On the Algebraicity of Special L-Values of Hermitian Modular Forms.

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Received: December 1, 2014
Revised: July 8, 2015

Communicated by Otmar Venjakob

Abstract. In this work we prove some results on the algebraicity of special L-values attached to Hermitian modular forms. Our work is based on techniques developed by Goro Shimura in his book "Arithmeticity in the Theory of Automorphic Forms", and our results are in some cases complementary to results obtained previously by Michael Harris on the same question.

2010 Mathematics Subject Classification: 11F55, 11F67, 11M36
Keywords and Phrases: Hermitian modular forms, Special L-Values, Periods.

1. Introduction

In his admirable book [24], Goro Shimura establishes various algebraicity results concerning special L-values of Siegel modular forms, both of integral and half-integral weight, as well as of Hermitian modular forms. All these results are over an algebraic closure of $\mathbb{Q}$, in the sense, that it is shown that the ratio of the special L-values over the Petersson inner product of the corresponding modular forms, defined over $\overline{\mathbb{Q}}$, is an algebraic number up to some powers of π (see also the discussion in [24, page 239]). In this work, similar to our previous works [2, 4], where the Siegel modular form situation was considered, we obtain, in some cases, more precise information about the field of definition of these ratios, and we establish a reciprocity law of the action of the absolute Galois group. Moreover, in some cases, we even extend some of the results of Shimura concerning the general algebraicity of these ratios. These can be achieved by employing a recent result due to Klosin in [18] (see Theorem 4.2 below).
We note that the questions addressed here have been considered by Michael Harris in [11, 13] where the situation of Hermitian modular forms attached to unitary groups over a quadratic imaginary field was considered. In the last section of this paper we will compare the results of this work with the ones

obtained by Harris. Here we only mention that our work differs from the ones of Harris in the method used. Harris uses the so-called doubling method to study the L-values while here we employ the Rankin-Selberg method. Harris' results are more general in the sense that the doubling method can cover Hermitian modular forms for unitary groups with archimedean components isomorphic to $U(n,m)$ with $n \neq m$, something which cannot be done by the Rankin-Selberg method. Moreover Harris considers vector valued Hermitian modular forms, while here we restrict ourselves to the scalar weight case. Having said that, there are some critical values in the scalar weight situation, where the results of our work are not covered by the ones obtained by Harris. We provide details on this point in the last section of the paper.
However the main motivation for our work springs from the analytic part of Iwasawa Theory. This work should be seen as the first step towards the construction of p-adic measures for Hermitian modular forms, which is the subject of our forthcoming work [5]. Actually the results obtained here will be used in [5] to determine the field of definition of the p-adic measures constructed there. The construction of such measures has been initiated by Harris, Li and Skinner in [15, 16] (the interested reader should also see important related work of Eischen [7, 8]), where an Eisenstein measure, which interpolates p-adic Siegel type Eisenstein series is constructed, and some hints toward the construction of the p-adic measure are given. Their work should be seen as a vast generalization of the work of Katz [17]. Actually in our work [3] we have constructed these measures for the case of Hermitian modular forms attached to definite unitary groups of one obtaining p-adic measures attached to Hecke Characters and two variables. However in all these works one needs to assume that the prime ideals of F above p split in K. We believe that this assumption is needed only in the case where the archimedean components of the unitary group is of the form $U(n,m)$ with $n \neq m$, that is the Witt signature is not trivial. We defer a more detailed discussion on this to our forthcoming work [5], but we only mention here, that this is related to the fact that in the case of non-trivial Witt signature, one needs to evaluate p-adic modular forms on CM points, and this can be done p-adically only when the, corresponding to these points, CM abelian varieties are ordinary at p. And the above condition on p guarantees this. Finally we mention that our approach using the Rankin-Selberg method should be seen as the unitary analogue of the work of Panchishkin, and of Courtieu and Panchishkin [6], who considered the symplectic case (Siegel modular forms over $\mathbb{Q}$ and of even genus).

Notation: As in our works [2, 4] also in this one we use the books of Shimura [23, 24] as our main references. For this reason we have decided to keep the notation used in these two books. The only important notational difference, is the use of the L-notation to denote the L-function instead of the Z used in Shimura's works.

2. Hermitian modular forms

In this section we introduce the notion of a Hermitian modular form, both classically and adelically. We follow closely the books of Shimura [23, Chapter II] and [24, Chapter I], and we remark that we adopt the convention done in the second book with respect to the weight of Hermitian modular forms (see the discussion on page 32, Section 5.4 in [24]).
Let K be a field equipped with an involution ρ. For a positive integer $n \in \mathbb{N}$ we define the matrix $\eta := \eta_n := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \in GL_{2n}(K)$, and the group $G := U(n,n) := \{\alpha \in GL_{2n}(K) | \alpha^* \eta \alpha = \eta\}$, where $\alpha^* := {}^t\alpha^\rho$. Moreover we define $\hat{\alpha} := (\alpha^*)^{-1}$ and $S := S^n := \{s \in M_n(K) | s^* = s\}$ for the set of Hermitian matrices with entries in K. If we take $K = \mathbb{C}$ and ρ denotes complex conjugation then the group $G(\mathbb{R}) = \{\alpha \in GL_{2n}(\mathbb{C}) | \alpha^* \eta \alpha = \eta\}$ acts on the symmetric space (Hermitian upper half space) $\mathbb{H}_n := \{z \in M_n(\mathbb{C}) | i(z^* - z) > 0\}$ by linear fractional transformations. That is for $\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \in G(\mathbb{R})$ and $z \in \mathbb{H}_n$ we have $\alpha \cdot z := (a_\alpha z + b_\alpha)(c_\alpha z + d_\alpha)^{-1} \in \mathbb{H}_n$.

Let now K be a CM field of degree $2d := [K : \mathbb{Q}]$ and we let ρ denote the complex conjugation. We write F for the maximal totally real subfield. Moreover we write $\mathcal{O}_K$ for the ring of integers of K, $\mathcal{O}_F$ for that of F, D_F and D_K for their discriminants and $\mathfrak{d}$ for the different ideal of F. We write $\mathbf{a}$ for the set of archimedean places of F. We now pick a CM type $(K, \{\tau_v\}_{v \in \mathbf{a}})$ of K. For an element $a \in K$ we write $a_v \in \mathbb{C}$ for $\tau_v(a)$. We will identify τ_v with v and also view $\mathbf{a}$ as archimedean primes of K. Finally we let $\mathbf{b}$ be the set of all complex embeddings of K, and we note that $\mathbf{b} = \mathbf{a} \coprod \mathbf{a}\rho$.
We define $G_\mathbb{A} := G(\mathbb{A})$, the adeles of G, and we write $G_\mathbf{h} = \prod_v' G_v$ for the finite part, and $G_\mathbf{a} = \prod_{v \in \mathbf{a}} G_v$ for the archimedean part. Note that we understand G as an algebraic group over F, and hence the finite places v above are finite places of F. For a description of G_v at a finite place we refer to [23, Chapter 2]. We define an action of $G_\mathbb{A}$ on $\mathcal{H}$ by $g \cdot z := g_\mathbf{a} \cdot z$, with $g \in G_\mathbb{A}$ and $z \in \mathcal{H}$. Following Shimura, we define for two fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$ of F such that $\mathfrak{a}\mathfrak{b} \subseteq \mathcal{O}_F$, the subgroup of $G_\mathbb{A}$,

$$D[\mathfrak{a}, \mathfrak{b}] := \left\{ \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix} \in G_\mathbb{A} | a_x \prec \mathcal{O}_{F_v}, b_x \prec \mathfrak{a}_v, c_x \prec \mathfrak{b}_v, d_x \prec \mathcal{O}_{F_v}, \forall v \in \mathbf{h} \right\}$$

where we use the notation $\prec$ in [24, page 11], where $x \prec \mathfrak{b}_v$ means that the v-component of the matrix x has are all its entries in $\mathfrak{b}_v$. For a finite adele $q \in G_\mathbf{h}$ we define $\Gamma^q = \Gamma^q(\mathfrak{b}, \mathfrak{c}) := G \cap D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$, a congruence subgroup of G. Given a Hecke character ψ of K of conductor dividing $\mathfrak{c}$ we define a character on $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ by $\psi(x) = \prod_{v|\mathfrak{c}} \psi_v(det(d_x)_v)^{-1}$, and a character ψ_q on Γ^q by $\psi_q(\gamma) = \psi(q^{-1}\gamma q)$.

We write $\mathbb{Z}^{\mathbf{a}} := \prod_{v\in\mathbf{a}}\mathbb{Z}$, $\mathbb{Z}^{\mathbf{b}} := \prod_{v\in\mathbf{b}}\mathbb{Z}$ and $\mathcal{H} := \prod_{v\in\mathbf{a}}\mathbb{H}_n$. For a function $f : \mathcal{H} \to \mathbb{C}$ and an element $k \in \mathbb{Z}^{\mathbf{b}}$ we define

$$(f|_k\alpha)(z) := j_\alpha(z)^{-k}f(\alpha z),\ \alpha \in G_{\mathbb{A}},\ z \in \mathcal{H}.$$

Here we write $z = (z_v)_{v\in\mathbf{a}}$ with $z_v \in \mathbb{H}_n$ and define

$$j_\alpha(z)^{-k} := \prod_{v\in\mathbf{a}} det(\mu(c_{\alpha_v}z_v + d_{\alpha_v})^{-k_v} det(\overline{c_{\alpha_v}}^t z_v + \overline{d_{\alpha_v}})^{-k_{v\rho}}.$$

For a fixed $\mathfrak{b}$ and $\mathfrak{c}$ as above, a $q \in G_{\mathbf{h}}$ and a Hecke character ψ of K, we define

Definition 2.1. [24, page 31] A function $f : \mathcal{H} \to \mathbb{C}$ is called a Hermitian modular form for the congruence subgroup Γ^q of weight $k \in \mathbb{Z}^{\mathbf{b}}$ and nebentype ψ_q if:

(1) f is holomorphic,
(2) $f|_k\gamma = \psi_q(\gamma)f$ for all $\gamma \in \Gamma^q$,
(3) f is holomorphic at cusps (see [24, page 31] for this notion).

The space of Hermitian modular forms of weight k for the congruences group Γ^q is denoted by $\mathcal{M}_k(\Gamma^q, \psi_q)$. For any $\gamma \in G$ we have a Fourier expansion of the form (see [24, page 33]

$$(f|_k\gamma)(z) = \sum_{s\in\mathfrak{S}} c(s, \gamma; f)e_{\mathbf{a}}(sz),$$

where $\mathfrak{S}$ a lattice in $S_+ := \{s \in S|\ s_v \geq 0,\ \forall v \in \mathbf{a}\}$, and $e_{\mathbf{a}}(x) = exp(2\pi i \sum_v tr(x_v))$. An f is called a cusp form if $c(s, \gamma; f) = 0$ for any $\gamma \in G$ and s with $det(s) = 0$. The space of cusp forms we will be denoted by $\mathcal{S}_k(\Gamma^q, \psi_q)$. Given an element $f \in \mathcal{S}_k(\Gamma^q, \psi_q)$, and a function g on $\mathcal{H}$ such that $g|_k\gamma = \psi_q(\gamma)f$ for all $\gamma \in \Gamma^q$ we define the Petersson inner product

$$\langle f, g\rangle := \langle f, g\rangle_{\Gamma^q} := \int_{\Gamma^q\backslash\mathcal{H}} f(z)\overline{g(z)}\delta(z)^m dz,$$

where $\delta(z) := det(\frac{i}{2}(z^* - z))$ and dz a measure on $\Gamma^q \setminus \mathcal{H}$ defined as in [24, Lemma 3.4] and $m = (m_v)_{v\in\mathbf{a}}$ with $m_v = k_v + k_{v\rho}$.

We now turn to the adelic Hermitian modular forms. If we write D for a group of the form $D[\mathfrak{b}^{-1}, \mathfrak{bc}]$, and ψ a Hecke character of finite order then we define,

Definition 2.2. [24, page 166]) A function $\mathbf{f} : G_{\mathbb{A}} \to \mathbb{C}$ is called an adelic Hermitian modular form if

(1) $\mathbf{f}(\alpha x w) = \psi(w)j_w^k(\mathbf{i})\mathbf{f}(x)$ for $\alpha \in G$, $w \in D$ with $w_{\mathbf{a}}(\mathbf{i}) = \mathbf{i}$,
(2) For every $p \in G_{\mathbf{h}}$ there exists $f_p \in \mathcal{M}_k(\Gamma^p, \psi_p)$, where $\Gamma^p := G\cap pCp^{-1}$ such that $\mathbf{f}(py) = (f_p|_k y)(\mathbf{i})$ for every $y \in G_{\mathbf{a}}$.

Here we write $\mathbf{i} := (i1_n, \dots, i1_n) \in \mathcal{H}$. We denote this space by $\mathcal{M}_k(D, \psi)$. Moreover there exists a finite set $\mathcal{B} \subset G_{\mathbf{h}}$ such that $G_{\mathbb{A}} = \coprod_{b\in\mathcal{B}} GbD$ and an isomorphism $\mathcal{M}_k(D, \psi) \cong \oplus_{b\in\mathcal{B}}\mathcal{M}_k(\Gamma^b, \psi_b)$ (see [23, Chapter 2]). Actually one can pick the elements of $\mathcal{B}$ to be of the form $dia[\hat{q},q]$ for $q \in GL_n(K)_{\mathbf{h}}$. For a $q \in GL_n(K)_{\mathbb{A}}$ and an $s \in S_{\mathbb{A}}$ we have (see [24, pages 167-168]

$$\mathbf{f}\left(\left(\begin{array}{cc} q & s\hat{q} \\ 0 & \hat{q}\end{array}\right)\right) = det(q_{\mathbf{a}})^{k\rho}\sum_{\tau \in S_+} c_{\mathbf{f}}(\tau,q)e_{\mathbf{a}}(\mathbf{i}q^*\tau q)e_{\mathbb{A}}(\tau s)$$

For the properties of $c_{\mathbf{f}}(\tau,q)$ we refer to the [24, Proposition 20.2] and for the definition of $e_{\mathbb{A}}$ to [24, page 127]. Finally one can extend the notion of the Petersson inner product to the adelic setting (see [23, page 81]. We write $\langle\cdot,\cdot\rangle$ for this.

For a subfield L of $\mathbb{C}$ we will be writing $\mathcal{M}_k(\Gamma^q,\psi,L)$ for the subspace of $\mathcal{M}_k(\Gamma^q,\psi)$ whose Fourier expansion at infinity has coefficients in L. For a fixed set $\mathcal{B}$, with elements of the form $diag[\hat{q},q]$ and $q \in GL_n(K)_{\mathbf{h}}$, we will be writing $\mathcal{M}_k(D,\psi,L)$ for the subspace of $\mathcal{M}_k(D,\psi)$ consisting of elements whose image under the above isomorphism lies in $\oplus_{b\in\mathcal{B}}\mathcal{M}_k(\Gamma^b,\psi_b,L)$. Finally we define the adelic cusp forms $\mathcal{S}_k(D,\psi)$ to be the subspace of $\mathcal{M}_k(D,\psi)$, which map to $\oplus_{b\in\mathcal{B}}\mathcal{S}_k(\Gamma^b,\psi_b)$.

In the rest of this section we obtain some results which we will use later. The first two are minor modification of two results in [24, Theorem 10.4 (3) and Theorem 7.11]. Since they are not stated there in the form we need for our purposes we have decided to provide the needed changes in the proofs in [24].

Lemma 2.3. *Let $q \in GL_n(K)_{\mathbf{h}}$ be a diagonal matrix and consider the space $\mathcal{M}_k(\Gamma^q,\psi)$, with ψ a character of finite order. We write Φ for the Galois closure of K over $\mathbb{Q}$ and Φ_ψ for the extension of Φ obtained by adjoining the values of the character ψ. Then we have that*

$$\mathcal{M}_k(\Gamma^q,\psi,\mathbb{C}) = \mathcal{M}_k(\Gamma^q,\psi,\Phi_\psi)\otimes_{\Phi_\psi}\mathbb{C}$$

Proof. This is in principle [24, Theorem 10.4 (3) and (4)]. The difference with the statement there is that we want to have a more precise base field in the presence of nebentype. Keeping the notation as in the proof in [24] we explain how we can obtain the result.

Our condition on q guarantees that if we write $D^q = qDq^{-1}$ then we have that $xD^qx^{-1} = D^q$ for every $x \in \imath(\mathbb{Z}_{\mathbf{h}}^{\times})$. Moreover, since we always take our D of a very specific type we have that $\mathcal{M}_k(\Gamma^q,\psi)^\tau = \mathcal{M}_k(\Gamma^q,\psi)$ for every $\tau \in Gal(\overline{\mathbb{Q}}/\Phi_\psi)$ (see the remark of Shimura after the proof of his Theorem 10.4 in [24]). But then the argument of in the proof of Theorem 10.4 (4) in [24] works also in this situation. Namely any f will be in some $\mathcal{M}_k(\Gamma^q,\Xi,\psi)$ and $f^\sigma \in \mathcal{M}_k(\Gamma^q,\Xi,\psi)$ for all $\sigma \in Gal(\Xi/\Phi_\psi)$. In particular for any $a \in \Xi$ we will have $\sum_\sigma (af)^\sigma \in \Phi_\psi$. □

The next result is a modification of [24, Theorem 7.11]. There are three differences: (i) we state it here for any weight, not necessarily parallel, (ii) our field Φ' contains the Galois closure of K over $\mathbb{Q}$, and (iii) the field Φ' contains a finite abelian extension of $\mathbb{Q}$ and not the whole $\mathbb{Q}_{ab}$ as in [24].

Lemma 2.4. *Let Φ denote the Galois closure of K over $\mathbb{Q}$, and let $\alpha \in G$. Consider an element $f \in \mathcal{M}_k(\Gamma, \chi, \Phi)$ for some $k \in \mathbf{b}$. Then $f|_k\alpha \in \mathcal{M}_k(\alpha^{-1}\Gamma\alpha, \chi_\alpha, \Phi')$, where Φ' is a finite extension of Φ obtained by adjoining roots of unity, and χ_α the character of the congruence subgroup $\alpha^{-1}\Gamma\alpha$ defined by $\chi_\alpha(\gamma) = \chi(\alpha\gamma\alpha^{-1})$ for $\gamma \in \alpha^{-1}\Gamma\alpha$.*

Proof. As in the proof of [24, Theorem 7.11] we can write α as a product of elements in the Siegel parabolic P and η. So it is enough to establish the claim for such elements. For the element $\eta \in SU(n,n)$ the claim follows from the fact that $(f^\sigma|\eta)^\sigma = \chi(t)^n f$ where $t \in \mathbb{Z}_{\mathbf{h}}^\times$, such that $[t, \mathbb{Q}] = \sigma$ on $\mathbb{Q}_{ab}$. For this we refer to [9]. So in particular we need to adjoin the values of the finite character χ. For the action of the elements in the parabolic we use the same argument as in the proof of Theorem 7.11 in [24], with the only difference that since we are also considering non-parallel weight we need to take the field Φ' to contain Φ. □

Lemma 2.5. *Write h_F and h_K for the class number of F and K respectively. Assume $h_F = 1$ and that $(2n, h_K) = 1$. Then we can pick a finite number of $q \in GL(K)_{\mathbf{h}}$ such that $\mathcal{B} = \{diag[q, \hat{q}]\}$ and $det(qq^*) = 1$.*

Proof. This lemma is in principle the one of Klosin in [18, Corollary 3.9] by observing that his argument generalizes to CM fields by assuming the class number of F is equal to one. □

For a fixed ideal $\mathfrak{b}$ we write $D(\mathfrak{c})$ for the group $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$.

