Picard Groups, Weight Structures, and (noncommutative) Mixed Motives

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Abstract. We develop a general theory which, under certain assumptions, enables the computation of the Picard group of a symmetric monoidal triangulated category equipped with a weight structure in terms of the Picard group of the associated heart. As an application, we compute the Picard group of several categories of motivic nature – mixed Artin motives, mixed Artin-Tate motives, bootstrap motivic spectra, noncommutative mixed Artin motives, noncommutative mixed motives of central simple algebras – as well as the Picard group of certain derived categories of symmetric ring spectra.

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1. Introduction and statement of results

The computation of the Picard group $\mathrm{Pic}(\mathcal{T})$ of a symmetric monoidal (triangulated) category $\mathcal{T}$ is, in general, a very difficult task. The goal of this article is to explain how the theory of weight structures allows us to greatly simplify this task.

Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a symmetric monoidal triangulated category equipped with a weight structure $w = (\mathcal{T}^{w \geq 0}, \mathcal{T}^{w \leq 0})$; consult §3 for details. Assume that the symmetric monoidal structure $- \otimes -$ (as well as the $\otimes$-unit $\mathbf{1}$) restricts to the heart $\mathcal{H} := \mathcal{T}^{w \geq 0} \cap \mathcal{T}^{w \leq 0}$ of the weight structure. We say that the

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category $\mathcal{T}$ has the *w-Picard property* if the group homomorphism $\mathrm{Pic}(\mathcal{H})\times\mathbb{Z} \to \mathrm{Pic}(\mathcal{T}), (a,n) \mapsto a[n]$, is invertible. Our first main result provides sufficient conditions for this property to hold:

Theorem 1.1. *Assume that the weight structure w on $\mathcal{T}$ is* bounded, *i.e. $\mathcal{T} = \cup_{n\in\mathbb{Z}}\mathcal{T}^{w\geq 0}[-n] = \cup_{n\in\mathbb{Z}}\mathcal{T}^{w\leq 0}[-n]$, and that there exists a full, additive, conservative, symmetric monoidal functor from $\mathcal{H}$ into a symmetric monoidal semi-simple abelian category* A *which is moreover* local *in the sense that if $a\otimes b = 0$ then $a = 0$ or $b = 0$. Under these assumptions, the category $\mathcal{T}$ has the w-Picard property.*

As explained in [7, §4.3], every bounded weight structure is uniquely determined by its heart. Concretely, given any additive subcategory $\mathcal{H}' \subset \mathcal{T}$ which generates $\mathcal{T}$ and for which we have $\mathrm{Hom}_{\mathcal{H}'}(a, b[n]) = 0$ for every $n > 0$ and $a, b \in \mathcal{H}'$, there exists a unique bounded weight structure on $\mathcal{T}$ with heart the Karoubi-closure of $\mathcal{H}'$ in $\mathcal{T}$. Roughly speaking, the construction of a bounded weight structure on a triangulated category amounts simply to the choice of an additive subcategory with trivial positive Ext-groups.
Our second main result formalizes the conceptual idea that the w-Picard property satisfies a "global-to-local" descent principle:

Theorem 1.2. *Assume the following:*

- (A1) *The heart $\mathcal{H}$ of the bounded weight structure w is essentially small and R-linear for some commutative indecomposable Noetherian ring R. Moreover, $\mathrm{Hom}_{\mathcal{H}}(a, b)$ is a finitely generated flat R-module for any two objects $a, b \in \mathcal{H}$;*
- (A2) *For every residue field $\kappa(\mathfrak{p})$, with $\mathfrak{p} \in \mathrm{Spec}(R)$, there exists a symmetric monoidal triangulated category $(\mathcal{T}_{\kappa(\mathfrak{p})}, \otimes, \mathbf{1})$ equipped with a weight structure $w_{\kappa(\mathfrak{p})}$ and with a weight-exact symmetric monoidal functor $\iota_{\kappa(\mathfrak{p})} \colon \mathcal{T} \to \mathcal{T}_{\kappa(\mathfrak{p})}$. Moreover, the functor $\iota_{\kappa(\mathfrak{p})}$ induces an equivalence of categories between the Karoubization of $\mathcal{H}\otimes_R \kappa(\mathfrak{p})$ and $\mathcal{H}_{\kappa(\mathfrak{p})}$.*

Under assumptions (A1)-(A2), if the categories $\mathcal{T}_{\kappa(\mathfrak{p})}$ have the $w_{\kappa(\mathfrak{p})}$-Picard property, then the category $\mathcal{T}$ has the w-Picard property.

Remark 1.3.
- (i) At assumption (A1) we can consider more generally the case where R is possibly decomposable; consult Remark 5.3(i).
- (ii) As it will become clear from the proof of Theorem 1.2, at assumption (A2) it suffices to consider the residue fields $\kappa(\mathfrak{m})$ associated to the maximal and minimal prime ideals of R; consult Remark 5.3(ii).

Due to their generality and simplicity, we believe that Theorems 1.1-1.2 will soon be part of the toolkit of every mathematician interested in Picard groups of triangulated categories. In the next section, we illustrate the usefulness of these results by computing the Picard group of several important categories of motivic nature; consult also §2.6 for a topological application.

2. Applications

Let k be a base field, which we assume perfect, and R a commutative ring of coefficients, which we assume indecomposable and Noetherian. Voevodsky's category of geometric mixed motives $\mathrm{DM}_{\mathrm{gm}}(k;R)$ (see [14, 24]), Morel-Voevodsky's stable $\mathbb{A}^1$-homotopy category $\mathrm{SH}(k)$ (see [26, 28, 40]), and Kontsevich's category of noncommutative mixed motives $\mathrm{KMM}(k;R)$ (see [19, 20, 21, 34]), play nowadays a central role in the motivic realm. A major challenge, which seems completely out of reach at the present time, is the computation of the Picard group of these symmetric monoidal triangulated categories[2]. In what follows, making use of Theorems 1.1-1.2, we achieve this goal in the case of certain important subcategories.

2.1. Mixed Artin motives. The category of *mixed Artin motives* $\mathrm{DMA}(k;R)$ is defined as the thick triangulated subcategory of $\mathrm{DM}_{\mathrm{gm}}(k;R)$ generated by the motives $M(X)_R$ of zero-dimensional smooth k-schemes X. The smallest additive, Karoubian, full subcategory of $\mathrm{DMA}(k;R)$ containing the objects $M(X)_R$ identifies with the (classical) category of Artin motives $\mathrm{AM}(k;R)$.

Theorem 2.1. *When the degrees of the finite separable field extensions of k are invertible in R, we have* $\mathrm{Pic}(\mathrm{DMA}(k;R)) \simeq \mathrm{Pic}(\mathrm{AM}(k;R)) \times \mathbb{Z}$.

Example 2.2. Theorem 2.1 holds, in particular, in the following cases:

(i) The field k is arbitrary and R is a $\mathbb{Q}$-algebra;
(ii) The field k is formally real (*e.g.* $k = \mathbb{R}$) and $1/2 \in R$;
(iii) Let p be a (fixed) prime number, l a perfect field, and H a Sylow pro-p-subgroup of $\mathrm{Gal}(\bar{l}/l)$. Theorem 2.1 also holds with $k := \bar{l}^H$ and $1/p \in R$.

Whenever R is a field, the R-linearized Galois-Grothendieck correspondence induces a symmetric monoidal equivalence of categories between $\mathrm{AM}(k;R)$ and the category $\mathrm{Rep}_R(\Gamma)$ of continuous finite dimensional R-linear representations of the absolute Galois group $\Gamma := \mathrm{Gal}(\overline{k}/k)$. Since the $\otimes$-invertible objects of $\mathrm{Rep}_R(\Gamma)$ are the 1-dimensional Γ-representations, $\mathrm{Pic}(\mathrm{AM}(k;R)) \simeq \mathrm{Pic}(\mathrm{Rep}_R(\Gamma))$ identifies with the group of continuous characters from Γ^{ab} to $R^\times$. In the particular case where $k = \mathbb{Q}$, the profinite group Γ^{ab} agrees with $\widehat{\mathbb{Z}}^\times$. Consequently, all the elements of $\mathrm{Rep}_R(\Gamma)$ can be represented by Dirichlet characters. Moreover, in the cases where $\mathrm{char}(k) \neq 2$ and $R = \mathbb{Q}$, we have the following computation

$$k^\times/(k^\times)^2 \xrightarrow{\simeq} \mathrm{Pic}(\mathrm{Rep}_{\mathbb{Q}}(\Gamma)) \qquad \lambda \mapsto (\Gamma \twoheadrightarrow \mathrm{Gal}(k(\sqrt{\lambda})/k) \xrightarrow{\sigma \mapsto -1} \mathbb{Q}^\times)\,,$$

where σ stands for the generator of the Galois group $\mathrm{Gal}(k(\sqrt{\lambda})/k) \simeq \mathbb{Z}/2\mathbb{Z}$; see Peter [30, pages 340-341]. A similar computation holds in characteristic 2 with $k^\times/(k^\times)^2$ replaced by $k/\{\lambda + \lambda^2 \,|\, \lambda \in k\}$.

[2] Consult Bachmann [4], resp. Hu [17], for the construction of $\otimes$-invertible objects in the motivic category $\mathrm{DM}_{\mathrm{gm}}(k;\mathbb{Z}/2\mathbb{Z})$, resp. $\mathrm{SH}(k)$, associated to quadrics.

Now, let $\mathcal{A}(k;R)$ be an additive, Karoubian, symmetric monoidal, full subcategory of $\mathrm{AM}(k;R)$, and $\mathrm{D}\mathcal{A}(k;R)$ the thick triangulated subcategory of $\mathrm{DMA}(k;R)$ generated by the motives associated to the objects of $\mathcal{A}(k;R)$. Under these notations, Theorem 2.1 admits the following generalization:

Theorem 2.3. *Assume that there exists a set of finite separable field extensions $l_i/k, i \in I$, such that the following two conditions hold:*

- (B1) *Every object of the category $\mathcal{A}(k;R)$ is isomorphic to a summand of a finite direct sum of the motives associated to the field extensions $l_i/k, i \in I$;*
- (B2) *For each $i \in I$, the degree of the finite field extension l_i/k is invertible in R.*

Under assumptions (B1)-(B2), we have $\mathrm{Pic}(\mathrm{D}\mathcal{A}(k;R)) \simeq \mathrm{Pic}(\mathcal{A}(k;R)) \times \mathbb{Z}$.

