HNN Extensions of Quasi-Lattice Ordered Groups and their Operator Algebras

Astrid an Huef[1], Iain Raeburn[2], and Ilija Tolich[3]

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Abstract. The Baumslag-Solitar group is an example of an HNN extension. Spielberg showed that it has a natural positive cone, and that it is then a quasi-lattice ordered group in the sense of Nica. We give conditions for an HNN extension of a quasi-lattice ordered group (G, P) to be quasi-lattice ordered. In that case, if (G, P) is amenable as a quasi-lattice ordered group, then so is the HNN extension.

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1 Introduction

Since they were introduced by Nica [9], quasi-lattice ordered groups and their C^*-algebras have generated considerable interest (see, for example, [5],[6]). The amenability of quasi-lattice ordered groups has been a deep subject (see, for example, [3],[4] and [7]). Quasi-lattice ordered groups are also examples of the more recent LCM semigroups [1], [13]. Here we generalise two recent results about the Baumslag-Solitar group.

First, Spielberg proved that the Baumslag-Solitar group is quasi-lattice ordered [11]. The Baumslag-Solitar group is an example of an HNN extension of $\mathbb{Z}$, and hence we wondered if HNN extensions could provide new classes of quasi-lattice ordered groups. Spielberg also showed that a groupoid associated to the Baumslag-Solitar semigroup is amenable [11, Theorem 3.22].

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Second, Clark, an Huef and Raeburn examined the phase-transitions of the Toeplitz algebra of the Baumslag-Solitar group [2]. As part of their investigation they proved that the Baumslag-Solitar group is amenable as a quasi-lattice ordered group. The standard way to prove amenability, introduced by Laca and Raeburn [5], is to use a "controlled map": an order-preserving homomorphism between quasi-lattice ordered groups. They observed that the height map, which counts the number of times the stable letter of the HNN extension appears in a word, is almost a controlled map, and then they adapted the standard proof in [2, Appendix A] to fit.

Our innovation in this paper is a more general definition of a controlled map. We prove in Theorem 3.2 that if (G, P) is a quasi-lattice ordered group and there is a controlled map μ into an amenable group, and if $\ker\mu$ is an amenable quasi-lattice ordered group, then (G, P) is amenable. The motivation for Theorem 3.2 was two-fold. First, if a normal subgroup N of a group G is amenable and G/N is amenable, then G is amenable, and second, Spielberg's result on amenability of groupoids [12, Proposition 9.3].

In Theorem 4.1 we give conditions under which an HNN extension of a quasi-lattice ordered group is quasi-lattice ordered. This result allows us to construct many new examples of quasi-lattice ordered groups. Finally, we use Theorem 3.2 to prove that an HNN extension of an amenable quasi-lattice ordered group is amenable (Theorem 5.1).

2 Preliminaries

Let P be a subsemigroup of a discrete group G such that $P \cap P^{-1} = \{e\}$. There is a partial order on G defined by

$$x \leq y \Leftrightarrow x^{-1}y \in P \Leftrightarrow y \in xP.$$

The order is left-invariant in the sense that $x \leq y$ implies $zx \leq zy$ for all $z \in G$. A partially ordered group (G, P) is *quasi-lattice ordered* if every finite subset of G with a common upper bound in P has a least common upper bound in P [9, Definitions 2.1]. By [3, Lemma 7], (G, P) is quasi-lattice ordered if and only if:

if $x \in PP^{-1}$, then there exist a pair $\mu, \nu \in P$ with $x = \mu\nu^{-1}$ such that $\gamma, \delta \in P$ and $\gamma\delta^{-1} = \mu\nu^{-1}$ imply $\mu \leq \gamma$ and $\nu \leq \delta$. (The pair μ, ν is unique.) (1)

Let (G, P) be a quasi-lattice ordered group, and let $x, y \in G$. If x and y have a common upper bound in P, then their least common upper bound in P is denoted $x \vee y$. We write $x \vee y = \infty$ when x and y have no common upper bound in P and $x \vee y < \infty$ when they have a common upper bound. An *isometric representation of* P in a C^*-algebra A is a map $W : P \to A$ such that $W_e = 1$, W_p is an isometry and $W_pW_q = W_{pq}$ for all $p, q \in P$. We say that W

is *covariant* if

$$W_pW_p^*W_qW_q^* = \begin{cases} W_{p\vee q}W_{p\vee q}^* & \text{if } p\vee q<\infty \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Equivalently, W is covariant if

$$W_p^*W_q = \begin{cases} W_{p^{-1}(p\vee q)}W_{q^{-1}(p\vee q)}^* & \text{if } p\vee q<\infty \\ 0 & \text{otherwise.} \end{cases}$$

An example of a covariant representation is $T : P \to B(\ell^2(P))$ characterised by $T_p\epsilon_x = \epsilon_{px}$ where $\{\epsilon_x : x \in P\}$ is the orthonormal basis of point masses in $\ell^2(P)$.

In [9, §§2.4 and 4.1] Nica examined two C^*-algebras associated to (G,P). The reduced C^*-algebra $C_r^*(G,P)$ of (G,P) is the C^*-subalgebra of $B(\ell^2(P))$ generated by $\{T_p : p \in P\}$. The universal C^*-algebra $C^*(G,P)$ of (G,P) is generated by a universal covariant representation w; it is universal for covariant representations of P in the following sense: for any covariant representation $W : P \to A$ there exists a unital homomorphism $\pi_W : C^*(G,P) \to A$ such that $\pi_W(w_p) = W_p$. It follows from (2) that

$$C^*(G,P) = \overline{\operatorname{span}}\{w_pw_q^* : p,q \in P\}.$$

Nica defined (G,P) to be *amenable* if the homomorphism $\pi_T : C^*(G,P) \to C_r^*(G,P)$ is faithful [9, §4.2]. He identified an equivalent condition: there exists a conditional expectation $E : C^*(G,P) \to \overline{\operatorname{span}}\{w_pw_p^* : p \in P\}$, and (G,P) is amenable if and only if E is faithful (that is, $E(a^*a) = 0$ implies $a^*a = 0$ for all $a \in C^*(G,P)$). Laca and Raeburn took this second condition as their definition of amenability [5, Definition 3.4].

3 Order-preserving maps and amenability

A key technique, introduced by Laca and Raeburn in [5, Proposition 6.6][4], is the use of an order-preserving homomorphism between two quasi-lattice ordered groups which preserves the least upper bound structure. Crisp and Laca called such a homomorphism a *controlled map* [4]. If (G,P) and (K,Q) are quasi-lattice ordered groups, $\mu : G \to K$ is a controlled map and K is an amenable group, then (G,P) is amenable as a quasi-lattice ordered group by [5, Proposition 6.6]. Motivated by work in [2, Appendix A] we now give a weaker definition for a controlled map. We then follow the program of [2] to generalise [5, Proposition 6.6]. We state this generalisation in Theorem 3.2 below; its proof will take up the remainder of this section.

[4]There is an error in the statement of [5, Proposition 6.6]: the final line should read "if $\mathcal{G}$ is an amenable group, then (G,P) is amenable".

Definition 3.1. Let (G,P) and (K,Q) be quasi-lattice ordered groups. Let $\mu : G \to K$ be an order-preserving group homomorphism. For each $k \in Q$, let Σ_k be the set of $\sigma \in \mu^{-1}(k) \cap P$ which are minimal, that is,

$$x \in \mu^{-1}(k) \cap P \text{ and } x \leq \sigma \Rightarrow \sigma = x.$$

We say μ is a *controlled map* if it has the following properties:

1. For all $x, y \in P$ such that $x \vee y < \infty$ we have $\mu(x) \vee \mu(y) = \mu(x \vee y)$.
2. For all $k \in Q$, Σ_k is complete in the following sense: for every $x \in \mu^{-1}(k) \cap P$ there exists $\sigma \in \Sigma_k$ such that $\sigma \leq x$.
3. For all $k \in Q$ and $\sigma, \tau \in \Sigma_k$ we have $\sigma \vee \tau < \infty \Rightarrow \sigma = \tau$.

If $\mu : G \to K$ is a controlled map in the sense of [5], then it is a controlled map in the sense of Definition 3.1.

Theorem 3.2. *Let (G,P) and (K,Q) be quasi-lattice ordered groups. Suppose that $\mu : G \to K$ is a controlled map. If K is an amenable group and $\big(\mu^{-1}(e), \mu^{-1}(e) \cap P\big)$ is an amenable quasi-lattice ordered group, then (G,P) is amenable.*

We start by showing that the kernel of a controlled map is a quasi-lattice ordered group.

Lemma 3.3. *Let (G,P) and (K,Q) be quasi-lattice ordered groups, and suppose that $\mu : G \to K$ is a controlled map. Then $\big(\mu^{-1}(e), \mu^{-1}(e) \cap P\big)$ is a quasi-lattice ordered group.*

Proof. It is clear that $\mu^{-1}(e)$ is a subgroup of G and that $\mu^{-1}(e) \cap P$ is a unital semigroup.
Recall from [9, Definitions 2.1] that (G,P) is quasi-lattice ordered if and only if

1. any $x \in PP^{-1}$ has a least upper bound in P, and
2. any $x, y \in P$ with a common upper bound have a least upper bound.

Let $x \in (\mu^{-1}(e) \cap P)(\mu^{-1}(e) \cap P)^{-1}$, say $x = yz^{-1}$ where $y, z \in \mu^{-1}(e) \cap P$. Then $x \in PP^{-1}$, and since $G,P)$ is quasi-lattice ordered, x has a least upper bound w in P. Since y is an upper bound for x in P, we have $w \leq y$, and since μ is order-preserving we have $\mu(w) \leq \mu(y) = e$. Then $\mu(w) = e$, and $w \in \mu^{-1}(e) \cap P$ is an upper bound for x. Let w' be any upper bound for x in $\mu^{-1}(e) \cap P$. Then $w \leq w'$ in P, and hence also in $\mu^{-1}(e) \cap P$. Thus w is the least upper bound for x in $\mu^{-1}(e) \cap P$.
Next, let $x, y \in \mu^{-1}(e) \cap P$, and suppose that x, y have a common upper bound $z \in \mu^{-1}(e) \cap P$. Then z is a common upper bound for x, y in P, and hence $x \vee y$ exists in P and $x \vee y \leq z$. Since $x, y \in P$, by Item 1 of Definition 3.1 we have $\mu(x \vee y) = \mu(x) \vee \mu(y) = e$. Hence $x \vee y \in \mu^{-1}(e) \cap P$, and it follows that $x \vee y$ is the least upper bound for x and y in $\mu^{-1}(e) \cap P$. Thus $\big(\mu^{-1}(e), \mu^{-1}(e) \cap P\big)$ is a quasi-lattice ordered group. □

To prove Theorem 3.2 we will show that the conditional expectation

$$E : C^*(G,P) \to \overline{\operatorname{span}}\{w_p w_p^* : p \in P\}$$

is faithful. We will use the amenability of K to construct a faithful conditional expectation $\Psi^\mu : C^*(G,P) \to \overline{\operatorname{span}}\{w_p w_q^* : \mu(p) = \mu(q)\}$, and then show that E is faithful when restricted to range Ψ^μ. To construct Ψ^μ we follow the method of [5, Lemma 6.5] which uses a coaction.
Let G be a discrete group and let A be a unital C^*-algebra. Let

$$\delta_G : C^*(G) \to C^*(G) \otimes_{\min} C^*(G)$$

be the comultiplication of G which is characterised by $\delta_G(u_g) = u_g \otimes u_g$ for $g \in G$. A *coaction* of G on A is a unital homomorphism $\delta : A \to A \otimes_{\min} C^*(G)$ such that

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta.$$

We say that δ is *nondegenerate* if $\delta(A)(1 \otimes C^*(G)) = A \otimes_{\min} C^*(G)$.

