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L_2 and L_∞ Estimates of the Solutions for the Compressible Navier-Stokes Equations in a 3D Exterior Domain

Dedicated to Professors Takaaki Nishida and Masayasu Mimura on their sixtieth birthdays

By

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Abstract

We consider the boundary value problem of the equation of motion of viscous compressible fluid in a 3D exterior domain. We shall give the L_∞ estimates of the solutions and L_2-estimates of the derivative with respect to space variable of the solutions.

§1. Introduction

In this paper, we consider the equation of the motion of compressible viscous fluid in a 3D exterior domain. The equation is given by the following system for the density $\rho(t,x)$ and the velocity $v(t,x) = (v_1(t,x), v_2(t,x), v_3(t,x))$,

(1.1)
$$\begin{aligned}
&\rho_t + \nabla\cdot(\rho v) = 0 && \text{in} \quad (0,\infty)\times\Omega,\\
&\rho(v_t + (v\cdot\nabla)v) + \nabla P(\rho) = \mu\Delta v + (\mu+\nu)\nabla(\nabla\cdot v) && \text{in} \quad (0,\infty)\times\Omega,\\
&v|_{\partial\Omega} = 0 && \text{on} \quad (0,\infty)\times\partial\Omega,\\
&\rho(0,x) = \rho_0(x), \quad v(0,x) = v_0(x) && \text{in} \quad \Omega,
\end{aligned}$$

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where Ω is an exterior domain in $\mathbf{R}^3$ with the compact smooth boundary $\partial\Omega$, $P = P(\rho)$ the pressure, $\mu \geq 0, (2/3)\mu + \nu > 0$ the viscosity coefficients. The unique existence of smooth solutions globally in time near constant state $(\bar{\rho}_0, 0)$, where $\bar{\rho}_0$ is a positive constant, was proved by the employing the same argument as in Matsumura and Nishida [11], [12] for the Cauchy problem in $\mathbf{R}^3$; Matsumura and Nishida [13], [14], [15] for the exterior domain in $\mathbf{R}^3$. Concerning the decay property of solutions $(\rho(t,x), v(t,x))$, if the initial data $(\rho_0(x) - \bar{\rho}_0, v_0(x))$ belongs to H^4 and L_1, then as $t \to \infty$

$$\begin{aligned}
\|(\rho(t,\cdot) - \bar{\rho}_0, v(t,\cdot))\|_{L_\infty} &= O(t^{-3/2}),\\
\|(\rho(t,\cdot) - \bar{\rho}_0, v(t,\cdot))\|_{L_2} &= O(t^{-3/4}),\\
\|(\rho(t,\cdot) - \bar{\rho}_0, v(t,\cdot))\|_{L_2} &= O(t^{1/2}).
\end{aligned}$$

This fact was investigated by Hoff and Zumbrun [3], [4], Liu and Wang [9], Matsumura and Nishida [11], [12], Ponce [16] and Weike [17] for the Cauchy problem case; Kobayashi [7], Kobayashi and Shibata [8] for the exterior domain case. On the other hand, if the initial data belongs to H^3 only, namely we do not assume that the initial data belongs to L_1, then, Deckelnick [1], [2] showed that as $t \to \infty$

$$\begin{aligned}
\|(\rho_t(t,\cdot), v_t(t,\cdot))\|_{L_2(\Omega)} &= O(t^{-1/2}),\\
\|\partial_x(\rho(t,\cdot), v(t,\cdot))\|_{L_2(\Omega)} &= O(t^{-1/4}),\\
\|(v(t,\cdot))\|_{C^0(\bar{\Omega})} &= O(t^{-1/4}),\\
\|\rho(t,\cdot) - \bar{\rho}_0\|_{C^0(\bar{\Omega})} &= O(t^{-1/8}),
\end{aligned}$$

in the exterior domain case; Matsumura [10] showed that as $t \to \infty$

$$\begin{aligned}
\|(\rho_t(t,\cdot), v_t(t,\cdot))\|_{L_2(\mathbf{R}^3)} &= O(t^{-1/2}),\\
\|\partial_x(\rho(t,\cdot), v(t,\cdot))\|_{L_2(\mathbf{R}^3)} &= O(t^{-1/2}),\\
\|\partial_x^2(\rho(t,\cdot), v(t,\cdot))\|_{L_2(\mathbf{R}^3)} &= O(t^{-1}),\\
\|(\rho(t,\cdot) - \bar{\rho}_0, v(t,\cdot))\|_{L_\infty(\mathbf{R}^3)} &= O(t^{-3/4}),
\end{aligned}$$

in the Cauchy problem case. In this paper, we shall investigate the exterior problem of the system (1.1) and give the better decay rate than the rate obtained by Deckelnick [1], [2] in the case that the initial data belongs to H^3 or H^4 only. In particular, the L_2-decay rate of the first derivative with respect to the spacial variable x for the solutions corresponds to the rate obtained by Matsumura [10].