Example 2.4 (Mixed Dirichlet motives). Let R be a field. Following Wildeshaus [41, Def. 3.4], a *Dirichlet motive* is an Artin motive for which the corresponding Γ-representation factors through an abelian (finite) quotient. Take $\mathcal{A}(k;R)$ to be the category of Dirichlet motives. In this case, the associated symmetric monoidal triangulated category $\mathrm{D}\mathcal{A}(k;R)$ is called the category of *mixed Dirichlet motives*. Since the $\otimes$-invertible objects of $\mathrm{Rep}_R(\Gamma)$ are the 1-dimensional representations, and all these representations factor through an abelian (finite) quotient, the inclusion of categories $\mathcal{A}(k;R) \subset \mathrm{AM}(k;R)$ yields an isomorphism $\mathrm{Pic}(\mathcal{A}(k;R)) \simeq \mathrm{Pic}(\mathrm{AM}(k;R))$. Consequently, in the case where R is of characteristic zero, Theorem 2.3 implies that $\mathrm{Pic}(\mathrm{D}\mathcal{A}(k;R)) \simeq \mathrm{Pic}(\mathrm{AM}(k;R)) \times \mathbb{Z}$. Intuitively speaking, the difference between (mixed) Dirichlet motives and (mixed) Artin motives is not detected by the Picard group.

2.2. Mixed Artin-Tate motives. The category $\mathrm{DMAT}(k;R)$ of *mixed Artin-Tate motives* is defined as the thick symmetric monoidal triangulated subcategory of $\mathrm{DM}_{\mathrm{gm}}(k;R)$ generated by the motives $M(X)_R$ of zero-dimensional smooth k-schemes X and by the Tate motives $R(m), m \in \mathbb{Z}$.

Theorem 2.5. *When the degrees of the finite separable field extensions of k are invertible in R, we have* $\mathrm{DMAT}(k;R) \simeq \mathrm{Pic}(\mathrm{AM}(k;R)) \times \mathbb{Z} \times \mathbb{Z}$.

Now, let $\mathcal{A}(k;R)$ be an additive, Karoubian, symmetric monoidal, full subcategory of $\mathrm{AM}(k;R)$, and $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$ the thick symmetric monoidal triangulated subcategory of $\mathrm{DMAT}(k;R)$ generated by the motives associated to the objects of $\mathcal{A}(k;R)$ and by the Tate motives $R(m), m \in \mathbb{Z}$. Theorem 2.5 admits the following generalization:

Theorem 2.6. *Assume that there exists a set of field extensions $l_i/k, i \in I$, as in Theorem 2.3. Under these assumptions, we have* $\mathrm{Pic}(\mathrm{D}\mathcal{A}\mathrm{T}(k;R)) \simeq \mathrm{Pic}(\mathcal{A}(k;R)) \times \mathbb{Z} \times \mathbb{Z}$.

Example 2.7 (Mixed Tate motives). Take $\mathcal{A}(k;R)$ to be the smallest additive, Karoubian, full subcategory of $\mathrm{AM}(k;R)$ containing the $\otimes$-unit. In this case, the associated symmetric monoidal triangulated category $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$ is called the category of *mixed Tate motives*. Since $\mathcal{A}(k;R)$ identifies with the category of finitely generated projective R-modules[3], we conclude from Theorem 2.6 that the Picard group of $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$ is isomorphic to $\mathrm{Pic}(R)\times\mathbb{Z}\times\mathbb{Z}$. Note that we are not imposing the invertibility of any integer in R.

Example 2.8 (Mixed Dirichlet-Tate motives). Take $\mathcal{A}(k;R)$ to be the category of Dirichlet motives. In this case, the associated symmetric monoidal triangulated category $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$ is called the category of *mixed Dirichlet-Tate motives*. Recall from Example 2.4 that the Picard group of $\mathcal{A}(k;R)$ is isomorphic to the Picard group of $\mathrm{AM}(k;R)$. Consequently, in the case where R is of characteristic zero, Theorem 2.6 implies that $\mathrm{Pic}(\mathrm{D}\mathcal{A}\mathrm{T}(k;R))\simeq \mathrm{Pic}(\mathrm{AM}(k;R))\times\mathbb{Z}\times\mathbb{Z}$.

2.3. Motivic spectra. The *bootstrap category* $\mathrm{Boot}(k)$ is defined as the thick triangulated subcategory of $\mathrm{SH}(k)$ generated by the $\otimes$-unit $\Sigma^\infty(\mathrm{Spec}(k)_+)$. The former category contains a lot of information. For example, as proved by Levine in [22, Thm. 1], whenever k is algebraically closed and of characteristic zero, the category $\mathrm{Boot}(k)$ identifies with the homotopy category of finite spectra $\mathcal{SH}_c$. In particular, we have non-trivial negative Ext-groups

$$(2.9)\quad \mathrm{Hom}_{\mathrm{Boot}(k)}(\Sigma^\infty(\mathrm{Spec}(k)_+),\Sigma^\infty(\mathrm{Spec}(k)_+)[-n])\simeq\pi_n(\mathbb{S})\qquad n>0\,,$$

where $\mathbb{S}$ stands for the sphere spectrum. Moreover, as proved by Morel in [25, Thm. 6.2.2], whenever k is of characteristic $\neq 2$, we have a ring isomorphism

$$(2.10)\qquad \mathrm{End}_{\mathrm{Boot}(k)}(\Sigma^\infty(\mathrm{Spec}(k)_+))\simeq GW(k)\,,$$

where $GW(k)$ stands for the Grothendieck-Witt ring of k.

Theorem 2.11. *Assume that* $\mathrm{char}(k)\neq 2$ *and that* $GW(k)$ *is Noetherian. Under these assumptions, we have* $\mathrm{Pic}(\mathrm{Boot}(k))\simeq\mathrm{Pic}(GW(k))\times\mathbb{Z}$.

Remark 2.12. The ring $GW(k)$ is Noetherian if and only if $k^\times/(k^\times)^2$ is finite.

Example 2.13. Theorem 2.11 holds, in particular, in the following cases:

(i) The field k is quadratically closed (*e.g.* k is algebraically closed or the field of constructible numbers). In this case, we have $GW(k)\simeq\mathbb{Z}$;
(ii) The field k is the field of real numbers $\mathbb{R}$. In this case, we have $GW(\mathbb{R})\simeq\mathbb{Z}[C_2]$, where C_2 stands for the cyclic group of order 2;
(iii) The field k is the finite field $\mathbb{F}_q$ with q odd. In this case, $k^\times/(k^\times)^2=C_2$.

Intuitively speaking, Theorem 2.11 shows that none of the motivic spectra which are built using the non-trivial Ext-groups (2.9) is $\otimes$-invertible!

[3]Recall that the Picard group $\mathrm{Pic}(R)$ of a Dedekind domain R is its ideal class group $C(R)$.

2.4. Noncommutative mixed Artin motives. The category of *noncommutative mixed Artin motives* $\mathrm{NMAM}(k;R)$ is defined as the thick triangulated subcategory of $\mathrm{KMM}(k;R)$ generated by the noncommutative motives $U(l)_R$ of finite separable field extensions l/k. The smallest additive, Karoubian, full subcategory of $\mathrm{NMAM}(k;R)$ containing the objects $U(l)_R$ identifies with $\mathrm{AM}(k;R)$.

The category of noncommutative mixed Artin motives is in general much richer than the category of mixed Artin motives. For example, whenever R is a $\mathbb{Q}$-algebra, $\mathrm{DMA}(k;R)$ identifies with the category $\mathrm{Gr}_{\mathbb{Z}}\mathrm{AM}(k;R)$ of $\mathbb{Z}$-graded objects in $\mathrm{AM}(k;R)$; see [39, page 217]. This implies that $\mathrm{DMA}(k;R)$ has trivial higher Ext-groups. On the other hand, given any two finite separable field extensions l_1/k and l_2/k, we have non-trivial negative Ext-groups (see [33, §4])

$$\mathrm{Hom}_{\mathrm{NMAM}(k;R)}(U(l_1)_R, U(l_2)_R[-n]) \simeq K_n(l_1 \otimes_k l_2)_R \qquad n > 0\,, \tag{2.14}$$

where $K_n(l_1 \otimes_k l_2)$ stands for the n^{th} algebraic K-theory group of $l_1 \otimes_k l_2$. Roughly speaking, $\mathrm{NMAM}(k;R)$ contains not only $\mathrm{AM}(k;R)$ but also all the higher algebraic K-theory groups of finite separable field extensions. For example, given a number field $\mathbb{F}$, we have the following computation (due to Borel [12, §12])

$$\mathrm{Hom}_{\mathrm{NMAM}(\mathbb{Q};\mathbb{Q})}(U(\mathbb{Q})_{\mathbb{Q}}, U(\mathbb{F})_{\mathbb{Q}}[-n]) \simeq \begin{cases} \mathbb{Q}^{r_2} & n \equiv 3 \pmod 4 \\ \mathbb{Q}^{r_1+r_2} & n \equiv 1 \pmod 4 \\ 0 & \text{otherwise} \end{cases} \qquad n \geq 2\,,$$

where r_1 (resp. r_2) stands for the number of real (resp. complex) embeddings of $\mathbb{F}$.

Theorem 2.15. *When the degrees of the finite separable field extensions of k are invertible in R, we have* $\mathrm{Pic}(\mathrm{NMAM}(k;R)) \simeq \mathrm{Pic}(\mathrm{AM}(k;R)) \times \mathbb{Z}$.

Example 2.16. Theorem 2.15 holds in the cases (i)-(iii) of Example 2.2.