Lemma 2.6. *Write $H := \{x \in K^\times | xx^\rho = 1\}$. Then*

$$det(D(\mathfrak{c})) = det(D(\mathcal{O}_F)) = U_0 := \{x^\rho/x \in H_{\mathbb{A}} | x_v \in \mathcal{O}_{K_v}^\times, \forall v \in \mathbf{h}\}.$$

In particular we have that there exists a set $\mathcal{B}$ such that

$$G_{\mathbb{A}} = \coprod_{b \in \mathcal{B}} GbD(\mathfrak{c}) = \coprod_{b \in \mathcal{B}} GbD(\mathcal{O}_F).$$

Moreover the elements $b \in \mathcal{B}$ can be taken in the form $b = \begin{pmatrix} \hat{q} & 0 \\ 0 & q \end{pmatrix}$ with $q \in GL_n(K)_{\mathbf{h}}$.

Proof. For the fact that $det(D(\mathcal{O}_F)) = U$ we refer to [23, Lemma 5.11]. The first equality then follows by observing that the map $q \mapsto \begin{pmatrix} \hat{q} & 0 \\ 0 & q \end{pmatrix}$ defined an embedding of $GL_n(K)$ into G. But then $det\left(\begin{pmatrix} \hat{q} & 0 \\ 0 & q \end{pmatrix}\right) = \frac{det(q)^\rho}{det(q)}$ and hence also $det(D(\mathfrak{c})) = U$ (see also the proof of Lemma 9.10 in [23]. The last statement of the lemma follows from [23, Lemma 9.8]. □

3. Eisenstein and Theta series for $U(n,n)_{/F}$

In this section we collect some results on theta series and Eisenstein series, which we will need later.

3.1. THETA SERIES. Shimura in [24, Appendix A.5] and [23, Appendix A.7] attach a theta series, $\theta_{\mathbb{A}}(x,\omega)$, to a Hecke character ω, a positive definite matrix τ, a matrix $r \in GL_n(K)_{\mathbf{h}}$ and an element $\mu \in \mathbb{Z}^{\mathbf{b}}$ with $\mu_v \mu_{v\rho} = 0$ and $\mu_v \geq 0$ for all $v \in \mathbf{b}$. His construction depends on a choice of a Hecke character ϕ of K with infinity type $\phi(x_{\mathbf{a}}) = x_{\mathbf{a}}^{-1}|x_{\mathbf{a}}|$, such that $\phi = \theta$ on $F_{\mathbb{A}}^{\times}$, where θ is the non-trivial character of the quadratic extension K/F. We summarize various results of Shimura in [23, 24] in the following theorem.

THEOREM 3.1 (Shimura, section A5.5 in [24] and Proposition A7.16 in [23]). *$\theta_{\mathbb{A}}(x,\omega)$ is an element in $M_l(C,\omega')$ with $C = D[\mathfrak{b}^{-1},\mathfrak{bc}]$ and $\omega' = \omega\phi^{-n}$, and $l = \mu + n\mathbf{a}$. Moreover $\theta_{\mathbb{A}}(x,\omega)$ is a cusp form if $\mu \neq 0$. The ideals $\mathfrak{b}$ and $\mathfrak{c}$ are given as follows. We define a fractional ideals $\mathfrak{y}$ and $\mathfrak{t}$ in F such that $g^*\tau g \in \mathfrak{y}$ and $h^*\tau^{-1}h \in \mathfrak{t}^{-1}$ for all $g \in r\mathcal{O}_K^n$ and $h \in \mathcal{O}_K^n$. Then we can take $\mathfrak{b} = \mathfrak{d}\mathfrak{y}$ and $\mathfrak{bc} = \mathfrak{d}(\mathfrak{t}\mathfrak{e}\mathfrak{f}^{\rho}\mathfrak{f} \cap \mathfrak{y}\mathfrak{e} \cap \mathfrak{y}\mathfrak{f})$, where $\mathfrak{e}$ is the relative discriminant of K over F. For an element $q \in GL_n(K)_{\mathbf{h}}$ we have that the q^{th} component of the theta series is given by*

$$\theta_{q,\omega}(z) = \omega'(det(q)^{-1})|det(q)|_K^{n/2} \times$$

$$\sum_{\xi \in V \cap rR^*q^{-1}} \omega_{\mathbf{a}}(det(\xi))\omega^*(det(r^{-1}\xi q)\mathcal{O}_K)det(\xi)^{\mu\rho}\mathbf{e}_{\mathbf{a}}(\xi^*\tau\xi z),$$

where $\xi \in V \cap rR^*q^{-1}$ such that $\xi^*\tau\xi = \sigma$. Here $V = M_n(K)$, $R^* = \{w \in M_n(K)_{\mathbb{A}} | w_v \prec \mathcal{O}_{Kv}, \forall v \in \mathbf{h}\}$, and ω^* denotes the ideal character associated to the Hecke character ω.

3.2. EISENSTEIN SERIES. In this section we introduce Siegel type Eisenstein series following closely Shimura [23, Chapter III] and [24, Chapter IV]. This Eisenstein series will be nearly holomorphic, and can be given an algebraic structure. In particular there is an action of the absolute Galois group on them and our goal is to obtain a reciprocity law of this action. This question has been considered by Feit in [9] for Eisenstein series of the special unitary group $SU(n,n)$, and hence our main aim here is to extend his results to the unitary group. The main difference of course is the lack of the strong approximation theorem which is available for the special unitary group. Our main contribution in this section is Lemma 3.8.

We consider a weight $k \in \mathbb{Z}^{\mathbf{b}}$ and a $\mathfrak{c} \subset O_F$, an integral ideal of F. Moreover we pick a Hecke character χ of K with infinity type $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{\ell}|x_{\mathbf{a}}|^{-\ell}$, where $\ell = (k_v - k_{v\rho})_{v\in\mathbf{a}}$ and of conductor dividing $\mathfrak{c}$. For a fractional ideal $\mathfrak{b}$ we write D for $D[\mathfrak{b}^{-1},\mathfrak{bc}]$. Then for a pair $(x,s) \in G_{\mathbb{A}} \times \mathbb{C}$, we denote by $E_{\mathbb{A}}(x,s) = E_{\mathbb{A}}(x,s;\chi,D)$ the Siegel type Eisenstein series associated to the character χ and the weight k. We recall here its definition, taken from [24, page 131],

$$E_{\mathbb{A}}(x,s) = \sum_{\gamma \in P\backslash G} \mu(\gamma x)\epsilon(\gamma x)^{-s}, \quad \Re(s) >> 0,$$

where P is the Siegel parabolic subgroup and the function $\mu : G_{\mathbb{A}} \to \mathbb{C}$ is supported on $P_{\mathbb{A}}D \subset G_{\mathbb{A}}$, defined for an $x = pw$ with $p \in P_{\mathbb{A}}$ and $w \in D$ by,

$$\mu(x) := \prod_v \mu_v(p_v w_v), \text{ with } \mu_v(p_v w_v) = \begin{cases} \chi_v(det(d_{p_v}))^{-1} & \text{if } v \in \mathbf{h},\ v \nmid \mathfrak{c} \\ \chi_v(det(d_{p_v} d_{w_v}))^{-1} & \text{if } v \in \mathbf{h},\ v \mid \mathfrak{c} \\ j_{x_v}^{k_v}(\mathbf{i})^{-1}|j_{x_v}(\mathbf{i})|^{m_v} & \text{if } v \in \mathbf{a}, \end{cases}$$

where $m = (k_v + k_{v\rho})_v$. The function $\epsilon : G_{\mathbb{A}} \to \mathbb{C}$ is defined as $\epsilon(x) = |det(d_p)|_{\mathbf{h}} j_x(\mathbf{i})^{2\mathbf{a}}$ where $x_{\mathbf{h}} = pw$ with $p \in P_{\mathbf{h}}$ and $w \in D[\mathfrak{b}^{-1}, \mathfrak{b}]_{\mathbf{h}}$. Moreover we define the normalized Eisenstein series

$$D_{\mathbb{A}}(x,s) = E_{\mathbb{A}}(x,s) \prod_{i=0}^{n-1} L_{\mathfrak{c}}(2s - i, \chi_1 \theta^i),$$

where we recall that θ is the non-trivial character associated to K/F and χ_1 is the restriction of the Hecke character χ to $F_{\mathbb{A}}^{\times}$. Here, for a Hecke character ϕ of F, we write $L_{\mathfrak{c}}(s,\phi)$ for the Dirichlet series associated to ϕ with the Euler factors at the primes dividing $\mathfrak{c}$ removed. For a $q \in GL_n(K)_{\mathbf{h}}$ we define $D_q(z,s;k,\chi,\mathfrak{c})$ a function on $(z,s) \in \mathcal{H} \times \mathbb{C}$ associated to $D_{\mathbb{A}}(x,s)$ by the rule (see [24, page 132])

$$D_q(x(\mathbf{i}), s; k, \chi, \mathfrak{c}) = j_x^k(\mathbf{i}) D_{\mathbb{A}}(diag[q, \hat{q}]x, s).$$

Even though the above defined Eisenstein series are the ones which are relevant to our applications, we need to introduce yet another kind of Eisenstein series for which we have explicit information about their Fourier expansion. In particular we define the $E_{\mathbb{A}}^*(x,s) := E_{\mathbb{A}}(x\eta_{\mathbf{h}}^{-1}, s)$ and $D_{\mathbb{A}}^*(x,s) := D_{\mathbb{A}}(x\eta_{\mathbf{h}}^{-1}, s)$, and as before we write $D_q^*(z,s;k,\chi,\mathfrak{c})$ for the series associated to $D_{\mathbb{A}}^*(x,s)$. We write the Fourier expansion of $E_{\mathbb{A}}^*(x,s)$ by

$$E_{\mathbb{A}}^*\left(\begin{pmatrix} q & \sigma\hat{q} \\ 0 & \hat{q} \end{pmatrix}, s\right) = \sum_{h \in S} c(h,q,s)\mathbf{e}_{\mathbb{A}}(h\sigma),$$

where $q \in GL_n(K)_{\mathbb{A}}$ and $\sigma \in S_{\mathbb{A}}$. For the coefficients $c(h,q,s)$ we have the following results of Shimura,

Proposition 3.2 (Shimura, Proposition 18.14 and Proposition 19.2 in [23]). *Suppose that $\mathfrak{c} \neq \mathcal{O}_F$. Then $c(h,q,s) \neq 0$ only if $({}^t qhq)_v \in (\mathfrak{d}\mathfrak{b}^{-1}\mathfrak{c}^{-1})_v \tilde{S}_v$ for every $v \in \mathbf{h}$. In this case*

$$c(h,q,s) = C(S)\chi(det(-q))^{-1}|det(qq^*)_{\mathbf{h}}|_{\mathbf{h}}^{n-s}|det(qq^*)_{\mathbf{a}}|^s N(\mathfrak{b}\mathfrak{c})^{-n^2} \times$$

$$\Xi(qq^*; h; s\mathbf{a} + (k_v - k_{\rho v})/2, s\mathbf{a} - (k_v + k_{\rho v})/2)\alpha_{\mathfrak{c}}(\omega \cdot {}^t qhq, 2s, \chi_1),$$

where $N(\cdot)$ denotes the norm from F to $\mathbb{Q}$, $|x|_{\mathbf{h}} := \prod_{v \in \mathbf{h}} |x_v|_v$ with $|\cdot|_v$ the normalized absolute value at the finite place v, ω is a finite idele such that $\omega O_F = \mathfrak{b}\mathfrak{d}$, $\tilde{S}_v$ the dual lattice to $S(O_{F_v})$, the Hermitian matrices with entries in O_{F_v}, and

$$C(S) := 2^{n(n-1)d}|D_F|^{-n/2}|D_K|^{-n(n-1)/4}.$$

Moreover if we write r for the rank of h and let $g \in GL_n(F)$ such that $g^{-1}hg = diag[h', 0]$ with $h' \in S^r$. Then

$$\alpha_{\mathfrak{c}}(\omega \cdot {}^t qhq, 2s, \chi_1) = \Lambda_{\mathfrak{c}}(s)^{-1}\Lambda_h(s) \prod_{v \in \mathbf{c}} f_{h,q,v}\left(\chi(\pi_v)|\pi_v|^{2s}\right),$$

where

$$\Lambda_{\mathfrak{c}}(s) = \prod_{i=0}^{n-1} L_{\mathfrak{c}}(2s-i, \chi_1\theta^i), \quad \Lambda_h(s) = \prod_{i=0}^{n-r+1} L_{\mathfrak{c}}(2s-n-i, \chi_1\theta^{n+i-1}).$$

Here $f_{h,q,v}$ are polynomials with coefficients in $\mathbb{Z}$, independent of χ_1. The set $\mathbf{c}$ consists of finitely many finite places of F which are prime to $\mathfrak{c}$. For the precise description we refer to [23, page 158-159]*, and for the function $\Xi(g;h;\alpha,\beta) = \prod_{v\in\mathbf{a}} \xi(y_v, h_v; \alpha_v, \beta_v)$ we refer to* [24, page 140].

For a number field W, a $k \in \mathbb{Z}^{\mathbf{b}}$ and $r \in \mathbb{Z}^{\mathbf{a}}$ we follow [24] and write $\mathcal{N}_k^r(W)$ for the space of W-rational nearly holomorphic modular forms of weight k (see [24, page 103 and page 110] for the definition). The index r should be thought as a degree of nearly holomorphicity, with $r = 0$ being holomorphic. Without going into much of details here on the definition of the nearly holomorphic forms, we just mention that even though these modular forms are not holomorphic, one can still impose an algebraic structure on them. In general this can be done by studying their values at CM points. However we give here an equivalent definition taken from [24, page 117], which is enough for our purposes, and it is based on the Fourier expansion. Namely an element $f \in \mathcal{N}_k^r(W)$ is a C^∞ function on $\mathcal{H}$, with the modularity property (i.e. $f|_k\gamma = f$ for all γ in some congruence subgroup Γ) and has an expansion of the form

$$f(z) = \sum_{h\in\mathfrak{S}} s_h(\pi^{-1}i({}^t z - z^\rho)^{-1})e_{\mathbf{a}}(hz),$$

where $\mathfrak{S}$ is a lattice of S_+, and $s_h(T)$ is a finite sum of elements of the form $\prod_{v\in\mathbf{a}} P_v(T_{ij}^{(v)})$ with $P_v(T_{ij}^{(v)})$ are homogeneous polynomials of degree r_v in the variables $T_{ij}^{(v)}$, $1 \le i, j \le n$, with coefficients in W. It turns out (see [24, Theorem 14.12]) that the absolute Galois group acts on them by acting on the coefficients of the polynomials $s_h(T)$.

For the Eisenstein series $D_q(z, s; \chi, \mathfrak{c})$ we have the following theorem of Shimura [24, Theorem 17.12], which tells us for which values of s the Eisenstein series introduced above are nearly holomorphic.

THEOREM 3.3 (Shimura, Theorem 17.12 in [24]). *We set $m := (k_v + k_{v\rho})_{v\in\mathbf{a}}$. Let K' be the reflex field of K with respect to the selected CM type and K_χ the field generated over K' by the values of χ. Let Φ be the Galois closure of K over $\mathbb{Q}$ and suppose $2n - m_v \le \mu \le m_v$ and $m_v - \mu \in 2\mathbb{Z}$ for every $v \in \mathbf{a}$. Then $D_q(z, \mu/2; k, \chi, \mathfrak{c})$ belongs to $\pi^\beta \mathcal{N}_k^r(\Phi K_\chi \mathbb{Q}_{ab})$, except when $0 \le \mu < n$, $\mathfrak{c} = \mathcal{O}_F$, and $\chi_1 = \theta^\mu$, where $\beta = (n/2)\sum_{v\in\mathbf{a}}(m_v + \mu) - dn(n-1)/2$ and*

$r = n(m-\mu+2)/2$ if $\mu = n+1$, $F = \mathbb{Q}$ and $\chi_1 = \theta^{n+1}$. Otherwise $r = (n/2)(m - |\mu - n|\mathbf{a} - n\mathbf{a})$.

We now explain how the nearly holomorphic Eisenstein series can be obtained from holomorphic ones (excluding some cases) by the use of the so-called Maasss-Shimura differential operators. An important property of these operators is that they are Galois equivariant in the sense explained in Equation 1 below.
For an element $p \in \mathbb{Z}^{\mathbf{a}}$ and a weight $q \in \mathbb{Z}^{\mathbf{b}}$ we write Δ_q^p for the differential operators defined in [24, page 146 and page 148]. Moreover it is shown in [24] that $\Delta_q^p \mathcal{N}_q^t(\Phi\mathbb{Q}_{ab}) \subset \pi^{n|p|}\mathcal{N}_{q'}^{t+np}(\Phi\mathbb{Q}_{ab})$, where $q' \in \mathbb{Z}^{\mathbf{b}}$ is defined by $q'_v = q_v + p_v$ and $q'_{\rho v} := q_{\rho v} + p_v$ for $v \in \mathbf{a}$, and for any $f \in \mathcal{N}_q^t(W)$ and $\sigma \in Gal(W/\Phi)$ we have that

$$\left(\pi^{-n|p|}\Delta_q^p(f)\right)^\sigma = \pi^{-n|p|}\Delta_q^p(f^\sigma) \tag{1}$$

Let $\mu \in \mathbb{Z}$ and $k \in \mathbb{Z}^{\mathbf{b}}$ be as in Theorem 3.3. If $\mu \geq n$ then by [24, page 148] we have that

$$\Delta_{\mu\mathbf{a}}^p D_q(z,\mu/2;k',\chi,\mathfrak{c}) =^{\mathbb{Q}^\times} (i/2)^{n|p|} D_q(z,\mu/2;k,\chi,\mathfrak{c}), \tag{2}$$

where $p = (m - \mu\mathbf{a})/2$ and $k' \in \mathbb{Z}^{\mathbf{b}}$ with $k'_v = k_v - p_v$ and $k_{\rho v} = k_{\rho v} - p_v$ for $v \in \mathbf{a}$. The notation $=^{\mathbb{Q}^\times}$ means equality up to elements in $\mathbb{Q}^\times$, and $|p| := \sum_{v \in \mathbf{a}} p_v$. Similarly if $\mu < n$ (see again [24, page 148]) then we have

$$\Delta_{\nu\mathbf{a}}^p D_q(z,\mu/2;k'',\chi,\mathfrak{c}) =^{\mathbb{Q}^\times} (i/2)^{n|p|} D_q(z,\mu/2;k\mathbf{a},\chi,\mathfrak{c}), \tag{3}$$

where $\nu = 2n - \mu$, $p = (m - \nu\mathbf{a})/2$ and $k''_v = k'_{\rho v}$ for all $v \in \mathbf{b}$. Now the following lemma is immediate from the above equations, and it reduces the study of the Galois equivariant properties of the nearly Eisenstein series to holomorphic ones of a very particular weight.