§2. Notation and Main Results

Let L_p denotes the usual L_p space on Ω with norm $\|\cdot\|_{L_p}$. Put

$$W_p^m = \{u \in L_p \mid \|u\|_{W_p^m} < \infty\}, \quad \|u\|_{W_p^m} = \sum_{|\alpha|\le m} \|\partial_x^\alpha u\|_{L_p},$$
$$H^m = W_2^m, \quad W_p^0 = L_p, \quad H^0 = L_2.$$

Set

$$W_p^{k,m} = \{(\rho, v) = (\rho, v_1, v_2, v_3) \mid \rho \in W_p^k,\ v_j \in W_p^m,\ j = 1, 2, 3\},$$
$$\|(\rho, v)\|_{W_p^{k,m}} = \|\rho\|_{W_p^k} + \|v\|_{W_p^m},$$

and

$$H^{k,m} = W_2^{k,m}, \quad \|u\|_{H^{k,m}} = \|u\|_{W_2^{k,m}}.$$

First, in order to state the existence of solutions according to Matsumura and Nishida [13], [14], [15], we introduce the assumptions and notations. Let $\bar\rho_0$ be a positive constant. We assume that

A1. P is a smooth function in a neighborhood of $\bar\rho_0$ and $\partial P/\partial\rho > 0$.

A2. The initial data (ρ_0, v_0) satisfies the compatibility condition and regularity, namely $(\rho_0 - \bar\rho_0, v_0) \in H^3$, $v_0|_{\partial\Omega} = 0$ and $(\rho_1, v_1) = (\rho_t, v_t)|_{t=0}$ satisfies

$$\begin{aligned}
&\rho_1 = -\nabla\cdot(\rho_0 v_0),\\
&v_1 = -(v_0\cdot\nabla)v_0 + \frac{\mu}{\rho_0}\Delta v_0 + \frac{\nu}{\rho_0}\nabla(\nabla\cdot v_0) - \frac{\nabla P(\rho_0)}{\rho_0},
\end{aligned}$$

and

$$\rho_1 \in H^2, \quad v_1 \in H^1, \quad v_1|_{\partial\Omega} = 0.$$

Put

$$\begin{aligned}
X(0,\infty) = \Bigg\{ & U = (\rho, v) \mid \rho - \bar\rho_0 \in \bigcap_{j=0}^{1} C^j([0,\infty); H^{3-j}),\\
& \partial_x\rho \in L_2((0,\infty); H^2), \quad \rho_t,\ v_t \in L_2((0,\infty); H^2),\\
& v \in \bigcap_{j=0}^{1} C^j([0,\infty); H^{3-2j}), \quad \partial_x v \in L_2((0,\infty); H^3) \Bigg\},
\end{aligned}$$

and

$$\begin{aligned}
N(0,\infty)^2 = &\sup_{0\le t<\infty} \left(\|U(t) - \bar U_0\|_{H^3}^2 + \|U_t(t)\|_{H^{2,1}}^2\right)\\
&+ \int_0^\infty \left(\|\partial_x U(s)\|_{H^{2,3}}^2 + \|U_s(s)\|_{H^2}^2\right) ds,
\end{aligned}$$

where $\bar U_0 = (\bar\rho_0, 0)$. Then, we have

Proposition 2.1 (*Matsumura and Nishida* [13], [14], [15]). *Assume that the assumptions* A.1 *and* A.2 *hold. Then, there exists an* ϵ_0 *such that if* $\|(\rho_0-\bar\rho_0, v_0)\|_{H^3} \le \epsilon_0$, *then* (1.1) *admits a unique solution* $(\rho, v) \in X(0,\infty)$.

Moreover, there exists a constant C *such that*

$$N(0,\infty) \le C\|(\rho_0-\bar\rho_0, v_0)\|_{H^3}. \tag{2.1}$$

Remark. If the initial data $(\rho_0-\bar\rho_0, v_0) \in H^4$ and satisfies the second order compatibility condition and regularity, namely $(\rho_2, v_2) = (\rho_{tt}, v_{tt})|_{t=0}$ is determined successively by the initial data (ρ_0, v_0) through the system (1.1), then we have

$$\begin{aligned}\tilde N(0,\infty)^2 &= \sup_{0\le t<\infty}\left(\|U(t)-\bar U_0\|^2_{H^4} + \|U_t(t)\|^2_{H^{3,2}}\right)\\ &\quad + \int_0^\infty \left(\|\partial_x U(s)\|^2_{H^{3,4}} + \|U_s(s)\|^2_{H^3}\right) ds,\\ &\le C\|(\rho_0-\bar\rho_0, v_0)\|^2_{H^4}.\end{aligned} \tag{2.2}$$

Now, we shall state our main results.

Theorem 2.1. *Assume that the assumptions* A.1 *and* A.2 *hold. Then, there exists an* ϵ_1 *such that if* $\|(\rho_0-\bar\rho_0, v_0)\|_{3,2} \le \epsilon_1$, *the solution* (ρ, v) *of the system* (1.1) *has the following asymptotic behavior as* $t \to \infty$:

$$\begin{aligned}\|(\rho_t(t,\cdot), v_t(t,\cdot))\|_{L_2} &= O(t^{-1/2}),\\ \|\partial_x v(t,\cdot)\|_{H^1} &= O(t^{-1/2}),\\ \|\partial_x \rho(t,\cdot)\|_{L_2} &= O(t^{-1/2}),\\ \|\partial_x^2 \rho(t,\cdot)\|_{L_2} &= O(t^{-3/4}\log t),\\ \|(\rho(t,\cdot)-\bar\rho_0, v(t,\cdot))\|_{L_\infty} &= O(t^{-3/4}\log t).\end{aligned}$$