Theorem 2.15 shows that although the category $\mathrm{NMAM}(k;R)$ is much richer than $\mathrm{DMA}(k;R)$, this richness is not detected by the Picard group.

Now, let $\mathcal{A}(k;R)$ be an additive, Karoubian, symmetric monoidal, full subcategory of $\mathrm{AM}(k;R)$, and $\mathrm{NM}\mathcal{A}(k;R)$ the thick triangulated subcategory of $\mathrm{NMAM}(k;R)$ generated by the noncommutative motives associated to the objects of $\mathcal{A}(k;R)$. Theorem 2.15 admits the following generalization:

Theorem 2.17. *Assume that there exists a set of field extensions $l_i/k, \in I$, as in Theorem 2.3. Under these assumptions, we have* $\mathrm{Pic}(\mathrm{NM}\mathcal{A}(k;R)) \simeq \mathrm{Pic}(\mathcal{A}(k;R)) \times \mathbb{Z}$.

Example 2.18 (Noncommutative mixed Dirichlet motives). Take $\mathcal{A}(k;R)$ to be the category of Dirichlet motives. In this case, the associated symmetric monoidal triangulated category $\mathrm{NM}\mathcal{A}(k;R)$ is called the category of *noncommutative mixed Dirichlet motives*. Recall from Example 2.4 that the Picard group of $\mathcal{A}(k;R)$ is isomorphic to $\mathrm{Pic}(\mathrm{AM}(k;R))$. Consequently, in the case where R is of characteristic zero, Theorem 2.15 implies that $\mathrm{Pic}(\mathrm{NM}\mathcal{A}(k;R)) \simeq \mathrm{Pic}(\mathrm{AM}(k;R)) \times \mathbb{Z}$. Roughly speaking, the difference between mixed Dirichlet motives and noncommutative mixed Dirichlet motives is not detected by the Picard group.

Example 2.19 (Bootstrap category). Take $\mathcal{A}(k;R)$ to be the smallest additive, Karoubian, full subcategory of $\mathrm{AM}(k;R)$ containing the $\otimes$-unit. In this case, the associated symmetric monoidal triangulated category $\mathrm{NM}\mathcal{A}(k;R)$ is called the *bootstrap category*. Since $\mathcal{A}(k;R)$ identifies with the category of finitely generated projective R-modules, we conclude from Theorem 2.17 that $\mathrm{Pic}(\mathrm{NM}\mathcal{A}(k;R)) \simeq \mathrm{Pic}(R) \times \mathbb{Z}$. Similarly to Example 2.7, we are not imposing the invertibility of any integer in R.

2.5. Noncommutative mixed motives of central simple algebras. Let us denote by $\mathrm{NMCSA}(k;R)$ the thick triangulated subcategory of $\mathrm{KMM}(k;R)$ generated by the noncommutative motives $U(A)_R$ of central simple k-algebras A. In the same vein, let $\mathrm{CSA}(k;R)$ be the smallest additive, Karoubian, full subcategory of $\mathrm{NMCSA}(k;R)$ containing the objects $U(A)_R$. As proved in [35, Thm. 9.1], given any two central simple k-algebras A and B, we have the following equivalence

$$U(A)_{\mathbb{Z}} \simeq U(B)_{\mathbb{Z}} \Leftrightarrow [A] = [B] \in \mathrm{Br}(k)\,, \tag{2.20}$$

where $\mathrm{Br}(k)$ stands for the Brauer group of k. Intuitively speaking, (2.20) shows that the noncommutative motive $U(A)_{\mathbb{Z}}$ and the Brauer class $[A]$ contain exactly the same information. We have moreover non-trivial negative Ext-groups:

$$\mathrm{Hom}_{\mathrm{NMCSA}(k;\mathbb{Z})}(U(A)_{\mathbb{Z}}, U(B)_{\mathbb{Z}}[-n]) \simeq \pi_n(K(A^{\mathrm{op}} \otimes_k B) \wedge H\mathbb{Z}) \;\; n > 0\,, \tag{2.21}$$

where $H\mathbb{Z}$ stands for the Eilenberg-MacLane spectrum of $\mathbb{Z}$. Roughly speaking, the category $\mathrm{NMCSA}(k;\mathbb{Z})$ contains information not only about the Brauer group but also about all the higher algebraic K-theory of central simple algebras.

Theorem 2.22. *The following holds:*

(i) *We have an isomorphism* $\mathrm{Pic}(\mathrm{NMCSA}(k;R)) \simeq \mathrm{Pic}(\mathrm{CSA}(k;R)) \times \mathbb{Z}$*;*
(ii) *We have an isomorphism* $\mathrm{Pic}(\mathrm{CSA}(k;\mathbb{Z})) \simeq \mathrm{Br}(k)$*.*

Remark 2.23. Let R be a field. As explained in Remark 10.6, the Picard group of the category $\mathrm{Pic}(\mathrm{CSA}(k;R))$ is trivial when $\mathrm{char}(R) = 0$ and isomorphic to $\mathrm{Br}(k)\{p\}$ when $\mathrm{char}(R) = p > 0$.

Intuitively speaking, Theorem 2.22 shows that none of the noncommutative mixed motives which are built using the non-trivial negative Ext-groups (2.21) is $\otimes$-invertible!

2.6. A topological application. Let E be a commutative symmetric ring spectrum and $\mathcal{D}_c(E)$ the associated derived category of compact E-modules; see [15, 31].

Theorem 2.24. *Assume that the ring spectrum E is* connective, *i.e.* $\pi_n(E) = 0$ *for every $n < 0$, and that $\pi_0(E)$ is an indecomposable Noetherian ring. Under these assumptions, we have* $\mathrm{Pic}(\mathcal{D}_c(E)) \simeq \mathrm{Pic}(\pi_0(E)) \times \mathbb{Z}$.

Example 2.25 (Finite spectra). Let E be the sphere spectrum $\mathbb{S}$. In this case, the category $\mathcal{D}_c(\mathbb{S})$ is equivalent to the homotopy category of finite spectra $\mathcal{SH}_c$ and $\pi_0(\mathbb{S}) \simeq \mathbb{Z}$. Consequently, we obtain $\mathrm{Pic}(\mathcal{SH}_c) \simeq \mathbb{Z}$. This computation was originally established by Hopkins-Mahowald-Sadofsky in [16] using different tools. Note that this computation may be understood as a particular case of Theorem 2.11.

Example 2.26 (Ordinary rings). Let E be the Eilenberg-MacLane spectrum HR of a commutative indecomposable Noetherian ring R. In this case, $\mathcal{D}_c(HR) \simeq \mathcal{D}_c(R)$ and $\pi_0(HR) \simeq R$. Consequently, we obtain $\mathrm{Pic}(\mathcal{D}_c(R)) \simeq \mathrm{Pic}(R) \times \mathbb{Z}$; consult Remark 5.3(i) for the case where R is decomposable. This computation was originally established in [13]. Although Fausk did not use weight structures, one observes that by applying our arguments (see §5) to the triangulated category $D_c(R)$, equipped with the weight structure whose heart consists of the finitely generated projective R-modules, one obtains a reasoning somewhat similar to his one.

3. Weight structures

In this section we briefly review the theory of weight structures. This will give us the opportunity to fix some notations that will be used throughout the article.

Definition 3.1. (see [7, Def. 1.1.1]) A *weight structure* w on a triangulated category $\mathcal{T}$, also known as a *co-t-structure* in the sense of Pauksztello [29], consists of a pair of additive subcategories $(\mathcal{T}^{w\geq 0}, \mathcal{T}^{w\leq 0})$ satisfying the following conditions[4]:

(i) The categories $\mathcal{T}^{w\geq 0}$ and $\mathcal{T}^{w\leq 0}$ are closed under taking summands in $\mathcal{T}$;
(ii) We have inclusions of categories $\mathcal{T}^{w\geq 0} \subset \mathcal{T}^{w\geq 0}[1]$ and $\mathcal{T}^{w\leq 0}[1] \subset \mathcal{T}^{w\leq 0}$;
(iii) For every $a \in \mathcal{T}^{w\geq 0}$ and $b \in \mathcal{T}^{w\leq 0}[1]$, we have $\mathrm{Hom}_{\mathcal{T}}(a, b) = 0$;
(iv) For every $a \in \mathcal{T}$ there exists a distinguished triangle $c[-1] \to a \to b \to c$ in $\mathcal{T}$ with $b \in \mathcal{T}^{w\leq 0}$ and $c \in \mathcal{T}^{w\geq 0}$.

Given an integer $n \in \mathbb{Z}$, let $\mathcal{T}^{w\geq n} := \mathcal{T}^{w\geq 0}[-n]$, $\mathcal{T}^{w\leq n} := \mathcal{T}^{w\leq 0}[-n]$, and $\mathcal{T}^{w=n} := \mathcal{T}^{w\geq n} \cap \mathcal{T}^{w\leq n}$. The objects belonging to $\cup_{n\in\mathbb{Z}}\mathcal{T}^{w=n}$ are called *w-pure* and the additive subcategory $\mathcal{H} := \mathcal{T}^{w=0}$ is called the *heart* of the weight structure. Finally, a weight structure w is called *bounded* if $\mathcal{T} = \cup_{n\in\mathbb{Z}}\mathcal{T}^{w\geq n} = \cup_{n\in\mathbb{Z}}\mathcal{T}^{w\leq n}$.

Assumption: Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a symmetric monoidal triangulated category

[4] Following [7], we will use the so-called *cohomological convention* for weight structures. This differs from the homological convention used in [8, 10, 11, 41].

equipped with a weight structure w. Throughout the article, we will always assume that the symmetric monoidal structure is *w-pure* in the sense that the tensor product $-\otimes -$ (as well as the $\otimes$-unit $\mathbf{1}$) restricts to the heart $\mathcal{H}$.

