Relative Property (T) for Nilpotent Subgroups

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Abstract. We show that relative Property (T) for the abelianization of a nilpotent normal subgroup implies relative Property (T) for the subgroup itself. This and other results are a consequence of a theorem of independent interest, which states that if H is a closed subgroup of a locally compact group G, and A is a closed subgroup of the center of H, such that A is normal in G, and $(G/A, H/A)$ has relative Property (T), then $(G, H^{(1)})$ has relative Property (T), where $H^{(1)}$ is the closure of the commutator subgroup of H. In fact, the assumption that A is in the center of H can be replaced with the weaker assumption that A is abelian and every H-invariant finite measure on the unitary dual of A is supported on the set of fixed points.

1. Introduction

Relative Property (T) is an analogue of Kazhdan's Property (T) for pairs (G, H), where H is a closed subgroup of the locally compact group G. More precisely, (G, H) has *relative Property (T)* if every unitary representation of G with almost-invariant vectors has H-invariant vectors. (See Definition 2.2. Additional information can be found in [3, pp. 41–43], [5], and [14].) This concept has proved useful for many purposes, including the study of finitely-additive measures on Euclidean spaces [17], the construction of II_1 factors with trivial fundamental group [22], the construction of new examples of groups with Kazhdan's Property (T) that satisfy the Baum-Connes Conjecture [26], and proving that particular groups have Kazhdan's Property (T). In particular, the usual proof that $\mathrm{SL}(3, \mathbb{R})$ has Kazhdan's Property (T) is based on the fact that the pair $\left(\mathrm{SL}(2, \mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2\right)$ has relative Property (T) [3, pp. 47–50].

The very basic case where the subgroup H is abelian and normal has been a focus of attention (see, for example, [6, 7, 10, 12, 26] and [28, Lem. 3.1]). We generalize the results that were obtained in this situation by allowing H

to be nilpotent, rather than abelian. Indeed, the following theorem provides a nilpotent analogue of any result that establishes relative Property (T) for abelian, normal subgroups.

NOTATION 1.1. For any topological group N, we let $N^{(1)} = \mathrm{cl}([N, N])$ be the closure of the commutator subgroup of N, and let $N^{ab} = N/N^{(1)}$ be the abelianization of N.

THEOREM 1.2. *Let N be a closed, nilpotent, normal subgroup of a locally compact group G. Then (G, N) has relative Property (T) if and only if $(G/N^{(1)}, N^{ab})$ has relative Property (T).*

As an example, consider a semidirect product $H \ltimes A$, where A is abelian. Y. Cornulier and R. Tessera [6] have characterized precisely when the pair $(H \ltimes A, A)$ has relative Property (T), so the theorem yields a characterization for pairs $(H \ltimes N, N)$, where N is nilpotent. The following corollary is a special case that is in a particularly usable form, and is based on work of Y. Cornulier and A. Valette [7].

NOTATION 1.3. Assume the locally compact group H acts on a 1-connected, nilpotent Lie group N, and L is a closed, connected, H-invariant subgroup of N, such that $[N, N] \subseteq L$. Then $N/L \cong \mathbb{R}^n$ for some n, so the action of H induces a homomorphism $\mathrm{Int}_{N/L} \colon H \to \mathrm{GL}(n, \mathbb{R})$. We use $\mathrm{Int}_{N/L}(H)^\bullet$ to denote the *closure* of the image of this homomorphism.

COROLLARY 1.4. *Assume the locally compact group H acts on a 1-connected, nilpotent Lie group N. The pair $(H \ltimes N, N)$ has relative Property (T) if and only if, for every closed, connected, H-invariant, proper subgroup L of N that contains $N^{(1)}$, the group $\mathrm{Int}_{N/L}(H)^\bullet$ is* <u>not</u> *amenable.*

A special case of Corollary 1.4, in which H is a Lie group and other assumptions are also made, was proved in [4, Prop. 4.1.4, p. 44].
The above results are consequences of the following theorem, which is of independent interest.

THEOREM 1.5. *Let H be a closed subgroup of a locally compact group G, and let A be a closed, abelian subgroup of H. Assume that A is normal in G, and that every H-invariant finite measure on the unitary dual $\widehat{A}$ is supported on the set of fixed points of H. If $(G/A, H/A)$ has relative Property (T), then $(G, H^{(1)})$ has relative Property (T).*

The (easy) proof of Theorem 1.2 does not require the full generality of Theorem 1.5, but only the following special case in which H acts trivially on A.

COROLLARY 1.6. *Let H be a closed subgroup of a locally compact group G, and let A be a closed subgroup of the center of H, such that A is normal in G. If $(G/A, H/A)$ has relative Property (T), then $(G, H^{(1)})$ has relative Property (T).*

REMARK 1.7. The special case of Corollary 1.6 in which $G = H$ is a well-known result of J.–P. Serre that appears in [3, Thm. 1.7.11, p. 66]. More generally, the special case where A is central in all of G, not merely in H, is a generalization of [5, Prop. 3.1.3].