Corollary 2.1. *The assumptions in Theorem* 2.1 *hold. Moreover, if the initial data* $(\rho_0-\bar\rho_0, v_0) \in H^4$ *satisfies the second order compatibility condition and regularity in Remark, then we have*

$$\|(\rho(t,\cdot)-\bar\rho_0, v(t,\cdot))\|_{L_\infty} = O(t^{-3/4}) \quad as \quad t\to\infty.$$

§3. Linearized Problem

In this section, we shall consider the following linearized problem associated to the problem (1.1) (see Section 4)

$$\begin{aligned} \rho_t + \gamma\nabla\cdot v &= 0 \quad \text{in} \quad [0,\infty)\times\Omega,\\ v_t - \alpha\Delta v - \beta\nabla(\nabla\cdot v) + \gamma\nabla\rho &= 0 \quad \text{in} \quad [0,\infty)\times\Omega,\\ v|_{\partial\Omega} &= 0 \quad \text{on} \quad [0,\infty)\times\partial\Omega,\\ \rho(0,x) = \rho_0(x), \quad v(0,x) &= v_0(x) \quad \text{in} \quad \Omega,\end{aligned} \tag{3.1}$$

where $\alpha > 0, \beta \geq 0$ and $\gamma > 0$. Let A be the 4×4 matrix of the differential operator of the form:

$$A = \begin{pmatrix} 0 & \gamma\nabla \\ \gamma\nabla & -\alpha\Delta - \beta\nabla\nabla \end{pmatrix}$$

with the domain:

$$D_p(A) = \{U = (\rho, v) \in W_p^{1,2} \mid \quad v|_\Omega = 0\}$$

for $1 < p < \infty$. Then, (3.1) is written in the form:

$$U_t + AU = 0 \quad \text{for} \quad t > 0, \quad U|_{t=0} = U_0,$$

where $U_0 = (\rho_0, v_0)$ and $U = (\rho, v)$. Then,

Proposition 3.1 (*Kobayashi* [5], [6], [7], *Kobayashi and Shibata* [8]). *The operator $-A$ generates an analytic semigroup $\{e^{-tA}\}_{t\geq 0}$ on $W_p^{1,0}$, $1 < p < \infty$ and the following properties hold.*

(I) *Let $1 < p < \infty$. For $0 < t \leq 2$, we have*

$$\|e^{-tA}U\|_{W_p^{1,0}} \leq C\|U\|_{W_p^{1,0}} \qquad \textit{for} \ \ U \in W_p^{1,0}, \tag{3.2}$$

$$\|e^{-tA}U\|_{W_p^1} \leq Ct^{-1/2}\|U\|_{W_p^{1,0}} \quad \textit{for} \ \ U \in W_p^{1,0}, \tag{3.3}$$

$$\|(\mathbf{I}-\mathbf{P})e^{-tA}U\|_{W_p^2} \leq Ct^{-1/2}\|U\|_{W_p^{2,1}} \quad \textit{for} \ \ U \in W_p^{2,1}. \tag{3.4}$$

Here and hereafter, we shall use the notations:

$$\mathbf{I}U = U, \quad \mathbf{P}U = v \quad \textit{and} \quad (\mathbf{I}-\mathbf{P})U = \rho \quad \textit{for} \quad U = (\rho, v).$$

(II) *Let $1 \leq q \leq 2 \leq p < \infty$. For $U \in W_p^{1,0} \cap L_q$ and $t \geq 1$*

(3.5)

$$\|e^{-tA}U\|_{L_p} \leq Ct^{-\sigma}\left(\|U\|_{L_q} + \|U\|_{W_p^{1,0}}\right), \quad 2 \leq p < \infty, \quad \sigma = \frac{3}{2}\left(\frac{1}{q} - \frac{1}{p}\right),$$

(3.6)

$$\|\partial_t e^{-tA}U\|_{L_p} + \|\partial_x e^{-tA}U\|_{L_p} \leq Ct^{-\sigma-\frac{1}{2}}\left(\|U\|_{L_q} + \|U\|_{W_p^{1,0}}\right), \quad 2 \leq p \leq 3,$$

(3.7)

$$\|e^{-tA}U\|_{W_\infty^{0,1}} \leq Ct^{-\frac{3}{2q}}\left(\|U\|_{L_q} + \|U\|_{W_p^{1,0}}\right), \quad 3 < p < \infty,$$

where $\partial_t = d/dt$ and $\partial_x^m u = (\partial_x^\alpha u \mid |\alpha| = m)$. Moreover, if $q > 1$ and $U \in W_p^{2,1} \cap W_q^{1,0}$, then for $t \leq 1$

(3.8)

$$\|\partial_x^2(\mathbf{I}-\mathbf{P})e^{-tA}U\|_{L_p} + \|\partial_t\partial_x e^{-tA}U\|_{L_p} \leq Ct^{-\frac{3}{2q}}\left(\|U\|_{W_q^{1,0}} + \|U\|_{W_p^{2,1}}\right),$$
$$2 \leq p < \infty.$$

