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The q-twisted Cohomology and the q-hypergeometric Function at $|q| = 1$

By

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Abstract

We construct the q-twisted cohomology associated with the q-multiplicative function of Jordan-Pochhammer type at $|q| = 1$. In this framework, we prove the Heine's relations and a connection formula for the q-hypergeometric function of the Barnes type. We also prove an orthogonality relation of the q-little Jacobi polynomials at $|q| = 1$.

§1. Introduction

In this paper we construct the q-twisted cohomology at $|q| = 1$ in Jordan-Pochhammer case and prove some properties of the q-hypergeometric function at $|q| = 1$ defined in [NU].

The basic hypergeometric function with $0 < |q| < 1$ [GR] is represented in terms of a Jackson integral. In [A1, AK], a formulation of Jackson integrals is given. Namely, for a q-multiplicative function defined by means of q-version of Sato's b-functions [Sa], the q-twisted cohomology is defined. In this approach, Jackson integrals can be regarded as a pairing between this cohomology and q-cycles.

We consider the case that $|q| = 1$ and q is not a root of unity. Then the structure of b-functions is the same as in the case of $0 < |q| < 1$ and associated q-multiplicative function can be constructed in terms of the double sine function [B]. The problem is to define a suitable integral which is a certain generalization of Jackson integrals to the case of $|q| = 1$.

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In [MT], a family of solutions to the quantum Knizhnik-Zamolodchikov equation (qKZ) at $|q| = 1$ was constructed. The solution is represented in terms of a pairing between two functional spaces, which is called the hypergeometric pairing. The hypergeometric pairing was defined by Tarasov and Varchenko in the study of the rational qKZ [TV1] and the trigonometric qKZ for $0 < |q| < 1$ [TV2]. In the trigonometric case, the hypergeometric pairing is a pairing between a space of trigonometric functions and that of elliptic functions. It is given by an integral over a closed contour with a kernel function defined by a product of the infinite product with step q. This kernel function is the q-multiplicative function mentioned above. By taking residues, we can represent this integral in terms of a Jackson integral.

In the case of $|q| = 1$, the hypergeometric pairing is a pairing between two spaces of trigonometric functions which depend on two respective values of a deformation parameter

$$q = e^{2\pi i\omega} \quad \text{and} \quad Q = e^{\frac{2\pi i}{\omega}}.$$

Hence the pairing induces certain duality of two functional spaces at $|q| = 1$. This type of duality has appeared in mathmatical physics: for example, modular double of quantum group [F] and matrix elements in quantum Toda chain [Sm].

In this paper we define an integral associated with the q-multiplicative function of Jordan-Pochhammer type at $|q| = 1$ in a similar way to [MT]. This integral is regarded as a pairing between two functional spaces which depend on q and Q, respectively. Then we can define a cohomology of these spaces associated with this integral. In this way we get the q-twisted cohomology at $|q| = 1$. From this point of view, we can prove some relations satisfied by the q-hypergeometric function of the Barnes type at $|q| = 1$.

The plan of the paper is as follows. In Section 2, we recall the result of q-analogue of b-functions following [A2] and define a q-multiplicative function at $|q| = 1$. In Section 3, we construct the q-twisted cohomology associated with the q-multiplicative function of Jordan-Pochhammer type. In this case we can find a basis of the cohomology. In order to prove linear independence, we write down the formula for a determinant, see (3.23). This formula gives us the q-Beta integral formula at $|q| = 1$ in a special case. In Section 4, we prove two properties of the q-hypergeometric function at $|q| = 1$: Heine's relations and a connection formula. Section 5 is additional one. We discuss the q-little Jacobi polynomials and their orthogonality with respect to the kernel of the q-Beta integral at $|q| = 1$ given in Section 3.

§2. The q-multiplicative Function at $|q| = 1$

Let q be a nonzero complex number. In this paper, we consider the case that $|q| = 1$ and q is not a root of unity. We put $q = e^{2\pi i\omega}$ $(\omega > 0, \omega \notin \mathbb{Q})$.

Let L be an l dimensional integer lattice in $\mathbb{C}^l$:

$$L := \{\chi = (\chi_1, \ldots, \chi_l) | \chi_j \in \mathbb{Z}, j = 1, \ldots, l\} \subset \mathbb{C}^l. \tag{2.1}$$

For a set of nonzero rational functions $\{b_\chi(t)\}_{\chi \in L}$, where

$$b_\chi(t) = b_\chi(t_1, \ldots, t_l) \in \mathbb{C}(t_1, \ldots, t_l)^\times, \tag{2.2}$$

we consider the following system of difference equations:

$$\Phi(z + \chi) = b_\chi(t)\Phi(z), \quad (\chi \in L), \tag{2.3}$$

where $z = (z_1, \ldots, z_l) \in \mathbb{C}^l$ and

$$t = (t_1, \ldots, t_l) := (e^{2\pi i\omega z_1}, \ldots, e^{2\pi i\omega z_l}). \tag{2.4}$$

The compatibility condition of (2.3) implies

$$b_0(t) = 1, \tag{2.5}$$

$$b_{\chi+\chi'}(t) = b_\chi(t) b_{\chi'}(q^\chi \cdot t) \quad \text{for any } \chi, \chi' \in L, \tag{2.6}$$

where $q^\chi \cdot t = (q^{\chi_1} t_1, \ldots, q^{\chi_l} t_l)$. The conditions (2.5) and (2.6) mean that the set $\{b_\chi(t)\}_{\chi \in L}$ defines a 1-*cocycle*. A set $\{b_\chi(t)\}_{\chi \in L}$ is said to be a 1-*coboundary* if and only if there exists a nonzero rational function $\varphi(t)$ such that

$$b_\chi(t) = \frac{\varphi(q^\chi \cdot t)}{\varphi(t)} \quad \text{for any } \chi \in L. \tag{2.7}$$

Let us consider the quotient $H^1 :=\{\text{1-cocycles}\}/\{\text{1-coboundaries}\}$. H^1 has a mutiplicative group structure. For $\mu \in L^* := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$, we put

$$\mu(m) = \mu(0, \ldots, \overset{m-\text{th}}{1}, \ldots, 0) \in \mathbb{Z} \tag{2.8}$$

and

$$t^\mu = t_1^{\mu(1)} \ldots t_l^{\mu(l)}. \tag{2.9}$$

Note that $\mu(\chi) = \sum_{m=1}^{l} \mu(m)\chi_m$ for $\chi = (\chi_1, \ldots, \chi_l) \in L$.

Then the following result holds.

