbolicity assures that it is indeed an equivalence relation. The set of all equivalent classes of X_∞ is called the *hyperbolic boundary* (*at infinity*) and denoted by ∂X. Next we define the Gromov product on $X \cup \partial X$. For $x, y \in X \cup \partial X$, we choose sequences $(x_n), (y_n)$ converging to x, y, respectively. Then we define $\langle x|y\rangle = \liminf_{n\to\infty}\langle x_n|y_n\rangle_w$. Note that this is well-defined and if $x, y \in X$ then the above product coincides with the Gromov product on X.

Definition 9.3. The topology of $X \cup \partial X$ is defined by the following neighborhood basis:

$$\{y \in X \mid d(x, y) < r\} \qquad \text{for } x \in X, r > 0,$$

$$\{y \in X \cup \partial X \mid \langle x|y\rangle > r\} \qquad \text{for } x \in \partial X, r > 0.$$

We remark that if X is a tree, then the hyperbolic boundary ∂X coincides with the natural boundary Ω in the sense of [16].

Finally we prove that an amalgamated free product $\Gamma = *_H G_i$, considered in this paper, is a hyperbolic group.

Lemma 9.4. *The group $\Gamma = *_H G_i$ is a hyperbolic group.*

Proof. Let $S = \{g \in \bigcup_i G_i \mid |g| \le 1\}$. Let $G(\Gamma, S)$ be the corresponding Cayley graph. It suffices to show (‡) for $w = e$. For $x, y, z \in \Gamma$, we can write uniquely as follows:

$$x = x_1 \cdots x_n h_x,$$
$$y = y_1 \cdots y_m h_y,$$
$$z = z_1 \cdots z_k h_z,$$

where

$$x_1 \in \Omega_{i(x_1)}, \ldots, x_n \in \Omega_{i(x_n)}, h_x \in H,$$
$$y_1 \in \Omega_{i(y_1)}, \ldots, y_m \in \Omega_{i(y_m)}, h_y \in H,$$
$$z_1 \in \Omega_{i(z_1)}, \ldots, z_k \in \Omega_{i(z_k)}, h_z \in H.$$

such that each element has length one. Then $d(x,e) = n$, $d(y,e) = m$ and $d(z,e) = k$. If $i(x_1) = i(y_1), \ldots, i(x_{l(x,y)}) = i(y_{l(x,y)})$ and $i(x_{l(x,y)+1}) \neq i(y_{l(x,y)+1})$, then $\langle x|y\rangle_e = l(x,y)$. Similarly, we obtain the positive integers $l(x,z), l(y,x)$ such that $\langle x|z\rangle_e = l(x,z), \langle y|z\rangle_e = l(y,z)$. We can have (‡) with $\delta = 0$. □

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