Lemma 3.4. *Let (G,P) be a quasi-lattice ordered group. Suppose that there exists a group K and a homomorphism $\mu : G \to K$. Then there exists an injective coaction*

$$\delta_\mu : C^*(G,P) \to C^*(G,P) \otimes_{\min} C^*(K)$$

characterised by $\delta_\mu(w_p) = w_p \otimes u_{\mu(p)}$ for all $p \in P$.

Proof. Let $W : P \to C^*(G,P) \otimes_{\min} C^*(K)$ be characterised by $W_p = w_p \otimes u_{\mu(p)}$. We will show that W is a covariant representation, and then take $\delta_\mu := \pi_W$. Unitaries are isometries and hence W_p is isometric for all $p \in P$. Observe that $W_e = w_e \otimes u_{\mu(e)} = 1 \otimes 1$, and

$$W_p W_q = w_p w_q \otimes u_{\mu(p)} u_{\mu(q)} = w_{pq} \otimes u_{\mu(pq)} = W_{pq} \text{ for all } p, q \in P.$$

Thus W is an isometric representation. To prove W is covariant, we fix $x, y \in P$ and compute:

$$\begin{aligned} W_x W_x^* W_y W_y^* &= w_x w_x^* w_y w_y^* \otimes u_{\mu(x)} u_{\mu(x)}^* u_{\mu(y)} u_{\mu(y)}^* \\ &= \begin{cases} w_{x\vee y} w_{x\vee y}^* \otimes 1 & \text{if } x \vee y < \infty \\ 0 \otimes 1 & \text{otherwise} \end{cases} \\ &= \begin{cases} w_{x\vee y} w_{x\vee y}^* \otimes w_{\mu(x\vee y)} w_{\mu(x\vee y)}^* & \text{if } x \vee y < \infty \\ 0 & \text{otherwise} \end{cases} \\ &= W_{x\vee y} W_{x\vee y}^*. \end{aligned}$$

Thus W is a covariant representation of P. By the universal property of $C^*(G,P)$, there exists a homomorphism $\delta_\mu := \pi_W$, which is characterised by $\delta_u(w_p) = W_p = w_p \otimes u_{\mu(p)}$. Since $W_e = 1 \otimes 1$ it follows that δ_μ is unital.

To prove the comultiplication identity, we compute on generators: for $p, q \in P$ we have

$$\begin{aligned}((\delta_\mu \otimes \mathrm{id}) \circ \delta_\mu)(w_p w_q^*) &= (\delta_\mu \otimes \mathrm{id})(w_p w_q^* \otimes u_{\mu(pq^{-1})}) \\ &= \delta_\mu(w_p w_q^*) \otimes \mathrm{id}(u_{\mu(pq^{-1})}) \\ &= w_p w_q^* \otimes u_{\mu(pq^{-1})} \otimes u_{\mu(pq^{-1})} \\ &= w_p w_q^* \otimes \delta_K(u_{\mu(pq^{-1})}) \\ &= \mathrm{id} \otimes \delta_K(w_p w_q^* \otimes u_{\mu(pq^{-1})}) \\ &= ((\mathrm{id} \otimes \delta_K) \circ \delta_\mu)(w_p w_q^*).\end{aligned}$$

Hence $(\delta_\mu \otimes \mathrm{id}) \circ \delta_\mu = (\mathrm{id} \otimes \delta_K) \circ \delta_\mu$. Thus δ_μ is a coaction.
To show that δ_μ is injective, let $\pi : C^*(G,P) \to B(H)$ be a faithful representation. We will show that π can be written as a composition of δ_μ and another representation. Let $\epsilon : C^*(K) \to \mathbb{C}$ be the trivial representation on $\mathbb{C}$ such that $\epsilon(u_k) = 1$ for all $k \in K$. By the properties of the minimal tensor product (see [10, Proposition B.13]) there exists a homomorphism

$$\pi \otimes \epsilon : C^*(G,P) \otimes_{\min} C^*(K) \to B(H) \otimes \mathbb{C} = B(H).$$

Since

$$(\pi \otimes \epsilon) \circ \delta_\mu(w_p) = (\pi \otimes \epsilon)(w_p \otimes u_{\mu(p)}) = \pi(w_p),$$

we have $\pi = (\pi \otimes \epsilon) \circ \delta_\mu$. Now suppose that $\delta_\mu(a) = 0$ for some $a \in C^*(G,P)$. Then $0 = (\pi \otimes \epsilon) \circ \delta_\mu(a) = \pi(a)$. Since π is faithful, $a = 0$. Hence δ_μ is injective.
To prove that δ_μ is a nondegenerate coaction we must show that

$$\delta_\mu(C^*(G,P))(1 \otimes C^*(K)) = C^*(G,P) \otimes_{\min} C^*(K).$$

It suffices to show that we can get the spanning elements $w_p w_q^* \otimes u_k$, and this is easy:

$$\delta_\mu(w_p w_q^*)(1 \otimes u_{\mu(qp^{-1})k}) = w_p w_q^* \otimes u_{\mu(pq^{-1})}(1 \otimes u_{\mu(qp^{-1})k}) = w_p w_q^* \otimes u_k.$$

Thus δ_μ is nondegenerate. □

Let λ be the left-regular representation of a discrete group K. There is a trace τ on $C^*(K)$ characterised by

$$\tau(u_k) = (\lambda_k \epsilon_e \mid \epsilon_e) = \begin{cases} 1 & \text{if } k = e \\ 0 & \text{otherwise.} \end{cases}$$

It is well-known that if K is an amenable group, then τ is faithful.

Lemma 3.5. *Let (G, P) be a quasi-lattice ordered group. Suppose that there exist a group K and a homomorphism $\mu : G \to K$. Let*

$$\delta_\mu : C^*(G, P) \to C^*(G, P) \otimes_{\min} C^*(K)$$

be the coaction of Lemma 3.4. Then

$$\Psi^\mu := (\mathrm{id} \otimes \tau) \circ \delta_\mu$$

is a conditional expectation of $C^*(G, P)$ with range $\overline{\mathrm{span}}\{w_p w_q^* : \mu(p) = \mu(q)\}$. If K is an amenable group, then Ψ^μ is faithful.

Proof. Since $\mathrm{id} \otimes \tau$ and δ_μ are linear and norm decreasing, so is Ψ^μ. Since $\Psi^\mu(w_e) = 1$ the norm of Ψ^μ is 1. We have

$$\Psi^\mu(w_p w_q^*) = \begin{cases} w_p w_q^* & \text{if } \mu(p) = \mu(q) \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

and hence $\Psi^\mu \circ \Psi^\mu = \Psi^\mu$. Thus Ψ^μ is a conditional expectation by [14].
From (3) we see that $\overline{\mathrm{span}}\{w_p w_q^* : \mu(q) = \mu(p)\} \subseteq \mathrm{range}\, \Psi^\mu$. To show the reverse inclusion, fix $b \in \mathrm{range}\, \Psi^\mu$, say $b = \Psi^\mu(a)$ for some $a \in C^*(G, P)$. Also fix $\epsilon > 0$. There exists a finite subset $F \subseteq P \times P$ such that $\|a - \sum_{(p,q)\in F} \lambda_{p,q} w_p w_q^*\| < \epsilon$. Since Ψ^μ is linear and norm-decreasing,

$$\begin{aligned} \epsilon > \Big\| a - \sum_{(p,q)\in F} \lambda_{p,q} w_p w_q^* \Big\| &\geq \Big\| \Psi^\mu\Big(a - \sum_{(p,q)\in F} \lambda_{p,q} w_p w_q^* \Big) \Big\| \\ = \Big\| \Psi^\mu(a) - \Psi^\mu\Big(\sum_{(p,q)\in F} \lambda_{p,q} w_p w_q^* \Big) \Big\| &= \Big\| b - \sum_{(p,q)\in F,\ \mu(p)=\mu(q)} \lambda_{p,q} w_p w_q^* \Big\|. \end{aligned}$$

Thus $b \in \overline{\mathrm{span}}\{w_p w_q^* : \mu(q) = \mu(p)\}$, and $\mathrm{range}\, \Psi^\mu = \overline{\mathrm{span}}\{w_p w_q^* : \mu(q) = \mu(p)\}$.
Now suppose that K is amenable. To see that Ψ^μ is faithful, we follow the proof of [5, Lemma 6.5]. Let $a \in C^*(G, P)$ and suppose that $\Psi^\mu(a^*a) = 0$. Let f be an arbitrary state on $C^*(G, P)$. Then

$$\begin{aligned} 0 = f(\Psi^\mu(a^*a)) &= f \circ (\mathrm{id} \otimes \tau) \circ \delta_\mu(a^*a) \\ &= (f \otimes \tau) \circ \delta_\mu(a^*a) = \tau \circ (f \otimes \mathrm{id}) \circ \delta_\mu(a^*a). \end{aligned}$$

Since K is amenable, τ is faithful. Hence $(f \otimes \mathrm{id}) \circ \delta(a^*a) = 0$. This implies that for all states f on $C^*(G, P)$ and states g on $C^*(K)$,

$$g \circ (f \otimes \mathrm{id}) \circ \delta_\mu(a^*a) = (f \otimes g) \circ \delta_\mu(a^*a) = 0.$$

To see that $\delta_\mu(a^*a) = 0$, let $\pi_1 : C^*(G, P) \to H_1$ and $\pi_2 : C^*(K) \to H_2$ be faithful representations. Then $\pi_1 \otimes \pi_2$ is a faithful representation of $C^*(G, P) \otimes_{\min} C^*(K)$ on $B(H_1 \otimes H_2)$ by [10, Corollary B.11]. Fix unit vectors

$h \in H_1$, $k \in H_2$. There exists a state $f_h \otimes f_k$ on $C^*(G,P) \otimes_{\min} C^*(K)$ defined by

$$f_h \otimes f_k(x) = (\pi_1 \otimes \pi_2(x)(h \otimes k) \mid h \otimes k).$$

Since $(f \otimes g) \circ \delta_\mu(a^*a) = 0$ for all states f of $C^*(G,P)$ and g of $C^*(K)$, we have

$$\begin{aligned} 0 &= f_h \otimes f_k(\delta_\mu(a^*a)) \\ &= (\pi_1 \otimes \pi_2(\delta_\mu(a^*a))(h \otimes k) \mid h \otimes k) \\ &= (\pi_1 \otimes \pi_2(\delta_\mu(a))(h \otimes k) \mid \pi_1 \otimes \pi_2(\delta_\mu(a))h \otimes k) \\ &= \|\pi_1 \otimes \pi_2(\delta_\mu(a))(h \otimes k)\|^2. \end{aligned}$$

Hence $\pi_1 \otimes \pi_2(\delta_\mu(a^*a)) = 0$. Since $\pi_1 \otimes \pi_2$ is faithful, $\delta_\mu(a^*a)) = 0$. But δ_μ is injective, and hence $a = 0$, and Ψ^μ is faithful. $\square$

Next we investigate the structure of

$$\operatorname{range} \Psi^\mu = \overline{\operatorname{span}}\{w_p w_q^* : \mu(p) = \mu(q)\}.$$

Lemma 3.6. *Let (G,P) and (K,Q) be quasi-lattice ordered groups, and suppose that $\mu : G \to K$ is a controlled map. Let $k \in Q$, and let F be a finite subset of Σ_k. Let*

$$B_{k,F} := \operatorname{span}\{w_\sigma \omega_\alpha w_\beta^* w_\tau^* : \sigma, \tau \in F, \alpha, \beta \in \mu^{-1}(e) \cap P\}.$$

Then $B_{k,F}$ is a closed C^*-subalgebra of $\overline{\operatorname{span}}\{w_p w_q^* : \mu(p) = \mu(q)\}$.