Remark 3.2 (Self-duality). The notion of weight structure is (categorically) self-dual. Given a triangulated category $\mathcal{T}$ equipped with a weight structure w, the opposite triangulated category $\mathcal{T}^{\mathrm{op}}$ inherits the opposite weight structure w^{op} with $(\mathcal{T}^{\mathrm{op}})^{w^{\mathrm{op}}\leq 0} := \mathcal{T}^{w\geq 0}$ and $(\mathcal{T}^{\mathrm{op}})^{w^{\mathrm{op}}\geq 0} := \mathcal{T}^{w\leq 0}$.

Definition 3.3. An exact functor $F\colon \mathcal{T} \to \mathcal{T}'$ between triangulated categories equipped with weight structures w and w', respectively, is called *weight-exact* if $F(\mathcal{T}^{w\leq 0}) \subseteq \mathcal{T}'^{w'\leq 0}$ and $F(\mathcal{T}^{w\geq 0}) \subseteq \mathcal{T}'^{w'\geq 0}$.

Remark 3.4. Whenever the weight structure w is bounded, an exact functor $F\colon \mathcal{T} \to \mathcal{T}'$ is weight-exact if and only if $F(\mathcal{T}^{w=0}) \subseteq \mathcal{T}^{w'=0}$; see [10, Prop. 1.2.3(5)].

3.1. Weight complexes. Let $\mathcal{T}$ be a triangulated category equipped with a weight structure w. Following [7, Def. 2.2.1] (see also [8, §2.2]), we can assign to every object $a \in \mathcal{T}$ a certain (cochain) weight $\mathcal{H}$-complex $t(a) : \cdots \to a^{m-1} \to a^m \to a^{m+1} \to \cdots$. For example, if $a \in \mathcal{T}^{w=0}$, then we can take for $t(a)$ the complex $\cdots \to 0 \to a \to 0 \to \cdots$ supported in degree 0. As explained in *loc. cit.*, the assignment $a \mapsto t(a)$ is well-defined only up to homotopy equivalence. Nevertheless, we will use the notation a^p for the p^{th} term of some choice of a weight $\mathcal{H}$-complex $t(a)$. This is justified by the next result:

Proposition 3.5. *(see* [10, Prop. 1.4.2(6)-(7)]*)*

(i) *Let $F\colon \mathcal{T} \to \mathcal{T}'$ be a weight-exact functor as in Definition 3.3. If $t(a)$ is a weight $\mathcal{H}$-complex for a, then $F(t(a))$ is a weight $\mathcal{H}'$-complex for $F(a)$;*

(ii) *Given an additive functor $G\colon \mathcal{H} \to \mathrm{A}$, with values in an abelian category, the assignment $a \mapsto H^0(G(t(a)))$ yields a well-defined (i.e. independent of the choice of $t(a)$) homological functor[5] $\mathrm{H}_0\colon \mathcal{T} \to \mathrm{A}$. Moreover, the assignment $G \mapsto \mathrm{H}_0$ is natural in the functor G.*

We denote by H_n the precomposition of H_0 with the n^{th} suspension functor of $\mathcal{T}$.

Remark 3.6. Note that if $a \in \mathcal{T}^{w=m}$, then $\mathrm{H}_n(a) = 0$ for every $n \neq m$.

Remark 3.7. Following the referee's suggestion, we recall here in an informal way the construction of weight complexes. Let $\mathcal{T}$ be a triangulated category equipped with a weight structure w. Given $a \in \mathcal{T}$ and $m \in \mathbb{Z}$, the axiom (iv) of Definition 3.1 implies the existence of a distinguished triangle $b^m \to a \to c^m \to b^m[1]$ with $b^m \in \mathcal{T}^{w\geq m}$ and $c^m \in \mathcal{T}^{w\leq m-1}$. These triangles are *not* determined (up to isomorphism) by the couple (a, m). Nevertheless, given a morphism $g\colon a \to a'$ and an integer $m' \leq m$, we can extend g to a morphism

[5]The homological functors obtained this way are called *pure* due to their relation with Deligne's theory of weights on cohomology; see [8, Rk. 2.4.5(5)].

between the corresponding triangles; this extension is unique whenever $m' < m$. This fact, applied to a fixed object a and to all integers m, yields connecting morphisms $\partial^m \colon b^{m+1} \to b^m$. If one shifts the cone of ∂^m by $[m]$, we then obtain a sequence of objects a^m in $\mathcal{T}^{w=0}$. Moreover, the corresponding triangles give rise to connecting morphisms which yield a weight complex for a. The above considerations show that weight complexes are naturally "respected" by weight-exact functors. This naturality easily carries over to the pure functors considered in the above Proposition 3.5(ii). However, these pure functors do not depend on any choices up to canonical isomorphisms.

3.2. Karoubization. Given a category $\mathcal{C}$, let us write $\mathrm{Kar}(\mathcal{C})$ for its Karoubization. Recall that the objects of $\mathrm{Kar}(\mathcal{C})$ are the pairs (a, e), with $a \in \mathcal{C}$ and e an idempotent of the ring of endomorphisms $\mathrm{End}_{\mathcal{C}}(a, a)$. The morphisms are given by $\mathrm{Hom}_{\mathrm{Kar}(\mathcal{C})}((a, e), (b, e')) := e \circ \mathrm{Hom}_{\mathcal{C}}(a, b) \circ e'$. By construction, $\mathrm{Kar}(\mathcal{C})$ comes equipped with the canonical functor $\mathcal{C} \to \mathrm{Kar}(\mathcal{C}), a \mapsto (a, \mathrm{id})$. Whenever $\mathcal{C}$ is symmetric monoidal, resp. triangulated, the category $\mathrm{Kar}(\mathcal{C})$ is also symmetric monoidal, resp. triangulated; see [6, Thm. 1.5]. Moreover, the canonical functor $\mathcal{C} \to \mathrm{Kar}(\mathcal{C})$ becomes symmetric monoidal, resp. exact.
The following result relates Karoubian categories to bounded weight structures.

Proposition 3.8. *Let $\mathcal{T}$ be a Karoubian triangulated category. Assume that there exists a full additive subcategory $\mathcal{H}' \subset \mathcal{T}$ that generates*[6] *$\mathcal{T}$ and which is* negative *in $\mathcal{T}$ in the sense that there are no $\mathcal{T}$-extensions of positive degrees between objects of $\mathcal{H}'$. Under these assumptions, there exists a unique bounded weight structure w on $\mathcal{T}$ such that its heart $\mathcal{H}$ contains $\mathcal{H}'$. Moreover, $\mathcal{H}$ is equivalent to* $\mathrm{Kar}(\mathcal{H}')$.

Proof. The proof is an immediate consequence of [7, Thm. 4.3.2 II and Prop. 5.2.2]; consult also [11, Cor. 2.1.2] for the generalization of this statement to the case where $\mathcal{T}$ is not necessarily Karoubian. □

4. Proof of Theorem 1.1

We start with the following auxiliary result:

Proposition 4.1. *A symmetric monoidal triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$, equipped with a weight structure w, has the w-Picard property (see §1) if and only if all its $\otimes$-invertible objects are w-pure.*

Proof. Let $(a, n), (b, m) \in \mathrm{Pic}(\mathcal{H}) \times \mathbb{Z}$. On the one hand, when $n = m$, we have $a[n] \simeq b[m]$ in $\mathcal{T}$ if and only if $a \simeq b$ in $\mathcal{H}$. This follows from the fact that the suspension functor is an auto-equivalence of $\mathcal{T}$. On the other hand, when $n \neq m$, we have $a[n] \not\simeq b[m]$ in $\mathcal{T}$. This follows from the fact that $\mathrm{Hom}_{\mathcal{T}}(a[n], b[m])$, resp. $\mathrm{Hom}_{\mathcal{T}}(b[m], a[n])$, is zero whenever $m < n$, resp. $n < m$; see Definition 3.1(iii). This implies that the canonical group homomorphism

$$\mathrm{Pic}(\mathcal{H}) \times \mathbb{Z} \longrightarrow \mathrm{Pic}(\mathcal{T}) \qquad (a, n) \mapsto a[n] \tag{4.2}$$

[6] *i.e.* the smallest thick triangulated subcategory of $\mathcal{T}$ containing $\mathcal{H}'$ is $\mathcal{T}$ itself.

is injective. Consequently, we conclude that the category $\mathcal{T}$ has the w-Picard property if and only if (4.2) is surjective. In other words, $\mathcal{T}$ has the w-Picard property if and only if all its $\otimes$-invertible objects are w-pure. $\square$

Remark 4.3. Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a symmetric monoidal triangulated category equipped with a weight structure w. The arguments used in the proof of Proposition 4.1 allow us to conclude that if by hypothesis $a[n] \otimes b[m] \simeq \mathbf{1}$ for certain objects $a, b \in \mathcal{H}$ and integers $n, m \in \mathbb{Z}$, then $n = -m$ and a is the $\otimes$-inverse of b.

