The Zeta Function of a Finite Category

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Abstract. We define the zeta function of a finite category. We prove a theorem that states a relationship between the zeta function of a finite category and the Euler characteristic of finite categories, called the series Euler characteristic [BL08]. Moreover, it is shown that for a covering of finite categories, $P : E \to B$, the zeta function of E is that of B to the power of the number of sheets in the covering. This is a categorical analogue of the unproved conjecture of Dedekind for algebraic number fields and the Dedekind zeta functions.

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Contents

1 Introduction

Euler characteristics and zeta functions are defined for various mathematical objects; for example, simplicial complexes, algebraic varieties, and graphs. In many cases, we can observe that the zeta function knows the Euler characteristic, as the following three examples suggest.

1. Let G be a finite connected graph. Then, the Ihara zeta function of G is defined by

$$Z_G(u) = \prod_{[C]} \frac{1}{1 - u^{\ell[C]}}$$

where $[C]$ is an equivalence class of certain paths in G and ℓ is the length function. The zeta function Z_G has the determinant expression

$$Z_G(u) = \frac{(1-u^2)^{1-r}}{|I - A_G u + Q_G u^2|}$$

for some matrices A_G and Q_G where I is the unit matrix and r is the rank of the fundamental group of G (Theorem 2 of [ST96]). It is clear that $1 - r$ is the Euler characteristic $\chi(G)$ of G.

2. Let Δ be a simplicial complex on a vertex set

$$\{1, 2, \ldots, N\}$$

and let $\mathbb{F}_q$ be a finite field. Björner and Sarkaria defined the zeta function of Δ over $\mathbb{F}_q$ by

$$Z_\Delta(q,t) = \exp\left(\sum_{m=1}^{\infty} \#V(\Delta, \mathbb{F}_{q^m}) \frac{t^m}{m}\right)$$

where $V(\Delta, \mathbb{F}_{q^m})$ is the set of points in the projective space $\mathbb{F}_{q^m}P^{N-1}$ whose support belongs to Δ [BS98]. By Theorem 2.2 of [BS98], the zeta function has a rational expression; that is,

$$Z_\Delta(q,t) = \prod_{k=0}^{d} \frac{1}{(1-q^k t)^{f_k^*}}$$

for some integers f_k^* where d is the dimension of Δ. Here, we obtain $\sum_{k=0}^{d} f_k^* = \chi(V(\Delta, \mathbb{C}))$ (Corollary 2.4 of [BS98]).

3. Let X be an n-dimensional smooth projective variety over a finite field $\mathbb{F}_q$. Then, the zeta function of X is defined by

$$Z_X(T) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m(X)}{m} T^m\right)$$

where $N_m(X)$ is the number of points in X over $\mathbb{F}_{q^m}$. One of the Weil conjectures states that Z_X has a rational expression of the following form:

$$Z_X(T) = \frac{P(T)}{Q(T)}$$

for some polynomials $P(T)$ and $Q(T)$ with coefficients in $\mathbb{Z}$, and we obtain $\chi(X) = \deg Q - \deg P$ (e.g., see [Har77]).

These examples tell us that the zeta function knows the Euler characteristic. In this paper, we define the zeta function of a finite category and we prove a theorem that states a relationship between the zeta function of a finite category and the Euler characteristic of a finite category, called the *series Euler characteristic* [BL08].

Let C be a finite category. A *finite category* is a category having finitely many objects and morphisms. Then, the *zeta function* of C is defined by

$$\zeta_C(z) = \exp\left(\sum_{m=1}^{\infty} \frac{\# N_m(C)}{m} z^m\right)$$

where

$$N_m(C) = \{\ (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_m} x_m)\ \text{ in } C\}.$$

The zeta function of a finite category introduced in this paper is different from the one introduced by Kurokawa [Kur96]. His zeta function is for a large category; for example, the category of Abelian groups.

Next, let us recall the Euler characteristics of categories. The *Euler characteristic of a finite category* was defined by Leinster [Lei08]. This was the first Euler characteristic for categories. Subsequently, there have emerged the *series Euler characteristic* by Berger-Leinster [BL08] and the L^2-*Euler characteristic* by Fiore-Lück-Sauer [FLS11] as well as the *extended* L^2-*Euler characteristic* [Nog] and the *Euler characteristic of* $\mathbb{N}$-*filtered acyclic categories* [Nog11] by the author. In this paper, we often use the series Euler characteristic, so we provide a more detailed explanation for the series Euler characteristic.

For a finite category C whose set of objects is $\{x_1, \ldots, x_N\}$, its *series Euler characteristic* $\chi_\Sigma(C)$ is defined by substituting $t = -1$ in

$$\frac{\mathrm{sum}(\mathrm{adj}(I - (A_C - I)t))}{|I - (A_C - I)t|}$$

if it exists, where $A_C = (\#\mathrm{Hom}(x_i, x_j))$ is called the *adjacency matrix* of C and sum means to take the sum of all the entries of a matrix. This rational function is a rational expression of the power series $\sum_{m=0}^{\infty} \#\overline{N_m}(C)t^m$ where $\overline{N_m}(C)$ is the set of *nondegenerate* chains of morphisms of length m in C

$$\overline{N_m}(C) = \{\ (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_m} x_m)\ \text{ in } C \mid f_i \neq 1\}.$$

This Euler characteristic is defined from the viewpoint of the classifying spaces. For a small category C, we can construct the topological space (in fact, a CW-complex) BC, called the *classifying space* of C. There is a one-to-one correspondence between the set of m-dimensional parts (m-cells) of BC and $\overline{N_m}(C)$ [Qui73]. The Euler characteristic of a cell-complex is defined by the alternating sum of the number of m-cells. Hence, the Euler characteristic of C should be defined by $\sum_{m=0}^{\infty}(-1)^m\#\overline{N_m}(C)$ in a topological sense. However, this series often fails to converge, so we substitute $t = -1$ in the rational expression instead of the power series $\sum_{m=0}^{\infty} \#\overline{N_m}(C)t^m$. For more details, see [BL08].
The following is our main theorem.

Main Theorem (Theorem 3.5). Suppose that C is a finite category with Euler characteristic $\chi_{\sum}(C)$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the nonzero eigenvalues of A_C whose algebraic multiplicities are $e_1, e_2, \ldots, e_n$. Then,

1. the zeta function of C is
$$\zeta_C(z) = \prod_{k=1}^{n} \frac{1}{(1-\lambda_k z)^{\beta_{k,0}}} \exp\left(\sum_{j=1}^{e_k-1} \frac{\beta_{k,j} z^j}{j(1-\lambda_k z)^j}\right)$$
for some complex numbers $\beta_{k,j}$,

2. the sum of all the indexes $\beta_{k,0}$ is the number of objects of C,

3. each λ_k is an algebraic integer, and

4. $\displaystyle\sum_{k=1}^{n}\sum_{j=0}^{e_k-1}(-1)^j \frac{\beta_{k,j}}{\lambda_k^{j+1}} = \chi_{\Sigma}(C) \in \mathbb{Q}.$

Part 3 is an analogue of the Weil conjecture and, in fact, it does not need the condition that C has Euler characteristic (see Theorem 3.3). Part 4 implies that, although each λ_k and $\beta_{k,j}$ is a complex number, this alternating sum is always rational. In this paper, we define $\mathrm{Log}\, z$ and the power functions by the principal value; that is,

$$\mathrm{Log}\, z = \log|z| + i\mathrm{Arg}(z) \quad (z \in \mathbb{C} - \{x \in \mathbb{R} \mid x \leq 0\}, -\pi < \mathrm{Arg}(z) < \pi)$$

and

$$z^{\alpha} = e^{\alpha\, \mathrm{Log}\, z} \quad (z, \alpha \in \mathbb{C}, z \neq 0).$$

If we do not assume the condition that C has Euler characteristic, Part 1 is given by the following theorem.

Theorem 1.1 (Theorem 3.3). *Let C be a finite category. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the nonzero eigenvalues of A_C and their algebraic multiplicities are $e_1, e_2, \ldots, e_n$. Then, the zeta function of C is*

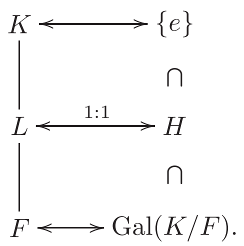
$$\zeta_C(z) = \prod_{k=1}^{n} \frac{1}{(1-\lambda_k z)^{\beta_{k,0}}} \exp\left(Q(z) + \sum_{j=1}^{e_k - 1} \frac{\beta_{k,j} z^j}{j(1-\lambda_k z)^j} \right)$$

for some complex numbers $\beta_{k,j}$ and a polynomial $Q(z)$ with $\mathbb{Q}$-coefficients whose constant term is zero.

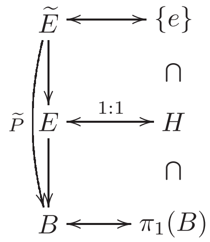
If we do not assume the condition that C has Euler characteristic, Part 2 fails (see Example 3.7).