§4. Proof of Theorem 2.1

First of all, we shall introduce the linearized equations. By the change of unknown functions: $(\rho, v) \to (\rho + \bar{\rho}_0, v)$, (1.1) is reduced to the following equation:

$$\begin{aligned} &\rho_t + \bar{\rho}_0 \nabla \cdot v = f_1 \\ &v_t - \hat{\mu}\Delta v - (\hat{\mu} + \hat{\nu})\nabla(\nabla \cdot v) + p_1 \nabla \rho = f_2, \\ &\rho(0,x) = \rho_0(x) - \bar{\rho}_0, \quad v(0,x) = v_0(x), \end{aligned} \tag{4.1}$$

where $\hat{\mu} = \mu/\bar{\rho}_0, \hat{\nu} = \nu/\bar{\rho}_0, p_1 = P_\rho(\bar{\rho}_0)/\bar{\rho}_0$,

$$\begin{aligned} f_1 &= -\rho\nabla \cdot v - \nabla\rho \cdot v, \\ f_2 &= -(v \cdot \nabla)v + \left(\frac{\mu}{\rho + \bar{\rho}_0} - \hat{\mu}\right)\Delta v + \left(\frac{\mu + \nu}{\rho + \bar{\rho}_0} - \hat{\mu} - \hat{\nu}\right)\nabla(\nabla \cdot v), \\ &\quad + \left(p_1 - \frac{P_\rho(\rho)}{\rho + \bar{\rho}_0}\right)\nabla\rho. \end{aligned} \tag{4.2}$$

If we put $\rho' = (p_1/\bar{\rho}_0)^{1/2}\rho$ and $v' = v$, then (4.1) is reduced to the symmetric form

$$\begin{aligned} &\rho'_t + \gamma\nabla \cdot v' = f'_1 \\ &v'_t - \alpha\Delta v' - \beta\nabla(\nabla \cdot v') + p_1\nabla\rho' = f_2, \\ &\rho'(0,x) = \rho'_0, \quad v'(0,x) = v_0(x), \end{aligned}$$

where $\alpha = \hat{\mu}, \beta = \hat{\mu} + \hat{\nu}$ and $\gamma = \sqrt{P_\rho(\bar{\rho}_0)}$. For the notational simplicity, we write: $\rho = \rho', v = v', f_1 = f'_1$, again. If we put $U = (\rho, v), U_0 = (\rho_0, v_0), F(U) = (f_1, f_2)$ and

$$A = \begin{pmatrix} 0 & \gamma\nabla \\ \gamma\nabla & -\alpha\Delta - \beta\nabla\nabla \end{pmatrix}$$

then (1.1) is reduced to the following equations

$$\begin{aligned} U_t + AU &= F(U), \\ U(0) &= U_0. \end{aligned} \tag{4.3}$$

Here $F(U) = (f_1, f_2)$ is written as follows:

$$\begin{aligned} f_1 &= -\frac{\gamma}{\bar{\rho}_0}(\rho\nabla \cdot v + \nabla\rho \cdot v), \\ f_2 &= -(v \cdot \nabla)v + a_1(\rho)\rho\Delta v + a_2(\rho)\rho\nabla(\nabla \cdot v) + a_3(\rho)\rho\nabla\rho, \end{aligned}$$

where $a_j(\rho)\,(j = 1,2,3)$ represent identities (4.2). To prove our main results, we shall estimate the following integral equation

$$U(t) = e^{-tA}U_0 - S(t), \quad S(t) = \int_0^t e^{-(t-s)A}F(U)(s)ds.$$

Let $N(0,\infty)$ be the quantity defined in Section 2. By choosing $\|(\rho_0 - \bar{\rho}_0, v_0)\|_{H^3}$ small enough, we can make $N(0,\infty)$ as small we want, and therefore we will state the smallness assumption in term of $N(0,\infty)$ instead of $\|(\rho_0 - \bar{\rho}_0, v_0)\|_{H^3}$ in the course of our proof of Theorem 2.1 below.

Step 1. Put

$$\begin{aligned} M_1(t) &= \sup_{0\le s\le t}(1+s)^{\frac{1}{2}}\|\partial_s U(s)\|_{L_2}, \\ M_2(t) &= \sup_{0\le s\le t}(1+s)^{\frac{1}{2}}\|\partial_x U(s)\|_{H^1}, \\ M_3(t) &= \sup_{0\le s\le t}(1+s)^{\frac{1}{2}}\|U(s)\|_{L_\infty}. \end{aligned}$$

Then, there exists an $\epsilon > 0$ such that if $N(0,\infty) \le \epsilon$, then

$$M_1(t) + M_2(t) + M_3(t) \le C\|U_0\|_{H^2}.$$

First, we shall show that

$$M_1(t) \le C((M_2(t) + M_3(t))N(0,\infty) + \|U_0\|_{H^{1,0}}), \tag{4.4}$$

$$\sup_{0\le s\le t}(1+s)^{1/2}\|\partial_x U(s)\|_{L_2} \le C((M_2(t) + M_3(t))N(0,\infty) + \|U_0\|_{H^{1,0}}), \tag{4.5}$$

$$M_3(t) \le C((M_2(t) + M_3(t))N(0,\infty) + \|U_0\|_{H^2}). \tag{4.6}$$

When $0 \le t \le 2$, by Proposition 2.1 and the inequality

$$\|u\|_{L_p} \le C\|u\|_{H^2}, \quad 6 < p \le \infty, \tag{4.7}$$

we have

$$M_1(t) + M_2(t) + M_3(t) \le CN(0,\infty), \tag{4.8}$$

and therefore we consider the case when $t \ge 1$, below. By (3.6) with $(p,q) = (2,2)$