Let us now prove Theorem 1.1. Let $b \in \mathcal{T}$ be a (fixed) $\otimes$-invertible object. Thanks to Proposition 4.1, it suffices to prove that b is w-pure. By assumption, there exists a full, additive, conservative, symmetric monoidal functor $G\colon \mathcal{H} \to \mathrm{A}$ into a symmetric monoidal semi-simple abelian category which is moreover local. Proposition 3.5(ii) applied to this functor G yields well-defined homological functors $\mathrm{H}_n\colon \mathcal{T} \to \mathrm{A}, n \in \mathbb{Z}$.
Consider the homological functor $\mathcal{T} \to \mathrm{A}, a \mapsto \mathrm{H}_0(a \otimes b)$. Since by assumption the weight structure w is bounded, [7, Thm. 2.3.2] applied to the preceding homological functor yields a convergent Künneth spectral sequence

$$E_1^{pq} = \mathrm{H}_q(a^p \otimes b) \Rightarrow \mathrm{H}_{p+q}(a \otimes b)\,. \tag{4.4}$$

The object a^p belongs to the heart $\mathcal{H}$ and the functor $a^p \otimes -\colon \mathcal{T} \to \mathcal{T}$ is weight-exact in the sense of Definition 3.3. Using the fact that $t(b)$ is a weight $\mathcal{H}$-complex for b, we conclude from Proposition 3.5(i) that $a^p \otimes t(b)$ is a weight $\mathcal{H}$-complex for $a^p \otimes b$. Therefore, the complex computing $\mathrm{H}_*(a^p \otimes b)$ can be obtained from the complex computing $\mathrm{H}_*(b)$ by tensoring with $G(a^p)$ (recall that G is symmetric monoidal). Since the category A is semi-simple, it follows then that $\mathrm{H}_q(a^p \otimes b) \simeq G(a^p) \otimes \mathrm{H}_q(b)$. Furthermore, the functoriality of the assignment $G \mapsto \mathrm{H}_0$ mentioned in Proposition 3.5(ii) implies that the differential $E_1^{pq} \to E_1^{(p+1)q}$ equals the corresponding morphism induced by the boundary $a^p \to a^{p+1}$ (tensored with b). Making use once again of the semi-simplicity of A, we conclude that $E_2^{pq} \simeq \mathrm{H}_p(a) \otimes \mathrm{H}_q(b)$. Recall from [7, Thm. 2.3.2] that, in contrast with the E_1-terms, the E_2-terms are essentially independent of the choice of (the terms of) the weight complex $t(a)$. Let us denote by m_a, resp. m_a', the smallest, resp. largest, integer such that $\mathrm{H}_n(a) = 0$ for every $n < m_a$, resp. $n > m_a'$; the existence of such integers follows from the fact that the weight structure w is bounded. Similarly, let m_b, resp. m_b', be the smallest, resp. largest, integer such that $\mathrm{H}_n(b) = 0$ for every $n < m_b$, resp. $n > m_b'$. Since by assumption the category A is local, we have $\mathrm{H}_{m_a}(a) \otimes \mathrm{H}_{m_b}(b) \neq 0$ and $\mathrm{H}_{m_a'}(a) \otimes \mathrm{H}_{m_b'}(b) \neq 0$. Using the second page of the spectral sequence (4.4), we conclude that

$$\mathrm{H}_{m_a+m_b}(a \otimes b) \neq 0 \quad \text{and} \quad \mathrm{H}_{m_a'+m_b'}(a \otimes b) \neq 0\,. \tag{4.5}$$

Now, recall that b is a $\otimes$-invertible object. Therefore, by definition, we have $a \otimes b \simeq \mathbf{1}$ for some ($\otimes$-invertible) object $a \in \mathcal{T}$. Since $\mathrm{H}_n(a \otimes b) \simeq \mathrm{H}_n(\mathbf{1}) = 0$ for every $n \neq 0$, we conclude from (4.5) that $m_b = m_b'$, $m_a = m_a'$, and $m_a = -m_b$.

Thanks to Proposition 4.6 below, this implies that $b \in \mathcal{T}^{w=m_b}$. In particular, the object b is w-pure, and so the proof is finished.

Proposition 4.6. *(Conservativity I) Let $\mathcal{T}$ be a triangulated category equipped with a bounded weight structure w. Assume that there exists a full, additive, conservative functor $G\colon \mathcal{H} \to \mathrm{A}$ from the heart of w into a semi-simple abelian category. Under this assumption, an object $b \in \mathcal{T}$ belongs to $\mathcal{T}^{w=m}$ if and only if $\mathrm{H}_n(b) = 0$ for every $n \neq m$.*

Proof. Consult [8, Cor. 2.3.5]. □

Remark 4.7 (Künneth spectral sequence). (i) Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a symmetric monoidal triangulated equipped with a bounded weight structure w, and $G\colon \mathcal{H} \to \mathrm{A}$ a symmetric monoidal additive functor. Consider the associated homological functors $\mathrm{H}_n\colon \mathcal{T} \to \mathrm{A}, n \in \mathbb{Z}$. The arguments used in the proof of Theorem 1.1 allow us to conclude that there exists a convergent Künneth spectral sequence

$$E_1^{pq} = \mathrm{H}_q(a^p \otimes b) \Rightarrow \mathrm{H}_{p+q}(a \otimes b)\,.$$

Assume that the (abelian) category A is moreover semi-simple and local. Then, given any $\otimes$-invertible object $b \in \mathcal{T}$, there exists an integer m_b such that $\mathrm{H}_n(b) = 0$ for every $n \neq m_b$ and $\mathrm{H}_{m_b}(b) \in \mathrm{A}$ is $\otimes$-invertible.
(ii) Given non-zero objects a and b as in item (i), Proposition 4.6 yields the existence of integers m_a and m_b satisfying the conditions described in the proof of Theorem 1.1. This implies that $\mathrm{H}_{m_a+m_b}(a \otimes b) \neq 0$, and consequently that $a \otimes b \neq 0$. In particular, $\mathcal{T}$ is *local* in the sense of [5, §4]; consult Proposition 4.2 from *loc. cit.*

5. Proof of Theorem 1.2

Let $b \in \mathcal{T}$ be a $\otimes$-invertible object. Thanks to Proposition 4.1, it suffices to prove that b is w-pure. Since the functors $\iota_{\kappa(\mathfrak{p})}\colon \mathcal{T} \to \mathcal{T}_{\kappa(\mathfrak{p})}$ are symmetric monoidal, and by assumption the categories $\mathcal{T}_{\kappa(\mathfrak{p})}$ have the $w_{\kappa(\mathfrak{p})}$-Picard property, the objects $\iota_{\kappa(\mathfrak{p})}(b)$ are $w_{\kappa(\mathfrak{p})}$-pure. Concretely, $\iota_{\kappa(\mathfrak{p})}(b)$ belongs to $\mathcal{T}_{\kappa(\mathfrak{p})}^{w=m_{\kappa(\mathfrak{p})}}$ for some integer $m_{\kappa(\mathfrak{p})} \in \mathbb{Z}$. Our goal is to prove that all the integers $m_{\kappa(\mathfrak{p})}$, with $\mathfrak{p} \in \mathrm{Spec}(R)$, are equal and that the object b belongs to $\mathcal{T}^{w=m_{k(\mathfrak{p})}}$. We start by addressing the first goal. Since by assumption the commutative ring R is indecomposable, its spectrum $\mathrm{Spec}(R)$ is connected. Hence, it suffices to verify that $m_{\kappa(\mathfrak{p})} = m_{\kappa(\mathfrak{P})}$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$ belonging to the closure of a prime ideal $\mathfrak{P} \in \mathrm{Spec}(R)$; in the particular case where R is moreover an integral domain we can simply take $\mathfrak{P} = \{0\}$. Note that the assumptions of Theorem 1.2, as well as the definition of the integers $m_{\kappa(\mathfrak{p})}$ and $m_{\kappa(\mathfrak{P})}$, are (categorically) self-dual; see Remark 3.2. Therefore, it is enough to verify the inequalities $m_{\kappa(\mathfrak{p})} \geq m_{\kappa(\mathfrak{P})}$.
Given an R-algebra S, consider the abelian category $\mathrm{PShv}^S(\mathcal{H})$ of R-linear functors from $\mathcal{H}^{\mathrm{op}}$ to the category of S-modules. Note that the Yoneda functor

$$(5.1) \qquad \mathcal{H} \longrightarrow \mathrm{PShv}^S(\mathcal{H}) \qquad a \mapsto (c \mapsto \mathrm{Hom}_{\mathcal{H}}(c, a) \otimes_R S)$$

induces a fully faithful embedding of $\mathcal{H} \otimes_R S$ into the full subcategory of $\mathrm{PShv}^S(\mathcal{H})$ consisting of projective objects; see [24, Lem. 8.1]. Note also that every R-algebra homomorphism $S \to S'$ gives rise to a functor $- \otimes_S S' \colon \mathrm{PShv}^S(\mathcal{H}) \to \mathrm{PShv}^{S'}(\mathcal{H})$. Since $\mathrm{PShv}^S(\mathcal{H})$ is abelian, Proposition 3.5(ii) yields a homological functor

$$\mathrm{H}_0^S \colon \mathcal{T} \longrightarrow \mathrm{PShv}^S(\mathcal{H}) \qquad a \mapsto \big(c \mapsto H^0(\mathrm{Hom}_{\mathcal{H}}(c, t(a)) \otimes_R S)\big)\ .$$

Recall from assumption (A2) that the functor $\iota_{\kappa(\mathfrak{p})}$ induces a $\otimes$-equivalence of categories $\mathrm{Kar}(\mathcal{H} \otimes_R \kappa(\mathfrak{p})) \simeq \mathcal{H}_{\kappa(\mathfrak{p})}$. This implies that $\mathrm{H}_0^{\kappa(\mathfrak{p})}$ factors through $\iota_{\kappa(\mathfrak{p})}$. Consequently, thanks to Remark 3.6, we have $\mathrm{H}_n^{\kappa(\mathfrak{p})}(b) = 0$ for every $n \neq m_{\kappa(\mathfrak{p})}$.
Let us denote by Q the localization of $R/\mathfrak{P}$ at the prime ideal $\mathfrak{p}$. Note that Q is a local Noetherian integral domain with fraction field $\kappa(\mathfrak{P})$. Recall from assumption (A1) that the commutative ring R is Noetherian and that the R-modules of morphisms of the heart $\mathcal{H}$ are finitely generated and flat. Thanks to the universal coefficients theorem, this implies that $\mathrm{H}_l^Q(b) \otimes_Q \kappa(\mathfrak{p}) = \mathrm{H}_l^{\kappa(\mathfrak{p})}(b)$, with l being the largest integer such that $\mathrm{H}_l^Q(b) \neq 0$. Consequently, by applying the Nakayama lemma to the local ring Q and to the (objectwise) finitely generated Q-module $\mathrm{H}_l^Q(b)$, we conclude that $\mathrm{H}_l^{\kappa(\mathfrak{p})}(b) \neq 0$. Hence, the equality $m_{\kappa(\mathfrak{p})} = l$ holds. Now, since $\kappa(\mathfrak{P})$ is a flat Q-module, the universal coefficients theorem yields that $\mathrm{H}_n^{\kappa(\mathfrak{P})}(b) = 0$ for every $n > l$. This allows us to conclude that $l = m_{k(\mathfrak{p})} \geq m_{k(\mathfrak{P})}$.
Let us now address the second goal, *i.e.* prove that $b \in \mathcal{T}^{w=m}$ with $m := m_{k(\mathfrak{p})}$. Making use of Remark 3.2 once again, we observe that it suffices to prove that $b \in \mathcal{T}^{w \leq m}$. Thanks to Proposition 5.2 below, it is enough to verify that $\mathrm{H}_n^R(b) = 0$ for every $n > m$. Let us denote by l the largest integer such that $\mathrm{H}_l^R(b) \neq 0$. An argument similar to the one used in the preceding paragraph, implies that $\mathrm{H}_l^R(b) \otimes_R \kappa(\mathfrak{p}) = \mathrm{H}_l^{\kappa(\mathfrak{p})}(b)$ for every $\mathfrak{p} \in \mathrm{Spec}(R)$. Since $\mathrm{H}_n^{\kappa(\mathfrak{p})}(b) = 0$ for all $n > m$ and $\mathfrak{p} \in \mathrm{Spec}(R)$, we then conclude that $\mathrm{H}_n^R(b) = 0$ for every $n > m$. This finishes the proof.

