Characterization of 2D Rational Local Conformal Nets and Its Boundary Conditions: the Maximal Case

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Abstract. Let $\mathcal{A}$ be a completely rational local Möbius covariant net on S^1, which describes a set of chiral observables. We show that local Möbius covariant nets $\mathcal{B}_2$ on 2D Minkowski space which contains $\mathcal{A}$ as chiral left-right symmetry are in one-to-one correspondence with Morita equivalence classes of Q-systems in the unitary modular tensor category DHR($\mathcal{A}$). The Möbius covariant boundary conditions with symmetry $\mathcal{A}$ of such a net $\mathcal{B}_2$ are given by the Q-systems in the Morita equivalence class or by simple objects in the module category modulo automorphisms of the dual category. We generalize to reducible boundary conditions.

To establish this result we define the notion of Morita equivalence for Q-systems (special symmetric $*$-Frobenius algebra objects) and non-degenerately braided subfactors. We prove a conjecture by Kong and Runkel, namely that Rehren's construction (generalized Longo-Rehren construction, α-induction construction) coincides with the categorical full center. This gives a new view and new results for the study of braided subfactors.

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Contents

1 Introduction

The subject of algebraic quantum field theory has led to many structural results and recently also to interesting constructions and classifications in quantum field theory. Conformal quantum field theory can be conveniently studied in this approach. In particular there is the notion of a conformal QFT on Minkowski space and boundary conformal QFT on Minkowski half-plane $x > 0$.

One can associate with a boundary conformal QFT (boundary theory) a conformal QFT on Minkowski space (bulk theory), but in general several boundary theories can have the same bulk theory, which correspond to different boundary conditions of the bulk theory.

In a different framework Fuchs, Runkel and Schweigert gave a general construction, the so-called TFT construction, of a (euclidean) rational full conformal field theory (CFT). The construction can be divided into two steps: first one chooses a certain vertex operator algebra (VOA), whose representation category $\mathcal{C}$ is a modular tensor category and which specifies chiral fields. This can be seen as the analytical part. Then with a choice of a special symmetric Frobenius algebra object $A \in \mathcal{C}$ one can construct correlators on an arbitrary Riemann surface. The bulk field content depends on the Morita equivalence class of A, while A itself fixes a boundary condition.

Carpi, and two of the authors gave a general procedure starting from an algebraic quantum field theory on the Minkowski space, to obtain all locally isomorphic boundary conformal QFT nets, in other words to find all possible boundary conditions (with unique vacuum). The main purpose of this paper is to show that there is a similar classification for the boundary conditions for maximal (full) (conformal) local nets on Minkowski space and its boundary conditions as in the afore mentioned TFT construction.

Let us consider more concretely a quantum field theory on Minkowski space. By introducing new coordinates $x_\pm = t \mp x$ we identify the two-dimensional Minkowski space $\mathbb{M} = \{(t,x) \in \mathbb{R}^2\}$ with metric $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2$ with the product $L_+ \times L_-$ of two light rays $L_\pm = \{(t,x) : t \pm x = 0\}$ with metric $\mathrm{d}s^2 = \mathrm{d}x_+ \mathrm{d}x_-$. The densities of conserved quantities (symmetries) are prescribed by left and right moving chiral fields, i.e. fields just depending on x_+ or x_-, respectively.

For example for the stress-energy tensor holds $T_{00,01} = T_+(x_+) \pm T_-(x_-)$ and for the conserved $\mathsf{U}(1)$-current holds $j_{0,1}(t,x) = j_+(x_+) \pm j_-(x_-)$. In the algebraic setting such conserved quantities are abstractly given by a net $\mathcal{A}_2(O) = \mathcal{A}_+(I) \otimes \mathcal{A}_-(J)$.

In general, there can be other local observables, so the net of observables is a local extension $\mathcal{B}(O) \supset \mathcal{A}_2(O)$ of $\mathcal{A}_2$. We ask this extension to be irreducible ($\mathcal{B}(O) \cap \mathcal{A}_2(O)' = \mathbb{C}\cdot 1$), which is for example true if we assume that $\mathcal{A}_2$ contains the stress energy tensor of $\mathcal{B}$.

We will also assume that the algebras of left and right moving chiral fields are isomorphic, in other words $\mathcal{A}_2(O) = \mathcal{A}(I) \otimes \mathcal{A}(J)$ where $O = I \times J \subset L_+ \times L_-$ and $\mathcal{A}$ is a local Möbius covariant net on $\mathbb{R}$. So in this case symmetries are prescribed by the net $\mathcal{A}$.

We further assume $\mathcal{A}$ to be completely rational, this is for example true for the net Vir_c generated by the stress energy tensor with central charge $c < 1$, $\mathsf{SU}(N)$ loop group models, or conformal nets associated with even lattices (lattice compactifications). The category of Doplicher–Haag–Roberts superselection sectors of a completely rational conformal net is a unitary modular tensor category [KLM01].

Fixing $\mathcal{A}$ we are, as a first step, interested in classifying all nets $\mathcal{B}$ "containing the symmetries described by $\mathcal{A}$", i.e. to classify all local extensions $\mathcal{B}_2 \supset \mathcal{A}_2$. It turns out that the maximal ones are classified by Morita equivalence classes of chiral extensions $\mathcal{A} \subset \mathcal{B}$.

Let us look a moment into nets defined on $\mathbb{M}_+ = \{(t,x) \in M : x > 0\}$, i.e. nets with a boundary at $x = 0$. We are interested to prescribe boundary conditions of $\mathcal{B}_2$ without flow of "charges" associated with $\mathcal{A}$. The vanishing of the chargeflow across the boundary of the charges associated with $\mathcal{A}$ is encoded in the algebraic framework via the trivial boundary net $\mathcal{A}_+(O) = \mathcal{A}(I) \vee \mathcal{A}(J)$ with $I \times J \in \mathbb{M}_+$. This net is locally isomorphic to $\mathcal{A}_2$ restricted to $\mathbb{M}_+$. In other words $\mathcal{A}_+$ prescribes the boundary condition of $\mathcal{A}_2$ such that there is no charge flow across the boundary.

Now given a two-dimensional net $\mathcal{B}_2$ which contains the given rational symmetries described by $\mathcal{A}$, i.e. a local irreducible extension $\mathcal{B}_2 \supset \mathcal{A}_2$, we are now interested in all boundary conditions with no charge flow associated with $\mathcal{A}$ as above. Such a boundary condition is abstractly given [LR04, CKL13] by a net $\mathcal{B}_+ \supset \mathcal{A}_+$ on $\mathbb{M}_+$ which is locally isomorphic to $\mathcal{B}_2$ such that this isomorphism restricts to an isomorphism of $\mathcal{A}_+ \cong \mathcal{A}_2$.

A classification gets feasibile by operator algebraic methods. Finite index subfactors $N \subset M$ are in one-to-one correspondence with algebra objects (Q-systems) in the unitary tensor category $\mathrm{End}(N)$ of endomorphisms of N.

Local irreducible extension $\mathcal{B} \supset \mathcal{A}$ of nets with finite index give rise to nets of subfactors $\mathcal{A}(O) \subset \mathcal{B}(O)$ and the corresponding Q-system (up to isomorphism) is independent of O and is in the category of localized DHR endomorphisms. Conversely, every such Q-system gives a relatively local extension, which is local if and only if the Q-system is commutative. In particular, one has a one-to-one correspondence between Q-systems and relatively local extensions.

This situation can be abstracted to the setting of braided subfactors, namely we fix an interval I, set $N = A(I)$ and denote by ${}_N\mathcal{C}_N$ the category of localized DHR endomorphisms which are localized in I. We can start with a type III factor N and a modular tensor category ${}_N\mathcal{C}_N \subset \mathrm{End}(I)$ and look into subfactors $N \subset M$ such that the corresponding Q-system is in ${}_N\mathcal{C}_N$. We introduce the notion of Morita equivalence of such braided subfactors. As a main technical result we show that a conjecture of Kong and Runkel [KR10] is true. Namely, we show in Prop. 4.18 that the generalized Longo–Rehren construction [Reh00] coincides with the full center construction in the categorical literature (e.g. [FFRS06, KR08]). We give some consequences on the study of braided subfactors and modular invariants. This result opens the possiblity to apply many results from the categorical literature to the braided subfactor and conformal net setting. In particular, we make use of the result that Q-systems are Morita equivalent if and only if they have the same full center [KR08].

Going back to the conformal net setting we get the main result. Namely, maximal 2D extensions $\mathcal{B}_2 \supset \mathcal{A}_2$ are classified by Morita equivalence classes of Q-systems in $\mathrm{Rep}(\mathcal{A})$ (see Prop. 6.7 and irreducible boundary conditions of $\mathcal{B}_2$ are classified by equivalence classes of irreducible Q-systems in the Morita class (see Prop.6.11). We also treat reducible boundary conditions, which were not conisidered before in the literature, and show that we get a classification by reducible Q-systems.

The article is structured as follows.

In Sec. 2 we give some background on the category of endomorphisms of a type III factor, Q-systems, unitary modular tensor categories (UMTC), braided subfactors and the α-induction construction.

In Sec. 3 we give a notion of Morita equivalence for subfactors and Q-systems in UMTCs. The Morita equivalence class of a subfactor in a UMTC can be described by irreducible sectors in the module category of the subfactor modulo automorphisms of some dual category.
In Sec. 4 we show that the α-induction construction in subfactors coincide with the full center construction in the categorical literature. This is the first main technical result.
In Sec. 5 we study maximal commutative Q-systems in the category ${}_N\mathcal{C}_N \boxtimes \overline{{}_N\mathcal{C}_N}$ (the Drinfel'd center of ${}_N\mathcal{C}_N$) and give a characterization of them. We give some application to the study of modular invariants and examples of inequivalent extensions with same modular invariant, i.e. example of non-vanishing second cohomology.
In Sec. 6 we apply our former results to the study of conformal field theory on the Minkowski space in the operator algebraic (Haag–Kastler) framework. We give a proof of a folk theorem about the representation theory of local extensions (Prop. 6.4). Given a completely rational conformal net $\mathcal{A}$, as the main result, we obtain a classification of maximal local CFTs containing the chiral observables described by $\mathcal{A}$ and all its boundary conditions. We also discuss reducible boundary conditions, i.e. we drop the assumption that the boundary condition possesses a unique vacuum. Finally, we give a relation to the construction of adding a boundary in [CKL13], which gives an alternative proof for the classification of boundary conditions.

2 Preliminaries

2.1 Endomorphisms of type III factors and Q-systems

Let us look into the following strict 2–C*-category $\mathcal{C}$. Its 0-cells $\mathrm{Ob}(\mathcal{C}) = \{N, M, P, \ldots\}$ are given by a (finite) set of type III factors. The 1-cells are given for $M, N \in \mathrm{Ob}(\mathcal{C})$ by $\mathrm{Mor}(M,N)$, i.e. the set of unital $*$-homomorphisms (morphism) from $\rho : M \to N$ with finite (statistical) dimension $d\rho \equiv d_\rho = [N : \rho(M)]^{\frac{1}{2}}$, where $[N : \rho(M)]$ denotes the minimal index [Jon83, Kos86]. The 2-cells are intertwiners, i.e. for $\lambda, \mu \in \mathrm{Mor}(M,N)$ we define $\mathrm{Hom}(\lambda,\mu) = \{t \in N : t\lambda(m) = \mu(m)t \text{ for all } m \in M\}$. Then $\mathrm{Hom}(\lambda,\mu)$ is a vector space and we write $\langle\lambda,\mu\rangle = \dim\mathrm{Hom}(\lambda,\mu)$ for its dimension. Let $\rho \in \mathrm{Mor}(M,N)$. We call ρ irreducible if $\rho(M)' \cap N = \mathbb{C} \cdot 1_N$. A sector is a unitary equivalence class $[\rho] = \{\mathrm{Ad}\, U \circ \rho : U \in N \text{ unitary}\}$. We denote by $\mathrm{End}(N) = \mathrm{Mor}(N,N)$, which is a 2–C*-category with only one 0-cell, so a C*-tensor category.
Let $\rho_1, \ldots, \rho_n \in \mathrm{Mor}(M,N)$, and let $r_i \in N$ be generators of the Cuntz algebra $\mathcal{O}_n$, i.e. $\sum_{i=1}^n r_i r_i^* = 1_N$ and $r_j^* r_i = \delta_{ij} \cdot 1_N$. The morphism

$$\rho = \sum_{i=1}^n \mathrm{Ad}\, r_i \circ \rho_i \in \mathrm{Mor}(M,N),$$

is called direct sum of $\rho_1, \ldots, \rho_n$ and we have $r_i \in \mathrm{Hom}(\rho_i, \rho)$. The direct sum is unique on sectors and we write it as

$$[\rho] =: [\rho_1] \oplus \cdots \oplus [\rho_n] =: \bigoplus_{i=1}^{n} [\rho_i]\,,$$

and for the multiple direct sum we introduce the notation:

$$n[\sigma] := \bigoplus_{i=1}^{n} [\sigma]\,, \qquad\qquad n \in \mathbb{N}, \sigma \in \mathrm{Mor}(M,N)\,.$$

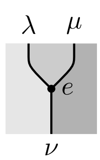
We say that a full and replete subcategory $\mathcal{C}$ of $\mathrm{Mor}(M,N)$ has SUBOBJECTS, if every object is a finite direct sum of irreducible sectors in $\mathcal{C}$. Similarly, we say it has DIRECT SUMS, if $\rho_1,\ldots,\rho_n \in \mathcal{C}$ implies that also their direct sum is in $\mathcal{C}$. Let us assume $\mathcal{C}$ has subobjects. If $e \in \mathrm{Hom}(\rho,\rho)$ is a (not necessarily orthogonal) projection (idempotent), then there exists a $\rho' \in \mathcal{C}$ and $s \in \mathrm{Hom}(\rho',\rho)$ and $t \in \mathrm{Hom}(\rho,\rho')$ such that $s \cdot t = e$ and $t \cdot s = 1_{\rho'} \equiv 1_N$. We note that if we have $e \in \mathrm{Hom}(\theta,\theta)$ we have an orthonormal projection $p = e(1+e-e^*)^{-1} \in \mathrm{Hom}(\theta,\theta)$ with the same range. If $[\rho] = \bigoplus_{i=1}^{m}[\rho_i]$ and $[\sigma] = \bigoplus_{j=1}^{n}[\sigma_j]$ we can decompose $t \in \mathrm{Hom}(\rho,\sigma)$ as

$$t = \bigoplus_{ij} t_{ij} := s_i \cdot t_{ij} \cdot r_i^*, \qquad\qquad t_{ij} \in \mathrm{Hom}(\rho_i,\sigma_j)\,,$$

where $r_i \in \mathrm{Hom}(\rho_i,\rho)$ and $s_j \in \mathrm{Hom}(\sigma_j,\sigma)$ are isometries as above. Similarly, one can decompose $t \in \mathrm{Hom}(\rho,\sigma\tau)$ etc.

Let us briefly explain the graphical notation (string diagrams) [JS91, BEK99, BEK00, Sel11, BDH14] which we will use. The 0-cells $N, M, \ldots$ are drawn as shaded two-dimensional regions, with different shadings for each factor. A 1-cell $\rho \in \mathrm{Mor}(N,M)$ is a vertical line (one dimensional) between the region M and N and composition of 1-cells correspond to horizontal concatenation. The identity $\mathrm{id}_N \in \mathrm{End}(N)$ is not drawn. The 2-cells $t \in \mathrm{Hom}(\rho,\sigma)$ are vertices between two lines. Sometimes we draw also boxes and again the identity $1_\rho \equiv 1 \in \mathrm{Hom}(\rho,\rho)$ is in general not drawn. The composition of intertwiners is vertical concatenation and the monoidal product horizontal concatenation.

We use a Frobenius rotation invariant convention for trivalent vertices, namely for an isometry $e \in \mathrm{Hom}(\nu,\lambda\mu)$ we introduce the diagram



$$=: \sqrt[4]{\frac{d\lambda d\mu}{d\nu}}\, e\,.$$

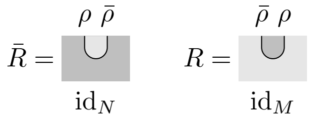
Let $\mathcal{C} \subset \mathrm{End}(N)$ and $\mathcal{D} \subset \mathrm{End}(M)$ be two full subcategories. We define the DELIGNE PRODUCT $\mathcal{C} \boxtimes \mathcal{D}$ to be the completion of $\mathcal{C} \otimes_{\mathbb{C}} \mathcal{D}$ under subobjects and direct sums cf. [LR97, Appendix].

A morphism $\bar\rho\colon N \to M$ is said to be a CONJUGATE to $\rho\colon M \to N$ if there exist intertwiners $R \in (\mathrm{id}_M, \bar\rho\rho)$ and $\bar R \in (\mathrm{id}_N, \rho\bar\rho)$ such that the CONJUGATE EQUATIONS hold:

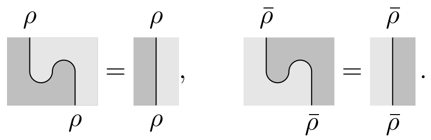
$$(1_\rho \otimes R^*)\cdot(\bar R \otimes 1_\rho) \equiv \rho(R^*)\cdot \bar R = 1_\rho \tag{1}$$

$$(1_{\bar\rho} \otimes \bar R^*)\cdot(R \otimes 1_{\bar\rho}) \equiv \bar\rho(\bar R^*)\cdot R = 1_{\bar\rho}\,. \tag{2}$$

The 2–morphisms $R, \bar R$ will graphically be represented by



and the above equations (1), (2) are sometimes called ZIG-ZAG IDENTITIES, because in diagrams they are given by



If ρ is irreducible we ask the solution $R, \bar R$ to be NORMALIZED, i.e. $\|R\| = \|\bar R\|$. In the case that ρ is not irreducible we further ask that $R, \bar R$ is a STANDARD solution of the conjugate equation, i.e. R (and similar $\bar R$) is of the form

$$R = \sum_i (\bar W_i \otimes W_i)\cdot R_i \equiv \bigoplus_i R_i\,,$$

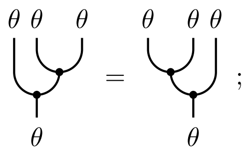
where $R_i \in (\mathrm{id}_M, \bar\rho_i\rho_i)$ is a normalized solution for an irreducible object $\rho_i \prec \rho$ and $W_i \in (\rho_i, \rho)$ and $\bar W_i \in (\bar\rho_i, \rho)$ are isometries expressing ρ and $\bar\rho$ as direct sums of irreducibles. We note that for the dimension $d_\rho \equiv d\rho$ of ρ we have $R^*R = d_\rho \cdot 1_M$ and $d\rho = d\bar\rho$. For $N \neq M$ we may always choose $\bar R_\rho = R_{\bar\rho}$. If we have a subcategory ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ we may choose a system ${}_N\Delta_N$ of representants for every sector in ${}_N\mathcal{C}_N$ and choose R_ρ for every $\rho \in {}_N\Delta_N$ such that for $[\rho] \neq [\bar\rho]$ we have $\bar R_\rho = R_{\bar\rho}$. For $[\bar\rho] = [\rho]$ the intertwiners R_ρ and $\bar R_\rho$ are intrinsically related, namely $\bar R_\rho = \pm R_\rho$ holds, where the sign ± 1 is called the Frobenius–Schur indicator. In this case the sector $[\rho]$ is called REAL for $+1$ and PSEUDO-REAL for -1. Although $[\rho]$ and $[\bar\rho]$ might be represented by the same $\rho \in {}_N\Delta_N$ we still use $\bar\rho$ in the diagrammatically notation to distinguish between R_ρ and $\bar R_\rho$.

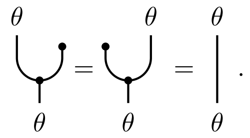
A triple $\Theta = (\theta, w, x)$ with $\theta \in \mathrm{End}(N)$ and isometries $w\colon \mathrm{id}_N \to \theta$ and $x\colon \theta \to \theta^2$, which we will graphically display as

is called a Q-sytem (cf. [Lon94, LR97]) if it fulfills

$$xx = \theta(x)x \qquad (x \otimes 1_\theta)x = (1_\theta \otimes x)x \qquad \text{(associativity)}$$
$$w^*x = \theta(w^*)x = \lambda 1_\theta \qquad (w^* \otimes 1_\theta)x = (1_\theta \otimes w^*)x = \lambda 1_\theta \qquad \text{(unit law)}$$

where $\lambda = \sqrt{d\theta}^{-1}$. In graphical notation this reads:

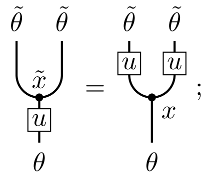


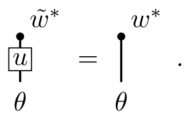


Two Q-systems $\Theta = (\theta, w, x)$ and $\tilde\Theta = (\tilde\theta, \tilde w, \tilde x)$ in $\mathrm{End}(N)$ are called equivalent, if there is a unitary $u \in \mathrm{Hom}(\theta, \tilde\theta)$, such that

$$\tilde x u = (u \otimes u)x \equiv u\theta(u)x\,; \qquad u\tilde w = w$$

hold, or graphically:

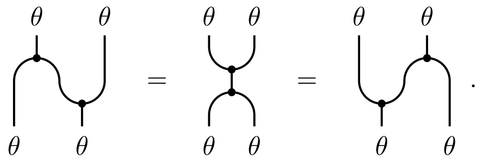




A Q-system in a C*-tensor category automatically [LR97] fulfills the "Frobenius law"

$$(x^* \otimes 1_\theta)(1_\theta \otimes x) \equiv x^*\theta(x) \quad = \quad xx^* \quad = \quad (1_\theta \otimes x^*)(x \otimes 1_\theta) \equiv \theta(x^*)x$$

or graphically:

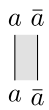
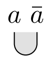
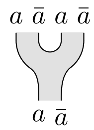


This means a Q-system is a special symmetric ∗-Frobenius algebra object, but we prefer to use the name Q-system which is most common in the subfactor context, (other names would be monoid, algebra object, monoidal algebra). We say a Q-system $\Theta = (\theta, w, x)$ is irreducible (called haploid in the Frobenius algebra context) if $\langle \mathrm{id}_N, \theta \rangle = 1$.