$$\|\partial_t e^{-tA}U_0\|_{L_2} + \|\partial_x e^{-tA}U_0\|_{L_2} \le Ct^{-1/2}\|U_0\|_{H^{1,0}}, \tag{4.9}$$

and by (3.7) with $(p,q) = (4,2)$ and the inequalities:

$$\|u\|_{L_p} \le C\|u\|_{H^1} \quad (2 \le p < 6) \quad \text{and} \quad \|u\|_{L_6} \le C\|\partial_x u\|_{L_2}, \tag{4.10}$$

we have

$$\|e^{-tA}U_0\|_{L_\infty} \le Ct^{-3/4}\|U_0\|_{H^2}. \tag{4.11}$$

The main task is the estimation of $S(t)$, which is divided into the two parts as follows:

$$S(t) = \left\{\int_{t-1}^{t} + \int_{0}^{t-1}\right\} e^{-(t-s)A}F(U)(s)\,ds = I(t) + II(t).$$

Before going further on the proof of (4.4), (4.5) and (4.6) we prepare the estimates of nonlinear term $F(U)$: By (4.7), (4.10) and the Hölder's inequality, we have

$$\begin{aligned} \|F(U)(s)\|_{L_1} &\le C\|U\|_2\|\partial_x U(s)\|_{H^1}, \\ \|F(U)(s)\|_{H^{2,1}} &\le C\left(\|U(s)\|_{L_\infty} + \|\partial_x U(s)\|_{H^1}\right)\|\partial_x U(s)\|_{H^2}, \\ \|F(U)(s)\|_{W_4^{1,0}} &\le C\left(\|U(s)\|_{L_\infty} + \|\partial_x U(s)\|_{H^1}\right)\|\partial_x U(s)\|_{H^2}, \\ \|\partial_s F(U)(s)\|_{H^{1,0}} &\le C\left(\|U(s)\|_{L_\infty} + \|\partial_x U(s)\|_{H^1}\right)\|\partial_s U(s)\|_{H^2}. \end{aligned} \tag{4.12}$$

Now, we return to estimate $S(t)$. By (3.6) with $(p,q) = (2,1)$ and (4.12), we have

$$\begin{aligned} &\|\partial_t II(t)\|_{L_2} + \|\partial_x II(t)\|_{L_2} \\ &\le C\int_0^{t-1}(t-s)^{-5/4}\left(\|F(U)(s)\|_{L_1} + \|F(U)(s)\|_{H^{1,0}}\right)\,ds \\ &\le C\int_0^{t-1}(t-s)^{-5/4}(1+s)^{-1/2}\,ds\,(M_2(t)+M_3(t))\,N(0,\infty) \\ &\le C(1+t)^{-1/2}\,(M_2(t)+M_3(t))\,N(0,\infty). \end{aligned} \tag{4.13}$$

On the other hand, by (3.3) and (4.12),

$$\begin{aligned} \|\partial_x I(t)\|_{L_2} &\le C\int_{t-1}^{t}(t-s)^{-1/2}\|F(U)(s)\|_{H^{1,0}}\,ds \\ &\le C\int_{t-1}^{t}(t-s)^{-1/2}(1+s)^{-1/2}\,ds\,(M_2(t)+M_3(t))\,N(0,\infty) \\ &\le C(1+t)^{-1/2}\,(M_2(t)+M_3(t))\,N(0,\infty). \end{aligned} \tag{4.14}$$

Combining (4.8), (4.9), (4.13) and (4.14), we have (4.5). By integration by parts,

$$\partial_t I(t) = \int_{t-1}^{t} e^{-(t-s)A}\partial_s F(U)(s)\,ds$$

and therefore by (3.2) and (4.12),

$$
\begin{aligned}
(4.15)\quad \|\partial_t I(t)\|_{L_2} &\leq C\int_{t-1}^{t} \|\partial_s F(U)(s)\|_{H^{1,0}}\, ds \\
&\leq C(1+t)^{-1/2}\left(\int_{t-1}^{t} \|\partial_s U(s)\|_{H^2}^2\, ds\right)^{1/2} (M_2(t)+M_3(t)) \\
&\leq C(1+t)^{-1/2}\,(M_2(t)+M_3(t))\,N(0,\infty).
\end{aligned}
$$

Combining (4.8), (4.9), (4.13) and (4.15), we have (4.4). By (3.7) with $(p,q) = (4,1)$ and (4.12),

$$
\begin{aligned}
(4.16)\ \|II(t)\|_{L_\infty} &\leq C\int_0^{t-1} (t-s)^{-3/2}\left(\|F(U)(s)\|_{L_1} + \|F(U)(s)\|_{W_4^{1,0}}\right) ds \\
&\leq C\int_0^{t-1} (t-s)^{-3/2}(1+s)^{-1/2}\, ds(M_2(t)+M_3(t))N(0,\infty) \\
&\leq C(1+t)^{-1/2}(M_2(t)+M_3(t))N(0,\infty).
\end{aligned}
$$

By (3.3), (4.12) and the Sobolev inequality

$$
\begin{aligned}
(4.17)\quad \|I(t)\|_{L_\infty} &\leq C\|I(t)\|_{W_4^1} \\
&\leq C\int_{t-1}^{t} (t-s)^{-1/2}\|F(U)(s)\|_{W_4^{1,0}}\, ds \\
&\leq C\int_{t-1}^{t} (t-s)^{-1/2}(1+s)^{-1/2}\, ds(M_2(t)+M_3(t))N(0,\infty) \\
&\leq C(1+t)^{-1/2}(M_2(t)+M_3(t))N(0,\infty).
\end{aligned}
$$

