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The Second Bounded Cohomology of 3-manifolds

By

Koji FUJIWARA* and Ken'ichi OHSHIKA**

Abstract

Let G be the fundamental group of a compact, orientable 3-manifold M. We show that if each piece of the canonical decomposition of M has a geometric structure (e.g. when G contains $\mathbb{Z}^2$), then either G has infinite dimensional second bounded cohomology or G is virtually solvable.

§1. Introduction

The bounded cohomology of a discrete group G is defined using a subcomplex of the ordinary cochain complex ([G1]). Set

$$C_b^k(G;\mathbb{R}) = \{f : G^k \to \mathbb{R} \mid f \text{ has bounded range}\}.$$

A coboundary operator $\delta : C_b^k(G;\mathbb{R}) \to C_b^{k+1}(G;\mathbb{R})$ is given by

$$\delta f(g_0, \dots, g_k) = f(g_1, \dots, g_k) + \sum_{i=1}^{k} (-1)^i f(g_0, \dots, g_{i-1}g_i, \dots, g_k) + (-1)^{k+1} f(g_0, \dots, g_{k-1}).$$

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*Mathematical Institute, Tohoku University, Sendai 980-8578, Japan.
e-mail: `fujiwara@math.tohoku.ac.jp`
**Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan.
e-mail: `ohshika@math.wani.osaka-u.ac.jp`

The cohomology of the complex $\{C_b^k(G;\mathbb{R}), \delta\}$ is called the (*real*) *bounded cohomology group* of G, and denoted by $H_b^*(G;\mathbb{R})$.

Gromov gave a definition of the bounded cohomology for topological spaces, and showed that it is isomorphic to the one for its fundamental group for a large class of spaces including all manifolds (not necessary $K(\pi,1)$). It is known that $H_b^1(G;\mathbb{R})$ is trivial for every group G, and that $H_b^n(G;\mathbb{R})$ is trivial for all $n \geq 1$ if G is amenable. There is a nice account on the subject by Ivanov [I].

Brooks [B] constructed countably many independent second bounded cohomology classes for free groups G of rank at least two (see also [Mi]). He used the canonical action of a free group on a simplicial tree. His idea applies to a group which acts on a geodesic space that is "hyperbolic" in the sense of Gromov [G2] ([BaGh], [EF], [F1], [F2]). The group G in the following list has infinite dimensional $H_b^2(G;\mathbb{R})$. We remark that if the dimension of $H_b^2(G;\mathbb{R})$ is infinite, then it is automatically the cardinal of the continuum since $H_b^2(G;\mathbb{R})$ is a Banach space ([I], [MaMo]).

List.

(a) Non-elementary word-hyperbolic groups ([EF]). (See [G2] for the definition of word-hyperbolic groups): e.g. free groups of rank at least two, the surface groups of genus at least two, and the fundamental group of a closed Riemannian manifold of negative sectional curvature.

(b) Groups G which decompose as $G = A *_C B$ such that $|C\backslash A/C| \geq 3$ and $|B/C| \geq 2$ or $G = A*_{C,\varphi}$ such that $|A/C| \geq 2$ and $|A/\varphi(C)| \geq 2$ ([F1]).

(c) Groups G which act properly discontinuously by isometries on a Gromov-hyperbolic space such that the limit set of the action has more than two points ([F2]): e.g. lattices in a rank-1 semi-simple Lie group, Kleinian groups which are not virtually nilpotent.

We apply those results to the fundamental group of a compact orientable 3-manifold. A group G is called *virtually* solvable if it contains a solvable subgroup of finite index. In particular $H_b^2(G;\mathbb{R})$ is then trivial. The following is our main theorem.

Theorem 1.1 (Algebraic). *Let G be the fundamental group of a compact, orientable 3-manifold M. Suppose that G decomposes non-trivially as $G = A *_C B$ or $A*_C$. Then either $H_b^2(G;\mathbb{R})$ is infinite dimensional or G is virtually solvable.*

Moreover G is virtually solvable if and only if G is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$ or the fundamental group of one of the following manifolds.