Our methods also apply to relative Property (T) for triples, rather than pairs.

DEFINITION 1.8 ([14, Rem. 0.2.2, p. 3]). Let H and M be closed subgroups of a locally compact group G. We say that the triple (G, H, M) has relative *Property (T)*, if for any unitary representation π of G, such that the restriction $\pi|_H$ has almost-invariant vectors, then there exist nonzero $\pi(M)$-invariant vectors.

For example, we prove the following result, which was conjectured by C. R. E. Raja [24, Conjecture 1 of §7] in the special case where N is required to be a connected Lie group (in addition to being nilpotent).

COROLLARY 1.9. *Suppose that H and N are locally compact groups, such that N is nilpotent and assume that H acts on N by automorphisms. Then the triple $(H \ltimes N, H, N)$ has relative Property (T) if and only if the triple $(H \ltimes N^{ab}, H, N^{ab})$ has relative Property (T).*

A modified version of Theorem 1.5 also yields a classification of Kazhdan sets in some groups.

DEFINITION 1.10. A subset Q of a locally compact group G is a *Kazhdan set* for G if there exists $\epsilon > 0$, such that every unitary representation of G with a nonzero (Q, ϵ)-invariant vector has a nonzero invariant vector. (See Definition 2.2(1) for the definition of a (Q, ϵ)-invariant vector.)

C. Badea and S. Grivaux [2, Thm. 6.1] obtained a Fourier-analytic characterization of Kazhdan sets in abelian groups (that are locally compact). The following corollary extends this to two other classes of groups. (The special case where G is a Heisenberg group was proved by C. Badea and S. Grivaux [2, Thm. 6.11].)

DEFINITION 1.11. A connected Lie group G is *real split* if every eigenvalue of $\operatorname{Ad} g$ is real, for every $g \in G$. For example, every connected, nilpotent Lie group is real split.

COROLLARY 1.12. *Let G be a locally compact group that either is nilpotent or is a connected, real split, solvable Lie group. Then a subset Q of G is a Kazhdan set for G if and only if the image of Q in G^{ab} is a Kazhdan set for G^{ab}.*

REMARK 1.13. Y. Cornulier [5, p. 302] has generalized the notion of relative Property (T) to pairs (G, H) in which H is a sub*set* of G, rather than a sub*group*. Corollary 1.6 extends to this setting in the obvious way (see Corollary 8.7), but the hypotheses of the corresponding generalization of Theorem 1.5 are not as clean (cf. Theorem 8.6).

Other consequences of Theorem 1.5 can be found in Sections 5, 7 and 9.

Here is an outline of the paper. Section 2 establishes some notation and recalls (or proves) several basic facts about relative Property (T), introducing the notion of relative Property (T) with approximation. Section 3 defines a tensor product that is fibered over the eigenspaces of an abelian normal subgroup, and discusses the associated invariant or almost-invariant vectors. Section 4 uses the results of Sections 2 and 3 to give a short proof of a generalization of Theorem 1.5 that applies to triples, rather than pairs. (The section also proves a slightly different result that also implies Theorem 1.2.) Section 5 uses Theorem 1.5 (and its generalizations) to prove the other results stated in the above introduction (plus some related results). Section 6 shows that if N is compactly generated, and nilpotent, then it has a unique largest subgroup $L^\dagger$, such that $(G, L^\dagger)$ has relative Property (T). Section 7 proves a generalization of Corollary 1.4 that does not require the subgroup N to be a Lie group. Section 8 presents results on relative Property (T) for triples (G, H, M) in which the subset M is not required to be a subgroup. Finally, Section 9 records a few other observations about relative Property (T).

2. Relative Property (T) for pairs and triples

Assumption 2.1. Hilbert spaces and locally compact groups are assumed to be second countable. (So all locally compact groups in this paper are σ-compact.)

Definition 2.2 ([3, Defns. 1.1.1 and 1.4.3, pp. 28 and 41]). Let π be a unitary representation of a locally compact group G on a Hilbert space $\mathcal{H}$, and let H be a closed subgroup of G.

(1) For a subset Q of G and $\epsilon > 0$, a vector $\xi \in \mathcal{H}$ is (Q, ϵ)-invariant if $\|\pi(g)\xi - \xi\| \leq \epsilon\|\xi\|$ for all $g \in Q$.
(2) π has almost-invariant vectors if π has nonzero (Q, ϵ)-invariant vectors, for every compact $Q \subseteq G$ and $\epsilon > 0$.
(3) The pair (G, H) has relative Property (T) if every unitary representation of G that has almost-invariant vectors, also has nonzero H-invariant vectors.