Proposition 2.1. *H^1 is represented by cocycles of the following form:*

$$b_\chi(t) = a_\chi \prod_{\nu=0}^{\mu_0(\chi)-1} (q^\nu t^{\mu_0}) \frac{\prod_{j=1}^{k} (q^{\gamma_j} t^{\mu_j}; q)_{\mu_j(\chi)}}{\prod_{j=1}^{k'} (q^{\gamma'_j} t^{\mu'_j}; q)_{\mu'_j(\chi)}} \tag{2.10}$$

for $\mu_0, \mu_j, \mu'_j \in L^*$ and $\gamma_j, \gamma'_j \in \mathbb{C}$. Here $\{a_\chi\}_{\chi \in L}$ is a set of nonzero constants satisfying $a_{\chi+\chi'} = a_\chi a_{\chi'}$ for any $\chi, \chi' \in L$, and

$$(x; q)_n := \begin{cases} \prod_{j=1}^{n-1} (1 - xq^j), & \text{for } n \geqslant 0, \\ \prod_{j=1}^{-n} (1 - xq^{-j})^{-1}, & \text{for } n < 0. \end{cases} \tag{2.11}$$

The expression (2.10) *is not unique.*

This result is a q-analogue of Sato's result [Sa] and was stated by Aomoto in [A2]. In [A2], Aomoto stated this result in the case of $0 < |q| < 1$. However, it can be checked that the proposition holds unless q is a root of unity.

Let us find a solution $\Phi(z)$ to (2.3) for the 1-cocycle $\{b_\chi(t)\}_{\chi \in L}$ given by (2.10). Following [N], we set

$$\langle x \rangle := \exp\left(\frac{\pi i}{2}\left((1+\omega)x - \omega x^2\right)\right) S_2\left(x \middle| 1, \frac{1}{\omega}\right), \tag{2.12}$$

where $S_2(x)$ is the double sine function. We refer the reader to [JM] for the double sine function. Moreover, we define a function $\sigma(x)$ by

$$\sigma(x) := \exp(\pi i \left((1+\omega)x - \omega x^2\right)) = \langle x \rangle \left\langle 1 + \frac{1}{\omega} - x \right\rangle. \tag{2.13}$$

These functions satisfy

$$\frac{\langle x+1 \rangle}{\langle x \rangle} = \frac{1}{1 - e^{2\pi i \omega x}}, \quad \frac{\sigma(x+1)}{\sigma(x)} = -e^{-2\pi i \omega x}. \tag{2.14}$$

For $\mu \in L^*$, we set

$$\mu(z) := \sum_{m=1}^{l} \mu(m) z_m. \tag{2.15}$$

Then we have

$$\frac{\langle\mu(z+\chi)+\gamma\rangle}{\langle\mu(z)+\gamma\rangle}=\frac{1}{(q^{\gamma}t^{\mu};q)_{\mu(\chi)}}, \tag{2.16}$$

$$\frac{\sigma(\mu(z+\chi))}{\sigma(\mu(z))}=(-1)^{\mu(\chi)}\prod_{\nu=0}^{\mu(\chi)-1}(q^{\nu}t^{\mu})^{-1}.$$

From (2.13) and (2.16), we can get a solution to (2.3) in the following form:

$$\Phi(z)=t_1^{\alpha_1}\dots t_l^{\alpha_l}\frac{\prod_{j=1}^{n'}\langle\mu'_j(z)+\gamma'_j\rangle}{\prod_{j=1}^{n}\langle\mu_j(z)+\gamma_j\rangle}, \tag{2.17}$$

where $\alpha_m,\gamma_j,\gamma'_j\in\mathbb{C},\mu_j,\mu'_j\in L^*$.

A function $\Phi(z)$ of type (2.17) is called a *q-multiplicative function* at $|q|=1$.

§3. The q-twisted Cohomology in Jordan-Pochhammer Case

Let us consider the q-multiplicative function of Jordan-Pochhammer type given by

$$\Phi(z)=t^{\alpha}\prod_{j=1}^{n}\frac{\langle z+\gamma'_j\rangle}{\langle z+\gamma_j\rangle}, \tag{3.1}$$

where $z\in\mathbb{C},t=e^{2\pi i\omega z}=q^z$ and $\alpha,\gamma_j,\gamma'_j\in\mathbb{C}$. We assume $\gamma_j\neq\gamma'_k$ for any $j,k=1,\dots,n$.

We denote by D,D_j and D'_j the difference operators corresponding to the displacements $\alpha\mapsto\alpha+1,\gamma_j\mapsto\gamma_j+1$ and $\gamma'_j\mapsto\gamma'_j+1$, respectively. Let $\mathcal{A}$ be the commutative algebra generated by $D^{\pm1},D_j^{\pm1}$ and $D_j'^{\pm1}$ $(j=1,\dots,n)$ over $\mathbb{C}$. We define a subspace Z of $\mathbb{C}(t)$ by

$$Z:=\{(\kappa\Phi)/\Phi|\kappa\in\mathcal{A}\}. \tag{3.2}$$

It is easy to see that

$$Z=\left\{\frac{f(t)}{\prod_{j=1}^{n}(c_jq^{-\ell_j}t;q)_{\ell_j}(c'_jt;q)_{\ell'_j}}\middle|f(t)\in\mathbb{C}[t,t^{-1}]\text{ and }\ell_j,\ell'_j\in\mathbb{Z}_{\geqslant0}\right\}, \tag{3.3}$$

where $c_j=e^{2\pi i\omega\gamma_j}$ and $c'_j=e^{2\pi i\omega\gamma'_j}$.

Next we define another space of rational functions. An important point is that the function $\langle x\rangle$ satisfies also the following functional relation:

$$\frac{\langle x+\frac{1}{\omega}\rangle}{\langle x\rangle}=\frac{1}{1-e^{2\pi i x}}. \tag{3.4}$$

We denote by $\widetilde{D}, \widetilde{D_j}$ and $\widetilde{D'_j}$ the difference operators corresponding to the displacements $\alpha \mapsto \alpha+\frac{1}{\omega}, \gamma_j \mapsto \gamma_j+\frac{1}{\omega}$ and $\gamma'_j \mapsto \gamma'_j+\frac{1}{\omega}$, respectively. In the same way as before, we consider the commutative algebra $\widetilde{\mathcal{A}}$ generated by $\widetilde{D}^{\pm 1}, \widetilde{D_j}^{\pm 1}$ and $\widetilde{D'_j}^{\pm 1}$ $(j=1,\ldots,n)$ over $\mathbb{C}$ and set

$$\widetilde{Z}:=\{(\widetilde{\kappa}\Phi)/\Phi | \widetilde{\kappa}\in\widetilde{\mathcal{A}}\}. \tag{3.5}$$

Then we have

(3.6)
$$\widetilde{Z}=\left\{\frac{\widetilde{f}(T)}{\prod_{j=1}^{n}(C_jQ^{-\widetilde{\ell}_j}T;Q)_{\widetilde{\ell}_j}(C'_jT;Q)_{\widetilde{\ell}'_j}}\middle| \widetilde{f}(T)\in\mathbb{C}[T,T^{-1}] \text{ and } \widetilde{\ell}_j,\widetilde{\ell}'_j\in\mathbb{Z}_{\geqslant 0}\right\},$$

where $Q=e^{\frac{2\pi i}{\omega}}, T=e^{2\pi i z}, C_j=e^{2\pi i\gamma_j}$ and $C'_j=e^{2\pi i\gamma'_j}$.