Our zeta function is related with coverings of small categories. We show that for a covering of finite categories, $P : E \to B$, the zeta function of E is that of B to the power of the number of sheets in the covering. This is an analogue of the unproved conjecture of Dedekind. The conjecture is that for a finite extension K_2 of an algebraic number field K_1 the Dedekind zeta function $\zeta_{K_1}(s)$ of K_1 divides that of K_2 [Waa75]. An *algebraic number field* is a finite extension of $\mathbb{Q}$.

A covering of small categories is an analogy of Galois theory. A fundamental theorem of Galois theory is that if K/F is a finite Galois extension, the set of intermediate fields of K and F is bijective to the set of subgroups of the Galois group $\mathrm{Gal}(K/F)$:



For a covering of small categories $\widetilde{P} : \widetilde{E} \to B$, where $\widetilde{E}$ is the universal covering of B, the set of the isomorphism classes of intermediate coverings of $\widetilde{P}$ is bijective to the set of subgroups of the fundamental group $\pi_1(B)$:



(see Corollary 2.24 of [Tan]). We have the following correspondences:

$$\begin{array}{rcl} \text{coverings} & \leftrightarrow & \text{extensions of fields} \\ \pi_1 & \leftrightarrow & \text{Galois groups} \\ \text{intermediate coverings} & \leftrightarrow & \text{intermediate fields} \\ \vdots & & \vdots \end{array}$$

For an analogy between coverings of spaces and extensions of fields, see [Mor12]. By the diagrams above, we can conclude that the relationship between our zeta functions and coverings is an analogue of the Dedekind conjecture. Graph theoretic analogue of this conjecture was considered in Corollary 1 of §2 of [ST96] ([ST00] and [ST07] are its continuation).
This remainder of this paper is organized as follows: In Section 2, the zeta function of a finite category is defined, and we compute the zeta functions of finite groupoids and finite acyclic categories. We classify the zeta functions of one-object finite categories and two-objects finite categories. In Section 3, we prove our main theorem, and we introduce four zeta functions of finite categories having three-objects. In Section 4, we prove that for a covering of finite categories, $P : E \to B$, the zeta function of E is that of B to the power of the number of sheets in the covering.

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2 Definition and examples

In this section, we define the zeta function of a finite category, and we compute zeta functions.

2.1 Definition

Before defining the zeta function of a finite category, we review the symbols that are often used in this paper.
Let C be a finite category. Then, let

$$N_m(C) = \{\ (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_m} x_m)\ \text{ in } C\}$$

and

$$\overline{N_m}(C) = \{\ (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_m} x_m)\ \text{ in } C \mid f_i \neq 1\}.$$

The difference between these is merely whether identity morphisms are used or not. For $m = 0$, we set $N_0(C) = \overline{N_0}(C) = \mathrm{Ob}(C)$. In this paper we have the important equality

$$\# N_m(C) = \mathrm{sum}(A_C^m).$$

Indeed, if

$$\mathrm{Ob}(C) = \{x_1, x_2, \ldots, x_N\}$$

and $A_C = (a_{ij})$, then the (i, j)-entry of A_C^m is

$$\sum_{1 \le k_1, k_2, \ldots, k_{m-1} \le N} a_{ik_1} a_{k_1 k_2} a_{k_2 k_3} \ldots a_{k_{m-1} j}.$$

This is the number of chains of morphisms of length m from x_i to x_j. Hence, we obtain the equality.

Definition 2.1. Let C be a finite category. Then, we define the *zeta function* $\zeta_C(z)$ of C by

$$\zeta_C(z) = \exp\left(\sum_{m=1}^{\infty} \frac{\# N_m(C)}{m} z^m \right).$$

This function belongs to the power series ring $\mathbb{Q}[[z]]$. If preferable, the zeta function can be considered a function of a complex variable by choosing z to be a sufficiently small complex number. Indeed, for a complex number z such that $|z| < \frac{1}{\mathrm{sum}(A_C)}$, the series absolutely converges; that is,

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{\# N_m(C)}{m} |z|^m &= \sum_{m=1}^{\infty} \frac{\mathrm{sum}(A_C^m)}{m} |z|^m \\ &\le \sum_{m=1}^{\infty} \frac{\{\mathrm{sum}(A_C)\}^m}{m} |z|^m < +\infty. \end{aligned}$$

Example 2.2. This is the simplest example. Let $*$ denote the terminal category. Then, its zeta function is

$$\begin{aligned} \zeta_*(z) &= \exp\left(\sum_{m=1}^{\infty} \frac{\# N_m(*)}{m} z^m \right) \\ &= \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} z^m \right) \\ &= \exp\left(-\mathrm{Log}(1 - z)\right) \\ &= \frac{1}{1 - z}. \end{aligned}$$

2.2 Groupoids

In this subsection, we compute the zeta functions of finite groupoids. First, we compute the zeta functions of connected finite groupoids.

A category C is *connected* if C is a nonempty category and there exists a zig-zag sequence of morphisms in C,

$$x \xrightarrow{f_1} x_1 \xleftarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xleftarrow{f_n} y \ ,$$

for any objects x and y of C. We do not have to consider the direction of the last morphism f_n, since we can insert an identity morphism into the sequence. A nonempty groupoid Γ is connected if and only if there exists a morphism $f : x \to y$ for any objects x and y of Γ.

Proposition 2.3. *Let Γ be a connected finite groupoid. Then, its zeta function is*

$$\zeta_\Gamma(z) = \frac{1}{(1 - \#N_0(\Gamma) o(\Gamma) z)^{\#N_0(\Gamma)}}$$

where $o(\Gamma)$ is the order of the automorphism group $\mathrm{Aut}(x)$ *for some object x of Γ.*

Proof. Let

$$\mathrm{Ob}(\Gamma) = \{x_1, x_2, \ldots, x_N\}.$$

We count the chains of morphisms of length m in Γ. To determine

$$y_0 \xrightarrow{f_1} y_1 \xrightarrow{f_2} \cdots \xrightarrow{f_m} y_m$$

we first determine the objects $y_0, y_1, \ldots, y_m$. There are N^{m+1} ways to determine these. There are $o(\Gamma)^m$ ways to determine the morphisms $f_1, f_2, \ldots, f_m$, since we have

$$\#\mathrm{Hom}(x, y) = \#\mathrm{Hom}(x', y') = o(\Gamma)$$

for any objects x, x', y, and y' of Γ. Hence, we obtain $\#N_m(\Gamma) = N^{m+1} o(\Gamma)^m$. Thus, we have

$$\begin{aligned} \zeta_\Gamma(z) &= \exp\left(\sum_{m=1}^{\infty} \frac{N^{m+1} o(\Gamma)^m}{m} z^m \right) \\ &= \exp\left(N \sum_{m=1}^{\infty} \frac{1}{m} (N o(\Gamma) z)^m \right) \\ &= \exp\left(-N \,\mathrm{Log}(1 - N o(\Gamma) z) \right) \\ &= \frac{1}{(1 - N o(\Gamma) z)^N}. \end{aligned}$$

□

Remark 2.4. 1. The zeta function of a finite category is not invariant under equivalence of categories. For example, let Γ_N be the following groupoid:

$$x_1 \rightleftarrows x_2 \rightleftarrows \cdots \rightleftarrows x_N$$

for any natural number N. Then, Γ_N is equivalent to Γ_M for any natural number M. Proposition 2.3 implies that their zeta functions are not the same if $N \neq M$.

2. The zeta function of a finite category depends only on the underlying graph, not on the composition of the finite category, and the zeta function of a finite category is not the same as the zeta function of its underlying graph. For a directed graph D, the zeta function $Z_D(u)$ of D is defined by the formal product of certain equivalence classes of paths (see [KS00] and [MS01] for more details). It has a determinant expression of the following form:
$$Z_D(u) = \frac{1}{|I - A_D u|}$$
where A_D is the adjacency matrix of D.

   For example, the zeta function of Γ_2 (see above) is
$$\frac{1}{(1-2z)^2},$$
but the zeta function of its underlying graph is
$$\frac{1}{1-u^2}.$$

The proposition above can be generalized to the following proposition.

Proposition 2.5. *Suppose that C is a finite category and its adjacency matrix $A_C = (a_{ij})$ satisfies the condition $\sum_i a_{ij} = \sum_i a_{ij'}$ for any j and j'. Then, its zeta function is*
$$\zeta_C(z) = \frac{1}{(1 - (\sum_i a_{ij})\, z)^{\#N_0(C)}}.$$

Proof. Under the condition that $\sum_i a_{ij} = \sum_i a_{ij'}$ for any j and j', we have
$$\#N_m(C) = \mathrm{sum}(A_C^m) = \#\mathrm{Ob}(C)\left(\sum_i a_{ij}\right)^m.$$
Hence, we obtain the result. □

This result is with respect to the columns of A_C, but it is clear that there is a similar result with respect to the rows of A_C.

Remark 2.6. A finite category and its opposite category have the same zeta function. Indeed, we have $A_C = {}^{\mathrm{t}}A_{C^{\mathrm{op}}}$, so $\mathrm{sum}(A_C^m) = \mathrm{sum}(A_{C^{\mathrm{op}}}^m)$. Hence, their zeta functions are the same.

By the following lemma, computing the zeta function of a finite category is reduced to computing the zeta functions of its connected components.