Definition 2.1. Every irreducible $a \in \mathrm{Mor}(M, N)$ defines an irreducible Q-system

$$\Theta_a = (\theta_a, w_a, x_a) := (a\bar{a}, \bar{r}_a, a(r_a))$$

in $\mathrm{End}(N)$, where $r_a \colon \mathrm{id}_M \to \bar{a}a$ and $\bar{r}_a \colon \mathrm{id}_N \to a\bar{a}$ are isometries such that $\bar{R}_a = \sqrt{da} \cdot \bar{r}_a$ and $R_a = \sqrt{da} \cdot r_a$ fulfill the conjugate equations (1,2) for a. In graphical notation:

$\theta_a =$  , $\sqrt{da}\, w_a =$  , $\sqrt{da}\, x =$  .

We remark that up to this point everything can abstractly be defined in a 2–C*-category.
Consider now a finite index irreducible subfactor $N \subset M$ with inclusion $\iota \colon N \to M$ then $\Theta := \Theta_{\bar{\iota}}$ gives dual canonical Q-system of $N \subset M$ (and $\Gamma = \Theta_\iota$ the canonical Q-system). The endomorphism $\theta \equiv \bar{\iota}\iota \in \mathrm{End}(N)$ is called the dual canonical endomorphism of $N \subset M$ ($\gamma \equiv \iota\bar{\iota} \in \mathrm{End}(M)$ is called the canonical endomorphism).
Conversely, starting from an irreducible Q-system Θ in $\mathrm{End}(N)$, there is a subfactor $N_1 \subset N$, where N_1 is defined to be the image $N_1 := E(N)$ of the conditional expectation $E(\ \cdot\ ) = x^*\theta(\ \cdot\ )x$ and there is subfactor (extension) $N \subset M$ defined by the Jones basic construction $N_1 \subset N \subset M$ (cf. [LR95]). One can make the construction of M explicit (cf. [BKLR15]) and obtains this way a dual morphism $\bar{\iota} \colon M \to N$ of the inclusion $\iota \colon N \to M$ such that $\Theta = \Theta_{\bar{\iota}}$. The upshot of this discussion is that there is a one-to-one correspondence (cf. [Lon94]) of

- Q-systems in $\mathrm{End}(N)$ up to equivalence.
- Irreducible finite index subfactors $N \subset M$ up to conjugation.

Remark 2.2. We note that θ alone does not fix $N \subset M$, which can be seen as a cohomological obstruction. Izumi and Kosaki [IK02] define the second cohomology $H^2(N \subset M)$ to be all equivalence classes of Q-systems $\Theta = (\theta, w, x)$ with θ the dual canonical endomorphism of $N \subset M$ (their definition uses actually the canonical endomorphism). We say the second cohomology of $N \subset M$ vanishes if there up to equivalence is just one Q-system $\Theta = (\theta, x, w)$, where θ is the dual canonical endomorphism of $N \subset M$.

We finally note that Θ is a Q-system in the full C^*-tensor subcategory with subobjects generated by θ. The Q-system becomes "trivial", i.e. is of the form $\Theta_{\bar{\iota}}$, in the 2–C*-category formed of 0-cells $\{N, M\}$ and full and replete subcategories ${}_L\mathcal{C}_P \subset \mathrm{Mor}(P, L)$ with subobjects and direct sums, which is generated by $\{\iota, \bar{\iota}\}$. We remark that this is actually a general feature of Frobenius algebra object in rigid tensor categors, in particular the obtained 2–C*-category together with the 1-morphisms $\iota\colon N \to M$ and $\bar\iota\colon M \to N$ appears in [Müg03a] under the name MORITA CONTEXT. In the general situation having a special symmetric Frobenius algebra A in a rigid tensor category $\mathcal{C}$ one can find a bi-category $\tilde{\mathcal{C}} \supset \mathcal{C}$ giving a Morita context in which the Frobenius algebra becomes trivial, cf. [Müg03a] for details.

2.2 UMTCs in End(N) and braided subfactors

Let us fix a type III factor N and write ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ for a full and replete subcategory ${}_N\mathcal{C}_N$ of $\mathrm{End}(N)$, such that each object is a finite direct sum of irreducible objects and ${}_N\mathcal{C}_N$ is closed under taking finite direct sums. We use this notation to stress that it is a category of N-N morphisms. We may choose an endomorphism for each irreducible sector and denote the set of these endomorphisms by ${}_N\Delta_N$. Let us assume the following properties:

1. $\mathrm{id}_N \in {}_N\Delta_N$.
2. There are only finitely many irreducible sectors in ${}_N\mathcal{C}_N$, i.e. $|{}_N\Delta_N| < \infty$.
3. If $\sigma \in {}_N\Delta_N$ then also a conjugate (dual) $\bar\sigma \in {}_N\Delta_N$.
4. If $\rho, \sigma \in {}_N\Delta_N$, then $\rho \circ \sigma \in {}_N\mathcal{C}_N$, in other words we have that
$$[\mu \circ \nu] = \bigoplus N_{\mu\nu}^{\rho}[\rho], \qquad N_{\mu\nu}^{\rho} = \langle \rho, \mu\nu \rangle,$$
where $N_{\mu\nu}^{\rho}$ are called FUSION RULE COEFFICIENTS.

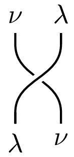
This means that ${}_N\mathcal{C}_N$ is a finite rigid C*–tensor category [LR97], i.e. a UNITARY FUSION CATEGORY. We associated with ${}_N\mathcal{C}_N$ a finite dimensional vector space $K_0({}_N\mathcal{C}_N) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^{|{}_N\Delta_N|}$, where $|{}_N\Delta_N|$ denotes the cardinality of the system ${}_N\Delta_N$ and $K_0({}_N\mathcal{C}_N)$ is the Grothendieck group of the monoidal category ${}_N\mathcal{C}_N$. We define the GLOBAL DIMENSION $\dim {}_N\mathcal{C}_N$ of ${}_N\mathcal{C}_N$ to be
$$\dim {}_N\mathcal{C}_N = \sum_{\rho \in {}_N\Delta_N} (d\rho)^2 \,.$$

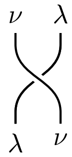
We remark that for convenience we assume ${}_N\mathcal{C}_N$ to be a subcategory of $\mathrm{End}(N)$. But it turns out that this is not a lost of generality, because every countable generated rigid C*–tensor can be embedded in $\mathrm{End}(N)$ by the result of [Yam03]. We will need more structure on ${}_N\mathcal{C}_N$, in particular we additionally assume:

5. There is a natural family $\{\varepsilon(\mu,\nu) \in \mathrm{Hom}(\mu\nu, \nu\mu) : \mu,\nu \in {}_N\mathcal{C}_N\}$ fulfilling:
$$\varepsilon(\lambda,\mu\nu) = (1_\mu \otimes \varepsilon(\lambda,\nu)) \cdot (\varepsilon(\lambda,\mu) \otimes 1_\nu) \equiv \mu(\varepsilon(\lambda,\nu)) \cdot \varepsilon(\lambda,\mu)$$
$$\varepsilon(\lambda\mu,\nu) = (\varepsilon(\lambda,\nu) \otimes 1_\mu) \cdot (1_\lambda \otimes \varepsilon(\mu,\nu)) \equiv \varepsilon(\lambda,\nu) \cdot \lambda(\varepsilon(\mu,\nu)).$$
Naturality means, that for $s\colon \sigma \to \sigma'$ and $t\colon \tau \to \tau'$
$$\begin{aligned}(t \otimes s) \cdot \varepsilon(\sigma,\tau) &\equiv t \cdot \tau(s) \cdot \varepsilon(\sigma,\tau)\\ &= \varepsilon(\sigma',\tau') \cdot (s \otimes t) \equiv \varepsilon(\sigma',\tau') \cdot s \cdot \sigma(t).\end{aligned}$$

We note that this family is determined by $\{\varepsilon(\mu,\nu) \in \mathrm{Hom}(\mu\nu,\nu\mu) : \mu,\nu \in {}_N\Delta_N\}$.

That means that ${}_N\mathcal{C}_N$ is a braided unitary fusion category which has automatically the structure of a unitary ribbon fusion category. We then say that ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ is a URFC. The braiding $\varepsilon^+(\lambda,\mu) := \varepsilon(\lambda,\mu)$ always comes along with an opposite braiding $\varepsilon^-(\lambda,\mu) := \varepsilon(\mu,\lambda)^*$ which in general is different from $\varepsilon^+(\lambda,\mu)$. We will graphically denote the braiding by:

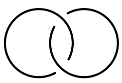
$\varepsilon^+(\lambda,\nu) =$ 

$\varepsilon^-(\lambda,\nu) =$  .

We denote by $\overline{{}_N\mathcal{C}_N}$ the braided category obtained by interchanging the braiding with the opposite braiding.
Finally, most of the time we will also use the following additional assumption:

6. The braiding is non-degenerate, i.e. $\varepsilon^+(\lambda,\mu) = \varepsilon^-(\lambda,\mu)$ for all $\mu \in {}_N\Delta_N$ implies $[\lambda] = [\mathrm{id}_N]$.

We then say ${}_N\mathcal{C}_N$ is modular. In other words ${}_N\mathcal{C}_N$ is a unitary modular tensor category (UMTC).
We define (see [BEK99]) for $\lambda,\mu \in {}_N\Delta_N$

$Y_{\lambda\mu} =$ $\bar\lambda$  $\bar\mu$; $\omega_\lambda \cdot 1_\lambda =$ 

and the following $|{}_N\Delta_N| \times |{}_N\Delta_N|$-matrices

$$S_{\lambda\mu} = (\dim {}_N\mathcal{C}_N)^{-\frac{1}{2}} Y_{\lambda,\mu}\,, \qquad T_{\lambda\mu} = \mathrm{e}^{-\pi \mathrm{i} c/12}\delta_{\lambda\mu}\omega_\lambda\,, \tag{3}$$

where

$$z = \sum_{\rho\in {}_N\Delta_N} (d\rho)^2\omega_\rho\,; \qquad c = 4\arg(z)/\pi\,.$$

They obey the relations of the partial Verlinde modular algebra: $TSTST = S$, $CTC = T$, and $CSC = S$, where $C_{\mu\nu} = \delta_{\mu,\bar\nu}$ is the charge conjugation matrix.
The property (6) is equivalent to:

(6') $Z({}_N\mathcal{C}_N) \cong {}_N\mathcal{C}_N \boxtimes \overline{{}_N\mathcal{C}_N}$, where $Z({}_N\mathcal{C}_N)$ is the Drinfeld center of ${}_N\mathcal{C}_N$ [Müg03b, Corollary 7.11] and

(6") the matrix $S = (S_{\lambda\mu})$ is unitary.

In particular, in the modular case we have ([BEK99, Prop. 2.5]):

$$S^*S = T^*T = 1\,, \qquad (ST)^3 = S^2 = C\,, \qquad CTC = T\,,$$

i.e. S and T define a unitary representation of $\mathsf{SL}(2,\mathbb{Z}) \cong \mathbb{Z}_6 *_{\mathbb{Z}_2} \mathbb{Z}_4$ on $\mathbb{C}^{|{}_N\Delta_N|}$ if and only if ${}_N\mathcal{C}_N$ is modular.

2.3 Braided subfactors and α-induction

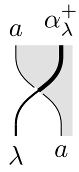
Let N be a type III factor, ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ a URFC and let $\iota(N) \subset M$ be an irreducible subfactor such that $\theta \equiv \bar\iota\iota \in {}_N\mathcal{C}_N$. We call the data $(\iota(N) \subset M, {}_N\mathcal{C}_N)$ a braided subfactor. If ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ happens to be a UMTC we call the braided subfactor a non-degenerately braided. There is an obvious one-to-one correspondence between (the equivalence classes of) braided subfactors in ${}_N\mathcal{C}_N$ and Q-systems in ${}_N\mathcal{C}_N$.
For $\rho \in {}_N\mathcal{C}_N$ we define its α-induction by

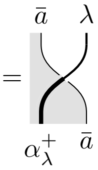
$$\alpha^\pm_\lambda = \bar\iota^{-1} \circ \mathrm{Ad}(\varepsilon^\pm(\lambda,\theta)) \circ \lambda \circ \bar\iota \in \mathrm{End}(M)\,.$$

We define the module category ${}_N\mathcal{C}_M$ to be the full subcategory with subobjects and direct sums of $\mathrm{Mor}(M,N)$, which is generated by ${}_N\mathcal{C}_N\bar\iota \equiv \{\rho\bar\iota : \rho \in {}_N\mathcal{C}_N\}$ and choose a set of representatives of irreducible sectors ${}_N\Delta_M$. In the same way we define ${}_M\mathcal{C}_N$ and the dual category ${}_M\mathcal{C}_M$ generated by $\iota_N\mathcal{C}_N$ and $\iota_N\mathcal{C}_N\bar\iota$, respectively. Finally we define ${}_M\mathcal{C}^\pm_M$ to be generated by $\alpha^\pm({}_N\mathcal{C}_N)$, respectively, and the ambichiral category ${}_M\mathcal{C}^0_M = {}_M\mathcal{C}^+_M \cap {}_M\mathcal{C}^-_M$. Again we choose a set of representatives of irreducible sectors ${}_M\Delta_N, {}_M\Delta_M, {}_M\Delta^\pm_M, {}_M\Delta^0_M$ in the respective categories.
It turns out that ${}_M\mathcal{C}^\pm_M \subset {}_M\mathcal{C}_M$ and that ${}_M\mathcal{C}^+_M \cup {}_M\mathcal{C}^-_M$ generates ${}_M\mathcal{C}_M$ [BEK99, Thm. 5.10]. It will be convenient to work in the 2-category generated by ${}_N\mathcal{C}_N \cup {}_N\mathcal{C}_M \cup {}_M\mathcal{C}_N \cup {}_M\mathcal{C}_M$.
As shown in [BEK99, Prop. 3.1], we have for $a \in {}_N\mathcal{C}_M$, $\lambda \in {}_N\mathcal{C}_N$:

$$\varepsilon^\pm(\lambda, a\iota) \in \mathrm{Hom}(\lambda a, a\alpha^\pm_\lambda) \qquad \mathcal{E}^\pm(\lambda,\bar a) \in \mathrm{Hom}(\alpha^\pm_\lambda \bar a, \bar a\lambda)\,,$$

where $\mathcal{E}^\pm(\lambda,\bar a) := T^*\iota(\varepsilon^\pm(\lambda,\bar\nu))\alpha^\pm_\lambda(T)$ for $a \in {}_N\mathcal{C}_M$ with $\bar a \prec \bar\iota\nu$ for some $\nu \in {}_N\mathcal{C}_N$ and $T \in (\bar a, \bar\iota\nu)$ an isometry. The definition does not depend on the choice of ν and T. We set $\mathcal{E}^\pm(\bar a,\lambda) := (\mathcal{E}^\mp(\lambda,\bar a))^*$. We represent this graphically—where we use thin lines for morphisms in ${}_M\mathcal{C}_N$ and ${}_N\mathcal{C}_M$, normal lines for endomorphisms in ${}_N\mathcal{C}_N$ and thick lines for endomorphisms in ${}_M\mathcal{C}_M$—as follows:

$\varepsilon^+(\lambda, a\iota) =$ 

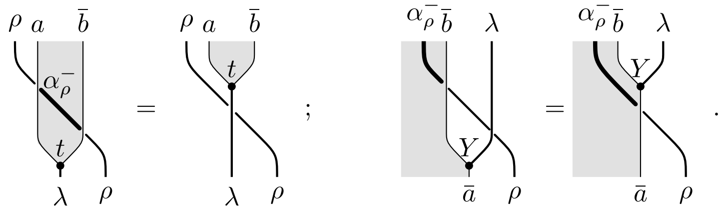
; $\qquad \mathcal{E}^+(\lambda,\bar a) =$ 

.

The intertwining braided fusion equations (IBFE's) [BEK99, Prop. 3.3] hold, namely

$$\begin{aligned}
\rho(t)\,\varepsilon^{\pm}(\lambda,\rho) &= \varepsilon^{\pm}(a\iota,\rho)\,a(\mathcal{E}^{\pm}(\bar b,\rho))\,t\,,\\
t\,\varepsilon^{\pm}(\rho,\lambda) &= a(\mathcal{E}^{\pm}(\rho,\bar b))\,\varepsilon^{\pm}(\rho,a\iota)\,\rho(t)\,,\\
\rho(y)\,\varepsilon^{\pm},(a\iota,\rho) &= \varepsilon^{\pm}(\lambda,\rho)\,\lambda(\varepsilon^{\pm}(b\iota,\rho))\,y\,,\\
y\,\varepsilon^{\pm}(\rho,a\iota) &= \lambda(\varepsilon^{\pm}(\rho,b\iota)\,\varepsilon^{\pm}(\rho,\lambda))\,\rho(y)\,,\\
\alpha^{\mp}(Y)\,\mathcal{E}^{\pm}(\bar a,\rho) &= \mathcal{E}^{\pm}(\bar b,\rho)\,\bar b(\varepsilon^{\pm}(\lambda,\rho))\,Y\,,\\
Y\,\mathcal{E}^{\pm}(\rho,\bar a) &= \bar b(\varepsilon^{\pm}(\rho,\lambda))\,\mathcal{E}^{\pm}(\rho,\bar b)\,\alpha^{\pm}_{\rho}\rho(Y)\,,
\end{aligned}$$

where $\lambda,\rho\in {}_N\mathcal{C}_N$, $a,b\in {}_N\mathcal{C}_M$ with conjugates $\bar a,\bar b\in {}_M\mathcal{C}_N$; $t\in\mathrm{Hom}(\lambda,a\bar b)$, $y\in\mathrm{Hom}(a,\lambda b)$ and $Y\in\mathrm{Hom}(\bar a,\bar b\lambda)$. The IBFE's have simple graphical interpretation, e.g. the first and sixth equations are represented by:



For details we refer to [BEK99, Sect. 3.3].
There is a relative braiding [BEK00, p. 738]

$$\mathcal{E}_{\mathrm{r}}(\beta_+,\beta_-) := S^*\alpha_\mu(T^*)\varepsilon(\lambda,\mu)\alpha^+_\lambda(S)T\in\mathrm{Hom}(\beta_+\beta_-,\beta_+\beta_-)\,,\tag{4}$$

where for fixed $\beta_\pm\in {}_M\mathcal{C}^{\pm}_M$, we choose $\lambda,\mu\in {}_N\mathcal{C}_N$, such that $\beta_+\prec\alpha^+_\lambda$, $\beta_-\prec\alpha^-_\nu$ and isometries S,T, such that $T\in\mathrm{Hom}(\beta_+,\alpha^+\mu)$ and $S\in\mathrm{Hom}(\beta_-,\alpha^-_\mu)$. The definition is independent of the particular choice of λ,μ,S,T.
The relative braidings give a non-degenerate braiding $\varepsilon(\ \cdot\ ,\ \cdot\ ):=\mathcal{E}_{\mathrm{r}}(\ \cdot\ ,\ \cdot\ )$ on ${}_M\mathcal{C}^0_M$ by [BEK00, Sec. 4], so in particular ${}_M\mathcal{C}^0_M$ becomes a UMTC.
In general for two braided subfactors $\iota_a(N)\subset M_a$ and $\iota_b(N)\subset M_b$ in ${}_N\mathcal{C}_N$ we define ${}_{M_a}\mathcal{C}_{M_b}$ as a full subcategory of $\mathrm{Mor}(M_b,M_a)$ with subobjects and direct sums generated by $\iota_a{}_N\mathcal{C}_N\bar\iota_b$.

3 Morita equivalence for braided subfactors

3.1 Module categories, modules and bimodules

In this section we give the notion of Morita equivalent non-degenerately braided subfactors.
We adapt the following definitions from [Ost03].