Combining (4.8), (4.11), (4.16) and (4.17), we have (4.6). Next, we shall show that

$$
(4.18)\quad \sup_{0\leq s\leq t}(1+s)^{1/2}\|\partial_x^2 U(t)\|_{L_2} \leq C((M_2(t)+M_3(t))N(0,\infty) + \|U_0\|_{H^2}).
$$

In order to prove (4.18), we shall use the following Proposition (see [8, Proposition A, p. 3])

Proposition 4.1. *Let b be an arbitrary number such that $B_{b-3} = \{x \in \mathbf{R}^3 \,|\, |x| \leq b-3\} \supset \partial\Omega$. Let $1 < p < \infty$ and m be an integer ≥ 0. Suppose that $u = (u_1, u_2, u_3) \in W_p^{m+2}(\Omega)$ and $f = (f_1, f_2, f_3) \in W_p^m(\Omega)$ satisfy the equation*:

$$
-\alpha\Delta u - \beta\nabla(\nabla\cdot u) = f \quad in \quad \Omega \quad and \quad u|_{\partial\Omega} = 0,
$$

where $\alpha > 0$ and $\alpha + \beta > 0$. Then, the following estimate holds:

$$\|\partial_x^{m+2} u\|_{L_p} \leq C_{m,p} \left(\|f\|_{W_p^m} + \|u\|_{W_p^1(\Omega\cap B_b)} \right).$$

Applying Proposition 4.1 to the second equation of (4.1), we have

$$\begin{aligned}&\|\partial_x^2 \mathbf{P}U(t)\|_{L_2}\\ &\leq C\left(\|\partial_t \mathbf{P}U(t)\|_{L_2} + \|\partial_x(\mathbf{I}-\mathbf{P})U(t)\|_{L_2} + \|\mathbf{P}F(U)(t)\|_{L_2} + \|\mathbf{P}U(t)\|_{H^1(\Omega\cap B_b)}\right),\end{aligned}$$

which together with (4.4), (4.5), (4.6) and (4.12) implies that

$$\|\partial_x^2 \mathbf{P}U(t)\|_{L_2} \leq C(1+t)^{-1/2}((M_2(t)+M_3(t))N(0,\infty) + \|U_0\|_{H^2}). \tag{4.19}$$

Therefore, our task is to estimate $\|\partial_x^2(\mathbf{I}-\mathbf{P})e^{-tA}U_0\|_{L_2}$. By (3.8) with $(p,q) = (2,2)$,

$$\|\partial_x^2(\mathbf{I}-\mathbf{P})e^{-tA}U_0\|_{L_2} \leq Ct^{-3/4}\|U_0\|_{H^{2,1}}. \tag{4.20}$$

By (3.8) with $(p,q) = (2,2)$ and (4.12),

$$\begin{aligned}&\|\partial_x^2(\mathbf{I}-\mathbf{P})II(t)\|_2\\ &\leq C\int_0^{t-1}(t-s)^{-3/4}\|F(U)(s)\|_{H^{2,1}}ds\\ &\leq C\int_0^{t-1}(t-s)^{-3/4}(1+s)^{-1/2}\|\partial_x U(s)\|_{H^2}\,ds(M_2(t)+M_3(t))\\ &\leq C\left(\int_0^{t-1}(t-s)^{-3/2}(1+s)^{-1}\,ds\right)^{1/2}(M_2(t)+M_3(t))N(0,\infty)\\ &\leq (1+t)^{-1/2}(M_2(t)+M_3(t))N(0,\infty).\end{aligned} \tag{4.21}$$

On the other hand, by (3.4) and (4.12)

(4.22)

$$\begin{aligned}\|\partial_x^2(\mathbf{I}-\mathbf{P})I(t)\|_{L_2} &\leq C\int_{t-1}^t (t-s)^{-1/2}\|F(U)(s)\|_{H^{2,1}}ds\\ &\leq C\int_{t-1}^t (t-s)^{-1/2}(1+s)^{-1/2}ds(M_2(t)+M_3(t))N(0,\infty)\\ &\leq C(1+t)^{-1/2}(M_2(t)+M_3(t))N(0,\infty).\end{aligned}$$

Combining (4.8), (4.20), (4.21), (4.22) with (4.19), we have (4.18). By (4.4), (4.5), (4.6) and (4.18), we have

$$M_1(t) + M_2(t) + M_3(t) \leq C\|U_0\|_{2,2} + C(M_2(t)+M_3(t))N(0,\infty).$$

If $CN(0,\infty) < 1$, then we have Step 1.