Lemma 2.7. *Let $C_1, C_2, \ldots, C_n$ be finite categories. Then, the zeta function of $C = \coprod_{k=1}^n C_k$ is*
$$\zeta_C(z) = \prod_{k=1}^n \zeta_{C_k}(z).$$

Proof. Since $N_m(C) = \coprod_{k=1}^n N_m(C_k)$, we obtain

$$\begin{aligned} \zeta_C(z) &= \exp\left(\sum_{m=1}^{\infty} \frac{\#N_m(C)}{m} z^m\right) \\ &= \prod_{k=1}^{n} \exp\left(\sum_{m=1}^{\infty} \frac{\#N_m(C_k)}{m} z^m\right) \\ &= \prod_{k=1}^{n} \zeta_{C_k}(z). \end{aligned}$$

□

Corollary 2.8. *Suppose that* Γ *is a finite groupoid and* $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ *are its connected components; that is,* $\Gamma = \coprod_{k=1}^n \Gamma_k$ *and each* Γ_k *is connected. Then, the zeta function of* Γ *is*

$$\zeta_\Gamma(z) = \prod_{k=1}^{n} \frac{1}{(1 - \#N_0(\Gamma_k) o(\Gamma_k) z)^{\#N_0(\Gamma_k)}}.$$

Proof. Lemma 2.7 and Proposition 2.3 directly imply the result. □

2.3 Acyclic categories

In this subsection, we compute the zeta functions of finite acyclic categories by using another expression for our zeta function.

Definition 2.9. A small category A is defined to be an *acyclic category* if all the endomorphisms are only identity morphisms and if there exists a morphism $f : x \to y$ such that $x \neq y$, then there does not exist a morphism $g : y \to x$.

Lemma 2.10. *Let* C *be a finite category. Then, we have*

$$\#N_m(C) = \sum_{j=0}^{m} \binom{m}{j} \#\overline{N_j}(C)$$

for any $m \geq 0$.

Proof. Suppose that $0 \leq j \leq m$ and take any $(f_1, f_2, \ldots, f_j)$ of $\overline{N_j}(C)$. Then, we can make $\binom{m}{j}$-elements of $N_m(C)$ by inserting identity morphisms. Hence, we obtain the result. □

Proposition 2.11. *Let* C *be a finite category. Then, we have*

$$\zeta_C(z) = \frac{1}{(1-z)^{\#\overline{N_0}(C)}} \exp\left(\sum_{j=1}^{\infty} \frac{\#\overline{N_j}(C) z^j}{j(1-z)^j}\right).$$

Proof. Lemma 2.10 implies

$$\begin{aligned}
\zeta_C(z) &= \exp\left(\sum_{m=1}^{\infty}\frac{1}{m}\#N_m(C)z^m\right)\\
&= \exp\left(\sum_{m=1}^{\infty}\frac{1}{m}\sum_{j=0}^{m}\binom{m}{j}\#\overline{N_j}(C)z^m\right)\\
&= \exp\left(\sum_{j=0}^{\infty}\#\overline{N_j}(C)\sum_{m=1}^{\infty}\frac{1}{m}\binom{m}{j}z^m\right)\\
&= \exp\left(\sum_{m=1}^{\infty}\frac{\#\overline{N_0}(C)}{m}z^m+\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\frac{\#\overline{N_j}(C)}{m}\binom{m}{j}z^m\right)\\
&= \frac{1}{(1-z)^{\#\overline{N_0}(C)}}\exp\left(\sum_{j=1}^{\infty}\frac{\#\overline{N_j}(C)z^j}{j(1-z)^j}\right).
\end{aligned}$$

Note that $\binom{m}{j}=0$ if $m<j$. The last equality is implied by the equality (1.5.5) in [Wil06]. □

Corollary 2.12. *Let A be a finite acyclic category. Then, the zeta function of A is*

$$\zeta_A(z)=\frac{1}{(1-z)^{\#\overline{N_0}(A)}}\exp\left(\sum_{j=1}^{M}\frac{\#\overline{N_j}(A)z^j}{j(1-z)^j}\right)$$

for a sufficiently large integer M.

Proof. By Lemma 3.5 of [Nog], there exists a sufficiently large integer M such that $\overline{N_j}(A)=\emptyset$ for any $j>M$. Proposition 2.11 completes this proof. □

2.4 Finite categories having one or two objects

In this subsection, we classify the zeta functions of finite categories having one or two objects. In all the zeta functions that we have already seen, only rational numbers appear, but irrational numbers appear in the classification.
First, we compute the zeta functions of one-object finite categories.

Proposition 2.13. *Let C be a one-object finite category. Then, its zeta function is*

$$\zeta_C(z)=\frac{1}{1-\#N_1(C)z}.$$

Proof. Since C has only one object, all the morphisms can be composed, so we have

$$\#N_m(C)=(\#N_1(C))^m.$$

Hence, we obtain the result. □

Lemma 2.14. *Suppose that C is a finite category having N objects and its adjacency matrix A_C is diagonalizable, with*

$$A_C = P \cdot \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N) \cdot P^{-1}.$$

Then, we have

$$\zeta_C(z) = \exp\left(\mathrm{sum}\left(P \cdot \mathrm{diag}\left(\mathrm{Log}\,\frac{1}{1-\lambda_1 z},\right.\right.\right.$$
$$\left.\left.\left.\mathrm{Log}\,\frac{1}{1-\lambda_2 z}, \ldots, \mathrm{Log}\,\frac{1}{1-\lambda_N z}\right) \cdot P^{-1}\right)\right).$$

Proof. We have

$$\begin{aligned}
\zeta_C(z) &= \exp\left(\mathrm{sum}\sum_{m=1}^{\infty}\frac{A_C^m}{m}z^m\right)\\
&= \exp\left(\mathrm{sum}\sum_{m=1}^{\infty}\frac{1}{m}P\cdot\mathrm{diag}(\lambda_1^m, \lambda_2^m, \ldots, \lambda_N^m)\cdot P^{-1}z^m\right)\\
&= \exp\left(\mathrm{sum}\left(P\cdot\mathrm{diag}\left(\mathrm{Log}\,\frac{1}{1-\lambda_1 z},\right.\right.\right.\\
&\quad \left.\left.\left.\mathrm{Log}\,\frac{1}{1-\lambda_2 z}, \ldots, \mathrm{Log}\,\frac{1}{1-\lambda_N z}\right)\cdot P^{-1}\right)\right).
\end{aligned}$$

□

Proposition 2.15. *Let C be a two-object finite category and let $A_C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, its zeta function is*

$$\zeta_C(z) = \begin{cases} \dfrac{1}{(1-az)^2}\exp\left(\dfrac{bz}{1-az}\right) & \text{if } a = d, b \neq 0, c = 0 \\ \dfrac{1}{(1-az)^2}\exp\left(\dfrac{cz}{1-az}\right) & \text{if } a = d, b = 0, c \neq 0 \\ \dfrac{1}{(1-\lambda^+ z)^{\beta_0^+}}\dfrac{1}{(1-\lambda^- z)^{\beta_0^-}} & \text{otherwise,} \end{cases}$$

where $\lambda^\pm$ are the eigenvalues of A_C and

$$\beta_0^\pm = \begin{cases} 1 & \text{if } a = d, b = c = 0 \\ 1 \pm \frac{b+c}{\sqrt{\Delta}} & \text{otherwise.} \end{cases}$$

Here, $\Delta = (a-d)^2 + 4bc$ is the discriminant of the characteristic polynomial of A_C.

Proof. If $a = d$, $b \neq 0$, and $c = 0$, then we have

$$\begin{aligned} \#N_m(C) &= a^m + a^{m-1}b + a^{m-2}bd + \cdots + abd^{m-2} + bd^{m-1} + d^m \\ &= 2a^m + mba^{m-1}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \zeta_C(z) &= \exp\left(\sum_{m=1}^{\infty} \frac{1}{m}(2a^m + mba^{m-1})z^m\right) \\ &= \frac{1}{(1-az)^2} \exp\left(\frac{bz}{1-az}\right). \end{aligned}$$

If $a = d$, $b = 0$, and $c \neq 0$, then we can similarly prove the result.
If $a = d$ and $b = c = 0$, then the category consists of one-object categories, so Lemma 2.7 and Proposition 2.13 imply the result.
In the other cases, A_C is diagonalizable over $\mathbb{R}$ since Δ is nonzero and is a nonnegative real number. We omit the process to compute P, since the calculation is routine. Lemma 2.14 completes the proof. □

Example 2.16. Let

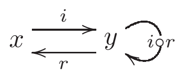
$$P = x \longrightarrow y\ .$$

Then, $A_P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, so Proposition 2.15 implies that the zeta function is

$$\zeta_P(z) = \frac{1}{(1-z)^2} \exp\left(\frac{z}{1-z}\right),$$

which is not a rational function. In the proof of Theorem 3.3, we will find the reason why the zeta function of a finite category has an exponential factor is that a nonzero eigenvalue of its adjacency matrix has algebraic multiplicity.