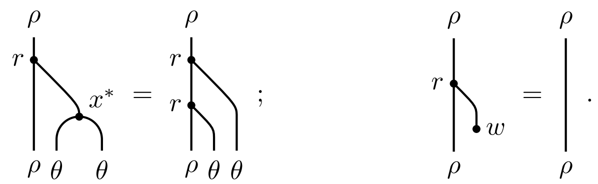
Definition 3.1. A (strict) module category over a tensor category $\mathcal{C}$ is a category $\mathcal{M}$ together with an exact bifunctor $\otimes\colon\mathcal{C}\times\mathcal{M}\to\mathcal{M}$ such that $(X\otimes Y)\otimes M = X\otimes(Y\otimes M)$ for all $X,Y\in\mathcal{C}$ and $M\in\mathcal{M}$.

Let $\mathcal{M}_1, \mathcal{M}_2$ be two module categories over $\mathcal{C}$. A (strict) MODULE FUNCTOR from $\mathcal{M}_1$ to $\mathcal{M}_2$ is a functor $F\colon \mathcal{M}_1 \to \mathcal{M}_2$ such that $F(X \otimes M) = X \otimes F(M)$. Two module categories $\mathcal{M}_1$ and $\mathcal{M}_2$ over $\mathcal{C}$ are called ISOMORPHIC if there exist a module functor, which is an isomorphism of categories.

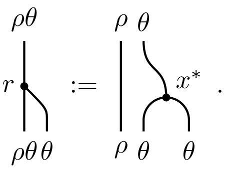
Let ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ be a UFC and let $\Theta = (\theta, w, x)$ be a Q-system in ${}_N\mathcal{C}_N$ corresponding to $N \subset M$. A (right) Θ-module (cf. [EP03]) is a pair (ρ, r) with $\rho \in {}_N\mathcal{C}_N$ and $\tilde{r} \in \mathrm{Hom}(\rho \circ \theta, \rho)$, such that r^* is an isometry and $\tilde{r} = \sqrt[4]{d\theta}\, r$ satisfies

$$\begin{aligned} \tilde{r} \cdot (1_\rho \otimes m) &= \tilde{r} \cdot (\tilde{e} \otimes 1_\theta) &\quad \Leftrightarrow \quad&& \tilde{r} \cdot \rho(m) &= \tilde{\rho}(\tilde{r}^2) \\ \tilde{r} \cdot (1_\rho \otimes r) &= 1_\rho &\quad \Leftrightarrow \quad&& \tilde{r} \cdot \rho(e) &= 1_\rho \end{aligned}$$

where $m = \sqrt[4]{d\theta} x^*$ the multiplication and $e = \sqrt[4]{d\theta} w$ the unit of the (Frobenius) algebra object corresponding to Θ. Graphically this means:



A left Θ-module can be defined similarly. We note that because we are working in C*-categories and ask r^* to be an isometry, that a module is also a co-module by the action r^*. The endomorphism $\rho\theta$ with $\rho \in {}_N\mathcal{C}_N$ has the structure of a right Θ-module, where the action is given by $\tilde{r} = 1_\rho \otimes m \equiv \rho(m) \equiv \sqrt[4]{d\theta} \cdot \rho(x^*) \in \mathrm{Hom}(\rho\theta\theta, \rho\theta)$ in other words $r = \rho(x^*)$, graphically:



It is called the INDUCED MODULE. Any irreducible right Θ-module is equivalent to a submodule of an induced module cf. [Ost03].

The Θ-modules form a category with $\mathrm{Hom}_\Theta(\rho, \sigma) \equiv \mathrm{Hom}_\Theta((\rho, r), (\sigma, s)) = \{t \in \mathrm{Hom}(\rho, \sigma) : tr = st\}$, so the arrows are arrows of the objects which intertwine the actions. There is a correspondence between projections $p \in \mathrm{Hom}_\Theta(\rho, \rho)$ and submodules, namely we can choose ρ_p and $t \in \mathrm{Hom}(\rho_p, \rho)$ with $t^*t = 1_{\rho_p}$, $tt^* = p$ and define $r_p = t^*rt$.

Let $\Theta_a = (\theta_a, w_a, x_a)$ and $\Theta_b = (\theta_b, w_b, x_b)$ be two Q-systems in ${}_N\mathcal{C}_N$. A Θ_a-Θ_b bimodule is a triple (ρ, r_a, r_b) with $\rho \in {}_N\mathcal{C}_N$ and $\rho_a \in \mathrm{Hom}(\theta_a\rho, \rho)$ and $\rho_b \in \mathrm{Hom}(\rho\theta_b, \rho)$, such that (ρ, r_a) is a left Θ_a-module and (ρ, r_b) is a (right) Θ_b-module and which commute, i.e.

$$r_a \cdot \theta_a(r_b) = r_b \cdot r_a.$$

We can define:

$$r := r_a \cdot (1_{\theta_a} \otimes r_b) = r_b \cdot (r_a \otimes 1_{\theta_a}) \in (\theta_a \circ \rho \circ \theta_b, \rho).$$

Let $\rho = (\rho, r_a, r_b)$ and $\sigma = (\sigma, s_a, s_b)$ be two Θ_a–Θ_b bimodules. An intertwiner $t\colon \rho \to \sigma$ is an Θ_a–Θ_b bimodule intertwiner, if t intertwines the actions r and s, i.e.

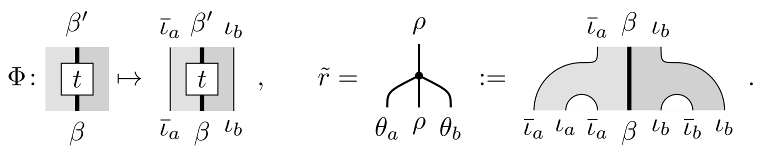
$$tr = s(1_{\theta_a} \otimes t \otimes 1_{\theta_b}) \equiv s\theta_a(t)\,.$$

Let us denote by $\mathrm{Bim}(\Theta_a, \Theta_b)$ the category of bimodules with $\mathrm{Hom}_{\Theta_a-\Theta_b}(\rho, \sigma)$ Θ_a-Θ_b bimodule intertwiner. We note that one can give Q-systems, bimodules and intertwiners the structure of a bicategory, by introducing a relative tensor product between bimodules.
We set $\mathrm{Mod}(\Theta) = \mathrm{Bim}(1, \Theta)$ to be the category of (right) Θ-modules.
The category $\mathrm{Mod}(\Theta)$ has a natural structure of a (strict) left ${}_N\mathcal{C}_N$ module category, where the functor ${}_N\mathcal{C}_N \times \mathrm{Mod}(\Theta)$ is given by $(\mu, \rho) \mapsto \mu\rho$ where $\mu\rho$ is a right-module with $r_{\mu\sigma} = \mu(r_\rho)$ and $\mathrm{Hom}_{\mathrm{Mod}(\Theta)}(\rho, \sigma) \ni T \mapsto \mu(T) \in \mathrm{Hom}_{\mathrm{Mod}(\Theta)}(\mu\rho, \mu\sigma)$.

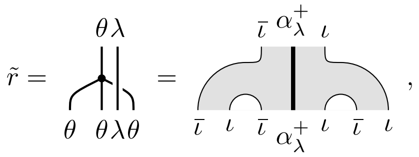
Proposition 3.2 ([EP03, Lemma 3.1.]). *Let ${}_N\mathcal{C}_N$ be a UMTC and Θ_a, Θ_b irreducible Q-systems in ${}_N\mathcal{C}_N$. The category of Θ_a-Θ_b bimodules is equivalent to the category ${}_{M_a}\mathcal{C}_{M_b}$. The functor Φ maps $\beta \in {}_{M_a}\mathcal{C}_{M_b}$ to $\bar\iota_a \circ \beta \circ \iota_b$ and $t \in \mathrm{Hom}(\beta, \beta')$ to $\bar\iota_a(t) \in \mathrm{Hom}_{\Theta_a\text{-}\Theta_b}(\Phi(\beta), \Phi(\beta'))$.*

Proof. In [EP03, Lemma 3.1.] is shown that the functor Φ is fully faithful. It is also shown that is is essentially surjective, so it gives an equivalence of categories. □

The functor Φ is graphically given as follows, where $\rho = \Phi(\beta)$ $\tilde r \in \mathrm{Hom}(\theta_a\rho\theta_b, \rho)$ the action:

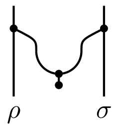


Remark 3.3. Let $\Theta = (\theta, w, x)$ be a Q-system in a UMTC ${}_N\mathcal{C}_N$ with corresponding subfactor $\iota(N) \subset M$. The bimodule $\Phi(\alpha^\pm_\lambda) \equiv \bar\iota\alpha^\pm_\lambda\iota \equiv \bar\iota\iota\lambda$ is the object $\theta\lambda$ with left action the induced action x^* and right action by $x^*\varepsilon^\pm(\lambda, \theta)$, namely for the $+$-case:



where equality can be seen easily using $\iota\lambda = \alpha^+_\lambda \iota$, $\Theta = \Theta_{\bar\iota}$ and the IBFEs by pulling the λ-string between $\bar\iota$ and ι. The $-$-case works analogous using the opposite braiding. The obtained bimodules coincide with the notion of α-induction in the categorical literature.

The category $\mathrm{Bim}(\Theta,\Theta)$ becomes a tensor category, where $\rho \otimes_\Theta \sigma$ is the object associated to the projection in $P_{\rho\otimes_\Theta\sigma} \in \mathrm{Hom}(\rho\sigma, \rho\sigma)$ given by:

$$P_{\rho\otimes_\Theta\sigma} = \frac{1}{\sqrt{d\theta}}$$  .

and it is easy to check that Φ is a tensor functor. Thus, $\mathrm{Bim}(\Theta,\Theta)$ and ${}_M\mathcal{C}_M$ are equivalent as tensor categories. We note that this category is non-strict. We can define the categories $\mathrm{Bim}^\pm(\Theta,\Theta)$ to be the image of ${}_M\mathcal{C}_M^\pm$ under Φ and $\mathrm{Bim}^0(\Theta,\Theta) = \mathrm{Bim}^+(\Theta,\Theta) \cap \mathrm{Bim}^-(\Theta,\Theta)$.
In the special case $M_a = N$ and $M_b = M$ and $\theta_a = \theta$ we have an equivalence of the category ${}_N\mathcal{C}_M$ and the category $\mathrm{Mod}(\Theta)$ of right Θ-modules given by $\bar a \mapsto \bar a\iota$. The category of right Θ-modules $\mathrm{Mod}(\Theta)$ becomes a module category over ${}_N\mathcal{C}_N$ using the monoidal structure inherent from $\mathrm{End}(N)$. The same is true for ${}_N\mathcal{C}_M$.
In particular, it follows:

Proposition 3.4. *Let ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ be a UMTC and Θ be a Q-system in ${}_N\mathcal{C}_N$ with corresponding subfactor $N \subset M$. Then* $\mathrm{Mod}(\Theta)$ *and* ${}_N\mathcal{C}_M$ *are equivalent as module categories.*

Proof. It follows directly from the properties of the monoidal structure, that the functor Φ (in the case of $M_a = N$ and $M_b = M$ and $\theta_a = \theta$) in the proof of Prop. 3.2 is a module functor, so in particular a module isomorphism, between the two module categories $\mathrm{Mod}(\Theta)$ and ${}_N\mathcal{C}_M$ over ${}_N\mathcal{C}_N$. □

We remark that in general in the definition of module it is not assumed that r is a (multiple) of an isometry, because the existence of a unitary structure is not assumed. But since every module in the general sense is equivalent to a submodule of an induced module and the submodule can chosen to have a multiple of an isometry as action, we can without lost of generality restrict to modules where r is a multiple of an isometry. This can also be shown directly [BKLR15].
Let $a \in {}_N\mathcal{C}_M$ be irreducible and consider the subfactor $N \subset M_a$ given by the Q-system Θ_a (see Def. 2.1). Let M_a be the factor which is given by Jones basic construction $a(M) \subset N \subset M_a$ and denote the inclusion map $\iota_a \colon N \hookrightarrow M_a$. Because the subfactors $\bar\iota_a(M_a) \subset N$ and $a(M) \subset N$ have by definition the same Q-system and thus are conjugated by a unitary in N, we may and do choose $\bar\iota_a \colon M_a \to N$, such that $\bar\iota_a(M_a) = a(M)$. This implies that $\alpha = \bar\iota_a^{-1} \circ a \colon M \to M_a$ is an isomorphism with conjugate $\alpha^{-1} = a^{-1} \circ \bar\iota_a \colon M_a \to M$.

Lemma 3.5 (cf. [LR04, Eva02]). *Let ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ be a UMTC and Θ be a Q-system in ${}_N\mathcal{C}_N$ with corresponding subfactor $N \subset M$.*
For $a \in {}_N\mathcal{C}_M$ irreducible let Θ_a be the canonical Q-system ($\Theta_a = a\bar{a}, w_a, x_a$) and $N \subset M_a$ the corresponding subfactor. Then ${}_N\mathcal{C}_M$ and ${}_N\mathcal{C}_{M_a}$ are isomorphic as module categories of ${}_N\mathcal{C}_N$. The isomorphism is given by $\Psi\colon b \mapsto b \circ a^{-1} \circ \iota_a$ and $\mathrm{Hom}_{{}_N\mathcal{C}_M}(b,c) \ni t \mapsto t \in \mathrm{Hom}_{{}_N\mathcal{C}_{M_a}}(\Psi(b), \Psi(c))$.

Remark 3.6. Given $a \in {}_N\mathcal{C}_M$ we have the Q-systen Θ_a with $\theta_a = a\bar{a}$. Let $\beta = \Phi(a) \in \mathrm{Mod}(\Theta)$, then $\bar{\beta}$ is a Θ left module and there is another way to construct a Q-system [KR08] denoted by $\bar{\beta} \otimes_\Theta \beta$, and it is easy to check that $\bar{\beta} \otimes_\Theta \beta \cong \bar{a}a$ and that the obtained Q-systems are equivalent.

3.2 The Morita equivalence class of a braided subfactor

In the following we use the definition of Morita equivalence for module categories as in [Ost03, Def. 3.3]. Let ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ be a UMTC. We remember that we call a pair $(N \subset M, {}_N\mathcal{C}_N)$ where $N \subset M$ is a subfactor whose Q-system Θ is in ${}_N\mathcal{C}_N$ a non-degenerately braided subfactor.

Definition 3.7. Let ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ be a UMTC. Two irreducible Q-systems Θ_a and Θ_b in ${}_N\mathcal{C}_N$ are called Morita equivalent if one of the following equivalent statements hold:

- $\mathrm{Mod}(\Theta_a)$ and $\mathrm{Mod}(\Theta_b)$ are equivalent as module categories over ${}_N\mathcal{C}_N$.
- ${}_N\mathcal{C}_{M_a}$ and ${}_N\mathcal{C}_{M_b}$ are equivalent as module categories over ${}_N\mathcal{C}_N$, where $N \subset M_\bullet$ is corresponding to $\Theta_\bullet$.

We say that the subfactors $N \subset M_a$ and $N \subset M_b$ are Morita equivalent if their Q-systems Θ_a and Θ_b, respectively, are Morita equivalent.

Let $(\iota(N) \subset M, {}_N\mathcal{C}_N)$ be a non-degenerately braided subfactor. It follows directly that for $a, b \in {}_N\mathcal{C}_M$ irreducible Θ_a and Θ_b are Morita equivalent and in particular are Morita equivalent to $\Theta_{\bar{\iota}}$. But it can also happen that Θ_a and Θ_b are equivalent for $[a] \neq [b]$. If $\mathcal{C}$ is a UTFC, we denote by $\mathrm{Pic}(\mathcal{C})$ the full and replete subcategory (2-group) with objects $\{\rho \in \mathcal{C} : d\rho = 1\}$ (not completed under direct sums).

Proposition 3.8 ([GS15]). *Given two irreducible objects $a, b \in {}_N\mathcal{C}_M$. Then the Q-systems Θ_a and Θ_b are equivalent if and only if there is an automorphism $\beta \in \mathrm{Pic}({}_M\mathcal{C}_M)$ such that $b\beta = a$.*

Now we can give a characterization of the Morita equivalence class of a non-degenerately braided subfactor.

Proposition 3.9. *Let ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ be a UMTC and let Θ be a Q-system in ${}_N\mathcal{C}_N$. Then there is a one-to-one correspondence between*

1. *equivalence classes $[\Theta_a]$ of irreducible Q-systems Morita equivalent to Θ,*

2. *irreducible sectors* $[a]$ *with* $a \in {}_N\mathcal{C}_M$ *up the identification:* $[a] \sim [b]$ *if there is an automorphism* $\beta \in {}_M\mathcal{X}_M$, *such that* $[a] = [\beta b]$,

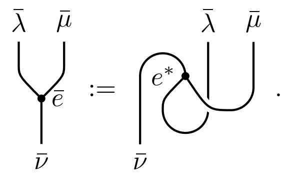
3. *elements in* ${}_N\Delta_M / \operatorname{Pic}({}_M\mathcal{C}_M)$.

Proof. Statement (3) is just a reformulation of (2). Let $a \in {}_N\mathcal{X}_M$ then we obtain a canonical Q-system Θ_a in ${}_N\mathcal{C}_N$ which is Morita equivalent to Θ by Lemma 3.5. Conversely given a Q-system Θ_a Morita equivalent to Θ then ${}_N\mathcal{C}_M$ is equivalent to ${}_N\mathcal{C}_{M_a}$. The element $a \in {}_N\mathcal{C}_M$ corresponding to $\iota_a \in {}_N\mathcal{C}_{M_a}$ under this equivalence is the corresponding element in ${}_N\mathcal{C}_M$, cf. [Ost03, Remark 3.5]. The rest follows by Prop. 3.8. □

4 α-induction construction and the full center

4.1 The full center and Rehren's construction coincide

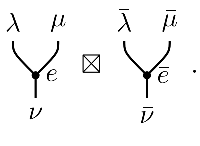
Let N be a type III factor and ${}_N\mathcal{C}_N \subset \operatorname{End}(N)$ a UMTC. As before let ${}_N\Delta_N = \{\mathrm{id}_N, \rho_1, \ldots, \rho_n\}$ a set of representatives for each sector.
Given $\nu, \lambda, \mu \in {}_N\Delta_N$, we can choose a set of isometries $B(\nu, \lambda\mu) := \{e_i\}_{i=1,\ldots,\langle\nu,\lambda\mu\rangle}$ with $e_i \in \operatorname{Hom}_{{}_N\mathcal{C}_N}(\nu, \lambda\mu)$, such that $\{e_i\}$ form an orthonormal basis with respect to the scalar product $(e, f) = \Phi_\nu(e^* f)$ defined by the left inverse Φ_ν of ν [LR97] or equivalently defined by $(e, f) \cdot 1_\nu = e^* f$. We define for an isometry $e \in \operatorname{Hom}_{{}_N\mathcal{C}_N}(\nu, \lambda\mu)$ an isometry $\bar{e} \in \operatorname{Hom}_{\overline{{}_N\mathcal{C}_N}}(\bar\nu, \bar\lambda\bar\mu)$ by



Definition 4.1 (Longo–Rehren construction). Let ${}_N\mathcal{C}_N \subset \operatorname{End}(N)$ a URFC. There is a Q-system $\Theta_{\mathrm{LR}} = (\theta_{\mathrm{LR}}, w_{\mathrm{LR}}, x_{\mathrm{LR}})$ in ${}_N\mathcal{C}_N \boxtimes \overline{{}_N\mathcal{C}_N}$ given by:

$$[\theta_{\mathrm{LR}}] = \bigoplus_{\rho \in {}_N\mathcal{C}_N} [\rho \boxtimes \bar\rho], \qquad x_{\mathrm{LR}} = \frac{1}{\sqrt{d\theta}} \bigoplus_{\lambda\mu\nu} \sum_{e \in B(\nu,\lambda\mu)} \sqrt{\frac{d\lambda d\mu}{d\nu d\theta}}\, e \boxtimes \bar e\,,$$

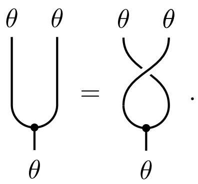
$$= \bigoplus_{\lambda\mu\nu} \sum_{e \in B(\nu,\lambda\mu)}$$



More general, for an equivalence of braided categories $\phi\colon {}_N\mathcal{C}_N \to {}_N\mathcal{C}'_N$, we define the Q-system $\Theta^\phi_{\mathrm{LR}} = (\theta^\phi_{\mathrm{LR}}, w^\phi_{\mathrm{LR}}, x^\phi_{\mathrm{LR}})$ in ${}_N\mathcal{C}_N \boxtimes \overline{{}_N\mathcal{C}'_N}$ by

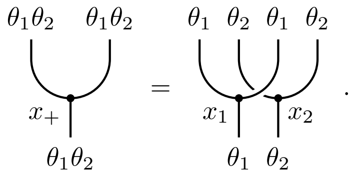
$$[\theta^\phi_{\mathrm{LR}}] = \bigoplus_{\rho \in {}_N\mathcal{C}_N} [\rho \boxtimes \phi(\bar\rho)], \qquad x^\phi_{\mathrm{LR}} = \bigoplus_{\lambda\mu\nu} \sum_{e \in B(\nu,\lambda\mu)} \sqrt{\frac{d\lambda d\mu}{d\nu d\theta}} e \boxtimes \overline{\phi(e)}\,.$$

Definition 4.2. Let ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ be a URFC. A Q-system $\Theta = (\theta, w, x)$ in ${}_N\mathcal{C}_N$ is called commutative if $\varepsilon(\theta,\theta)x = x$. Diagrammatically:

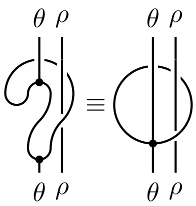


Proposition 4.3 ([LR95]). *The Q-system obtained by the Longo–Rehren construction is commutative.*

Definition 4.4 (Product Q-system). Let $\Theta_i = (\theta_i, w_i, x_i)$ with $i = 1, 2$ be two Q-systems in a URFC category ${}_N\mathcal{C}_N$. Then we define two Q-systems $\Theta_1 \circ^{\pm} \Theta_2 = (\theta_1 \circ \theta_2, w_1 w_2, x_\pm)$ in ${}_N\mathcal{C}_N$, where $x_\pm = \theta_1(\varepsilon^{\pm}(\theta_1,\theta_2))x_1\theta_1(x_2)$, graphically:



Definition 4.5. For $\Theta \equiv (\theta, w, x)$ a Q-system in ${}_N\mathcal{C}_N$ and $\rho \in {}_N\mathcal{C}_N$, we define

$P^{\mathrm{l}}_\Theta(\rho) = \frac{1}{\sqrt{d\theta}} \cdot$  $\in \mathrm{Hom}(\theta\rho, \theta\rho)$

and $P^{\mathrm{l}}_\Theta := P^{\mathrm{l}}_\Theta(\mathrm{id}_N)$. Similarly, we define $P^{\mathrm{r}}_\Theta(\rho)$ and P^{r}_Θ by interchanging the braiding with the opposite braiding.