Now we define a pairing between Z and $\widetilde{Z}$. For

$$\varphi(t)=\frac{t^m}{\prod_{j=1}^{n}(c_jq^{-\ell_j}t;q)_{\ell_j}(c'_jt;q)_{\ell'_j}}\in Z \tag{3.7}$$

and

$$\widetilde{\varphi}(T)=\frac{T^{\widetilde{m}}}{\prod_{j=1}^{n}(C_jQ^{-\widetilde{\ell}_j}T;Q)_{\widetilde{\ell}_j}(C'_jT;Q)_{\widetilde{\ell}'_j}}\in\widetilde{Z}, \tag{3.8}$$

we set

$$I(\varphi,\widetilde{\varphi}):=\int_C dz\Phi(z)\varphi(t)\widetilde{\varphi}(T). \tag{3.9}$$

Here the contour C is taken to be the imaginary axis $(-i\infty, i\infty)$ except that the poles at

$$-\gamma_j+\ell_j+\frac{\widetilde{\ell}_j}{\omega}+\mathbb{Z}_{\leqslant 0}+\frac{1}{\omega}\mathbb{Z}_{\leqslant 0} \quad (j=1,\ldots,n) \tag{3.10}$$

are on the left of C and the poles at

$$-\gamma'_j-\ell'_j-\frac{\widetilde{\ell}'_j}{\omega}+\mathbb{Z}_{\geqslant 1}+\frac{1}{\omega}\mathbb{Z}_{\geqslant 1} \quad (j=1,\ldots,n) \tag{3.11}$$

are on the right of C. Then the integral (3.9) is absolutely convergent if

$$0 < \mathrm{Re}\alpha + m + \frac{\widetilde{m}}{\omega} < \sum_{j=1}^{n}(\mathrm{Re}\gamma_j - \mathrm{Re}\gamma'_j) + \sum_{j=1}^{n}(\ell_j + \ell'_j) + \frac{1}{\omega}\sum_{j=1}^{n}(\widetilde{\ell_j} + \widetilde{\ell'_j}). \tag{3.12}$$

Under the condition (3.12) the integrand of (3.9) decreases exponentially as $z \to \pm i\infty$.

Let us consider cohomologies of Z and $\widetilde{Z}$ associated with the integral (3.9). We set

$$B := \mathrm{span}_{\mathbb{C}}\left\{\psi(t) - b_\chi(t)\psi(q^\chi t) | \psi(t) \in Z, \chi \in \mathbb{Z}\right\}, \tag{3.13}$$

where $b_\chi(t) = \Phi(z+\chi)/\Phi(z)$. Note that for any $\psi(t) \in Z, \widetilde{\varphi}(T) \in \widetilde{Z}$ and $\chi \in \mathbb{Z}$ we can deform the contour C so that there are no poles of the function $\Phi(z)\psi(t)\widetilde{\varphi}(T)$ between C and $C+\chi$. Thus we have

$$\begin{aligned}\int_C dz\Phi(z)\{\psi(t) - b_\chi(t)\psi(q^\chi t)\}\widetilde{\varphi}(T) &= \left(\int_C - \int_{C+\chi}\right) dz\Phi(z)\psi(t)\widetilde{\varphi}(T)\\ &= 0,\end{aligned} \tag{3.14}$$

if all the integrals are convergent. Here we used the fact that $T = e^{2\pi i z}$ is invariant under the change $z \to z - \chi$. Hence, we find

$$I(\varphi_0, \widetilde{\varphi}) = 0 \quad \text{for}\, \varphi_0 \in B\, \text{and}\, \widetilde{\varphi} \in \widetilde{Z}. \tag{3.15}$$

Similarly, we set

$$\widetilde{B} := \mathrm{span}_{\mathbb{C}}\left\{\widetilde{\psi}(T) - \widetilde{b_\chi}(T)\widetilde{\psi}(Q^\chi T) | \widetilde{\psi}(T) \in \widetilde{Z}, \chi \in \mathbb{Z}\right\}, \tag{3.16}$$

where $\widetilde{b_\chi}(T) := \Phi(z + \frac{\chi}{\omega})/\Phi(z)$. Then we have

$$I(\varphi, \widetilde{\varphi_0}) = 0 \quad \text{for}\, \varphi \in Z\, \text{and}\, \widetilde{\varphi_0} \in \widetilde{B}. \tag{3.17}$$

From these relations, we define the *q-twisted cohomology* H and $\widetilde{H}$ by

$$H := Z/B \quad \text{and} \quad \widetilde{H} := \widetilde{Z}/\widetilde{B}. \tag{3.18}$$

We note that the structure of the cohomology H is determined by the parameters q, q^α, c_j and $c'_j\, (j = 1, \ldots, n)$. We write down this dependence explicitly as

$$H = \mathrm{H}(q|q^\alpha; c_1, \ldots, c_n; c'_1, \ldots, c'_n). \tag{3.19}$$

Then $\widetilde{H}$ is written as

$$\widetilde{H} = \mathrm{H}(Q|Q^{\omega\alpha}; C_1, \ldots, C_n; C'_1, \ldots, C'_n). \tag{3.20}$$

It is easy to see that the following proposition holds.

Proposition 3.1. *The cohomology* $\mathrm{H}(q|q^{\alpha}; c_1, \ldots, c_n; c'_1, \ldots, c'_n)$ *is generated by*

$$\left\{ \frac{1}{1 - c'_j t} \middle| j = 1, \ldots, n \right\}, \tag{3.21}$$

if the parameters q, q^{α}, c_j *and* c'_j $(j = 1, \ldots, n)$ *are generic.*

Moreover, we see that the set $\{\frac{1}{1-c'_j t}\}_{j=1,\ldots,n}$ is a basis of H from the following determinant formula.