Step 2. Put

$$M_4(t) = \sup_{0\le s\le t} \frac{(1+s)^{3/4}}{\log(2+s)} \|\partial_x^2(\mathbf{I}-\mathbf{P})U(s)\|_{L_2},$$
$$M_5(t) = \sup_{0\le s\le t} \frac{(1+s)^{3/4}}{\log(2+s)} \|U(s)\|_{L_\infty}.$$

Then, there exists an $\epsilon' > 0$ such that if $N(0,\infty) \le \epsilon'$, then

$$M_4(t) + M_5(t) \le C(\|U_0\|_{H^2} + M_2(t)^2).$$

By (4.7), (4.10) and the Hölder's inequality,

(4.23)
$$\|F(U)(s)\|_{H^{2,1}} \le C(\|U(s)\|_{L_\infty} + \|\partial_x^2\rho(s)\|_{L_2})\|\partial_x U(s)\|_{H^2} + C\|\partial_x U(s)\|_{H^1}^2.$$

Therefore, by (3.8) with $(p,q) = (2,2)$ and (4.23),

(4.24)
$$\begin{aligned}
&\|\partial_x^2(\mathbf{I}-\mathbf{P})II(t)\|_2 \\
&\le C\int_0^{t-1}(t-s)^{-3/4}\|F(U)(s)\|_{H^{2,1}}ds \\
&\le C\left(\int_0^{t-1}(t-s)^{-3/2}(1+s)^{-3/2}(\log(1+s))^2\,ds\right)^{1/2}(M_4(t)+M_5(t))N(0,\infty) \\
&\quad +C\int_0^{t-1}(t-s)^{-3/4}(1+s)^{-1}\,dsM_2(t)^2 \\
&\le C(1+t)^{-3/4}\log(1+t)\left((M_4(t)+M_5(t))N(0,\infty)+M_2(t)^2\right);
\end{aligned}$$

and by (3.4) and (4.23),

(4.25)
$$\begin{aligned}
&\|\partial_x^2(\mathbf{I}-\mathbf{P})I(t)\|_{L_2} \\
&\le C\int_{t-1}^{t}(t-s)^{-1/2}\|F(U)(s)\|_{H^{2,1}}ds \\
&\le C\int_{t-1}^{t}(t-s)^{-1/2}(1+s)^{-3/4}\log(1+s)ds(M_4(t)+M_5(t))N(0,\infty) \\
&\quad +C\int_{t-1}^{t}(t-s)^{-1/2}(1+s)^{-1}\,dsM_2(t)^2 \\
&\le C(1+t)^{-3/4}\left((M_4(t)+M_5(t))N(0,\infty)+M_2(t)^2\right).
\end{aligned}$$

By (4.7), (4.10) and the Hölder's inequality

$$\begin{aligned}(4.26)\qquad &\|F(U)(s)\|_{L_2} \le C\|U(s)\|_{L_\infty}\|\partial_x U(s)\|_{H^1},\\ &\|F(U)(s)\|_{W_4^{1,0}} \le C\left(\|U(s)\|_{L_\infty} + \|\partial_x^2\rho(s)\|_{L_2}\right)\|\partial_x U(s)\|_{H^2}.\end{aligned}$$

Therefore, by (3.7) with $(p,q) = (4,2)$ and (4.26),

(4.27)

$$\begin{aligned}&\|II(t)\|_{L_\infty}\\ &\le C\int_0^{t-1}(t-s)^{-3/4}\left(\|F(U)(s)\|_2 + \|F(U)(s)\|_{W_4^{1,0}}\right)ds\\ &\le C\left(\int_0^{t-1}(t-s)^{-3/2}(1+s)^{-3/2}(\log(1+s))^2\,ds\right)^{1/2}(M_4(t)+M_5(t))N(0,\infty)\\ &\le C(1+t)^{-3/4}\log(1+t)(M_4(t)+M_5(t))N(0,\infty);\end{aligned}$$

and by (3.3), (4.26) and the Sobolev inequality

(4.28)

$$\begin{aligned}\|I(t)\|_{L_\infty} &\le C\|I(t)\|_{W_4^1}\\ &\le C\int_{t-1}^{t}(t-s)^{-1/2}\|F(U)(s)\|_{W_4^{1,0}}\,ds\\ &\le C\int_{t-1}^{t}(t-s)^{-1/2}(1+s)^{-3/4}\log(1+s)\,ds(M_4(t)+M_5(t))N(0,\infty)\\ &\le C(1+t)^{-3/4}\log(1+t)(M_4(t)+M_5(t))N(0,\infty).\end{aligned}$$

Combining (4.11), (4.24), (4.25), (4.11), (4.27), (4.28) with (4.8), we have

$$M_4(t) + M_5(t) \le C(M_4(t)+M_5(t))N(0,\infty) + CM_2(t)^2 + C\|U_0\|_{H^2},$$

which means the Step 2. By Steps 1 and 2, the proof of Theorem 3 is completed.

§5. Proof of Corollary 2.1

Let $\tilde{N}(0,\infty)$ be the quantity defined in Section 2. Put

$$\begin{aligned}M_6(t) &= \sup_{0\le s\le t}(1+s)^{-1/2}\|\partial_t\partial_x U(t)\|_{L_2},\\ M_7(t) &= \sup_{0\le s\le t}(1+s)^{-1/2}\|\partial_x^3\mathbf{P}U(t)\|_{L_2},\\ M_8(t) &= \sup_{0\le s\le t}(1+s)^{-3/4}\|U(t)\|_{L_\infty}.\end{aligned}$$

By (2.2), when $0 \leq t \leq 2$, we have

$$\sum_{j=1}^{8} M_j(t) \leq C\tilde{N}(0,\infty). \tag{5.1}$$

Therefore we consider the case when $t \leq 1$, below. By (3.8) with $(p,q) = (2,2)$,

$$\|\partial_t\partial_x e^{-tA}U_0\|_{L_2} \leq Ct^{-3/4}\|U_0\|_{H^{2,1}}. \tag{5.2}$$