If the pair (G, H) has relative Property (T), then (Q, ϵ)-invariant vectors can be approximated by H-invariant vectors:

Theorem 2.3 (Jolissaint [15, Thm. 1.2 (a2 $\Rightarrow$ b2)]). *Assume H is a closed subgroup of a locally compact group G, such that (G, H) has relative Property (T). Then, for every $\delta > 0$, there exist a compact subset Q of G, and $\epsilon > 0$, such that if π is any unitary representation of G on a Hilbert space $\mathcal{H}$, and ξ is a nonzero (Q, ϵ)-invariant vector in $\mathcal{H}$, then $\|\xi - \eta\| < \delta\|\xi\|$, for some H-invariant vector $\eta \in \mathcal{H}$.*

This result does not extend to triples with relative Property (T), because the following is an example in which the triple (G, H, M) has relative Property (T), but there are almost-invariant vectors for H that cannot be approximated by M-invariant vectors.

Example 2.4. Let $G = \mathrm{O}(n) \ltimes \mathbb{R}^n$, and let H and M be the stabilizers in G of two different points x and y in $\mathbb{R}^n$ (so H and M are two different conjugates of $\mathrm{O}(n)$). Then it is not difficult to see that (G, H, M) has relative Property (T). (Namely, note that H has Property (T), because it is compact, and that every representation of G with an H-invariant vector must also have an M-invariant vector, because M is conjugate to H.)
Let π be the natural representation of G on $L^2(\mathbb{R}^n)$. There is a nonzero H-invariant function ξ in $L^2(\mathbb{R}^n)$ whose support is contained in a small disk centered at x (small enough that the disk does not contain y). Then ξ is (Q, ϵ)-invariant for every $Q \subseteq H$ and $\epsilon > 0$, but ξ is not well approximated by any M-invariant function.

This observation motivates the following definition, which identifies the cases where the approximation is always possible:

Definition 2.5. Let H and M be closed subgroups of a locally compact group G. We say that the triple (G, H, M) has *relative Property (T) with approximation* if, for every $\delta > 0$, there exist a compact subset Q of H and $\epsilon > 0$, such that if ξ is any (Q, ϵ)-invariant vector of any unitary representation of G, then there is an M-invariant vector η, such that $\|\eta - \xi\| \le \delta\|\xi\|$.

It is obvious that relative Property (T) with approximation implies relative Property (T). The converse is not true, as Example 2.4 gives a triple that has relative Property (T) but not relative Property (T) with approximation. However, Theorem 2.3 tells us that the two properties are equivalent when $G = H$. They are also equivalent when the third group in the triple is normal:

Lemma 2.6. *Assume H and M are closed subgroups of a locally compact group G, such that (G, H, M) has relative Property (T). If M is normal in G, then (G, H, M) has relative Property (T) with approximation.*

Proof. This is a standard argument (cf. [3, Prop. 1.1.9, p. 31]). Let $\delta > 0$ be arbitrary. Since (G, H, M) has relative Property (T), there exist a compact subset Q of H and $\epsilon' > 0$, such that every unitary representation of G with nonzero (Q, ϵ')-invariant vectors has nonzero M-invariant vectors.
Let $\epsilon = \delta\epsilon'/2$, and suppose that ξ is a (Q, ϵ)-invariant unit vector for a unitary representation π of G on a Hilbert space $\mathcal{H}$. We need to find an M-invariant vector η that is δ-close to ξ.

Let $P\colon \mathcal{H} \to (\mathcal{H}^M)^\perp$ be the projection onto the orthogonal complement of the space of M-invariant vectors. We may assume $P(\xi) \neq 0$ (otherwise ξ is invariant and we take $\eta = \xi$). Since M is normal in G, we know that $\mathcal{H}^M$ is G-invariant, so π restricts to a representation of G on $(\mathcal{H}^M)^\perp$. For all $q \in Q$, we have

$$\|\pi(q)\, P(\xi) - P(\xi)\| = \|P\big(\pi(q)\xi - \xi\big)\| \leq \|\pi(q)\xi - \xi\| \leq \epsilon.$$

However, $P(\xi)$ cannot be (Q, ϵ')-invariant, because $(\mathcal{H}^M)^\perp$ has no nonzero M-invariant vectors. Therefore $\epsilon > \epsilon'\|P(\xi)\|$, which means

$$\|P(\xi)\| < \epsilon/\epsilon' = \delta/2 < \delta\|\xi\|.$$

Hence $\eta = \xi - P(\xi) \neq 0$ is M-invariant and $\|\eta - \xi\| \leq \delta\|\xi\|$ as desired. $\qquad\square$