Proposition 5.2 (Conservativity II). *Let $\mathcal{T}$ be a triangulated category equipped with a bounded weight structure w whose heart $\mathcal{H}$ is R-linear and small. Consider the associated homological functors $\mathrm{H}_n^R \colon \mathcal{T} \to \mathrm{PShv}^R(\mathcal{H}), n \in \mathbb{Z}$. Under these assumptions, an object $b \in \mathcal{T}$ belongs to $\mathcal{T}^{w \leq m}$ if and only if $\mathrm{H}_n^R(b) = 0$ for every $n > m$.*

Proof. Combine [8, Prop. 2.3.4] with [8, Rk. 2.3.6(2)]. □

Remark 5.3. (i) Suppose that in Theorem 1.2 the commutative ring R is of the form $\Pi_{j=1}^r R_j$, with R_j an indecomposable Noetherian ring. In this case, the corresponding idempotents $e_j \in R$ give naturally rise to categorical decompositions $\mathcal{T} \simeq \Pi_{j=1}^r \mathcal{T}_j$ and $\mathcal{H} \simeq \Pi_{j=1}^r \mathcal{H}_j$. By applying Theorem 1.2 to each one of the categories $\mathcal{T}_j$, we conclude that

$$\mathrm{Pic}(\mathcal{T}) \simeq \Pi_{j=1}^r \mathrm{Pic}(\mathcal{T}_j) \simeq \Pi_{j=1}^r (\mathrm{Pic}(\mathcal{H}_j) \times \mathbb{Z}) \simeq \mathrm{Pic}(\mathcal{H}) \times \mathbb{Z}^r$$

whenever all the triangulated categories $\mathcal{T}_j$ are non-zero;

(ii) At assumption (A2) of Theorem 1.2, instead of working with all prime ideals $\mathfrak{p} \in \mathrm{Spec}(R)$, note that it suffices to consider any connected subset of $\mathrm{Spec}(R)$ that contains all maximal ideals of R. For example, in the particular case where R is local, it suffices to consider the (unique) closed point $\mathfrak{p}_o$ of $\mathrm{Spec}(R)$.

6. Proof of Theorem 2.3

Recall from [24, Part 4 and Lecture 20][39] the construction of the symmetric monoidal triangulated category $\mathrm{DM}_{\mathrm{gm}}(k; R)$. Given any two zero-dimensional smooth k-schemes X and Y, we have trivial positive Ext-groups:

$$\mathrm{Hom}_{\mathrm{DMA}(k;R)}(M(X)_R, M(Y)_R[n]) = 0 \qquad n > 0\,.$$

This implies that the subcategory $\mathrm{AM}(k; R) \subset \mathrm{DMA}(k; R)$ is *negative* in the sense of Proposition 3.8. Consequently, the subcategory $\mathcal{A}(k; R) \subset \mathrm{D}\mathcal{A}(k; R)$ is also negative. Making use of Proposition 3.8, we then conclude that the $\mathrm{D}\mathcal{A}(k; R)$ carries a bounded weight structure w_R with heart $\mathcal{A}(k; R)$.

Let us now show that the category $\mathrm{D}\mathcal{A}(k; R)$ has the w_R-Picard property; note that this automatically concludes the proof. By construction, $\mathcal{A}(k; R)$ is essentially small. Moreover, we have natural isomorphisms

$$\mathrm{Hom}_{\mathrm{D}\mathcal{A}(k;R)}(M(X)_R, M(Y)_R) \simeq CH^0(X \times Y)_R\,.$$

Since the R-modules $CH^0(X \times Y)_R$ are free, assumptions (A1) of Theorem 1.2 are verified. In what concerns assumptions (A2), take for $\mathcal{T}_{\kappa(\mathfrak{p})}$ the category $\mathrm{D}\mathcal{A}(k; \kappa(\mathfrak{p}))$ and for $\iota_{\kappa(\mathfrak{p})}$ the functor $- \otimes_R \kappa(\mathfrak{p})\colon \mathrm{D}\mathcal{A}(k; R) \to \mathrm{D}\mathcal{A}(k; \kappa(\mathfrak{p}))$. By construction, the latter functor is weight-exact (see Remark 3.4), symmetric monoidal, and induces an equivalence of symmetric monoidal categories between $\mathrm{Kar}(\mathcal{A}(k; R) \otimes_R \kappa(\mathfrak{p}))$ and $\mathcal{A}(k; \kappa(\mathfrak{p}))$. This shows that assumptions (A2) are also verified.

Let us now prove that the categories $\mathrm{D}\mathcal{A}(k; \kappa(\mathfrak{p}))$ have the $w_{\kappa(\mathfrak{p})}$-Picard property; thanks to Theorem 1.2 this implies that $\mathrm{D}\mathcal{A}(k; R)$ has the w_R-Picard property. In order to do so, we will make use of Theorem 1.1. Concretely, we will prove that the categories $\mathcal{A}(k; \kappa(\mathfrak{p}))$ are abelian semi-simple and local. Let us write L for the composite of the finite separable field extensions $l_i/k, i \in I$, inside $\overline{k}$, G for the profinite Galois group $\mathrm{Gal}(L/k)$, and G_i for the finite Galois group $\mathrm{Gal}(l_i/k)$. Thanks to assumption (B1), there is a $\otimes$-equivalence between $\mathcal{A}(k; \kappa(\mathfrak{p}))$ and the category of finite dimensional $\kappa(\mathfrak{p})$-linear continuous G-representations $\mathrm{Rep}_{\kappa(\mathfrak{p})}(\mathrm{G})$. Consequently, since $\mathrm{G} \simeq \lim_{i \in I} \mathrm{G}_i$, we conclude that $\mathcal{A}(k; \kappa(\mathfrak{p})) \simeq \mathrm{colim}_{i \in I} \mathrm{Rep}_{\kappa(\mathfrak{p})}(\mathrm{G}_i)$. Now, since the group G_i is finite, the category $\mathrm{Rep}_{\kappa(\mathfrak{p})}(\mathrm{G}_i)$ may be identified with the category of finitely generated (right) $\kappa(\mathfrak{p})[\mathrm{G}_i]$-modules. Thanks to assumption (B2), the degree of the field extension l_i/k is invertible in R and hence in $\kappa(\mathfrak{p})$. The (classical) Maschke theorem then implies that the category $\mathrm{Rep}_{\kappa(\mathfrak{p})}(\mathrm{G}_i)$ is abelian semi-simple. Note that this category is moreover local since the tensor product is defined on the underlying $\kappa(\mathfrak{p})$-vector spaces. The proof follows now automatically from the above description of the categories $\mathcal{A}(k;\kappa(\mathfrak{p}))$.

7. Proof of Theorem 2.6

Let us denote by $\mathcal{A}\mathrm{T}(k;R)$ the smallest additive, Karoubian, full subcategory of $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$ containing the objects $M(X)_R(m)[2m]$, with $M(X)_R \in \mathcal{A}$ and $m \in \mathbb{Z}$. Under these notations, we have trivial positive Ext-groups:

$$\mathrm{Hom}_{\mathrm{D}\mathcal{A}\mathrm{T}(k;R)}(M(X)_R(m)[2m], M(Y)_R(m')[2m'][n]) = 0 \qquad n > 0\,.$$

This implies that the subcategory $\mathcal{A}\mathrm{T}(k;R) \subset \mathrm{D}\mathcal{A}\mathrm{T}(k;R)$ is negative in the sense of Proposition 3.8. The motives of the zero-dimensional smooth k-schemes, as well as the Tate motives, are stable under tensor product. Therefore, $\mathcal{A}\mathrm{T}(k;R)$ generates[7] the triangulated category $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$. Making use of Proposition 3.8 once again, we then conclude that $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$ carries a bounded weight structure w_R with heart $\mathcal{A}\mathrm{T}(k;R)$. Thanks to the equivalence of categories

$$\mathrm{Gr}_{\mathbb{Z}}\mathcal{A}(k;R) \xrightarrow{\simeq} \mathcal{A}\mathrm{T}(k;R) \qquad \{M(X_m)\}_{m\in\mathbb{Z}} \mapsto \bigoplus_{m\in\mathbb{Z}} M(X_m)(m)[2m]\,,$$

an argument similar to the one of the proof of Theorem 2.3 implies that the category $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$ has the w_R-Picard property. Consequently, we have $\mathrm{Pic}(\mathrm{D}\mathcal{A}\mathrm{T}(k;R)) \simeq \mathrm{Pic}(\mathcal{A}\mathrm{T}(k;R)) \times \mathbb{Z}$. The proof follows now from the natural isomorphisms

$$\mathrm{Pic}(\mathcal{A}\mathrm{T}(k;R)) \simeq \mathrm{Pic}(\mathrm{Gr}_{\mathbb{Z}}\mathcal{A}(k;R)) \simeq \mathrm{Pic}(\mathcal{A}(k;R)) \times \mathbb{Z}\,.$$