(i) *A torus bundle over S^1.*

(ii) *A Seifert fibred space with a Euclidean base orbifold.*

(iii) *A manifold obtained by gluing two twisted S^1-bundles over Möbius bands along their boundaries. Then G admits an exact sequence*

$$1 \to \mathbb{Z} \times \mathbb{Z} \to G \to \mathbb{Z}_2 * \mathbb{Z}_2 \to 1.$$

Theorem 1.1 applies to all Haken manifolds. More generally one can show the following.

Theorem 1.2 (Topological)**.** *Let G be the fundamental group of a compact orientable 3-manifold M. Suppose that M has no boundary component homeomorphic to S^2. Assume that each piece of the canonical decomposition of M has a geometric structure. Then $H_b^2(G;\mathbb{R})$ is infinite dimensional or G is virtually solvable.*

Moreover, G is virtually solvable if and only if M is homeomorphic to one of the manifolds in (i), (iii) *in Theorem* 1.1, *or one of the following.*

(ii)' *A Seifert fibred space with a Euclidean or spherical base orbifold.*

(iv) *A manifold which is finitely covered by S^3. (Then G is finite).*

(v) *$S^2 \times S^1$ and $D^2 \times S^1$. (Then $G \cong \mathbb{Z}$).*

(vi) *$PR^3 \sharp PR^3$. (Then $G \cong \mathbb{Z}_2 * \mathbb{Z}_2$).*

The geometrization conjecture ([T]) says that if M is a compact orientable 3-manifold, each piece of the canonical decomposition of M has a geometric structure (see Section 2).

Corollary 1.3. *Let G be the fundamental group of a compact, orientable 3-manifold M. If G contains $\mathbb{Z} \times \mathbb{Z}$, then $H_b^2(G;\mathbb{R})$ is infinite dimensional or G is virtually solvable.*

It follows that if G is a knot group, then $H_b^2(G;\mathbb{R})$ is infinite dimensional or the knot is trivial ([F2]).

Theorem 1.2 or Corollary 1.3 applies to any compact orientable 3-manifold M unless it is irreducible, non-Haken (in particular closed) with infinite fundamental group G without $\mathbb{Z} \times \mathbb{Z}$ subgroups. Hyperbolization conjecture ([T]) says that such 3-manifold is hyperbolic, so that G is word-hyperbolic.

Let M be an orientable, non-compact 3-manifold. If the fundamental group is finitely generated, then M has a compact submanifold, called a *core*, which is homotopy equivalent to M ([Sc2]). One can apply our results to a core of M.

§2. Canonical Decompositions of 3-manifolds

We review standard facts about 3-manifolds (see [Sc1]). Let M be a compact orientable 3-manifold with no boundary component homeomorphic to a 2-sphere. A 2-sphere embedded in M is called *essential* unless it bounds a 3-ball in M. Consider a maximal system (always finite) of disjoint, non-parallel, essential, separating, embedded spheres $\{S_i\}$ in M. Cutting M along $\{S_i\}$ gives a connected sum decomposition such that each summand does not contain essential, separating, embedded spheres (called *prime*). The decomposition is unique and called the *prime decomposition* of M.

Let M be a compact, orientable, prime 3-manifold. Unless M is $S^2 \times S^1$, M does not contain a non-separating embedded 2-sphere. Such manifolds are called *irreducible*.

Let M be a compact, orientable, irreducible 3-manifold. If ∂M is compressible in M, then by the loop theorem, one can find a compressing disc D for ∂M. One cuts M along D and obtains N. If ∂N is still compressible, we continue the same operation. This process must stop in finite steps and gives a splitting of M into pieces with incompressible boundary.

Let M be a compact, orientable, irreducible, 3-manifold with possibly empty incompressible boundary. An embedded torus in a 3-manifold is said to be *essential* if it is incompressible and not parallel to a boundary component. The theory of Jaco-Shalen-Johannson ([JSh], [Jo]) says that there is a finite collection of disjoint, incompressible, embedded tori $\{T_i\}$ such that by cutting M along T_i, we obtain a family of 3-manifolds each of which is either a Seifert fibred space or admits no essential tori (called *atoroidal*). A Seifert fibred space is an S^1-bundle whose base space is a 2-orbifold. The process and its outcome we described of cutting M along embedded 2-spheres, 2-discs, and tori is called a *canonical decomposition*.