It is immediate from the definitions that the pair (G, H) has relative Property (T) if and only if the triple (G, G, H) has relative Property (T). Now, suppose $M \subseteq H \subseteq G$. It is obvious that if the pair (H, M) has relative Property (T), then the triple (G, H, M) has relative Property (T). However, the converse is not true, even if M is contained in H and is normal in G:

EXAMPLE 2.7. Fix $n \geq 4$, and embed $\mathrm{SL}(3, \mathbb{R})$ in $\mathrm{SL}(n, \mathbb{R})$, in such a way that $\mathrm{SL}(3, \mathbb{R})$ fixes a nonzero vector $v \in \mathbb{R}^n$. Then

    (1) the triple $\big(\mathrm{SL}(n, \mathbb{R}) \ltimes \mathbb{R}^n, \mathrm{SL}(3, \mathbb{R}) \ltimes \mathbb{R}^n, \mathbb{R}^n\big)$ has relative Property (T), but

    (2) the pair $\big(\mathrm{SL}(3, \mathbb{R}) \ltimes \mathbb{R}^n, \mathbb{R}^n\big)$ does not have relative Property (T).

Proof. (1) Let π be a unitary representation of $\mathrm{SL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$, such that the restriction of π to $\mathrm{SL}(3, \mathbb{R}) \ltimes \mathbb{R}^n$ has nonzero almost-invariant vectors. Since $\mathrm{SL}(3, \mathbb{R})$ has Property (T) [3, Thm, 1.4.15, p. 49], we know that π has nonzero $\mathrm{SL}(3, \mathbb{R})$-invariant vectors. The Moore Ergodicity Theorem (or Mautner phenomenon) [19, Cor. 11.2.8, p. 216] tells us that every $\mathrm{SL}(3, \mathbb{R})$-invariant vector is $\mathrm{SL}(n, \mathbb{R})$-invariant. Since the triple $\big(\mathrm{SL}(n, \mathbb{R}) \ltimes \mathbb{R}^n, \mathrm{SL}(n, \mathbb{R}), \mathbb{R}^n\big)$ has relative Property (T) (see, for example, [24, Thm. 1.1]), these vectors are $\mathbb{R}^n$-invariant. (2) Since $\mathrm{SL}(3, \mathbb{R})$ fixes v (and $\mathrm{SL}(3, \mathbb{R})$ is simple, so its representation on $\mathbb{R}^n$ is completely reducible), we see that the abelianization of $\mathrm{SL}(3, \mathbb{R}) \ltimes \mathbb{R}^n$ is noncompact. So nontrivial 1-dimensional representations of $\mathrm{SL}(3, \mathbb{R}) \ltimes \mathbb{R}^n$ approximate the trivial representation, and are trivial on $\mathrm{SL}(3, \mathbb{R})$, but have no $\mathbb{R}^n$-invariant vectors. $\qquad\square$

3. Invariant vectors and tensor products

As was mentioned in the introduction, Theorem 1.5 is a generalization of a theorem of Serre. The proof of Serre's result in [3, Thm. 1.7.11, p. 66] is based on the fact that if A is central in G, and π is irreducible, then $\pi(A)$ consists of scalar matrices, so A is in the kernel of $\pi \otimes \overline{\pi}$ (see Notation 3.1 for the definition of the conjugate representation $\overline{\pi}$). To generalize this proof, we construct a different representation, denoted $\pi \otimes_{\widehat{A}} \overline{\pi}$, that is trivial on A, even if $\pi(A)$ does not consist of scalars (see Definition 3.4). In geometric terms, π can be realized

as an action on the L^2-sections of a vector bundle over the unitary dual of A, and the representation $\pi \otimes_{\widehat{A}} \overline{\pi}$ is constructed by tensoring this vector bundle with its conjugate. However, the official definition of $\pi \otimes_{\widehat{A}} \overline{\pi}$ in Section 3A uses the terminology of real analysis and representation theory, instead of the language of vector bundles.

For the proof of Theorem 1.5, it is important to know that almost-invariant vectors for π yield almost-invariant vectors for $\pi \otimes_{\widehat{A}} \overline{\pi}$. That is the point of Proposition 3.6 below. Conversely, Proposition 3.8 will be used to obtain invariant vectors for π from invariant vectors for $\pi \otimes_{\widehat{A}} \overline{\pi}$.

NOTATION 3.1. We use:
- $\overline{x}$ for the complex conjugate of the number x,
- $\overline{\mathcal{H}}$ for the conjugate of the Hilbert space $\mathcal{H}$ [3, p. 293],
- $\overline{\xi}$ for the element of $\overline{\mathcal{H}}$ corresponding to the element ξ of $\mathcal{H}$ (so $\overline{x\xi} = \overline{x}\,\overline{\xi}$ for $x \in \mathbb{C}$), and
- $\overline{\pi}$ for the unitary representation on $\overline{\mathcal{H}}$ that is obtained from the unitary representation π on $\mathcal{H}$ [3, Defn. A.1.10, p. 294].