LEMMA 3.4. *Assume there exists $A(\chi), B(\chi) \in \mathbb{Q}_{ab}$ and $\beta_1, \beta_2 \in \mathbb{N}$ such that for all $\sigma \in Gal(K_\chi/\mathbb{Q})$*

$$\left(\frac{D_q(z,\mu/2;k',\chi,\mathfrak{c})}{\pi^{\beta_1}A(\chi)}\right)^\sigma = \frac{D_q(z,\mu/2;k',\chi^\sigma,\mathfrak{c})}{\pi^{\beta_1}A(\chi^\sigma)}, \ \ \mu \geq n, \text{ and } k'_v + k'_{\rho v} = \mu$$

and

$$\left(\frac{D_q(z,\mu/2;k'',\chi,\mathfrak{c})}{\pi^{\beta_2}B(\chi)}\right)^\sigma = \frac{D_q(z,\mu/2;k'',\chi^\sigma,\mathfrak{c})}{\pi^{\beta_2}B(\chi^\sigma)}, \ \ \mu < n, \text{ and } k''_v + k''_{\rho v} = \nu.$$

Then we have for $\mu \geq n$ that

$$\left(\frac{D(z,\mu/2;k,\chi,\mathfrak{c})}{\pi^{\beta_1+n|p|}i^{n|p|}A(\chi)}\right)^\sigma = \frac{D(z,\mu/2;k,\chi^\sigma,\mathfrak{c})}{\pi^{\beta_1+n|p|}i^{n|p|}A(\chi^\sigma)}, \ \ p = (m-\mu\mathbf{a})/2 \in \mathbb{Z}^{\mathbf{a}},$$

and for $\mu < n$ that

$$\left(\frac{D(z,\mu/2;k,\chi,\mathfrak{c})}{\pi^{\beta_2+n|p|}i^{n|p|}B(\chi)}\right)^\sigma = \frac{D(z,\mu/2;k,\chi^\sigma,\mathfrak{c})}{\pi^{\beta_2+n|p|}i^{n|p|}B(\chi^\sigma)}, \ \nu = 2n-\mu \ \ p = (m-\nu\mathbf{a})/2 \in \mathbb{Z}^{\mathbf{a}},$$

We are interested in algebraicity statements for Eisenstein series with the property that $k_v + k_{v\rho} \geq n$ for all $v \in \mathbf{a}$, where k is the weight of the Eisenstein series. The following lemma indicates that for these Eisenstein series it is enough to study the effect of the action of the Galois group on the full rank Fourier coefficients. The proof of the lemma can be found in [2], where the similar situation of Siegel modular forms was considered. Its proof relies on the fact (see [24, Proposition 6.16]) that for the weights under consideration there does not exist singular Hermitian modular forms, that is Hermitian modular forms whose Fourier coefficients are supported on singular Hermitian matrices.

LEMMA 3.5. *Let* $f(z) = \sum_{h \in S} c(h)\mathbf{e}^n_{\mathbf{a}}(hz) \in \mathcal{M}_k(\overline{\mathbb{Q}})$ *with* $k_v + k_{\rho v} \geq n$. *Assume that for an element* $\sigma \in Gal(\overline{\mathbb{Q}}/\Phi)$ *we have* $c(h)^\sigma = ac(h)$ *for all* h *with* $det(h) \neq 0$ *for some* $a \in \mathbb{C}$. *Then* $c(h)^\sigma = ac(h)$ *for all* $h \in S$. *In particular* $f^\sigma = af$.

We can now consider the action of $Gal(\overline{\mathbb{Q}}/\Phi)$ on the Eisenstein series. Thanks to the Lemma 3.4 above it is enough to consider the Galois action on the holomorphic ones. That is, we consider the following two Eisenstein series

(1) $D_q(z, \mu/2; k, \chi, \mathfrak{c}) \in \pi^\beta \mathcal{M}_k(\overline{\mathbb{Q}})$ for $\mu \geq n$ and $k_v + k_{\rho v} = \mu$,
(2) $D_q(z, \nu/2; k, \chi, \mathfrak{c}) \in \pi^\beta \mathcal{M}_k(\overline{\mathbb{Q}})$ for $\nu = 2n - (k_v + k_{\rho v})$ for all $v \in \mathbf{a}$ and $\nu \leq n$,

where β is determined by Theorem 3.3.
For these values of s we collect in the following lemma some properties that we will need concerning the functions $\Xi(y, h; \alpha, \beta) = \prod_{v \in \mathbf{a}} \xi(y, h; \alpha, \beta)$. For the proof, which can be obtained from the study of this function in [21], we refer to the similar proof done in the symplectic case in [2].

LEMMA 3.6. *Let* $h \in S$ *with* $det(h) \neq 0$ *and* $y \in S^{\mathbf{a}}_+(\mathbb{R})$. *Then for* $\mu \in \mathbb{Z}$

$$\Xi(y, h; \mu, 0) = 2^{d(1-n)} i^{-dn\mu} (2\pi)^{dn\mu} \Gamma_n(\mu)^{-d} N(det(h))^{\mu - n} e_{\mathbf{a}}(iyh)$$

and

$$\Xi(y, h; n, n-\mu) = i^{-dn\mu} 2^{-(dn(\mu+1))} \pi^{dn^2} \Gamma_n(n)^{-d} \left(\prod_{v \in \mathbf{a}} det(y_v)^{-(n-\mu)} \right) e_{\mathbf{a}}(iyh).$$

GALOIS RECIPROCITY OF EISENSTEIN SERIES. We start by considering first the holomorphic Eisenstein series $D^*_q(z, \mu/2; k, \chi, \mathfrak{c}) \in \pi^\beta \mathcal{M}_k(\overline{\mathbb{Q}})$ for $\mu \geq n$ and $k_v + k_{\rho v} = \mu$. If we write $D^*_q(z, \mu/2; k, \chi, \mathfrak{c}) = \sum_{h \in S} b(h, q, \chi) e_{\mathbf{a}}(hz)$ then we have that for full rank h,

$$b(h, q, \chi) = \prod_{i=0}^{n-1} L_{\mathfrak{c}}(\mu - i, \chi_1 \tau^i) det(y)^{-\mu/2} c(h, q, \mu/2), \;\; z = x + \mathbf{i}y.$$

In particular we conclude that

$$b(h, q, \chi) = det(y)^{-\mu/2} C(S) \chi(det(-q))^{-1} |det(qq^*)_{\mathbf{h}}|_{\mathbf{h}}^{n-\mu/2} det(qq^*)_{\mathbf{a}}|^{\mu/2} \times$$
$$N(\mathfrak{bc})^{-n^2} 2^{d(1-n)} i^{-dn\mu} (2\pi)^{dn\mu} \Gamma_n(\mu)^{-d} N(det(h))^{\mu-n} \prod_{v \in \mathbf{c}} f_{h,q,v}\left(\chi_1(\pi_v)|\pi_v|^\mu\right)$$

and hence

$$\left(\frac{b(h,q,\chi)}{c(S)|det(qq^*)_{\mathbf{h}}|_{\mathbf{h}}^{n-\mu/2} i^{-dn\mu}(2\pi)^{dn\mu}}\right)^{\sigma} = \frac{b(h,q,\chi^{\sigma})}{c(S)|det(qq^*)_{\mathbf{h}}|_{\mathbf{h}}^{n-\mu/2} i^{-dn\mu}(2\pi)^{dn\mu}}$$

for all $\sigma \in Gal(\overline{\mathbb{Q}}/\Phi)$. In particular we conclude that

$$\left(\frac{D_q^*(z,\mu/2;k,\chi,\mathfrak{c})}{c(S)|det(qq^*)_{\mathbf{h}}|_{\mathbf{h}}^{n-\mu/2} i^{-dn\mu}(2\pi)^{dn\mu}}\right)^{\sigma} = \frac{D_q^*(z,\mu/2;k,\chi^{\sigma},\mathfrak{c})}{c(S)|det(qq^*)_{\mathbf{h}}|_{\mathbf{h}}^{n-\mu/2} i^{-dn\mu}(2\pi)^{dn\mu}}$$

for all $\sigma \in Gal(\overline{\mathbb{Q}}/\Phi)$. Similarly we consider the Eisenstein series

$$D_q^*(z,\nu/2;k,\chi,\mathfrak{c}) \in \pi^{\beta}\mathcal{M}_k(\overline{\mathbb{Q}})$$

for $\nu = 2n - (k_v + k_{\rho v})$ for all $v \in \mathbf{a}$, $k_v + k_{v\rho} = \mu \in \mathbb{N}$ and $\nu \leq n$. If we write $\sum_{h\in S} a(h,q,\chi)e_{\mathbf{a}}(hz)$ then we have that for full rank h,

$$a(h,q,\chi) = \prod_{i=0}^{n-1} L_{\mathfrak{c}}(\nu - i, \chi_1\tau^i) det(y)^{-\mu/2} c(h,q,\nu/2).$$

In particular we have

$$a(h,q,\chi) = det(y)^{-\mu/2} C(S)\chi(det(-q))^{-1} |det(qq^*)_{\mathbf{h}}|_{\mathbf{h}}^{n-\nu/2} det(qq^*)_{\mathbf{a}}|^{\nu/2} i^{-dn\mu} \times$$

$$N(\mathfrak{bc})^{-n^2} 2^{-(dn(\mu+1))} \pi^{dn^2} \Gamma_n(n)^{-d} \prod_{v\in\mathbf{a}} det(y_v)^{-(n-\mu)} \prod_{v\in\mathbf{c}} f_{h,q,v}\left(\chi_1(\pi_v)|\pi_v|^{\nu}\right) =$$

$$C(S)\chi(det(-q))^{-1} |det(qq^*)_{\mathbf{h}}|_{\mathbf{h}}^{n-\nu/2} N(\mathfrak{bc})^{-n^2} \times i^{-dn\mu} 2^{-(dn(\mu+1))} \pi^{dn^2} \Gamma_n(n)^{-d} \times$$

$$\prod_{v\in\mathbf{c}} f_{h,q,v}\left(\chi_1(\pi_v)|\pi_v|^{\nu}\right).$$

In particular as before we have that

$$\left(\frac{a(h,q,\chi)}{c(S)|det(qq^*)_{\mathbf{h}}|_{\mathbf{h}}^{n-\nu/2} i^{-dn\mu} 2^{-(dn(\mu+1))} \pi^{dn^2}}\right)^{\sigma} =$$

$$\frac{a(h,q,\chi^{\sigma})}{c(S)|det(qq^*)_{\mathbf{h}}|_{\mathbf{h}}^{n-\nu/2} i^{-dn\mu} 2^{-(dn(\mu+1))} \pi^{dn^2}}$$

for all $\sigma \in Gal(\overline{\mathbb{Q}}/\Phi)$. In particular we conclude that

$$\left(\frac{D_q^*(z,\nu/2;k,\chi,\mathfrak{c})}{c(S)|det(qq^*)_{\mathbf{h}}|_{\mathbf{h}}^{n-\nu/2} i^{-dn\mu} 2^{-(dn(\mu+1))} \pi^{dn^2}}\right)^{\sigma} =$$

$$\frac{D_q^*(z,\nu/2;k,\chi^{\sigma},\mathfrak{c})}{c(S)|det(qq^*)_{\mathbf{h}}|_{\mathbf{h}}^{n-\nu/2} i^{-dn\mu} 2^{-(dn(\mu+1))} \pi^{dn^2}}$$

Remark 3.7. We note here that the appearance of the term $|det(qq^*)_{\mathbf{h}}|_{\mathbf{h}}$ makes the statement of the reciprocity law dependent on the choice of the component q, when of course the exponents have roots. It is possible, under some assumptions, to force $det(qq^*) = 1$. For example, as it was shown in the previous section, we could take that the class number of K is prime to $2n$ and the class number of F is one. Or we could write the reciprocity law not over Φ but some extension which contains the possible roots of $|det(qq^*)_{\mathbf{h}}|_{\mathbf{h}}$ of the finitely many selected q's. However as we have seen, these terms show up also on the Fourier expansion of the theta series, and later we will be considering products of the theta series with Eisenstein series, and these terms will cancel.

Our next aim is to obtain information about the action of $Gal(\overline{\mathbb{Q}}/\Phi)$ on $D_{\mathbb{A}}(x,s)$ from the information we have obtained from the action on the components of $D^{\star}(x,s)$ for $s = \mu/2$ or $\nu/2$ as above. We start with the following lemma.

Lemma 3.8. *Let* $\mathbf{f} \in \mathcal{M}_k(C,\chi,\overline{\mathbb{Q}})$ *and define* $\mathbf{g}(x) := \mathbf{f}(x\eta_{\mathbf{h}}) \in \mathcal{M}_k(C',\psi,\overline{\mathbb{Q}})$, *where* $C' := \eta_{\mathbf{h}} C \eta_{\mathbf{h}}^{-1}$ *and some character* ψ *related to* χ. *For* $\sigma \in Gal(\overline{\mathbb{Q}}/\Phi)$ *we have*

$$\mathbf{g}^{\sigma}(x) = \chi(a^{-n})^{\sigma}\mathbf{f}^{\sigma}(x\eta_{\mathbf{h}}),$$

where $a \in \mathbb{Z}_{\mathbf{h}}^{\times}$ *such that with respect to the reciprocity law we have* $[a,\mathbb{Q}] = \sigma_{|\mathbb{Q}^{ab}}$. *In particular we have that*

$$\left(\frac{\mathbf{g}(x)}{\tau(\chi_1^n)}\right)^{\sigma} = \frac{\mathbf{f}^{\sigma}(x\eta_{\mathbf{h}})}{\tau((\chi_1^n)^{\sigma})},$$

where $\tau(\chi_1)$ *is the Gauss sum associated to the Hecke character* χ_1 *of* F *defined by taking the restriction of* χ *to* F.

Proof. We can pick a finite set $Q \subset GL_n(K)_{\mathbf{h}}$ such that $G_{\mathbb{A}} = \coprod_{q\in Q} G diag[q,\hat{q}]C$, and moreover we can pick this set so that the matrices q are diagonal (see for example the proof of [23, Lemma 9.8 (3)]). Then we know that the adelic form $\mathbf{f}$ corresponds to the the array of modular forms (f_p) for $p = diag[q,\hat{q}]$ and $q \in Q$. We have $f_p \in M_k(\Gamma^p,\chi_p)$. We now fix yet another decomposition of $G_{\mathbb{A}}$ by picking $Q' = \{q' := \hat{q} | q \in Q\}$ We note that if we write $p' := diag[q',\widehat{q'}]$ then $p' = \eta_{\mathbf{h}} p \eta_{\mathbf{h}}^{-1}$. In particular we see that indeed the set Q' gives a decomposition $G_{\mathbb{A}} = \coprod_{q'\in Q'} G diag[q',\widehat{q'}]C'$. Indeed we have

$$G_{\mathbb{A}} = \coprod_{q\in Q} G diag[q,\hat{q}]C = \coprod_{q\in Q} G diag[q,\hat{q}]C\eta_{\mathbf{h}}^{-1} = \coprod_{q\in Q} G\eta_{\mathbf{h}}^{-1}\eta_{\mathbf{h}} diag[q,\hat{q}]\eta_{\mathbf{h}}^{-1}C' =$$

$$\coprod_{q'\in Q'} G\eta_{\mathbf{h}}^{-1} diag[q',\widehat{q'}]C' = \coprod_{q'\in Q'} G\eta_{\mathbf{a}} diag[q',\widehat{q'}]C' = \coprod_{q'\in Q'} G diag[q',\widehat{q'}]\eta_{\mathbf{a}}C' =$$

$$\coprod_{q'\in Q'} G diag[q',\widehat{q'}]C'.$$

We now claim that $g_{p'}(z) = f_p(z)|_k\eta^{-1}$. It is enough to show that $(g_{p'}|_k y)(\mathbf{i}) = (f_p|_k\eta^{-1}|_k y)(\mathbf{i})$ for all $y \in G_{\mathbf{a}}$. But we have

$$(g_{p'}|_k y)(\mathbf{i}) = \mathbf{g}(p'y) = \mathbf{g}(\eta_{\mathbf{h}} p \eta_{\mathbf{h}}^{-1} y) = \mathbf{f}(\eta_{\mathbf{h}} p \eta_{\mathbf{h}}^{-1} y \eta_{\mathbf{h}}) = \mathbf{f}(\eta p \eta_{\mathbf{a}}^{-1} y) = \mathbf{f}(p\eta_{\mathbf{a}}^{-1} y),$$

and

$$(f_p|_k\eta^{-1}|_k y)(\mathbf{i}) = (f_p|\eta^{-1}y)(\mathbf{i}) = \mathbf{f}(p\eta_{\mathbf{a}}^{-1}y).$$

Hence we conclude that the p'-component of $\mathbf{g}$ is given by $f_p|_k\eta^{-1}$. By [24, Lemma 23.14] we have that $\mathbf{g}^\sigma$ has p'-component $g^\sigma_{p'} = (f_p|_k\eta^{-1})^\sigma$. But this has been established by Feit [9], (note that $\eta_n \in SU(n,n)$ and hence we have Shimura's reciprocity law also here, but see also [25, Lemma 5] and [2]) to conclude that $g^\sigma_{p'} = (f_p|_k\eta^{-1})^\sigma = \chi^\sigma(a^{-n})(f_p)^\sigma|_k\eta^{-1}$. A last remark here is that, since we are taking the q's diagonal matrices we have that $diag[q,\hat{q}]diag[a,a^{-1}]diag[q^{-1},\hat{q}^{-1}] = diag[a,a^{-1}]$, and hence the value of the nebentype character of each component at $diag[q,\hat{q}]diag[a,a^{-1}]diag[\hat{q},q]$ is the same. But, by the same argument as above, the p'-component of $\mathbf{f}^\sigma(x\eta_{\mathbf{h}})$ is equal to $(f_p)^\sigma|_k\eta^{-1}$ and hence we conclude the first part of the proof. The last follows by standard properties of Gauss sums over totally real fields (see for example [20, page 657]). Note that in the reciprocity law, the values of χ restricted to F (actually to $\mathbb{Q}$) matter. ☐

