Minimax Principles, Hardy-Dirac Inequalities, and Operator Cores for Two and Three Dimensional Coulomb-Dirac Operators

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Abstract. For $n \in \{2, 3\}$ we prove minimax characterisations of eigenvalues in the gap of the n dimensional Dirac operator with an potential, which may have a Coulomb singularity with a coupling constant up to the critical value $1/(4-n)$. This result implies a so-called Hardy-Dirac inequality, which can be used to define a distinguished self-adjoint extension of the Coulomb-Dirac operator defined on $\mathsf{C}_0^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{2(n-1)})$, as long as the coupling constant does not exceed $1/(4-n)$. We also find an explicit description of an operator core of this operator.

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1 Introduction

The relativistic dynamics of an electron moving in an atomic field is described by a Dirac operator with potential V having a Coulomb singularity. Since we want to consider such Dirac operators in two and three dimensions simultaneously, we assume throughout the text that $n \in \{2, 3\}$. In n dimensions the relativistic electron corresponds to a $2(n-1)$ component spinor and V is a $2(n-1) \times 2(n-1)$ hermitian matrix function on $\mathbb{R}^n$. We say that V belongs to $\mathfrak{P}_n$ if for some $\nu \in [0, 1/(4-n))$ the inequality $0 \geq V \geq -\nu/|\cdot| \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}}$ holds.

This motivates the following question. Does the Dirac operator with potential $V \in \mathfrak{P}_n \cup \{-1/\big((4-n)|\cdot|\big) \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}}\}$

$$\tilde{D}_n(V) := \begin{cases} -\mathrm{i}\boldsymbol{\sigma}\cdot\nabla + \sigma_3 + V \text{ if } n=2 \\ -\mathrm{i}\boldsymbol{\alpha}\cdot\nabla + \beta + V \text{ if } n=3 \end{cases} \text{ defined on } \mathsf{C}_0^\infty(\mathbb{R}^n\setminus\{0\};\mathbb{C}^{2(n-1)}), \tag{1}$$

have a unique self-adjoint extension? In (1) are $\boldsymbol{\sigma} = (\sigma_1, \sigma_2), \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ vectors; $\sigma_1, \sigma_2, \sigma_3$ the standard Pauli matrices; $\boldsymbol{\alpha}_i = \begin{pmatrix} 0_{\mathbb{C}^2} & \sigma_i \\ \sigma_i & 0_{\mathbb{C}^2}\end{pmatrix}$ for $i \in \{1,2,3\}$ and $\beta = \begin{pmatrix} \mathbb{I}_{\mathbb{C}^2} & 0_{\mathbb{C}^2} \\ 0_{\mathbb{C}^2} & -\mathbb{I}_{\mathbb{C}^2}\end{pmatrix}$. It is the uniqueness not the existence of a self-adjoint extension that is doubtful. For example the Coulomb-Dirac operator $\tilde{D}_n(-\nu/|\cdot| \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}})$ is essentially self-adjoint if $n=2, \nu=0$ or $n=3, \nu \in [0, \sqrt{3}/2]$ but for $n=2, \nu \in (0,1/2]$ or $n=3, \nu \in (\sqrt{3}/2, 1]$ there are infinitely many self-adjoint extensions (see Lemma 14). Thus it is also natural to ask, whether there is a physically distinguished self-adjoint extension? In fact for $V \in \mathfrak{P}_n$ there is a unique self-adjoint extension $D_n(V)$ of $\tilde{D}_n(V)$, for which the wave functions in its domain possess finite mean kinetic energy, i.e. $\mathfrak{D}\big(D_n(V)\big) \subset \mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{2(n-1)})$. The existence of this distinguished self-adjoint extension is proven in Section 3. There we apply some general results developed in [15]. Note that for $\nu \in \big[0, 1/(4-n)\big)$ the domain of the Coulomb-Dirac operator $D_n(-\nu/|\cdot| \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}})$ is contained in $\mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{2(n-1)})$ and for $\tilde{D}_n\big(((n-4)|\cdot|)^{-1} \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}}\big)$ there is no self-adjoint extension with this property. In this sense $1/(4-n)$ is a critical constant. At this point we want to mention that in the context of Theorem 5 we define a distinguished self-adjoint extension of $\tilde{D}_n(-\nu/|\cdot| \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}})$ for $\nu \in [0, 1/(4-n)]$, i.e. the case of a Coulomb potential with the critical coupling constant $1/(4-n)$ is in particular included here.

Let $V \in \mathfrak{P}_n$. As in Proposition 1 in [4] one can prove that there is a gap in the essential spectrum of $D_n(V)$. To be more precise

$$\sigma_{\mathrm{ess}}\big(D_n(V)\big) = (-\infty, -1] \cup [1, \infty).$$

In 1986 James D. Talman proposed in [17] a formal minimax characterisation of the lowest eigenvalue in the gap of the essential spectrum of the operator $D_3(V)$. In this work we prove a minimax characterisation of eigenvalues in the gap of $D_3(V)$ in the spirit of Talman and an analogous result for $D_2(V)$. The exact result is:

THEOREM 1 (Talman minimax principle). *Let $V \in \mathfrak{P}_n$. If the k^{th} eigenvalue μ_k of $D_n(V)$ in $(-1,1)$, counted from below with multiplicity, exists, then it is given by*

$$\mu_k = \inf_{\substack{\mathfrak{M} \subset \mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{n-1}) \\ \dim \mathfrak{M} = k}} \sup_{\psi \in (\mathfrak{M} \oplus \mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{n-1}))\setminus\{0\}} \frac{\mathsf{d}_n[\psi] + \mathsf{v}[\psi]}{\|\psi\|^2}.$$

Here $\mathtt{d}_n$ *and* $\mathtt{v}$ *are the quadratic forms associated to the operators* $D_n(0)$ *and* V*.*

About Theorem 1 we want to remark that for $n = 3$ there is an historical overview of results of the same type in [13] and that for $n = 2$ there is no comparable result known. Moreover, Theorem 1 improves in the three dimensional case Theorem 3 in [13], which is the best known result for a Dirac operator with an electrostatic potential having strong Coulomb singularity.
Furthermore, we give a different proof of the Esteban-Séré minimax principle (see Theorem 2 in [13] and [9]) and prove an analogous result for two dimensional Dirac operators:

Theorem 2 (Esteban-Séré minimax principle). *Let* $V \in \mathfrak{P}_n$*. If the* k^{th} *eigenvalue* μ_k *of* $D_n(V)$ *in* $(-1,1)$*, counted from below with multiplicity, exists, then it is given by*

$$\mu_k = \inf_{\substack{\mathfrak{M}\subset P_n^+ \mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{2(n-1)}) \\ \dim \mathfrak{M}=k}} \ \sup_{\psi\in(\mathfrak{M}\oplus P_n^- \mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{2(n-1)}))\setminus\{0\}} \frac{\mathtt{d}_n[\psi]+\mathtt{v}[\psi]}{\|\psi\|^2}.$$

Here P_n^+ *is the projector on the non-negative spectral subspace of* $D_n(0)$ *and* $P_n^- := \mathbb{I} - P_n^+$*.*

A direct application of Theorem 1 is:

Theorem 3 (Hardy-Dirac inequality). *Let* v *be a scalar function on* $\mathbb{R}^n$ *such that* $v \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}} \in \mathfrak{P}_n$*. Moreover, we define the operator:*

$$K_n := \begin{cases} -\mathrm{i}\partial_1 - \partial_2 \ \textit{if}\ n = 2, \\ -\mathrm{i}\boldsymbol{\sigma}\cdot\nabla \ \textit{if}\ n = 3, \end{cases}$$

and denote by $\lambda(v)$ *the smallest eigenvalue of* $D_n(v\otimes\mathbb{I}_{\mathbb{C}^{2(n-1)}})$ *in the gap* $(-1,1)$*. Then for all* $\varphi \in \mathsf{H}^1(\mathbb{R}^n;\mathbb{C}^{n-1})$ *the inequality*

$$0 \le \int_{\mathbb{R}^n} \frac{|K_n\varphi(\mathbf{x})|^2}{1+\lambda(v)-v(\mathbf{x})}\mathrm{d}\mathbf{x} + \int_{\mathbb{R}^n} \big(1-\lambda(v)+v(\mathbf{x})\big)|\varphi(\mathbf{x})|^2\mathrm{d}\mathbf{x} \qquad (2)$$

holds.