Proof. It is straightforward to see that $B_{k,F}$ is contained in $\overline{\operatorname{span}}\{w_p w_q^* : \mu(p) = \mu(q)\}$. Let $A = \overline{\operatorname{span}}\{\omega_\alpha w_\beta^* : \alpha, \beta \in \mu^{-1}(e) \cap P\}$. We will prove the lemma by showing that $B_{k,F}$ is isomorphic to

$$M_F(\mathbb{C}) \otimes A.$$

By Item 3 of Definition 3.1, the elements of F have no common upper bound unless they are equal. So

$$(w_\sigma w_\tau^*)(w_\gamma w_\delta^*) = w_{\sigma\tau^{-1}(\tau \vee \gamma)} w^*_{\delta\gamma^{-1}(\tau\vee\gamma)} = \begin{cases} w_\sigma w_\delta^* & \text{if } \tau = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\{w_\sigma w_\tau^* : \sigma, \tau \in F\}$ is a set of matrix units in the C^*-algebra $\overline{B_{k,F}}$. This gives a homomorphism $\theta : M_F(\mathbb{C}) \to \overline{B_{k,F}}$ which maps the matrix units $\{E_{\sigma,\tau} : \sigma, \tau \in F\}$ in $M_F(\mathbb{C})$ to $\{w_\sigma w_\tau^* : \sigma, \tau \in F\} \subset B_{k,F}$. It is easy to check that the formula

$$\psi(D) = \sum_{\gamma \in F} w_\gamma D w_\gamma^*$$

gives a homomorphism $\psi : A \to B_{k,F}$. We have

$$\begin{aligned}\theta(E_{\sigma,\tau})\psi(D) &= w_\sigma w_\tau^* \sum_{\gamma \in F} w_\gamma D w_\gamma^* \\ &= w_\sigma w_\tau^* w_\tau D w_\tau^* \quad (w_\tau^* w_\gamma = 0 \text{ unless } \tau = \gamma) \\ &= w_\sigma D w_\tau^* \\ &= (\sum_{\gamma \in F} w_\gamma D w_\gamma^*) w_\sigma w_\tau^* \\ &= \psi(D)\theta(E_{\sigma,\tau}).\end{aligned}$$

Each $M \in M_F(\mathbb{C})$ is a linear combination of the $E_{\sigma,\tau}$, and hence $\psi(D)\theta(M) = \theta(M)\psi(D)$ for all $M \in M_F(\mathbb{C})$ and $D \in A$. Since the ranges of θ and ψ commute, the universal property of the maximal tensor product gives a homomorphism $\theta \otimes_{\max} \psi$ of $M_F(\mathbb{C}) \otimes_{\max} A$ into $\overline{B_{k,F}}$.
By [10, Theorem B.18]

$$M_F(\mathbb{C}) \otimes_{\max} A = \operatorname{span}\{E_{\sigma,\tau} \otimes D : \sigma, \tau \in F \text{ and } D \in A\},$$

with no closure. So the range of $\theta \otimes_{\max} \psi$ is spanned by $\theta(E_{\sigma,\tau})\psi(D) = w_\sigma D w_\tau^*$ and hence is $B_{k,F}$. Thus $B_{k,F}$ is a closed C^*-subalgebra of $\overline{\operatorname{span}}\{w_p w_q^* : \mu(p) = \mu(q)\}$. ☐

Let $\{\epsilon_x : x \in P\}$ be the usual basis for $\ell^2(P)$. Let T be the covariant representation of (G, P) on $\ell^2(P)$ such that $T_p \epsilon_x = \epsilon_{px}$, and let π_T be the corresponding homomorphism of $C^*(G, P)$ onto $C_r^*(G, P)$ such that $\pi_T(w_p) = T_p$. For $k \in Q$ we consider the subspaces

$$H_k := \overline{\operatorname{span}}\{\epsilon_{\gamma z} : \gamma \in \Sigma_k, z \in \mu^{-1}(e) \cap P\}.$$

Lemma 3.7. *Let (G, P) and (K, Q) be quasi-lattice ordered groups, and suppose that $\mu : G \to K$ is a controlled map. Let $k \in Q$, and let F be a finite subset of Σ_k. Then*

1. *H_k is invariant for $\pi_T|_{B_{k,F}}$;*
2. *if $\big(\mu^{-1}(e), \mu^{-1}(e) \cap P\big)$ is amenable, then $\pi_T(\cdot)|_{H_k}$ is isometric on $B_{k,F}$.*

Proof. For Item 1, let $\sigma, \tau \in F$ and let $x, y \in \mu^{-1}(e) \cap P$ and let $\epsilon_{\gamma z} \in H_k$. Then $w_\sigma w_x w_y^* w_\tau^*$ is a spanning element of $B_{k,F}$. Since $\mu(\tau) = k = \mu(\gamma)$ we have

$$\pi_T(w_\sigma w_x w_y^* w_\tau^*)\epsilon_{\gamma z} = \begin{cases} \epsilon_{\sigma x y^{-1} z} & \text{if } \gamma = \tau \text{ and } y \leq z \\ 0 & \text{otherwise.} \end{cases}$$

If $\pi_T(w_\sigma w_x w_y^* w_\tau^*)\epsilon_{\gamma z} = 0$ we are done. Otherwise, to see that $\epsilon_{\sigma x y^{-1} z}$ is back in H_k, suppose that $y \leq z$. Then $y^{-1}z \in P$. Since $\sigma x \in \mu^{-1}(k) \cap P$ we have $\epsilon_{(\sigma x)(y^{-1}z)} \in H_k$. It follows that H_k is invariant for $\pi_T|_{B_{k,F}}$.
For Item 2 suppose that $\big(\mu^{-1}(e), \mu^{-1}(e) \cap P\big)$ is amenable. We will show that $\pi_T(\cdot)|_{H_k}$ is faithful on $B_{k,F}$. Take $B = \sum_{\sigma,\tau \in F} w_\sigma D_{\sigma,\tau} w_\tau^* \in B_{k,F}$ and suppose that $\pi_T(B)|_{H_k} = 0$. Fix $\gamma, \delta \in F$. Then

$$T_\gamma^* \pi_T(B) T_\delta = \pi_T(w_\gamma^*)\pi_T(B)\pi_T(w_\delta) = \pi_T(D_{\gamma,\delta}).$$

Since T_δ is an injection from H_e to H_k and $\pi_T(B)|_{H_k} = 0$, it follows that $\pi_T(B)T_\delta|_{H_e} = 0$. Thus $\pi_T(D_{\gamma,\delta})|_{H_e} = 0$.
But the restriction

$$(\pi_T|_{C^*\left(\mu^{-1}(e), \mu^{-1}(e)\cap P\right)} \cdot)|_{H_e}$$

is the Toeplitz representation of $\big(\mu^{-1}(e), \mu^{-1}(e) \cap P\big)$, and hence is faithful by amenability. Thus $D_{\gamma,\delta} = 0$. Repeating the argument finitely many times shows that all the $D_{\sigma,\tau} = 0$ and hence that $B = 0$. Thus $\pi_T(\cdot)|_{H_k}$ is faithful on $B_{k,F}$, and therefore is isometric. $\square$

Lemma 3.8. *Let (G, P) and (K, Q) be quasi-lattice ordered groups, and suppose that $\mu : G \to K$ is a controlled map. Let*

$$B_k = \overline{\operatorname{span}}\{w_p w_q^* : \mu(p) = \mu(q) = k\}.$$

Let $\mathcal{F}$ be the set of all finite sets $F \subseteq \Sigma_k$. Then $B_k = \overline{\cup_{F\in\mathcal{F}} B_{k,F}}$. If $\big(\mu^{-1}(e), \mu^{-1}(e) \cap P\big)$ is amenable, then $\pi_T(\cdot)|_{H_k}$ is isometric on B_k.

Proof. Observe that $\mathcal{F}$ is a directed set partially ordered by inclusion with $E, F \in \mathcal{F}$ majorismajorizeded by $E \cup F$. If $E \subseteq F$, then $B_{k,E} \subseteq B_{k,F}$. Thus $\{B_{k,F} : F \in \mathcal{F}\}$ is an inductive system with limit $\overline{\cup_{F\in\mathcal{F}} B_{k,F}}$.
For each $F \in \mathcal{F}$ we have $B_{k,F} \subseteq B_k$, and B_k is closed. Therefore $\overline{\cup_{F\in\mathcal{F}} B_{k,F}} \subseteq B_k$. To prove the reverse inclusion it suffices to show that the spanning elements of B_k are in $B_{k,F}$ for some F. Fix $p, q \in P$ such that $\mu(p) = \mu(q) = k$ and consider $w_p w_q^*$. By Item 2 of Definition 3.1, the set Σ_k of minimal elements is complete, and there exists $\sigma, \tau \in \Sigma_k$ such that $\sigma \leq p$ and $\tau \leq q$. Hence there exists $x, y \in P$ such that $p = \sigma x$ and $q = \tau y$. Thus $w_p w_q^* = w_{\sigma x} w_{\tau y}^* = w_\sigma(w_x w_y^*) w_\tau^*$ and $w_x w_y^* \in C^*\big(\mu^{-1}(e), \mu^{-1}(e) \cap P\big)$. Since $\{\sigma, \tau\} \in \mathcal{F}$ we have $w_p w_q^* \in B_{k,\{\sigma,\tau\}}$. Thus $B_k \subseteq \overline{\cup_{F\in\mathcal{F}} B_{k,F}}$, and equality follows.
Finally, suppose that $\big(\mu^{-1}(e), \mu^{-1}(e) \cap P\big)$ is amenable. Then $\pi_T(\cdot)|_{H_k}$ is isometric on $B_{k,F}$ for all $F \in \mathcal{F}$ by Item 2 of Lemma 3.7. Since π_T is isometric on every $B_{k,F}$, its extension to the closure is also isometric. $\square$

Let $\mathcal{I}$ be the set of all finite sets $I \subset Q$ that are closed under $\vee$ in the sense that $s, t \in I$ and $s \vee t < \infty$ implies that $s \vee t \in I$.