Lemma 4.6. $P^{\mathrm{l/r}}_\Theta(\rho)$ *is a projection.*

Proof. That $P^{\mathrm{l}}_\Theta(\rho)^2 = P^{\mathrm{l}}_\Theta(\rho)$ is proven as in [FRS02, Lemma 5.2], see also [BKLR15]. We just remark that we have a prefactor due to another normalization and that one can check that $P^{\mathrm{l}}_\Theta(\rho)$ is selfadjoint. □

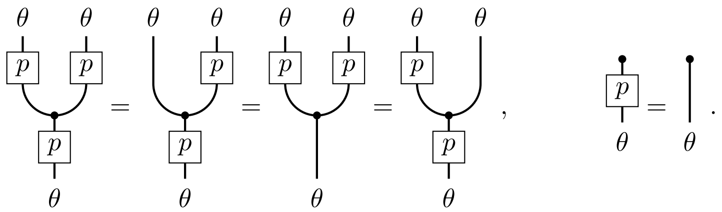
Proposition 4.7 (Sub-Q-system cf. [BKLR15]). *Let $p \in \mathrm{Hom}(\theta,\theta)$ be an orthogonal projection satisfying $p\theta(p)xp = \theta(p)xp = pxp = p\theta(p)x$ and $w^*p = w^*$.*

Let $\theta_p \prec \theta$ corresponding to p, i.e. there a isometry $s \in \mathrm{Hom}(\theta_p, \theta)$, such that $s^*s = 1_{\theta_p}$ and $ss^* = p$. Then $\Theta_p = (\theta_p, w_p, x_p)$ with

$$w_p := s^*w, \qquad x_p := \sqrt{\frac{d\theta}{d\theta_p}} \cdot s^*\theta(s^*)xs$$

is a Q-system.

Graphically, the conditions are given by:



Remark 4.8. The notion of sub-Q-system Θ_p of Θ corresponds to the notion of intermediate subfactor L with $N \subset L \subset M$ where Θ is the dual canonical Q-system of $N \subset M$. Namely, the properties of the sub-Q-system are just a reformulation of [ILP98, Corollary 3.10]. Namely, they consider subspaces $K_\rho \subset \mathrm{Hom}(\iota, \iota\rho)$ for each $\rho \in {}_N\Delta_N$, which correspond to a projection $p \in \mathrm{Hom}(\theta, \theta)$ if we identify the Hilbert spaces $\mathrm{Hom}(\rho, \theta)$ and $\mathrm{Hom}(\iota, \iota\rho)$ by Frobenius reciprocity.

Remark 4.9 (cf. [BKLR15]). If one drops the condition $w^*p = w^*$ in Prop. 4.7 then we obtain a more general "sub" Q-system $\Theta_p = (\theta_p, w_p, x_p)$ with

$$w_p := \lambda^{-1} \cdot s^*w, \qquad x_p := \lambda \cdot \sqrt{\frac{d\theta}{d\theta_p}} \cdot s^*\theta(s^*)xs$$

where $\lambda = \sqrt{w^*pw}$.

Definition 4.10. We denote by $C_\mathrm{l}(\Theta) = (C_\mathrm{l}(\theta), C_\mathrm{l}(w), C_\mathrm{l}(x))$ the left center of Θ, which is defined to be the sub-Q-sytem associated with the projection $P^\mathrm{l}_\Theta \in \mathrm{Hom}(\theta, \theta)$. Analogously, the right center $C_\mathrm{r}(\Theta)$ is defined using P^r_Θ.

Remark 4.11 ([FFRS06, Lemma 2.30]). The Q-system $C_{\mathrm{l/r}}(\Theta)$ is a maximal commutative sub-Q-system of Θ.

Remark 4.12. The intermediate factor $N \subset M_+ \subset M$ defined in [BE00] is given by the Q-system $C_\mathrm{l}(\Theta)$. Namely, the characterization of P^l_Θ in [FFRS06, Lemma 2.30] is the characterization in [BE00, Lemma 4.1] in terms of subspaces $\mathcal{H}_\rho \subset \mathrm{Hom}(\iota, \iota\rho)$ of "charged intertwiners". Similarly, $N \subset M_- \subset M$ is given by $C_r(\Theta)$.

Definition 4.13 (cf. [FFRS08]). Let ${}_N\mathcal{C}_N$ be a UMTC. The full center of a Q-system Θ is defined to be the Q-system $Z(\Theta) \equiv (Z(\theta), Z(w), Z(x)) = C_\mathrm{l}((\Theta \boxtimes \mathrm{id}_N) \circ^+ \Theta_{\mathrm{LR}})$ in ${}_N\mathcal{C}_N \boxtimes \overline{{}_N\mathcal{C}_N}$.

In particular we have $Z(\mathrm{id}_N) = \Theta_{\mathrm{LR}}$.

Definition 4.14. Let ${}_N\mathcal{C}_N$ be a URFC and $\Theta = (\theta, w, x)$ a Q-system in ${}_N\mathcal{C}_N$. We define

$$\mathrm{Hom}_{\mathrm{loc}}(\theta\rho, \sigma) = \{t \in \mathrm{Hom}(\theta\rho, \sigma) : t \cdot P^{\mathrm{l}}_\theta(\rho) = t\}\,,$$
$$\mathrm{Hom}_{\mathrm{loc}}(\sigma, \theta\rho) = \{t^* \in \mathrm{Hom}(\sigma, \theta\rho) : P^{\mathrm{l}}_\theta(\rho) \cdot t^* = t^*\}\,.$$

In particular, the spaces $\mathrm{Hom}_{\mathrm{loc}}(\theta\rho, \sigma)$ and $\mathrm{Hom}_{\mathrm{loc}}(\sigma, \theta\rho)$ are anti-isomorphic, due to the self-adjointness of $P^{\mathrm{l}}_\theta(\rho)$.

Lemma 4.15. *The isometry* $\psi \in \mathrm{Hom}\left(Z(\theta), (\theta \boxtimes \mathrm{id}_N)\theta_{\mathrm{LR}}\right)$ *with* $\psi\psi^* = P^{\mathrm{l}}_{(\Theta\boxtimes\mathrm{id}_N)\circ^+\Theta_{\mathrm{LR}}}$ *and* $\psi^*\psi = 1$ *is of the form:*

$$\psi = \bigoplus_{\lambda_1,\lambda_2 \in {}_N\Delta_N} \bigoplus_{m \in B(\theta\lambda_2,\lambda_1)_{\mathrm{loc}}} m^* \boxtimes \mathrm{id}_{\lambda_2} \in \mathrm{Hom}\left(Z(\theta), (\theta \boxtimes \mathrm{id}_N)\theta_{\mathrm{LR}}\right)\,,$$

where the sum over m *goes over an ONB of* $\mathrm{Hom}_{\mathrm{loc}}(\theta\lambda_2, \lambda_1)$*. In particular:*

$$[Z(\theta)] = \bigoplus_{\lambda_1,\lambda_2 \in {}_N\Delta_N} \langle \theta\lambda_2, \lambda_1\rangle_{\mathrm{loc}} \left[\lambda_1 \boxtimes \overline{\lambda}_2\right]\,,$$

where $\langle\ \cdot\ ,\ \cdot\ \rangle_{\mathrm{loc}} = \dim \mathrm{Hom}_{\mathrm{loc}}(\ \cdot\ ,\ \cdot\ )$.

Proof. We first note that $u \in \mathrm{Hom}\left(R(\theta), (\theta \boxtimes 1)\theta_{\mathrm{LR}}\right)$ given by

$$u := \bigoplus_{\lambda_1,\lambda_2 \in {}_N\Delta_N} \bigoplus_{m \in B(\theta\lambda_2,\lambda_1)} m^* \boxtimes \mathrm{id}_{\lambda_2} \in \mathrm{Hom}\left(R(\theta), (\theta \boxtimes \mathrm{id}_N)\theta_{\mathrm{LR}}\right)\,,$$
$$R(\theta) := \bigoplus_{\lambda_1,\lambda_2 \in {}_N\Delta_N} \langle \theta\lambda_2, \lambda_1\rangle \lambda_1 \boxtimes \overline{\lambda}_2$$

is a unitary interwiner. It can be shown that

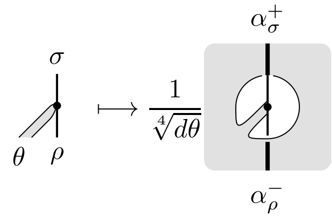
$$P^{\mathrm{l}}_{(\Theta\boxtimes\mathrm{id}_N)\circ^+\Theta_{\mathrm{LR}}} \cdot u = P^{\mathrm{l}}_{\Theta\boxtimes\mathrm{id}_N}(\theta_{\mathrm{LR}}) \cdot u \equiv \left(\bigoplus_{\lambda \in {}_N\Delta_N} P^{\mathrm{l}}_\Theta(\lambda) \boxtimes 1_{\overline{\lambda}}\right) \cdot u\,.$$

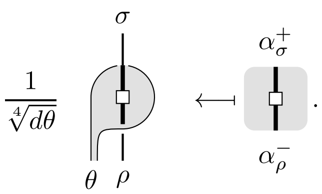
The equality is the statement [FFRS06, Prop. 3.14(i)], namely it is proven that $C_{\mathrm{l}}((\Theta \boxtimes \mathrm{id}_N) \circ^+ \Theta_{\mathrm{LR}})$ which is associated with $P^{\mathrm{l}}_{(\Theta\boxtimes\mathrm{id}_N)\circ^+\Theta}$ is associated with the projection $P^{\mathrm{l}}_{\Theta\boxtimes\mathrm{id}_N}(C_{\mathrm{l}}(\theta_{\mathrm{LR}})) \equiv P^{\mathrm{l}}_{\Theta\boxtimes\mathrm{id}_N}(\theta_{\mathrm{LR}})$. We can conclude by eventually choosing another basis that a maximal isometry invariant w.r.t. $P^{\mathrm{l}}_{(\Theta\boxtimes\mathrm{id}_N)\circ^+\Theta_{\mathrm{LR}}}$ is given by summing just over ONB's of $\mathrm{Hom}_{\mathrm{loc}}(\theta\lambda_2, \lambda_1)$. ☐

Given a Q-system Θ in ${}_N\mathcal{C}_N$ and $\iota(N) \subset M$ its associated subfactor with the inclusion map $\iota\colon N \to M$, we will constantly use that the Q-system Θ is of the form $\Theta_{\bar\iota}$ as in Def. 2.1, in other words the Q-system Θ becomes trivial in the 2–C*-category generated by ${}_N\mathcal{C}_N, \iota, \bar\iota$. This simplifies many graphical proofs.

Lemma 4.16. *Let ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ be a UMTC, Θ a Q-system in ${}_N\mathcal{C}_N$ and $N \subset M$ the corresponding subfactor. Let $\rho, \sigma \in {}_N\mathcal{C}_N$ be irreducible. The spaces $\mathrm{Hom}_{\mathrm{loc}}(\theta\rho, \sigma)$ and $\mathrm{Hom}(\alpha^-_\rho, \alpha^+_\sigma)$ are isomorphic by the map:*

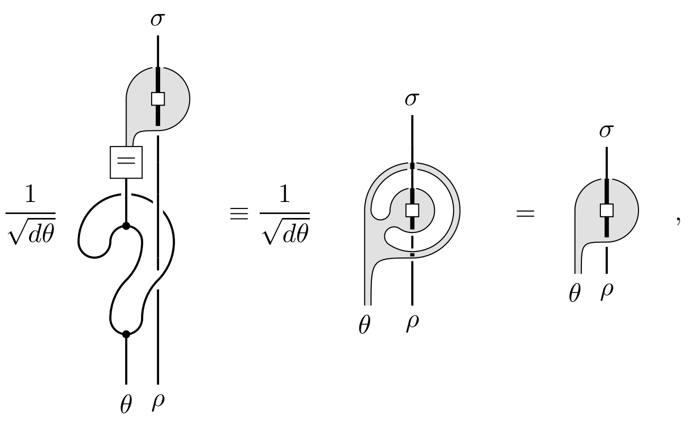
$$\mathrm{Hom}_{\mathrm{loc}}(\theta\rho, \sigma) \longrightarrow \mathrm{Hom}(\alpha^-_\rho, \alpha^+_\sigma)$$





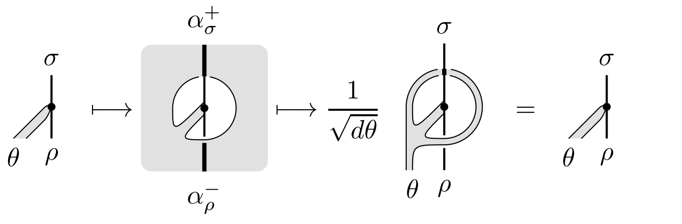
In the same way $\mathrm{Hom}_{\mathrm{loc}}(\rho, \theta\sigma)$ is isomorphic to $\mathrm{Hom}(\alpha^+_\rho, \alpha^-_\sigma)$. This gives a unitary equivalence between the Hilbert spaces $\mathrm{Hom}_{\mathrm{loc}}(\rho, \theta\sigma)$ with scalar product $(e, f) = \Phi_\sigma(e^* f)$ and $\mathrm{Hom}(\alpha^+_\rho, \alpha^-_\sigma)$ with scalar product $(e', f') = \Phi_{\alpha^+_\sigma}(e'^* f')$, where Φ_σ and $\Phi_{\alpha^+_\sigma}$ denote the unique left inverse and unique standard left inverse, respectively.

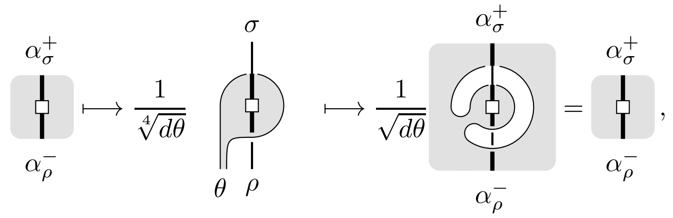
Proof. We first check that the map is well defined, namely the image is an element in $\mathrm{Hom}_{\mathrm{loc}}(\theta\rho, \sigma)$ and we have ("=" denotes the trivial intertwiner identifying $\theta = \bar{\iota}\iota$)



where we used in the first equation that Θ is of the form $\Theta_{\bar{\iota}}$ and in the second equation that the closed string can be contracted which cancels the prefactor. So we conclude that the image is actually in $\mathrm{Hom}_{\mathrm{loc}}(\theta\rho, \sigma)$.

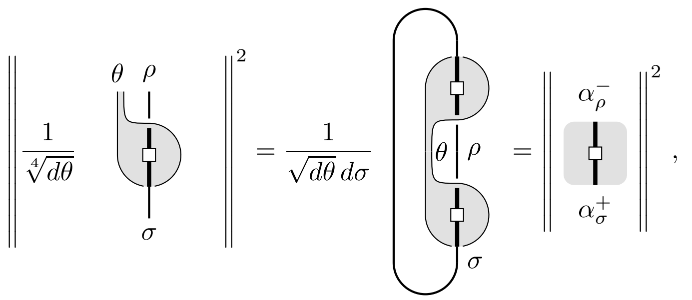
We have to show that both maps are inverse to each other:





where the last equation in the first line is exactly the fact that the intertwiner is in $\mathrm{Hom}_{\mathrm{loc}}(\theta\rho,\sigma)$, namely the diagram can be deformed to obtain $P^{\mathrm{l}}_{\Theta}(\rho)$ which can be omitted; in the last equation of the second line the closed string can again be contracted to a dimension cancelling the prefactor.

Finally, unitarity can be seen as follows:

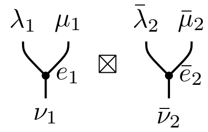


where in the last equation we use that the string diagram can be deformed to give the standard left inverse for α^+_σ (cf. [Reh00, Lemma 2.2]). □

Definition 4.17 (α-induction construction [Reh00]). For a braided subfactor $\iota(N)\subset M$ in ${}_N\mathcal{C}_N$ there is a Q-system $\Theta_M=(\theta_M,w_M,x_M)$ in ${}_N\mathcal{C}_N\boxtimes\overline{{}_N\mathcal{C}_N}$

given by:

$$
\begin{aligned}
[\theta_M] &= \bigoplus_{\rho,\sigma\in {}_N\Delta_N} Z_{\mu\nu}[\mu \boxtimes \bar\nu],\\
Z_{\mu\nu} &= \langle \alpha^+_\mu, \alpha^-_\nu\rangle\\
x_M &= \bigoplus_{lmn}\sum_{e_1,e_2} \sqrt{\frac{d\lambda_2 d\mu_2}{d\theta_M d\nu_2}}\Phi^1_{\nu_1}[\iota(e_1{}^*)(\phi^*_l\otimes\phi^*_m)\iota(e_2)\phi_n]\cdot e_1\boxtimes \bar e_2,\\
&= \bigoplus_{lmn}\sum_{e_1,e_2} \frac{1}{\sqrt{d\theta_M}}\sqrt[4]{\frac{d\lambda_2 d\mu_2 d\nu_1}{d\lambda_1 d\mu_1 d\nu_2}}\Phi^1_{\nu_1}[\cdots]
\end{aligned}
$$



where l is considered as a multi-index $(\lambda_1 \in {}_N\Delta_N, \lambda_2 \in {}_N\Delta_N, l = 1,\cdots, Z_{\lambda_1,\lambda_2})$ and e_i stands for an ONB in $\mathrm{Hom}(\nu_i, \lambda_i\mu_i)$ and ϕ_l an ONB in $\mathrm{Hom}(\alpha^+_{\lambda_1}, \alpha^-_{\lambda_2})$ with respect to the induced left inverse $\Phi^1_{\lambda_1}$.

The following result was conjectured in [KR10]. It can be seen as the main technical result. It allows to apply a lot of results obtained in the categorical literature to the braided subfactor and conformal net setting.