We begin by recalling some basic facts of functional analysis.

LEMMA 3.2.
  (1) [29, §3.4, pp. 42–49] *If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces, then there is a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, such that*

$$\|v_1 \otimes v_2\| = \|v_1\| \, \|v_2\|$$

  for all $v_1 \in \mathcal{H}_1$ and $v_2 \in \mathcal{H}_2$.
  (2) (*cf.* [29, p. 267 and Thm. 3.12(b), p. 49]) *If U_1 and U_2 are unitary operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, then there is a unitary operator $U_1 \otimes U_2$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$, such that*

$$(U_1 \otimes U_2)(v_1 \otimes v_2) = U_1 v_1 \otimes U_2 v_2.$$

  (3) *The natural map $\mathcal{U}(\mathcal{H}_1) \times \mathcal{U}(\mathcal{H}_2) \to \mathcal{U}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is continuous when the unitary groups are given the strong operator topology.*

3A. A FIBERED TENSOR PRODUCT.

NOTATION 3.3. Assume
- π is a unitary representation of a locally compact group G, and
- A is an abelian, normal subgroup of G.

Applying the representation theory of abelian groups [3, Thm. D.3.1(i), p. 375] to the restriction $\pi|_A$ provides a unique projection-valued measure $\mathbb{P}$ on the unitary dual $\widehat{A}$, such that, for $a \in A$, we have

$$\pi(a) = \int_{\widehat{A}} \lambda(a) \, d\mathbb{P}(\lambda).$$

The uniqueness implies that

$$\mathbb{P}_{gE} = \pi(g) \, \mathbb{P}_E \, \pi(g)^{-1} \text{ for } g \in G \text{ and } E \subseteq \widehat{A},$$

so this is a system of imprimitivity for π (as defined in [27, top of page 203]). If this system of imprimitivity is homogeneous (as defined in [27, p. 218]), then [27, Theorem 6.11, pp. 220–221] tells us there is a measure μ on $\widehat{A}$, a Hilbert space $\mathcal{H}$, and a Borel cocycle $\alpha \colon G \times \widehat{A} \to \mathcal{U}(\mathcal{H})$, such that (up to isomorphism) π is the representation on $L^2(\widehat{A}, \mu; \mathcal{H})$ given by

$$\big(\pi(g)f\big)(\lambda) = \sqrt{D(g,\lambda)}\, \alpha(g,\lambda)\, f(g^{-1}\lambda)$$

for $g \in G$, $f \in L^2(\widehat{A}, \mu; \mathcal{H})$, and $\lambda \in \widehat{A}$, and where $D(g,\lambda)$ is the Radon-Nikodym derivative of the action of g on $\widehat{A}$.

DEFINITION 3.4. With the above notation (and assuming that $\mathbb{P}$ is homogeneous), we define $\pi' = \pi \otimes_{\widehat{A}} \overline{\pi}$ to be the unitary representation of G on $L^2(\widehat{A}, \mu; \mathcal{H} \otimes \overline{\mathcal{H}})$ that is defined by replacing α with $\alpha \otimes \overline{\alpha}$ in the formula for $\pi(g)$:

$$\big(\pi'(g)f\big)(\lambda) = \sqrt{D(g,\lambda)}\,\big(\alpha(g,\lambda) \otimes \overline{\alpha(g,\lambda)}\big)\, f(g^{-1}\lambda).$$

(Lemma 3.2(3) implies that the cocycle $\alpha \otimes \overline{\alpha}$ is Borel measurable.)

REMARK 3.5. Notice that A is in the kernel of $\pi \otimes_{\widehat{A}} \overline{\pi}$, which means that $\pi \otimes_{\widehat{A}} \overline{\pi}$ is a representation of G/A.

An important feature of the fibered tensor product is that it preserves almost-invariant vectors, and more precisely we have the following.

PROPOSITION 3.6. If $f \in L^2(\widehat{A}, \mu; \mathcal{H})$ is a $(Q, \epsilon/3)$-invariant unit vector for π, then the function $f'(\lambda) = \frac{1}{\|f(\lambda)\|} f(\lambda) \otimes \overline{f(\lambda)}$ is a (Q, ϵ)-invariant unit vector for $\pi \otimes_{\widehat{A}} \overline{\pi}$. (We use the convention that $f'(\lambda) = 0$ if $f(\lambda) = 0$.)

Before giving the proof we need the following simple lemma.