8. Proof of Theorem 2.11

Recall from Ayoub [2, §4][3, §2.1.1] the construction of the symmetric monoidal triangulated category $\mathbf{DA}(k;\mathbb{Z})$ (with respect to the Nisnevich topology); in what follows, we write $\mathrm{Boot}(k;\mathbb{Z})$ for the thick triangulated subcategory generated by the $\otimes$-unit $\Sigma^\infty(\mathrm{Spec}(k)_+)_{\mathbb{Z}}$. By construction, we have an exact symmetric monoidal functor $(-)_{\mathbb{Z}}\colon \mathrm{SH}(k) \to \mathbf{DA}(k;\mathbb{Z})$ which restricts to the bootstrap categories. Let $\mathrm{P}(k)$, resp. $\mathrm{P}(k;\mathbb{Z})$, be the smallest additive, Karoubian, full subcategory of $\mathrm{Boot}(k)$, resp. $\mathrm{Boot}(k;\mathbb{Z})$, containing the $\otimes$-unit $\Sigma^\infty(\mathrm{Spec}(k)_+)$, resp. $\Sigma^\infty(\mathrm{Spec}(k)_+)_{\mathbb{Z}}$. We have trivial positive Ext-groups (see [40, Thm. 4.14]):

$$\mathrm{Hom}_{\mathrm{Boot}(k)}(\Sigma^\infty(\mathrm{Spec}(k)_+), \Sigma^\infty(\mathrm{Spec}(k)_+)[n]) = 0 \quad n > 0\,;$$

similarly for $\mathrm{Boot}(k;\mathbb{Z})$. This implies that the subcategory $\mathrm{P}(k) \subset \mathrm{Boot}(k)$, resp. $\mathrm{P}(k;\mathbb{Z}) \subset \mathrm{Boot}(k;\mathbb{Z})$, is negative in the sense of Proposition 3.8. Making use of this latter proposition, we then conclude that the category $\mathrm{Boot}(k)$, resp. $\mathrm{Boot}(k;\mathbb{Z})$, carries a bounded weight structure w, resp. $w_{\mathbb{Z}}$, with heart $\mathrm{P}(k)$, resp. $\mathrm{P}(k;\mathbb{Z})$.

[7] *i.e.* the smallest thick triangulated subcategory containing $\mathcal{A}\mathrm{T}(k;R)$ is $\mathrm{D}\mathcal{A}\mathrm{T}(k;R)$.

Let us now show that the category $\mathrm{Boot}(k)$ has the w-Picard property. Thanks to the ring isomorphism (2.10), $\mathrm{P}(k)$ identifies with the category $\mathrm{Proj}(GW(k))$ of finitely generated projective $GW(k)$-modules. Moreover, the functor $(-)_{\mathbb{Z}}$ restricts to an equivalence of categories $\mathrm{P}(k) \stackrel{\simeq}{\to} \mathrm{P}(k;\mathbb{Z})$; this is an immediate consequence of [9, Prop. 2.3.7] (this equivalence also follows easily from [27, Thm. 6.37]). Consequently, since the Grothendieck-Witt ring $GW(k)$ is indecomposable (see [18, Prop. 2.2]), all the assumptions (A1) of Theorem 1.2 (with $R = GW(k)$) are verified. In what concerns assumptions (A2), take for $\mathcal{T}_{\kappa(\mathfrak{p})}$ the bounded derived category $\mathcal{D}^b(\kappa(\mathfrak{p}))$ of finite dimensional $\kappa(\mathfrak{p})$-vector spaces $\mathrm{Vect}(\kappa(\mathfrak{p}))$ and for $\iota_{\kappa(\mathfrak{p})}$ the composed functor:

$$\mathrm{Boot}(k) \xrightarrow{(-)_{\mathbb{Z}}} \mathrm{Boot}(k;\mathbb{Z}) \xrightarrow{t(-)} K^b(\mathrm{Proj}(GW(k))) \xrightarrow{-\otimes_{GW(k)}\kappa(\mathfrak{p})} \mathcal{D}^b(\kappa(\mathfrak{p}))\,. \tag{8.1}$$

Some explanations are in order: since the category $\mathbf{DA}(k;\mathbb{Z})$ is defined as the localization of a certain category of complexes, it admits a tensor differential graded (=dg) enhancement. Making use of [4, Lem. 18], we then conclude that the weight complex construction gives rise to an exact symmetric monoidal functor $t(-)$ with values in the bounded homotopy category of $\mathrm{Proj}(GW(k))$. By construction, the composed functor (8.1) is weight-exact, symmetric monoidal, and induces a $\otimes$-equivalence of categories between $\mathrm{Kar}(\mathrm{P}(k)\otimes_{GW(k)}\kappa(\mathfrak{p}))$ and $\mathrm{Vect}(\kappa(\mathfrak{p}))$. This shows that the assumptions (A2) are also verified. Finally, since the categories $\mathcal{D}^b(\kappa(\mathfrak{p}))$ clearly have the $w_{\kappa(\mathfrak{p})}$-Picard property, we conclude from Theorem 1.2 that $\mathrm{Boot}(k)$ has the w-Picard property. This finishes the proof.

9. Proof of Theorem 2.17

Recall from [34, §9][33, §4] the construction of the symmetric monoidal triangulated category $\mathrm{KMM}(k;R)$. Given any two finite separable field extensions l_1/k and l_2/k, we have trivial positive Ext-groups (see [33, Prop. 4.4]):

$$\mathrm{Hom}_{\mathrm{NMAM}(k;R)}(U(l_1)_R, U(l_2)_R[n]) \simeq \pi_{-n}(K(l_1\otimes_k l_2)\wedge HR) = 0 \qquad n>0\,.$$

This implies that the subcategory $\mathrm{AM}(k;R)\subset \mathrm{NMAM}(k;R)$ is negative in the sense of Proposition 3.8. Consequently, the subcategory $\mathcal{A}(k;R)\subset \mathrm{NM}\mathcal{A}(k;R)$ is also negative. Making use of Proposition 3.8, we then conclude that the category $\mathrm{NM}\mathcal{A}(k;R)$ carries a bounded weight structure[8] w_R with heart $\mathcal{A}(k;R)$. Now, a proof similar to the one of Theorem 2.3, with $\mathrm{D}\mathcal{A}(k;R)$ and $\mathrm{D}\mathcal{A}(k;\kappa(\mathfrak{p}))$ replaced by $\mathrm{NM}\mathcal{A}(k;R)$ and $\mathrm{NM}\mathcal{A}(k;\kappa(\mathfrak{p}))$, respectively, allows us to conclude that the category $\mathrm{NM}\mathcal{A}(k;R)$ has the w_R-Picard property. Consequently, we have $\mathrm{Pic}(\mathrm{NM}\mathcal{A}(k;R))\simeq \mathrm{Pic}(\mathcal{A}(k;R))\times\mathbb{Z}$.

[8] A bounded weight structure on the category of noncommutative mixed motives was originally constructed in [36, Thm. 1.1].

10. Proof of Theorem 2.22

Item (i). Similarly to the proof of Theorem 2.17, given any two central simple k-algebras A and B, we have trivial positive Ext-groups (see [33, Prop. 4.4]):

$$\mathrm{Hom}_{\mathrm{NMCSA}(k;R)}(U(A)_R, U(B)_R[n]) \simeq \pi_{-n}(K(A^{\mathrm{op}} \otimes_k B) \wedge HR) = 0 \quad n > 0\,.$$

This implies that the subcategory $\mathrm{CSA}(k;R) \subset \mathrm{NMCSA}(k;R)$ is negative in the sense of Proposition 3.8. Making use of this latter proposition, we then conclude that $\mathrm{NMCSA}(k;R)$ carries a bounded weight structure w_R with heart $\mathrm{CSA}(k;R)$.

Let us now show that the category $\mathrm{NMCSA}(k;R)$ has the w_R-Picard property. By construction, the category $\mathrm{CSA}(k;R)$ is essentially small. Moreover, since the K-theory spectrum $K(A^{\mathrm{op}} \otimes_k B)$ is connective, we have natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathrm{CSA}(k;R)}(U(A)_R, U(B)_R) &\simeq \pi_0(K(A^{\mathrm{op}} \otimes_k B) \wedge HR) \\ &\simeq \pi_0(K(A^{\mathrm{op}} \otimes_k B)) \otimes_{\mathbb{Z}} R \qquad (10.1) \\ &\simeq K_0(A^{\mathrm{op}} \otimes_k B) \otimes_{\mathbb{Z}} R \simeq R\,, \end{aligned}$$

where (10.1) follows from the stable Hurewicz theorem. This implies, in particular, that the assumptions (A1) of Theorem 1.2 are verified. In what concerns assumptions (A2), take for $\mathcal{T}_{\kappa(\mathfrak{p})}$ the category $\mathrm{NMCSA}(k;\kappa(\mathfrak{p}))$ and for $\iota_{\kappa(\mathfrak{p})}$ the functor $-\otimes_R \kappa(\mathfrak{p})\colon \mathrm{NMCSA}(k;R) \to \mathrm{NMCSA}(k;\kappa(\mathfrak{p}))$. By construction, the latter functor is weight-exact (see Remark 3.4), symmetric monoidal, and induces an equivalence of symmetric monoidal categories between $\mathrm{Kar}(\mathrm{CSA}(k;R) \otimes_R \kappa(\mathfrak{p}))$ and $\mathrm{CSA}(k;\kappa(\mathfrak{p}))$. This shows that the assumptions (A2) are also verified.