We follow the tradition of [5] and call these type of inequality Hardy-Dirac inequality. In [6] it is demonstrated, how one can prove Hardy-Dirac inequalities for $n = 3$ with the help of the Talman minimax principle.
We know that the lowest eigenvalue of $D_n(-\nu/|\cdot|\otimes\mathbb{I}_{\mathbb{C}^{2(n-1)}})$ in $(-1,1)$ is $\sqrt{1-\big((4-n)\nu\big)^2}$ for $\nu \in \big(0, 1/(4-n)\big)$ (see [7] and [19]). Thus Theorem 3 implies with a simple limiting argument

Corollary 4. *Let* $\nu \in [0, 1/(4-n)]$. *Then*

$$0 \leq \int_{\mathbb{R}^n} \left(\frac{|K_n\varphi|^2}{1+\sqrt{1-\big((4-n)\nu\big)^2}+\frac{\nu}{|x|}} + \left(1-\sqrt{1-\big((4-n)\nu\big)^2}-\frac{\nu}{|x|}\right)|\varphi|^2 \right) \mathrm{d}\mathbf{x}$$

holds for all $\varphi \in \mathsf{H}^1(\mathbb{R}^n;\mathbb{C}^{n-1})$.

Let $\nu \in \big[0, 1/(4-n)\big]$. With the help of Corollary 4 and Theorem 1 in [8] ($\tilde{D}_n(-\nu/|\cdot| \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}})$ corresponds to H there) we know that there is only one self-adjoint extension of $\tilde{D}_n(-\nu/|\cdot| \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}})$ with a positive Schur complement. We denote this distinguished self-adjoint extension by D_n^ν. Now we want to give an explicit description of an operator core of D_n^ν. For this purpose we introduce polar and spherical coordinates. We denote by the coordinate pair $(\rho, \vartheta) \in [0,\infty) \times [0, 2\pi)$ the radial and angular polar coordinates in $\mathbb{R}^2$ and by the coordinate triplet $(r, \theta, \phi) \in [0,\infty) \times [0,\pi) \times [0, 2\pi)$ the radial, inclination and azimuthal spherical coordinates in $\mathbb{R}^3$. For $m \in \{-1/2, 1/2\}^{n-1}$ we define the function $\zeta_{n,m}^\nu$ in polar coordinates for $n = 2$

$$\zeta_{2,m}^\nu(\rho,\vartheta) := \xi(\rho)\rho^{\sqrt{1/4-\nu^2}-1/2} \begin{pmatrix} \nu\frac{\mathrm{e}^{-\mathrm{i}(1/2+m)\vartheta}}{\sqrt{2\pi}} \\ -\mathrm{i}\big(\sqrt{1/4-\nu^2}+(-1)^{1/2-m}/2\big)\frac{\mathrm{e}^{\mathrm{i}(1/2-m)\vartheta}}{\sqrt{2\pi}} \end{pmatrix}; \tag{3}$$

and in spherical coordinates for $n = 3$

$$\zeta_{3,m}^\nu(r,\theta,\phi) := \xi(r) r^{\sqrt{1-\nu^2}-1} \begin{pmatrix} \nu\Omega_{\frac{1}{2}+m_2, m_1, -m_2}(\theta,\phi) \\ -\mathrm{i}\big(\sqrt{1-\nu^2}+(-1)^{\frac{1}{2}-m_2}\big)\Omega_{\frac{1}{2}-m_2, m_1, m_2}(\theta,\phi) \end{pmatrix}; \tag{4}$$

with the spherical spinor $\Omega_{l,m,s}$ (see Relation (7) in [10]) and the smooth cut-off function ξ (i.e., $\xi \in \mathsf{C}^\infty(\mathbb{R}_+;\mathbb{R}_+)$, $\xi(t) = 1$ for $t \in (0,1)$ and $\xi(t) = 0$ for $t > 2$). In the next theorem we give a characterisation of an operator core of D_n^ν with the help of the functions $\zeta_{n,m}^\nu$ introduced in (3) and (4).

Theorem 5 (Operator core). *Let* $\nu \in \big[0, 1/(4-n)\big]$. *The set*

$$\mathfrak{C}_n^\nu := \mathsf{C}_0^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{2(n-1)}) \dot{+} \begin{cases} \{0\}, & \text{if } n=2,\ \nu=0 \text{ or } n=3,\ \nu \in \big[0, \frac{\sqrt{3}}{2}\big]; \\ \operatorname{span}\{\zeta_{n,m}^\nu : m \in \{-1/2,1/2\}^{n-1}\}, & \text{else}; \end{cases} \tag{5}$$

is an operator core for D_n^ν.

The knowledge of the operator core of D_n^ν is important for the proof of estimates on the square of the operator, see e.g. [14]. In Remark 15 we show that for $\nu \in \big[0, 1/(4-n)\big)$ the set $\mathfrak{C}_n^\nu$ is an operator core for $D_n(-\nu/|\cdot| \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}})$. A direct consequence is:

Corollary 6. *Let $\nu \in [0, 1/(4-n))$. The distinguished self-adjoint extensions of $\tilde{D}_n(-\nu/|\cdot| \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}})$ in the sense of [15] and [8] coincide, i.e.,*

$$D_n^\nu = D_n(-\nu/|\cdot| \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}}).$$

The proofs of the minimax characterisations rely on the angular momentum channel decomposition of the Coulomb-Dirac operator in the momentum space. This representation and the corresponding unitary transformations are introduced in the next section. In the remaining sections we prove in the order of enumeration: Theorems 1, 2, 3 and 5.

2 Angular momentum channel decomposition in the momentum space

The Fourier transform connects the quantum mechanical descriptions of a particle in the configuration and momentum space. We use the standard unitary Fourier transform $\mathcal{F}_n$ in $\mathsf{L}^2(\mathbb{R}^n)$ given for $\varphi \in \mathsf{L}^1(\mathbb{R}^n) \cap \mathsf{L}^2(\mathbb{R}^n)$ by

$$\mathcal{F}_n\varphi := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{-\mathrm{i}\langle \cdot, \mathbf{x}\rangle} \varphi(\mathbf{x}) \mathrm{d}\mathbf{x}. \tag{6}$$

For the angular momentum channel decomposition in n dimensions we use an orthonormal basis in $\mathsf{L}^2(\mathbb{S}^{n-1}; \mathbb{C}^{n-1})$. For $n = 2$ this orthonormal basis is $\left((2\pi)^{-1/2}\mathrm{e}^{\mathrm{i}m(\cdot)}\right)_{m\in\mathbb{Z}}$. In three dimensions we use spherical spinors $\Omega_{l,m,s}$, which are defined in Relation (7) in [10], with $l \in \mathbb{N}_0$, $m \in \{-l-1/2, \ldots, l+1/2\}$ and $s \in \{-1/2, 1/2\}$. The corresponding index sets are denoted by

$$\mathfrak{T}_2 := \mathbb{Z}; \tag{7}$$

and

$$\mathfrak{T}_3 := \left\{(l, m, s) : l \in \mathbb{N}_0, m \in \left\{-l - \frac{1}{2}, \ldots, l + \frac{1}{2}\right\}, s = \pm\frac{1}{2}, \Omega_{l,m,s} \neq 0\right\}. \tag{8}$$

Furthermore, we define subsets $\mathfrak{T}_n^\pm$ of $\mathfrak{T}_n$:

$$\mathfrak{T}_n^a := \begin{cases} 2\mathbb{Z} & \text{if } n = 2,\ a = +; \\ 2\mathbb{Z}+1 & \text{if } n = 2,\ a = -; \\ \{(l, m, s) \in \mathfrak{T}_3 : s = \pm 1/2\} & \text{if } n = 3,\ a = \pm. \end{cases} \tag{9}$$