Proposition 3.2. *Set*

$$\varphi_j(t) = \frac{1}{1 - c'_j t}, \quad \widetilde{\varphi}_j(T) = \frac{1}{1 - C'_j T}, \quad (j = 1, \ldots, n). \tag{3.22}$$

Then

$$\begin{aligned} \det\left(I(\varphi_j, \widetilde{\varphi}_k)\right)_{j,k=1,\ldots,n} &= \langle 1 \rangle^n \exp\left(-2\pi i \omega\alpha \sum_{j=1}^{n} \gamma'_j\right) \\ &\quad \times \frac{\langle \alpha + \sum_{j=1}^{n} \gamma_j - \sum_{j=1}^{n} \gamma'_j \rangle}{\langle \alpha \rangle \prod_{j,k=1}^{n} \langle \gamma_j - \gamma'_k \rangle} \\ &\quad \times \prod_{1 \leqslant j < k \leqslant n} \frac{(1 - e^{2\pi i \omega(\gamma'_j - \gamma'_k)})(1 - e^{2\pi i(\gamma'_j - \gamma'_k)})}{\sigma(\gamma'_j - \gamma'_k)}. \end{aligned} \tag{3.23}$$

Proof. First we set

$$\begin{aligned} \psi_k(t) &= \frac{1}{1 - c'_k t} \prod_{j=1}^{k-1} \frac{1 - c_j t}{1 - c'_j t}, \\ \widetilde{\psi}_k(T) &= \frac{1}{1 - C'_k T} \prod_{j=1}^{k-1} \frac{1 - C_j T}{1 - C'_j T}, \ (j = 1, \ldots, n). \end{aligned} \tag{3.24}$$

Then we have

$$\psi_k(t) = \sum_{p=1}^{k-1} \frac{1 - c_p/c'_p}{1 - c'_k/c'_p} \prod_{\substack{j=1 \\ j \neq p}}^{k-1} \frac{1 - c_j/c'_p}{1 - c'_j/c'_p} \varphi_p(t) + \prod_{j=1}^{k-1} \frac{1 - c_j/c'_k}{1 - c'_j/c'_k} \varphi_k(t) \tag{3.25}$$

and a similar formula for $\widetilde{\psi}_k(T)$.

Hence we find

$$\det\left(I(\varphi_j,\widetilde{\varphi}_k)\right)=\prod_{1\leqslant j<k\leqslant n}\left(\frac{1-c'_j/c'_k}{1-c_j/c'_k}\frac{1-C'_j/C'_k}{1-C_j/C'_k}\right)\det\left(I(\psi_j,\widetilde{\psi}_k)\right). \tag{3.26}$$

The determinant in the right hand side of (3.26) is a special case of the determinant discussed in [MT]. Combining the result in [MT] and (3.26), we get the formula (3.23). □

Remark. In the case of $n=1$ and $\gamma'_1=0$, the formula (3.23) is represented as follows:

$$\int_C q^{\alpha z}\frac{\langle z\rangle}{\langle z+\beta\rangle}\frac{1}{1-t}\frac{1}{1-T}dz=\int_C q^{\alpha z}\frac{\langle z+1+\frac{1}{\omega}\rangle}{\langle z+\beta\rangle}dz=\frac{\langle 1\rangle\langle\alpha+\beta\rangle}{\langle\alpha\rangle\langle\beta\rangle}. \tag{3.27}$$

We may call (3.27) the *q-Beta integral formula* at $|q|=1$.

To finish this section we find a system of difference equation in α satisfied by the function

$$\Psi(\alpha)=\Psi(\alpha|\widetilde{\varphi}):=\int_C dz\Phi(z)\widetilde{\varphi}(T)\quad\text{for }\widetilde{\varphi}(T)\in\widetilde{H} \tag{3.28}$$

in a similar manner to [AK].

For the q-multiplicative function (3.1), we can represent the function $b_\chi(t)=\frac{\Phi(z+\chi)}{\Phi(z)}$ $(\chi\in\mathbb{Z})$ as follows:

$$b_\chi(t)=q^{\chi\alpha}\frac{b_\chi^+(t)}{b_\chi^-(t)}, \tag{3.29}$$

where $b_\chi^+(t)$ and $b_\chi^-(t)$ are polynomials in t and have no common factor. For example, if $\chi=1$ we have

$$b_1^+(t)=\prod_{j=1}^n(1-c_jt),\quad b_1^-(t)=\prod_{j=1}^n(1-c'_jt). \tag{3.30}$$

By setting $\psi(t)=b_\chi^-(q^{-\chi}t)$ in (3.13), we find

$$b_\chi^-(q^{-\chi}t)-q^{\chi\alpha}b_\chi^+(t)\in B. \tag{3.31}$$

Note that $t\,\Phi=D\Phi$, where D is the difference operator defined by $\alpha\mapsto\alpha+1$. Therefore, we get

$$\left\{b_\chi^-(q^{-\chi}D)-q^{\chi\alpha}b_\chi^+(D)\right\}\Psi=0 \tag{3.32}$$

for $\chi \in \mathbb{Z}$ such that $b_\chi^-(q^{-\chi}D)\Psi$ and $b_\chi^+(D)\Psi$ are defined. These equations are q-analogues of Mellin-Sato hypergeometric equations [AK] at $|q| = 1$. In the case of $\chi = 1$, the equation (3.32) is given by

$$\left\{ \prod_{j=1}^{n} (1 - q^{-1} c_j' D) - q^{\alpha} \prod_{j=1}^{n} (1 - c_j D) \right\} \Psi = 0. \tag{3.33}$$

§4. Application to the q-hypergeometric Function

§4.1. Preliminaries

Following [GR], we recall some properties of the basic hypergeometric series with $0 < |q| < 1$ given by

$$\phi(a, b, c; t) := \sum_{k=0}^{\infty} \frac{(a;q)_k (b;q)_k}{(q;q)_k (c;q)_k} t^k \tag{4.1}$$

for $|t| < 1$.

This function satisfies the Heine's relations:

$$\phi(a, b, q^{-1}c) - \phi(a, b, c) = tc \frac{(1-a)(1-b)}{(q-c)(1-c)} \phi(qa, qb, qc), \tag{4.2}$$

$$\phi(qa, b, c) - \phi(a, b, c) = ta \frac{1-b}{1-c} \phi(qa, qb, qc), \tag{4.3}$$

$$\phi(qa, q^{-1}b, c) - \phi(a, b, c) = q^{-1} t \frac{aq - b}{1-c} \phi(qa, b, qc). \tag{4.4}$$

Here we abbreviated $\phi(a, b, c; x)$ to $\phi(a, b, c)$.