Lemma 3.9. *Let (G,P) and (K,Q) be quasi-lattice ordered groups, and suppose that $\mu : G \to K$ is a controlled map. For each $I \in \mathcal{I}$ let*

$$C_I = \overline{\operatorname{span}}\{w_p w_q^* : \mu(p) = \mu(q) \in I\}.$$

Then C_I is a C^*-subalgebra of $\overline{\operatorname{span}}\{w_p w_q^* : \mu(p) = \mu(q)\}$, $C_I = \operatorname{span}\{B_k : k \in I\}$ and $\overline{\operatorname{span}}\{w_p w_q^* : \mu(p) = \mu(q)\} = \overline{\cup_{I\in\mathcal{I}} C_I}$.

Proof. Fix $I \in \mathcal{I}$. To see that C_I is a C^*-subalgebra, it suffices to show that $\operatorname{span}\{w_p w_q^* : \mu(p) = \mu(q) \in I\}$ is a $*$-subalgebra. It's clearly closed under taking adjoints. Let $p, q, r, s \in P$ such that $\mu(p) = \mu(q) \in I$ and $\mu(r) = \mu(s) \in I$. Then

$$w_p w_q^* w_r w_s^* = \begin{cases} w_{pq^{-1}(q\vee r)} w^*_{sr^{-1}(q\vee r)} & \text{if } q \vee r < \infty \\ 0 & \text{otherwise.} \end{cases}$$

If $w_p w_q^* w_r w_s^* = 0$ we are done. So suppose that $w_p w_q^* w_r w_s^* \neq 0$. Then $q \vee r < \infty$. Since μ is a controlled map and $\mu(p) = \mu(q)$, by Item 1 of Definition 3.1,

$$\mu(pq^{-1}(q \vee r)) = \mu(q \vee r) = \mu(q) \vee \mu(r).$$

Similarly, $\mu(sr^{-1}(q \vee r)) = \mu(q) \vee \mu(r)$. Since I is closed under $\vee$ we have $\mu(q) \vee \mu(r) \in I$, and hence $w_p w_q^* w_r w_s^* \in \operatorname{span}\{w_p w_q^* : \mu(p) = \mu(q) \in I\}$. It follows that C_I is a C^*-subalgebra.

For each $k \in I$, we have $B_k \subseteq C_I$, and so $\operatorname{span}\{B_k : k \in I\} \subseteq C_I$. To show the reverse inclusion observe that for $w_p w_q^* \in C_I$ we have $w_p w_q^* \in B_{\mu(p)}$. Since the finite span of closed subalgebras is closed, $\overline{\operatorname{span}}\{w_p w_q^* : \mu(p) = \mu(q) \in I\} \subseteq \operatorname{span}\{B_k : k \in I\}$. Thus $C_I = \operatorname{span}\{B_k : k \in I\}$. $\square$

Proposition 3.10. *Let (G,P) and (K,Q) be quasi-lattice ordered groups, and suppose that $\mu : G \to K$ is a controlled map. If $\left(\mu^{-1}(e), \mu^{-1}(e) \cap P\right)$ is amenable, then π_T is faithful on $\overline{\operatorname{span}}\{w_p w_q^* : \mu(p) = \mu(q)\}$.*

Proof. By Lemma 3.9, $\overline{\operatorname{span}}\{w_p w_q^* : \mu(p) = \mu(q)\} = \overline{\cup_{I\in\mathcal{I}} C_I}$. Thus it suffices to show that π_T is isometric on each C_I. Fix $I \in \mathcal{I}$. Suppose that $\pi_T(R) = 0$ for some $R \in C_I$. Then there exist $R_k \in B_k$ such that $R = \sum_{k\in I} R_k$ and then $\sum_{k\in I} \pi_T(R_k) = 0$.

We claim that if $k \not\leq j$, then $\pi_T(B_k)|_{H_j} = 0$ (it then follows that $\pi_T(R_k)|_{H_j} = 0$). To prove the claim, it suffices to show that $\pi_T(w_q^*)\epsilon_{\gamma z} = 0$ for all $q \in \mu^{-1}(k) \cap P$ and $\epsilon_{\gamma z} \in H_k$. We have

$$\pi_T(w_q^*)\epsilon_{\gamma z} = T_q^* \epsilon_{\gamma z} = \begin{cases} \epsilon_{q^{-1}\gamma z} & \text{if } q \leq \gamma z \\ 0 & \text{otherwise.} \end{cases}$$

But $q \leq \gamma z$ implies $k = \mu(q) \leq \mu(\gamma z) = \mu(\gamma) = j$. So $k \not\leq j$ implies $\pi_T(w_q^*)\epsilon_{\gamma z} = 0$. Hence $\pi_T(B_k)|_{H_j} = 0$ if $k \not\leq j$ as claimed.

Let l be a minimal element of I in the sense that $x \leq l$ implies $x = l$. Then for $k \in I$, we have $k \not\leq l$ unless $k = l$. Now

$$0 = \sum_{k \in I} \pi_T(R_k)|_{H_l} = \pi_T(R_l)|_{H_l}.$$

Since $\big(\mu^{-1}(e), \mu^{-1}(e) \cap P\big)$ is amenable, $\pi_T(\cdot)|_{B_l}$ is isometric on B_l by Lemma 3.8. Thus $R_l = 0$.
Let l_2 be a minimal element of $I \backslash \{l\}$. Then we can repeat the above argument to get $R_{l_2} = 0$. Since I is finite, we can continue to conclude that $R = 0$. □

We can now prove Theorem 3.2

Proof of Theorem 3.2. Suppose that K is an amenable group. To see (G, P) is amenable, we will show that the conditional expectation

$$E : C^*(G, P) \to \overline{\operatorname{span}}\{w_p w_p^* : p \in P\}$$

is faithful. Let Ψ^μ be the conditional expectation of Lemma 3.5. We have

$$\begin{aligned} E(\Psi^\mu(w_p w_q^*)) &= \begin{cases} \Psi(w_p w_q^*) & \text{if } \mu(p) = \mu(q) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} w_p w_p^* & \text{if } p = q \\ 0 & \text{otherwise} \end{cases} \\ &= \Psi(w_p w_q^*), \end{aligned}$$

and hence $E = E \circ \Psi^\mu$.
Since K is an amenable group, Ψ^μ is faithful by Lemma 3.5. Let $P_z \in B(\ell^2(P))$ be the orthogonal projection onto $\operatorname{span}\{\epsilon_z\}$. It is straightforward to show that the diagonal map $\Delta : B(\ell^2(P)) \to B(\ell^2(P))$ given by

$$\Delta(T) = \sum_{z \in P} P_z T P_z$$

is a conditional expectation such that $\Delta \circ \pi_T = \pi_T \circ E$ and is faithful.
Now suppose that $R \in C^*(G, P)$ and $E(R^*R) = 0$. Then $E(\Psi^\mu(R^*R)) = 0$ and so $\pi_T \circ E(\Psi^\mu(R^*R)) = 0$. This gives $\Delta \circ \pi_T(\Psi^\mu(R^*R)) = 0$. Since Δ is faithful, it follows that $\pi_T(\Psi^\mu(R^*R)) = 0$. Since $\big(\mu^{-1}(e), \mu^{-1}(e) \cap P\big)$ is amenable, Lemma 3.10 implies that π_T is faithful on $\overline{\operatorname{span}}\{w_p w_q^* : \mu(p) = \mu(q)\} = \operatorname{range} \Psi^\mu$. Thus $\Psi^\mu(R^*R) = 0$, and then $R = 0$ since Ψ^μ is faithful. Hence E is faithful and (G, P) is amenable. □

4 Quasi-lattice ordered HNN extensions

Let G be a group, let A and B be subgroups of G, and let $\phi : A \to B$ be an isomorphism. The group with presentation

$$G^* = \langle G, t \mid t^{-1} a t = \phi(a), a \in A \rangle$$

is the *HNN extension of* G with respect to A, B and ϕ. For every HNN extension G^* the *height map* is the homomorphism $\theta : G^* \to \mathbb{Z}$ such that $\theta(g) = 0$ for all $g \in G$ and $\theta(t) = 1$.

Example. Let $c, d \in \mathbb{Z} \setminus \{0\}$. The Baumslag-Solitar group

$$\mathrm{BS}(c,d) = \langle x, t \mid t^{-1}x^d t = x^c \rangle = \langle x, t \mid tx^c = x^d t \rangle$$

is an HNN extension of $\mathbb{Z}$ with respect to $A = d\mathbb{Z}$, $B = c\mathbb{Z}$ and $\phi : A \to B$ given by $\phi(dn) = cn$ for all $n \in \mathbb{Z}$. Then $\mathbb{Z}^*$ satisfies the relation $t^{-1}dt = \phi(d) = c$. Let $\mathrm{BS}(c,d)^+$ be the subsemigroup of $\mathrm{BS}(c,d)$ generated by x and t. Spielberg showed in [11, Theorem 2.11] that $(\mathrm{BS}(c,d), \mathrm{BS}(c,d)^+)$ is quasi-lattice ordered for all $c, d > 0$; he also proved in [11, Lemma 2.12] that $(\mathrm{BS}(c,-d), \mathrm{BS}(c,-d)^+)$ is not quasi-lattice ordered unless $c = 1$.

To work with an HNN extension we use a normal form for its elements from [8, Theorem 2.1]. We choose X to be a complete set of left coset representatives for G/A, that is, $xA \neq x'A$ for $x \neq x' \in X$. Similarly, choose a complete set Y of left coset representatives for G/B. Then a (right) *normal form relative to* X *and* Y of $g \in G$ is a product

$$g = g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \dots g_{n-1} t^{\epsilon_n} g_n$$

where:

1. g_n is an arbitrary element of G.
2. If $\epsilon_i = 1$, then g_{i-1} is an element of X
3. If $\epsilon_i = -1$, then g_{i-1} is an element of Y.