Note that if $(l, m, -1/2) \in \mathfrak{T}_3^-$ then $l \in \mathbb{N}$.
Moreover, we introduce bijective maps

$$T_2 : \mathfrak{T}_2 \to \mathfrak{T}_2,\ T_2 k := k + 1; \tag{10}$$

and

$$T_3 : \mathfrak{T}_3 \to \mathfrak{T}_3,\ T_3(l,m,s) := (l+2s, m, -s). \tag{11}$$

We can represent any $\varphi \in \mathsf{L}^2(\mathbb{R}^2;\mathbb{C})$ in polar coordinates and $\zeta \in \mathsf{L}^2(\mathbb{R}^3;\mathbb{C}^2)$ in spherical coordinates as

$$\varphi(\rho,\vartheta) = \sum_{k\in\mathfrak{T}_2} (2\pi\rho)^{-1/2}\varphi_k(\rho)\mathrm{e}^{\mathrm{i}k\vartheta}; \tag{12}$$

$$\zeta(r,\theta,\phi) = \sum_{(l,m,s)\in\mathfrak{T}_3} r^{-1}\zeta_{(l,m,s)}(r)\Omega_{l,m,s}(\theta,\phi); \tag{13}$$

with

$$\varphi_k(\rho) := \sqrt{\frac{\rho}{2\pi}}\int_0^{2\pi} \varphi(\rho,\vartheta)\mathrm{e}^{-\mathrm{i}k\vartheta}\mathrm{d}\vartheta; \tag{14}$$

$$\zeta_{(l,m,s)}(r) := r\int_0^{2\pi}\int_0^{\pi} \big\langle \Omega_{l,m,s}(\theta,\phi), \zeta(r,\theta,\phi)\big\rangle_{\mathbb{C}^2} \sin(\theta)\mathrm{d}\theta\mathrm{d}\phi. \tag{15}$$

With the help of (14) and (15) we define the unitary operator

$$\mathcal{U}_n : \mathsf{L}^2(\mathbb{R}^n;\mathbb{C}^{n-1}) \to \bigoplus_{j\in\mathfrak{T}_n} \mathsf{L}^2(\mathbb{R}_+); \quad \psi \mapsto \bigoplus_{j\in\mathfrak{T}_n} \psi_j. \tag{16}$$

For the proof of the following lemma see Theorem 2.2.5 in [1] (based on Lemmata 2.1, 2.2 of [2]) for $n=2$ and Section 2.2 in [1] for $n=3$.

Lemma 7. *For $j \in \big(\mathbb{N}_0/2 - 1/2\big)$ and $z \in (1,\infty)$ let*

$$Q_j(z) = 2^{-j-1}\int_{-1}^{1} (1-t^2)^j (z-t)^{-j-1}\,\mathrm{d}t \tag{17}$$

be a Legendre function of the second kind (see Section 15.3 in [21]). Let the sesquilinear form q_j be defined on $\mathsf{L}^2\big(\mathbb{R}_+,(1+p^2)^{1/2}\mathrm{d}p\big)\times\mathsf{L}^2\big(\mathbb{R}_+,(1+p^2)^{1/2}\mathrm{d}p\big)$ by

$$\mathsf{q}_j[f,g] := \pi^{-1}\int_0^\infty\int_0^\infty \overline{f(p)}Q_j\Big(\frac{1}{2}\Big(\frac{q}{p}+\frac{p}{q}\Big)\Big)g(q)\,\mathrm{d}q\,\mathrm{d}p. \tag{18}$$

For the special case $f=g$ we introduce $\mathsf{q}_j[f] := \mathsf{q}_j[f,f]$.
Then for every $\zeta,\eta \in \mathsf{H}^{1/2}(\mathbb{R}^n)$ the relation

$$\int_{\mathbb{R}^n} \frac{\overline{\zeta}(\mathbf{x})\cdot\eta(\mathbf{x})}{|\mathbf{x}|}\mathrm{d}\mathbf{x} = \begin{cases} \sum_{k\in\mathfrak{T}_2} \mathsf{q}_{|k|-1/2}\big[(\mathcal{F}_2\zeta)_k,(\mathcal{F}_2\eta)_k\big] & \text{if } n=2,\\ \sum_{(l,m,s)\in\mathfrak{T}_3} \mathsf{q}_l\big[(\mathcal{F}_3\zeta)_{(l,m,s)},(\mathcal{F}_3\eta)_{(l,m,s)}\big] & \text{if } n=3, \end{cases} \tag{19}$$

holds.

The operators $-\mathrm{i}\boldsymbol{\sigma}\cdot\nabla$ and $-\mathrm{i}\boldsymbol{\alpha}\cdot\nabla$ are partially diagonalised in the momentum space by the unitary transforms

$$\mathcal{W}_2 : \mathsf{L}^2(\mathbb{R}^2;\mathbb{C}^2) \to \bigoplus_{k\in\mathfrak{T}_2} \mathsf{L}^2(\mathbb{R}_+;\mathbb{C}^2); \quad \begin{pmatrix}\varphi\\ \psi\end{pmatrix} \mapsto \bigoplus_{k\in\mathfrak{T}_2}\begin{pmatrix}\varphi_k\\ \psi_{T_2 k}\end{pmatrix} \tag{20}$$

and

$$\mathcal{W}_3 : \mathsf{L}^2(\mathbb{R}^3;\mathbb{C}^4) \to \bigoplus_{(l,m,s)\in\mathfrak{T}_3} \mathsf{L}^2(\mathbb{R}_+;\mathbb{C}^2); \quad \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} \mapsto \bigoplus_{(l,m,s)\in\mathfrak{T}_3}\begin{pmatrix}\psi^+_{(l,m,s)}\\ \psi^-_{T_3(l,m,s)}\end{pmatrix} \tag{21}$$

with

$$\psi^+_{(l,m,s)} := \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_{(l,m,s)} \text{ and } \psi^-_{(l,m,s)} := \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}_{(l,m,s)} \tag{22}$$

for $(l,m,s)\in\mathfrak{T}_3$. To be more precise:

Lemma 8. *For the self-adjoint operators $-\mathrm{i}\boldsymbol{\sigma}\cdot\nabla$ and $-\mathrm{i}\boldsymbol{\alpha}\cdot\nabla$ the relations*

$$(\mathcal{W}_n\mathcal{F}_n)^*\left(\bigoplus_{j\in\mathfrak{T}_n}\begin{pmatrix}0 & (\cdot)\\ (\cdot) & 0\end{pmatrix}\right)(\mathcal{W}_n\mathcal{F}_n) = \begin{cases}-\mathrm{i}\boldsymbol{\sigma}\cdot\nabla & \text{if } n=2,\\ -\mathrm{i}\boldsymbol{\alpha}\cdot\nabla & \text{if } n=3,\end{cases} \tag{23}$$

hold.

Proof. By a straightforward calculation and Relation 2.1.28 in [1] the relations

$$\boldsymbol{\sigma}\cdot\mathbf{x} = \begin{pmatrix}0 & \mathrm{e}^{-\mathrm{i}\vartheta}\rho\\ \mathrm{e}^{\mathrm{i}\vartheta}\rho & 0\end{pmatrix} \text{ for } \mathbf{x}\in\mathbb{R}^2; \tag{24}$$

$$\boldsymbol{\sigma}\cdot\frac{\mathbf{x}}{|\mathbf{x}|}\Omega_{l,m,s} = \Omega_{l+2s,m,-s} \text{ for } \mathbf{x}\in\mathbb{R}^3 \text{ and } (l,m,s)\in\mathfrak{T}_3; \tag{25}$$

hold.

The set $\mathsf{C}_0^\infty(\mathbb{R}^n;\mathbb{C}^{2(n-1)})$ is dense in $\mathsf{H}^1(\mathbb{R}^n;\mathbb{C}^{2(n-1)})$. Thus it is enough to work with $\psi\in\mathsf{C}_0^\infty(\mathbb{R}^2;\mathbb{C}^2)$ and $\zeta\in\mathsf{C}_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$.