It also satisfies a connection formula:

$$\begin{aligned} \phi(a, b, c; t) = {} & \frac{(b)_\infty (c/a)_\infty}{(c)_\infty (b/a)_\infty} \frac{\Theta(at)}{\Theta(t)} \phi(a, qa/c, qa/b; qc/abt) \\ & + \frac{(a)_\infty (c/b)_\infty}{(c)_\infty (a/b)_\infty} \frac{\Theta(bt)}{\Theta(t)} \phi(b, qb/c, qb/a; qc/abt), \end{aligned} \tag{4.5}$$

where $(a)_\infty := \prod_{j=1}^{\infty} (1 - q^{j-1} a)$ and $\Theta(x) := (q)_\infty (x)_\infty (q/x)_\infty$.

Now we consider the q-hypergeometric function of the Barnes type at $|q| = 1$ [NU], which is defined as follows in our notation:

$$\Psi(\alpha, \beta, \gamma; x) := \frac{\langle \alpha \rangle \langle \beta \rangle}{\langle 1 \rangle \langle \gamma \rangle} \left(-\frac{1}{2\pi i} \right) \int_{C_0} \frac{\langle z+1 \rangle \langle z+\gamma \rangle}{\langle z+\alpha \rangle \langle z+\beta \rangle} \frac{\pi (-q^x)^z}{\sin \pi z} dz, \tag{4.6}$$

where $-q^x = e^{2\pi i\omega x - \pi i}$ and the contour C_0 is the imaginary axis $(-i\infty, i\infty)$ except that the poles at

$$-\alpha + \mathbb{Z}_{\leqslant 0} + \frac{1}{\omega}\mathbb{Z}_{\leqslant 0}, \quad -\beta + \mathbb{Z}_{\leqslant 0} + \frac{1}{\omega}\mathbb{Z}_{\leqslant 0} \tag{4.7}$$

are on the left of C_0 and the poles at

$$\mathbb{Z}_{\geqslant 0} + \frac{1}{\omega}\mathbb{Z}_{\geqslant 0}, \quad -\gamma + \mathbb{Z}_{\geqslant 1} + \frac{1}{\omega}\mathbb{Z}_{\geqslant 1} \tag{4.8}$$

are on the right of C_0.

By using

$$\left(-\frac{1}{2\pi i}\right) \frac{\pi(-q^x)^z}{\sin \pi z} = q^{xz} \frac{1}{1 - e^{2\pi i z}}, \tag{4.9}$$

we can rewrite (4.6) as follows:

$$\Psi(\alpha, \beta, \gamma; x) = \frac{\langle\alpha\rangle\langle\beta\rangle}{\langle 1\rangle\langle\gamma\rangle} \int_{C_0} q^{xz} \frac{\langle z + 1 + \frac{1}{\omega}\rangle\langle z + \gamma\rangle}{\langle z + \alpha\rangle\langle z + \beta\rangle} dz. \tag{4.10}$$

Now we denote by $\Phi(z)$ the integrand of (4.10):

$$\Phi(z) = q^{xz} \frac{\langle z + 1 + \frac{1}{\omega}\rangle\langle z + \gamma\rangle}{\langle z + \alpha\rangle\langle z + \beta\rangle}. \tag{4.11}$$

This function $\Phi(z)$ is the q-multiplicative function of Jordan-Pochhammer type. From (3.12), the integral (4.10) is absolutely convergent if

$$0 < \mathrm{Re}x < 1 + \frac{1}{\omega} + \mathrm{Re}\gamma - \mathrm{Re}\alpha - \mathrm{Re}\beta. \tag{4.12}$$

In this case, the equation (3.33) is nothing but the hypergeometric difference equation at $|q| = 1$:

$$\left\{(1 - D)(1 - q^{\gamma - 1}D) - q^x(1 - q^\alpha D)(1 - q^\beta D)\right\} \Psi = 0, \tag{4.13}$$

where D is the difference operator defined by $x \mapsto x + 1$.

For the q-multiplicative function (4.10), we define two functional spaces Z and $\widetilde{Z}$ as in the previous section, and set

$$\Psi(\alpha, \beta, \gamma; x|\widetilde{\varphi}) := \frac{\langle\alpha\rangle\langle\beta\rangle}{\langle 1\rangle\langle\gamma\rangle} \int_{C_0} \Phi(z)\widetilde{\varphi}(T) dz \qquad \text{for} \quad \widetilde{\varphi} \in \widetilde{Z}. \tag{4.14}$$

Note that $\Psi(\alpha, \beta, \gamma; x) = \Psi(\alpha, \beta, \gamma; x|1)$. Then the function (4.14) also satisfies (4.13).

For simplicity's sake, hereafter we use the following notation:

$$t = e^{2\pi i\omega z},\ a = e^{2\pi i\omega\alpha},\ b = e^{2\pi i\omega\beta},\ c = e^{2\pi i\omega\gamma} \tag{4.15}$$

and

$$T = e^{2\pi iz},\ A = e^{2\pi i\alpha},\ B = e^{2\pi i\beta},\ C = e^{2\pi i\gamma}. \tag{4.16}$$

§4.2. Heine's relations

Proposition 4.1. *We abbreviate* $\Psi(\alpha,\beta,\gamma;x)$ *to* $\Psi(\alpha,\beta,\gamma)$. *Then the following equalities hold*:

$$\Psi(\alpha,\beta,\gamma-1) - \Psi(\alpha,\beta,\gamma) = q^x c\frac{(1-a)(1-b)}{(q-c)(1-c)}\Psi(\alpha+1,\beta+1,\gamma+1), \tag{4.17}$$

$$\Psi(\alpha+1,\beta,\gamma) - \Psi(\alpha,\beta,\gamma) = q^x a\frac{1-b}{1-c}\Psi(\alpha+1,\beta+1,\gamma+1), \tag{4.18}$$

$$\Psi(\alpha+1,\beta-1,\gamma) - \Psi(\alpha,\beta,\gamma) = q^{x-1}\frac{aq-b}{1-c}\Psi(\alpha+1,\beta,\gamma+1). \tag{4.19}$$

Proof. For $f(t) \in Z$, we set

$$[f] := \frac{\langle\alpha\rangle\langle\beta\rangle}{\langle 1\rangle\langle\gamma\rangle}\int_{C_0}\Phi(z)f(t)dz. \tag{4.20}$$

First we prove (4.17). It is easy to see that

$$\Psi(\alpha,\beta,\gamma-1) = \left[\frac{1-q^{-1}ct}{1-q^{-1}c}\right], \quad \Psi(\alpha,\beta,\gamma) = [1]. \tag{4.21}$$

Hence, we have

$$\Psi(\alpha,\beta,\gamma-1) - \Psi(\alpha,\beta,\gamma) = \left[\frac{c(1-t)}{q-c}\right]. \tag{4.22}$$

On the other hand, by changing the variable $z \to z-1$, we find

$$\Psi(\alpha+1,\beta+1,\gamma+1) = \left[q^{-x}\frac{(1-c)(1-t)}{(1-a)(1-b)}\right]. \tag{4.23}$$

From (4.22) and (4.23), we get (4.17). We can prove (4.18) in the same way as above.