Example 2.17. Let $\mathbb{F}$ be the following category:



where $r \circ i = 1_x, i \circ r \neq 1_y$. Then, $A_{\mathbb{F}} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Proposition 2.15 implies that the zeta function is

$$\zeta_{\mathbb{F}}(z) = \frac{1}{\left(1 - \left(\frac{3+\sqrt{5}}{2}\right)z\right)^{1+\frac{2}{\sqrt{5}}}} \frac{1}{\left(1 - \left(\frac{3-\sqrt{5}}{2}\right)z\right)^{1-\frac{2}{\sqrt{5}}}}.$$

The reason that $\sqrt{5}$ appears is that the sequence $(\#\overline{N_m}(\mathbb{F}))_{m\geq 0}$ is a subsequence of the Fibonacci sequence $(F_m)_{m\geq 1}$; that is, we have $\#\overline{N_m}(\mathbb{F}) = F_{m+3}$ and

$$F_m = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right)$$

(see §1.3 of [Wil06]). Hence, Proposition 2.11 also implies the result.
Here, let us confirm that Theorem 3.5 holds for this zeta function.

1. The zeta function of $\mathbb{F}$ is
$$\zeta_{\mathbb{F}}(z) = \frac{1}{\left(1-\left(\frac{3+\sqrt{5}}{2}\right)z\right)^{1+\frac{2}{\sqrt{5}}}} \frac{1}{\left(1-\left(\frac{3-\sqrt{5}}{2}\right)z\right)^{1-\frac{2}{\sqrt{5}}}}.$$

2. The sum of the indexes is the number of objects in $\mathbb{F}$, which is
$$\left(1+\frac{2}{\sqrt{5}}\right)+\left(1-\frac{2}{\sqrt{5}}\right)=2.$$

3. The numbers $\frac{3\pm\sqrt{5}}{2}$ are algebraic integers. More precisely, they are integers in the real quadratic number field $\mathbb{Q}(\sqrt{5})$. The ring of integers in $\mathbb{Q}(\sqrt{5})$ is
$$\left\{\frac{a+b\sqrt{5}}{2}\middle| a,b\in\mathbb{Z}, a\equiv b \mod 2\right\}.$$

4. We obtain
$$\frac{1+\frac{2}{\sqrt{5}}}{\frac{3+\sqrt{5}}{2}}+\frac{1-\frac{2}{\sqrt{5}}}{\frac{3-\sqrt{5}}{2}}=1=\chi_\Sigma(\mathbb{F}).$$

3 Main theorem

In this section, we prove our main theorem.

3.1 Preparations for our main theorem

Throughout this section, we will use the following notation.

1. Unless otherwise stated, C is a finite category having N objects.

2. The two polynomials $|A_C - Iz|$ and $\mathrm{sum}(\mathrm{adj}(A_C - Iz))$ that will often be used are expressed in the following forms:
$$|A_C - Iz| = a_0 + a_1 z + \cdots + a_N z^N$$
and
$$\mathrm{sum}(\mathrm{adj}(A_C - Iz)) = b_0 + b_1 z + \cdots + b_{N-1} z^{N-1}.$$

3. We denote the codegrees of $|A_C - Iz|$ and $\mathrm{sum}(\mathrm{adj}(A_C - Iz))$ by the following:
$$\mathrm{codeg}\,|A_C - Iz| = r$$
and
$$\mathrm{codeg}(\mathrm{sum}(\mathrm{adj}(A_C - Iz))) = s.$$

The *codegree* of a polynomial $f(z)$ is the smallest n such that the coefficient of z^n is nonzero. The coefficients a_N, a_{N-1}, and a_0 are $(-1)^N$, $(-1)^{N-1}\mathrm{Tr}(A_C)$, and $|A_C|$, respectively, and b_{N-1} is $(-1)^{N-1}N$. Hence, the codegree of $|A_C - Iz|$ is less than or equal to $N-1$ if C is a nonempty category, since $\mathrm{Tr}(A_C) \geq N$.

Remark 3.1. The category C has Euler characteristic if and only if $s \geq r$. In this case, we have

$$\chi_\Sigma(C) = \frac{b_r}{a_r}.$$

(See the bottom of p. 46 in [BL08].)

Lemma 3.2. *If C has Euler characteristic, then we have*

$$\deg\big(\mathrm{sum}(\mathrm{adj}(I - A_C z)A_C)\big) = \deg|I - A_C z| - 1 = N - r - 1.$$

Proof. Lemma 2.2 of [NogA] and Remark 3.1 imply this result. ◻

To finish this subsection, we prepare some symbols that are needed to state our main theorem.

Suppose that the characteristic polynomial $|A_C - Iz|$ is factored as follows:

$$|A_C - Iz| \quad = \quad z^r a_N (z-\lambda_1)^{e_1}(z-\lambda_2)^{e_2}\cdots(z-\lambda_n)^{e_n}$$

where $e_i \geq 1$ for any i and $\lambda_i \neq \lambda_j$ if $i \neq j$. Namely, each λ_k is a nonzero eigenvalue of A_C and e_k is its algebraic multiplicity. Then, $|I - A_C z|$ is factored as follows:

$$|I - A_C z| \quad = \quad (-1)^N a_r \left(z - \frac{1}{\lambda_1}\right)^{e_1}\left(z - \frac{1}{\lambda_2}\right)^{e_2}\cdots\left(z - \frac{1}{\lambda_n}\right)^{e_n}.$$

Suppose that

$$\mathrm{sum}(\mathrm{adj}(I - A_C z)A_C) = q(z)|I - A_C z| + r(z),$$

where $r(z)$ and $q(z)$ are polynomials with $\mathbb{Z}$-coefficients and

$$\deg r(z) < \deg|I - A_C z|.$$

Then, $\frac{r(z)}{|I - A_C z|}$ has a partial fraction decomposition to the following form:

$$\frac{r(z)}{|I - A_C z|} = \frac{(-1)^N}{a_r}\sum_{k=1}^{n}\sum_{i=1}^{e_k}\frac{A_{k,i}}{(z - \frac{1}{\lambda_k})^i} \tag{1}$$

for some complex numbers $A_{k,i}$.

3.2 A proof of our main theorem

In this subsection, we give a proof of our main theorem. The symbols without explanations are explained in the previous subsection.

Theorem 3.3. *Suppose that* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the nonzero eigenvalues of* A_C *and* $e_1, e_2, \ldots, e_n$ *are their algebraic multiplicities. Then,*

1. *the zeta function of* C *is*

$$\zeta_C(z) = \prod_{k=1}^{n} \frac{1}{(1-\lambda_k z)^{\beta_{k,0}}} \exp\left(Q(z) + \sum_{j=1}^{e_k-1} \frac{\beta_{k,j} z^j}{j(1-\lambda_k z)^j} \right),$$

where $\beta_{k,0} = (-1)^{N-1} \frac{A_{k,1}}{a_r}$,

$$\beta_{k,j} = \frac{(-1)^{N-1}}{a_r} \sum_{i=j}^{e_k-1} \binom{i-1}{j-1} (-1)^i \lambda_k^{i+j} A_{k,i+1}$$

for $j \geq 1$, *and* $Q(z) = \frac{1}{n} \int q(z)\, dz$ *is a polynomial of* $\mathbb{Q}[z]$ *whose constant term is zero, and*

2. *each* λ_k *is an algebraic integer.*

To prove this theorem, we use the following proposition.

Proposition 3.4 (Proposition 2.1 of [NogA]). *Let* C *be a finite category. Then, the zeta function of* C *is*

$$\zeta_C(z) = \exp\left(\int \frac{\mathrm{sum}(\mathrm{adj}(I - A_C z) A_C)}{|I - A_C z|} dz \right).$$

Proposition 2.1 of [NogA] assumes the invertibility of A_C, but that hypothesis is not used in the proof. Hence, we can abandon that hypothesis, and the same proof can be used for this proposition.

Proof of Theorem 3.3. Proposition 3.4 implies

$$\begin{aligned}
\zeta_C(z) &= \exp\left(\int q(z)\ dz + \int \frac{(-1)^N}{a_r} \sum_{k=1}^{n}\sum_{i=1}^{e_k} \frac{A_{k,i}}{(z-\frac{1}{\lambda_k})^i}\ dz\right) \\
&= \exp\left(\int q(z)\ dz + \frac{(-1)^N}{a_r}\int \sum_{k=1}^{n} \frac{A_{k,1}}{(z-\frac{1}{\lambda_k})}\ dz + \right.\\
&\quad \left.\frac{(-1)^N}{a_r}\int \sum_{k=1}^{n}\sum_{i=2}^{e_k} \frac{A_{k,i}}{(z-\frac{1}{\lambda_k})^i}\ dz\right) \\
&= \exp\left(Q(z) + \frac{(-1)^N}{a_r}\sum_{k=1}^{n} A_{k,1}\,\mathrm{Log}\left(z-\frac{1}{\lambda_k}\right) + \right.\\
&\quad \left.\frac{(-1)^N}{a_r}\sum_{k=1}^{n}\sum_{i=2}^{e_k} -\frac{A_{k,i}}{(i-1)}\frac{1}{(z-\frac{1}{\lambda_k})^{i-1}} + B\right) \\
&= \prod_{k=1}^{n} \frac{1}{(z-\frac{1}{\lambda_k})^{(-1)^{N-1}\frac{A_{k,1}}{a_r}}} \times \\
&\quad \exp\left(Q(z) + \frac{(-1)^{N-1}}{a_r}\sum_{k=1}^{n}\sum_{i=1}^{e_k-1} \frac{A_{k,i+1}}{i(z-\frac{1}{\lambda_k})^i}\right)B' \\
&= \prod_{k=1}^{n} \frac{1}{(-\frac{1}{\lambda_k})^{(-1)^{N-1}\frac{A_{k,1}}{a_r}}(1-\lambda_k z)^{(-1)^{N-1}\frac{A_{k,1}}{a_r}}} \times \\
&\quad \exp\left(Q(z) + \frac{(-1)^{N-1}}{a_r}\sum_{k=1}^{n}\sum_{i=1}^{e_k-1} \frac{A_{k,i+1}}{i(z-\frac{1}{\lambda_k})^i}\right)B' \\
&= \prod_{k=1}^{n} \frac{1}{(1-\lambda_k z)^{(-1)^{N-1}\frac{A_{k,1}}{a_r}}} \times \\
&\quad \exp\left(Q(z) + \frac{(-1)^{N-1}}{a_r}\sum_{k=1}^{n}\sum_{i=1}^{e_k-1} \frac{A_{k,i+1}}{i(z-\frac{1}{\lambda_k})^i}\right)B'',
\end{aligned}$$