A locally homogeneous Riemannian metric on a manifold with a certain maximal property is called a *geometric structure*. In dimension 3, there are eight geometric structures which are modeled on $\mathbf{S}^3, \mathbf{E}^3, \mathbf{H}^3, \mathbf{H}^2 \times \mathbf{E}, \mathbf{S}^2 \times \mathbf{E}, \widetilde{SL_2}(\mathbb{R}), Nil$, and $Solv$, respectively.

The *geometrization conjecture* ([T]) is that each piece in the canonical decomposition admits a geometric structure. It is known that the Seifert fibred spaces admit geometric structures. A compact irreducible 3-manifold is called *Haken* if it contains a properly embedded two-sided incompressible surface. By Thurston's uniformization theorem, every atoroidal, Haken manifold admits a geometrically finite hyperbolic structure on its interior (called *hyperbolic*).

§3. Proofs

The following result by Bouarich is useful.

Theorem 3.1 ([Bo]). *Let $G_3 \to G_2 \to G_1 \to 1$ be an exact sequence of groups. Then the induced sequence of the second bounded cohomology is exact*;

$$0 \to H_b^2(G_1;\mathbb{R}) \to H_b^2(G_2;\mathbb{R}) \to H_b^2(G_3;\mathbb{R}).$$

In particular $H_b^2(G_1;\mathbb{R})$ and $H_b^2(G_2;\mathbb{R})$ are isomorphic if G_3 is amenable.

We prove Theorems 1.1 and 1.2 simultaneously.

Proof of Theorems 1.1 *and* 1.2. We start with the assumption of Theorem 1.1. We may assume that there is no connected component of ∂M homeomorphic to S^2. We do not lose generality since we can change M by filling in 3-balls without changing G.

Let $M = M_1 \sharp \cdots \sharp M_n$ be the prime decomposition of M. For our purpose, we may assume that $\pi_1(M_i)$ is not trivial for each i. If the prime decomposition of M has more than one summands, then G is freely decomposable. It follows from List (b) that $H_b^2(G;\mathbb{R})$ is infinite dimensional unless G is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$.

Let M be prime. If M is $S^2 \times S^1$, then $G \cong \mathbb{Z}$ and we are done. Therefore we suppose that M is irreducible. If there is a component of ∂M which is not π_1-injective, then by the loop theorem, G decomposes as $G = A*_{\{e\}}$ or $G = A * B$. In the first case $H_b^2(G)$ is infinite dimensional unless $G \cong \mathbb{Z}$ (List (b)). We have already discussed the second case. Therefore we suppose that M is boundary irreducible. (∂M might be empty). Then M is Haken. Indeed, if ∂M is not empty, then M is Haken. If M is closed, by a standard technique in 3-dimensional topology, it follows from the assumption that $G = A*_C B$ or $G = A*_C$ and that one can find a closed, embedded, 2-sided, incompressible surface F in M so that $\pi_1(F) \subset C$. Since M is irreducible, F is not homeomorphic to S^2, hence M is Haken.

If M is Haken, each piece of the canonical decomposition of M admits a geometric structure, which is the assumption in Theorem 1.2. Now we start the proof of Theorem 1.2. At the end we obtain Theorem 1.1 too. We see that G is either finite or isomorphic to $\mathbb{Z}$, or isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$ if and only if M is homeomorphic to the manifolds in (iv), (v), or (vi) in Theorem 1.2, respectively.

In the rest, we assume that M is irreducible with (possibly empty) incompressible boundary. If the torus decomposition of M is trivial, M is homeomorphic to one of the following manifolds.

(1) A torus bundle over S^1.
(2) A Seifert fibred space with a spherical or Euclidean base orbifold. (If M is Haken, a spherical base does not appear.)
(3) A Seifert fibred space with a hyperbolic base orbifold O.
(4) A geometrically finite hyperbolic manifold (we consider the interior of M).

In the case of (1) or (2), G is virtually solvable. In the case of (3), there is an exact sequence

$$1 \to \mathbb{Z} \to G \to \pi_1^{\mathrm{orb}}(O) \to 1.$$

Since $H_b^2(G) \cong H_b^2(\pi_1^{\mathrm{orb}}(O))$, (by Theorem 3.1) and $\pi_1^{\mathrm{orb}}(O)$ is a non-elementary word-hyperbolic group, $H_b^2(G)$ is infinite dimensional (List (a)). For (4), since G is a Kleinian group which is not virtually nilpotent, $H_b^2(G)$ is infinite dimensional (List (c)).