Next we prove (4.19). By changing the variable $z \to z+1$, we have

$$\Psi(\alpha+1,\beta-1,\gamma) = \left[q^{x-1}\frac{(q-b)(1-at)(1-qat)}{(1-a)(1-qt)(1-ct)}\right]. \tag{4.24}$$

It is easy to see that

$$\Psi(\alpha,\beta,\gamma)=[1],\quad \Psi(\alpha+1,\beta,\gamma+1)=\left[\frac{(1-c)(1-at)}{(1-a)(1-ct)}\right]. \tag{4.25}$$

By using this, we can find the following:

$$\begin{aligned}\Psi(\alpha+1,\beta-1,\gamma)-\Psi(\alpha,\beta,\gamma)-q^{x-1}\frac{aq-b}{1-c}\Psi(\alpha+1,\beta,\gamma+1)\\ =\left[-1+q^x\frac{(1-at)(1-bt)}{(1-qt)(1-ct)}\right].\end{aligned} \tag{4.26}$$

Note that

$$-1+q^x\frac{(1-at)(1-bt)}{(1-qt)(1-ct)}=-\{1-b_1(t)\cdot 1\}\in B. \tag{4.27}$$

Therefore, (4.26) equals to 0. This completes the proof of (4.19). □

In the proof above, we see that Heine's relations come from some relations in H. Hence, we find that the function $\Psi(\alpha,\beta,\gamma;x|\widetilde{\varphi})$ (4.14) also satisfies Heine's relations.

§4.3. Connection formula

Proposition 4.2.

(4.28)

$$\begin{aligned}&\Psi(\alpha,\beta,\gamma;x)\\ &\quad=\frac{\langle\beta\rangle\langle\gamma-\alpha\rangle}{\langle\gamma\rangle\langle\beta-\alpha\rangle}\frac{\sigma(x+\alpha)}{\sigma(x)}\Psi(\alpha,1+\alpha-\gamma,1+\alpha-\beta;1+\frac{1}{\omega}+\gamma-\alpha-\beta-x)\\ &\qquad+\frac{\langle\alpha\rangle\langle\gamma-\beta\rangle}{\langle\gamma\rangle\langle\alpha-\beta\rangle}\frac{\sigma(x+\beta)}{\sigma(x)}\Psi(\beta,1+\beta-\gamma,1+\beta-\alpha;1+\frac{1}{\omega}+\gamma-\alpha-\beta-x).\end{aligned}$$

Proof. We rewrite the integral

$$\begin{aligned}\Psi(\alpha,1+\alpha-\gamma,1+\alpha-\beta;1+\frac{1}{\omega}+\gamma-\alpha-\beta-x)\\ =\frac{\langle\alpha\rangle}{\langle 1\rangle}\int_{C_0'}q^{(1+\frac{1}{\omega}+\gamma-\alpha-\beta-x)z}\\ \times\frac{\langle 1+\alpha-\gamma\rangle}{\langle 1+\alpha-\beta\rangle}\frac{\langle z+1+\frac{1}{\omega}\rangle\langle z+1+\alpha-\beta\rangle}{\langle z+\alpha\rangle\langle z+1+\alpha-\gamma\rangle}dz,\end{aligned} \tag{4.29}$$

where C_0' is the contour associated with the set of parameters $(\alpha, 1+\alpha-\gamma, 1+\alpha-\beta)$.

By changing the variable $z \to -z-\alpha$, we have

$$(4.30)\qquad (4.29) = \frac{\langle\alpha\rangle}{\langle 1\rangle}\int_{C_0} q^{(1+\frac{1}{\omega}+\gamma-\alpha-\beta-x)(-z-\alpha)} \times \frac{\langle 1+\alpha-\gamma\rangle}{\langle 1+\alpha-\beta\rangle}\frac{\langle -z-\alpha+1+\frac{1}{\omega}\rangle\langle -z+1-\beta\rangle}{\langle -z\rangle\langle -z+1-\gamma\rangle}dz,$$

where C_0 is the contour defined in (4.6). By using (2.13), we have

$$(4.31)\qquad \text{the integrand of (4.30)} = q^{(1+\frac{1}{\omega}+\gamma-\alpha-\beta-x)(-z-\alpha)} \times \frac{\sigma(1+\alpha-\gamma)}{\sigma(1+\alpha-\beta)}\frac{\sigma(-z-\alpha+1+\frac{1}{\omega})\sigma(-z+1-\beta)}{\sigma(-z)\sigma(-z+1-\gamma)} \times \frac{\langle\frac{1}{\omega}+\beta-\alpha\rangle}{\langle\frac{1}{\omega}+\gamma-\alpha\rangle}\frac{\langle z+1+\frac{1}{\omega}\rangle\langle z+\gamma+\frac{1}{\omega}\rangle}{\langle z+\alpha\rangle\langle z+\beta+\frac{1}{\omega}\rangle}.$$

It can be shown that

$$(4.32)\qquad q^{(1+\frac{1}{\omega}+\gamma-\alpha-\beta-x)(-z-\alpha)} \times \frac{\sigma(1+\alpha-\gamma)}{\sigma(1+\alpha-\beta)}\frac{\sigma(-z-\alpha+1+\frac{1}{\omega})\sigma(-z+1-\beta)}{\sigma(-z)\sigma(-z+1-\gamma)} = q^{xz}\frac{\sigma(x)}{\sigma(x+\alpha)}.$$

From (3.4), we have

$$(4.33)\qquad \frac{\langle\frac{1}{\omega}+\beta-\alpha\rangle}{\langle\frac{1}{\omega}+\gamma-\alpha\rangle}\frac{\langle z+1+\frac{1}{\omega}\rangle\langle z+\gamma+\frac{1}{\omega}\rangle}{\langle z+\alpha\rangle\langle z+\beta+\frac{1}{\omega}\rangle} = \frac{\langle\beta-\alpha\rangle}{\langle\gamma-\alpha\rangle}\frac{\langle z+1+\frac{1}{\omega}\rangle\langle z+\gamma\rangle}{\langle z+\alpha\rangle\langle z+\beta\rangle}\frac{(A-C)(1-BT)}{(A-B)(1-CT)}.$$