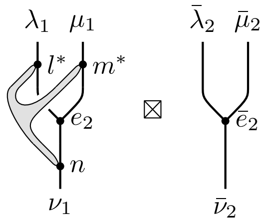
Proposition 4.18. *Let ${}_N\mathcal{C}_N$ be a UMTC. The α-induction construction for $(\iota(N) \subset M, {}_N\mathcal{C}_N)$ coincides with the full center $Z(\Theta)$ of the corresponding Q-system Θ.*

Proof. It is already clear that the two constructions give equivalent objects, namely

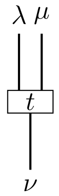
$$
[Z(\theta)] = \bigoplus_{\lambda_1,\lambda_2\in {}_N\Delta_N} \langle\theta\lambda_2,\lambda_1\rangle_{\mathrm{loc}}[\lambda_1\boxtimes\bar\lambda_2] = \bigoplus_{\lambda_1,\lambda_2\in {}_N\Delta_N}\langle\alpha^+_{\lambda_1},\alpha^-_{\lambda_2}\rangle[\lambda_1\boxtimes\bar\lambda_2] = [\theta_M]
$$

follows from Lemma 4.15 and Lemma 4.16. We have to show that the two intertwiners $Z(x)$ and x_M of the two respective constructions are equivalent. We decompose $Z(x)$ w.r.t. an ONB to show that we obtain the same coefficients as in the α-induction construction for x_M. Using Lemma 4.15 we have:

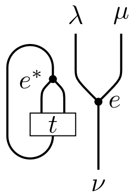
$$
\sqrt{d\theta_{\mathrm{LR}}}\sqrt{d\theta}Z(x) = \bigoplus_{lmn}\sum_{e_2}\sqrt[4]{\frac{d\lambda_1}{d\lambda_2}\frac{d\mu_1}{d\mu_2}\frac{d\nu_1}{d\nu_2}} \quad , \qquad (5)
$$



where l, m, n run over an ONB of $\mathrm{Hom}_{\mathrm{loc}}(\lambda_1, \theta\lambda_2)$, $\mathrm{Hom}_{\mathrm{loc}}(\mu_1, \theta\mu_2)$ and $\mathrm{Hom}_{\mathrm{loc}}(\nu_1, \theta\nu_2)$, respectively. We use the following expansion of an arbitrary intertwiner $t \in \mathrm{Hom}(\nu, \lambda\mu)$ with respect to an ONB $\{e\}$ of

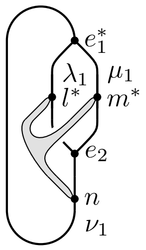


$$= \sum_e \Phi_\nu(e^* t) e = \frac{1}{\sqrt{d\lambda d\mu d\nu}} \sum_e$$

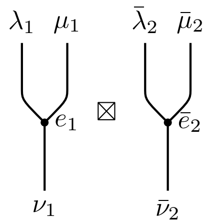


with respect to an orthonormal basis $\{e\}$ of $\mathrm{Hom}(\nu, \lambda\mu)$. The rhs of Eq. (5) becomes

$$= \bigoplus_{lmn} \sum_{e_1, e_2} \frac{\sqrt[4]{\frac{d\lambda_1}{d\lambda_2}\frac{d\mu_1}{d\mu_2}\frac{d\nu_1}{d\nu_2}}}{\sqrt{d\lambda_1 d\mu_1 d\nu_1}}$$



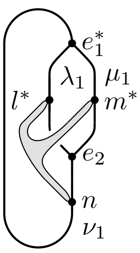
$\cdot$



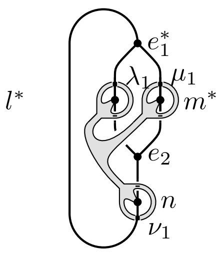
$\cdot$

We calculate:

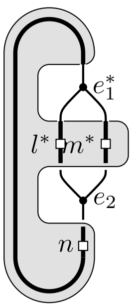
$$\sqrt[4]{\frac{d\lambda_1}{d\lambda_2}\frac{d\mu_1}{d\mu_2}\frac{d\nu_1}{d\nu_2}}$$



$$= \sqrt[4]{\frac{d\lambda_1}{d\lambda_2}\frac{d\mu_1}{d\mu_2}\frac{d\nu_1}{d\nu_2}} (d\theta)^{-\frac{3}{2}}$$



$=$



$$= d\nu_1 \sqrt{d\theta} \sqrt[4]{\frac{d\lambda_1 d\lambda_2 d\mu_1 d\mu_2}{d\nu_1 d\nu_2}} \Phi^1_{\nu_1}[\cdots],$$

where we first use that the intertwiners l, m, n are in $\mathrm{Hom}_{\mathrm{loc}}(\ \cdot\ ,\ \cdot\ )$ and then replace by Lemma 4.16 with an orthonormal basis in $\mathrm{Hom}(\alpha^+_{\lambda_1}, \alpha^-_{\lambda_2})$ and in the second step deform the ι string to obtain the left inverse of $\alpha^+_{nu_1}$ and $\Phi^1_{\nu_1}[\cdots]$ is the expression of Def. 4.17. This shows that $Z(x)$ has the same coefficients as x_M from the α-induction construction. □

We need the following general result as a main tool in the following sections.

Proposition 4.19 (cf. [KR08]). *Let Θ_a and Θ_b be irreducible in a UMTC ${}_N\mathcal{C}_N$. Then Θ_a and Θ_b are Morita equivalent if and only if $Z(\Theta_a)$ and $Z(\Theta_b)$ are equivalent.*

4.2 The adjoint functor of the full center

We have a tensor functor T as follows: the map

$$T\left(\bigoplus_i \lambda_i \boxtimes \bar{\mu}_i\right) = \bigoplus_i \lambda_i \circ \bar{\mu}_i \tag{6}$$

is an extension of the monoidal product (which by definition is a bifunctor). We have $T(\mathrm{id}_N \boxtimes \mathrm{id}_N) = \mathrm{id}_N$ and the family of morphisms

$$\begin{aligned} \mu_{(\rho_1\boxtimes\bar\sigma_1),(\rho_2\boxtimes\bar\sigma_2)} &\colon T(\rho_1 \boxtimes \bar\sigma_1) \circ T(\rho_2 \boxtimes \bar\sigma_2) \longrightarrow T(\rho_1\rho_2 \boxtimes \bar\sigma_1\bar\sigma_2) \\ \mu_{(\rho_1\boxtimes\bar\sigma_1),(\rho_2\boxtimes\bar\sigma_2)} &:= (1_{\rho_1} \otimes \varepsilon(\rho_2,\bar\sigma_1)^* \otimes 1_{\bar\sigma_2}) \equiv \rho_1(\varepsilon(\rho_2,\bar\sigma_1)^*) \end{aligned} \tag{7}$$

extends to a family

$$\mu_{(\beta_1),(\beta_2)} \colon T(\beta_1) \circ T(\beta_2) \longrightarrow T(\beta_1 \circ \beta_2), \qquad \beta_1, \beta_2 \in {}_N\mathcal{C}_N \boxtimes \overline{{}_N\mathcal{C}_N}$$

and makes the following diagram commute:

$$\begin{array}{ccc} T(\beta_1)\circ T(\beta_2)\circ T(\beta_3) & \longrightarrow & T(\beta_1)\circ T(\beta_2\circ\beta_3) \\ \downarrow & & \downarrow \\ T(\beta_1\circ\beta_2) & \longrightarrow & T(\beta_1\circ\beta_2\circ\beta_3) \end{array}.$$

This means T is a (strict with respect to the unity but in general non-strict for associativity, i.e. $\mu_{\bullet,\bullet} \neq 1$) strong monoidal functor (tensor functor). It is well known that strong monoidal functors map monoids into monoids, by this we can conclude that for $\Theta_2 = (\theta_2, w_2, x_2)$ a Q-system in ${}_N\mathcal{C}_N \boxtimes \overline{{}_N\mathcal{C}_N}$ we obtain a (reducible) Q-system $T(\Theta_2) = (T(\theta_2), w_{T(\Theta_2)}, x_{T(\Theta_2)})$ by

$$w_{T(\Theta_2)} = T(w_2), \qquad\qquad x_{T(\Theta_2)} = \mu^*_{\theta_2,\theta_2} \cdot T(x_2)$$

or explicitely by ($t_i^{jk} \in \mathrm{Hom}(\rho_i \boxtimes \bar\sigma_i, \rho_j\rho_k \boxtimes \bar\sigma_j\bar\sigma_k)$)

$$\begin{aligned} \theta &= \bigoplus_i \rho_i \boxtimes \bar\sigma_i & x &= \bigoplus_{ijk} t_i^{jk} \\ T(\theta_2) &= \bigoplus_i \rho_i\bar\sigma_i & x_{T(\Theta)} &= \bigoplus_{ijk} \underbrace{\rho_j(\varepsilon(\rho_k,\bar\sigma_j)) \cdot T(t_i^{jk})}_{\in \mathrm{Hom}(\rho_i\bar\sigma_i, \rho_j\bar\sigma_j\rho_k\bar\sigma_k)} . \end{aligned}$$

We note that even if Θ is commutative $T(\Theta)$ is in general not commutative, because the functor is not braided.

We introduce the notion of a direct sum for Q-systems (cf. [EP03, p. 321]). Let ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ be a URFC and $\{\Theta_i = (\theta_i, w_i, x_i)\}_{i=1,\ldots,n}$ be Q-systems in ${}_N\mathcal{C}_N$. The direct sum Q-system $\Theta = (\theta, w, x)$ with $\theta = \bigoplus_{i=1}^n \theta_i$ is defined by

$$\theta = \sum_{i=1}^n \mathrm{Ad}\, T_i \circ \theta_i\,, \quad w = \frac{1}{\sqrt{d(\theta)}} \sum_{i=1}^n d_i \cdot T_i \cdot w_i\,, \quad x = \sum_{i=1}^n \theta(T_i) T_i x_i T_i^*\,,$$

where $d_i = \sqrt{d(\theta_i)} = d(\iota_i)$ and T_i are generators of the Cuntz algebra with n elements, i.e. $T_i^* T_j = \delta_{ij} \cdot 1$ and $\sum_i T_i T_i^* = 1$. If (θ_i, w_i, x_i) corresponds to the subfactor $N \subset M_i$ with inclusion map ι_i, then (θ, w, x) corresponds to the inclusion $N \subset \bigoplus_{i=1}^n M_i$. The $p_i = T_i T_i^*$ give a decomposition in the sense of Remark 4.9.
The following identity has been proven on the level of objects in [Eva02, Prop. 3.3.]. We remark that a priori it is not clear that this "curious identity" holds also on the level of Q-systems. It is directly related to the adding the boundary construction in [CKL13] as we discuss in Sect. 6.6.

Proposition 4.20 (cf. [KR08, Prop. 4.3]). *Let ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ be a UMTC and Θ a Q-system in ${}_N\mathcal{C}_N$ with corresponding subfactor $N \subset M$. Then we have an equivalence of Q-systems:*

$$T(Z(\Theta)) \cong \bigoplus_{a \in {}_N\Delta_M} \Theta_a.$$

Our first aim was to prove this identity directly for the α-induction construction. We had a graphical proof for the trivial Q-system. Because the α-induction construction coincides with the full center it follows now easily from the general results of [KR08].

Proof. We note (see Rem. 3.6) that the Q-system Θ_a for some $a \in {}_N\mathcal{C}_M$ or equivalently $\bar{a} \in {}_M\mathcal{C}_N$ corresponds on the nose with the Q-system $\Phi(\bar{a})^\vee \otimes_\Theta \Phi(\bar{a}) = \Phi(a) \otimes_\Theta \Phi(\bar{a})$ constructed in [KR08], where $\Phi\colon {}_M\mathcal{C}_N \to \mathrm{Bim}(\Theta, \mathrm{id})$ is the functor in Prop. 3.2. Then one can directly apply [KR08, Prop. 4.3]. □

As a corollary this implies the "curious identity" which was proven in [Eva02, Prop. 3.3.] and shows that behind this identity indeed sits more structure.

Corollary 4.21 (cf. [Eva02, Prop. 3.3.], see also [BEK99, Cor 6.13.]). *Let $N \subset M$ be a non-degenerately braided type III subfactor and $Z_{\lambda\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle$ for $\lambda, \mu \in {}_N\Delta_N$. Then we have*

$$\bigoplus_{a \in {}_N\Delta_M} [a\bar{a}] = \bigoplus_{\rho,\sigma \in {}_N\Delta_N} Z_{\rho\sigma} [\rho\bar{\sigma}] \tag{8}$$

and in particular the number of elements in ${}_N\Delta_M$ or ${}_M\Delta_N$ is given by

$$|{}_N\Delta_M| = |{}_M\Delta_N| = \sum_{\rho \in {}_N\Delta_N} Z_{\rho\rho}\,.$$

Remark 4.22. The functor $T(\ \cdot\ )$ gives a (left) adjoint to the full center $Z(\ \cdot\ )$, namely Θ is a sub-Q-system of $T(Z(\Theta))$.

5 Modular invariance and Q-systems in ${}_N\mathcal{C}_N \boxtimes \overline{{}_N\mathcal{C}_N}$

5.1 Characterization of modular invariant Q-systems

Let ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ be a UMTC. Given a Q-system Θ and the corresponding extension $\iota(N) \subset M$ let $Z_{\mu\nu} = \langle \alpha^+_\mu, \alpha^-_\nu \rangle$ for $\mu, \nu \in {}_N\Delta_N$. The matrix $Z = (Z_{\mu\nu})_{\mu,\nu \in {}_N\Delta_N}$ is a MODULAR INVARIANT [BEK99], i.e. it commutes with S and T from (3). It is called normalized because $Z_{00} = 1$ and sufferable because it comes from an inclusion $\iota(N) \subset M$. The α-induction construction or equivalently the full center gives a Q-system Θ_2 in ${}_N\mathcal{C}_N \boxtimes \overline{{}_N\mathcal{C}_N}$ with $[\theta_2] = \bigoplus_{\mu,\nu \in {}_N\Delta_N} Z_{\mu\nu}[\mu \boxtimes \bar\nu]$. It is sometimes convenient to write the matrix $(Z_{\mu\nu})$ formally in character form as $Z = \sum_{\mu,\nu \in {}_N\Delta_N} Z_{\mu,\nu}\chi_\mu\bar\chi_\nu$.

Lemma 5.1 ([BEK00], see also [KO02, Thm 4.5]). *Let ${}_N\mathcal{C}_N$ be a UMTC. If Θ is an irreducible commutative Q-system in ${}_N\mathcal{C}_N$, then $\dim {}_M\mathcal{C}^0_M = \dim {}_N\mathcal{C}_N/(d\Theta)^2$. In particular, $d\Theta \leq \dim({}_N\mathcal{C}_N)^{\frac{1}{2}}$.*

Proof. The first statement is a combination of Thm. 4.2 and Prop. 3.1 in [BEK00]. The second statement follows from the first, using $\dim {}_M\mathcal{C}^0_M \geq 1$. Using Remark 3.3 and 5.6, this also follows from [KO02, Thm 4.5]. □

Proposition 5.2 ([KR09, Thm. 3.4, Prop. 3.22]). *Let Θ_2 be an irreducible commutative Q-system in ${}_N\mathcal{C}_N \boxtimes \overline{{}_N\mathcal{C}_N}$, then the following are equivalent:*

1. $d\Theta_2 = \dim({}_N\mathcal{C}_N)$
2. $Z = (Z_{\mu\nu})$ *is a modular invariant*
3. $\Theta_2 \equiv Z(\Theta)$ *for some irreducible Q-system Θ in ${}_N\mathcal{C}_N$.*

Proof. (3) are equivalent (1) by [KR09, Thm. 3.4, Prop. 3.22] (see also [Müg10, Thm 3.4], [DMNO13]).
The notion of modular invariance in [KR09, Thm. 3.4] is a bit different. But by [LR04, Appendix C] we obtain that (2) implies (1), namely the argument shows that if $d\theta < \dim({}_N\mathcal{C}_N)$ then Z cannot be modular invariant. Together with Lemma 5.1 this gives the statement.
(3) implies (2) is clear by the fact that $Z_{\mu\nu} = \langle \alpha^+_\mu, \alpha^-_\nu \rangle$ defines a modular invariant and that $Z(\Theta)$ coincides with the α-induction construction Prop. 4.18. □

5.2 Permutation modular invariants

Let ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ be a UMTC. A non-negative integer valued matrix $Z = (Z_{\mu\nu})_{\mu,\nu \in {}_N\Delta_N}$ with $Z_{\mathrm{id}_N,\mathrm{id}_N} = 1$ is called a MODULAR INVARIANT if it commutes with the matrices S and T constructed in Subsect. 2.2. It is called REALIZABLE (sufferable) if there exists a braided subfactor $(\iota(N) \subset M, {}_N\mathcal{C}_N)$ such that $Z_{\mu\nu} = \langle \alpha^+_\mu, \alpha^-_\nu \rangle$.

Proposition 5.3. *Let ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ be a UMTC and $\phi \in \mathrm{Aut}({}_N\Delta_N)$ which only fixes the sector $[\mathrm{id}_N]$ and which extends to a braided automorphism of ${}_N\mathcal{C}_N$. Then there is a braided subfactor $N \subset M_\phi$ in ${}_N\mathcal{C}_N$ with*

$$[\theta_\phi] = \bigoplus_\nu n_\nu[\nu], \qquad n_\nu = \sum_\mu \langle \mu\phi(\bar\mu), \nu\rangle$$

which realizes the permutation modular invariant $Z_{\mu\nu} = \delta_{\nu,\phi(\mu)}$.

Proof. By the Longo–Rehren construction Def. 4.1 there is a Q-system $\Theta^\phi_{\mathrm{LR}}$ with:

$$[\theta^\phi_{\mathrm{LR}}] = \bigoplus_\mu [\mu \boxtimes \phi(\bar\mu)]\,.$$

We define the Q-system $\Theta_\phi := T(\Theta^\phi_{\mathrm{LR}})$ in ${}_N\mathcal{C}_N$ with

$$[\theta_\phi] := \bigoplus_\mu [\mu\phi(\bar\mu)] = \bigoplus_\nu n_\nu[\nu], \qquad n_\nu = \sum_\mu \langle \mu\phi(\bar\mu), \nu\rangle$$

as above which is irreducible because $0 = \langle \mu\phi(\bar\mu), \mathrm{id}_N\rangle$ for $[\mu] \neq [\mathrm{id}_N]$ by the assumption about ϕ not having non-trivial fixed points. Because $T(\;\cdot\;)$ is left-adjoint to $Z(\;\cdot\;)$ the subfactor $N \subset M_\phi$ given by the Q-system Θ_ϕ has the modular invariant $Z_{\mu\nu} = \delta_{\nu,\phi(\mu)}$. ☐

A particular case is, if ${}_N\mathcal{C}_N$ has no non-trivial self-conjugate sectors besides the trivial sector, in this case the charge conjugation C might fulfill the assumptions and the obtained subfactor realizes the charge conjugation modular invariant $Z = C$. We therefore can answer a particular case of the question how $Z = C$ is realized, namely the case that there are no non-trivial self-conjugate charges.

Example 5.4. The UMTC $E_{6,1}$ for example obtained by positive energy representation of loop groups, has 3 sectors $\{\rho_0, \rho_1, \rho_2\}$ with $\mathbb{Z}_3$ fusion rules, i.e. $[\rho_i\rho_j] = [\rho_{i+j \mod 3}]$ for $0 \leq i,j \leq 2$, and the charge conjugation transposes the two non-trivial charges. Then Prop. 5.3 yields a Q-system with $[\theta] = [\rho_0] \oplus [\rho_1] \oplus [\rho_2]$ which realizes $Z = C$, i.e. $Z = |\chi_0|^2 + \chi_1\bar\chi_2 + \chi_2\bar\chi_1$.

If there is fixed point in the permutation the same construction as in the proof of Prop. 5.3 is possible but we do not know how a dual canonical endomorphism of an irreducible Q-system giving the modular invariant would look, because the "adjoint functor" gives a reducible Q-system. Nevertheless, we can conclude that for a permutation matrix Z of ${}_N\Delta_N$ which gives rise to a braided automorphism, there exists a braided subfactor $\iota(N) \subset M$ in ${}_N\mathcal{C}_N$ which has Z as a modular invariant, i.e. such permutation modular invariants are realizable. The category ${}_N\mathcal{C}_N$ is called pointed if all irreducible objects are invertible, i.e. have dimension 1 or in other words ${}_N\mathcal{C}_N = \mathrm{Pic}({}_N\mathcal{C}_N)$.

Lemma 5.5. *Let ${}_N\mathcal{C}_N \in \mathrm{End}(N)$ be a pointed UMTC and let Θ_1 and Θ_2 be Q-systems. If Θ_1 and Θ_2 are Morita equivalent, then they are equivalent.*

Proof. Let Θ_1 and Θ_2 be irreducible Q-systems in ${}_N\mathcal{C}_N$ which are Morita equivalent. Without lost of generatlity, we may assume that $\Theta_1 = \Theta_{\bar\iota}$ comes from a subfactor $\iota(N) \subset M$ and $\Theta_2 = \Theta_a$ with $a \in {}_N\mathcal{C}_M$ irreducible.
Because ${}_N\mathcal{C}_N$ is pointed the sectors form an abelian (due to the braiding) group denoted G. The multiplication in G is given by the fusion rules, i.e. ${}_N\Delta_N = \{\lambda_g : g \in G\}$ with $[\lambda_g\lambda_h] = [\lambda_{gh}]$ for all $g, h \in G$ and $[\lambda_{g^{-1}}] = [\bar\lambda_g]$. We note that $\iota\lambda_g$ is irreducible, namely by Frobenius reciprocity $\langle \iota\lambda_g, \iota\lambda_g\rangle = \langle \theta, \lambda_g\bar\lambda_g\rangle = \langle \theta, \mathrm{id}_N\rangle = 1$. Therefore ${}_N\Delta_M \subset \{\lambda_g\bar\iota : g \in G\}$ (because there can be $[\lambda_g\bar\iota] = [\lambda_h\bar\iota]$). So we may assume that $a = \lambda_g\bar\iota$ and can conclude that $[\theta_a] = [\lambda_g\bar\iota\iota\bar\lambda_g] = [\theta\bar\lambda_g\lambda_g] = [\theta]$. It is easy to check that using $\varepsilon(\lambda_g, \theta)$ we can construct a unitary intertwiner $\theta_a \to \theta\lambda_g\bar\lambda_g \to \theta$, which gives an equivalence of the two Q-systems.
Alternatively, we can use that $\bar\alpha^{\pm}_{\lambda_{g^{-1}}}$ is an automorphism satisfying $a\bar\alpha^{\pm}_{\lambda_{g^{-1}}} = \lambda_g\bar\iota\alpha^{\pm}_{\lambda_{g^{-1}}} = \lambda_g\lambda_{g^{-1}}\bar\iota = \bar\iota$. Then Prop. 3.8 gives an alternative proof of the statement. □

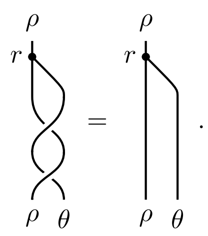
Let ${}_N\mathcal{C}_N \subset \mathrm{End}(N)$ be a pointed UMTC and Θ be a Q-system and $Z_{\mu\nu} = \langle \alpha^+_\mu, \alpha^-_\nu\rangle$. Then Lemma 5.5 shows that $T(Z(\Theta))$ is equivalent to $\bigoplus_{i=1}^{\mathrm{tr}\, Z}\Theta$. Therefore in this case we obtain an easy formula for θ in terms of its modular invariant matrix $Z = (Z_{\mu\nu})$:

$$[\theta] = \frac{1}{\mathrm{tr}\, Z}\bigoplus_{\rho\in{}_N\Delta_N}\sum_{\mu,\nu\in{}_N\Delta_N} Z_{\mu\nu}N^{\rho}_{\mu\nu}[\rho]\,,$$

see also [Pin07].