By [8, Theorem 2.1], for every choice of complete left coset representatives X and Y, each $g \in G^*$ has a unique normal form.
Our goal is to generalise the properties of the Baumslag-Solitar group with $c, d > 0$ to construct quasi-lattice ordered HNN extensions of other quasi-lattice ordered groups.
Let (G, P) be a quasi-lattice ordered group. Let G^* be the HNN extension of G with respect to subgroups A and B with an isomorphism $\phi : A \to B$. Let P^* be the subsemigroup of G^* generated by P and t. In general, (G^*, P^*) is not a quasi-lattice ordered group. For example, if $c > 1$, then $(\mathrm{BS}(c,-d), \mathrm{BS}(c,-d)^+)$ is not quasi-lattice ordered by [11, Lemma 2.12]. We need some conditions on our subgroups A and B and on the isomorphism ϕ which ensure that (G^*, P^*) is quasi-lattice ordered.
There are two reasons why $(\mathrm{BS}(c,d), \mathrm{BS}(c,d)^+)$ is easy to work with. The first is that there are natural choices of coset representatives: $\{0, \dots, d-1\}$ for $A = d\mathbb{Z}$ and $\{0, \dots, c-1\}$ for $B = c\mathbb{Z}$. The second is that the subgroup isomorphism $\phi : md \mapsto mc$ takes positive elements to positive elements. In

particular, using this choice of representatives, every element $\omega \in \mathrm{BS}(c,d)^+$ has a unique normal form

$$\omega = x^{m_0}tx^{m_1}t\dots x^{m_{n-1}}tx^{m_n}$$

where $0 \le m_i < d$ for $i < n$ and $m_n \in \mathbb{N}$. This choice of coset representatives is associated to the division algorithm on $\mathbb{N}$: for every $n \in \mathbb{N}$ we can uniquely write $n = md + r$ for some $m \in \mathbb{N}$ and $0 \le r \le d-1$.
In general, for G^* we would like a natural choice of coset representatives for G/A and G/B so that every element of P^* has a unique normal form that is a sequence of elements in P and t.

THEOREM 4.1. *Let (G,P) be a quasi-lattice ordered group with subgroups A and B. Suppose that:*

1. *There is an isomorphism $\phi : A \to B$ such that $\phi(A \cap P) = B \cap P$.*
2. *Every left coset $gA \in G/A$ such that $gA \cap P \ne \emptyset$ has a smallest representative in P, that is, there exists $p \in P$ such that $q \in gA \cap P \Rightarrow p \le q$.*
3. *For every $x, y \in B$, $x \vee y < \infty \Rightarrow x \vee y \in B$.*

Let $G^* = \langle G, t \mid t^{-1}at = \phi(a), a \in A\rangle$ be the HNN extension of G and let P^* be the subsemigroup of G^* generated by $\{P, t\}$. Then (G^*, P^*) is quasi-lattice ordered.

Before we can prove Theorem 4.1, we need to prove two lemmas. The first shows that elements of P^* are guaranteed to have normal forms made up of elements of P and t if and only if Item 2 of Theorem 4.1 holds. The second is a technical lemma which we will use several times in Theorem 4.1 and in later proofs.

LEMMA 4.2. *Suppose that (G,P) is a quasi-lattice ordered group with subgroups A and B. Suppose that $\phi : A \to B$ is a group isomorphism such that $\phi(A\cap P) = B \cap P$. Let $G^* = \langle G, t \mid t^{-1}at = \phi(a), a \in A\rangle$ be the corresponding HNN extension of G and let P^* be the subsemigroup of G^* generated by $P \cup \{t\}$. Let*

$$L_A := \{p \in P : q \in pA \cap P \Rightarrow p \le q\}.$$

The following two statements are equivalent:

1. *Every left coset $gA \in G/A$ such that $gA \cap P \ne \emptyset$ has a smallest coset representative $p \in P$;*
2. *There exists a complete set X of left coset representatives such that $L_A \subseteq X$ and every $\alpha \in P^*$ has normal form*

$$\alpha = p_0tp_1t\dots p_{n-1}tp_n \text{ where } p_i \in L_A \text{ for all } 0 \le i < n,\ p_n \in P. \quad (4)$$

Proof. Assume Item 1. Choose a complete set X of coset representatives for G/A which contains L_A. Let $\alpha \in P^*$. If $\theta(\alpha) = 0$ then $\alpha \in P$, and α has form (4) trivially.
We proceed by induction on $\theta(\alpha) \geq 1$. Suppose that $\theta(\alpha) = 1$. We may write $\alpha = q_0 t q_1$ for some $q_0, q_1 \in P$. Then $q_0 A \cap P \neq \emptyset$, and there exists $p_0 \in L_A$ such that $p_0 A = q_0 A$ and $p_0 \leq q_0$. Thus$p_0^{-1} q_0 \in P \cap A$. Hence $q_0 = p_0 a$ for some $a \in A \cap P$. Thus α has normal form

$$\alpha = p_0 a t q_1 = p_0 t \phi(a) q_1.$$

Since $\phi(A \cap P) = B \cap P$ we have $\phi(a) \in P$ and so X satisfies Item 2.
Suppose that all α with $1 \leq \theta(\alpha) \leq k$ have normal form (4). Consider α with $\theta(\alpha) = k + 1$. We write

$$\alpha = q_0 t q_1 t \dots t q_k t q_{k+1}.$$

By assumption, we can write the first $2k + 2$ terms of α in normal form

$$p_0 t p_1 t \dots p_{k-1} t r_k$$

where $p_i \in L_A$ for $0 \leq i < k$ and $r_k \in P$. There exists $p_k \in L_A$ such that $r_k A = p_k A$ and $p_k \leq r_k$. As above, we can write $r_k = p_k a$ for some $a \in A \cap P$. Then

$$\begin{aligned} \alpha &= p_0 t p_1 t \dots t r_k t q_{k+1} \\ &= p_0 t p_1 t \dots t p_k a t q_{k+1} \\ &= p_0 t p_1 t \dots t p_k t \phi(a) q_{k+1}. \end{aligned}$$

We set $p_{k+1} = \phi(a) q_{k+1}$, which is in P because $\phi(a)$ is. Then $\alpha = p_0 t p_1 t \dots t p_k t p_{k+1}$ has form (4). By induction, every α has normal form (4). This implies Item 2.
For Item 2 $\Rightarrow$ Item 1, we argue by contradiction: we will assume Item 2 holds but Item 1 doesn't. Let X be a set of coset representatives satisfying Item 2, and suppose that there exists a coset gA such that $gA \cap P \neq \emptyset$ which has no smallest coset representative in P.
Let $x \in X$ be the coset representative of gA in X. First, suppose that $x \in P$. Then p is not smallest, and there exists $q \in gA \cap P$ with $x \not\leq q$. Thus $x^{-1} q \notin P$. Consider $qt \in P^*$ in normal form:

$$qt = x x^{-1} q t = x t \phi(x^{-1} q). \tag{5}$$

(Since $q \in gA \cap P$ and $p \in L_A$, we have $q \notin L_A$. Thus the normal form for qt must be $xt\phi(x^{-1}q)$.) Since $\phi(A \cap P) = B \cap P$ and $x^{-1} q \notin P$, we have $\phi(x^{-1} q) \notin P$, which contradicts that the right-hand-side of (5) is in normal form.
Second, assume that $x \notin P$. By assumption $gA \cap P \neq \emptyset$. Let $q \in gA \cap P$. Then the right-hand-side of (5) is again the normal form for qt. Either $x \leq q$

or $x \not\leq q$. Suppose that $x \leq q$. Then $q^{-1}x \in P$ and $\phi(x^{-1}q) \in P$. Then (5) implies that $x \in L_A$, that is $x \in P$, a contradiction. Finally, suppose that $x \not\leq q$. Then $x^{-1}q \notin P$ implies that $\phi(x^{-1}q) \notin P$, again contradicting the normal form for qt. Thus Item 2 $\Rightarrow$ Item 1. $\square$

Lemma 4.3. *Let (G, P) be a quasi-lattice ordered group and let B be a subgroup of G. Suppose that for every $x, y \in B$, $x \vee y < \infty \Rightarrow x \vee y \in B$. Then for all $x \in B \cap PP^{-1}$, there exist $\mu, \nu \in B \cap P$ such that $x = \mu\nu^{-1}$ and for all $p, q \in P$ with $pq^{-1} = x$ we have $\mu \leq p$ and $\nu \leq q$.*

The lemma says that if $x \in PP^{-1} \cap B$, then the minimal elements of (1) must also be contained in B. In particular, if $\phi : A \to B$ is an isomorphism such that $\phi(A \cap P) = B \cap P$, then $\phi^{-1}(x) \in PP^{-1}$.

Proof of Lemma 4.3. Fix $x \in B \cap PP^{-1}$. Say $x = st^{-1}$ with $s, t \in P$. Then $x^{-1}s \in P$ and $x \leq s$. Also $e \leq s$, and so $x \vee e < \infty$. Since $e, x \in B$ we get $x \vee e \in B$. Let $\mu = x \vee e$ and $\nu = x^{-1}(x \vee e)$. Then $\mu\nu^{-1} = x \vee e(x^{-1}(x \vee e))^{-1} = x$.

Fix $p, q \in P$ such that $x = pq^{-1}$. Then $x^{-1}p = (pq^{-1})^{-1}p = q$, and so $x \leq p$. Therefore $\mu = x \vee e \leq p$. Now $\mu^{-1}p \in P$, and then $\nu^{-1}q = \mu^{-1}p \in P$ gives $\nu \leq q$. $\square$

We can now prove Theorem 4.1. Its proof is based on [11, Theorem 2.11], and our presentation is helped by Emily Irwin's treatment of [11, Theorem 2.11] in her University of Otago Honours thesis.

Proof of Theorem 4.1. Fix $x \in P^*P^{*-1}$. We shall prove that there exist $\mu, \nu \in P^*$ with $x = \mu\nu^{-1}$ such that whenever $\gamma\delta^{-1} = x$ we have $\mu \leq \gamma$ and $\nu \leq \delta$ (see Equation 1).

Choose $\alpha, \beta \in P^*$ such that $x = \alpha\beta^{-1}$. By Items 1-2, Lemma 4.2 applies. Thus there exists a complete set X of left coset representatives of G/A that contains

$$L_A := \{p \in P : q \in pA \cap P \Rightarrow p \leq q\},$$

and we can write α and β in unique normal form:

$$\alpha = p_0tp_1t \dots tp_mtr \text{ where } p_i \in L_A \text{ and } r \in P;$$

$$\beta = q_0tq_1t \dots tq_nts \text{ where } q_i \in L_A \text{ and } s \in P.$$

Now $x = \alpha\beta^{-1}$ is equal to

$$\alpha\beta^{-1} = p_0tp_1t \dots tp_mtrs^{-1}t^{-1}q_n^{-1}t^{-1} \dots t^{-1}q_0^{-1}.$$

First we look for initial cancellations in the middle of $\alpha\beta^{-1}$: if $rs^{-1} \in B$, then we can replace $trs^{-1}t^{-1}$ with $\phi^{-1}(rs^{-1})$. By Item 3, Lemma 4.3 applies and there exist $b_1, b_2 \in P \cap B$ such that $rs^{-1} = b_1b_2^{-1}$. Then

$$\phi^{-1}(rs^{-1}) = \phi^{-1}(b_1b_2^{-1}) = \phi^{-1}(b_1)\phi^{-1}(b_2)^{-1}.$$

Since $\phi(A \cap P) = B \cap P$ we have $\phi^{-1}(b_1)\phi^{-1}(b_2)^{-1} \in PP^{-1}$. Then

$$x = \alpha\beta^{-1} = p_0 t p_1 t \dots t p_m \phi^{-1}(b_1)\phi^{-1}(b_2)^{-1} q_n^{-1} t^{-1} \dots t^{-1} q_0^{-1}. \qquad (6)$$

We can repeat this process until there are no more cancellations available in the middle, and so we assume this is the case for the expression (6). This gives the following cases:

(a) there are no more t and no more t^{-1},

(b) there are no more t^{-1},

(c) there are no more t,

(d) there are t and t^{-1}, and then the term with t to the left and t^{-1} to its right is not in B.