LEMMA 3.7. Suppose
- $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces,
- $v_i, w_i \in \mathcal{H}_i$ for $i = 1, 2$,
- $\|w_1\| = \|w_2\|$,
- for $z \in \{v_1, v_2, w_1, w_2\}$, $\widehat{z}$ is a unit vector such that $z = \|z\|\,\widehat{z}$, and
- $D \geq 0$.

Then

$$\|Dw_1 \otimes \widehat{w}_2 - \widehat{v}_1 \otimes v_2\| \leq 2\|Dw_1 - v_1\| + \|Dw_2 - v_2\|.$$

Proof. We have

$$\|Dw_1 \otimes \widehat{w}_2 - \widehat{v}_1 \otimes v_2\|$$
$$\leq \|Dw_1 \otimes \widehat{w}_2 - v_1 \otimes \widehat{w}_2\| + \|v_1 \otimes \widehat{w}_2 - \widehat{v}_1 \otimes (Dw_2)\| + \|\widehat{v}_1 \otimes (Dw_2) - \widehat{v}_1 \otimes v_2\|$$
$$= \|(Dw_1 - v_1) \otimes \widehat{w}_2\| + \left\|\big(\|v_1\| - D\|w_2\|\big)\widehat{v}_1 \otimes \widehat{w}_2\right\| + \|\widehat{v}_1 \otimes (Dw_2 - v_2)\|$$
$$= \|Dw_1 - v_1\| + \Big|\|v_1\| - D\|w_2\|\Big| + \|Dw_2 - v_2\|.$$

Since $\|w_1\| = \|w_2\|$, the conclusion now follows from the fact that $\big|\|v\| - \|w\|\big| \leq \|v - w\|$ for all vectors v and w in any Hilbert space.   □

Proof of Proposition 3.6. First, note that for $\lambda \in \widehat{A}$, we have

$$\|f'(\lambda)\| = \left\|\frac{1}{\|f(\lambda)\|}\, f(\lambda) \otimes \overline{f(\lambda)}\right\| = \frac{1}{\|f(\lambda)\|} \cdot \|f(\lambda)\| \cdot \|\overline{f(\lambda)}\| = \|\overline{f(\lambda)}\| = \|f(\lambda)\|.$$

Therefore $\|f'\|_2 = \|f\|_2$, so f' is a unit vector for the representation $\pi' = \pi \otimes_{\widehat{A}} \overline{\pi}$. For $g \in Q$ and $\lambda \in \widehat{A}$, let

$$v_1 = f(\lambda), \quad w_1 = \alpha(g, \lambda)\, f(g^{-1}\lambda), \quad v_2 = \overline{v_1},$$
$$w_2 = \overline{w_1}, \quad \text{and} \quad D = \sqrt{D(g, \lambda)}.$$

Then

$$f'(\lambda) = \frac{1}{\|f(\lambda)\|}\, f(\lambda) \otimes \overline{f(\lambda)} = \widehat{v_1} \otimes v_2.$$

and

$$\begin{aligned}
\bigl(\pi'(g)f'\bigr)(\lambda) &= \sqrt{D(g, \lambda)}\, \alpha'(g, \lambda)\, f'(g^{-1}\lambda) \\
&= \frac{\sqrt{D(g, \lambda)}}{\|f(g^{-1}\lambda)\|}\, \alpha(g, \lambda)\, f(g^{-1}\lambda) \otimes \overline{\alpha(g, \lambda)}\, \overline{f(g^{-1}\lambda)} \\
&= Dw_1 \otimes \widehat{w_2}
\end{aligned}$$

Therefore, Lemma 3.7 tells us that

$$\begin{aligned}
\bigl\|\bigl(\pi'(g)f'\bigr)(\lambda) - f'(\lambda)\bigr\| &\leq 2\|Dw_1 - v_1\| + \|Dw_2 - v_2\| \\
&= 2\|Dw_1 - v_1\| + \|D\overline{w_1} - \overline{v_1}\| \\
&= 3\|Dw_1 - v_1\| \\
&= 3\,\bigl\|\bigl(\pi(g)f\bigr)(\lambda) - f(\lambda)\bigr\|,
\end{aligned}$$

since

$$\bigl(\pi(g)f\bigr)(\lambda) = \sqrt{D(g, \lambda)}\, \alpha(g, \lambda)\, f(g^{-1}\lambda) = Dw_1.$$

So

$$\|\pi'(g)f' - f'\|_2 \leq 3\,\|\pi(g)f - f\|_2 < 3 \cdot \frac{\epsilon}{3} = \epsilon. \qquad \square$$

3B. Obtaining invariant vectors from a tensor product. The following result is based on ideas of Jolissaint [15, Thm. 1.2] and Nicoara-Popa-Sasyk [20, proof of Lem. 1].