Combining (4.32) and (4.33), we get

$$(4.34)\qquad \Psi\left(\alpha, 1+\alpha-\gamma, 1+\alpha-\beta; 1+\frac{1}{\omega}+\gamma-\alpha-\beta-x\right) = \frac{\langle\alpha\rangle\langle\beta-\alpha\rangle}{\langle 1\rangle\langle\gamma-\alpha\rangle}\frac{\sigma(x)}{\sigma(x+\alpha)}\int_{C_0}\Phi(z)\frac{(A-C)(1-BT)}{(A-B)(1-CT)}dz.$$

By exchanging α and β, we find

$$(4.35)\qquad \Psi\left(\beta, 1+\beta-\gamma, 1+\beta-\alpha; 1+\frac{1}{\omega}+\gamma-\alpha-\beta-x\right) = \frac{\langle\beta\rangle\langle\alpha-\beta\rangle}{\langle 1\rangle\langle\gamma-\beta\rangle}\frac{\sigma(x)}{\sigma(x+\beta)}\int_{C_0}\Phi(z)\frac{(B-C)(1-AT)}{(B-A)(1-CT)}dz.$$

Therefore, we get

(4.36) the rhs of (4.28)

$$= \frac{\langle\alpha\rangle\langle\beta\rangle}{\langle 1\rangle\langle\gamma\rangle}\int_{C_0}\Phi(z)\left\{\frac{(A-C)(1-BT)}{(A-B)(1-CT)}+\frac{(B-C)(1-AT)}{(B-A)(1-CT)}\right\}dz$$

$$= \frac{\langle\alpha\rangle\langle\beta\rangle}{\langle 1\rangle\langle\gamma\rangle}\int_{C_0}\Phi(z)\cdot 1dz = \Psi(\alpha,\beta,\gamma;x). \qquad \square$$

In the proof above, we see that the formula (4.28) comes from the following simple relation in $\widetilde{H}$:

$$\frac{(A-C)(1-BT)}{(A-B)(1-CT)}+\frac{(B-C)(1-AT)}{(B-A)(1-CT)} = 1. \tag{4.37}$$

If we consider H as a cohomology and $\widetilde{H}$ as its dual, that is a homology, then the relation (4.37) is a relation among some homologies, and the formula (4.28) is a linear relation among the integrals associated with different homologies.

§5. The q-little Jacobi Polynomials at $|q| = 1$

First we recall the definition of the q-little Jacobi polynomials in the case of $0 < |q| < 1$ [GR]:

$$p_n^{(\alpha,\beta)}(t) := \phi(q^{-n}, q^{\alpha+\beta+n+1}, q^{\alpha+1}; qt), \quad (n = 0, 1, \ldots). \tag{5.1}$$

The following orthogonality relation holds [AA, GR]:

$$\int_0^1 t^{\alpha-1}\frac{(tq)_\infty}{(tq^\beta)_\infty}p_m^{(\alpha-1,\beta-1)}(t)p_n^{(\alpha-1,\beta-1)}(t)d_qt = \delta_{m,n}c_n, \tag{5.2}$$

where

$$c_n = (1-q)\frac{(q)_\infty(q^{\alpha+\beta})_\infty}{(q^\alpha)_\infty(q^\beta)_\infty}\frac{1-q^{\alpha+\beta-1}}{1-q^{\alpha+\beta+2n-1}}\frac{(q)_n(q^\beta)_n}{(q^{\alpha+\beta-1})_n(q^\alpha)_n}q^{n\alpha}. \tag{5.3}$$

In (5.2), the integral is a Jackson integral defined by

$$\int_0^1 f(t)d_qt := (1-q)\sum_{k=0}^{\infty} f(q^n)q^n, \tag{5.4}$$

and $(a)_n = (a;q)_n$.

The formula (5.2) means that the q-little Jacobi polynomials are orthogonal polynomials with respect to the kernel of the q-Beta integral given by

$$\int_0^1 t^{\alpha-1}\frac{(tq)_\infty}{(tq^\beta)_\infty}d_qt=(1-q)\frac{(q)_\infty(q^{\alpha+\beta})_\infty}{(q^\alpha)_\infty(q^\beta)_\infty}. \tag{5.5}$$

Let us consider the case of $|q|=1$. We can get the q-little Jacobi polynomials at $|q|=1$ from the q-hypergeometric function (4.10) as follows.

Proposition 5.1. *For $n\in\mathbb{Z}_{\geqslant 0}$, we have*

$$\lim_{\alpha\to -n}\Psi(\alpha,\beta,\gamma;x)=\phi(q^{-n},q^\beta,q^\gamma;q^x). \tag{5.6}$$

Note that the right hand side of (5.6) *is a polynomial in q^x and so makes sense at $|q|=1$.*

Proof. Recall the definition of $\Psi(\alpha,\beta,\gamma;x)$:

$$\Psi(\alpha,\beta,\gamma;x):=\frac{\langle\alpha\rangle\langle\beta\rangle}{\langle 1\rangle\langle\gamma\rangle}\int_{C_0}\Phi(z)dz. \tag{5.7}$$

At $\alpha=-n$, the coefficient $\langle\alpha\rangle$ has a zero and the integral has a pole because of pinches of the contour C_0 by poles at $z=0,1,\ldots,n$ and $z=-\alpha-n,-\alpha-n+1,\ldots,-\alpha$, respectively. In order to avoid these pinches, we take the residues at $z=-\alpha-n,\ldots,-\alpha$. Then we get

$$\begin{aligned}\Psi(\alpha,\beta,\gamma;x)=&\frac{\langle\alpha\rangle\langle\beta\rangle}{\langle 1\rangle\langle\gamma\rangle}\sum_{k=0}^{n}2\pi i\mathrm{res}_{z=-\alpha-k}\Phi(z)dz\\&+\frac{\langle\alpha\rangle\langle\beta\rangle}{\langle 1\rangle\langle\gamma\rangle}\times(\text{regular at}\,\alpha=-n).\end{aligned} \tag{5.8}$$

The second term of the rhs of (5.8) equals zero at $\alpha=-n$. Hence it suffices to calculate the limit of the first term.