We now fix a set $\mathcal{B} \subset G_{\mathbf{h}}$ such that $G_{\mathbb{A}} = \coprod_{b\in\mathcal{B}} GbD$. We pick this set to be of the form $diag[q,\hat{q}]$ with $q \in GL_n(K_{\mathbf{h}})$ and diagonal. We define the Eisenstein series $\mathcal{D}(x,\frac{\mu}{2},\chi)$ to correspond to q-components $\frac{1}{|det(qq)_{\mathbf{h}}|_{\mathbf{h}}^{\delta_1/2}}D_q(z,\mu/2;k,\chi,\mathfrak{c})$ and similarly $\mathcal{D}(x,\frac{\nu}{2},\chi)$ to correspond to q-components $\frac{1}{|det(qq)_{\mathbf{h}}|_{\mathbf{h}}^{\delta_2/2}}D_q(z,\nu/2;k,\chi,\mathfrak{c})$, where $\delta_1 = 0$ if $n-\mu/2 \in \mathbb{Z}$ and 1 otherwise. Similarly $\delta_2 = 0$ if $n-\nu/2 \in \mathbb{Z}$ and 1 otherwise. Then from the Lemma 3.8 above and the reciprocity on the D_q^* Eisenstein series we conclude that,

PROPOSITION 3.9. *For $\sigma \in Gal(\overline{\mathbb{Q}}/\Phi)$ we have that*

$$\left(\frac{\mathcal{D}(x,\mu/2,\chi)}{C(S)\tau(\chi_1^n)i^{-dn\mu}(\pi)^{dn\mu}}\right)^\sigma = \frac{\mathcal{D}(x,\mu/2,\chi^\sigma)}{C(S)\tau((\chi_1^n)^\sigma)i^{-dn\mu}(\pi)^{dn\mu}},$$

and

$$\left(\frac{\mathcal{D}(x,\nu/2,\chi)}{C(S)\tau(\chi_1^n)i^{-dn\mu}(\pi)^{dn^2}}\right)^\sigma = \frac{\mathcal{D}(x,\nu/2,\chi^\sigma)}{C(S)\tau((\chi_1^n)^\sigma)i^{-dn\mu}(\pi)^{dn^2}}.$$

4. THE L-FUNCTION ATTACHED TO A HERMITIAN MODULAR FORM

In this section we introduce the L-functions, whose special values we will study, and present an integral representation of them by using the Rankin-Selberg method. Up to a result of Klosin [18, Equation (7.28)], presented as Theorem 4.2 below, everything else in this section is taken from [24, Chapter V].

We start by defining the L-functions associated to eigenforms, and introduce Shimura's generalization of the so-called Adrianov-Kalinin identity in the unitary case. In particular this identity will allow us to obtain a relation of the L-function with another Dirichlet series (in the notation below $D'_{r,\tau}(s,\mathbf{f},\chi)$), which even though itself does not have an Euler product representation, can be written as a Rankin-Selberg type integral. This Rankin-Selberg representation is the content of Theorem 4.1 below, which is due to Shimura. However this form is not enough for our purposes, the reason being that we do not have a good understanding of the Fourier expansion of the Eisenstein series involved. For this reason we will use an identity proved by Klosin (Theorem 4.2 below) to obtain a Rankin-Selberg representation involving the Siegel type Eisenstein series introduced in the last section, and whose Galois reciprocity we have studied.

4.1. THE STANDARD L-FUNCTION. We fix a fractional ideal $\mathfrak{b}$ and an integral ideal $\mathfrak{c}$ of F. We set $C = D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$. For an integral $\mathcal{O}_K$-ideal $\mathfrak{a}$ we write $T(\mathfrak{a})$ for the Hecke operator defined by Shimura in [24, page 162].
We consider a non-zero adelic Hermitian modular form $\mathbf{f} \in \mathcal{M}_k(C, \psi)$ and assume that we have $f|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ with $\lambda(\mathfrak{a}) \in \mathbb{C}$ for all integral $\mathcal{O}_K$-ideals $\mathfrak{a}$. If χ denotes a Hecke character of K of conductor $\mathfrak{f}$, then in [24, page 171] it is shown that the Dirichlet series

$$Z(s, \mathbf{f}, \chi) := \left(\prod_{i=1}^{2n} L_{\mathfrak{c}}(2s - i + 1, \chi_1 \theta^{i-1}) \right) \times \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) \chi^*(\mathfrak{a}) N(\mathfrak{a})^{-s}, \quad \Re(s) >> 0,$$

has an Euler product representation which we write as $Z(s, \mathbf{f}, \chi) = \prod_{\mathfrak{q}} Z_{\mathfrak{q}} \left(\chi^*(\mathfrak{q}) N(\mathfrak{q})^{-s} \right)$, where we recall χ^* is the ideal character associated to the Hecke character χ. The sum runs over all integral ideals of K and the product is over all prime ideals of K. For the description of the Euler factors $Z_{\mathfrak{q}}$ at the prime ideal $\mathfrak{q}$ of K we refer to [24, page 171]. We will need another L-function which we will denote by $L(s, \mathbf{f}, \chi)$ and we define by

$$L(s, \mathbf{f}, \chi) := \prod_{\mathfrak{q}} Z_{\mathfrak{q}} \left(\chi^*(\mathfrak{q}) (\psi/\psi_{\mathfrak{c}})(\pi_{\mathfrak{q}}) N(\mathfrak{q})^{-s} \right), \tag{4}$$

where $\pi_{\mathfrak{q}}$ a uniformizer of $K_{\mathfrak{q}}$. We note here that we may obtain the first from the second up to a finite number of Euler factors by setting $\chi\psi^{-1}$ for χ.

4.2. THE RANKIN-SELBERG METHOD. For $\tau \in S_+$ and $r \in GL_n(K)_{\mathbf{h}}$ we define, following [24, page 180], the Dirichlet series,

$$D'_{r,\tau}(s, \mathbf{f}, \chi) := \sum_{x \in B/E} \psi(det(rx)) \chi^*(det(x)\mathcal{O}_K) c_{\mathbf{f}}(\tau, rx) |det(x)|_K^{s-n}. \tag{5}$$

Here $B = GL_n(K)_{\mathbf{h}} \cap \prod_v M_n(\mathcal{O}_{K_v})$, $E = \prod_{v \in \mathbf{h}} GL_n(\mathcal{O}_{K_v})$ and for an idele x of K we write $x\mathcal{O}_K$ for the fractional ideal of K corresponding to x. Moreover $|\cdot|_K$ denotes the adelic absolute values of K, and, for later use, we denote by $|\cdot|_F$ the one of F. This Dirichelt series cannot be written in an Euler product form, but it has the advantage that it can be written as a Rankin-Selberg type integral as we will see later. However first we give the relation of this Dirichlet series and the L-function introduced before. The following equation, which is taken from [24, Theorem 20.4], is often called an Adrianov-Kalinin type equation, since it was first observed in the symplectic case by Adrianov and Kalinin in [1]. We have,

$$(6)\quad D'_{r,\tau}(s,\mathbf{f},\chi)\Lambda_{\mathfrak{c}}(s)\prod_{v\in\mathbf{b}} g_v(\chi(\psi/\psi_{\mathfrak{c}})(\pi_v)|\pi_v|^s) = L(s,\mathbf{f},\chi)(\psi/\psi_{\mathfrak{c}})^2(det(r))\times$$
$$\sum_{L<M\in\mathcal{L}_\tau} \mu(M/L)(\psi_{\mathfrak{c}}^2/\psi)(det(y))\chi^*(det(r^*\hat{y})\mathcal{O}_K)|det(r^*\hat{y})|_K^s c_{\mathbf{f}}(\tau,y),$$

where $\Lambda_{\mathfrak{c}}(s) := \prod_{i=1}^n L_{\mathfrak{c}}(2s+1-n-i,\psi_1\chi_1\theta^{n+i-1})$.

Let us now explain the notation in the above equation (see [24, page 164] for more details). $\mathcal{L}_\tau$ is the set of O_K-lattices L in K^n such that $\ell^*\tau\ell \in \mathfrak{b}\mathfrak{d}^{-1}$ for all $\ell \in L$. Moreover for the chosen ideal $\mathfrak{c}$ above, and for two O_K lattices M, N we write $M < N$ if $M \subset N$ and $M \otimes_{O_K} O_{K_v} = N \otimes_{O_K} O_{K_v}$ for every $v \mid \mathfrak{c}O_K$. In particular in the sum above we take $L := rO_K^n$, and in the sum, over the M's, we take $y \in GL_n(K)_{\mathbf{h}}$ such that $M = yO_K^n$ and $y^{-1}r \in B$. Moreover we define $\mu(A)$, for a torsion O_K-module A, recursively by $\sum_{B\subset A}\mu(B) = 1$ if A is the trivial module and 0 otherwise. This is a generalized Möbius function and we refer to [24, Lemma 19.10] for details. The important fact for our purposes is that the function μ is $\mathbb{Z}$-valued. Finally the g_v are Siegel-series related to the polynomials $f_v(x)$ mentioned in Proposition 3.2 above, and we refer to [24, Theorem 20.4] for the precise definition.

We now fix a Hecke character ϕ of K such that $\phi(y) = y_{\mathbf{a}}^{-1}|y_{\mathbf{a}}|$ for $y \in K_{\mathbf{a}}^\times$ and the restriction of ϕ to $F_{\mathbb{A}}^\times$ is the non-trivial Hecke character of F corresponding to the extension K/F. The existence of this character follows from [24, Lemma A5.1]. Keeping the notations from above we let $t \in \mathbb{Z}^{\mathbf{a}}$ be the infinity type of χ and define $\mu \in \mathbb{Z}^{\mathbf{b}}$ (see [24, page 181]) by

$$\mu_v = t_v - k_{v\rho} + k_v, \text{ and } \mu_{v\rho} = 0 \text{ if } t_v \geq k_{v\rho} - k_v,$$

and

$$\mu_v = 0, \text{ and } \mu_{v\rho} = k_{v\rho} - k_v - t_v \text{ if } t_v < k_{v\rho} - k_v.$$

We moreover define $l = \mu + n\mathbf{a}$ and $\psi' := \chi^{-1}\phi^{-n}$. Given μ, ϕ, τ and χ as above we write $\theta_\chi(x) := \theta_{\mathbb{A}}(x,\lambda) \in \mathcal{M}_l(C',\psi')$ for the theta series that we can associate to $(\mu,\phi,\tau,\chi^{-1})$ by taking $\omega := \chi^{-1}$ in Theorem 3.1. We write $\mathfrak{c}'$ for the integral ideal defined by $C' = D[\mathfrak{b}'^{-1}, \mathfrak{b}'\mathfrak{c}']$.

We now fix a decomposition $GL_n(K)_{\mathbb{A}} = \coprod_{q\in Q} GL_n(K)qEGL_n(K)_{\mathbf{a}}$. In particular the size of the set Q is nothing else than the class number of K. Then we have the following integral expression for the Dirichlet series $D'_{r,\tau}(s,\mathbf{f},\chi)$.

Theorem 4.1 (Shimura, pages 179-181 in [24]). *With notation as above we have*

$$\prod_{v\in\mathbf{a}}(4\pi)^{-n(s+h_v)}\Gamma_n(s+h_v)D'_{r,\tau}(s+3n/2,\mathbf{f},\chi) = det(\tau)^{s\mathbf{a}+h}|det(r)|_K^{-s-n/2}\times$$
$$\sum_{q\in Q}(\psi/\psi')(det(q))|det(qq^*)|_F^s \times A\langle f_q(z), \theta_{q,\chi}E(z,\bar{s}+n; m-m',\Gamma^q)\rangle_{\Gamma^q},$$

where $h := 1/2(k_v + k_{v\rho} + l_v + l_{v\rho})_{v\in\mathbf{a}}$, $m = (k_v + k_{v\rho})_{v\in\mathbf{a}}$, $m' = (l_v + l_{v\rho})_{v\in\mathbf{a}}$ *and* $\Gamma^q := q\Gamma q^{-1}$, *with* Γ *a suitably chosen congruence subgroup of* $SU(n,n)$.

A is a rational number times $C(S)^{-1}$ (see Proposition 3.2 for the definition). Here $\theta_{q,\chi}$ is the theta series introduced in section 3,

$$E(z, \bar{s}+n; m-m', \Gamma^q) = \sum_{\gamma \in \Gamma^q \cap P \backslash \Gamma^q} det(i(z^* - z)/2)^{s - \frac{m-m'}{2}}|_{m-m'}\gamma$$

is the Eisenstein series of $SU(n,n)$ defined in [24, page 137] *with $\kappa = 0$.*

Let us remark here that in the form given in [24, pages 179-181] only one group Γ appears. However it is easily seen that our choices can be made as above by picking $\Gamma \subset SU(n,n) \cap C''$ for C'' deep enough which is contained in both C and C'.
As we mentioned at the beginning of this section, the next step is to replace the Eisenstein series $E(z,s,m-m',\Gamma^q)$ in the expression above, with an Eisenstein series of the form appeared in the previous section. The two kinds of Eisenstein series are related as for example it is explained in [24, Lemma 17.2]. In particular we want to have an Eisenstein series for which we have i) a good understanding of its Galois reciprocity laws and ii) a good understanding of nearly holomorphicity (see also the remark after Theorem 6.2 on this). This is the reason of the importance for our purposes of the following result, shown by Klosin in [18, Equation (7.28)] in a slightly different form.

THEOREM 4.2 (Klosin, Equation (7.28) in [18]). *With notation as above we have*

$$\frac{|X|}{[\Gamma_0(\mathfrak{c}''):\Gamma]}\langle f_q(z), \theta_{q,\chi}(z)E(z,\bar{s},m-m',\Gamma^q)\rangle_{\Gamma^q} =$$

$$(\psi'/\psi)(det(q))|det(qq^*)|_F^{-s}\langle f_q(z), \theta_{q,\chi}(z)E_q(z,\bar{s};k-l,(\psi'/\psi)^c,\mathfrak{c}'')\rangle_{\Gamma_0^q(\mathfrak{c}'')},$$

where $(\psi'/\psi)^c(x) := (\psi'/\psi)(x^c)$ and X denotes the number of Hecke characters of infinity type t and conductor dividing $\mathfrak{f}_\chi$, and $\mathfrak{c}''$ any non-trivial integral ideal such that $\mathfrak{c}\mathfrak{c}'|\mathfrak{c}''$.

Proof. The proof is in principle done in [loc. cit.]. Here we only make a few remarks in order to justify the slightly different formula.
We remark here that in [18] the case of $F = \mathbb{Q}$ is considered, but it is easy to see that his result holds for totally real fields. Moreover the assumption $(|Cl_K|, 2n) = 1$ which Klosin makes at the beginning of his section 7 in [18], where the above equation is shown, it is used only later and not for the above equation.
Moreover we have also considered the case of $\mathbf{f}$ with non-trivial nebentype and this is why our formula differs slightly from these in [loc. cit.]. We also comment on the fact that we use more general weights than in [loc. cit]. Indeed one needs to observe that in Lemma 17.13 in [24] the identity used in [18] and cited as formula (17.5) of [24] is extended to the case of weights $k \in \mathbb{Z}^{\mathbf{b}}$. But then one needs only to observe that for $\Gamma \subset SU(n,n)$ we have the equality of the Eisenstein series

$$E_q(z,s;m-m',\Gamma) = E_q(z,s;k-l,\Gamma),$$

where on the left side we have $m-m' \in \mathbb{Z}^{\mathbf{a}}$ and on the right we have $k-l \in \mathbb{Z}^{\mathbf{b}}$. This point is explained in [24, page 32, paragraph 5.4]. Here we simply note that the Eisenstein series on the right is defined by

$$E_q(z,s;k-l,\Gamma) = \sum_{\alpha \in \Gamma \cap P \backslash \Gamma} det(i(z^*-z)/2)^{s-\frac{m-m'}{2}}|_{k-l}\alpha,$$

since $m-m' = ((k_v - l_v) + (k_{v\rho} - l_{v\rho}))_v$.

□

Putting all the above results together we can conclude the following theorem (see also [18, Theorem 7.8]),

Theorem 4.3 (Shimura, Klosin). *Let $0 \neq \mathbf{f} \in \mathcal{M}_k(C,\psi))$ such that $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ for every $\mathfrak{a}$. Then*

$$\Gamma((s))L(s+3n/2,\mathbf{f},\chi)(\psi/\psi_{\mathfrak{c}})^2(det(r))\times$$

$$\sum_{L<M\in\mathcal{L}_\tau} \mu(M/L)(\psi_{\mathfrak{c}}^2/\psi)(det(y))\chi^*(det(r^*\hat{y})\mathcal{O}_K)|det(r^*\hat{y})|_K^{s+3n/2}c_{\mathbf{f}}(\tau,y) =$$

$$\Lambda_{\mathfrak{c}}(s+3n/2,\theta(\psi\chi)_1)\cdot\prod_{v\in\mathbf{b}} g_v(\chi(\psi/\psi_{\mathfrak{c}})(\pi_v)|\pi_v|^{2s+3n})det(\tau)^{s\mathbf{a}+h}|det(r)|_K^{-s-n/2}\times$$

$$C_0\sum_{q\in Q}|det(qq^*)|_F^{-n}vol(\Phi_q)\langle f_q(z),\theta_{q,\chi}(z)\mathcal{E}_q(z,\bar{s}+n)\rangle_{\Gamma_0^q(\mathfrak{c}'')},$$

where $\mathcal{E}_q(z,\bar{s}+n) := E_q(z,\bar{s}+n;k-l,(\psi'/\psi)^c,\mathfrak{c}'')$, $\mathfrak{c}''$ any non-trivial integral ideal of F such that $\mathfrak{c}\mathfrak{c}'|\mathfrak{c}''$. If moreover we assume that $k_v+k_{v\rho}\geq n$ for some $v\in\mathbf{a}$, then there exists $\tau\in S_+\cap GL_n(K)$ and $r\in GL_n(K)_{\mathbf{h}}$ such that

$$\Gamma((s))\psi_{\mathfrak{c}}(det(r))c_{\mathbf{f}}(\tau,r)L(s+3n/2,\mathbf{f},\chi) =$$

$$\Lambda_{\mathfrak{c}}(s+3n/2,\theta(\psi\chi)_1)\cdot\left(\prod_{v\in\mathbf{b}} g_v(\chi(\pi_{\mathfrak{p}})N(\mathfrak{p})^{-2s-3n})\right)det(\tau)^{s\mathbf{a}+h}|det(r)|_K^{-s-n/2}\times$$

$$C_0\sum_{q\in Q}|det(qq^*)|_F^{-n}vol(\Phi_q) < f_q(z),\theta_{q,\chi}(z)\mathcal{E}_q(z,\bar{s}+n) >_{\Gamma_0^q(\mathfrak{c}'')},$$

where

$$\Gamma((s)) := \prod_{v\in\mathbf{a}}(4\pi)^{-n(s+h_v)}\Gamma_n(s+h_v), \ \ \textit{and}\ C_0 := \frac{[\Gamma_0(\mathfrak{c}''):\Gamma]A}{|X|}.$$

We close this section by remarking that $vol(\Phi_q)$ is independent from q (see for example [23, page 67]) and hence we will be writing simply $vol(\Phi)$. Moreover we have that this is equal to π^{dn^2} times a rational number [24, Proposition 24.9].