where we replaced (and will replace) the constant term by B, B', B''.... Lemma 2.7 of [NogA] implies

$$\begin{aligned}
\zeta_C(z) &= \prod_{k=1}^{n} \frac{1}{(1-\lambda_k z)^{(-1)^{N-1}\frac{A_{k,1}}{a_r}}} \times \\
&\quad \exp\left(Q(z) + \frac{(-1)^{N-1}}{a_r}\sum_{k=1}^{n}\sum_{i=1}^{e_k-1} \frac{A_{k,i+1}}{i}\sum_{j=1}^{i} \frac{\binom{i}{j}(-\lambda_k)^i(-z)^j}{(z-\frac{1}{\lambda_k})^j}\right)B'''.
\end{aligned}$$

Here, we use the boundary condition $\zeta_C(0) = 1$. This condition is directly implied by the definition of the zeta function. Hence, we obtain $B''' = 1$. By exchanging $\sum_i$ and $\sum_j$, we have

$$\zeta_C(z) = \prod_{k=1}^{n} \left(\frac{1}{(1-\lambda_k z)^{(-1)^{N-1}\frac{A_{k,1}}{a_r}}} \right) \exp\Bigg(Q(z) +$$

$$\frac{(-1)^{N-1}}{a_r} \sum_{k=1}^{n} \sum_{j=1}^{e_k-1} \frac{z^j}{j(1-\lambda_k z)^j} \left(\sum_{i=j}^{e_k-1} \binom{i-1}{j-1} (-1)^i \lambda_k^{i+j} A_{k,i+1} \right) \Bigg).$$

Hence, we obtain the first result.
Since $(-1)^N|A_C - Iz|$ is a monic polynomial with coefficients in $\mathbb{Z}$, it follows that λ_k is an algebraic integer, so we obtain the second result.

□

Theorem 3.5. *Suppose that C has Euler characteristic and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the nonzero eigenvalues of A_C and $e_1, e_2, \dots, e_n$ are their algebraic multiplicities. Then, we obtain the following results.*

1. *The zeta function of C is*

$$\zeta_C(z) = \prod_{k=1}^{n} \frac{1}{(1-\lambda_k z)^{\beta_{k,0}}} \exp\left(\sum_{j=1}^{e_k-1} \frac{\beta_{k,j} z^j}{j(1-\lambda_k z)^j} \right)$$

where $\beta_{k,0} = (-1)^{N-1}\frac{A_{k,1}}{a_r}$ and

$$\beta_{k,j} = \frac{(-1)^{N-1}}{a_r} \sum_{i=j}^{e_k-1} \binom{i-1}{j-1} (-1)^i \lambda_k^{i+j} A_{k,i+1}$$

for $j \geq 1$.

2. *The sum of all the indexes $\beta_{k,0}$ is the number of objects of C; that is,*

$$\sum_{k=1}^{n} \beta_{k,0} = N.$$

3. *Each λ_k is an algebraic integer.*

4.

$$\sum_{k=1}^{n} \sum_{j=0}^{e_k-1} (-1)^j \frac{\beta_{k,j}}{\lambda_k^{j+1}} = \chi_\Sigma(C) \in \mathbb{Q}. \quad (2)$$

Proof. Since C has Euler characteristic, Lemma 3.2 implies

$$\deg\big(\mathrm{sum}(\mathrm{adj}(I - A_C z)A_C)\big) < \deg |I - A_C z|.$$

Hence, we have $q(z) = 0$ and $r(z) = \mathrm{sum}(\mathrm{adj}(I - A_C z)A_C)$, so we obtain the first result by Theorem 3.3 as $Q(z) = 0$.
By elementary calculation, we have

$$\frac{\mathrm{sum}(\mathrm{adj}(A_C - Iz))}{|A_C - Iz|} = -\frac{N}{z} - \frac{1}{z^2}\frac{\mathrm{sum}(\mathrm{adj}(I - \frac{1}{z}A_C)A_C)}{|I - \frac{1}{z}A_C|}.$$

Since $r(z) = \mathrm{sum}(\mathrm{adj}(I - A_C z)A_C)$, the partial fraction decomposition (1) tells us that this is equal to

$$-\frac{N}{z} + \frac{(-1)^{N-1}}{a_r}\sum_{k=1}^{n}\sum_{i=1}^{e_k}\frac{A_{k,i}z^{i-2}}{(1 - \frac{z}{\lambda_k})^i}.$$

This, in turn, is equal to the Laurent series,

$$\left(\sum_{k=1}^{N}\beta_{k,0} - N\right)\frac{1}{z} + \frac{(-1)^{N-1}}{a_r}\sum_{k=1}^{n}\left(\frac{A_{k,1}}{\lambda_k} + A_{k,2}\right) + \sum_{m=1}^{\infty} c_m z^m,$$

for some complex numbers $c_1, c_2, \ldots$. Since C has Euler characteristic, the rational function $\frac{\mathrm{sum}(\mathrm{adj}(A_C - Iz))}{|A_C - Iz|}$ is defined at zero (see p. 45 of [BL08]), so $\sum_{k=1}^{n}\beta_{k,0} = N$, proving the second result.
We have already shown the third result in Theorem 3.3.
Finally, we show the fourth result. The left hand side of (2) is

$$\begin{aligned}
(2) \quad = \quad & \sum_{k=1}^{n}(-1)^{N-1}\frac{A_{k,1}}{\lambda_k a_r} \\
& + \frac{(-1)^{N-1}}{a_r}\sum_{k=1}^{n}\sum_{j=1}^{e_k-1}(-1)^j\frac{\sum_{i=j}^{e_k-1}\binom{i-1}{j-1}(-1)^i\lambda_k^{i+j}A_{k,i+1}}{\lambda_k^{j+1}} \\
= \quad & \sum_{k=1}^{n}\Bigg((-1)^{N-1}\frac{A_{k,1}}{\lambda_k a_r} \\
& + \frac{(-1)^{N-1}}{a_r}\sum_{j=1}^{e_k-1}\sum_{i=j}^{e_k-1}(-1)^{j+i}\lambda_k^{i-1}\binom{i-1}{j-1}A_{k,i+1}\Bigg) \\
= \quad & \sum_{k=1}^{n}\Bigg((-1)^{N-1}\frac{A_{k,1}}{\lambda_k a_r} \\
& + \frac{(-1)^{N-1}}{a_r}\sum_{i=1}^{e_k-1}(-1)^i\lambda_k^{i-1}A_{k,i+1}\left(\sum_{j=1}^{i}(-1)^j\binom{i-1}{j-1}\right)\Bigg) \\
= \quad & \frac{(-1)^{N-1}}{a_r}\left(\sum_{k=1}^{n}\frac{A_{k,1}}{\lambda_k} + A_{k,2}\right).
\end{aligned}$$

The Laurent series implies

$$\begin{aligned}\chi_{\sum}(C) &= \left.\frac{\mathrm{sum}(\mathrm{adj}(A_C - Iz))}{|A_C - Iz|}\right|_{z=0} \\ &= \frac{(-1)^{N-1}}{a_r}\sum_{k=1}^{n}\left(\frac{A_{k,1}}{\lambda_k} + A_{k,2}\right).\end{aligned}$$

Hence, we obtain the result. □

We give an interpretation of Part 2 and 4 of Theorem 3.5 by residues. Let f be a holomorphic function on the whole complex plane with the exception of finitely many poles $p_1, p_2, \ldots, p_j$. Then the *residue of f at infinity* is defined by

$$\mathrm{Res}(f(z) : \infty) = -\sum_{i=1}^{j}\mathrm{Res}(f(z) : p_i).$$

Corollary 3.6. *If C has Euler characteristic, then we have*

$$\mathrm{Res}\left(\frac{\zeta_C'(z)}{\zeta_C(z)} : \infty\right) = N$$

and

$$\mathrm{Res}\left(z\frac{\zeta_C'(z)}{\zeta_C(z)} : \infty\right) = \chi_{\sum}(C).$$

Proof. By Proposition 3.4, the logarithmic derivative of $\zeta_C(z)$ is

$$\frac{\mathrm{sum}(\mathrm{adj}(I - A_C z)A_C)}{|I - A_C z|}.$$

Lemma 3.2, the partial fraction decomposition (1), and Part 2 of Theorem 3.5 imply the first result. Moreover, by elementary calculation, we have

$$z\frac{\zeta_C'(z)}{\zeta_C(z)} = -N + \frac{(-1)^N}{a_r}\sum_{k=1}^{n}\frac{\frac{A_{k,1}}{\lambda_k} + A_{k,2}}{z - \frac{1}{\lambda_k}} + \sum_{k=1}^{n}\sum_{i=2}^{e_k}\frac{c_{k,i}}{(z - \frac{1}{\lambda_k})^i}$$

for some complex numbers $c_{k,i}$. Hence we obtain

$$\mathrm{Res}\left(z\frac{\zeta_C'(z)}{\zeta_C(z)} : \infty\right) = \frac{(-1)^{N-1}}{a_r}\left(\sum_{k=1}^{n}\frac{A_{k,1}}{\lambda_k} + A_{k,2}\right) = \chi_{\sum}(C).$$

The last equality follows from one of the equations at the bottom of the proof of Theorem 3.5. □

3.3 Examples

In this subsection, we introduce four examples of zeta functions. These are implied, for example, by routine calculations to solve the characteristic polynomial of each adjacency matrix and compute a partial fraction decomposition. Since the calculations are routine, only the results are shown.