By using (2.14), we have

$$\begin{aligned}&2\pi i\mathrm{res}_{z=-\alpha-(n-k)}\Phi(z)dz\\&=2\pi i\mathrm{res}_{z=-\alpha}\Phi(z)\prod_{j=1}^{n-k}\frac{(1-q^{-j+1}t)(1-q^{-j}ct)}{(1-q^{-j}at)(1-q^{-j}bt)}q^{-(n-k)x}dz\\&=\langle 1\rangle\langle 1+\frac{1}{\omega}-\alpha\rangle\frac{\langle\gamma-\alpha\rangle}{\langle\beta-\alpha\rangle}\prod_{j=1}^{n-k}\frac{(1-q^{-j+1}/a)(1-q^{-j}c/a)}{(1-q^{-j})(1-q^{-j}b/a)}q^{-(\alpha+n-k)x}.\end{aligned} \tag{5.9}$$

Here we used the notation (4.15) and

$$2\pi i \mathrm{res}_{z=0} \frac{dz}{\langle z\rangle} = \frac{i}{\sqrt{\omega}} = \langle 1\rangle. \tag{5.10}$$

Therefore, we get

$$\begin{aligned}
&\lim_{\alpha\to -n} \Psi(\alpha,\beta,\gamma;x) \\
&= \lim_{\alpha\to -n} \langle\alpha\rangle\langle 1+\frac{1}{\omega}-\alpha\rangle \frac{\langle\beta\rangle}{\langle\beta-\alpha\rangle}\frac{\langle\gamma-\alpha\rangle}{\langle\gamma\rangle} \\
&\quad\times \sum_{k=0}^{n} q^{-(\alpha+n-k)x} \prod_{j=1}^{n-k} \frac{(1-q^{-j+1}/a)(1-q^{-j}c/a)}{(1-q^{-j})(1-q^{-j}b/a)} \\
&= \sigma(-n)\frac{\langle\beta\rangle}{\langle\beta+n\rangle}\frac{\langle\gamma+n\rangle}{\langle\gamma\rangle} \sum_{k=0}^{n} q^{kx} \prod_{j=1}^{n-k} \frac{(1-q^{n+1-j})(1-q^{n-j}c)}{(1-q^{-j})(1-q^{n-j}b)} \\
&= (-1)^n q^{-\frac{n(n+1)}{2}} \prod_{j=1}^{n} \frac{1-q^j}{1-q^{-j}} \sum_{k=0}^{n} q^{kx} \prod_{j=0}^{k-1} \frac{(1-q^{-n+j})(1-q^j b)}{(1-q^{j+1})(1-q^j c)} \\
&= \phi(q^{-n},q^{\beta},q^{\gamma};q^x).
\end{aligned} \tag{5.11}$$

□

From this proposition, we get

$$p_n^{(\alpha,\beta)}(q^x) = \Psi(-n,\alpha+\beta+n+1,\alpha+1;x+1), \quad (n=0,1,\ldots). \tag{5.12}$$

Then we find that the q-little Jacobi polynomials (5.12) satisfy the orthogonal relation associated with the q-Beta integral at $|q|=1$ (3.27).

Proposition 5.2.

$$\int_C q^{\alpha z} \frac{\langle z+1+\frac{1}{\omega}\rangle}{\langle z+\beta\rangle} p_m^{(\alpha-1,\beta-1)}(t) p_n^{(\alpha-1,\beta-1)}(t)dz = \delta_{m,n}c_n, \tag{5.13}$$

where $t=q^z=e^{2\pi i\omega z}$ and

$$c_n = \frac{\langle 1\rangle\langle\alpha+\beta\rangle}{\langle\alpha\rangle\langle\beta\rangle} \frac{1-q^{\alpha+\beta-1}}{1-q^{\alpha+\beta+2n-1}} \frac{(q)_n(q^{\beta})_n}{(q^{\alpha+\beta-1})_n(q^{\alpha})_n} q^{n\alpha}. \tag{5.14}$$

In the left hand side of (5.13)*, the contour C is the imaginary axis* $(-i\infty,i\infty)$ *except that the poles at $\mathbb{Z}_{\geqslant 0}+\frac{1}{\omega}\mathbb{Z}_{\geqslant 0}$ are on the right of C and the poles at $-\beta+\mathbb{Z}_{\leqslant 0}+\frac{1}{\omega}\mathbb{Z}_{\leqslant 0}$ are on the left of C.*

Proof. First we rewrite the orthogonality relation with $0<|q|<1$ (5.2) as follows. We expand the product

$$p_m^{(\alpha-1,\beta-1)}(t)p_n^{(\alpha-1,\beta-1)}(t)=\sum_{k=0}^{m+n}A_k^{m,n}t^k,\quad A_k^{m,n}\in\mathbb{C}. \tag{5.15}$$

By using (5.5), we have

$$\begin{aligned}\int_0^1 t^{\alpha-1}\frac{(tq)_\infty}{(tq^\beta)_\infty}p_m^{(\alpha-1,\beta-1)}(t)p_n^{(\alpha-1,\beta-1)}(t)d_qt \\ =\sum_{k=0}^{m+n}A_k^{m,n}\int_0^1 t^{\alpha+k-1}\frac{(tq)_\infty}{(tq^\beta)_\infty}d_qt \\ =\sum_{k=0}^{m+n}A_k^{m,n}(1-q)\frac{(q)_\infty(q^{\alpha+\beta+k})_\infty}{(q^{\alpha+k-1})_\infty(q^\beta)_\infty} \\ =(1-q)\frac{(q)_\infty(q^{\alpha+\beta})_\infty}{(q^\alpha)_\infty(q^\beta)_\infty}\sum_{k=0}^{m+n}A_k^{m,n}\prod_{j=0}^{k-1}\frac{1-q^{\alpha+j}}{1-q^{\alpha+\beta+j}}.\end{aligned} \tag{5.16}$$

Hence the relation (5.2) is equivalent to

$$\sum_{k=0}^{m+n}A_k^{m,n}\prod_{j=0}^{k-1}\frac{1-q^{\alpha+j}}{1-q^{\alpha+\beta+j}}=\delta_{m,n}\frac{1-q^{\alpha+\beta-1}}{1-q^{\alpha+\beta+2n-1}}\frac{(q)_n(q^\beta)_n}{(q^{\alpha+\beta-1})_n(q^\alpha)_n}q^{n\alpha}. \tag{5.17}$$

Note that (5.17) is an algebraic equality and holds also in the case of $|q|=1$.

On the other hand, we find the following from (3.27) in the same way as (5.16):

$$\begin{aligned}\int_C q^{\alpha z}\frac{\langle z+1+\frac{1}{\omega}\rangle}{\langle z+\beta\rangle}p_m^{(\alpha-1,\beta-1)}(t)p_n^{(\alpha-1,\beta-1)}(t)dz \\ =\frac{\langle 1\rangle\langle\alpha+\beta\rangle}{\langle\alpha\rangle\langle\beta\rangle}\sum_{k=0}^{m+n}A_k^{m,n}\prod_{j=0}^{k-1}\frac{1-q^{\alpha+j}}{1-q^{\alpha+\beta+j}}.\end{aligned} \tag{5.18}$$

From (5.17) and (5.18), we get (5.13). □

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