5. Petersson Inner Product and Periods

In this section we define some archimedean periods, which we will use to normalize the special values of the function $L(s, \mathbf{f}, \chi)$. The definition of these periods is inspired by the work of Sturm [25] (building on previous work of Shimura), for integral weight Siegel modular forms of even genus over the rationals. In our works [2, 4] we have extended also these results to totally real fields, considered odd genus and the case of half-integral weight Siegel modular forms.

We start by proving a lemma with respect to the action of "good" Hecke operator $T(\mathfrak{a})$, relative to the group $C = D[\mathfrak{b}^{-1}, \mathfrak{bc}]$. Here good means that $\mathfrak{a}$ is prime to $\mathfrak{c}$.

Lemma 5.1. *Let W be a number field, which contains the values of the finite character ψ. Then the operators $T(\mathfrak{a})$ preserve $\mathcal{M}_k(C, \psi, W)$.*

Proof. Following Shimura in [24, page 161], and using the same notation as in there, we consider the formal Dirichlet series $\mathbf{f}|\mathfrak{J} := \sum_{\mathfrak{a}}[\mathfrak{a}]\mathbf{f}|T(\mathfrak{a})$. For $\tau \in S_+$ and $q \in GL_n(K)_{\mathbf{h}}$ Shimura shows in (page 170, loc. cit.) that $c(\tau, q; \mathbf{f}|\mathfrak{J})$ is equal to

$$\sum_{g,h} \psi_{\mathfrak{c}}(det(h^{-1}g))|det(g)|_K^{-n} c(\tau, qh^{-1}g; \mathbf{f})\alpha_{\mathfrak{c}}(\hat{h}q^*\tau qh^{-1})[det(gh)\mathcal{O}_F].$$

The point which is important here is that Shimura shows (see Theorem 16.2 in (loc. cit.) that $\alpha_{\mathfrak{c}}(\cdot)$ is a rational formal Dirichlet series (i.e. has coefficients in $\mathbb{Q}$). In particular by the equation above we conclude that the $c(\tau, q; \mathbf{f}|T(\mathfrak{a}))$, which is obtained by equating the $[\mathfrak{a}]$ coefficient in the formal Dirichlet series above, is a $\mathbb{Q}(\psi)$ linear combination of the Fourier coefficients of $\mathbf{f}$. Hence we conclude the lemma. $\square$

We now fix a set $\mathcal{B} \subset G_{\mathbf{h}}$ such that $G_{\mathbb{A}} = \coprod GbC$. Note that this does not depend on the ideal $\mathfrak{c}$ thanks to Lemma 2.6. Moreover we can take the elements b to be of the form $b = diag[b_1, \hat{b_1}]$ with $b_1 \in GL_n(K_{\mathbf{h}})$. Keeping the notation of the previous section where we wrote $Q \subset GL_n(K)_{\mathbf{h}}$, we define, following Shimura [24, Lemma 28.4], the set $\mathcal{A}_b := \{q \in Q | diag[q, \hat{q}] \in GbC\}$, and the map

$$\pi_b : \oplus_{q \in \mathcal{A}_b} \mathcal{M}_k(\Gamma^q, \psi_q) \to \mathcal{M}_k(\Gamma^b, \psi_b), \quad (f_q)_q \mapsto h_b := B \sum_{q \in \mathcal{A}_b} f_q|\alpha_q,$$

with $diag[q, \hat{q}] \in \alpha_q bC$ and B the size of the set $\mathcal{B}$. As explained in [24] for any cusp form $\mathbf{g} \in \mathcal{S}_k(C, \psi))$ we have that $\langle g_b, h_b \rangle = B \sum_{q \in \mathcal{A}_b} \langle g_q, f_q \rangle$. In particular if we define the form $\mathbf{h}$ with local components h_b we have that $\langle \mathbf{g}, \mathbf{h} \rangle = \sum_{q \in Q} \langle g_q, f_q \rangle$. From Lemma 2.4 we have that the map π_b is defined over some finite extension of Φ, the Galois closure of K. We write L for the minimal field over which all π_b for $b \in \mathcal{B}$ are defined. Clearly this field is equal to Φ in the situation where the set $\mathcal{B}$ and the set Q have the same size. This, for example, can happen when the class number of F is one (see [23, page 66].

As we mentioned above, the following theorem should be seen as the unitary analogue of a theorem by Sturm [25] for Siegel modular forms of integral weight, even genus over the rationals (see also related work of Harris [14]), building on ideas of Shimura. Our proof combines ideas taken from the proof of Sturm in [25] as well as from the proof of a theorem of Shimura [24, Theorem 28.5]. This last one provides a result of the form $\frac{\langle \mathbf{f},\mathbf{g}\rangle}{\langle \mathbf{f},\mathbf{f}\rangle} \in \overline{\mathbb{Q}}$, for $\mathbf{f}$ a cusp form and $\mathbf{g}$ modular forms defined over $\overline{\mathbb{Q}}$. We also mention that similar theorems have been also proved in [2, 4] for Siegel modular forms over totally real fields of integral and half-integral weight.

Theorem 5.2. *Let $\mathbf{f} \in \mathcal{S}_k(C,\psi,\overline{\mathbb{Q}})$ be an eigenform for all the good Hecke operators of C, and define $m_v := k_v + k_{\rho v}$ for all $v \in \mathbf{a}$. Let Φ be the Galois closure of K over $\mathbb{Q}$ and write W for the extension of Φ generated by the Fourier coefficients of $\mathbf{f}$ and their complex conjugation. Assume $m_0 := min_v(m_v) > 3n+2$. Then there exists a period $\Omega_{\mathbf{f}} \in \mathbb{C}^\times$ and a finite extension Ψ of Φ such that for any $\mathbf{g} \in \mathcal{S}_k(\overline{\mathbb{Q}})$ we have*

$$\left(\frac{\langle \mathbf{f},\mathbf{g}\rangle}{\Omega_{\mathbf{f}}}\right)^\sigma = \frac{\langle \mathbf{f}^\sigma,\mathbf{g}^{\sigma'}\rangle}{\Omega_{\mathbf{f}^\sigma}},$$

for all $\sigma \in Gal(\overline{\mathbb{Q}}/\Psi)$, with $\sigma' := \rho\sigma\rho$. Here $\Omega_{\mathbf{f}^\sigma}$ is the period attached to the eigenform $\mathbf{f}^\sigma$. Moreover $\Omega_{\mathbf{f}}$ depends only on the eigenvalues of $\mathbf{f}$ and we have $\frac{\langle \mathbf{f},\mathbf{f}\rangle}{\Omega_{\mathbf{f}}} \in (W\Psi)^\times$. In particular we have $\frac{\langle \mathbf{f},\mathbf{g}\rangle}{\langle \mathbf{f},\mathbf{f}\rangle} \in (W\Psi)(\mathbf{g},\mathbf{g}^\rho)$, where $(W\Psi)(\mathbf{g},\mathbf{g}^\rho)$ denotes the extension of $W\Psi$ obtained by adjoining the values of the Fourier coefficients of $\mathbf{g}$ and $\mathbf{g}^\rho$.

Remark 5.3. Before we give the rather long proof of the above theorem we would like to indicate some of the ideas that allow us to obtain the above theorem.

(1) For the proof of Theorem 5.2 we will make use of Theorem 4.3 of the previous section. In particular, the fact that for the Eisenstein series involved in Theorem 4.3 we have a very good understanding of the Galois reciprocity obtained in section 3, will play a key role. Note that this is not the case for the Eisenstein series involved in the original expression of Shimura in Theorem 4.1, which of course is good enough if one is only interested in obtaining results over an algebraic closure of $\mathbb{Q}$, but not for the results which we are aiming here.
(2) The second observation is related to the bound imposed on the weight of the Hermitian modular form $\mathbf{f}$. In particular this is a bit weaker than the one appearing in [24, Theorem 28.5], where the bound is taken to be $3n$. The reason for this difference is to give us some freedom in selecting a particular character (in the notation of the proof χ), which will make the associated theta series (which we denote $\theta_{q,\chi}$ in the proof below) a cusp form. This will allow us not to have to worry about the

field of definition under which we have a splitting of the cuspidal part from the Eisenstein part. Note that this is quite important, since we are considering weights, which may be below the range of absolute convergent for Eisenstein series, which is $4n-2$ in the unitary case, and hence we do not have an explicit description of the field over which this decomposition is defined (see also Theorem 5.5 below).

(3) The extension Ψ will become explicit in the proof of the theorem. We will see that it is a finite extension of the field L defined above (the field of definition of the maps $\pi'_b s$) obtained by adjoining the values of two, once and for all fixed, characters, which are denoted χ and ϕ in the proof.

Proof of Theorem 5.2. We first consider the case where m_0 is even. We define $\mu \in \mathbb{Z}^{\mathbf{b}}$ by setting $\mu_{\rho v} = 0$ and $\mu_v = m_v - m_0 + 2$ for all $v \in \mathbf{a}$. We now set $t' := \mu_v - k_v + k_{v\rho}$, and consider a Hecke character χ of K of conductor $\mathfrak{f}_\chi$ such that $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{-t'}|x_{\mathbf{a}}|^{t'}$, and $\mathfrak{c}|\mathfrak{f}_\chi$. Later we explain more on the choice of the character χ. We recall here that we are taking ψ to be of finite order, so the infinity type is trivial. We now set $s := \frac{m_0-3n}{2} - 1$ in Theorem 4.3. We get

$$C(S)\Gamma((\frac{m_0-3n}{2}-1))L(\frac{m_0}{2}-1,\mathbf{f},\chi)(\psi/\psi_{\mathfrak{c}})^2(det(r))\times$$

$$\sum_{L<M\in\mathcal{L}_\tau}\mu(M/L)(\psi_{\mathfrak{c}}^2/\psi)(det(y))\chi^*(det(r^*\hat{y})\mathcal{O}_K)|det(r^*\hat{y})|_K^{\frac{m_0}{2}-1}c_{\mathbf{f}}(\tau,y) =$$

$$\Lambda_{\mathfrak{c}}(\frac{m_0}{2}-1,\theta(\psi\chi)_1)\cdot\prod_{v\in\mathbf{b}}g_v(\chi(\psi/\psi_{\mathfrak{c}})(\pi_v)|\pi_v|^{2m_0-1})det(\tau)^{m-n}|det(r)|_K^{-\frac{m_0+2n+2}{2}}\times$$

$$C_0\sum_{q\in Q}|det(qq^*)|_F^{-n}vol(\Phi)\langle f_q(z),\theta_{q,\chi}(z)E_q(z,\frac{\nu-2}{2};k-\ell,\xi,\mathfrak{c}'')\rangle_{\Gamma_0^q(\mathfrak{c}'')},$$

where $\xi := (\psi'/\psi)^c$ and we set $\nu := m_0 - n$. We note that $\frac{m_0}{2} - 1 > \frac{3n}{2}$ and that $(k_v - \ell_v) + (k_{v\rho} - \ell_{v\rho}) = \nu - 2 > 2n$. In particular the bound on $\nu - 1$ implies that $E_q(z, \frac{\nu-2}{2}; k-\ell, (\psi'/\psi)^c, C'')$ is holomorphic.

If m_0 is odd we define $\mu \in \mathbb{Z}^{\mathbf{b}}$ by setting $\mu_{\rho v} = 0$ and $\mu_v = m_v - m_0 + 1$ for all $v \in \mathbf{a}$. We now set $t' := \mu_v - k_v + k_{v\rho}$, and consider a Hecke character χ of K of conductor $\mathfrak{f}_\chi$ such that $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{-t'}|x_{\mathbf{a}}|^{t'}$.

We now set $s := \frac{m_0+1-3n}{2} - 1$ in Theorem 4.3 and get

$$C(S)\Gamma((\frac{m_0+1-3n}{2}-1))L(\frac{m_0+1}{2}-1,\mathbf{f},\chi)(\psi/\psi_{\mathfrak{c}})^2(det(r))\times$$

$$\sum_{L<M\in\mathcal{L}_\tau}\mu(M/L)(\psi_{\mathfrak{c}}^2/\psi)(det(y))\chi^*(det(r^*\hat{y})\mathcal{O}_K)|det(r^*\hat{y})|_K^{\frac{m_0-1}{2}}c_{\mathbf{f}}(\tau,y) =$$

$$\Lambda_{\mathfrak{c}}(\frac{m_0-1}{2},\theta(\psi\chi)_1)\cdot\prod_{v\in\mathbf{b}}g_v(\chi(\psi/\psi_{\mathfrak{c}})(\pi_v)|\pi_v|^{2m_0+1})det(\tau)^{m-n}|det(r)|_K^{\frac{2n+1-m_0}{2}}\times$$

$$C_0\sum_{q\in Q}|det(qq^*)|_F^{-n}vol(\Phi)\langle f_q(z),\theta_{q,\chi}(z)E_q(z,\frac{\nu-1}{2};k-\ell,\xi,\mathfrak{c}'')\rangle_{\Gamma_0^q(\mathfrak{c}'')},$$

where $\xi := (\psi'/\psi)^c$ and we set $\nu := m_0 - n$. We note that $\frac{m_0+1}{2} - 1 > \frac{3n}{2}$ and that $(k_v - \ell_v) + (k_{v\rho} - \ell_{v\rho}) = \nu - 1 > 2n$. In particular the bound on $\nu - 1$ implies that $E_q(z, \frac{\nu-1}{2}; k - \ell, (\psi'/\psi)^c, C'')$ is holomorphic.
In the case of m_0 is even we have

$$\Gamma((\frac{m_0 - 3n}{2} - 1)) = \prod_{v \in \mathbf{a}} (4\pi)^{-n(m_v - n)} \Gamma_n(m_v - n),$$

and a similar equality holds for m_0 odd, namely

$$\Gamma((\frac{m_0 + 1 - 3n}{2} - 1)) = \prod_{v \in \mathbf{a}} (4\pi)^{-n(m_v - n)} \Gamma_n(m_v - n).$$

We now notice that

$$\prod_{v \in \mathbf{a}} \Gamma_n(m_v - n) = \prod_{v \in \mathbf{a}} \pi^{n(n-1)/2} \prod_{j=0}^{n-1} \Gamma(m_v - n - j) \in \pi^{dn(n-1)/2} \mathbb{Q}^\times,$$

where $d = [F : \mathbb{Q}]$. In particular for m_0 even we have,

$$\Gamma((\frac{m_0 - 3n}{2} - 1)) \in \pi^{dn(n-1)/2 + dn^2 - n \sum_v m_v} \mathbb{Q}^\times$$

and similar equality holds for $\Gamma((\frac{m_0+1-3n}{2} - 1))$ when m_0 is odd. Recalling that $vol(\Phi) \in \pi^{dn^2} \mathbb{Q}^\times$ we conclude that for m_0 even,

$$\frac{\Gamma((\frac{m_0-3n}{2} - 1))}{vol(\Phi)} \in \pi^{dn(n-1)/2 - n \sum_v m_v} \mathbb{Q}^\times,$$

and similarly for $\frac{\Gamma((\frac{m_0+1-3n}{2} - 1))}{vol(\Phi)}$ when m_0 odd.