5.3 Maximal chiral subalgebras and second cohomology for modular invariant Q-systems

Let us assume that Θ is a commutative Q-system in ${}_N\mathcal{C}_N$ and $N \subset M$ the associated subfactor.
The category $\mathrm{Mod}(\Theta)$ forms a (non-strict) tensor category as follows. Let ρ, σ be two right Θ-modules. Because Θ is commutative, we obtain a left action on ρ and σ using the braiding, which makes them bimodules. Then the tensor product $\rho \otimes \sigma$ is defined to be the object $\rho \otimes_\Theta \sigma$ as in Remark 3.3, which we see as right module by forgetting the left action.
Let $\mathrm{Mod}_0(\Theta)$ the subcategory of dyslectic modules (see [Par95, KO02]), i.e. modules (ρ, r), such that $r\varepsilon(\theta, \rho)\varepsilon(\rho, \theta) = r$, graphically:



It can easily be seen that if we give the induced right Θ-module $\rho\theta$ the structure of a bimodule using the braiding that it becomes equivalent to the α-induction $\Phi(\alpha^\pm_\rho)$ in Remark 3.3, where the sign is depending on the choice of the braiding. We obtain that $\mathrm{Bim}^\pm(\Theta,\Theta)\cong\mathrm{Mod}(\Theta)$ as tensor categories, but we will just need the following fact.

Remark 5.6. The map obtained by restricting bimodules to right modules

$$\mathrm{Bim}^0(\Theta,\Theta)\to\mathrm{Mod}_0(\Theta)$$

is an equivalence of categories. Namely, an object in $\mathrm{Bim}^0(\Theta,\Theta)$ gives a dyslectic module, because using the fact that it is contained both, in the image of α^+ and α^-, we can "unwind" the double braid. Conversely, if a module is dyslectic, the left action obtained by the both braidings coincide, so it must come from $\mathrm{Bim}^0(\Theta,\Theta)$.

For $\beta\in{}_M\mathcal{C}_M$ we define the σ-restriction $\sigma_\beta=\bar\iota\beta\iota\in{}_N\mathcal{C}_N$.
Given $\Theta_\pm$ commutative Q-systems corresponding to $N\subset M_\pm$ it follows that ${}_{M_\pm}\mathcal{C}^0_{M_\pm}$ are again UMTCs. Let us assume there is a braided equivalence $\phi\colon {}_{M_+}\mathcal{C}^0_{M_+}\to{}_{M_-}\mathcal{C}^0_{M_-}$. Now we consider the Q-system $\Theta^\phi_{\mathrm{LR}}$ in ${}_{M_+}\mathcal{C}^0_{M_+}\boxtimes\overline{{}_{M_-}\mathcal{C}^0_{M_-}}$. By composing ι_{LR} with $\iota_1\boxtimes\iota_2$ we obtain a Q-system

$$\Theta_{(\Theta_+,\Theta_-,\phi)}=\Theta_{(\bar\iota_1\boxtimes\bar\iota_2)\circ\bar\iota^\phi_{\mathrm{LR}}}$$

with

$$[\theta^\phi_{\mathrm{LR}}]=\bigoplus_{\alpha\in{}_{M_+}\Delta^0_{M_+},\beta\in{}_{M_-}\Delta^0_{M_-}}\tilde Z_{\alpha\beta}[\alpha\boxtimes\bar\beta],\quad \tilde Z_{\alpha\beta}=\delta_{\alpha,\phi(\beta)}$$

$$[\theta_{(\Theta_+,\Theta_-,\phi)}]=\bigoplus_{\mu,\nu\in{}_N\Delta_N}Z_{\mu\nu}[\mu\boxtimes\bar\nu],\qquad Z_{\mu\nu}=\sum_{\alpha\beta}Z_{\alpha\beta}\langle\sigma^+_\alpha,\mu\rangle\langle\sigma^-_{\bar\beta},\bar\mu\rangle=\sum_\tau b^+_{\tau,\mu}b^-_{\phi(\tau),\nu}$$

where $b^\pm_{\tau,\mu}=\langle\sigma^\pm_\tau,\mu\rangle$ for $\tau\in{}_{M_\pm}\mathcal{C}^0_{M_\pm}$. All maximal commutative Q-systems in ${}_N\mathcal{C}_N\boxtimes\overline{{}_N\mathcal{C}_N}$ are of this form:

Proposition 5.7 ([DNO13, Prop. 3.7, Cor. 3.8]). *There is a one-to-one correspondence between*

1. *Equivalence classes of commutative irreducible Q-systems Θ_2 in ${}_N\mathcal{C}_N\boxtimes\overline{{}_N\mathcal{C}_N}$ with $d\theta_2=\dim({}_N\mathcal{C}_N)$.*
2. *Isomorphism classes of triples (Θ_+,Θ_-,ϕ) where $\Theta_\pm$ are commutative irreducible Q-systems in ${}_N\mathcal{C}_N$ and $\phi\colon{}_{M_+}\mathcal{C}^0_{M_+}\to{}_{M_-}\mathcal{C}^0_{M_-}$ is an equivalence of braided categories.*
3. *Indecomposable module categories over ${}_N\mathcal{C}_N$.*

Proof. This statement is proven in a more general setting in [DNO13, Prop. 3.7, Cor. 3.8]. They call the objects in point 1) Lagrangian algebras. We use that by Remark 3.3 and 5.6 (see also [Müg10, Thm 3.1]) the category ${}_{M_+}\mathcal{C}^0_{M_+}$ is equivalent to the category of dyslectic modules. □

We note that there can exist inequivalent ϕ_1, ϕ_2 giving the same modular invariant $Z = (Z_{\mu\nu})$. Namely if $\langle \sigma_{\phi_1(\tau)}, \mu\rangle = \langle \sigma_{\phi_2(\tau)}, \mu\rangle$ holds for all $\tau \in {}_{M_+}\mathcal{C}^0_{M_+}$ and $\mu \in {}_N\mathcal{C}_N$ for which $b^+_{\tau,\mu} \neq 0$. Because ϕ_1 and ϕ_2 are inequivalent the Q-systems $\Theta^{\phi_1}_{\mathrm{LR}}$ and $\Theta^{\phi_2}_{\mathrm{LR}}$ are inequivalent. This (or using Prop. 5.7) implies that also $\Theta_{(\Theta_+,\Theta_-,\phi_1)}$ and $\Theta_{(\Theta_+,\Theta_-,\phi_2)}$ are inequivalent. This means that the second cohomology (see Rem. 2.2) of $\Theta_{(\Theta_+,\Theta_-,\phi_{1,2})}$ does not vanish in this case.

Example 5.8. Let us consider for ${}_N\mathcal{C}_N$ the UMTC obtained by $\mathsf{SU}(3)_9$ and Θ_+ coming from the conformal inclusion $\mathsf{SU}(3)_9 \subset E_{6,1}$.
As in Ex. 5.4 the UMTC category $E_{6,1}$ has three sectors ${}_{M_+}\Delta^0_{M_+} = \{\beta_0, \beta_1, \beta_2\}$ and we obtain an extension $M_+ \subset \tilde{M}$ with $[\tilde{\theta}] = [\beta_0] \oplus [\beta_1] \oplus [\beta_2]$, which gives the permutation modular invariant interchanging $\beta_1 \leftrightarrow \beta_2$. Now $\sigma^+_{\beta_1} = \sigma^+_{\beta_2}$, so both inclusions $N \subset M_+$ and $N \subset \tilde{M}$ give by the above discussion the same modular invariant with respect to $\mathsf{SU}(3)_9$, which is $Z = |\chi_{0,0} + \chi_{9,0} + \chi_{0,9} + \chi_{4,1} + \chi_{1,4} + \chi_{4,4}|^2 + 2|\chi_{2,2} + \chi_{5,2} + \chi_{2,5}|^2$. This example appeared in [BE01], cf. [EP09, EP11].
So we can conclude that $\Theta_{(\Theta_+,\Theta_+,\mathrm{id})}$ and $\Theta_{(\Theta_+,\Theta_+,\phi)}$ in ${}_N\mathcal{C}_N \boxtimes \overline{{}_N\mathcal{C}_N}$ have isomorphic endomorphisms $[\theta_{(\Theta_+,\Theta_+,\mathrm{id})}] = [\theta_{(\Theta_+,\Theta_+,\phi)}]$ but the Q-systems are not equivalent. So we have an example where the second cohomology does not vanish.
The same happens for the inclusion[4] $G_{2,3} \subset E_{6,1}$ where $Z = |\chi_{00} + \chi_{11}|^2 + 2|\chi_{02}|^2$.

6 Conformal nets

We now apply the results to conformal nets.
Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be the one-point compactification of the real line $\mathbb{R}$, which we can by the Cayley map $\overline{\mathbb{R}} \ni x \mapsto z = \frac{\mathrm{i}-x}{\mathrm{i}+x} \in \mathbb{S}^1$ identify with the circle $\mathbb{S}^1 \subset \mathbb{C}$. We denote by $\mathsf{M\ddot{o}b}$ the Möbius group which is isomorphic to both:

- $\mathsf{PSL}(2,\mathbb{R})$, which acts naturally on the real line $\overline{\mathbb{R}}$, and
- $\mathsf{PSU}(1,1)$, which acts naturally on the circle $\mathbb{S}^1 \subset \mathbb{C}$.

The universal covering group of $\mathsf{M\ddot{o}b}$ is denoted by $\widetilde{\mathsf{M\ddot{o}b}}$. We denote by $\mathsf{M\ddot{o}b}_\pm = \mathsf{M\ddot{o}b} \rtimes \mathbb{Z}_2$ where the action of $\mathbb{Z}_2$ is given by the reflection $r\colon z \mapsto \bar{z}$ on $\mathbb{S}^1$. The rotations $R(\vartheta)z = \mathrm{e}^{\mathrm{i}\vartheta z}$ on $\mathbb{S}^1$, the dilations $\delta(s)x = \mathrm{e}^s x$ on $\mathbb{R}$, and the translations $\tau(t)x = x + t$ on $\mathbb{R}$ give three distinguished one-parameter subgroups of $\mathsf{M\ddot{o}b}$ which generate $\mathsf{M\ddot{o}b}$.

[4] This was told to us by V. Ostrik via mathoverflow

We denote by $I \in \mathcal{I}$ the set of all PROPER INTERVALS on $\mathbb{S}^1$, i.e. all open, connected, non-dense, non-empty intervals $I \subset \mathbb{S}^1$.

DEFINITION 6.1. A *local Möbius covariant net (conformal net)* $\mathcal{A}$ on $\mathbb{S}^1$ is a family $\{\mathcal{A}(I)\}_{I\in\mathcal{I}}$ of von Neumann algebras on a Hilbert space $\mathcal{H}_\mathcal{A}$, with the following properties:

A. ISOTONY. $I_1 \subset I_2$ implies $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$.

B. LOCALITY. $I_1 \cap I_2 = \emptyset$ implies $[\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$.

C. MÖBIUS COVARIANCE. There is a unitary representation U of $\mathsf{Möb}$ on $\mathcal{H}$ such that $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$.

D. POSITIVITY OF ENERGY. U is a positive energy representation, i.e. the generator L_0 (conformal Hamiltonian) of the rotation subgroup $U(R(\theta)) = \mathrm{e}^{\mathrm{i}\theta L_0}$ has positive spectrum.

E. VACUUM. There is a (up to phase) unique rotation invariant unit vector $\Omega \in \mathcal{H}$ which is cyclic for the von Neumann algebra $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I)$.

The REEH–SCHLIEDER PROPERTY automatically holds [FJ96], i.e. Ω is cyclic and separating for any $\mathcal{A}(I)$ with $I \in \mathcal{I}$. Furthermore, we have the BISOGNANO–WICHMANN PROPERTY [GF93, BGL93] saying that the modular operators with respect to Ω have geometric meaning; e.g. the modular operators for the upper circle I_0 are given by the dilation $\Delta^{\mathrm{i}t} = U(\delta(-2\pi t))$ and reflection $J = U(r)$, where here U is extended to $\mathsf{Möb}_\pm$. For a general interval $I \in \mathcal{I}$ the modular operators are given by a special conformal transformation δ_I and a reflection r_I both fixing the endpoints of I. The Bisognano–Wichmann property implies HAAG DUALITY

$$\mathcal{A}(I)' = \mathcal{A}(I') \qquad\qquad I \in \mathcal{I}$$

and it can be shown (see e.g. [GF93]) that each $\mathcal{A}(I)$ is a type III_1 factor in Connes' classification [Con73]. A conformal net is ADDITIVE [FJ96], i.e. for intervals $I \in \mathcal{I}$ and $I_1, \ldots, I_n \in \mathcal{I}$ we have

$$I \subset \bigcup_i I_i \quad \Longrightarrow \quad \mathcal{A}(I) \subset \bigvee_i \mathcal{A}(I_i)\,.$$

A local Möbius covariant net on $\mathcal{A}$ on $\mathbb{S}^1$ is called COMPLETELY RATIONAL if it

F. fulfills the SPLIT PROPERTY, i.e. for $I_0, I \in \mathcal{I}$ with $\overline{I_0} \subset I$ the inclusion $\mathcal{A}(I_0) \subset \mathcal{A}(I)$ is a split inclusion, namely there exists an intermediate type I factor M such that $\mathcal{A}(I_0) \subset M \subset \mathcal{A}(I)$.

G. is STRONGLY ADDITIVE, i.e. for $I_1, I_2 \in \mathcal{I}$ two adjacent intervals obtained by removing a single point from an interval $I \in \mathcal{I}$ the equality $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ holds.

H. for $I_1, I_3 \in \mathcal{I}$ two intervals with disjoint closure and $I_2, I_4 \in \mathcal{I}$ the two components of $(I_1 \cup I_3)'$, the μ-INDEX of $\mathcal{A}$

$$\mu(\mathcal{A}) := [(\mathcal{A}(I_2) \vee \mathcal{A}(I_4))' : \mathcal{A}(I_1) \vee \mathcal{A}(I_3)] \tag{9}$$

(which does not depend on the intervals I_i) is finite.

Example 6.2. Examples of completely rational local Möbius covariant nets are:

- Diffeomorphism covariant nets with central charge $c < 1$ [KL04a].
- The nets $\mathcal{A}_L$ where L is a positive even lattice [DX06] which contain as a special case [Bis12] loop group nets $\mathcal{A}_{G,1}$ at level 1 for G a compact connected, simply connected simply-laced Lie group.
- The loop group nets $\mathcal{A}_{\mathsf{SU}(n),\ell}$ for $\mathsf{SU}(n)$ at level ℓ. [Xu00].

Further examples of rational conformal nets can be obtained from these as follows:

- Finite index extensions and subnets of completely rational conformal nets. Namely, let $\mathcal{A} \subset \mathcal{B}$ be a finite subnet i.e. $[\mathcal{B}(I) : \mathcal{A}(I)] < \infty$ for some (then all) $I \in \mathcal{I}$, then $\mathcal{A}$ is completely rational iff $\mathcal{B}$ is completely rational [Lon03], in particular orbifolds $\mathcal{A}^G$ of completely rational nets $\mathcal{A}$ with G a finite group are completely rational.
- Let $\mathcal{A} \subset \mathcal{B}$ be a co-finite subnet , i.e. $[\mathcal{B}(I), \mathcal{A}(I) \vee \mathcal{A}^{\mathrm{c}}(I)] < \infty$ for some (then all) $I \in \mathcal{I}$, where the COSET NET $\mathcal{A}^{\mathrm{c}}$ is defined by $\mathcal{A}^{\mathrm{c}}(I) = \mathcal{A}' \cap \mathcal{B}(I)$ with $\mathcal{A}' = (\vee_{I \in \mathcal{I}} \mathcal{A}(I))'$. Then $\mathcal{B}$ is completely rational iff $\mathcal{A}$ and $\mathcal{A}^{\mathrm{c}}$ are completely rational [Lon03]. This gives many example of completely rational nets coming from the coset construction.

A SEPARABLE (NON-DEGENERATED) REPRESENTATION of a strongly additive local Möbius covariant net is a family $\pi = \{\pi_I : \mathcal{A}(I) \to \mathrm{B}(\mathcal{H}_\pi)\}_{I \in \mathcal{I}}$ of unital representations ($*$-homomorphisms) π_I of $\mathcal{A}(I)$ on a common separable Hilbert space $\mathcal{H}_\pi$, which are compatible, i.e.

$$\pi_{I_2} \restriction \mathcal{A}(I_1) = \pi_{I_1}, \qquad\qquad I_1 \subset I_2 .$$

Such a representation is automatically normal, i.e. all π_I are strongly continuous. We denote by $\mathrm{DHR}(\mathcal{A})$ the category of separable representations, where morphisms in $\mathrm{Hom}(\pi^1, \pi^2)$ are given by intertwiners $V \in \mathrm{B}(\mathcal{H}_{\pi^1}, \mathcal{H}_{\pi^2})$, such that $V\pi^1_I(a) = \pi^2_I(a)V$ for all $I \in \mathcal{I}$ and $a \in \mathcal{A}(I)$. Let us denote by $\mathrm{DHR}^0(\mathcal{A})$ the representations π with finite statistical dimension $d\pi$, which is defined to be

$$d\pi := [\pi_{I'}(\mathcal{A}(I'))' : \pi_I(\mathcal{A}(I))]^{\frac{1}{2}}$$

for some $I \in \mathcal{I}$, where $[M : N]$ is the minimal index. The definition of $d\pi$ does not depend on the choice of I.

Let us from now on fix a completely rational local Möbius covariant net $\mathcal{A}$ on $\mathbb{S}^1$. The category $\mathrm{DHR}^0(\mathcal{A})$ is a (unitary) modular tensor category [KLM01]. Every $\pi \in \mathrm{DHR}^0(\mathcal{A})$ is equivalent to a representation localized in a given $I_0 \in \mathcal{I}$, i.e. it exists a $\rho \cong \pi$ such that $\mathcal{H}_\rho = \mathcal{H}_\mathcal{A}$ and $\rho_{I_0'} = \mathrm{id}_{\mathcal{A}(I_0')}$. Namely, $\pi_{I_0'}(\mathcal{A}(I_0'))$ on $\mathcal{H}_\pi$ is spatially isomorphic to $\mathcal{A}(I_0')$ on $\mathcal{H}_\mathcal{A}$, by the type III property. Let $U\colon \mathcal{H}_\pi \to \mathcal{H}_\mathcal{A}$ be a unitary implementing this isomorphism, then $\rho = \{\rho_I := \mathrm{Ad}\, U \circ \pi_I\}_{I\in\mathcal{I}}$ does the job.
This implies that the category $\mathrm{DHR}^{I_0}(\mathcal{A})$ of representations with finite statistical dimensions which are localized in I_0 has the same irreducible sectors as $\mathrm{DHR}^0(\mathcal{A})$.
By Haag duality $\rho \in \mathrm{DHR}^{I_0}(\mathcal{A})$ implies $\rho_I(\mathcal{A}(I)) \subset \mathcal{A}(I)$ for every $I \supset I_0$, that means such a representation is an endomorphism and $d\rho = [\mathcal{A}(I_0) : \rho_{I_0}(\mathcal{A}(I_0))]^{\frac{1}{2}}$ equals the dimension of the endomorphism. Together with strong additivity it follows that all intertwiners are in $\mathcal{A}(I_0)$. In particular, this means that $\mathrm{DHR}^{I_0}(\mathcal{A})$ can naturally be seen as a full subcategory of $\mathrm{End}(\mathcal{A}(I_0))$ and that $\mathrm{DHR}^{I_0}(\mathcal{A})$ is equivalent to $\mathrm{DHR}^0(\mathcal{A})$. We note that the family $\{\rho_I\}$ is determined by ρ_{I_0} by using strong additivity and it is really enough to consider $\mathrm{DHR}^{I_0}(\mathcal{A})$ as a full and replete subcategory of $\mathrm{End}(\mathcal{A}(I_0))$ and we will drop the index I_0. Repleteness is just the fact that for $U \in \mathcal{A}(I_0)$ also $\mathrm{Ad}_U \circ\rho$ is localized in I_0.
The braiding (also called statistics operator) is given by:

$$\varepsilon(\rho_1, \rho_2) = \rho_2(U_1^*)U_2^*U_1\rho_1(U_2)\,,$$

where $U_i \in \mathrm{Hom}(\rho_i, \tilde\rho_i)$ and $\tilde\rho_i \in [\rho_i]$ is localized in I_i. Here $I_1, I_2 \subset I_0$ are two disjoint intervals such that $I_1 > I_2$ (I_2 sits clockwise after I_1 inside I_0). We also write ε^+ for ε and define the opposite braiding by $\varepsilon^-(\rho_1,\rho_2) = \varepsilon^+(\rho_2,\rho_1)^*$.
We will interpret $\mathcal{A}$ as the chiral observables or as chiral symmetries. For example $\mathcal{A} = \mathrm{Vir}_c$ with $c < 1$ is the net generated by the chiral stress energy tensor $T(x)$. We want to look into CFTs on Minkowski space containing the chiral observables $\mathcal{A}$ and boundary conditions on $\mathbb{M}_+$ which "preserve" these observables.