Proposition 3.8. *Let ρ be a unitary representation of a locally compact group M on a Hilbert space $\mathcal{H}$, and suppose $\xi \in \mathcal{H}$. If η' is any $(\rho \otimes \overline{\rho})$-invariant vector in $\mathcal{H} \otimes \overline{\mathcal{H}}$, then there is a $\rho(M^{(1)})$-invariant vector $\eta \in \mathcal{H}$, such that*

$$\|\eta - \xi\|\, \|\xi\| \leq 7\, \|\eta' - \xi \otimes \overline{\xi}\|.$$

Moreover, η can be chosen so that the subspace $\mathbb{C}\eta$ is $\rho(M)$-invariant.

Proof. Assume, without loss of generality, that $\|\xi\| = 1$, and, for convenience, let $\delta = \|\eta' - \xi \otimes \overline{\xi}\|$. We may assume that $\delta < 1/7$. (Otherwise, the desired inequality is satisfied with $\eta = 0$.)

Let $\mathcal{H}' = \mathcal{H} \otimes \overline{\mathcal{H}}$, and note that $\mathcal{H}'$ can be identified with the space of Hilbert-Schmidt operators on $\mathcal{H}$, which are compact operators with finite trace (see

[3, the discussion on page 294]). In this identification, the vector $\xi' = \xi \otimes \overline{\xi}$ corresponds to the rank-one orthogonal projection P_ξ on the line $\mathbb{C}\xi$, defined by

$$P_\xi(\eta) = \langle \eta, \xi \rangle \xi,$$

where the inner product is from $\mathcal{H}$. In particular, P_ξ is a self-adjoint operator with trace 1 and its spectrum is in $\{0, 1\}$. Therefore, all the Hilbert-Schmidt operators corresponding to elements in the closed convex hull of $(\rho \otimes \overline{\rho})(M)\xi'$ are self-adjoint operators with trace 1 and their spectrum is contained in $[0, 1]$. Let T be the Hilbert-Schmidt operator corresponding to η'. We may assume that η' is the projection of ξ' onto the space of $(\rho \otimes \overline{\rho})$-invariant vectors, so η' is in the closed convex hull of $(\rho \otimes \overline{\rho})(M)\xi'$. Then T is a self-adjoint Hilbert-Schmidt operator with trace 1 and whose spectrum is contained in $[0, 1]$.

As T is invariant under $\rho(M)$, so are all of its eigenspaces, and we claim that T has a one-dimensional eigenspace. Let $\{c_i\}$ be the eigenvalues of T, and assume $c_1 > c_i$ for all $i \neq 1$. Then $\sum c_i = 1$, and the Hilbert-Schmidt norm $\|T\|_{HS}$ of T satisfies

$$\|T\|_{HS} = \left(\sum c_i^2 \right)^{1/2} \leq 1 \quad \text{and} \quad \|T\|_{HS} \geq \|T\| = c_1.$$

From the definition of T, we have

$$\|T - P_\xi\|_{HS} = \|\eta' - \xi'\| = \delta.$$

Also

$$\|T^2 - P_\xi\|_{HS} = \|T^2 - P_\xi^2\|_{HS} \leq \|T + P_\xi\|_{HS} \|T - P_\xi\|_{HS} \leq 2\delta.$$

Then

$$\|T - T^2\|_{HS} \leq \|T - P_\xi\|_{HS} + \|P_\xi - T^2\|_{HS} \leq \delta + 2\delta = 3\delta.$$

As $c_1 > c_i$ for all $i \neq 1$, we have

$$3\delta \geq \|T\|_{HS} - \|T^2\|_{HS} = \left(\sum c_i^2 \right)^{1/2} - \left(\sum c_i^4 \right)^{1/2}$$

$$\geq (1 - c_1) \left(\sum c_i^2 \right)^{1/2} = (1 - c_1)\|T\|_{HS} \geq (1 - c_1)(1 - \delta).$$

Therefore

$$c_1 \geq 1 - \frac{3\delta}{1 - \delta} = \frac{1 - 4\delta}{1 - \delta} > \frac{1}{2},$$

since $\delta < 1/7$. Since the trace of T is 1, we conclude that the eigenspace corresponding to the eigenvalue c_1 has dimension 1, as claimed.