Moreover, the Fourier transform diagonalises differential operators:

$$\langle\psi,-\mathrm{i}\boldsymbol{\sigma}\cdot\nabla\psi\rangle = \langle\mathcal{F}_2\psi,\boldsymbol{\sigma}\cdot\boldsymbol{p}\mathcal{F}_2\psi\rangle, \tag{26}$$

$$\langle\zeta,-\mathrm{i}\boldsymbol{\alpha}\cdot\nabla\zeta\rangle = \langle\mathcal{F}_3\zeta,\boldsymbol{\alpha}\cdot\boldsymbol{p}\mathcal{F}_3\zeta\rangle. \tag{27}$$

Here we denote by $\boldsymbol{p}$ the independent variable of multiplication in $\mathsf{L}^2(\mathbb{R}^n;\mathrm{d}\mathbf{p})$. Now we prove (23) for $n=3$. We obtain by the representation (13) of the upper and lower bispinor of $\mathcal{F}_3\zeta$ and the notation introduced in (22) that the right hand side of (27) is equal to

$$2\sum_{\substack{(l',m',s')\in\mathfrak{T}_3\\ (l,m,s)\in\mathfrak{T}_3}} \Re\left(\left\langle|\boldsymbol{p}|^{-1}\left(\mathcal{F}_3\zeta\right)^+_{(l',m',s')}\Omega_{l',m',s'},(\boldsymbol{\sigma}\cdot\boldsymbol{p})|\boldsymbol{p}|^{-1}\left(\mathcal{F}_3\zeta\right)^-_{(l,m,s)}\Omega_{l,m,s}\right\rangle\right). \tag{28}$$

The expression in (28) is equal to

$$\begin{aligned} &2 \sum_{(l,m,s)\in\mathfrak{T}_3} \Re\Big(\big\langle (\mathcal{F}_3\zeta)^+_{(l+2s,m,-s)}, (\cdot)(\mathcal{F}_3\zeta)^-_{(l,m,s)}\big\rangle\Big) \\ &= \sum_{(l,m,s)\in\mathfrak{T}_3} \left\langle \begin{pmatrix} (\mathcal{F}_3\zeta)^+_{(l,m,s)} \\ (\mathcal{F}_3\zeta)^-_{T_3(l,m,s)} \end{pmatrix}, \begin{pmatrix} 0 & (\cdot) \\ (\cdot) & 0 \end{pmatrix} \begin{pmatrix} (\mathcal{F}_3\zeta)^+_{(l,m,s)} \\ (\mathcal{F}_3\zeta)^-_{T_3(l,m,s)} \end{pmatrix} \right\rangle \\ &= \left\langle \mathcal{W}_3\mathcal{F}_3\zeta, \left(\bigoplus_{(l,m,s)\in\mathfrak{T}_3} \begin{pmatrix} 0 & (\cdot) \\ (\cdot) & 0 \end{pmatrix} \right) \mathcal{W}_3\mathcal{F}_3\zeta \right\rangle \qquad (29) \end{aligned}$$

by the sequential application of (25), (21) and (6). Thus the claim of Lemma 8 is a consequence of (27), (28) and (29).
For $n = 2$ we obtain (23) by an analogous procedure, i.e., we represent the upper and lower component of $\mathcal{F}_2\psi$ by (12) in (26) and perform a calculation, which involves (24). □

3 Proof of Theorem 1

Let $V \in \mathfrak{P}_n$. We use the abstract minimax principle Theorem 1 of [13] to prove the Talman minimax principle. We apply the theorem with $q := \mathsf{d}_n$ (quadratic form associated to $D_n(0)$), $B := D_n(V)$ and $\Lambda_\pm$ as the projector $T_n^\pm$ on the upper and lower $(n-1)$ components of a $2(n-1)$ spinor, i.e.,

$$T_n^+ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \varphi \\ 0 \end{pmatrix}, \qquad T_n^- \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \psi \end{pmatrix}, \text{ for } \varphi, \psi \in \mathsf{L}^2(\mathbb{R}^n; \mathbb{C}^{n-1}).$$

That $D_n(V)$ plays the role of B in [13] is a consequence of Theorem 2.1 in [15] and the following lemma.

Lemma 9. *Let $V \in \mathfrak{P}_n$. Then the quadratic form $\mathtt{v}$ associated to the operator V is a form perturbation of $D_n(0)$ in the sense of Definition 2.1 in [15].*

Proof. V is $D_n(0)$ form bounded by the Herbst inequality (see Theorem 2.5 in [11]). Moreover, the inequality

$$\|r^{-1/2} D_n(0)^{-1} r^{-1/2}\| \leq 4 - n$$

holds. This is proven in Section 2 in [12] for $n = 3$. The same arguments also apply for $n = 2$ (see Step 1 in the proof of Theorem 1 in [4]). Thus

$$\|V^{1/2} D_n(0)^{-1} V^{1/2}\| \leq \|V^{1/2} r^{1/2}\|^2 \cdot \|r^{-1/2} D_n(0)^{-1} r^{-1/2}\| < 1.$$

Hence $1 + V^{1/2} D_n(0)^{-1} V^{1/2}$ has a bounded inverse by the Neumann series. Now the claim follows from Theorem 2.2 in [15] with $A := D_n(0)$ and $t := 0$. □

Since the assumptions *(i)* and *(ii)* of Theorem 1 in [13] are obviously fulfilled, it remains to check assumption *(iii)*. Thus it is enough to find an operator $L_n : \mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{n-1}) \to \mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{n-1})$ such that

$$\inf_{\varphi\in\mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{n-1})\setminus\{0\}} \frac{\mathtt{d}_n\left[\binom{\varphi}{L_n\varphi}\right] + \mathtt{v}\left[\binom{\varphi}{L_n\varphi}\right]}{\left\|\binom{\varphi}{L_n\varphi}\right\|^2} > -1.$$

Now we give in three steps an explicit construction of L_n and show that L_n satisfies the requirements. For $k \in \mathfrak{T}_2$ and $(l,m,s) \in \mathfrak{T}_3$ we define in the first step various constants:

$$c_n := 2(4-n)\frac{\Gamma(\frac{n+1}{4})^2}{\Gamma(\frac{n-1}{4})^2}; \tag{30}$$

$$c_{2,k} := \begin{cases} c_2^{-1} \text{ if } k \in \mathfrak{T}_2^-, \\ c_2 \text{ if } k \in \mathfrak{T}_2^+; \end{cases} \tag{31}$$

$$c_{3,(l,m,s)} := c_3^{2s}. \tag{32}$$

In the second step we define the operator R_n

$$R_n : \bigoplus_{j\in\mathfrak{T}_n} \mathsf{L}^2(\mathbb{R}_+) \to \bigoplus_{j\in\mathfrak{T}_n} \mathsf{L}^2(\mathbb{R}_+);\ \bigoplus_{j\in\mathfrak{T}_n} \psi_j \mapsto \bigoplus_{j\in\mathfrak{T}_n} c_{n,j}\psi_{T_n^{-1}j}. \tag{33}$$

Finally we define

$$L_n := (\mathcal{U}_n\mathcal{F}_n)^* R_n (\mathcal{U}_n\mathcal{F}_n). \tag{34}$$

The desired properties of L_n are proven in the following lemma:

Lemma 10. *Let $\varphi \in \mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{n-1})$ then $L_n\varphi \in \mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{n-1})$ and the following inequality*

$$\frac{c_n^2-1}{c_n^2+1}\left\|\begin{pmatrix}\varphi\\ L_n\varphi\end{pmatrix}\right\|^2 \le \mathtt{d}_n\left[\begin{pmatrix}\varphi\\ L_n\varphi\end{pmatrix}\right] - \frac{1}{4-n}\int_{\mathbb{R}^n}\frac{1}{|\mathbf{x}|}\left|\begin{pmatrix}\varphi(\mathbf{x})\\ (L_n\varphi)(\mathbf{x})\end{pmatrix}\right|^2 \mathrm{d}\mathbf{x} \tag{35}$$

holds.