In each case, we will write down our candidates for μ and ν and prove that they are the required minimums.

(a) Suppose that after initial cancellations, there are no more t and no more t^{-1}. Then $\alpha\beta^{-1} = p_0 q_0^{-1}$ is already in normal form. By Equation 1 there exist $\sigma, \tau \in P$ such that $p_0 q_0^{-1} = \sigma\tau^{-1}$ and for all $c, d \in P$ such that $cd^{-1} = \sigma\tau^{-1}$ we have $\sigma \leq c$ and $\tau \leq d$. So we write $x = \sigma\tau^{-1}$ and choose as our candidates $\mu = \sigma$ and $\nu = \tau$.
Let $\gamma, \delta \in P^*$ such that $x = \gamma\delta^{-1}$. Let θ be the height map. Then $\theta(x) = 0$ and hence $\theta(\gamma) = \theta(\delta)$. We will prove that $\mu \leq \gamma$ and $\nu \leq \delta$ by induction on $\theta(\gamma)$.
For $\theta(\gamma) = 0$ we have $\gamma, \delta \in P$, and then $\mu = \sigma \leq \gamma$ and $\nu = \tau \leq \delta$. Let $k \geq 0$ and suppose that for all $\gamma, \delta \in P^*$ such that $\theta(\gamma) = \theta(\delta) = k$ and $x = \gamma\delta^{-1}$ we have $\mu \leq \gamma$ and $\nu \leq \delta$. Now consider $\gamma, \delta \in P^*$ such that $x = \gamma\delta^{-1}$ and $\theta(\gamma) = \theta(\delta) = k + 1$. We write $\gamma = m_0 t \dots m_k t m_{k+1}$ and $\delta = n_0 t \dots n_k t n_{k+1}$ in normal form where $m_i, n_i \in L_A$ for $0 \leq i \leq k$ and $m_{k+1}, n_{k+1} \in P$. Next we reduce $x = \gamma\delta^{-1}$ towards normal form. We have

$$x = \gamma\delta^{-1} = m_0 t \dots m_k t m_{k+1} n_{k+1}^{-1} t^{-1} n_k \dots t n_0^{-1}.$$

Since x has a unique normal form with no t or t^{-1}, there must be some cancellation. Since the $m_i, n_i \in L_A$ for $0 \leq i \leq k$, the cancellation must occur across $m_{k+1} n_{k+1}^{-1}$. So $m_{k+1} n_{k+1}^{-1} \in B$ and $t m_{k+1} n_{k+1}^{-1} t^{-1} = \phi^{-1}(m_{k+1} n_{k+1}^{-1})$, and

$$\begin{aligned} x = \gamma\delta^{-1} &= m_0 t \dots t m_k (t m_{k+1} n_{k+1}^{-1} t^{-1}) n_k t^{-1} \dots t n_0^{-1} \\ &= m_0 t \dots t m_k \phi^{-1}(m_{k+1} n_{k+1}^{-1}) n_k t^{-1} \dots t n_0^{-1} \end{aligned}$$

By Item 3, Lemma 4.3 applies, and there exists $b_m, b_n \in B \cap P$ such that $m_{k+1} n_{k+1}^{-1} = b_m b_n^{-1}$ and $b_m \leq m_{k+1}$ and $b_n \leq n_{k+1}$. Then

$$x = m_0 t \dots t m_k \phi^{-1}(b_m)\phi^{-1}(b_n^{-1}) n_k t^{-1} \dots t n_0^{-1}.$$

Since $\phi(A\cap P) = B\cap P$ we have $m_{k+1}\phi^{-1}(b_m), n_{k+1}\phi^{-1}(b_n) \in P$. But now we have $\gamma' = m_0t\dots tm_k\phi^{-1}(b_m)$ and $\delta' = n_0t\dots tn_k\phi^{-1}(b_n)$ such that $\gamma'(\delta')^{-1} = x$ and $\theta(\gamma) = \theta(\delta') = k$. By our induction hypothesis we have $\mu \le \gamma'$ and $\nu \le \delta'$. To show that $\mu \le \gamma$ we compute:

$$\begin{aligned}
\gamma &= m_0t\dots tm_ktm_{k+1}\\
&= m_0t\dots tm_ktb_mb_m^{-1}m_{k+1}\\
&= m_0t\dots tm_k\phi^{-1}(b_m)tb_m^{-1}m_{k+1} \quad \text{(replacing } tb_m \text{ with } \phi^{-1}(b_m)t)\\
&= \gamma' tb_m^{-1}m_{k+1}.
\end{aligned}$$

Since $b_m^{-1}m_{k+1} \in P$ we have $tb_m^{-1}m_{k+1} \in P^*$. Now $(\gamma')^{-1}\gamma \in P^*$ and $\gamma' \le \gamma$. Since $\mu \le \gamma'$ this gives $\mu \le \gamma$. Similarly, $\delta = \delta' tb_n^{-1}n_{k+1}$ and so $\nu \le \delta$. By induction, for all $\gamma, \delta \in P^*$ such that $x = \gamma\delta^{-1}$ we have $\mu \le \gamma$ and $\nu \le \delta$.

(b) Suppose that after the initial cancellations there are no more t^{-1} left. Then we have x in normal form:

$$x = \alpha\beta^{-1} = p_0tp_1t\dots tp_{m-n-1}trs^{-1}$$

where $m \ge n$. By Equation 1 there exist $\sigma, \tau \in P$ such that whenever $rs^{-1} = r'(s')^{-1} = \sigma\tau^{-1}$, we have $\sigma \le r'$ and $\tau \le s'$. So

$$x = p_0tp_1t\dots tp_{m-n-1}t\sigma\tau^{-1}. \tag{7}$$

Our candidates are

$$\mu = p_0tp_1t\dots tp_{m-n-1}t\sigma \quad \text{and} \quad \nu = \tau.$$

Fix $\gamma, \delta \in P^*$ such that $x = \gamma\delta^{-1}$. Say $\gamma = m_0tm_1\dots tm_i$ and $\delta = n_0tn_1\dots n_j$ in normal form. Then

$$\gamma\delta^{-1} = m_0tm_1\dots tm_in_j^{-1}t^{-1}\dots n_1t^{-1}n_0.$$

Since $\theta(\gamma\delta^{-1}) = \theta(\mu\nu^{-1}) = m - n$ we get

$$i = \theta(\gamma) = \theta(\delta) + m - n$$

and hence $i \ge m - n$. It follows from the uniqueness of normal form that there exists $\gamma' \in P^*$ such that

$$\gamma\delta^{-1} = p_0tp_1t\dots tp_{m-n-1}t\gamma'\delta^{-1}.$$

Thus $p_0tp_1t\dots tp_{m-n-1}t \le \gamma$. Equation 7 gives $(p_0tp_1t\dots tp_{m-n-1}t)^{-1}x = \sigma\tau^{-1}$. Since $(p_0tp_1t\dots tp_{m-n-1}t)^{-1}\gamma$ and δ are both in P^* and since $(p_0tp_1t\dots tp_{m-n-1}t)^{-1}\gamma\delta^{-1} = \sigma\tau^{-1}$, we can apply case (a) above with $\mu' = \sigma$ and $\nu' = \tau$ to see that $\sigma \le (p_0tp_1t\dots tp_{m-n-1}t)^{-1}\gamma$ and $\tau \le \delta$. Hence $\mu = p_0tp_1t\dots tp_{m-n-1}t\sigma \le \gamma$ and $\nu \le \delta$ as required.

(c) Suppose that after the cancellations, there are no more t left. Then

$$x = rs^{-1}t^{-1}q_{n-m-1}^{-1}t^{-1}\dots t^{-1}q_0^{-1}$$

for some $r, s \in P$. Consider

$$x^{-1} = q_0tq_1t\dots tq_{n-m-1}tsr^{-1}.$$

By Equation 1 there exist $\sigma, \tau \in P$ such that whenever $r', s' \in P$ such that $rs^{-1} = r'(s')^{-1} = \sigma\tau^{-1}$, we have $\sigma \le r'$ and $\tau \le s'$. Let

$$\mu = \sigma \quad \text{and} \quad \nu = q_0tq_1t\dots tq_{n-m-1}t\tau.$$

By case (b), they have the property that $x^{-1} = \nu\mu^{-1}$ and that for all $\gamma, \delta \in P^*$ such that $x^{-1} = \delta\gamma^{-1}$ we have $\mu \le \gamma$ and $\nu \le \delta$. Taking inverses, $x = \mu\nu^{-1}$ and for all $\gamma, \delta \in P^*$ such that $x = \gamma\delta^{-1}$ we have $\mu \le \gamma$ and $\nu \le \delta$.

(d) Suppose that after the initial cancellations there are both t and t^{-1} left. Then the term with t to the left and t^{-1} to its right is not in B. There exist $k \le m$ and $l \le n$ such that

$$x = p_0tp_1t\dots tp_ktrs^{-1}t^{-1}q_l^{-1}t^{-1}\dots t^{-1}q_0^{-1}.$$

By Equation 1 there exist $\sigma, \tau \in P$ such that whenever $r', s' \in P$ such that $rs^{-1} = r'(s')^{-1} = \sigma\tau^{-1}$, we have $\sigma \le r'$ and $\tau \le s'$. Our candidates for μ, ν are

$$\mu = p_0tp_1t\dots tp_kt\sigma \quad \text{and} \quad \nu = q_0tq_1t\dots tq_lt\tau.$$

Fix $\gamma, \delta \in P^*$ such that $x = \gamma\delta^{-1}$. By the argument used in case (b), there exists $\gamma' \in P^*$ such that $\gamma\delta^{-1} = p_0tp_1t\dots tp_jt\gamma'\delta^{-1}$. Hence $\gamma' = (p_0tp_1t\dots tp_jt)^{-1}\gamma \in P^*$ and $p_0tp_1t\dots tp_jt \le \gamma$.
Consider $\gamma'\delta^{-1} = \sigma\tau^{-1}t^{-1}q_k^{-1}t^{-1}\dots t^{-1}q_0^{-1}$. Here $\gamma', \delta \in P^*$ and there are no t in $\gamma'\delta^{-1}$ after cancellation. Applying case (c) with $\mu' = \sigma$ and $\nu' = \nu$ to get $\mu' = \sigma \le \gamma'$ and $\nu' = \nu \le \delta$. Then $\mu = p_0tp_1t\dots tp_jt\sigma \le p_0tp_1t\dots tp_jt\gamma' = \gamma$. ☐

Theorem 4.1 gives new examples of quasi-lattice ordered HNN extensions.