We now describe the extension Ψ of the theorem. We first note that we can pick the characters χ and ϕ so that $\chi(x)$ and $\theta(x)$ belong in a finite extension of Φ for any $x \in K_{\mathbf{h}}^\times$. We start with the character ϕ. We note in the proof of lemma A5.1 in [24] that $\phi(x) = b_{\mathbf{a}}^{-\mathbf{a}} |b_{\mathbf{a}}|^{\mathbf{a}} \theta(c)$ if $x = abc$ with $a \in K^\times$, $b \in U$ (as in Shimura) and $c \in F_{\mathbb{A}}^\times$. In particular $\phi(x)^2 = 1$ if $x \in K_{\mathbf{h}}^\times \cap K^\times U F_{\mathbb{A}}^\times$. But then by looking at the proof of Lemma 11.15 in [23], where the extension of ϕ to $K_{\mathbb{A}}^\times$ one sees that we need to extend the values of ϕ by a finite number of roots of 1 or -1. Similarly in order to obtain the character χ we can start by a quadratic Hecke character χ_1 of F with infinity type t' mod 2. Note that this is always possible, since we can take χ_1 to be the quadratic character corresponding to a quadratic extension of F that is imaginary when $t_v \equiv 1$ mod 2 and real otherwise. Then we can apply the same argument as we did with the character ϕ. We now define the field Ψ to be the finite extension of Φ obtained by adjoining the values of the characters χ and ϕ on finite adeles, and such that the maps π_b discussed before the theorem are all defined over Ψ. For every q we write Γ'_q and Γ_q for the groups $\Gamma_0^q(\mathfrak{c}'')$ and $\Gamma_0^q(\mathfrak{c}'')$ respectively. We first consider the case of m_0 being even. We now write δ for the rational part of $\Gamma((\frac{m_0-3n}{2}-1)) \in \pi^{dn(n-1)/2+dn^2-n\sum_v m_v}\mathbb{Q}^\times$ and take $\beta \in \mathbb{N}$ so that $\pi^{-\beta}D_q(\nu-2/2) \in \mathcal{M}_{k-l}(\Phi\mathbb{Q}_{ab})$ where

$$D_q(\frac{\nu-2}{2}) := \Lambda_{\mathfrak{c}''}(\frac{m_0+1}{2}-1), \theta(\overline{\psi\chi})_1)E_q(z,\frac{\nu-2}{2};k-\ell,(\psi'/\psi)^c,\mathfrak{c}'').$$

We further set

$$B(\chi,\psi,\tau,r,\mathbf{f}) := \delta(\psi/\psi_{\mathfrak{c}})^2(det(r))\times$$

$$\sum_{L<M\in\mathcal{L}_\tau} \mu(M/L)(\psi_{\mathfrak{c}}^2/\psi)(det(y))\chi^*(det(r^*\hat{y})\mathcal{O}_K)|det(r^*\hat{y})|_K^{\frac{m_0}{2}-1}c_{\mathbf{f}}(\tau,y)$$

and

$$C(\chi,\psi,\tau,r) := C_0\frac{\Lambda_{\mathfrak{c}}(\frac{m_0-1}{2})}{\Lambda_{\mathfrak{c}''}(\frac{m_0-1}{2})}det(\tau)^{m-n}|det(r)|_K^{\frac{2n+2-m_0}{2}}\times$$

$$\prod_{v\in\mathbf{b}} g_v(\chi(\psi/\psi_{\mathfrak{c}})(\pi_v)|\pi_v|^{2m_0-1}).$$

We remark here that $\frac{\Lambda_{\mathfrak{c}}(\frac{m_0+1}{2}-1)}{\Lambda_{\mathfrak{c}''}(\frac{m_0+1}{2}-1)}$ is a product of finite many Euler factors, none of which is zero, thanks to the bound on m_0. We then have for every $\sigma \in Gal(\overline{\mathbb{Q}}/\Phi_\chi)$ that

$$B(\chi,\psi,\tau,r,\mathbf{f})^\sigma = B(\chi^\sigma,\psi^\sigma,\tau,r,\mathbf{f}^\sigma) \text{ and } C(\chi,\psi,\tau,r)^\sigma = C(\chi^\sigma,\psi^\sigma,\tau,r).$$

Keeping now the character χ fixed, we define the space $\mathcal{V} := \{\mathbf{g} \in \mathcal{S}_k(C,\psi)|\mathbf{g}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{g},\ (\mathfrak{a},\mathfrak{c}) = 1\}$, where $\lambda(\mathfrak{a})$ is the eigenvalue of $\mathbf{f}$ with respect to the good Hecke operators $T(\mathfrak{a})$. From above we have that for any given $\mathbf{g} \in \mathcal{V}$ there exists (τ,r) such that

$$B(\chi,\psi,\tau,r,\mathbf{g}) = \delta\psi_{\mathfrak{c}}(det(r))c_{\mathbf{g}}(\tau,r) \neq 0.$$

We note here that the same pair (τ,r) can be used for the form $\mathbf{g}^\sigma$, as it follows from the proof of Theorem 20.9 in [24]. As in [24, page 233] we write $\mathfrak{G}$ for the set of pairs (τ,r) for which such an $\mathbf{g}$ exists. From the observation above the set $\mathfrak{G}$ is the same also for the system of eigenvalues $\lambda(\mathfrak{a})^\sigma$, for all $\sigma \in Gal(\overline{\mathbb{Q}}/\Psi)$. In particular for such an (τ,r)

$$0 \neq C(S)\pi^\gamma L(\sigma_0,\mathbf{g},\chi)\delta\psi_{\mathfrak{c}}(det(r))c_{\mathbf{g}}(\tau,r) = \tag{7}$$

$$\left(\prod_{v\in\mathbf{b}} g_v(\chi(\pi_{\mathfrak{p}})N(\mathfrak{p})^{2m_0-1})\right)det(\tau)^{m-n}|det(r)|_K^{-\frac{m_0}{2}+n+1}C_0\times$$

$$\frac{\Lambda_{\mathfrak{c}}(\frac{m_0+1}{2}-1)}{\Lambda_{\mathfrak{c}''}(\frac{m_0+1}{2}-1)}\times\sum_{q\in Q}|det(qq^*)|_F^{-n}\langle g_q(z),\theta_{q,\chi}(z)D_q(z,\frac{\nu-2}{2})\rangle_{\Gamma'_q},$$

where we have set $\sigma_0 = \frac{m_0}{2}-1$. The fact that $L(\sigma_0,\mathbf{g},\chi) \neq 0$ is in principle [24, Theorem 20.13]. Indeed in page 183 of (loc. cit) Shimura first proves the non-vanishing of $Z'(\sigma_0,\mathbf{g},\chi)$ for any character χ with $\mu \neq 0$, as it is the case that we consider. Further we note that this in particular implies also that $C(\chi,\psi,\tau,r) \neq 0$ for all $(\tau,r) \in \mathfrak{G}$.

We note that

$$\langle g_q, \theta_{q,\chi} D_q \left(\frac{\nu-2}{2}\right)\rangle_{\Gamma_q'} = \langle g_q, Tr_{\Gamma_q'}^{\Gamma_q}\left(\theta_{q,\chi} D \left(\frac{\nu-2}{2}\right)\right)\rangle_{\Gamma_q},$$

and we define an element $\mathbf{g}_{\tau,r,\psi} \in \mathcal{S}_k(C,\psi)$ by defining its components as

$$\mathbf{g}_{b,\tau,r,\psi} := \pi_b \left(C(S)^{-1} |det(qq^*)|_F^{-n} \pi^{-\beta} Tr_{\Gamma_q'}^{\Gamma_q} \left(\theta_{q,\chi} \pi^{-\beta} D_q \left(\frac{\nu-2}{2}\right)\right)\right)$$

where we take $diag[q,\hat{q}] \in \mathcal{A}_b$. We note that it is a cusp form since we are taking $\mu \neq 0$ and hence the theta series is a cusp form. We now define $\mathcal{W}$ to be the space generated by $\mathbf{g}_{\tau,r,\psi}$ for $(\tau,r) \in \mathfrak{G}$.

We now claim that there exists an $\Omega_{\mathbf{f}} \in \mathbb{C}^\times$ such that for any $\mathbf{g}_{\tau,r,\psi}$

$$\left(\frac{\langle \mathbf{f}, \mathbf{g}_{\tau,r,\psi}\rangle}{\Omega_{\mathbf{f}}}\right)^\sigma = \frac{\langle \mathbf{f}^\sigma, \mathbf{g}_{\tau,r,\psi}^{\sigma'}\rangle}{\Omega_{\mathbf{f}^\sigma}},$$

where $\sigma' = \rho\sigma\rho$, with $\sigma \in Gal(\overline{\mathbb{Q}}/\Psi)$. We first note that the action of σ on $\mathbf{g}_{\tau,r,\psi}$ can be understood by the action on $\theta_{q,\chi}$ and $\pi^{-\beta} D_q\left(\frac{\nu-2}{2}\right)$. This follows from the fact that the maps π_b are defined over Ψ and the Galois equivariance of the trace map, which is proved right after this theorem. We now have $\left(|det(qq^*)|_{\mathbf{h}}^{-n/2}\theta_{q,\chi}\right)^{\sigma'} = |det(qq^*)|_{\mathbf{h}}^{-n/2}\theta_{q,\chi^\sigma}$ where we have used the Fourier expansion of $\theta_{q,\chi}$ given in Proposition 3.1 as well as the fact that χ is a unitary character, hence $\chi^\rho = \chi^{-1}$. For the Eisenstein series we first note that for m_0 even we have that $n - (\frac{\nu-2}{2}) = n - \frac{m_0-n-2}{2} \equiv \frac{n}{2} \mod \mathbb{Z}$. In particular for n odd, we have that $\delta_1 = 1$ in Proposition 3.9. Since we are really interested in the product $\theta_{q,\chi} D_q(\frac{\nu-2}{2})$ we have this factor cancelled out from the theta series. So in particular we can conclude that $\mathbf{g}_{\tau,r,\psi}^{\sigma'} = \frac{P(\xi)^{\sigma'}}{P(\xi^{\sigma'})}\mathbf{g}_{\tau,r,\psi^\sigma}$, where $P(\xi) := \tau(\xi_1^n) i^{-dn\mu}$. For any $\mathbf{g}_{\tau,r\psi}$ we have

$$\pi^\gamma L(\sigma_0, \mathbf{f}, \chi) B(\chi,\psi,\tau,r,\mathbf{f}) =$$

$$C(\chi,\psi,\tau,r)\langle \mathbf{f}, \mathbf{g}_{\tau,r,\psi}\rangle.$$

For any $(\tau,r) \in \mathfrak{G}$ we have seen that $C(\chi,\psi,\tau,r) \neq 0$. We obtain

$$\frac{\langle \mathbf{f}, \mathbf{g}_{\tau,r,\psi}\rangle}{\pi^\gamma L(\sigma_0,\mathbf{f},\chi)} = \frac{B(\chi,\psi,\tau,r,\mathbf{f})}{C(\chi,\psi,\tau,r)}.$$

For any $\sigma \in Gal(\overline{\mathbb{Q}}/\Psi)$ we have then

$$\left(\frac{\langle \mathbf{f}, \mathbf{g}_{\tau,r,\psi}\rangle}{\pi^\gamma L(\sigma_0,\mathbf{f},\chi)}\right)^\sigma = \left(\frac{B(\chi,\psi,\tau,r,\mathbf{f})}{C(\chi,\psi,\tau,r)}\right)^\sigma =$$

$$\frac{B(\chi^\sigma,\psi^\sigma,\tau,r,\mathbf{f}^\sigma)}{C(\chi^\sigma,\psi^\sigma,\tau,r)} = \frac{\langle \mathbf{f}^\sigma, \mathbf{g}_{\tau,r,\psi^\sigma}\rangle}{\pi^\gamma L(\sigma_0,\mathbf{f}^\sigma,\chi)}.$$

In particular we conclude that

$$\left(\frac{\langle \mathbf{f}, \mathbf{g}_{\tau,r,\psi}\rangle}{\pi^{\gamma} L(\sigma_0,\mathbf{f},\chi)}\right)^{\sigma} = \left(\overline{\left(\frac{P(\xi)^{\sigma'}}{P(\xi^{\sigma'})}\right)}\right)^{-1} \frac{\langle \mathbf{f}^{\sigma}, \mathbf{g}_{\tau,r,\psi^{\sigma'}}\rangle}{\pi^{\gamma} L(\sigma_0,\mathbf{f}^{\sigma},\chi)}.$$

In particular if we set $\Omega_{\mathbf{f}} := \pi^{\gamma} L(\sigma_0,\mathbf{f},\chi)\overline{P(\xi)}^{-1}$ then we conclude the claim. One should note here that $P(\xi^{\sigma'}) = P(\xi^{\sigma})$.

The case of m_0 odd can of course be done similarly. Namely if we set $\Omega_{\mathbf{f}} = \pi^{\gamma} L(\sigma_0,\mathbf{f},\chi)\overline{P(\xi)}^{-1}$ with $\sigma_0 = \frac{m_0+1}{2} - 1$, with $P(\xi)$ defined as above. then for any $\mathbf{f} \in \mathcal{V}$ and any $\mathbf{g}_{\tau,r,\psi}$

$$\left(\frac{\langle \mathbf{f}, \mathbf{g}_{\tau,r,\psi}\rangle}{\Omega_{\mathbf{f}}}\right)^{\sigma} = \frac{\langle \mathbf{f}^{\sigma}, \mathbf{g}^{\sigma'}_{\tau,r,\psi}\rangle}{\Omega_{\mathbf{f}^{\sigma}}},$$

where $\sigma' = \rho\sigma\rho$ and $\sigma \in Gal(\overline{\mathbb{Q}}/\Psi)$. Here one needs to observe that for m_0 odd we have that $n - \frac{m_0-n-1}{2} = n - \frac{m_0-1}{2} - \frac{n}{2} \equiv \frac{n}{2} \mod \mathbb{Z}$ and hence again the roots of $|det(qq^*|_{\mathbf{h}}$ (when n odd) of the Eisenstein series will cancel with the ones of the theta series.

With $\mathcal{W}'$ we denote the space generated by the projection of $\mathcal{W}$ on $\mathcal{V}$. By definition $\mathcal{W}' = \mathcal{V}$. Indeed for any element $\mathbf{g} \in \mathcal{V}$ there exists $\mathbf{h} \in \mathcal{W}'$ such that $\langle \mathbf{g}, \mathbf{h}\rangle_{\Gamma} \neq 0$, simply by taking the projection of the corresponding $\mathbf{g}_{\tau,r} := \mathbf{g}_{\tau,r,\psi}$ to $\mathcal{W}'$. So the $\mathbb{C}$ span of $\mathbf{g}_{\tau,r}$ with $(\tau,r) \in \mathfrak{G}$ is equal to $\mathcal{V}$. Since $\mathbf{g}_{\tau,r}$ have algebraic coefficients we have that the $\overline{\mathbb{Q}}$-span is equal to $\mathcal{V}(\overline{\mathbb{Q}})$. We can now establish the theorem for any $\mathbf{g} \in \mathcal{V}(\overline{\mathbb{Q}})$ since after writing $\mathbf{g} = \sum_j c_j \mathbf{g}_{\tau_j,r_j,\mathcal{V}} \in \mathcal{V}(\overline{\mathbb{Q}})$, where $\mathbf{g}_{\tau_j,r_j,\mathcal{V}}$ is the projection of $\mathbf{g}_{\tau_j,r_j}$ to $\mathcal{V}$, we have

$$\left(\frac{\langle \mathbf{f}, \mathbf{g}\rangle}{\Omega_{\mathbf{f}}}\right)^{\sigma} = \sum_j \overline{c_j}^{\sigma} \left(\frac{\langle \mathbf{f}^{\sigma}, \mathbf{g}^{\sigma'}_{\tau_j,r_j,\mathcal{V}}\rangle}{\Omega_{\mathbf{f}^{\sigma}}}\right) = \frac{\langle \mathbf{f}^{\sigma}, \mathbf{g}^{\sigma'}\rangle}{\Omega_{\mathbf{f}^{\sigma}}}.$$

Here we note that we make use of the important fact that $\mathbf{g}_{\tau_j,r_j}$ are cusp forms. We now take any $\mathbf{g} \in \mathcal{S}_k(C,\psi;\overline{\mathbb{Q}})$. The good Hecke operators act as commutative semi-simple linear transformations hence we have $\mathcal{S}_k(C,\psi,\overline{\mathbb{Q}}) = \mathcal{V} \oplus U$, with U a vector space which is stable under the action of the good Hecke operators. We write $\mathbf{g} = \mathbf{g}_1 + \mathbf{g}_2$ with $\mathbf{g}_1 \in \mathcal{V}$ and $\mathbf{g}_2 \in U$. Then we have that

$$\left(\frac{\langle \mathbf{f}, \mathbf{g}\rangle}{\Omega_{\mathbf{f}}}\right)^{\sigma} = \left(\frac{\langle \mathbf{f}, \mathbf{g}_1\rangle}{\Omega_{\mathbf{f}}}\right)^{\sigma} = \frac{\langle \mathbf{f}^{\sigma}, \mathbf{g}_1^{\sigma'}\rangle}{\Omega_{\mathbf{f}^{\sigma}}} = \frac{\langle \mathbf{f}^{\sigma}, \mathbf{g}^{\sigma'}\rangle}{\Omega_{\mathbf{f}^{\sigma}}}$$

where the first and the last equality follows from the fact that $< \mathbf{f}, \mathbf{g} >= 0$ and $< \mathbf{f}^{\sigma}, \mathbf{g}^{\sigma'} >= 0$ for $\mathbf{g} \in U$. It is enough to show this for $\mathbf{g}$ an eigenform for all the good Hecke operators with eigenvalues different from that of $\mathbf{f}$'s. That is, there exists an ideal $\mathfrak{a}$ with $(\mathfrak{a},\mathfrak{c}) = 1$ so that $T(\mathfrak{a})\mathbf{f} = \lambda_{\mathbf{f}}(\mathfrak{a})\mathbf{f}$ and $T(\mathfrak{a})\mathbf{g} = \lambda_{\mathbf{g}}(\mathfrak{a})\mathbf{g}$ such that $\lambda_{\mathbf{f}}(\mathfrak{a}) \neq \lambda_{\mathbf{g}}(\mathfrak{a})$. But then we have

$$\lambda_{\mathbf{f}}(\mathfrak{a})^{\sigma}\langle \mathbf{f}^{\sigma}, \mathbf{g}^{\sigma'}\rangle = \langle T(\mathfrak{a})\mathbf{f}^{\sigma}, \mathbf{g}^{\sigma'}\rangle =$$

$$\langle \mathbf{f}^{\sigma}, T(\mathfrak{a}^{\rho})\mathbf{g}^{\sigma'}\rangle = \langle \mathbf{f}^{\sigma}\lambda_{\mathbf{g}}(\mathfrak{a})^{\sigma\rho}\mathbf{g}^{\sigma'}\rangle = \langle \mathbf{f}^{\sigma}, \mathbf{g}^{\sigma'}\rangle\lambda_{\mathbf{g}}(\mathfrak{a})^{\sigma}$$

and hence we conclude that $< \mathbf{f}^\sigma, \mathbf{g}^{\sigma'} >= 0$. Here we have used the fact (see [24, Lemma 23.15]) that the adjoint of a good Hecke operator $T(\mathfrak{a})$ is $T(\mathfrak{a}^\rho)$ and that $\lambda_{\mathbf{g}}(\mathfrak{a})^\rho = \lambda_{\mathbf{g}}(\mathfrak{a}^\rho)$. In particular $T(\mathfrak{a})\mathbf{g}^{\sigma'} = \lambda_{\mathbf{f}}(\mathfrak{a}^\rho)^{\sigma\rho}\mathbf{g}^{\sigma'}$.
Finally taking $\mathbf{g}$ equal to $\mathbf{f}$ we obtain that $\Omega_{\mathbf{f}}$ is equal to $< \mathbf{f}, \mathbf{f} >$ up to a non-zero element $(W\Psi)^\times$. □

We now prove the lemma on the Galois equivariance property of the trace map, which was used in the proof above.

Lemma 5.4. *With notation as in the theorem we consider the trace map* $Tr : \mathcal{S}_k(\Gamma'_q, \psi) \to \mathcal{S}_k(\Gamma_q, \psi)$. *Then for any* $\sigma \in Gal(\overline{\mathbb{Q}}/\Phi)$ *we have that* $Tr(f)^\sigma = Tr(f^\sigma)$.