Proof. We recall that

$$\mathsf{H}^{1/2}(\mathbb{R}^n) = \{\psi \in \mathsf{L}^2(\mathbb{R}^n) : (1+|\cdot|^2)^{1/4}\mathcal{F}_n\psi \in \mathsf{L}^2(\mathbb{R}^n)\}.$$

Thus the unitarity of $\mathcal{U}_n$ implies

$$\mathsf{H}^{1/2}(\mathbb{R}^n) = \{\psi \in \mathsf{L}^2(\mathbb{R}^n) : \bigoplus_{j\in\mathfrak{T}_n}(1+(\cdot)^2)^{1/4}\big(\mathcal{F}_n\psi\big)_j \in \bigoplus_{j\in\mathfrak{T}_n}\mathsf{L}^2(\mathbb{R}_+)\}. \tag{36}$$

Moreover we observe that the operator R_n is bounded, which together with (36) and (34) implies that $L_n\varphi \in \mathsf{H}^{1/2}(\mathbb{R}^n)$.
Now we define the quadratic form p on $\mathsf{L}^2(\mathbb{R}_+, (1+p^2)^{1/2}\mathrm{d}p)$ by

$$\mathsf{p}[\chi] := \int_0^\infty p|\chi(p)|^2 \mathrm{d}p.$$

For the proof of (35) we recall that the quadratic form (18) satisfy the inequalities

$$\begin{aligned} \mathsf{q}_{k+1/2}[\zeta] &\le \mathsf{q}_{k-1/2}[\zeta]; \\ \mathsf{q}_{k+1}[\zeta] &\le \mathsf{q}_k[\zeta]; \\ \mathsf{q}_0[\zeta] &\le c_3^{-1}\mathsf{p}[\zeta], \quad \mathsf{q}_1[\zeta] \le c_3\mathsf{p}[\zeta]; \\ \mathsf{q}_{-1/2}[\zeta] &\le 2c_2^{-1}\mathsf{p}[\zeta], \quad \mathsf{q}_{1/2}[\zeta] \le 2c_2\mathsf{p}[\zeta]; \end{aligned} \tag{37}$$

for $k \in \mathbb{N}_0$ and $\zeta \in \mathsf{L}^2(\mathbb{R}_+, (1+p^2)^{1/2}\mathrm{d}p)$ (see [2] and [10]).
By Lemma 7 we obtain

$$\int_{\mathbb{R}^n} \frac{|\varphi(\mathbf{x})|^2}{|\mathbf{x}|}\mathrm{d}\mathbf{x} = \begin{cases} \sum\limits_{k\in\mathfrak{T}_2} \mathsf{q}_{|k|-1/2}\big[(\mathcal{F}_2\varphi)_k\big] \text{ if } n=2; \\ \sum\limits_{(l,m,s)\in\mathfrak{T}_3} \mathsf{q}_l\big[(\mathcal{F}_3\varphi)_{(l,m,s)}\big] \text{ if } n=3; \end{cases} \tag{38}$$

and by (31) - (34)

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{|(L_n\varphi)(\mathbf{x})|^2}{|\mathbf{x}|}\mathrm{d}\mathbf{x} \\ &= \begin{cases} \sum\limits_{k\in\mathfrak{T}_2^+} c_2^2\mathsf{q}_{|k|-\frac{1}{2}}\big[(\mathcal{F}_2\varphi)_{k-1}\big] + \sum\limits_{k\in\mathfrak{T}_2^-} c_2^{-2}\mathsf{q}_{|k|-\frac{1}{2}}\big[(\mathcal{F}_2\varphi)_{k-1}\big] \text{ if } n=2; \\ \sum\limits_{(l,m,\frac{1}{2})\in\mathfrak{T}_3^+} c_3^2\mathsf{q}_l\big[(\mathcal{F}_3\varphi)_{(l+1,m,-\frac{1}{2})}\big] + \sum\limits_{(l,m,-\frac{1}{2})\in\mathfrak{T}_3^-} c_3^{-2}\mathsf{q}_l\big[(\mathcal{F}_3\varphi)_{(l-1,m,\frac{1}{2})}\big] \text{ if } n=3. \end{cases} \end{aligned} \tag{39}$$

Note that $(l,m,s)\in\mathfrak{T}_3^-$ implies $l\in\mathbb{N}$. Hence (37) implies that the right hand sides of (38) can be estimated by

$$(4-n)\left(\sum_{j\in\mathfrak{T}_n^+} c_n^{-1}\mathsf{p}\big[(\mathcal{F}_n\varphi)_j\big] + \sum_{j\in\mathfrak{T}_n^-} c_n\mathsf{p}\big[(\mathcal{F}_n\varphi)_j\big]\right); \tag{40}$$

and the right hand side of (39) by

$$(4-n)\left(\sum_{j\in\mathfrak{T}_n^+} c_n\mathsf{p}\big[(\mathcal{F}_n\varphi)_{T_n^{-1}j}\big] + \sum_{j\in\mathfrak{T}_n^-} c_n^{-1}\mathsf{p}\big[(\mathcal{F}_n\varphi)_{T_n^{-1}j}\big]\right). \tag{41}$$

By $T_n(\mathfrak{T}_n^{\pm}) = \mathfrak{T}_n^{\mp}$ we conclude that (41) is equal to (40). This together with the relation

$$(\mathcal{F}_n L_n \varphi)_{T_n j} = c_{n,T_n j} (\mathcal{F}_n \varphi)_j \text{ for all } j \in \mathfrak{T}_n,$$

implies

$$\begin{aligned} &\frac{1}{4-n}\int_{\mathbb{R}^n} \frac{1}{|\mathbf{x}|}\left|\begin{pmatrix}\varphi(\mathbf{x}) \\ (L_n\varphi)(\mathbf{x})\end{pmatrix}\right|^2 \mathrm{d}\mathbf{x} \leq \\ &\sum_{j\in\mathfrak{T}_n} \int\limits_{\mathbb{R}_+} \left\langle \begin{pmatrix} (\mathcal{F}_n\varphi)_j(p) \\ (\mathcal{F}_n L_n \varphi)_{T_n j}(p)\end{pmatrix}, \begin{pmatrix} 0 & p \\ p & 0\end{pmatrix}\begin{pmatrix} (\mathcal{F}_n\varphi)_j(p) \\ (\mathcal{F}_n L_n \varphi)_{T_n j}(p)\end{pmatrix}\right\rangle_{\mathbb{C}^2} \mathrm{d}p. \end{aligned} \tag{42}$$

A straightforward calculation using (31) - (34) gives

$$\begin{aligned} &\left\langle \begin{pmatrix}\varphi \\ L_n\varphi\end{pmatrix}, \begin{pmatrix}\mathbb{I}_{\mathbb{C}^{n-1}} & 0 \\ 0 & \mp\mathbb{I}_{\mathbb{C}^{n-1}}\end{pmatrix}\begin{pmatrix}\varphi \\ L_n\varphi\end{pmatrix}\right\rangle \\ &= \left(1 \mp c_n^{-2}\right)\sum_{j\in\mathfrak{T}_n^+} \|(\mathcal{F}_n\varphi)_j\|^2 + \left(1\mp c_n^2\right)\sum_{j\in\mathfrak{T}_n^-}\|(\mathcal{F}_n\varphi)_j\|^2. \end{aligned} \tag{43}$$

By Lemma 8 we know that the right hand side of Relation (42) plus the minus case of the left hand side of (43) is equal to $\mathtt{d}_n\left[\binom{\varphi}{L_n\varphi}\right]$. Thus we obtain (35) by (42) and (43). □

4 Proof of Theorem 2

We proceed analogously to the proof of Theorem 1. Thus it is enough to find an operator $G_n : P_n^+ \mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{2(n-1)}) \to P_n^- \mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{2(n-1)})$ such that

$$\inf_{\varphi\in P_n^+\mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{2(n-1)})\setminus\{0\}} \frac{\mathtt{d}_n\left[\varphi + G_n\varphi\right] + \mathtt{v}\left[\varphi + G_n\varphi\right]}{\left\|\varphi + G_n\varphi\right\|^2} > -1 \tag{44}$$

holds. In the following lemma we prove that a possible choice of G_n is

$$G_n := (\mathcal{W}_n\mathcal{F}_n)^* E_n (\mathcal{W}_n\mathcal{F}_n), \tag{45}$$

with

$$E_n : \bigoplus_{j\in\mathfrak{T}_n} \mathsf{L}^2(\mathbb{R}_+;\mathbb{C}^2) \to \bigoplus_{j\in\mathfrak{T}_n} \mathsf{L}^2(\mathbb{R}_+;\mathbb{C}^2); \tag{46}$$

$$\bigoplus_{j\in\mathfrak{T}_n}\Psi_j \mapsto \bigoplus_{j\in\mathfrak{T}_n} \frac{1 - c_{n,j}(\cdot) + \sqrt{1+(\cdot)^2}}{c_{n,j} + (\cdot) + c_{n,j}\sqrt{1+(\cdot)^2}}\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}\Psi_j. \tag{47}$$

Lemma 11. *Let $\varphi \in P_n^+\mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{2(n-1)})$ then $G_n\varphi \in P_n^-\mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{2(n-1)})$ and the relation*

$$L_n(\varphi + G_n\varphi)_1 = (\varphi + G_n\varphi)_2 \tag{48}$$

holds.