Example 3.7. Let C be a finite category whose adjacency matrix is $\begin{pmatrix} 2 & 3 & 5 \\ 2 & 3 & 5 \\ 2 & 1 & 3 \end{pmatrix}$. This is Example 4.7 of [BL08]. The existence of such a category is assured by Lemma 4.1 of [BL08]. Then, $\chi_\Sigma(C)$ is not defined. Its zeta function is

$$\zeta_C(z) = \frac{1}{(1-8z)^{\frac{13}{4}}}.$$

We note that the index is not the number of objects of C; that is, $\frac{13}{4} \neq \frac{12}{4} = 3$. Therefore, we cannot abandon the hypothesis in Theorem 3.5 that C has Euler characteristic.

Example 3.8. Let C be a finite category whose adjacency matrix is $\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 8 & 5 \end{pmatrix}$. This is Example 4.5 of [BL08]. Then, both $\chi_L(C)$ and $\chi_\Sigma(C)$ are defined. Here, χ_L is the *Euler characteristic of a finite category* by Leinster [Lei08]. We have

$$\chi_L(C) = \frac{1}{2}, \chi_\Sigma(C) = \frac{1}{3}.$$

Its zeta function is

$$\zeta_C(z) = \frac{1}{(1-9z)^3}.$$

Note that $\frac{3}{9} = \chi_{\sum}(C)$, but $\frac{3}{9} \neq \chi_L(C)$, so our zeta function does not recover χ_L.

Remark 3.9. Our zeta function also does not recover the L^2*-Euler characteristic* $\chi^{(2)}$ [FLS11], since the zeta function of a finite category does not depend on its composition, but the L^2-Euler characteristic does. Indeed, let C_1 be a one-object category whose set of morphisms is $\{1, f\}$, where $f \circ f = f$, and let C_2 be almost the same category as C_1, with the only difference that $f \circ f = 1$ in C_2. Then, Proposition 2.13 implies that their zeta functions are

$$\zeta_{C_1}(z) = \zeta_{C_2}(z) = \frac{1}{1-2z},$$

but $\chi^{(2)}(C_2) = \frac{1}{2}$ and $\chi^{(2)}(C_1) \neq \frac{1}{2}$ by Example 5.12 and Remark 7.2 of [FLS11].

The zeta functions in the following two examples use nonreal numbers.

Example 3.10. Let C be a finite category whose adjacency matrix is $\begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 6 \\ 1 & 1 & 2 \end{pmatrix}$. Since A_C has an inverse matrix, Theorem 3.2 of [BL08] implies that C has Euler characteristic given by

$$\chi_{\sum}(C) = \mathrm{sum}(A_C^{-1}) = \frac{5}{6}.$$

Let us confirm that this zeta function satisfies the statement of Theorem 3.5.

1. The zeta function is
$$\zeta_C(z) = \frac{1}{(1-6z)^{\frac{125}{37}}(1-iz)^{\frac{-7+5i}{37}}(1+iz)^{\frac{-7-5i}{37}}}.$$
2. The sum of indexes is
$$\frac{125}{37} + \frac{-7+5i}{37} + \frac{-7-5i}{37} = 3.$$
3. The numbers 6 and $\pm i$ are algebraic integers. In particular, they are integers in $\mathbb{Q}(\sqrt{-1})$; that is, they belong to the ring of Gaussian integers $\mathbb{Z}[\sqrt{-1}]$.
4. Moreover, we have
$$\frac{1}{6}\frac{125}{37} + \frac{1}{i}\frac{-7+5i}{37} + \frac{1}{-i}\frac{-7-5i}{37} = \frac{5}{6}.$$

Example 3.11. Let C be a finite category whose adjacency matrix is $\begin{pmatrix} 4 & 7 & 8 \\ 1 & 4 & 5 \\ 1 & 1 & 3 \end{pmatrix}$. Since A_C has an inverse matrix, its Euler characteristic is given by

$$\chi_{\sum}(C) = \mathrm{sum}(A_C^{-1}) = 0.$$

Let us confirm that this zeta function satisfies the statement of Theorem 3.5.

1. The zeta function is
$$\zeta_C(z) = \frac{1}{(1-9z)^{\frac{252}{65}}(1-(1+i)z)^{\frac{-57+i}{130}}(1-(1-i)z)^{\frac{-57-i}{130}}}.$$
2. The sum of indexes is
$$\frac{252}{65} + \frac{-57+i}{130} + \frac{-57-i}{130} = 3.$$
3. The numbers 6 and $1 \pm i$ belong to the ring of Gaussian integers $\mathbb{Z}[\sqrt{-1}]$.
4. Moreover, we have
$$\frac{1}{9}\frac{252}{65} + \frac{1}{1+i}\frac{-57+i}{130} + \frac{1}{1-i}\frac{-57-i}{130} = 0.$$

4 Coverings of small categories

The aim of this section is to prove that for a covering of finite categories, $P : E \to B$, the zeta function of E is that of B to the power of the number of sheets in the covering. Some examples are given in the last subsection of this section.

4.1 Coverings and zeta functions

In this subsection, we show that for a covering of finite categories, $P : E \to B$, the zeta function of E is that of B to the power of the number of sheets in the covering.

Here, let us recall a covering of small categories [BH99].

Let C be a small category. For an object x of C, let $S(x)$ be the set of morphisms of C whose source is x,

$$S(x) = \{f : x \to * \in \mathrm{Mor}(C)\},$$

and let $T(x)$ be the set of morphisms of C whose target is x,

$$T(x) = \{g : * \to x \in \mathrm{Mor}(C)\}.$$

For the rest of this section, we assume that E and B are small categories and B is connected. A functor $P : E \to B$ is a *covering* if the following two restrictions of P are bijections for any object x of E:

$$P : S(x) \longrightarrow S(P(x))$$

$$P : T(x) \longrightarrow T(P(x)).$$

This condition is an analogue of the condition on an unramified covering of graphs (see [ST96]). A functor P is called a *discrete fibration* if the restriction $P : T(x) \longrightarrow T(P(x))$ is a bijection for any object x of E, and P is called a *discrete opfibration* if the restriction $P : S(x) \longrightarrow S(P(x))$ is a bijection for any object x of E. Thus, a functor is a covering if and only if it is both a discrete fibration and a discrete opfibration.

For an object b of B, the inverse image $P^{-1}(b)$ of the restriction of P with respect to objects,

$$P^{-1}(b) = \{x \in \mathrm{Ob}(E) \mid P(x) = b\},$$

is called the *fiber* of b by P. The cardinality of $P^{-1}(b)$ is called the *number of sheets in* P, and it does not depend on the choice of b since the base category B is connected (see Proposition 4.1).

Applying the classifying space functor B to a covering $P : E \to B$, we have the covering space BP in the topological sense (see [Tan]).

There has been much work on coverings of small categories; for example, see [BH99], [CM], and [Tan]. In particular, coverings of groupoids were studied in [May99].

The following proposition was briefly introduced without proof on p. 581 of [BH99]. However, the proposition is very important in this paper, so we give a proof.