If there are more than one piece in the torus decomposition of M, it suffices to discuss the following three cases.

(5) There is a piece which is a geometrically finite hyperbolic manifold.
(6) There is a piece which is a Seifert fibred space with a hyperbolic base orbifold Σ.
(7) Each piece is a Seifert fibred space with a Euclidean base orbifold.

In the case of (5), $H_b^2(G)$ is infinite dimensional. Indeed, let H be one of the pieces which are hyperbolic, and T one of the torus cusps of H. To apply List (b) to G, it suffices to show the following proposition (essentially shown in [F2]).

Proposition 3.2. *Let Γ be a non-elementary Kleinian group, and P a parabolic subgroup. Then $|P\backslash\Gamma/P| = \infty$.*

Proof. Let p be the point at infinity fixed by P. For each element $g \in \Gamma$, we have $\overline{PgP(p)} = \overline{Pg(p)}$, which is a countable set since p is the only accumulation point for the set $P(g(p))$. Now, if $|P\backslash\Gamma/P|$ were finite, then $\overline{\Gamma(p)}$ would be a countable set as well. It is a standard fact that $\overline{\Gamma(p)}$ coincides with the limit set of Γ, which is an uncountable infinite set when Γ is non-elementary. Therefore $|P\backslash\Gamma/P| = \infty$. $\square$

In the case of (6), $H_b^2(G)$ is infinite dimensional. We can argue in the same way as (5) since the double coset space of G with respect to one of the torus boundary groups T is infinite. To see this, it suffices to show that the double coset space of $\pi_1^{\mathrm{orb}}(O)$ with respect to the boundary subgroup C (isomorphic to $\mathbb{Z}$) which corresponds to T is infinite. Since O is hyperbolic, $\pi_1^{\mathrm{orb}}(O)$ is non-elementary word-hyperbolic group. We apply the following fact ([F2]). (Alternatively, one can argue using points at infinity as (5) since $\pi_1^{\mathrm{orb}}(O)$ is a Fuchsian group.)

Proposition 3.3. *Let Γ be a non-elementary word-hyperbolic group, and C a subgroup isomorphic to $\mathbb{Z}$. Then $|C\backslash\Gamma/C| = \infty$.*

In the case of (7), the 3-manifold M is obtained by pasting two twisted S^1-bundles over Möbius bands at their boundaries, and G is virtually solvable. Indeed, the base orbifold of each piece has to be either an annulus, a Möbius band, or a disc with two cone points of index 2 since it is Euclidean. The latter two give manifolds which are homeomorphic to each other, but with different fibrations. Since the torus decomposition is minimal, an S^1-bundle over an annulus does not appear. Thus the only case that remains to be considered is when M is a graph manifold obtained by pasting two twisted S^1-bundles over a Möbius band at their boundaries (so that the fibration cannot extend to the entire manifold). In this case, we have an exact sequence, $1 \to \mathbb{Z}\times\mathbb{Z} \to \pi_1(M) \to \mathbb{Z}_2 * \mathbb{Z}_2 \to 1$. Thus G contains a subgroup of index 2 which is solvable.

As a conclusion, among the cases (1)–(7), $H_b^2(G)$ is infinite dimensional in the cases of (3), (4), (5) and (6), and G is virtually solvable in the cases of (1), (2) and (7), which corresponds to (i), (ii)', (iii) in Theorem 1.2. As we already discussed, the other possibilities for G to be virtually solvable are when G is finite or isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_2 * \mathbb{Z}_2$, which exactly correspond to the manifolds in (iv), (v) or (vi). $\square$

We prove Corollary 1.3.

Proof. If M is reducible or boundary reducible, Theorem 1.1 applies. If ∂M is not empty, then Theorem 1.2 applies. Suppose M is irreducible and without boundary. Since G contains a subgroup isomorphic to $\mathbb{Z}\times\mathbb{Z}$, M either contains an embedded, incompressible torus or is homotopy equivalent to a Seifert fibred space ([Sc3], [CJu], [Ga]). In either case, Theorem 1.2 applies to M. $\square$

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