Proof. The proof of this is similar to the one given by Sturm in [25] for Siegel modular forms and extended in [2]. All we need to observe is that if we write $\Gamma_q = \coprod \Gamma'_q y$, then we can pick $y \in SU(n,n)$. Indeed we have by Lemma 2.6 that $det(\Gamma_q) = det(\Gamma'_q)$. In particular if we fix any decomposition $\Gamma_q = \coprod_x \Gamma'_q x$, then $det(x) \in det(\Gamma_q) = det(\Gamma'_q)$. That is there exists $\gamma \in \Gamma'_q$ such that $det(x) = det(\gamma)$. So if we consider the elements $y = \gamma^{-1}x$ then we have $det(y) = 1$ and so $y \in SU(n,n)$ and they form a set of representatives, which concludes our claim. Then we can follow an argument similar to the proof of [25, Lemma 11] or in [2, Lemma 8], since now we have the reciprocity law for elements in $SU(n,n)$ and the strong approximation holds. □

The above theorem can in some cases be stated in a stronger form, namely we can take that $\mathbf{g}$ above is actually in $\mathcal{M}_k(\overline{\mathbb{Q}})$. Of course this question is meaningful only when $\mathcal{S}_k \neq \mathcal{M}_k$, that is if $m_v = m_0$ for all v. However for this we need to know the rational decomposition of the Eisenstein part. This is known in the case of absolute convergence by a result of Michael Harris [12, Main Theorem 3.2.1]. Actually his result implies that,

Theorem 5.5 (Harris, Corollary to Theorem 3.2.1 in [12]). *Assume that* $m_0 > 4n - 2$. *Then there exists a projection operator*

$$\mathfrak{p} : \mathcal{M}_k(C, \psi, \overline{\mathbb{Q}}) \to \mathcal{S}_k(C, \psi, \overline{\mathbb{Q}}),$$

such that $\mathfrak{p}(\mathbf{g})^\sigma = \mathfrak{p}(\mathbf{g}^\sigma)$ *for all* $\sigma \in Gal(\overline{\mathbb{Q}}/\Phi)$.

In particular if we assume that $m_0 > 4n-2$ then we can take $\mathbf{g}$ in the Theorem 5.2 to be in $\mathbf{g} \in \mathcal{M}_k(\overline{\mathbb{Q}})$. Actually an improvement of this bound will allow us to remove some of the assumptions made in the theorems below. We note here that Shimura in [24, Theorem 27.14] has results towards this direction, but his results are only over an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$.

6. Algebraicity Results for Special L-Values and Reciprocity Laws

In this section we present various results regarding special values of the function $L(s, \mathbf{f}, \chi)$, with $\mathbf{f} \in \mathcal{S}_k(C, \psi)$, an eigenform for all Hecke operators. We recall that we have also considered the function $Z(s, \mathbf{f}, \chi)$. The two coincide when the nebentype of $\mathbf{f}$ is trivial. Indeed if we write $Z_{\mathfrak{q}}(\chi^*(\mathfrak{q})N(\mathfrak{q})^{-s})$ for the Euler factor of $Z(s,\mathbf{f},\chi)$ at some prime $\mathfrak{q}$ of K then the corresponding Euler factor of $L(s,\mathbf{f},\chi)$ is equal to $Z_{\mathfrak{q}}((\psi/\psi_{\mathfrak{c}})(\pi)\chi^*(\mathfrak{q})N(\mathfrak{q})^{-s})$, where π is a uniformizer of $K_{\mathfrak{q}}$. We note the equation

$$L(s,\mathbf{f},\chi\psi^{-1}) = Z_{\mathfrak{c}}(s,\mathbf{f},\chi),$$

where the subindex on the right hand side indicates that we have removed the Euler factors at all primes in the support of $\mathfrak{c}$. In particular if we take the character χ trivial at the primes dividing $\mathfrak{c}$ (the character χ may not be primitive) then we have that the two functions are the same.

We start by stating a result of Shimura [24, Theorem 28.8]. We take $\mathbf{f} \in \mathcal{S}_k(C;\overline{\mathbb{Q}})$, where

$$C = \{x \in D[\mathfrak{b}^{-1},\mathfrak{bc}] | a_x - 1 \prec \mathfrak{c}\}.$$

We moreover take $\mathbf{f}$ of trivial Nebentypus and assume that it is an eigenform for all Hecke operators away from the primes in the support of $\mathfrak{c}$. In the notation of [24, Chapter V], we take $\mathfrak{e} = \mathfrak{c}$, and not $\mathfrak{e} = \mathcal{O}_F$. In particular here we take the Euler factors Z_v trivial for v in the support of $\mathfrak{c}$.

Theorem 6.1 (Shimura, Theorem 28.8 in [24]). *With notation as above define $m_0 := min\{m_v := k_v + k_{v\rho} | v \in \mathbf{a}\}$ and assume $m_0 > 3n$. Let χ be a Hecke character of K such that $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^t |x_{\mathbf{a}}|^{-t}$ with $t \in \mathbb{Z}^{\mathbf{a}}$. Let $\sigma_0 \in \frac{1}{2}\mathbb{Z}$ such that*

$$4n - (2k_{v\rho} + t_v) \leq 2\sigma_0 \leq m_v - |k_v - k_{v\rho} - t_v|,$$

and

$$2\sigma_0 - t_v \in 2\mathbb{Z}, \ \ \forall v \in \mathbf{a}.$$

We exclude the cases

(1) $2\sigma_0 = 2n+1$, $F = \mathbb{Q}$ and $\chi_1 = \theta$, and $k_v - k_{v\rho} = t_v$,
(2) $0 < 2\sigma_0 \leq 2n$, $\mathfrak{c} = \mathcal{O}_F$, $\chi_1 = \theta^{2\sigma_0}$ and the conductor of χ is $\mathcal{O}_K$.

Then we have

$$\frac{Z(\sigma_0,\mathbf{f},\chi)}{\langle \mathbf{f},\mathbf{f}\rangle} \in \pi^{n(\sum_v m_v) + d(2n\sigma_0 - 2n^2 + n)}\overline{\mathbb{Q}},$$

where $d = [F:\mathbb{Q}]$.

We now take $\mathbf{f} \in \mathcal{S}_k(C,\psi;\overline{\mathbb{Q}})$ with C of the form $D[\mathfrak{b}^{-1},\mathfrak{bc}]$ (i.e. the standard setting in this paper). Then, with notation as in the previous theorem,

Theorem 6.2. *Let $\mathbf{f} \in \mathcal{S}_k(C,\psi;\overline{\mathbb{Q}})$ be an eigenform for all Hecke operators, and assume that $m_0 \geq 3n+2$. Let χ be a character of K such that $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^t |x_{\mathbf{a}}|^{-t}$ with $t \in \mathbb{Z}^{\mathbf{a}}$, and define $\mu \in \mathbb{Z}^{\mathbf{b}}$ by $\mu_v := -t_v - k_{v\rho} + k_v$ and $\mu_{v\rho} = 0$ if $k_{v\rho} - k_v + t_v \leq 0$, and $\mu_v = 0$ and $\mu_{v\rho} = k_{v\rho} - k_v + t_v$, if $k_{v\rho} - k_v + t_v > 0$. Assume moreover that either*

(1) there exists $v, v' \in \mathbf{a}$ such that $m_v \neq m_{v'}$, or
(2) $m_v = m_0$ for all v and $m_0 > 4n-2$, or
(3) $\mu \neq 0$.

Then let $\sigma_0 \in \frac{1}{2}\mathbb{Z}$ *such that*

$$4n - m_v + |k_v - k_{v\rho} - t_v| \leq 2\sigma_0 \leq m_v - |k_v - k_{v\rho} - t_v|,$$

and,

$$2\sigma_0 - t_v \in 2\mathbb{Z}, \ \ \forall v \in \mathbf{a}.$$

We exclude the σ_0*'s in* $n \leq 2\sigma_0 < 2n$*, for which there is no choice of the integral ideal* $\mathfrak{c}''$ *in Theorem 4.3 such that for any prime ideal* $\mathfrak{q}$ *of* F*,* $\mathfrak{q}|\mathfrak{c}''\mathfrak{c}^{-1}$ *implies either* $\mathfrak{q}|\mathfrak{f}'$ *or* $\mathfrak{q}$ *ramifies in* K*. Here* $\mathfrak{f}'$ *denotes the conductor of the character* χ_1*.*

Then we have

$$\frac{L(\sigma_0, \mathbf{f}, \chi)}{\langle \mathbf{f}, \mathbf{f}\rangle} \in \pi^{n(\sum_v m_v)+d(2n\sigma_0-2n^2+n)}\overline{\mathbb{Q}}.$$

Moreover, if we take a number field W *so that* $\mathbf{f}, \mathbf{f}^\rho \in \mathcal{S}_k(W)$ *and* $\Psi\Phi \subset W$*, where* Φ *is the Galois closure of* K *in* $\overline{\mathbb{Q}}$*, and* Ψ *as in the Theorem 5.2 then*

$$\frac{L(\sigma_0, \mathbf{f}, \chi)}{\pi^\beta \tau(\chi_1^n \psi_1^n \theta^{n^2}) i^{n\sum_{v\in\mathbf{a}} p_v} \langle \mathbf{f}, \mathbf{f}\rangle} \in \mathcal{W} := W(\chi),$$

where $\beta = n(\sum_v m_v) + d(2n\sigma_0 - 2n^2 + n)$*,* $W(\chi)$ *obtained from* W *by adjoining the values of* χ *on finite adeles, and* $p \in \mathbb{Z}^{\mathbf{a}}$ *is defined for* $v \in \mathbf{a}$ *as* $p_v = \frac{m_v - |k_v - k_{v\rho} - t_v| - 2\sigma_0}{2}$ *if* $\sigma_0 \geq n$*, and* $p_v = \frac{m_v - |k_v - k_{v\rho} - t_v| - 4n + 2\sigma_0}{2}$ *if* $\sigma_0 < n$*.*

Remark 6.3. With respect to the cases excluded in the above theorem we would like to make the following comment. As it will become clear in the proof, we need to make sure that the finite product $\frac{\Lambda_{\mathfrak{c}}(\sigma_0)}{\Lambda_{\mathfrak{c}''}(\sigma_0)}$ does not have a pole. But if it is possible to pick the integral ideal $\mathfrak{c}''$ such that for any prime ideal $\mathfrak{q}$ of F, $\mathfrak{q}|\mathfrak{c}''\mathfrak{c}^{-1}$ implies either $\mathfrak{q}|\mathfrak{f}'$ or $\mathfrak{q}$ ramifies in K, then the finite product $\frac{\Lambda_{\mathfrak{c}}(\sigma_0)}{\Lambda_{\mathfrak{c}''}(\sigma_0)}$ is equal to one, where we recall that for an integral ideal $\mathfrak{h}$ of F we write

$$\Lambda_{\mathfrak{h}}(s) = \prod_{i=1}^{n} L_{\mathfrak{h}}(2s - n + 1 - i, \psi_1\chi_1\theta^{n+i-1}).$$

Actually even a weaker condition is needed in order to guarantee that $\frac{\Lambda_{\mathfrak{c}}(\sigma_0)}{\Lambda_{\mathfrak{c}''}(\sigma_0)}$ does not have a pole. Indeed this can only happen for σ_0 in the range $n \leq 2\sigma_0 \leq 2n-1$ and if we have $(\psi_1\chi_1\theta^{2\sigma_0})(\mathfrak{q}) = 1$ for some integral ideal $\mathfrak{q}$ of F with $\mathfrak{q} \not| \mathfrak{c}$ and $\mathfrak{q}\,|\mathfrak{c}''$. Since there are only finite many primes $\mathfrak{q}$ dividing $\mathfrak{c}''\mathfrak{c}^{-1}$, this is a condition in finitely many integral ideals imposed on the character $\psi_1\chi\theta^{2\sigma_0}$, which is equal to $\psi_1\chi_1$ if $\sigma_0 \in \mathbb{Z}$ and $\psi_1\chi_1\theta$ otherwise. Finally we mention that a similar condition appears in the Siegel modular forms case in [22, Proposition 8.3].

Remark 6.4. Before giving the proof of the Theorem 6.2, we would like to make some comments on the differences and new input in comparison to the Theorem 6.2 [24, Theorem 28.8]. Of course as we mentioned the L-functions appearing in the two theorems will differ in the presence of nebentype. Even if we take the nebentype to be trivial, then one could try to compare the results, but one should keep in mind, as we indicated above, we obtain the

Z-function from the L-function after removing the finitely many bad Euler factors. However one should exclude then the values of s where some of these Euler factors are zero. Moreover the main aim of Theorem 6.2 is to provide more precise information about the field where the normalized L-values lie. Moreover:

(i) We would like to remark that our theorem (Theorem 6.2) provides some algebraic results in cases, which are not covered by the one of Shimura (Theorem 6.1), since the excluded values of σ_0 in Theorem 6.1 do not coincide with the ones excluded in Theorem 6.2. We would like to explain briefly why this is happening. In the proof of Theorem 6.1 [24, Theorem 28.8] two methods are provided. One is based on the doubling method (1st method in the notation there) and the other is based on the Rankin-Selberg method (2nd method in the notation there). The cases excluded in Theorem 6.1, are the ones where the Eisenstein series involved in the doubling method are not nearly holomorphic. Of course the question is whether one could avoid these restrictions by working with the Rankin-Selberg method (2nd method of proof). However in the proof of Theorem 6.1, the Rankin-Selberg method is applicable only to the case of $\sigma_0 \geq n$. The reason for this is that Shimura is working with the form given in Theorem 4.1. The Eisenstein series involved there does not allow one to consider the case of $\sigma_0 < n$, since in this case they are not nearly holomorphic. However in our proof we employ the form of Theorem 4.3, in which the nearly holomorphicity of the Eisenstein series involved is better understood, even when we take $\sigma_0 < n$.

(ii) The second point we would like to emphasize is related to the results about the field in which the L-values (after divided with the appropriate periods) lie. We start by explaining the assumptions (i),(ii) and (iii) of the Theorem 6.2. The reason for these assumptions is the lack of precise information about the field over which we have a decomposition of the Hermitian modular forms to the Eisenstein part and to the cuspidal part. As we mentioned above in Theorem 5.5, we have such an information in the case of absolute convergent, which is case (ii) in our theorem. In case (i), the non parallel situation we know that there is no Eisenstein part [23, Proposition 10.6]. Finally case (iii) will make the theta series, involved in the Rankin-Selberg method, a cusp form, and hence also the product with a nearly holomorhic Eisenstein series will be again cuspidal. In particular, after taking holomorphic projection, we will not have to worry about splitting from the Eisenstein part. We also would like to emphasize here that even though the cases (i) and (ii) could also be available using the doubling method, case (iii) is possible only when one uses the Rankin-Selberg method, due to the presence of the theta series.

(iii) Finally our proof relies heavily on the various results which we proved about the Galois reciprocity of Siegel-type Eisenstein series in section 3, as well as Theorem 5.2, which provides precise information about the ratio of Petersson inner products needed in the proof of Theorem 6.2.

Proof of Theorem 6.2. We set $s := \sigma_0 - \frac{3n}{2}$, and $\ell := \mu + n\mathbf{a}$. By Theorem 4.3 we know that there exists $\tau \in S_+ \cap GL_n(K)$ and $r \in GL_n(K)_\mathbf{h}$ such that

$$C(S)\Gamma((\sigma_0 - \frac{3n}{2}))\psi_\mathfrak{c}(det(r))c_\mathbf{f}(\tau, r)L(\sigma_0, \mathbf{f}, \chi) =$$

$$\Lambda_\mathfrak{c}(\sigma_0, \theta(\psi\chi)_1) \cdot \left(\prod_{v\in\mathbf{b}} g_v(\chi(\pi_\mathfrak{p})N(\mathfrak{p})^{-2\sigma_0})\right) det(\tau)^{\sigma_0 - \frac{3n}{2}\mathbf{a}+h}|det(r)|_K^{-\sigma_0+n} C_0 \times$$

$$\sum_{q\in Q} |det(qq^*)|_F^{-n} vol(\Phi_q)\langle f_q(z), \theta_{q,\chi}(z)E_q(z, \frac{\nu}{2}, (\psi'/\psi)^c, C'')\rangle_{\Gamma_0^q(\mathfrak{c}'')},$$

where we have set $\nu := 2\sigma_0 - n$ a,d hence $s = \frac{\nu}{2} - n$.

We first consider the conditions under which the gamma factors do no have any poles. We first recall that

$$\Gamma_n(s) = \pi^{n(n-1)/2} \prod_{j=0}^{n-1} \Gamma(s-j).$$

Hence for $\prod_{v\in\mathbf{a}} \Gamma_n(\sigma_0 - \frac{3n}{2} + \frac{k_v + k_{v\rho} + \ell_v + \ell_{v\rho}}{2})$. we have

$$\sigma_0 - \frac{3n}{2} + \frac{k_v + k_{v\rho} + \ell_v + \ell_{v\rho}}{2} = \sigma_0 - n + \frac{k_v + k_{v\rho} + \mu_v + \mu_{v\rho}}{2} =$$

$$\sigma_0 - n + \frac{k_v + k_{v\rho} + |k_v - k_{v\rho} - t_v|}{2}.$$

Hence we need that $\sigma_0 - n + \frac{k_v + k_{v\rho} + |k_v - k_{v\rho} - t_v|}{2} > n - 1$ or equivalently

$$2\sigma_0 > 4n - 2 - m - |k_v - k_{v\rho} - t_v|$$

We now consider the Eisenstein series $D_q(\frac{\nu}{2}) := \Lambda_{\mathfrak{c}''}(\sigma_0, \theta\overline{(\psi\chi)}_1)E_q(z, \frac{\nu}{2}; k - \ell, (\psi'/\psi)^c, C'')$ of weight $k-\ell$. We check for which values is nearly holomorphic. It is nearly holomorphic if and only if $2n - (k_v - \ell_v) - (k_{v\rho} - \ell_{v\rho}) \leq \nu \leq (k_v - \ell_v) + (k_{v\rho} - \ell_{v\rho})$ or equivalently

$$4n - m_v + |k_v - k_{v\rho} - t_v| \leq 2\sigma_0 \leq m_v - |k_v - k_{v\rho} - t_v|.$$