Example 4.4. We can now use Theorem 4.1 to show that the Baumslag-Solitar group $(\mathrm{BS}(c,d), \mathrm{BS}(c,d)^+)$ with $c, d > 0$ is quasi-lattice ordered. Since $(\mathbb{Z}, \mathbb{N})$ is totally ordered it is quasi-lattice ordered. Let $A = \{dm : m \in \mathbb{Z}\}$ and $B = \{cm : m \in \mathbb{Z}\}$. Every element $n \in \mathbb{N}$ has a unique decomposition $n = r + md$ where $m \in \mathbb{N}$ and $r \in \{0, 1, \dots, d-1\}$. The remainder r is a choice of coset representative $n + A = r + A$. For all $n' \in (n + A) \cap \mathbb{N}$ we have $n' = r + md + kd$ where $k \in \mathbb{Z}$ and $m + k \ge 0$. Thus $r \le n'$. Hence every coset of $\mathbb{Z}/A$ has nontrivial intersection with $\mathbb{N}$, and has a smallest coset representative in $\mathbb{N}$. Since B is totally ordered it is closed under taking least upper bounds. Define $\phi : A \to B$ by $\phi(dm) = cm$. Then $\phi(A \cap \mathbb{N}) = B \cap \mathbb{N}$. So Theorem 4.1 applies and gives that $(\mathbb{Z}^*, \mathbb{N}^*)$ is quasi-lattice ordered.

Example 4.5. We can generalise the previous example to $(\mathbb{Z}^2, \mathbb{N}^2)$, which is quasi-lattice ordered by [9, Example 2.3(2)]. Fix $a, b, c, d \in \mathbb{N} \setminus \{0\}$. Then $A = \{(am, bn) : m, n \in \mathbb{Z}\}$ and $B = \{(cm, dn) : m, n \in \mathbb{Z}\}$ are subgroups of $\mathbb{N}^2$. Let $\phi : A \to B$ be defined by $\phi((am, bn)) = (cm, dn)$. This ϕ satisfies $\phi(A \cap \mathbb{N}^2) = B \cap \mathbb{N}^2$. For all $(m, n) \in \mathbb{N}^2$, the division algorithm on $\mathbb{N}$ gives a unique decomposition

$$(m, n) = (r_1, r_2) + (ja, kb)$$

for $j, k \in \mathbb{N}$ and $r_1 \in \{0, \dots a-1\}$, $r_2 \in \{0, \dots, b-1\}$. Thus (r_1, r_2) is a minimal left coset representative of $(m, n) + A$. For all $(m, n), (p, q) \in B$, we have

$$(m, n) \vee (p, q) = (\max\{m, p, 0\}, \max\{n, q, 0\})$$

and hence $(m, n) \vee (p, q) \in B$. So B is closed under $\vee$. By Theorem 4.1, $(\mathbb{Z}^{2*}, \mathbb{N}^{2*})$ is a quasi-lattice ordered group with presentation

$$\mathbb{Z}^{2*} = \langle \mathbb{Z}^2 \cup \{t\} \mid (am, bn)t = t(cm, dn)\rangle.$$

It is straightforward to extend this construction to $(\mathbb{Z}^n, \mathbb{N}^n)$.

Example 4.6. Consider the free group $\mathbb{F}_2$ on 2 generators $\{a, b\}$ and let $\mathbb{F}_2^+$ be the subsemigroup generated by e, a and b. The pair $(\mathbb{F}_2, \mathbb{F}_2^+)$ is quasi-lattice ordered by [9, Example 2.3(4)]. Let $A = \{a^n : n \in \mathbb{Z}\}$, $B = \{b^n : n \in \mathbb{Z}\}$ and $\phi : A \to B$ defined by $\phi(a^n) = b^n$. Every $x \in F_2^+$ can be written as a product of $y \in \mathbb{F}_2^+$ which does not end in a followed by a^n for some $n \geq 0$. Then $y \in xA$. Every $z \in yA \cap \mathbb{F}_2^+$ begins with the word y which is in $\mathbb{F}_2^+$. It follows that $y \leq z$. Thus A has minimal left coset representatives in $\mathbb{F}_2^+$. Since B is totally ordered, it is trivially closed under $\vee$. It follows from Theorem 4.1 that $(\mathbb{F}_2^*, \mathbb{F}_2^{+*})$ is a quasi-lattice ordered group with presentation

$$\mathbb{F}_2^* = \langle \{a, b, t\} \mid at = tb\rangle.$$

Example 4.7. Building on $(\mathbb{F}_2, \mathbb{F}_2^+)$ again, fix $s, u \in \mathbb{N} \setminus \{0\}$, and let $A = \{a^{ms} : m \in \mathbb{Z}\}$, $B = \{b^{mu} : m \in \mathbb{Z}\}$ and $\phi : A \to B$ be $\phi(a^{ms}) = b^{mu}$. Then B is totally ordered and hence is closed under $\vee$. To see that A has smallest coset representatives, we observe that every $x \in \mathbb{F}_2^+$ is a product of a $y \in \mathbb{F}_2^+$ that does not end in a followed by a^n for some $n \in \mathbb{N}$. We write $n = r + js$ for some $j \in \mathbb{N}$ and $r \in \{0, 1, \dots, s-1\}$. We choose ya^r as our coset representative. Then for all $z \in ya^r A \cap \mathbb{F}_2^+$ we have $ya^r \leq z$. It follows from Theorem 4.1 that $(\mathbb{F}_2^*, \mathbb{F}_2^{+*})$ is a quasi-lattice ordered group with presentation

$$\mathbb{F}_2^* = \langle \{a, b, t\} \mid a^s t = tb^u\rangle.$$

Taking $u = s = 1$ gives Example 4.6.
Replacing B by $B' = \{a^{mu} : m \in \mathbb{Z}\}$ gives a quasi-lattice ordered group $(\mathbb{F}_2^*, \mathbb{F}_2^{+*})$ with presentation

$$\mathbb{F}_2^* = \langle \{a, b, t\} \mid a^s t = ta^u\rangle.$$

In the next two examples we show that it is easy to find subgroups which do not have minimal left coset representatives.

Example 4.8. Consider the group

$$G = \mathbb{Z}(\sqrt{2}) = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$$

with subsemigroup $\mathbb{Z}(\sqrt{2})^+ = \mathbb{Z}(\sqrt{2}) \cap [0, \infty)$. Let $A = \mathbb{Z}(2\sqrt{2})$. We claim there are no smallest coset representatives for G/A in $\mathbb{Z}(\sqrt{2})^+$. Suppose, aiming for a contradiction, that there exists some coset representative $p \in \mathbb{Z}(\sqrt{2})^+$ such that

$$q \in [p + \mathbb{Z}(2\sqrt{2})] \cap \mathbb{Z}(\sqrt{2})^+ \Rightarrow p \leq q.$$

Recall that $\mathbb{Z}(2\sqrt{2})$ is dense in $\mathbb{R}$.[5] Thus there exists some $a \in \mathbb{Z}(2\sqrt{2}) \cap (0, p)$. Thus $p - a \in [p + \mathbb{Z}(2\sqrt{2})] \cap \mathbb{Z}(\sqrt{2})^+$. But $p - a < p$, giving a contradiction.

Example 4.9. Consider $(\mathbb{Z}^2, \mathbb{N}^2)$, and let A be the subgroup generated by $\{(1, 2), (2, 1)\}$. Consider the coset

$$(1, 0) + A = (0, 1) + A.$$

Since $(1, 0)$ and $(0, 1)$ have no nonzero lower bound, there can be no choice of smallest coset representative.

5 Amenability of (G^*, P^*)

In this section we prove the following theorem.

Theorem 5.1. *Let (G, P) be a quasi-lattice ordered group with subgroups A and B. Suppose that $\phi : A \to B$ is an isomorphism which satisfies the hypotheses of Theorem 4.1. Let (G^*, P^*) be the corresponding HNN extension. If (G, P) is amenable, then (G^*, P^*) is amenable.*

To prove the theorem we will show that the height map $\theta : G^* \to \mathbb{Z}$ is a controlled map, that $(\theta^{-1}(e), \theta^{-1}(e) \cap P^*)$ is amenable, and then apply Theorem 3.2. To prove that $(\theta^{-1}(e), \theta^{-1}(e) \cap P^*)$ is amenable, we start by investigating order-preserving isomorphisms between the semigroups of quasi-lattice ordered groups.

The following lemma is well-known, and is straightforward to prove.

Lemma 5.2. *Let (G, P) and (K, Q) be quasi-lattice ordered groups. Suppose that there is a semigroup isomorphism $\phi : P \to Q$. Then ϕ is order-preserving. In particular, for $x, y \in P$, $x \vee y < \infty$ if and only if $\phi(x) \vee \phi(y) < \infty$. If $x \vee y < \infty$ then $\phi(x \vee y) = \phi(x) \vee \phi(y)$.*

[5] To see denseness observe that $0 < (-2 + 2\sqrt{2}) < 1$ and $(-2 + 2\sqrt{2})^n \in \mathbb{Z}(\sqrt{2})$ for all $n \in \mathbb{N}$. Thus for every open interval (u, v) there exists n such that $(-2 + 2\sqrt{2})^n < v - u$. Hence there exists $k \in \mathbb{Z}$ such that $k(-2 + 2\sqrt{2})^n \in (u, v)$.

Proposition 5.3. *Let (G, P) and (K, Q) be quasi-lattice ordered groups. Let $\{v_p : p \in P\}$ and $\{w_q : q \in Q\}$ be the generating elements of $C^*(G, P)$ and $C^*(K, Q)$, respectively. Suppose that there is a semigroup isomorphism $\phi : P \to Q$.*

1. *There exists an isomorphism $\pi_\phi : C^*(G, P) \to C^*(K, Q)$ such that $\pi_\phi(v_p) = w_{\phi(p)}$.*
2. *(G, P) is amenable if and only if (K, Q) is amenable.*

Proof. For 1, define $T^\phi : P \to C^*(K, Q)$ by $T^\phi_p = w_{\phi(p)}$. It is straightforward to show that T^ϕ is a covariant representation of P. Take $\pi_\phi := \pi_{T^\phi}$. Then $\pi_\phi : C^*(G, P) \to C^*(K, Q)$ satisfies $\pi_\phi(v_p) = w_{\phi(p)}$, and is an isomorphism with inverse $\pi_{\phi^{-1}}$.
For 2, let E_Q and E_P be the conditional expectations on $C^*(K, Q)$ and $C^*(G, P)$, respectively. It is straightforward to check that $E_P = \pi_\phi^{-1} \circ E_Q \circ \pi_\phi$. Suppose that (K, Q) is amenable, that is, E_Q is faithful. Let $a \geq 0$ such that $E_P(a) = 0$. Then $\pi_\phi(a) \geq 0$, and

$$0 = E_P(a) = \pi_\phi^{-1} \circ E_Q \circ \pi_\phi(a) \Rightarrow 0 = E_Q \circ \pi_\phi(a) \Rightarrow 0 = \pi_\phi(a)$$

because E_Q is faithful. Since π_ϕ is faithful, $a = 0$. Now E_P is faithful, and hence (G, P) is amenable. Symmetry gives the other direction. □

Next we need some lemmas which will be used to show that the height map θ is a controlled map. In particular we need to identify the minimal elements of Definition 3.1. If $x \in P^*$ has normal form

$$x = p_0 t p_1 t \dots p_{n-1} t p_n$$

we call $p_0 t p_1 t \dots p_{n-1} t$ the *stem* of x and write

$$\operatorname{stem}(x) = p_0 t p_1 t \dots p_{n-1} t.$$

The set of stems is our candidate for the minimal elements.