6.1 Extensions and Q-systems

Let M be a spacetime, e.g. Minkowski space and $\mathcal{K}$ a set of open spacetime regions in M, e.g. the set of double cones. Let G be a group acting locally on M and let $G(O)$ be the set of all $g \in G$, such that there is a continuous path γ in G from the identity to g such that $\gamma(t)O \in \mathcal{K}$.

Definition 6.3. A *local G-covariant net* $\mathcal{A}$ on M is a family $\{\mathcal{A}(O)\}_{O\in\mathcal{K}}$ of von Neumann algebras on a Hilbert space $\mathcal{H}$, with the following properties:

A. Isotony. $O_1 \subset O_2$ implies $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.

B. Locality. $[\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$ for all pairwise spacelike separated $O_1, O_2 \in \mathcal{K}$.

C. G-covariance. There is a unitary positive energy representation U of G on $\mathcal{H}$, such that $U(g)\mathcal{A}(O)U(g)^* = \mathcal{A}(gO)$ for all $g \in G(O)$

D. Vacuum. There is a (up to phase) unique G- invariant unit vector $\Omega \in \mathcal{H}$ which is cyclic and separating for $\mathcal{A}(O)$ for all $O \in \mathcal{K}$.

A G-covariant DHR representation of $\mathcal{A}$ is a compatible family $\pi = \{\pi_O \colon \mathcal{A}(O) \to \mathcal{B}(\mathcal{H}_\pi)\}_{O\in\mathcal{K}}$ of representations on a Hilbert space $\mathcal{H}_\pi$, such that for all $O \in \mathcal{K}$ there exists a unitary $V \colon \mathcal{H}_\pi \to \mathcal{H}$, such that the representation $\rho := \mathrm{Ad}\, V \circ \pi$ is localized in O, i.e. $\rho_{O_0} = \mathrm{id}_{\mathcal{A}(O_0)}$ for O_0 spacelike to O, and that there is a unitary projective representation U_π of G, such that $\mathrm{Ad}\, U_\pi(g) \circ \pi_O = \pi_{gO} \circ \mathrm{Ad}\, U(g)$ for all $g \in G(O)$.
Given two local G-covariant nets $\mathcal{A}$ and $\mathcal{B}$ on Hilbert spaces $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$, respectively, an arrow $\mathcal{A} \to \mathcal{B}$ is an isometry $V \colon \mathcal{H}_\mathcal{A} \to \mathcal{H}_\mathcal{B}$ and a compatible family of embeddings (representation) $\{\pi_O \colon \mathcal{A}(O) \hookrightarrow \mathcal{B}(O)\}$ such that for all $O \in \mathcal{K}$ we have $Va = \pi_O(a)V$, $VU_\mathcal{A}(g) = U_\mathcal{B}(g)V$ for all $g \in G$ and $V\Omega_\mathcal{A} = \Omega_\mathcal{B}$. $\mathcal{A}$ and $\mathcal{B}$ are called unitary equivalent if V is a unitary and π_O are isomorphisms.
Let us assume that we have a subnet $\mathcal{A}_0$ of $\mathcal{B}$, i.e. $\mathcal{A}_0(O) \subset \mathcal{B}(O)$ for all O and $U(g)\mathcal{A}_0(O)U(g)^* = \mathcal{A}_0(gO)$. Then $\mathcal{A} = \mathcal{A}_0 e$ with e the Jones projection on $\overline{\vee\mathcal{A}_0(O)\Omega}$ is a G-local net on $\mathcal{H}_\mathcal{A} := e\mathcal{H}$, in other words we have an arrow $\mathcal{A} \to \mathcal{B}$ in the above sense. We say that $\mathcal{A}$ is a subnet of $\mathcal{B}$ and $\mathcal{B}$ is a local extension of $\mathcal{A}$. By abuse of notation we will not distinguish between the net $\mathcal{A}$ and its representation on the bigger Hilbert space $\mathcal{H}$ and write $\mathcal{A} \subset \mathcal{B}$ or $\mathcal{B} \supset \mathcal{A}$ for an inclusion/extension of nets.
For every connected region we have a subfactor $\mathcal{A}(O) \subset \mathcal{B}(O)$. If the subfactor is irreducible, we call the extension irreducible and if the index is finite we call the extension finite. If we have a finite irreducible extension $\mathcal{B}$ of $\mathcal{A}$ then the corresponding Q-system of $\mathcal{A}(O) \subset \mathcal{B}(O)$ is a commutative irreducible Q-system in $\mathrm{DHR}^O(\mathcal{A})$ and conversely if we have a commutative irreducible Q-system Θ in $\mathrm{DHR}^O(\mathcal{A})$ we obtain a finite local extension $\mathcal{B}$ of $\mathcal{A}$. In particular we have a one-to-one correspondence between [LR95]:

- local finite irreducible extensions $\mathcal{B} \supset \mathcal{A}$ up to unitary equivalence and
- commutative irreducible Q-systems Θ in $\mathrm{DHR}^O(\mathcal{A})$ up to equivalvence.

If we assume Θ to be only irreducible, we still have a relatively local extension, i.e. $[\mathcal{A}(O_1), \mathcal{B}(O_2)] = \{0\}$ for O_1 and O_2 spacelike separated. We call such an extension $\mathcal{B} \supset \mathcal{A}$ also non-local extension to stress the fact that we do not assume locality of $\mathcal{B}$. There is a one-to-one correspondence between [LR95]:

- finite irreducible extensions $\mathcal{B} \supset \mathcal{A}$ up to unitary equivalence and
- irreducible Q-systems Θ in $\mathrm{DHR}^O(\mathcal{A})$ up to equivalence.

6.2 Representation theory of local extensions

The following is well-known to experts [Müg10].

Proposition 6.4. *Let $\mathcal{A} \subset \mathcal{B}$ a finite index inclusion of local Möbius covariant nets on $\mathbb{S}^1$ and let either net be completely rational. Then $\mathcal{A}$ and $\mathcal{B}$ are both completely rational and the inclusion is irreducible.*
Further, let $I \in \mathcal{I}$ be an interval $N := \mathcal{A}(I) \subset \mathcal{B}(I) =: M$ and ${}_N\mathcal{C}_N = \mathrm{DHR}^I(\mathcal{A})$, and Θ be the Q-system in ${}_N\mathcal{C}_N$ associated with $N \subset M$. Then $\mathrm{DHR}^I(\mathcal{B}) = {}_M\mathcal{C}_M^0$ as UMTCs and in particular $\mathrm{DHR}(\mathcal{B})$ is equivalent to $\mathrm{Mod}^0(\Theta)$ and $\mathrm{Bim}^0(\Theta,\Theta)$.

Proof. Both ${}_M\mathcal{C}_M^0$ and $\mathrm{DHR}^I(\mathcal{B})$ being full and replete subcategories of $\mathrm{End}(M)$, the only thing which needs to be checked is that both have the same irreducible sectors. The braiding on ${}_M\mathcal{C}_M^0$ can be checked to give the braiding on $\mathrm{Rep}(\mathcal{B})$ since the braiding is fixed by the universal property $\varepsilon(\rho_1,\rho_2) = 1$ if I_2 sits clockwise after I_1 inside I. A sector $[\beta] \in {}_M\Delta_M$ is a DHR sector if and only it is in ${}_M\Delta_M^0$ (see [LR95, BE98]), which implies ${}_M\mathcal{C}_M^0 \subset \mathrm{DHR}^I(\mathcal{B})$. To see equality, we realize that global dimensions coincide, namely $\dim \mathrm{DHR}^I(\mathcal{B}) \equiv \mu(\mathcal{B}) = [M:N]^{-2}\mu(\mathcal{A}) \equiv \dim {}_N\mathcal{C}_N/(d\theta)^2$ by [KLM01] and $\dim {}_M\mathcal{C}_M^0 = \dim {}_N\mathcal{C}_N/(d\theta)^2$ by Lemma 5.1. □

Remark 6.5. Commutative Q-systems Θ in a UMTC ${}_N\mathcal{C}_N$ are also called quantum subgroups, so finding quantum subgroups in a given UMTCs ${}_N\mathcal{C}_N$ and finding finite index local extensions of a local Möbius covariant $\mathcal{A}$ net with $\mathrm{DHR}^0(\mathcal{A}) \cong {}_N\mathcal{C}_N$ is equivalent. The representation theory of the extensions can be completely understood on a categorical level.
An analogous statement for inclusions of rational VOAs appeared recently in [HKL15].

6.3 Maximal 2D nets with chiral observables $\mathcal{A}$

Let $\mathcal{A}$ be a local Möbius covariant net on $\mathbb{S}^1 \cong \overline{\mathbb{R}}$. By restriction we can and will see $\mathcal{A}$ as a net on $\mathbb{R}$. Then Haag duality of $\mathcal{A}$ on $\mathbb{R}$ is equivalent to strong additivity of $\mathcal{A}$. We will assume that $\mathcal{A}$ is completely rational, therefore this holds automatically.
We denote by $\mathbb{M}$ the two-dimensional Minkowski space and by $\mathcal{K}$ the set of double cones $O \subset \mathbb{M}$. Each double cone is of the form

$$O = I \times J := \{(t,x) : t-x \in I,\ t+x \in J\},$$

where $I, J \in \mathcal{I}_0$ are two intervals on the light-rays $L_\pm = \{(t,x) : t \pm x = 0\}$.
The action of $\mathsf{M\ddot{o}b} \cong \mathsf{PSL}(2,\mathbb{R})$ on $\overline{\mathbb{R}}$ gives a local action of $\widetilde{\mathsf{M\ddot{o}b}}$ on $\mathbb{R}$ as in [KL04a]. We define $\mathsf{G}_2 = \widetilde{\mathsf{M\ddot{o}b}} \times \widetilde{\mathsf{M\ddot{o}b}}$ which acts locally on Minkowski space $\mathbb{M}$.
For $O \in \mathcal{K}$ we denote by $\mathsf{G}_2(O)$ all $g \in \mathsf{G}_2$ such that there is a path $\gamma\colon [0,1] \to \mathsf{G}_2$ from the identity element e to g with $\gamma(t)O \subset M$ for all $t \in [0,1]$.

We denote by $\mathcal{A}_2$ the net on $\mathcal{H}_\mathcal{A}\otimes\mathcal{H}_\mathcal{A}$ given by

$$\mathcal{A}_2(I\times J) := \mathcal{A}(I)\otimes\mathcal{A}(J).$$

It is a local Möbius covariant net on M as in [KL04a]. Every DHR representation of $\mathcal{A}_2$ with finite index is a direct sum of representations of the form $\rho\otimes\sigma$ where $\rho\in\mathrm{DHR}(\mathcal{A})$ and $\sigma\in\mathrm{DHR}(\mathcal{A})$. The braiding is given by $\varepsilon(\rho_1\otimes\sigma_1,\rho_2\otimes\sigma_2)=\varepsilon^+(\rho_1,\rho_2)\otimes\varepsilon^-(\sigma_1,\sigma_2)$. Therefore the category of DHR representations of $\mathcal{A}_2$ with finite statistical dimensions is equivalent to $\mathrm{DHR}^I(\mathcal{A})\boxtimes\overline{\mathrm{DHR}^J(\mathcal{A})}$.
Let us write $\mathcal{B}_2\supset\mathcal{A}_2$ for a local, Möbius covariant, irreducible extension of $\mathcal{A}_2$, i.e. a local Möbius covariant net $\mathcal{B}_2$ on Minkowski space $\mathbb{M}$ on the Hilbert space $\mathcal{H}_{\mathcal{B}_2}$ with irreducible vacuum vector Ω which is extending $\mathcal{A}_2\cong\mathcal{A}\otimes\mathcal{A}$, more precisely there is a representation π of $\mathcal{A}_2$ on $\mathcal{H}_{\mathcal{B}_2}$, such that $\pi(\mathcal{A}_2(O))\subset\mathcal{B}_2(O)$ is an irreducible inclusion of factors and $U(g)\pi(\mathcal{A}(O))U(g)^*=\pi(\mathcal{A}(gO))$ for all double cones $O\in\mathcal{K}$ and all $g\in\mathsf{G}(O)$. By abuse of notation we will omit the π.
We remember that there is a one-to-one correspondence between local irreducible extensions $\mathcal{B}_2\supset\mathcal{A}_2$ (up to unitary equivalence) and irreducible commutative Q-systems Θ_2 in $\mathrm{DHR}^I(\mathcal{A})\boxtimes\overline{\mathrm{DHR}^J(\mathcal{A})}$ (up to equivalence).

Proposition 6.6. *Let $\mathcal{B}_2\supset\mathcal{A}_2$ be a local extension. Then the following statements are equivalent:*

1. *The net $\mathcal{B}_2$ is a maximal local irreducible extension, i.e. if $\tilde{\mathcal{B}}_2\supset\mathcal{B}_2$ is a local irreducible extension, then $\mathcal{B}_2=\tilde{\mathcal{B}}_2$.*
2. *The index $[\mathcal{B}_2:\mathcal{A}_2]=\mu_2(\mathcal{A})\equiv\dim(\mathrm{DHR}(\mathcal{A}))$.*
3. *The matrix $(Z_{\lambda\mu})$ is a modular invariant.*
4. *The μ-index of $\mathcal{B}_2$ is 1.*
5. *The net $\mathcal{B}_2$ has no non-trivial superselection sectors.*

Proof. To show (2) ⇒ (1) let Θ_2 be a Q-system in $\mathrm{DHR}^I(\mathcal{A})\boxtimes\overline{\mathrm{DHR}^J(\mathcal{A})}$ giving the extension $\mathcal{A}(I)\otimes\mathcal{A}(J)\subset\mathcal{B}_2(I\times J)$ and let us assume that $[\mathcal{B}_2(I\times J):\mathcal{A}(I)\otimes\mathcal{A}(J)]=\mu_2(\mathcal{A})$. By Lemma 5.1 we have the following inequality:

$$\begin{aligned}d\Theta_2\equiv[\mathcal{B}_2:\mathcal{A}_2]&\leq\dim(\mathrm{DHR}(\mathcal{A}\otimes\mathcal{A}))^{\frac12}\equiv\dim\left(\mathrm{DHR}(\mathcal{A})\boxtimes\overline{\mathrm{DHR}(\mathcal{A})}\right)^{\frac12}\\&=\dim(\mathrm{DHR}(\mathcal{A}))\equiv\mu_2(\mathcal{A})\,.\end{aligned}$$

This implies maximality.
For showing (1) ⇒ (2), let us assume that $[\mathcal{B}_2:\mathcal{A}_2]<\mu_2(\mathcal{A})$. We need to show that there is an extension $\tilde{\mathcal{B}}_2\supsetneq\mathcal{B}_2$. This we obtain by adding the boundary [CKL13], i.e. from $\mathcal{B}_2$ we obtain a possible reducible boundary net (see Subsec. 6.6) of which we choose an irreducible subnet $\mathcal{B}_+$. We claim $\mathcal{B}_+$

cannot be Haag dual, but this follows because $[\mathcal{B}_+ : \mathcal{A}_+] = [\mathcal{B}_2 : \mathcal{A}_2] < \mu_2(\mathcal{A})$ and then [LR04, Prop. 2.13] implies $[\mathcal{B}_+^{\mathrm{d}} : \mathcal{B}_+] > 1$. So we have an inclusion $\mathcal{A}_+ \subset \mathcal{B}_+ \subsetneq \mathcal{B}_+^{\mathrm{d}}$ and a corresponding locally isomorphic inclusion $\mathcal{A}_2 \subset \mathcal{B}_2 \subsetneq \tilde{\mathcal{B}}_2$ as in [LR04], in particular $\mathcal{B}_2$ was not maximal.
The statements (2) and (3) are equivalent by Prop. 5.2 and the implication (5) $\Rightarrow$ (1) is clear.
(2) $\Rightarrow$ (4) follows by calculating the μ index [KLM01] and likewise the implication (4) $\Rightarrow$ (5) is [KLM01, Corollary 32]. □

Proposition 6.7. *There is a one-to-one correspondence between:*

1. *maximal local irreducible extensions $\mathcal{B}_2 \supset \mathcal{A}_2$ up to unitary equivalence.*
2. *Θ_2 commutative irreducible Q-systems in $\mathrm{DHR}^I(\mathcal{A}) \boxtimes \overline{\mathrm{DHR}^I(\mathcal{A})}$ with $d\theta_2 = \mu_2(\mathcal{A})$ up to equivalence.*
3. *(Non-local) irreducible extensions $\mathcal{B} \supset \mathcal{A}$ up to Morita equivalence.*
4. *Irreducible Q-systems Θ in $\mathrm{DHR}^I(\mathcal{A})$ up to Morita equivalence.*
5. *Indecomposable ${}_N\mathcal{C}_N$ module categories, where $N = \mathcal{A}(I)$ and ${}_N\mathcal{C}_N = DHR^I(\mathcal{A})$.*
6. *Local chiral extensions $\mathcal{A}_{\mathrm{L}} \supset \mathcal{A}$, $\mathcal{A}_{\mathrm{R}} \supset \mathcal{A}$ together with a braided equivalence $\phi\colon \mathrm{DHR}(\mathcal{A}_{\mathrm{L}}) \to \mathrm{DHR}(\mathcal{A}_{\mathrm{R}})$.*

Proof. The correspondence between (1) and (2) is Prop. 6.6, the one between (3) and (4) [LR95]. Starting with (4) we obtain (2) by applying the full center and it is well defined on Morita equivalence classes and injective by Prop. 4.19. It is surjective by Prop. 5.2, so (2) and (4) are equivalent. Equivalently, one can start with $\mathcal{B}_2$ and add the boundary to obtain a Haag dual boundary net (as in the proof before) which correspond to a non-local extension. The α-induction construction gives back the original net.
The correspondence between (4), (5) and (6) is just Prop. 5.7, where (6) is (2) of Prop. 5.7 reformulated in the language of nets, cf. [Müg10]. □

Remark 6.8. We know how the Morita equivalence looks like, see Subsec. 3.2.

6.4 Boundary conditions

Let $\mathcal{A}$ be a completely rational local Möbius covariant net on $\mathbb{S}^1$, which we will see as a net on $\mathbb{R}$ by restriction. Let $\mathbb{M}_+ = \{(t,x) \in \mathbb{M} : x > 0\}$ be Minkowski half-plane and let $\mathcal{K}_+$ be the set of double cones $O \Subset \mathbb{M}_+$. Double cones $O \in \mathcal{K}_+$ are in one-to-one correspondence with pairs of proper intervals $I, J \subset \mathbb{R}$ such that $I < J$. We write $O = I \times J$.
Let $\mathcal{A}_+$ be the net on $\mathbb{M}_+$ given by

$$\mathcal{A}_+(O) = \mathcal{A}(I) \vee \mathcal{A}(J) \qquad\qquad O = I \times J$$

which is locally covariant w.r.t. G_+ the universal covering of Möb, namely

$$U(g)\mathcal{A}_+(O)U(g)^* = \mathcal{A}_+(gO) \qquad\qquad g \in G_+(O)$$

where G_+ acts locally on $O = I \times J \in \mathcal{K}_+$ by $gO = gI \times gJ$ and $G_+(O)$ is the set of all $g \in G_+$ such that there is a continuous path γ from the identity to g such that $\gamma(t)O \in \mathcal{K}_+$.

By the split property it follows that $\mathcal{A}_+(O)$ is spatially isomorphic to $\mathcal{A}_2(O) \equiv \mathcal{A}(I) \otimes \mathcal{A}(J)$. This implies that the net $\mathcal{A}_+$ is locally isomorphic to the net $\mathcal{A}_2$ restricted to $\mathbb{M}_+$.