Moreover we need that $m_v - |k_v - k_{v\rho} - t_v| - 2\sigma_0 \in 2\mathbb{Z}$ or equivalently $t_v - 2\sigma_0 \in 2\mathbb{Z}$ for all $v \in \mathbf{a}$.
For the values at which the series $D_q(\frac{\nu}{2})$ is nearly holomorphic we know that

$$\pi^{-\gamma} D_q(\frac{\nu}{2}) \in \mathcal{N}_{k-\ell}(\overline{\mathbb{Q}}),$$

for $\gamma = (n/2)\sum_{v\in\mathbf{a}}(m_v - |k_v - k_{v\rho} - t_v| - 2n + 2\sigma_0) - dn(n-1)/2$. Moreover we note that the condition $t_v - 2\sigma_0 \in 2\mathbb{Z}$ implies that

$$\sigma_0 - n + \frac{k_v + k_{v\rho} + |k_v - k_{v\rho} - t_v|}{2} \in \mathbb{Z}$$

in particular $\Gamma_n(\sigma_0 - \frac{3n}{2} + \frac{k_v+k_{v\rho}+\ell_v+\ell_{v\rho}}{2}) \in \pi^{n(n-1)/2}\mathbb{Q}^\times$ and so $\prod_{v\in\mathbf{a}} \Gamma_n(\sigma_0 - \frac{3n}{2} + \frac{k_v+k_{v\rho}+\ell_v+\ell_{v\rho}}{2}) \in \pi^{d(n(n-1)/2}\mathbb{Q}^\times$. We then conclude that

$$\Gamma((\sigma_0 - \frac{3n}{2})) = \prod_{v\in\mathbf{a}} (4\pi)^{-n(\sigma_0-\frac{3n}{2}+h_v)}\Gamma_n(\sigma_0 - \frac{3n}{2} + h_v) \in \pi^{\epsilon}\mathbb{Q}^\times,$$

where $\epsilon = dn(n-1)/2 - n\sum_{v\in\mathbf{a}}(\sigma_0 - n + \frac{k_v+k_{v\rho}+|k_v-k_{v\rho}-t_v|}{2})$. Finally we recall that $vol(\Phi) \in \pi^{dn^2}\mathbb{Q}^\times$. Putting this together we have that up to elements of $\mathbb{Q}^\times$

$$C(S)\pi^{-\beta}L(\sigma_0, \mathbf{f}, \chi) =^{\mathbb{Q}^\times}$$

$$\psi_{\mathfrak{c}}^{-1}(det(r))c_{\mathbf{f}}(\tau, r)^{-1}\left(\prod_{v\in\mathbf{b}} g_v(\chi(\pi_{\mathfrak{p}})N(\mathfrak{p})^{-2\sigma_0})\right) det(\tau)^{\sigma_0-\frac{3n}{2}\mathbf{a}+h}\times$$

$$|det(r)|_K^{-\sigma_0+n}C_0\frac{\Lambda_{\mathfrak{c}}(\sigma_0, \theta(\psi\chi)_1)}{\Lambda_{\mathfrak{c}''}(\sigma_0, \theta(\psi\chi)_1)}\sum_{q\in Q}|det(qq^*)|_F^{-n}\langle f_q(z), \theta_{q,\chi}(z)\frac{D_q(\frac{\nu}{2})}{\pi^\gamma}\rangle,$$

where $\beta = n(\sum_v m_v) + d(2n\sigma_0 - 2n^2 + n)$. We have moreover established that $\sigma_0 - \frac{3n}{2}\mathbf{a} + h_v \in \mathbb{Z}$ for all $v \in \mathbf{a}$. Moreover we notice that because of the assumption in the theorem, the factor $\frac{\Lambda_{\mathfrak{c}}(\sigma_0,\theta(\psi\chi)_1)}{\Lambda_{\mathfrak{c}''}(\sigma_0,\theta(\psi\chi)_1)}$ does not have a pole and belongs to $\mathcal{W}$. Hence in order to conclude the theorem it is enough to show that

$$\frac{\sum_{q\in Q}|det(qq^*)|_F^{-n}\langle f_q(z), \theta_{q,\chi}(z)\frac{D_q(\frac{\nu}{2})}{\pi^\gamma})\rangle}{\pi^\gamma\tau(\chi_1^n\psi_1^n\theta^{n^2})i^{n\sum_{v\in\mathbf{a}}p_v}\langle\mathbf{f},\mathbf{f}\rangle} \in \mathcal{W}.$$

First we assume that $\sigma_0 \in \mathbb{Z}$. We note that for every q component

$$\frac{|det(qq^*)|_F^{-n}\theta_{q,\chi}(z)D_q(\frac{\nu}{2})}{\tau(\overline{\chi_1^n\psi_1^n\theta^{n^2}})i^{n\sum_{v\in\mathbf{a}}p_v}\pi^\gamma} \in \mathcal{N}_k(\mathcal{W}).$$

Indeed this follows from Lemma 1 and Proposition 3.9 combined with the observation that $n - \frac{\nu}{2} = n - \sigma_0 + \frac{n}{2} \equiv \frac{n}{2} \mod \mathbb{Z}$, and hence we do not have to worry about the $|det(qq^*)|_{\mathbf{h}}^{n/2}$, since the will cancelled out by the theta series. If we are in the cases (i) or (iii) then we know that $\theta_{q,\chi}(z)D_q(\frac{\nu}{2}) \in \mathcal{R}_k$ as in the notation of Shimura in [24, page 124] (cuspidal nearly holomorphic forms), and by [24, Proposition 15.6] we have that there exists $g_q \in S_k(\mathcal{W})$ such that

$$\langle f_q(z), \frac{|det(qq^*)|_F^{-n}\theta_{q,\chi}(z)D_q(\frac{\nu}{2})}{\tau(\overline{\chi_1^n\psi_1^n\theta^{n^2}})i^{n\sum_{v\in\mathbf{a}}p_v}\pi^\gamma}\rangle = \langle f_q, g_q\rangle.$$

Using the fact that the maps π_q are defined over $\mathcal{W}$ we have that there exists a cusp form $\mathbf{g}$ defined over $\mathcal{W}$ such that $\sum_{q\in Q}\langle f_q, g_q\rangle = \langle\mathbf{f}, \mathbf{g}\rangle$, and then using Theorem 5.2 we have that

$$\frac{\langle\mathbf{f},\mathbf{g}\rangle}{\langle\mathbf{f},\mathbf{f}\rangle} \in \mathcal{W},$$

and hence we conclude the theorem. In case (ii) we can use [24, Lemma 15.8] to conclude that there exists $g_q \in \mathcal{M}_q(\mathcal{W})$ such that

$$\langle f_q(z), \frac{|det(qq^*)|_F^{-n}\theta_{q,\chi}(z)D_q(\frac{\nu}{2})}{\tau(\overline{\chi_1^n\psi_1^n\theta^{n^2}})i^{n\sum_{v\in\mathbf{a}}p_v}\pi^\gamma}\rangle = \langle f_q, g_q\rangle,$$

and then use the theorem of Harris, stated in the previous section, and write $g_q = E_q + g'_q$ with $g'_q \in S_k(\mathcal{W})$ and E_q an Eisenstein series. And so we obtain $< f_q, g_q >=< f_q, g'_q >$. Then arguing as before we conclude the Theorem also in this case.

For the case $\sigma_0 \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$ the argument is almost identical, but now we have that $n - \frac{\nu}{2} \not\equiv \frac{n}{2} \mod \mathbb{Z}$. That is, we may have to worry about square roots of $|det(qq^*)|_{\mathbf{h}}$. But we note that the final expression is independent of the choice of $q \in Q$. As before we can establish that

$$\frac{L(\sigma_0, \mathbf{f}, \chi)}{\pi^\beta\tau(\chi_1^n\psi_1^n\theta^{n^2})i^{n\sum_{v\in\mathbf{a}}p_v}\langle\mathbf{f},\mathbf{f}\rangle} \in \mathcal{W}_Q,$$

where $\mathcal{W}_Q$ is the field obtained by adjoining $|det(qq^*)|_{\mathbf{h}}^{1/2}$ to $\mathcal{W}$. Let us pick another set $Q' \subset GL_n(K_{\mathbf{h}})$ so that for all $Q' \in Q'$ we have $q'_v = 1$ for all v that ramify in $\mathcal{W}_Q$. In particular $\mathcal{W}_{Q'} \cap \mathcal{W}_Q = \mathcal{W}$. But we also have that

$$\frac{L(\sigma_0, \mathbf{f}, \chi)}{\pi^\beta\tau(\chi_1^n\psi_1^n\theta^{n^2})i^{n\sum_{v\in\mathbf{a}}p_v}\langle\mathbf{f},\mathbf{f}\rangle} \in \mathcal{W}_{Q'}$$

by the same argument as for Q. But then we have that the values must actually lie in the intersection, namely $\mathcal{W}$. □

We now obtain also some results with reciprocity laws.

THEOREM 6.5. *Let $\mathbf{f} \in \mathcal{S}_k(C, \psi; \overline{\mathbb{Q}})$ be an eigenform for all Hecke operators. With notation as before we take $m_0 > 3n + 2$. Let χ be a Hecke character of K such that $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{-t}|x_{\mathbf{a}}|^t$ with $t \in \mathbb{Z}^{\mathbf{a}}$. Define $\mu \in \mathbb{Z}^{\mathbf{b}}$ as in the previous theorem. With the same assumptions as in the previous theorem and with $\Omega_{\mathbf{f}} \in \mathbb{C}^\times$ as defined in the previous section in Theorem 5.2 we have for all $\sigma \in Gal(\overline{\mathbb{Q}}/\Psi_Q)$ that*

$$\left(\frac{L(\sigma_0, \mathbf{f}, \chi)}{\pi^\beta\tau(\chi_1^n\psi_1^n\theta^{n^2})i^{n\sum_{v\in\mathbf{a}}p_v}\Omega_{\mathbf{f}}}\right)^\sigma = \frac{L(\sigma_0, \mathbf{f}^\sigma, \chi^\sigma)}{\pi^\beta\tau((\chi_1^n\psi_1^n\theta^{n^2})^\sigma)i^{n\sum_{v\in\mathbf{a}}p_v}\Omega_{\mathbf{f}^\sigma}},$$

where $\Psi_Q = \Psi$ if σ_0 is an integer and it is the algebraic extension of Ψ obtained by adjoining $|det(qq^*)|_{\mathbf{h}}^{1/2}$ for all $q \in Q$, for a fixed set Q, if σ_0 is a half integer.

Proof. This follows exactly in the same way as we argued in the theorem above, so we just mention the necessary additional observations. First we see that

$$\left(|det(qq^*)|_F^{-n}\theta_{q,\chi}(z)\frac{D_q(\frac{\nu}{2}, \overline{\chi\psi\phi^n})}{\pi^\gamma\tau(\overline{\chi_1^n\psi_1^n\theta^{n^2}})i^{n\sum_{v\in\mathbf{a}}p_v}})\right)^{\sigma'} =$$

$$|det(qq^*)|_F^{-n}\theta_{q,\chi^\sigma}(z)\frac{D_q(\frac{\nu}{2}, \overline{\chi^\sigma\psi^\sigma\phi^n})}{\pi^\gamma\tau(\overline{\chi^\sigma{}_1^n\psi^\sigma{}_1^n\theta^{n^2}})i^{n\sum_{v\in\mathbf{a}}p_v}}.$$

and the fact that the holomorphic projection we used Proposition 15.6 in [24] for cuspidal nearly holomorphic modular forms is Galois equivariant. This will settle the cases (i) and (iii). For (ii) we use as before Lemma 15.8 of [24] and Harris's projector from Theorem 5.5 which is Galois equivariant. We remark here that in Lemma 15.8 of [24] the Galois equivariance is not explicitly mentioned. However it follows from the fact that in our situation the space of nearly holomorphic modular forms has a basis over $\Phi \subset \Psi_Q$ (see [24, Proposition 14.13]), and the map will preserve the field of definition of the basis. Finally using now Theorem 5.2 we can finish the proof of the theorem. □

Remark 6.6. The only difficulty to obtain stronger results in the case where $\mu = 0$ and the $(m_v)_v$ is parallel is the lack of result as in [12], on the rationality of Eisenstein series beyond the absolute convergence. A strengthen of the Theorem 5.5 of Harris, will allow to obtain stronger results.

COMPARISON WITH THE RESULTS OF HARRIS IN [11, 13]: Let us now compare the above results with the rationality results obtained by Harris in [11, Theorem 3.5.13]. As we indicated in the introduction the results of Harris are much more general than the ones obtained here. For example, Harris establishes his results using the doubling method, which allows him to consider more general Hermitian modular forms, namely forms attached to unitary groups of the form $U(n,m)$ at infinity, where $n \neq m$. This cannot be done using the Rankin-Selberg method. Another direction in which the results of Harris are more general is the fact that he considers vector valued Hermitian modular forms, where in this work we have restricted ourselves to the scalar weight situation. So, in what follows we will be comparing results in the case of scalar weight modular forms, associated to a unitary group of trivial Witt signature (i.e of the form $U(n,n)$ at infinity).
We start by establishing a dictionary between unitary Hecke characters and Hecke characters of type A_0 in the sense of Weil. We recall the definition,

DEFINITION 6.7. A Grössencharacter of type A_0, in the sense of Weil, of conductor dividing a given integral ideal $\mathfrak{m}$ of K, is a homomorphism $\chi : I(\mathfrak{m}) \to \overline{\mathbb{Q}}$ such that there exist integers $\lambda(\tau)$ for each $\tau : K \hookrightarrow \overline{\mathbb{Q}}$, such that for each $\alpha \in K^\times$ we have

$$\chi((\alpha)) = \prod_\tau \tau(\alpha)^{\lambda(\tau)}, \text{ if } \alpha \equiv 1 \mod {}^\times\mathfrak{m}, \text{ and } \alpha >> 0.$$

It is well known (see for example [17]) if we take K to be a CM field then the above $\lambda(\tau)$ must satisfy some conditions. In particular if we select a CM type Σ of K then there exists integers $d(\sigma)$ for each $\sigma \in \Sigma$ and an integer k such that

$$\chi((\alpha)) = \prod_{\sigma\in\Sigma} \left(\frac{1}{\sigma(\alpha)^k} \left(\frac{\sigma(\bar{\alpha})}{\sigma(\alpha)} \right)^{d(\sigma)} \right), \text{ if } \alpha \equiv 1 \mod {}^\times\mathfrak{m}.$$

We now keep writing χ for the associated by class field theory adelic character to χ. As it is explained in [17, page 286] the infinity type is of the form (after we identify $\mathbf{a}$ with Σ),

$$\chi_{\mathbf{a}}(x) = \prod_{v \in \mathbf{a}} \left(\frac{x_v^{k+d_v}}{\bar{x_v}^{d_v}} \right).$$

We now consider the unitary character $\psi := \chi|\cdot|_{\mathbb{A}_K}^{-k/2}$, where $|x|_{\mathbf{a}} = \prod_{v \in \mathbf{a}} |x_v|_v^k$, where $|\cdot|_v$ is the standard absolute value as in Shimura, not the normalized (i.e. the square). We then have that

$$\psi_{\mathbf{a}}(x_{\mathbf{a}}) = \prod_{v \in \mathbf{a}} \left(\frac{x_v^{k/2+d_v}}{\bar{x_v}^{k/2+d_v}} \right) = \prod_{v \in \mathbf{a}} \left(\frac{x_v^{k+2d_v}}{(x_v \bar{x_v})^{k/2+d_v}} \right) = \prod_{v \in \mathbf{a}} \left(\frac{x_v^{k+2d_v}}{|x_v|^{k+2d_v}} \right).$$

In particular to a Grössencharacter χ of type A_0 of infinity type $-k\Sigma_\sigma \sigma + \Sigma_\sigma d(\sigma)\,(\sigma - \bar{\sigma})$ we can associate a unitary character ψ of weight $\{m_v\}_{v \in \mathbf{a}}$ with $m_v := k + 2d(\sigma)$ after identifying Σ with $\mathbf{a}$. The relation between the associated L functions is given by

$$L(s, \chi) = L(s + k/2, \psi).$$

Now we turn to the paper of Harris [11], and we use his notation. First we observe that the Dirichlet series relevant to our discussion is the function $L^{mot}(s, \pi, St)$ in the notation of Harris and not the function $L(s, \pi, St)$, which is related to the previous one by $L^{mot}(s, \pi, St) = L(s - n + \frac{1}{2})$. Here we note that Harris' n is equal $2n$ with our notation. In the work of Harris n is the dimension of the Hermitian space i.e. $n = r + s$ for $U(r, s)$. Now as Harris writes (page 154) the function $L^{mot}(s, \pi, St)$ is absolutely convergent for $Re(s) > 2n$, which means $Re(s) > n + \frac{1}{2}$ for the other function. By ([23, 24, Theorem 20.13 and Theorem 22.11]) we then conclude that $L(s, \mathbf{f}) = L^{mot}(s, \pi)$, where $L(s, \mathbf{f})$ is the L-functions considered in this paper. Now we consider twists by Hecke characters. Here we remark that in this paper we follow Shimura and we consider unitary Hecke characters. Harris considers Hecke characters of a particular type at infinity i.e. $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^k$ (see [11, page 136] for the definition of η_k that shows up in the main theorem). By the discussion above we have then that if we write ψ for the corresponding unitary character defined by $\psi := \chi|\cdot|_{\mathbb{A}_K}^{-k/2}$ then

$$L(s, \chi) = L(s + k/2, \psi).$$

In particular we have the equality

$$L^{mot}(s, \pi, St, \chi) = L(s + k/2, \mathbf{f}, \psi).$$

That means that our variable σ_0 is related to the variable m of Harris by $\sigma_0 = m + k/2$. Hence in Harris [11] the results are for $\sigma_0 > 2n$ or equivalently $2\sigma_0 > 4n$.

We now move to the more recent paper [13] of Harris, in which he explains how the results of his first work in [11] could be extended to cover cases beyond the absolute convergence range. Indeed in his Theorem 4.3 in [13] he obtains results which are beyond the range of absolute convergence. However he puts an assumption on the twisting character χ [13, equation 4.3.2], which allows the use of the Siegel-Weil formula. This assumption excludes various cases

considered here, so there are still quite a few cases where the results of Harris and ours do not overlap. Moreover, since Harris is using the doubling method to obtain his results, the limitations of the method, as they were explained in Remark 6.4 (i), are present also here, and hence there are some special values which can be only considered by employing the Rankin-Selberg method.
Let us add at this point that in the case of $F \neq \mathbb{Q}$ we obtained some results in the non-parallel weight situation without any assumption on the twists. We do not know whether the results in [11, 13] could be strengthen in the case of $F \neq \mathbb{Q}$ and non-parallel weight (both works are written in the case of $F = \mathbb{Q}$). Finally we mention that a result on the ratio of Petersson inner products, as it is stated in our Theorem 5.2, could be of independent interest.
A LAST COMMENT ON SCALAR WEIGHTS. We finish this paper by making a last comment on the fact that we restrict ourselves here to the scalar weight situation. To the best of author's knowledge the Rankin-Selberg method has been utilized towards algebraicity results only for scalar weight modular forms. And this both in the Hermitian and in the Siegel modular form case. However the work of Piatetski-Shapiro and Rallis in [19] indicates that the Rankin-Selberg method could be used to study also the L-values of vector valued Hermitian or Siegel modular forms. Needless to say that that there are numerous technicalities to be worked out in order to get from the analytic continuation results, which is one of the main aims in [19], to algebraicity results. Perhaps the most difficult of them seems to be to define the appropriate theta series, and especially the defining Schwartz function at infinity (note that the result in [19] guarantees the existence but it is not constructive). And, of course, then also worked the algebraicity of these theta functions. This could be the aim of a future work.

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