Proposition 4.1. *Let $P : E \to B$ be a covering. Then, $P^{-1}(b)$ is bijective to $P^{-1}(b')$ for any objects b and b' of B.*

Proof. It suffices to show that $P^{-1}(b)$ is bijective to $P^{-1}(b')$ if there exists a morphism $f : b \to b'$. Indeed, if this is proven, then for any objects b and b' we have a zig-zag sequence

$$b \longrightarrow b_1 \longleftarrow b_2 \longrightarrow \cdots \longleftarrow b' ,$$

so we obtain

$$P^{-1}(b) \cong P^{-1}(b_1) \cong \cdots \cong P^{-1}(b').$$

Suppose that there exists a morphism $f : b \to b'$. By the definition of a covering, there exist induced functions

$$f_* : P^{-1}(b) \longrightarrow P^{-1}(b'),\ f^* : P^{-1}(b') \longrightarrow P^{-1}(b).$$

Here, $f_*(x)$ is the target x' of the unique morphism $g : x \to x'$ such that $P(g) = f$, and similarly with f^*. It follows immediately from the uniqueness that $f^* f_* = 1$, and similarly with f^* and f_* reversed. Hence, f_* and f^* are inverse to one another. □

Definition 4.2. Let C be a small category and x be an object of C. Then, let $N_m(C)_x$ be the set of chains of morphisms of length m in C and whose target is x:

$$N_m(C)_x = \{\ (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_m} x_m)\ \text{ in } C \mid x_m = x\}.$$

Proposition 4.3. *Let $P : E \to B$ be a covering. Then, $N_m(E)_x$ is bijective to $N_m(B)_b$ for any object b of B, any x of $P^{-1}(b)$, and $m \geq 0$.*

Proof. Given a sequence of morphisms in B,

$$\mathbf{g} = \big(\ b_0 \xrightarrow{g_1} b_1 \xrightarrow{g_2} \cdots \xrightarrow{g_m} b_m = b\ \big),$$

there exists a unique morphism $f_m : x_{m-1} \to x$ such that $P(f_m) = g_m$ since P is a covering. If we repeat this process, we get a unique sequence of morphisms in E,

$$\mathbf{f} = \big(\ x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_m} x_m = x\ \big)$$

such that $P(\mathbf{f}) = \mathbf{g}$. This correspondence gives a bijection between $N_m(E)_x$ and $N_m(B)_b$. □

Proposition 4.4. *Let $P : E \to B$ be a covering and let b be an object of B. Then, $N_m(E)$ is bijective to $P^{-1}(b) \times N_m(B)$ for any $m \geq 0$.*

Proof. Proposition 4.3 implies

$$\begin{aligned} N_m(E) &= \coprod_{x\in \mathrm{Ob}(E)} N_m(E)_x \\ &= \coprod_{b\in \mathrm{Ob}(B)} \coprod_{x\in P^{-1}(b)} N_m(E)_x \\ &\cong \coprod_{b\in \mathrm{Ob}(B)} \coprod_{x\in P^{-1}(b)} N_m(B)_b \\ &\cong P^{-1}(b)\times N_m(B). \end{aligned}$$

□

The following theorem is an analogue of an unproved conjecture of Dedekind and is the main result of this section. The conjecture is that for algebraic number fields $K_1 \subset K_2$, the Dedekind zeta function $\zeta_{K_1}(s)$ of K_1 divides that of K_2 [Waa75]. The graph theoretic analogue of this conjecture was considered in Corollary 1 of §2 of [ST96].

Theorem 4.5. *Let $P : E \to B$ be a covering of finite categories and let b be an object of B. Then, we have*

$$\zeta_E(z) = \zeta_B(z)^{\#P^{-1}(b)}.$$

Proof. Proposition 4.4 and the definition of the zeta function of a finite category directly imply this fact; that is,

$$\begin{aligned} \zeta_E(z) &= \exp\left(\sum_{m=1}^{\infty} \frac{\#N_m(E)}{m} z^m\right) \\ &= \exp\left(\sum_{m=1}^{\infty} \frac{\#P^{-1}(b)\#N_m(B)}{m} z^m\right) \\ &= \zeta_B(z)^{\#P^{-1}(b)}. \end{aligned}$$

□

4.2 Coverings and Euler characteristics

Our main purpose in this section has already been accomplished in Theorem 4.5. Aside from the main topic of this section, we investigate the relationships between coverings and Euler characteristics of categories.

Let $p : X \to Y$ be a topological *fibration*, which is one of the generalized notions of covering spaces (e.g., see [Hat02] and [May99]). Under a suitable hypothesis, we have the formula

$$\chi(X) = \chi(F)\chi(Y),$$

where F is the fiber of p.

A categorical analogue of this formula was considered in [Lei08] and [FLS11]. Proposition 2.8 of [Lei08] is an analogue for Grothendieck fibrations and the Euler characteristic χ_L. Theorems 5.30 and 5.37 of [FLS11] are analogues for isofibrations, coverings of groupoids, and the L^2-Euler characteristic $\chi^{(2)}$.
In this subsection, we consider such analogues for coverings, the Euler characteristic $\chi_{\sum}$, and the Euler characteristic of $\mathbb{N}$-filtered acyclic categories χ_{fil} [Nog11].
Here, we recall the *Euler characteristic of an* $\mathbb{N}$*-filtered acyclic category* [Nog11]. Let A be an acyclic category. We define an order on the set $\text{Ob}(A)$ of objects of A by $x \leq y$ if there exists a morphism $x \to y$. Then, $\text{Ob}(A)$ is a poset; that is, $\text{Ob}(A)$ is acyclic and each hom-set has at most one morphism.

Definition 4.6. Let A be an acyclic category. A functor $\mu : A \to \mathbb{N} \cup \{0\}$ satisfying $\mu(x) < \mu(y)$ for $x < y$ in $\text{Ob}(A)$ is called *an* $\mathbb{N}$*-filtration of* A. A pair (A, μ) is called *an* $\mathbb{N}$*-filtered acyclic category*.

Definition 4.7. Let (A, μ) be an $\mathbb{N}$-filtered acyclic category. For nonnegative integers i and m, let

$$\overline{N_m}(A)_i = \{\mathbf{f} \in \overline{N_m}(A) \mid \mu(t(\mathbf{f})) = i\},$$

where $t(\mathbf{f}) = x_m$ if

$$\mathbf{f} = \left(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_m} x_m\right).$$

Suppose that each $\overline{N_m}(A)_i$ is finite. We define the formal power series $f_\chi(A, \mu)(t)$ over $\mathbb{Z}$ by

$$f_\chi(A, \mu)(t) = \sum_{i=0}^{\infty} (-1)^i \left(\sum_{m=0}^{i} (-1)^m \# \overline{N_m}(A)_i \right) t^i.$$

Then, we define

$$\chi_{\text{fil}}(A, \mu) = f_\chi(A, \mu)|_{t=-1}$$

if $f_\chi(A, \mu)(t)$ is rational and has a nonvanishing denominator at $t = -1$.

We first demonstrate the formula for coverings and the Euler characteristic $\chi_{\sum}$. Propositions 4.3 and 4.4 hold when nerves are *nondegenerate*, which means that we do not use identity morphisms. Let C be a small category and let x and y be objects of C. We define the following symbols:

$$\overline{S}(x) = S(x) \setminus \{1_x\}, \overline{T}(x) = T(x) \setminus \{1_x\},$$

$$\overline{\text{Hom}_C}(x, y) = \begin{cases} \text{Hom}_C(x, y) \setminus \{1_x\} & \text{if } x = y \\ \text{Hom}_C(x, y) & \text{if } x \neq y \end{cases}$$

and

$$\overline{N_m}(C)_x = \{ \left(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_m} x_m\right) \text{ in } C \mid f_i \neq 1, x_m = x\}.$$

Proposition 4.8. *Let $P : E \to B$ be a covering. Then, $\overline{N_m}(E)_x$ is bijective to $\overline{N_m}(B)_b$ for any object b of B, any x of $P^{-1}(b)$, and $m \geq 0$.*

Proof. If we replace the symbols in the proof of Proposition 4.3 by the above symbols with bars, we can use the same proof. Note that for any morphism f of E it follows that f is an identity morphism if and only if $P(f)$ is an identity morphism. □

Proposition 4.9. *Let $P : E \to B$ be a covering and b be an object of B. Then, $\overline{N_m}(E)$ is bijective to $P^{-1}(b) \times \overline{N_m}(B)$ for any $m \geq 0$.*

A *discrete category* consists of only objects and identity morphisms. The fiber of a covering $P : E \to B$ is a discrete category when we regard it as a category.

Proposition 4.10. *Let $P : E \to B$ be a covering of finite categories and let b be an object of B. Then, E has Euler characteristic if and only if B has Euler characteristic. In this case, we have*

$$\chi_{\sum}(E) = \chi_{\sum}(P^{-1}(b))\chi_{\sum}(B).$$

Proof. Theorem 2.2 of [BL08] and Proposition 4.9 imply

$$\begin{aligned} \sum_{m=0}^{\infty} \#\overline{N_m}(E)t^m &= \#P^{-1}(b) \sum_{m=0}^{\infty} \#\overline{N_m}(B)t^m \\ &= \#P^{-1}(b) \frac{\mathrm{sum}(\mathrm{adj}(I - (A_B - I)t))}{|I - (A_B - I)t|}. \end{aligned}$$

Hence, E has Euler characteristic if and only if we can substitute $t = -1$ in

$$\#P^{-1}(b) \frac{\mathrm{sum}(\mathrm{adj}(I - (A_B - I)t))}{|I - (A_B - I)t|}$$

if and only if we can substitute $t = -1$ in

$$\frac{\mathrm{sum}(\mathrm{adj}(I - (A_B - I)t))}{|I - (A_B - I)t|}$$

if and only if B has Euler characteristic. Thus, we have proven the first claim. If E has Euler characteristic, then we have

$$\begin{aligned} \chi_{\sum}(E) &= \#P^{-1}(b)\chi_{\sum}(B) \\ &= \chi_{\sum}(P^{-1}(b))\chi_{\sum}(B). \end{aligned}$$

□

Next, we demonstrate the formula for coverings and the Euler characteristic of $\mathbb{N}$-filtered acyclic categories χ_{fil}.