A boundary net $\mathcal{B}_+$ associated with $\mathcal{A}$ is a local, (locally) G_+-covariant net $\mathcal{B}_+$, which is an irreducible extension $\mathcal{B}_+ \supset \mathcal{A}_+$.

Starting with $\mathcal{B}_+ \supset \mathcal{A}_+$, we define the generated net $\mathcal{B}_+^{\mathrm{gen}} \supset \mathcal{A}$ on $\mathbb{R}$ by

$$\mathcal{B}_+^{\mathrm{gen}}(I) = \bigvee_{\substack{O\in\mathcal{K}_+\\ O\subset W_I}} \mathcal{B}_+(O) \supset \mathcal{A}(I)\,,$$

where $W_I = \{(t,x) : t \pm x \in I\}$ is the left wedge, such that its intersection on the t-axis is I.

Conversely, given $\mathcal{B} \supset \mathcal{A}$ a (non-local) extension on $\mathbb{R}$, we define

$$\mathcal{B}_+^{\mathrm{ind}}(O) = \mathcal{B}(L) \cap \mathcal{B}(K)'\,,$$

where $O = I \times J$ and $L \Subset K$, such that $L \cap K' = I \cup J$ or equivalently $O = W_L \cap W_K'$.

The dual net is defined by $\mathcal{B}_+^{\mathrm{d}}(O) = \mathcal{B}_+(O')'$ and $\mathcal{B}_+^{\mathrm{d}} = \mathcal{B}_+$ if and only if $\mathcal{B}_+$ is Haag dual.

Then $(\mathcal{B}_+^{\mathrm{ind}})^{\mathrm{gen}} = \mathcal{B}$ and $(\mathcal{B}_+^{\mathrm{gen}})_+^{\mathrm{ind}} = \mathcal{B}_+^{\mathrm{d}} = \mathcal{B}_+$ provided $\mathcal{B}_+$ was already Haag dual.

Together we have:

Proposition 6.9 ([LR04, LR95]). *There is a one-to-one correspondence between the equivalence classes of:*

1. *boundary nets $\mathcal{B}_+$ associated with $\mathcal{A}$, such that $\mathcal{B}_+$ is Haag dual.*
2. *boundary nets $\mathcal{B}_+$ associated with $\mathcal{A}$, such that $\mathcal{A}_+ \subset \mathcal{B}_+$ is maximal.*
3. *(Non-local) extensions $\mathcal{B} \supset \mathcal{A}$ on $\mathbb{R}$.*
4. *Q-systems in ${}_N\mathcal{C}_N$, where $N = \mathcal{A}(I)$ and ${}_N\mathcal{C}_N = \mathrm{DHR}^I(\mathcal{A})$.*

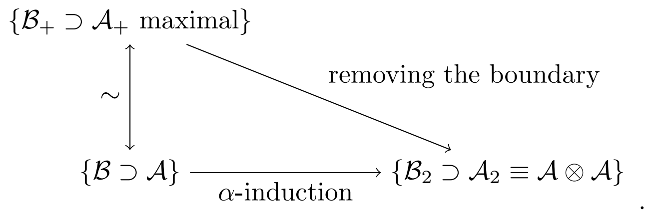
Definition 6.10. Let $\mathcal{B}_2 \supset \mathcal{A}_2$ be local extension, i.e. a CFT on Minkowski space. A (Möbius covariant) boundary condition of $\mathcal{B}_2 \supset \mathcal{A}_2$ with chiral symmetry $\mathcal{A}$ is a unitary equivalence class of boundary nets $\mathcal{B}_+ \supset \mathcal{A}_+$, where $\mathcal{B}_2 \restriction \mathbb{M}_+$ is locally covariantly isomorphic to $\mathcal{B}_+$, more precisely there is a compatible family of isomorphisms $\Phi_O \colon \mathcal{B}_+(O) \to \mathcal{B}_2(O)$ such that it restricts to an isomorphism $\mathcal{A}_+(O) \to \mathcal{A}_2(O)$ for all $O \in \mathcal{K}_+$ and that Φ is covariant respect to the covariance $U_{\mathcal{B}_+}$ of $\overline{\mathsf{M\ddot{o}b}}$ and $U_{\mathcal{B}_2}$ of $\overline{\mathsf{M\ddot{o}b}} \times \overline{\mathsf{M\ddot{o}b}}$ (where $\overline{\mathsf{M\ddot{o}b}}$ is the diagonal subgroup of $\overline{\mathsf{M\ddot{o}b}} \times \overline{\mathsf{M\ddot{o}b}}$).

Proposition 6.11. *Let $\mathcal{B}_2 \supset \mathcal{A}_2$ maximal and let $\mathcal{A} \subset \mathcal{B}$ given by Prop. 6.7. Then there is a one-to-one correspondence between:*

1. *Boundary conditions of $\mathcal{B}_2 \supset \mathcal{A}_2$ with chiral symmetry $\mathcal{A}$.*
2. *Unitary equivalence classes of $\mathcal{B}_a \supset \mathcal{A}$ Morita equivalent to $\mathcal{B} \supset \mathcal{A}$.*
3. *Sectors in*
$$ {}_N\mathcal{C}_M/\mathrm{Pic}({}_M\mathcal{C}_M)\,, $$
where $N = \mathcal{A}(I)$, $M = \mathcal{B}(I)$ and ${}_N\mathcal{C}_N = \mathrm{DHR}^I(\mathcal{A})$.

In particular the number of boudary conditions of $\mathcal{B}_2 \supset \mathcal{A}_2$ with chiral symmetry $\mathcal{A}$ is less or equal than
$$ |{}_N\Delta_M| \equiv \sum_{\lambda \in {}_N\Delta_N} Z_{\lambda\lambda}\,. $$

Proof. The following diagram commutes [LR09, Cor. 2]



Given a boundary condition, i.e. a boundary net $\mathcal{B}_{a,+} \supset \mathcal{A}_+$ let $\mathcal{B}_a \supset \mathcal{A}$ be the corresponding chiral extension. We note that $\mathcal{B}_{a,+}$ is Haag dual (cf. [LR09, App. C]), because $\mathcal{B}_2$ is modular invariant. If we remove the boundary we obtain $\mathcal{B}_2 \supset \mathcal{A}_2$, because the extensions are locally isomorphic and therefore isomorphic, see [LR09].
We conclude by commutativity of the above diagram that $\mathcal{B} \supset \mathcal{A}$ and $\mathcal{B}_a \supset \mathcal{A}$ are Morita equivalent, namely the α-induction construction gives equivalent two-dimensional extensions, which means the full centers are equivalent, which is equivalent to the Morita equivalence of $\mathcal{B} \supset \mathcal{A}$ and $\mathcal{B}_a \supset \mathcal{A}$.
Conversely, if we have given a chiral extension $\mathcal{B}_b \supset \mathcal{A}$ Morita equivalent to $\mathcal{B} \supset \mathcal{A}$, then $\mathcal{B}_{b,+} \supset \mathcal{A}_+$ is locally equivalent to $\mathcal{B}_{b,2} \supset \mathcal{A}_2 \restriction \mathbb{M}_+$ obtained by α-induction. But $\mathcal{B}_{2,b} \supset \mathcal{A}_2$ is isomorphic to $\mathcal{B}_2 \supset \mathcal{A}_2$ by Morita equivalence, so we get a boundary condition (this follows also from [LR04], realizing that the DHR orbit exhausts the Morita equivalence class).
Choosing $N = \mathcal{A}(I)$, $M = \mathcal{B}(I)$ and ${}_N\mathcal{C}_N = \mathrm{DHR}^I(\mathcal{A})$ the Q-systems Θ_a corresponding to $\mathcal{B}_a \supset \mathcal{A}$ which is Morita equivalent to $\mathcal{B} \supset \mathcal{A}$ are in one-to-one correspondence with ${}_N\mathcal{C}_M/\mathrm{Pic}({}_M\mathcal{C}_M)$ by Prop. 3.9. ∎

Example 6.12. We can give several cases as an example.

- If $\mathcal{A}$ is holomorphic, i.e. $\mathrm{DHR}(\mathcal{A})$ just contains the vacuum sector or equivalently $\mu(\mathcal{A}) = 1$, then $\mathcal{B}_2 = \mathcal{A}_2$ is maximal and the only 2D net and $\mathcal{A}_+$ is the only boundary condition. The family of holomorphic nets contains for example the conformal nets $\mathcal{A}_L$ associated with even selfdual lattices [DX06] like the E_8 lattice, Leech lattice etc., the Moonshine net $\mathcal{A}_\natural$ [KL06] and certain framed nets [KS14].

- For $\mathcal{A}$ from the family of conformal nets, for which $\mathrm{DHR}(\mathcal{A})$ is pointed, it follows from Lemma 5.5 that there is always just one boundary condition for each $\mathcal{B}_2 \supset \mathcal{A}_2$. This family for example contains all conformal nets $\mathcal{A}_L$ coming from an even lattice L [DX06], which include all loop group conformal nets $\mathcal{A}_{G,1}$ of compact, connected, simply connected, simply laced Lie groups G (the simple one being in one-to-one correspondence with A-D-E Dynkin diagrams) at level 1 [Bis12].

- If $\mathcal{A}$ is any completely rational net and $\mathcal{B}_2 = \mathcal{A}_{\mathrm{LR}} \supset \mathcal{A}_2$ given by the trivial Longo-Rehren extension, then ${}_N\mathcal{C}_M \cong {}_N\mathcal{C}_N \cong \mathrm{DHR}(\mathcal{A})$ and the boundary conditions are given by DHR sectors of $\mathcal{A}$ modulo DHR automorphisms of $\mathcal{A}$. This case is sometimes also called the Cardy case.

- For $\mathcal{A} = \mathcal{A}_{\mathsf{SU}(2),k}$ the two-dimensional extensions are in one-to-one correspondence with Dynkin diagrams of A-D-E type with Coxeter number $k+2$. The boundary conditions are given by orbits $[\nu]$ of a marked vertex ν under the automorphism group of the Dynkin diagram cf. [KLPR07].

- For $\mathcal{A} = \mathrm{Vir}_c$ with $c < 1$, the only possible values for c are $c = 1-6/m(m+1)$ with $m = 2, 3, 4, \ldots$. The maximal two-dimensional extensions are in one-to-one correspondence with pairs (G_1, G_2) of Dynkin diagrams of A-D-E type with Coxeter number m and $m+1$, respectively, cf. [KL04b]. The boundary conditions are given by pairs $([\nu_1], [\nu_2])$ with $[\nu_i]$ the orbit of a marked vertex on G_i under the automorphism group of G_i $(i = 1, 2)$. This result now follows also from [KLPR07].

The invertible objects (automorphisms) in ${}_M\mathcal{C}_M$ have to do with invertible defects (see for an interpretation of invertible defects in a different framework [DKR11]).
The difference between two inequivalent $a, b \in {}_N\mathcal{C}_M$ related by an invertible $\beta \in {}_M\mathcal{C}_M$ gets important if we also consider also reducible boundary conditions in the next section.

6.5 Reducible boundary conditions

With the notation as before, let us assume $\mathcal{B}_2 \supset \mathcal{A}_2$ is a maximal extension of $\mathcal{A}_2$. Using Prop. 6.7 we can choose a (non-local) extension $\mathcal{B} \supset \mathcal{A}$ such that $\mathcal{B}_2$ is given by the α-induction construction of $\mathcal{B} \supset \mathcal{A}$.
Let I be an interval, $N = \mathcal{A}(I)$, ${}_N\mathcal{C}_N = \mathrm{DHR}^I(\mathcal{A})$, $M = \mathcal{B}(I)$ and Θ the Q-system in ${}_N\mathcal{C}_N$ giving $N \subset M$. Then every $a \in {}_N\mathcal{C}_M$ gives a in general reducible Q-system Θ_a and an extension $\mathcal{B}_a \supset \mathcal{A}$.

We can define as before

$$\mathcal{B}_{a+}(O) = \mathcal{B}_a(L) \cap \mathcal{B}_a(K)'\,.$$

This net fulfills all the properties of a boundary CFT in [LR04], but the uniqueness of the vacuum and the joint irreducibility.

Proposition 6.13. *Let $a \in {}_N\mathcal{C}_M$ possibly reducible. Then the (reducible) boundary net $\mathcal{B}_{a,+} \supset \mathcal{A}_+$ is a (reducible) boundary condition for $\mathcal{B}_2 \supset \mathcal{A}_2$, which is given by the Q-system $Z(\Theta_a)$.*

Proof. If a is irreducible this is already proven.
Let a be reducible and let $\Theta_a = \bar\iota\iota$ be the Q-system with inclusion $\iota(\mathcal{A}(I)) \subset \mathcal{B}_a(I)$. Let $\{p_i\}_{i=1}^n$ be a set of minimal projections in $\iota(\mathcal{A}(I))' \cap \mathcal{B}_a(I) = \mathrm{Hom}(\iota,\iota)$ with $\sum_{i=1}^n p_i = 1$ with corresponding morphisms $\iota_i \prec \iota$. By the usually Reeh–Schlieder argument, the projection do not depend on the choice of I. The inclusion $\iota(\mathcal{A}(I)) \subset \mathcal{B}_a(I)$ is conjugated to

$$\left\{\begin{pmatrix}\iota_1(a) & & \\ & \ddots & \\ & & \iota_n(a)\end{pmatrix} : a \in \mathcal{A}(I)\right\} \subset \mathcal{B}_a(I)\otimes M_n(\mathbb{C}) \cong \mathcal{B}_a(I)\,.$$

With the same notation $\mathcal{A}_+(O) \subset \mathcal{B}_{a,+}(O)$ is conjugated to:

$$\left\{\begin{pmatrix}\iota_1(a) & & \\ & \ddots & \\ & & \iota_n(a)\end{pmatrix} : a \in \mathcal{A}_+(O)\right\} \subset \left\{\begin{pmatrix}b & & \\ & \ddots & \\ & & b\end{pmatrix} : b \in \mathcal{B}_{a,+}(O)\right\}. \tag{10}$$

Because $\Theta_2 := Z(\Theta_a)$ and $Z(\Theta_{\bar\iota_i})$ are equivalent (by Prop. 4.19) every $\mathcal{B}_{i,+} \supset \mathcal{A}_+$ is a boundary condition for $\mathcal{B}_2 \supset \mathcal{A}_2$. But then also the inclusion $\mathcal{B}_2 \supset \mathcal{A}_2$ is locally isomorphic to $\mathcal{B}_{a,+} \supset \mathcal{A}_+$ by (10) and the isomorphism restricted to $\mathcal{A}_2$ gives a local isomorphism of $\mathcal{A}_2$ restricted to $\mathbb{M}_+$ and $\mathcal{A}_+$. □

Note that in the reducible case the vacuum Ω of $\mathcal{B}_+$ is neither cyclic nor unique and that $\Omega = \sum_{i=1}^n \Omega_i$ with $\Omega_i = p_i\Omega$. The restriction of $\mathcal{B}_+$ to the subspace $\overline{\mathcal{B}_+(O)\Omega_i}$ is unitarily equivalent to the boundary condition coming from ι_i. In other words, ${}_N\mathcal{C}_M \ni a \mapsto \mathcal{B}_{a,+}$ maps direct sums of sectors to direct sums of boundary conditions.

Example 6.14. Consider $a, b \in {}_N\mathcal{C}_M$ irreducible and mutually inequivalent but related by an automorphism $\beta \in {}_M\mathcal{C}_M$, or equivalently $\Theta_a \cong \Theta_b$. This means the boundary conditions coming from a and b are the same, but for example the boundary conditions coming from $c := a \oplus a$ and $d := a \oplus b$ are different. This can be seen for example by regarding the relative commutants of the subfactors associated with Θ_c and Θ_d, namely $\bar c(N)' \cap N \cong \mathbb{C}\oplus\mathbb{C}$, while $\bar d(N)' \cap N \cong M_2(\mathbb{C})$.

6.6 Adding the boundary

In [CKL13] a purely operator algebraic construction of all boundary conditions is given. As a result a boundary net is obtained which is the direct sum of all boundary conditions.
Let us consider the inclusion

$$\mathcal{A}(I) \otimes \mathcal{A}(J) \subset \mathcal{B}_2(O)$$

for some fixed $O = I \times J \Subset W$ and let Θ_2 be the associated Q-system in $\mathrm{DHR}^I(\mathcal{A}) \boxtimes \overline{\mathrm{DHR}^J(\mathcal{A})}$. Let Ω be the vacuum in $\mathcal{H}_\mathcal{A}$ and let us define the state $\varphi_0(x \otimes y) = (\Omega, xy\Omega)$ for $x \in \mathcal{A}(I)$, $y \in \mathcal{A}(J)$ and let $\varepsilon_O \colon \mathcal{B}_2(O) \to \mathcal{A}_2(O) \cong \mathcal{A}_+(O)$ be the conditional expectation. This gives a state $\varphi = \varphi_0 \circ \varepsilon_0$ on $\mathcal{B}_2(O)$ (which can be extended to a state on $\mathfrak{A}_2(W)$). Using the GNS representation one get an inclusion $\mathcal{A}_+(O) \subset \mathcal{B}_+(O)$ on a bigger Hilbert space and which is by construction isomorphic to $\mathcal{A}_2(O) \subset \mathcal{B}_2(O)$. This construction extends to $\mathfrak{A}_2(W)$ and gives a (reducible) boundary net $\{\mathcal{B}_+(O)\}_{O \in \mathcal{K}_+}$. Let us define $\mathcal{B}(I) = \bigvee_{\mathcal{K}_+ \ni O \subset W(I)} \mathcal{B}_+(O)$ where $W(I)$ is the left wedge such that its intersection with the time axis $x = 0$ is equals I. This gives a non-local extension $\mathcal{B} \supset \mathcal{A}$. Let us fix $L \supset I \cup J$, then the Q-system of $\mathcal{B}(L) \supset \mathcal{A}(L)$ can be chosen to be localized in $I \cup J$ and it can be in particular trivially extended from the inclusion $\mathcal{A}_+(O) \subset \mathcal{B}_+(O)$ using strong additivity. Let's denote its Q-system by $\tilde{\Theta}$.

Proposition 6.15. *Let $\mathcal{B}_2 \supset \mathcal{A}_2$ be a local irreducible extension with Q-system Θ_2. The Q-system of the inclusion $\mathcal{A}(I) \subset \mathcal{B}(I)$, where $\mathcal{B} = \mathcal{B}_+^{\mathrm{gen}}$ and $\mathcal{B}_+$ is obtained by adding the boundary is equivalent to the Q-system $T(\Theta_2)$.*

Proof. We have to show that $\tilde{\Theta}$ is equivalent to $T(\Theta_2)$, where we see Θ_2 as a Q-system by the equivalence ${}_N\mathcal{C}_N \boxtimes \overline{{}_N\mathcal{C}_N} \cong \mathrm{DHR}^O(\mathcal{A}_2)$.
An endomorphism $\rho^I \boxtimes \bar{\sigma}^J$ gives an endomorphism $\rho^I \bar{\sigma}^J \in \mathrm{End}(\mathcal{A}(I) \vee \mathcal{A}(J))$ and this gives actually an isomorphism of tensor categories

$$\mathrm{End}(\mathcal{A}(I) \otimes \mathcal{A}(J)) \cong \mathrm{End}(\mathcal{A}(I) \vee \mathcal{A}(J))\,.$$

Starting from an object in $\mathrm{DHR}^O(\mathcal{A}_2)$ the image is a localized endomorphism of $\mathcal{A}(I) \vee \mathcal{A}(J)$ which can by strong additivity be extended to a localized endomorphism of $\mathrm{End}(\mathcal{A}(L))$, so we get a tensor functor

$$\tilde{T} \colon \mathrm{DHR}^I(\mathcal{A}_2) \to \mathrm{DHR}^L(\mathcal{A}) \equiv {}_N\mathcal{C}_N$$

where we choose $N := \mathcal{A}(L)$ and ${}_N\mathcal{C}_N = \mathrm{DHR}^L(\mathcal{A})$. We note that the μ from (7) is trivial as is $\varepsilon(\rho_2, \bar{\sigma}_1)$ because of the order of localization.
So the functor

$${}_N\mathcal{C}_N \boxtimes \overline{{}_N\mathcal{C}_N} \cong \mathrm{DHR}^O(\mathcal{A}_2) \to \mathrm{DHR}^L(\mathcal{A}) \equiv {}_N\mathcal{C}_N$$

is by construction equivalent to the tensor T from Subsec. 4.2 and, in particular $\tilde{\Theta}$ is equivalent to $T(\Theta_2)$. ∎

This gives as an alternative proof of Prop. 6.11. Let us assume $\mathcal{B}_2$ was modular invariant/maximal. All boundary conditions are obtained by the adding the boundary construction, and by Prop. 4.20 we can conclude:

Corollary 6.16. *All boundary conditions of $\mathcal{B}_2$ come from an $a \in {}_N\Delta_M$, where $N = \mathcal{A}(I)$, $M = \mathcal{B}(I)$, ${}_N\mathcal{C}_N = \mathrm{DHR}^I(\mathcal{A})$ and $\mathcal{B} \subset \mathcal{A}$ is any (non-local) extension giving $\mathcal{B}_2$ by the α-induction construction.*

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