By (3.8) with $(p,q) = (2,2)$ and (4.12),

(5.3)

$$\begin{aligned}
\|\partial_t\partial_x II(t)\|_{L_2} &\leq C\int_0^{t-1}(t-s)^{-3/4}\|F(U)(s)\|_{H^{2,1}}\,ds \\
&\leq C\int_0^{t-1}(t-s)^{-3/4}(1+s)^{-1/2}\|\partial_x U(s)\|_{H^2}\,ds(M_2(t)+M_3(t)) \\
&\leq C\left(\int_0^{t-1}(t-s)^{-3/2}(1+s)^{-1}\,ds\right)^{1/2}(M_2(t)+M_3(t))N(0,\infty) \\
&\leq C(1+t)^{-1/2}\left(M_2(t)+M_3(t)\right)N(0,\infty).
\end{aligned}$$

By the equation

$$\partial_t\partial_x I(t) = \int_{t-1}^{t}\partial_x e^{-(t-s)A}\partial_s F(U)(s)\,ds$$

and by (3.2) and (4.12),

$$\begin{aligned}
\|\partial_t\partial_x I(t)\|_{L_2} &\leq C\int_{t-1}^{t}(t-s)^{-1/2}\|\partial_s F(U)(s)\|_{H^{1,0}}\,ds \\
&\leq C\int_{t-1}^{t}(t-s)^{-1/2}(1+s)^{-1/2}\,ds\left(M_2(t)+M_3(t)\right)\tilde{N}(0,\infty) \\
&\leq C(1+t)^{-1/2}\left(M_2(t)+M_3(t)\right)\tilde{N}(0,\infty).
\end{aligned} \tag{5.4}$$

Combining (5.1) through (5.4), we have

$$M_6(t) \leq C(M_2(t)+M_3(t))\tilde{N}(0,\infty) + C\|U_0\|_{H^{2,1}}. \tag{5.5}$$

Applying Proposition 4.1 to the second equation of (4.1), we have

$$\begin{aligned}
\|\partial_x^3\mathbf{P}U(t)\|_{L_2} \leq C\big(&\|\partial_t\partial_x\mathbf{P}U(t)\|_{L_2} + \|\partial_x^2(\mathbf{I}-\mathbf{P})U(t)\|_{L_2} \\
&+ \|\partial_x\mathbf{P}F(U)(t)\|_{L_2} + \|\mathbf{P}U(t)\|_{H^1(\Omega\cap B_b)}\big),
\end{aligned}$$

which together with Steps 1 and 2 in Section 4, (5.5) and (4.12) implies that

$$M_7(t) \le C(M_2(t) + M_3(t))\tilde{N}(0,\infty) + C\|U_0\|_{H^2}. \tag{5.6}$$

Finally, we shall estimate $M_8(t)$. By (4.7), (4.10) and the Hölder's inequality

$$\|F(U)(s)\|_{W_4^{1,0}} \le C\left(\|U(s)\|_{L_\infty}\|\partial_x U(s)\|_{H^2} + \|\partial_x^2\rho(s)\|_{L_2}\|\partial_x v(s)\|_{H^2}\right). \tag{5.7}$$

Therefore, by (3.7) with $(p,q) = (4,2)$, (4.26) and (5.7)

$$\begin{aligned} \|II(t)\|_{L_\infty} &\le C\int_0^{t-1}(t-s)^{-3/4}\left(\|F(U)(s)\|_{L_2} + \|F(U)(s)\|_{W_4^{1,0}}\right)ds \\ &\le C\left(\int_0^{t-1}(t-s)^{-3/2}(1+s)^{-3/2}\,ds\right)^{1/2} M_8(t)N(0,\infty) \\ &\quad + C\int_0^{t-1}(t-s)^{-3/4}(1+s)^{-5/4}\log(1+s)ds M_4(t)M_7(t) \\ &\le C(1+t)^{-3/4}(M_8(t)N(0,\infty) + M_4(t)M_7(t)); \end{aligned} \tag{5.8}$$

and by (3.3), (5.7) and the Sobolev inequality

$$\begin{aligned} \|I(t)\|_{L_\infty} \le C\|I(t)\|_{W_4^1} &\le C\int_{t-1}^{t}(t-s)^{-1/2}\|F(U)(s)\|_{W_4^{1,0}}\,ds \\ &\le C\int_{t-1}^{t}(t-s)^{-1/2}(1+s)^{-3/4}\,ds M_8(t)N(0,\infty) \\ &\quad + C\int_{t-1}^{t}(t-s)^{-1/2}(1+s)^{-5/4}\log(1+s)\,ds M_4(t)M_7(t) \\ &\le C(1+t)^{-3/4}(M_8(t)N(0,\infty) + M_4(t)M_7(t)). \end{aligned} \tag{5.9}$$

Combining (5.1), (4.11), (5.8) and (5.9), we have

$$M_8(t) \le C(M_8(t)N(0,\infty) + M_4(t)M_7(t)) + C\|U_0\|_{H^2}.$$

If $CN(0,\infty) < 1$, then we have

$$M_8(t) \le CM_4(t)M_7(t) + C\|U_0\|_{H^2},$$

which completes the proof.

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