Remark 12. *By Lemma 10 and Relation* (48) *we conclude* (44).

Proof of Lemma 11. By Lemma 8 we deduce that $\psi \in P_n^{\pm}\mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{2(n-1)})$ if and only if there exists $\bigoplus_{j\in\mathfrak{T}_n}\zeta_j \in \bigoplus_{j\in\mathfrak{T}_n}\mathsf{L}^2(\mathbb{R}_+;(1+p^2)^{1/2}\mathrm{d}p)$ such that

$$\big(\mathcal{W}_n\mathcal{F}_n\psi\big)_j(p) = \begin{cases} \zeta_j(p)\begin{pmatrix} 1 \\ \frac{p}{1+\sqrt{1+p^2}} \end{pmatrix} \text{ ("+" case);} \\ \zeta_j(p)\begin{pmatrix} \frac{-p}{1+\sqrt{1+p^2}} \\ 1 \end{pmatrix} \text{ ("-" case);} \end{cases} \tag{49}$$

holds for every $j \in \mathfrak{T}_n$ and $p \in \mathbb{R}_+$. Hence we get $G_n\varphi \in P_n^{-}\mathsf{H}^{1/2}(\mathbb{R}^n;\mathbb{C}^{2(n-1)})$. By (49),(46) we obtain that there exists $\bigoplus_{j\in\mathfrak{T}_n}\chi_j \in \bigoplus_{j\in\mathfrak{T}_n}\mathsf{L}^2(\mathbb{R}_+;(1+p^2)^{1/2}\mathrm{d}p)$ such that

$$\big(\mathcal{W}_n\mathcal{F}_n\varphi\big)_j(p) = \chi_j(p)\begin{pmatrix} 1 \\ \frac{p}{1+\sqrt{1+p^2}} \end{pmatrix}$$

and

$$\begin{aligned} &\big((\mathbb{I}+E_n)\mathcal{W}_n\mathcal{F}_n\varphi\big)_j = \begin{pmatrix} \tilde{\chi}_j \\ c_{n,T_nj}\tilde{\chi}_j \end{pmatrix} \text{ with} \\ &\tilde{\chi}_j(p) := \frac{c_{n,j}\Big(p^2+(1+\sqrt{1+p^2})^2\Big)}{(1+\sqrt{1+p^2})(c_{n,j}+p+c_{n,j}\sqrt{1+p^2})}\chi_j(p) \text{ for } p \in \mathbb{R}_+, \end{aligned} \tag{50}$$

hold for every $j \in \mathfrak{T}_n$. Hence we get by (45),(33) and (34) the relation

$$\varphi + G_n\varphi = (\mathcal{W}_n\mathcal{F}_n)^*\bigoplus_{j\in\mathfrak{T}_n}\begin{pmatrix} \tilde{\chi}_j \\ c_{n,T_nj}\tilde{\chi}_j \end{pmatrix} = \begin{pmatrix} (\mathcal{U}_n\mathcal{F}_n)^*\bigoplus_{j\in\mathfrak{T}_n}\tilde{\chi}_j \\ L_n(\mathcal{U}_n\mathcal{F}_n)^*\bigoplus_{j\in\mathfrak{T}_n}\tilde{\chi}_j \end{pmatrix}.$$

Thus we have proven Relation (48). ☐

5 Proof of Theorem 3

Since the right hand side of (2) is continuous in the $\mathsf{H}^1(\mathbb{R}^n;\mathbb{C}^{n-1})$ norm (see Theorem 2.5 in [11]), we can assume that $\varphi \in \mathsf{C}_0^\infty(\mathbb{R}^n\setminus\{0\};\mathbb{C}^{n-1})\setminus\{0\}$ by the density of $\mathsf{C}_0^\infty(\mathbb{R}^n\setminus\{0\};\mathbb{C}^{n-1})$ in $\mathsf{H}^1(\mathbb{R}^n;\mathbb{C}^{n-1})$.

By the application of Theorem 1 we obtain

$$\lambda(v) \leq \sup_{\psi \in \mathsf{H}^1(\mathbb{R}^n\setminus\{0\};\mathbb{C}^{n-1})} I_{n,v,\varphi}(\psi) \text{ with} \tag{51}$$

$$I_{n,v,\varphi} : \mathsf{H}^1(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{n-1}) \to \mathbb{R}; \tag{52}$$

$$I_{n,v,\varphi}(\psi) := \frac{\left\langle \begin{pmatrix}\varphi\\ \psi\end{pmatrix}, \begin{pmatrix}(1+v)\otimes \mathbb{I}_{\mathbb{C}^{n-1}} & K_n\\ K_n & (-1+v)\otimes \mathbb{I}_{\mathbb{C}^{n-1}}\end{pmatrix}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\rangle}{\left\| \begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|^2}. \tag{53}$$

Note that we calculate the suprema in (51) over $\mathsf{H}^1(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{n-1})$ instead of $\mathsf{H}^{1/2}(\mathbb{R}^n; \mathbb{C}^{n-1})$. This is justified by a density argument, which makes use of the form boundedness of $v \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}}$ with respect to $D_n(0)$ (see Lemma 9) and the density of $\mathsf{H}^1(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{n-1})$ in $\mathsf{H}^{1/2}(\mathbb{R}^n; \mathbb{C}^{n-1})$.
Thus the proof of Theorem 3 basically follows from the following lemma.

Lemma 13. *We define*

$$J_{n,v,\varphi} : (-1,\infty) \to \mathbb{R};$$
$$J_{n,v,\varphi}(\lambda) := \int_{\mathbb{R}^n} \left(\frac{|K_n\varphi(\mathbf{x})|^2}{1+\lambda-v(\mathbf{x})} + \big(1-\lambda+v(\mathbf{x})\big)|\varphi(\mathbf{x})|^2 \right) \mathrm{d}\mathbf{x}.$$

For $\lambda \in (-1,\infty)$, $J_{n,v,\varphi}(\lambda) \leq 0$ *implies*

$$\sup_{\psi \in \mathsf{H}^1(\mathbb{R}^n\setminus\{0\};\mathbb{C}^{n-1})} I_{n,v,\varphi}(\psi) \leq \lambda.$$

Proof. We introduce

$$\psi_{n,v,\varphi} : (-1,\infty) \to \mathsf{H}^1(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{n-1}); \quad \psi_{n,v,\varphi}(\lambda) := \frac{K_n\varphi}{1+\lambda-v}. \tag{54}$$

For every $\zeta \in \mathsf{H}^1(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{n-1})$ the inequality

$$\begin{aligned}&\big(I_{n,v,\varphi}\big(\psi_{n,v,\varphi}(\lambda)+\zeta\big)-\lambda\big)\big(\|\varphi\|^2+\|\psi_{n,v,\varphi}(\lambda)+\zeta\|^2\big)\\ &= J_{n,v,\varphi}(\lambda) + 2\Re\langle \zeta, K_n\varphi-(1+\lambda-v)\psi_{n,v,\varphi}(\lambda)\rangle +\\ &\langle K_n\varphi-(1+\lambda-v)\psi_{n,v,\varphi}(\lambda), \psi_{n,v,\varphi}(\lambda)\rangle - \langle \zeta, (1+\lambda-v)\zeta\rangle \leq J_{n,v,\varphi}(\lambda)\end{aligned}$$

holds, and thus we conclude the claim. □

By Lemma 13 and (51) we obtain

$$J_{n,v,\varphi}\big(\lambda(v)-\varepsilon\big) > 0 \text{ for } \varepsilon \in \big(0, 1+\lambda(v)\big). \tag{55}$$

Letting $\varepsilon \searrow 0$ in (55) we obtain Theorem 3.