We now claim that the categories $\mathrm{NMCSA}(k;\kappa(\mathfrak{p}))$ have the $w_{\kappa(\mathfrak{p})}$-Picard property; thanks to Theorem 1.2 this implies that the category $\mathrm{NMCSA}(k;R)$ has the w_R-Picard property. Since the categories of finite dimensional (graded) vector spaces are local, our claim follows then from the combination of Theorem 1.1 with the following general result (with $R = \kappa(\mathfrak{p})$):

Proposition 10.2. *Let R be a field.*

(a) *When $\mathrm{char}(R) = 0$, the category $\mathrm{CSA}(k;R)$ is $\otimes$-equivalent to the category of finite dimensional R-vector spaces $\mathrm{vect}(R)$;*

(b) *When $\mathrm{char}(R) = p > 0$, there exists a full, additive, conservative, symmetric monoidal functor from $\mathrm{CSA}(k;R)$ into the category of finite dimensional $\mathrm{Br}(k)\{p\}$-graded R-vector spaces $\mathrm{Gr}_{\mathrm{Br}(k)\{p\}}\mathrm{vect}(R)$.*

Proof. Given an R-linear, additive, Karoubian, rigid[9] symmetric monoidal category $(\mathcal{C}, \otimes, \mathbf{1})$, with $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) \simeq R$, recall from [1, §1.4.1 and §1.7.1] the construction of the following categorical ideals

$$\begin{aligned} \mathcal{N}(a,b) &:= \{f\colon a \to b \,|\, \forall g\colon b \to a \;\; \mathrm{tr}(g \circ f) = 0\} \\ \mathcal{R}(a,b) &:= \{f\colon a \to b \,|\, \forall g\colon b \to a \;\; \mathrm{id}_a - (g \circ f) \text{ is invertible}\}\,, \end{aligned}$$

[9] Recall that a symmetric monoidal category is called *rigid* if all its objects are dualizable.

where $\mathrm{tr}(g \circ f)$ stands for the categorical trace of the endomorphism $g \circ f$. As explained in *loc. cit.*, the categorical ideal $\mathcal{N}$ is moreover symmetric monoidal.
Item (a). As proved in [38, Thm. 2.1], we have $U(A)_R \simeq U(k)_R$ for every central simple k-algebra A. Using the natural ring isomorphism $\mathrm{End}(U(k)_R) \simeq R$, we then conclude that the category $\mathrm{CSA}(k;R)$ is $\otimes$-equivalent to the category of finite dimensional R-vector spaces $\mathrm{vect}(R)$.
Item (b). By construction, the category $\mathrm{CSA}(k;R)$ is R-linear, additive, and symmetric monoidal. Moreover, all its objects are dualizable and $\mathrm{End}(U(k)_R) \simeq R$; see [34, §1.7.1]. As proved in [32, Prop. 6.11], the quotient $\mathrm{CSA}(k;R)/\mathcal{N}$ is $\otimes$-equivalent to the category $\mathrm{Gr}_{\mathrm{Br}(k)\{p\}}\mathrm{vect}(R)$. Consequently, we have an induced full, additive, and symmetric monoidal functor

$$\mathrm{CSA}(k;R) \longrightarrow \mathrm{Gr}_{\mathrm{Br}(k)\{p\}}\mathrm{vect}(R)\,. \tag{10.3}$$

It remains then only to prove that the functor (10.3) is moreover conservative. In order to do so, we will show the inclusion $\mathcal{N} \subseteq \mathcal{R}$. Thanks to [1, Prop. 7.1.6], this implies that the quotient functor (10.3) is conservative. By definition, the categorical ideals $\mathcal{N}$ and $\mathcal{R}$ are compatible with direct sums and summands. Hence, given central simple k-algebras A and B, it suffices to show that the inclusion $\mathcal{N}(U(A)_R, U(B)_R) \subseteq \mathcal{R}(U(A)_R, U(B)_R)$ holds. This inclusion follows now from the combination of the definitions of $\mathcal{N}$ and $\mathcal{R}$ with Lemma 10.4 below. □

Lemma 10.4. *Given a central simple k-algebra A, the following morphism*

$$\mathrm{End}_{\mathrm{CSA}(k;R)}(U(A)_R) \longrightarrow \mathrm{End}_{\mathrm{CSA}(k;R)}(U(k)_R) \simeq R \quad h \mapsto \mathrm{tr}(h)\,, \tag{10.5}$$

induced by the categorical trace construction, is invertible.

Proof. By construction, the induced morphism (10.5) is R-linear. Therefore, thanks to the natural isomorphism $\mathrm{End}(U(A)_R) \simeq R$, (10.5) is completely determined by the image of the identity of $U(A)_R$. In other words, (10.5) reduces to the morphism $R \to R, r \mapsto r \cdot \chi(U(A)_R)$, where $\chi(U(A)_R)$ stands for the Euler characteristic of the noncommutative motive $U(A)_R$. As proved in [34, Prop. 2.24], the Euler characteristic $\chi(U(A)_R)$ agrees with the Grothendieck class $[HH(A)]_R \in K_0(k)_R \simeq R$ of the Hochschild homology $HH(A)$ of A. Since $HH(A) \simeq A/[A,A] \simeq k$ (see [23, §1.2.12]), we then conclude that (10.5) is the identity. This finishes the proof. □

Remark 10.6. It follows from the proof of Proposition 10.2 that the Brauer group of the symmetric monoidal category $\mathrm{CSA}(k;R)$ is trivial when $\mathrm{char}(R) = 0$ and isomorphic to $\mathrm{Br}(k)\{p\}$ when $\mathrm{char}(k) = p > 0$.

Item (ii). Thanks to equivalence (2.20), we have an injective group homomorphism

$$\mathrm{Br}(k) \longrightarrow \mathrm{Pic}(\mathrm{CSA}(k;\mathbb{Z})) \qquad [A] \mapsto U(A)_{\mathbb{Z}}\,. \tag{10.7}$$

It remains then only to prove that (10.7) is moreover surjective. Recall from [34, §9][36] the construction of the symmetric monoidal triangulated category

$\mathrm{KMM}(k)$ and of the full subcategories $\mathrm{NMCSA}(k)$ and $\mathrm{CSA}(k)$. By construction, we have an exact symmetric monoidal functor $(-)_{\mathbb{Z}}\colon \mathrm{KMM}(k) \to \mathrm{KMM}(k;\mathbb{Z})$ which restricts to a $\otimes$-equivalence $\mathrm{CSA}(k) \simeq \mathrm{CSA}(k;\mathbb{Z})$. Therefore, making use [37, Thm. 2.20(iv)], we observe that the objects $U(A_1)_{\mathbb{Z}} \oplus \cdots \oplus U(A_m)_{\mathbb{Z}}$ of $\mathrm{CSA}(k;\mathbb{Z})$, with $m > 1$ are *not* $\otimes$-invertible. Since the category $\mathrm{CSA}(k;\mathbb{Z})$ is Karoubian (see [37, Thm. 2.20(i)]), we then conclude that (10.7) is moreover surjective.

Remark 10.8. Given any two central simple k-algebras A and B, we have

$$\mathrm{Hom}_{\mathrm{NMCSA}(k)}(U(A)_R, U(B)_R[n]) \simeq K_{-n}(A^{\mathrm{op}} \otimes_k B) = 0 \qquad n > 0\,.$$

Therefore, a proof similar to the one of Theorem 2.22, with $\mathrm{NMCSA}(k;\mathbb{Z})$ replaced by $\mathrm{NMCSA}(k)$, allows us to conclude that $\mathrm{Pic}(\mathrm{NMCSA}(k)) \simeq \mathrm{Br}(k) \times \mathbb{Z}$. In conclusion, although the categories $\mathrm{NMCSA}(k)$ and $\mathrm{NMCSA}(k;\mathbb{Z})$ are not equivalent, they have nevertheless the same Picard group!

11. Proof of Theorem 2.24

Let us denote by $\mathrm{P}(E)$ the smallest additive, Karoubian, full subcategory of $\mathcal{D}_c(E)$ containing the E-module E. Since by assumption the ring spectrum E is connective, we have trivial positive Ext-groups:

$$\mathrm{Hom}_{\mathcal{D}_c(E)}(E, E[n]) \simeq \pi_{-n}(E) = 0 \qquad n > 0\,.$$

This implies that the subcategory $\mathrm{P}(E) \subset \mathcal{D}_c(E)$ is negative in the sense of Proposition 3.8. Making use of this latter proposition, we then conclude that the category $\mathcal{D}_c(E)$ carries a bounded weight structure w with heart $\mathrm{P}(E)$.
Let us now show that the category $\mathcal{D}_c(E)$ has the w-Picard property. By construction, $\mathrm{P}(E)$ identifies with the category of finitely generated projective $\pi_0(R)$-modules. Therefore, by taking $R := \pi_0(E)$, all the assumptions (A1) of Theorem 1.2 are verified. In what concerns assumptions (A2), take for $\mathcal{T}_{k(\mathfrak{p})}$ the category $\mathcal{D}^b(k(\mathfrak{p}))$, equipped with the canonical bounded weight structure with heart $\mathrm{Vect}(k(\mathfrak{p}))$, and for $\iota_{k(\mathfrak{p})}$ the (composed) base-change functor

$$\mathcal{D}_c(E) \xrightarrow{-\wedge_E H\pi_0(E)} \mathcal{D}_c(H\pi_0(E)) \simeq \mathcal{D}_c(R) \xrightarrow{-\otimes_R k(\mathfrak{p})} \mathcal{D}^b(k(\mathfrak{p}))\,.$$

By construction, the latter functor is weight-exact (see Remark 3.4), symmetric monoidal, and induces a $\otimes$-equivalence of categories between $\mathrm{Kar}(\mathrm{P}(E)\otimes_R \kappa(\mathfrak{p}))$ and $\mathrm{Vect}(k(\mathfrak{p}))$. Since the categories $\mathcal{D}^b(k(\mathfrak{p}))$ clearly have the $w_{k(\mathfrak{p})}$-property, we conclude from Theorem 1.2 that $\mathcal{D}_c(E)$ has the w-Picard property.
Finally, since the category $\mathcal{D}_c(E)$ has the w-Picard property, we have an isomorphism $\mathrm{Pic}(\mathcal{D}_c(E)) \simeq \mathrm{Pic}(\mathrm{P}(E)) \times \mathbb{Z}$. The proof follows now from the fact that $\mathrm{Pic}(\mathrm{P}(E))$ is isomorphic to $\mathrm{Pic}(\pi_0(E))$.

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