Lemma 5.4. *Let (G, P) be a quasi-lattice ordered group with subgroups A and B. Suppose that $\phi : A \to B$ is an isomorphism which satisfies the hypotheses of Theorem 4.1. Let $p, q \in P$. Then p and q have a common upper bound in P^* if and only if p and q have a common upper bound in P.*

Proof. First suppose that p and q have a common upper bound $r \in P$. Then $r \in P^*$ and so r is a common upper bound for p and q in P^*.
Second, suppose that p and q have a common upper bound $x \in P^*$. If $\theta(x) = 0$, then $x \in P$ and we are done. Suppose, aiming for a contradiction, that $\theta(x) = k$ for some $k \geq 1$, and that p, q have no common upper bound y with $\theta(y) < k$. Observe that $p^{-1}x, q^{-1}x \in P^*$, and that $\theta(p^{-1}x) = \theta(x) = \theta(q^{-1}x) = k$. The hypothesis of Theorem 4.1 ensure that Lemma 4.2 applies, and we can write $p^{-1}x$ and $q^{-1}x$ in their normal forms

$$p^{-1}x = p_0tp_1t\dots p_{k-1}tp_k \quad \text{and} \quad q^{-1}x = q_0tq_1t\dots q_{k-1}tq_k,$$

where $p_i, q_i \in L_A$ for $i < k$ and $p_k, q_k \in P$. Consider

$$p^{-1}q = p^{-1}x(q^{-1}x)^{-1} = p_0tp_1t\dots p_{k-1}tp_kq_k^{-1}t^{-1}q_{k-1}^{-1}\dots t^{-1}q_0^{-1}.$$

Since $p^{-1}x(q^{-1}x)^{-1} = p^{-1}q$ and $p^{-1}q$ is in normal form we must have some cancellation. Since the first k terms are already in normal form, $p_kq_k^{-1} \in B$. By Lemma 4.3, there exist $b_1, b_2 \in B \cap P$ such that $b_1 \leq p_k$, $b_2 \leq q_k$ and $b_1b_2^{-1} = p_kq_k^{-1}$. Then

$$\begin{aligned} p^{-1}q = p^{-1}x(q^{-1}x)^{-1} &= p_0t\dots p_{k-1}t(p_kq_k^{-1})t^{-1}q_{k-1}^{-1}\dots t^{-1}q_0^{-1} \\ &= p_0t\dots p_{k-1}t(b_1b_2^{-1})t^{-1}q_{k-1}^{-1}\dots t^{-1}q_0^{-1} \\ &= p_0t\dots p_{k-1}\phi^{-1}(b_1)tt^{-1}\phi^{-1}(b_2)^{-1}q_{k-1}^{-1}\dots t^{-1}q_0^{-1} \\ &= p_0t\dots p_{k-1}\phi^{-1}(b_1)\phi^{-1}(b_2)^{-1}q_{k-1}^{-1}\dots t^{-1}q_0^{-1}. \end{aligned}$$

Rearranging gives

$$p(p_0t\dots tp_{k-1}\phi^{-1}(b_1)) = q(q_0tq_1t\dots tq_{k-1}\phi^{-1}(b_2)) \in P^*.$$

Therefore $y = p(p_0tp_1t\dots p_{k-1}\phi^{-1}(b_1))$ is a common upper bound for p and q in P^* and $\theta(y) = k-1$, giving us the contradiction we sought. Therefore p and q have a common upper bound y with $\theta(y) = 0$, and hence they have a common upper bound in P. □

The statement of Lemma 5.5 is adapted from [2, Lemma 3.4].

Lemma 5.5. *Suppose that $\phi : A \to B$ is an isomorphism which satisfies the hypotheses of Theorem 4.1. Let $x, y \in P^*$ such that $x \vee y < \infty$. Write*

$$x = \operatorname{stem}(x)p \text{ and } y = \operatorname{stem}(y)q \text{ where } p, q \in P.$$

1. *If $\theta(x) = \theta(y)$, then $\operatorname{stem}(x) = \operatorname{stem}(y)$ and $p \vee q < \infty$. In particular, $x \vee y = \operatorname{stem}(x)(p \vee q)$ and $\theta(x \vee y) = \theta(x) = \theta(y)$.*
2. *If $\theta(x) < \theta(y)$, then there exists $r \in P$ such that $x \vee y = yr$ and $\theta(x \vee y) = \theta(y)$.*

In particular, $\theta(x \vee y) = \max\{\theta(x), \theta(y)\}$.

Proof. For 1, suppose that $\theta(x) = \theta(y)$. We know that $x \leq x \vee y$ and $y \leq x \vee y$. Thus, by the uniqueness of normal forms, $\operatorname{stem}(x) = \operatorname{stem}(y)$. Now by left invariance of the partial order we see that

$$\begin{aligned} p &= \operatorname{stem}(x)^{-1}x \leq \operatorname{stem}(x)^{-1}(x \vee y) \quad \text{and} \\ q &= \operatorname{stem}(x)^{-1}y \leq \operatorname{stem}(x)^{-1}(x \vee y). \end{aligned}$$

Therefore p and q have a common left upper bound in P^* and hence, by Lemma 5.4, they have a common left upper bound in P and $p \vee q$ exists in P. By left invariance $x \vee y = \mathrm{stem}(x)p \vee q$. Further, $\theta(x \vee y) = \theta(x) = \theta(y)$.
For 2, suppose that $\theta(x) < \theta(y)$. Since $x \leq x \vee y$ we have $x^{-1}(x \vee y) \in P^*$. We can write $x^{-1}(x \vee y) = \tau\gamma u$ for some $u \in P$ and $\tau, \gamma \in P^*$ with $\theta(\tau) = \theta(y) - \theta(x)$ and $\theta(\gamma) = \theta(x \vee y) - \theta(y)$. Then $x \vee y = x\tau\gamma u$.
Now we have $x\tau \leq x \vee y$ and $\theta(x\tau) = \theta(x) + (\theta(y) - \theta(x)) = \theta(y)$. Write $x\tau = \mathrm{stem}(x\tau)w$ for some $w \in P$. Therefore $x\tau \vee y < \infty$ and $\theta(x\tau) = \theta(y)$ so we can apply Item 1 to see that $\mathrm{stem}(x\tau) = \mathrm{stem}(y)$ and $x\tau \vee y = \mathrm{stem}(y)(q \vee w)$. Now $x \vee y \leq \mathrm{stem}(y)(q \vee w)$. Therefore there exists some $r \in P$ such that $x \vee y = \mathrm{stem}(y)qr = yr$. Then $\theta(x \vee y) = \theta(y)$.
By Items 1 and 2 we see that

$$\theta(x \vee y) = \begin{cases} \theta(x) & \text{if } \theta(x) = \theta(y) \\ \theta(y) & \text{if } \theta(x) < \theta(y). \end{cases}$$

Thus $\theta(x \vee y) = \max\{\theta(x), \theta(y)\}$. □

Proof of Theorem 5.1. We will use Theorem 3.2; to do so we need to show that the height map $\theta : (G^*, P^*) \to (\mathbb{Z}, \mathbb{N})$ is a controlled map in the sense of Definition 3.1, and that $(\theta^{-1}(e), \theta^{-1}(e) \cap P^*)$ is amenable.
To see that θ is order-preserving, let $x, y \in P^*$ such that $x \leq y$. Then $x^{-1}y \in P^*$ and $\theta(x^{-1}y) \geq 0$. So $0 \leq \theta(x^{-1}y) = -\theta(x) + \theta(y)$ and hence $\theta(x) \leq \theta(y)$. By Lemma 5.5, if $x \vee y < \infty$, then $\theta(x \vee y) = \max\{\theta(x), \theta(y)\} = \theta(x) \vee \theta(y)$.
For every $k \in \mathbb{N}$, Σ_k is complete: if $x \in \theta^{-1}(k) \cap P^*$, then $\mathrm{stem}(x) \in \Sigma_k$ and $x = \mathrm{stem}(x)p$ for some $p \in P$. Hence $\mathrm{stem}(x) \leq x$. By the uniqueness of normal forms, if $\sigma, \tau \in \Sigma_k$ and $\sigma \vee \tau < \infty$ then $\sigma = \tau$. Therefore θ is a controlled map into the amenable group $\mathbb{Z}$.
Suppose that (G, P) is amenable. Then $\theta^{-1}(0) \cap P^*$ is the set of elements of P^* with height 0, and hence they all have normal form p_0 for some $p_0 \in P$. Thus $\theta^{-1}(0) \cap P^*$ is isomorphic to P. Since (G, P) is amenable, so is $(\theta^{-1}(0), \theta^{-1}(0) \cap P^*)$ by Lemma 5.3. Since $(\mathbb{Z}, \mathbb{N})$ is amenable, it now follows from Theorem 3.2 that (G^*, P^*) is an amenable quasi-lattice ordered group. □

EXAMPLE 5.6. Since $(\mathbb{Z}^2, \mathbb{N}^2)$ and $(\mathbb{F}_2, \mathbb{F}_2^+)$ are amenable quasi-lattice ordered groups [9, §5.1], Theorem 5.1 implies that the HNN extensions $(\mathbb{Z}^{2*}, \mathbb{N}^{2*})$ and $(\mathbb{F}_2^*, \mathbb{F}_2^{+*})$ in Examples 4.5-4.7 are amenable quasi-lattice ordered groups.

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Astrid an Huef
School of Mathematics & Statistics
Victoria University of Wellington
Wellington, New Zealand
astrid.anhuef@vuw.ac.nz

Iain Raeburn
School of Mathematics & Statistics
Victoria University of Wellington
Wellington, New Zealand
iain.raeburn@vuw.ac.nz

Ilija Tolich
Dept. of Mathematics & Statistics
University of Otago
Dunedin, New Zealand
itolich@maths.otago.ac.nz