6 Proof of Theorem 5

The proof is based on:

Lemma 14. *Let $\nu \in [0, 1/(4-n)]$. The restriction of $\big(\tilde{D}_n(-\nu/|\cdot| \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}})\big)^*$ to $\mathfrak{C}_n^\nu$ is essentially self-adjoint.*

Proof. For $m \in \mathfrak{T}_2$ and $(l, m, s) \in \mathfrak{T}_3$ we define

$$\kappa_m := m + 1/2;$$
$$\kappa_{(l,m,s)} := 2sl + s + 1/2.$$

Furthermore we introduce for every $j \in \mathfrak{T}_n$ the operator $D^{j,\nu}$ in $\mathsf{L}^2(\mathbb{R}_+; \mathbb{C}^2)$ by the differential expression

$$d^{j,\nu} := \begin{pmatrix} -\frac{\nu}{r} & -\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa_j}{r} \\ \frac{\mathrm{d}}{\mathrm{d}r} - \frac{\kappa_j}{r} & -\frac{\nu}{r} \end{pmatrix}$$

on $\mathsf{C}_0^\infty(\mathbb{R}_+; \mathbb{C}^2)$. Now we observe that any solution of the equation $d^{j,\nu}\varphi = 0$ in $\mathbb{R}_+$ is a linear combination of the two functions

$$\varphi_{j,1}^\nu(r) := \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} r^{\kappa_j} \text{ if } \nu = 0, \\ \begin{pmatrix} \nu \\ \sqrt{\kappa_j^2 - \nu^2} - \kappa_j \end{pmatrix} r^{\sqrt{\kappa_j^2 - \nu^2}} \text{ else,} \end{cases}$$

and

$$\varphi_{j,2}^\nu(r) := \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix} r^{-\kappa_j} \text{ if } \nu = 0, \\ \begin{pmatrix} \nu \\ -\sqrt{\kappa_j^2 - \nu^2} - \kappa_j \end{pmatrix} r^{-\sqrt{\kappa_j^2 - \nu^2}} \text{ if } 0 < \nu^2 < \kappa_j^2, \\ \begin{pmatrix} \nu \ln(r) \\ 1 - \kappa_j \ln(r) \end{pmatrix} \text{ if } \nu^2 = \kappa_j^2. \end{cases}$$

Through the application of the results of [20] as in Section 2 in [14] we obtain that the closure $D_{\text{ex}}^{j,\nu}$ of the restriction of $(D^{j,\nu})^*$ to $\mathfrak{C}^{j,\nu}$ is self-adjoint with

$$\mathfrak{C}^{j,\nu} := \begin{cases} \mathsf{C}_0^\infty(\mathbb{R}_+; \mathbb{C}^2) \dot{+} \operatorname{span}\{\xi \varphi_{j,1}^\nu\} \text{ if } \kappa_j^2 - \nu^2 < 1/4; \\ \mathsf{C}_0^\infty(\mathbb{R}_+; \mathbb{C}^2) \text{ else.} \end{cases}$$

Here ξ is a smooth cut-off function with $\xi \in \mathsf{C}^\infty(\mathbb{R}_+; \mathbb{R}_+)$, $\xi(t) = 1$ for $t \in (0, 1)$

and $\xi(t) = 0$ for $t > 2$. Thus we conclude the claim by

$$\big(\tilde{D}_n(-\nu/|\cdot| \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}})\big)^* = (\mathcal{W}_n\mathcal{M}_n)^* \left(\bigoplus_{j\in\mathfrak{T}_n} \big(D^{j,\nu} + \sigma_3\big)^*\right) \mathcal{W}_n\mathcal{M}_n \text{ with,} \tag{56}$$

$$\mathcal{M}_n := \operatorname{diag}(1, \mathrm{i}) \otimes \mathbb{I}_{\mathbb{C}^{n-1}}$$

(see Section 7.3.3 in [19] for $n = 2$ and Section 2.1 in [1] for $n = 3$) and the fact that σ_3 is a bounded operator in $\mathsf{L}^2(\mathbb{R}_+; \mathbb{C}^2)$. □

Remark 15. *Let $\nu \in \big[0, 1/(4-n)\big)$ and $j \in \mathfrak{T}_n$. By the embedding*

$$\mathsf{H}^{1/2}(\mathbb{R}^n) \subset \mathsf{L}^2(\mathbb{R}^n, (1 + |\mathbf{x}|^{-1})\mathrm{d}\mathbf{x})$$

and (56) *we obtain that the domain of $\big(\mathcal{W}_n M_n D_n(-\nu/|\cdot|\otimes\mathbb{I}_{\mathbb{C}^{2(n-1)}})\,(\mathcal{W}_n M_n)^*\big)_j$ is in $\mathsf{L}^2(\mathbb{R}_+, (1 + r^{-1})\mathrm{d}r)$. Hence there is a self-adjoint extension of $D^{j,\nu}$ with domain in $\mathsf{L}^2(\mathbb{R}_+, (1 + r^{-1})\mathrm{d}r)$. By $\xi\varphi^{\nu}_{j,2} \notin \mathsf{L}^2(\mathbb{R}_+, (1 + r^{-1})\mathrm{d}r)$ for $\nu > 0$ and Theorem 1.5 in [20] we get that $D^{j;\nu}_{\mathrm{ex}}$ is the unique self-adjoint extension of $D^{j,\nu}$ with domain in $\mathsf{L}^2(\mathbb{R}_+, (1 + r^{-1})\mathrm{d}r)$. Therefore, we obtain*

$$\big(\mathcal{W}_n M_n D_n(-\nu/|\cdot| \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}})\,(\mathcal{W}_n M_n)^*\big)_j = D^{j,\nu}_{\mathrm{ex}}.$$

We conclude that the closure of $\big(\tilde{D}_n(-\nu/|\cdot| \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}})\big)^*$ restricted to $\mathfrak{C}^{\nu}_n$ is $D_n(-\nu/|\cdot| \otimes \mathbb{I}_{\mathbb{C}^{2(n-1)}})$.

As a consequence of Lemma 14 it remains to prove that $\zeta^{\nu}_{n,m} \in \mathfrak{D}\big(D^{\nu}_n\big)$ for $m \in \{-1/2, 1/2\}^{n-1}$ and $(n, \nu) \in \big(\{2\} \times (0, 1/2]\big) \cup \big(\{3\} \times (\sqrt{3}/2, 1]\big)$. We introduce the symmetric and non-negative (by Corollary 4) quadratic form $\mathfrak{q}^{\nu}_n$ on $\mathsf{C}^{\infty}_0(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{n-1})$ by

$$\mathfrak{q}^{\nu}_n[\varphi] := \int_{\mathbb{R}^n} \Bigg(\frac{|K_n\varphi|^2}{1 + \sqrt{1 - \big((4-n)\nu\big)^2} + \frac{\nu}{|\mathbf{x}|}} + \left(1 - \sqrt{1 - \big((4-n)\nu\big)^2} - \frac{\nu}{|\mathbf{x}|}\right)|\varphi|^2\Bigg)\mathrm{d}\mathbf{x}.$$

Note that $\mathfrak{q}^{\nu}_n$ is closable by Theorem X.23 in [16]. We denote the domain of the closure of $\mathfrak{q}^{\nu}_n$ by $\mathfrak{Q}^{\nu}_n$.