Proposition 4.11. *Suppose that (A,μ_A) and (B,μ_B) are $\mathbb{N}$-filtered acyclic categories, b_0 is an object of B, and $P : A \to B$ is a covering whose fiber is finite, satisfying $\mu_A(x) = \mu_B(P(x))$ for any object x of A. Then, (A,μ_A) has Euler characteristic $\chi_{\mathrm{fil}}(A,\mu_A)$ if and only if B has Euler characteristic $\chi_{\mathrm{fil}}(B,\mu_B)$. In this case, we have*

$$\chi_{\mathrm{fil}}(A,\mu_A) = \chi_{\mathrm{fil}}(P^{-1}(b_0),\mu)\chi_{\mathrm{fil}}(B,\mu_B)$$

for any $\mathbb{N}$-filtration μ of $P^{-1}(b_0)$.

Proof. We have

$$\mu_A^{-1}(i) = \coprod_{b\in\mu_B^{-1}(i)} P^{-1}(b)$$

for any $i \geq 0$. Propositions 4.1 and 4.8 imply

$$\begin{aligned}
\overline{N_m}(A)_i &= \coprod_{x\in\mu_A^{-1}(i)} \overline{N_m}(A)_x \\
&\cong \coprod_{b\in\mu_B^{-1}(i)} \coprod_{x\in P^{-1}(b)} \overline{N_m}(A)_x \\
&\cong \coprod_{b\in\mu_B^{-1}(i)} P^{-1}(b) \times \overline{N_m}(B)_b \\
&\cong P^{-1}(b_0) \times \overline{N_m}(B)_i .
\end{aligned}$$

Hence, we have

$$\begin{aligned}
f_\chi(A,\mu_A)(t) &= \sum_{i=0}^{\infty}(-1)^i \left(\sum_{m=0}^{i}(-1)^m \#\overline{N_m}(A)_i\right) t^i \\
&= \sum_{i=0}^{\infty}(-1)^i \left(\sum_{m=0}^{i}(-1)^m \#P^{-1}(b_0)\#\overline{N_m}(B)_i\right) t^i \\
&= \#P^{-1}(b_0) f_\chi(B,\mu_B)(t).
\end{aligned}$$

Accordingly, $\chi_{\mathrm{fil}}(A,\mu_A)$ exists if and only if the power series $f_\chi(A,\mu_A)(t)$ is rational and we can substitute $t = -1$ in the rational function if and only if the power series $f_\chi(B,\mu_B)(t)$ is rational and we can substitute $t = -1$ in the rational function if and only if $\chi_{\mathrm{fil}}(B,\mu_B)$ exists. Thus, the first claim has been proven.

If $\chi_{\mathrm{fil}}(A,\mu_A)$ exists, then we have

$$\begin{aligned}
\chi_{\mathrm{fil}}(A,\mu_A) &= \#P^{-1}(b_0)\chi_{\mathrm{fil}}(B,\mu_B) \\
&= \chi_{\mathrm{fil}}(P^{-1}(b_0),\mu)\chi_{\mathrm{fil}}(B,\mu_B).
\end{aligned}$$

It is clear that $\chi_{\mathrm{fil}}(P^{-1}(b_0),\mu) = \#P^{-1}(b_0)$ for any $\mathbb{N}$-filtration μ. We can provide a filtration to $P^{-1}(b_0)$; for example, we can define $\mu : P^{-1}(b_0) \to \mathbb{N}\cup\{0\}$ by $\mu(x) = 0$ for any x of $P^{-1}(b_0)$. □

4.3 Examples

We give three examples of coverings of small categories.

Example 4.12. Let

$$\Gamma = \; x \underset{f^{-1}}{\overset{f}{\rightleftarrows}} y$$

and $B = \mathbb{Z}_2 = \{1, -1\}$. A group can be regarded as a category whose object is just one object (denoted by an asterisk), whose morphisms are the elements of G, and whose composition is the operation of G. We define $P : \Gamma \to B$ by $P(f) = P(f^{-1}) = -1$. Then, P is a covering that was studied in Example 5.33 of [FLS11]. Since Γ and B are finite connected groupoids, Proposition 2.3 implies

$$\zeta_\Gamma(z) = \frac{1}{(1-2z)^2}, \; \zeta_B(z) = \frac{1}{1-2z}.$$

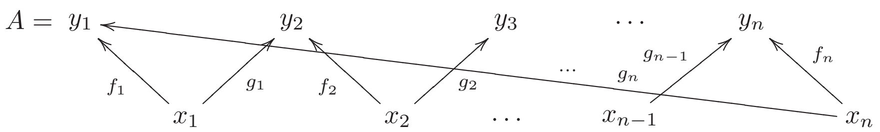
The number of sheets in P is two. We have $\zeta_\Gamma(z) = \zeta_B(z)^2$. Example 2.7 of [Lei08] and Theorem 3.2 of [BL08] imply

$$\chi_\sum(\Gamma) = 1, \chi_\sum(B) = \frac{1}{2}, \chi_\sum(P^{-1}(*)) = 2,$$

and hence we have

$$\chi_\sum(\Gamma) = \chi_\sum(P^{-1}(*))\chi_\sum(B).$$

Example 4.13. Let



and

$$B = \; x \underset{g}{\overset{f}{\rightrightarrows}} y \; .$$

We define a functor $P : A \to B$ by $P(x_i) = x$, $P(y_i) = y$, $P(f_i) = f$, and $P(g_i) = g$ for any i. Then, P is a covering. By Corollary 2.12, we have

$$\zeta_A(z) = \frac{1}{(1-z)^{2n}} \exp\left(\frac{2nz}{1-z}\right), \; \zeta_B(z) = \frac{1}{(1-z)^2} \exp\left(\frac{2z}{1-z}\right).$$

The number of sheets in P is n. We have $\zeta_A(z) = \zeta_B(z)^n$. Since A and B are finite acyclic categories, $\sum_{m=0}^\infty \#\overline{N_m}(A)t^m$ and $\sum_{m=0}^\infty \#\overline{N_m}(B)t^m$ are polynomials by Lemma 3.5 of [Nog]. Hence, we have

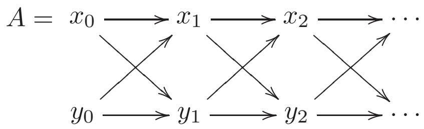
$$\chi_\sum(A) = 2n - 2n = 0, \; \chi_\sum(B) = 2 - 2 = 0, \; \chi_\sum(P^{-1}(x)) = n,$$

and then we have

$$\chi_\sum(A) = \chi_\sum(P^{-1}(x))\chi_\sum(B).$$

We introduce an example of a covering of infinite categories.

Example 4.14. Suppose that



and

$$B = \; b_0 \rightrightarrows b_1 \rightrightarrows b_2 \rightrightarrows \cdots$$

where A is a poset. For $n < m, b_n$ and b_m, we define

$$\mathrm{Hom}_B(b_n, b_m) = \{\varphi^0_{n,m}, \varphi^1_{n,m}\},$$

and a composition of B is defined by $\varphi^j_{m,\ell} \circ \varphi^i_{n,m} = \varphi^k_{n,\ell}$, where $k = 0$ or $k = 1$ and $k \equiv i + j \mod 2$ for $n < m < \ell$. We define $P : A \to B$ by $P(x_i) = P(y_i) = b_i$, with $P((x_n, x_m)) = P((y_n, y_m)) = \varphi^0_{n,m}$ and $P((y_n, x_m)) = P((x_n, y_m)) = \varphi^1_{n,m}$ for $n < m$. Then, P is a covering. The indexes of objects of A and B give $\mathbb{N}$-filtrations μ_A and μ_B to A and B, respectively. We have

$$f_\chi(A, \mu_A)(t) = \sum_{i=0}^{\infty} (-1)^i \left(\sum_{m=0}^{i} (-1)^m 2^{m+1} \binom{i}{m} \right) t^i = \frac{2}{1-t},$$

so $\chi_{\mathrm{fil}}(A, \mu_A) = 1$. We have

$$f_\chi(B, \mu_B)(t) = \sum_{i=0}^{\infty} (-1)^i \left(\sum_{m=0}^{i} (-1)^m 2^{m} \binom{i}{m} \right) t^i = \sum_{i=0}^{\infty} t^i = \frac{1}{1-t},$$

so $\chi_{\mathrm{fil}}(B, \mu_B) = \frac{1}{2}$. We obtain

$$\chi_{\mathrm{fil}}(A, \mu_A) = \chi_{\mathrm{fil}}(P^{-1}(b_0), \mu)\chi_{\mathrm{fil}}(B, \mu_B)$$

for any $\mathbb{N}$-filtration μ of $P^{-1}(b_0)$.
In fact, the category A is the barycentric subdivision of Γ in Example 4.12 and the category B is that of $\mathbb{Z}_2$ (see [Nog11] and [Nog]). Hence, Theorem 4.9 of [Nog11] and Example 4.12 directly imply their Euler characteristics $\chi_{\mathrm{fil}}(A, \mu_A)$ and $\chi_{\mathrm{fil}}(B, \mu_B)$.

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