By the characterisation of $\mathfrak{D}\big(D^{\nu}_n\big)$ in Theorem 1 in [8], it is enough to show that for all $m \in \{-1/2, 1/2\}^{n-1}$ the upper $(n-1)$ spinor of $\zeta^{\nu}_{n,m}$ is in $\mathfrak{Q}^{\nu}_n$, i.e., $\varsigma^{\nu}_{n,m} \in \mathfrak{Q}^{\nu}_n$ with $\varsigma^{\nu}_{2,m}$ given in polar coordinates by

$$\varsigma^{\nu}_{2,m}(\rho, \vartheta) := \xi(\rho)\rho^{\sqrt{1/4-\nu^2}-1/2}\mathrm{e}^{-\mathrm{i}(m+1/2)\vartheta};$$

and $\varsigma^{\nu}_{3,m}$ in spherical coordinates by

$$\varsigma^{\nu}_{3,m}(r, \theta, \phi) := \xi(r)r^{\sqrt{1-\nu^2}-1}\Omega_{1/2+m_2,m_1,-m_2}(\theta, \phi).$$

We achieve this goal by the application of the following abstract lemma

Lemma 16. *Let* q *be a closable and non-negative quadratic form on a dense linear subspace* $\mathfrak{Q}$ *of the Hilbert space* $\mathfrak{H}$ *and* $\psi \in \mathfrak{H}$. *If there is a sequence* $(\psi_n)_{n\in\mathbb{N}} \subset \mathfrak{Q}$ *with* $\sup_{n\in\mathbb{N}} \mathsf{q}[\psi_n] < \infty$ *which converges weakly in* $\mathfrak{H}$ *to* ψ, *then* ψ *is in the domain of the closure of* q.

Proof. We denote by $\overline{\mathsf{q}}$ the closure of q and by $\overline{\mathfrak{Q}}$ the domain of $\overline{\mathsf{q}}$. There is a unique self-adjoint operator $B : \overline{\mathfrak{Q}} \to \mathfrak{H}$ with

$$\overline{\mathsf{q}}[\varphi] = \|B\varphi\|^2 \text{ for all } \varphi \in \overline{\mathfrak{Q}}$$

by Theorem 2.13 in [18] (B^2 corresponds to A there). Thus we know that

$$\sup_{n\in\mathbb{N}} \|B\psi_n\|^2 < \infty.$$

Hence there is a $\Psi \in \mathfrak{H}$ and a subsequence $(B\psi_{n_k})_{n_k\in\mathbb{N}}$ of $(B\psi_n)_{n\in\mathbb{N}} \subset \mathfrak{H}$ that converges weakly to Ψ by the Banach-Alaoglu Theorem. This implies that $\big((\psi_{n_k}, B\psi_{n_k})\big)_{n_k\in\mathbb{N}}$ converges weakly to $(\psi, \Psi) \in \mathfrak{H} \oplus \mathfrak{H}$. By the closedness of the graph of B and Theorem 8 in Chapter 1 of [3] we deduce the claim. □

Now we pick $\upsilon \in \mathsf{C}_0^\infty(\mathbb{R}_+)$ such that $\upsilon(r) = \xi(r)$ for all $r \in [1, \infty)$ and $0 \leq \upsilon(r) \leq 1$ for $r \in (0, 1)$. Let $k \in \mathbb{N}$. We define

$$\upsilon_k(r) := \begin{cases} \upsilon(kr) & \text{if } r \in (0, 1/k]; \\ 1 & \text{if } r \in (1/k, 1]; \\ \xi(r) & \text{else ;} \end{cases}$$

and the function $\varsigma^\nu_{2,m,k}$ in the polar coordinates by

$$\varsigma^\nu_{2,m,k}(\rho, \vartheta) := \upsilon_k(\rho)\rho^{\sqrt{1/4-\nu^2}-1/2}\mathrm{e}^{-\mathrm{i}(m+1/2)\vartheta},$$

and $\varsigma^\nu_{3,m,k}$ in the spherical coordinates by

$$\varsigma^\nu_{3,m,k}(r, \theta, \phi) := \upsilon_k(r)r^{\sqrt{1-\nu^2}-1}\Omega_{1/2+m_2,m_1,-m_2}(\theta, \phi).$$

The sequence $(\varsigma^\nu_{n,m,k})_{k\in\mathbb{N}}$ converges to $\varsigma^\nu_{n,m}$ in $\mathsf{L}^2(\mathbb{R}^n; \mathbb{C}^{n-1})$. By Lemma 16 it is thus enough to prove that

$$\sup_{k\in\mathbb{N}} \mathsf{q}^\nu_n[\varsigma^\nu_{n,m,k}] < \infty. \tag{57}$$

Let $\varphi \in \mathsf{C}_0^\infty(\mathbb{R}^n \setminus \{0\}; \mathbb{C}^{n-1})$. At first we observe that

$$\mathsf{q}^\nu_n[\varphi] \leq \int_{\mathbb{R}^n} \left(\frac{|x|}{\nu}|K_n\varphi|^2 - \frac{\nu}{|x|}|\varphi|^2 + |\varphi|^2\right)\mathrm{d}\mathbf{x}. \tag{58}$$

A tedious calculation shows

$$K_n = \begin{cases} -\mathrm{i}\mathrm{e}^{i\vartheta}(\partial_\varrho - \frac{1}{\rho}A_2) \text{ with } A_2 := -\mathrm{i}\partial_\vartheta \text{ if } n = 2; \\ -\mathrm{i}\left(\boldsymbol{\sigma} \cdot \frac{x}{|x|}\right)\left(\partial_r - \frac{1}{r}A_3\right) \text{ with } A_3 := \boldsymbol{\sigma} \cdot \big(-\mathrm{i}\mathbf{x} \wedge \nabla\big) \text{ if } n = 3. \end{cases} \tag{59}$$

Using (59) and integration by parts we obtain that the right hand side of (58) is equal to

$$\int_{\mathbb{R}^n}\left(\frac{|\mathbf{x}|}{\nu}\left|\partial_{|\mathbf{x}|}\varphi\right|^2+\frac{1}{\nu|\mathbf{x}|}\left|(1/(4-n)+A_n)\varphi\right|^2-\frac{\left(\nu+\frac{1}{(4-n)^2\nu}\right)}{|\mathbf{x}|}\left|\varphi\right|^2+\left|\varphi\right|^2\right)\mathrm{d}\mathbf{x}. \tag{60}$$

By (60) and Relation 2.1.37 in [1] we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(\frac{|x|}{\nu}|K_n\varsigma^{\nu}_{n,m,k}|^2-\frac{\nu}{|x|}|\varsigma^{\nu}_{n,m,k}|^2+|\varsigma^{\nu}_{n,m,k}|^2\right)\mathrm{d}\mathbf{x}\\ &=\int_0^\infty\left(\frac{t^n}{\nu}\left|\partial_t\upsilon_k(t)t^{\sqrt{(4-n)^{-2}-\nu^2}-(4-n)^{-1}}\right|^2\right.\\ &\quad\left.-\nu\upsilon_k(t)^2t^{2\sqrt{(4-n)^{-2}-\nu^2}-1}+\upsilon_k(t)^2t^{2\sqrt{(4-n)^{-2}-\nu^2}}\right)\mathrm{d}t.\end{aligned} \tag{61}$$

A straightforward calculation shows that (61) is equal to

$$\begin{aligned}&\int_0^\infty\left(\nu^{-1}\upsilon_k'(t)^2t^{2\sqrt{(4-n)^{-2}-\nu^2}+1}+\upsilon_k(t)^2t^{2\sqrt{(4-n)^{-2}-\nu^2}}\right)\mathrm{d}t\\ &=\int_0^{1/k}\nu^{-1}k^2\upsilon'(kt)^2t^{2\sqrt{(4-n)^{-2}-\nu^2}+1}\mathrm{d}t+\int_1^\infty\nu^{-1}\upsilon'(t)^2r^{2\sqrt{(4-n)^{-2}-\nu^2}+1}\mathrm{d}t\\ &+\int_0^\infty\upsilon_k(t)^2t^{2\sqrt{(4-n)^{-2}-\nu^2}}\mathrm{d}t.\end{aligned} \tag{62}$$

An upper bound for the expression in (62) is

$$\int_0^\infty\nu^{-1}\upsilon'(t)^2t^{2\sqrt{(4-n)^{-2}-\nu^2}+1}\mathrm{d}t+\int_0^\infty\xi(t)^2t^{2\sqrt{(4-n)^{-2}-\nu^2}}\mathrm{d}t. \tag{63}$$

The combination of (63), (62) (61) and (58) implies (57).

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