Note that the c_1-eigenspace of T is not orthogonal to ξ: otherwise, if η_0 is a unit vector in the eigenspace, then

$$\frac{1}{2} < c_1 = \|c_1 \eta_0\| = \|T\eta_0 - P_\xi(\eta_0)\| \leq \|T - P_\xi\| \leq \delta < \frac{1}{7}.$$

Therefore, we may let η be the (unique) vector in the c_1-eigenspace of T, such that $P_\xi(\eta) = c_1 \xi$. Then

$$\|\eta - \xi\| = \frac{1}{c_1}\|T(\eta) - P_\xi(\eta)\| \leq \frac{1}{c_1}\|T - P_\xi\| \|\eta\| < \frac{\delta}{c_1}\|\eta\| < 2\delta\|\eta\|.$$

Since $\|\xi\| = 1$ and $2\delta < 2/7 < 1/3$, this implies $\|\eta\| < 3/2$, so $\|\eta - \xi\| < 3\delta < 7\delta$. Also, since $\mathbb{C}\eta$ is a 1-dimensional $\rho(M)$-invariant subspace, we know that $\rho(M^{(1)})$ acts trivially on it, so η is $\rho(M^{(1)})$-invariant. $\square$

4. Proof of the main theorem

Recall that all locally compact groups are assumed to be second countable (see Assumption 2.1).

Theorem 1.5 is the special case of the following result in which $G = H$. (Recall that if either $G = H$ or M is normal in G, then relative Property (T) for the triple (G, H, M) is equivalent to relative Property (T) with approximation, by Theorem 2.3 and Lemma 2.6.)

THEOREM 4.1. *Let H and M be closed subgroups of a locally compact group G, and let A be a closed, abelian subgroup of M that is normal in G. Assume that every M-invariant finite measure on $\widehat{A}$ is supported on the set of fixed points of M. If HA is closed and $(G/A, HA/A, M/A)$ has relative Property (T) with approximation, then $(G, H, M^{(1)})$ has relative Property (T) with approximation.*

Proof. Let $\delta > 0$ be arbitrary. Since $(G/A, HA/A, M/A)$ has relative Property (T) with approximation (and HA is closed), there is a compact subset Q of H and $\epsilon > 0$, such that if ξ' is any (Q, ϵ)-invariant vector for a unitary representation of G/A, then there is an M-invariant vector η', such that $\|\eta' - \xi'\| < \delta/7$.

Now, suppose π is a unitary representation of G, such that π has a $(Q, \epsilon/3)$-invariant vector f. (We wish to show that f is well-approximated by an M-invariant vector.) By replacing π with the direct sum $\pi \oplus \pi \oplus \cdots$ of infinitely many copies of itself, we may assume that all irreducible representations appearing in the direct integral decomposition of $\pi|_A$ have the same multiplicity (namely, ∞). By definition, this means that $\pi|_A$ is homogeneous, so Section 3A provides a measure μ on $\widehat{A}$, a Borel cocycle $\alpha \colon G \times \widehat{A} \to \mathcal{U}(\mathcal{H})$, a corresponding realization of π as a representation on the Hilbert space $L^2(\widehat{A}, \mu; \mathcal{H})$, and a unitary representation $\pi' = \pi \otimes_{\widehat{A}} \overline{\pi}$ of G.

Since π has been realized as a representation on $L^2(\widehat{A}, \mu; \mathcal{H})$, we know that $f \in L^2(\widehat{A}, \mu; \mathcal{H})$. Then Proposition 3.6 provides a (Q, ϵ)-invariant unit vector f' for π'. By the choice of Q and ϵ (and Remark 3.5), we know that there is a $\pi'(M)$-invariant vector $f'_M \in L^2(\widehat{A}, \mu; \mathcal{H} \otimes \overline{\mathcal{H}})$, such that $\|f'_M - f'\|_2 < \delta/7$. Since f'_M is M-invariant, we have

$$\|f'_M(\lambda)\| = \sqrt{D(m, \lambda)}\, \|f'_M(m^{-1}\lambda)\| \quad \text{for } m \in M \text{ and a.e. } \lambda \in \widehat{A},$$

so it is straightforward to check that $\|f'_M\|^2\, \mu$ is an M-invariant measure on $\widehat{A}$. Furthermore, this measure is finite because f'_M is in $L^2(\widehat{A}, \mu; \mathcal{H} \otimes \overline{\mathcal{H}})$. By assumption, this implies that (up to modifying f'_M on a set of measure zero) we may choose the support of f'_M to be contained in the set $\widehat{A}^M$ of fixed points of M in $\widehat{A}$. For each fixed $\lambda \in \widehat{A}^M$, the function $\alpha(m, \lambda)$ is a representation ρ_λ of M

on $\mathcal{H}$ (and $D(m, \lambda) \equiv 1$ on $\widehat{A}^M$), so M acts on the subspace $L^2(\widehat{A}^M, \mu|_{\widehat{A}^M}; \mathcal{H} \otimes \overline{\mathcal{H}})$ by

$$\big(\pi'(m)f'\big)(\lambda) = \big((\rho_\lambda \otimes \overline{\rho_\lambda}(m)))\big)f'(\lambda).$$

Note that $f'_M(\lambda)$ must be an M-invariant vector in $\mathcal{H} \otimes \overline{\mathcal{H}}$, for a.e. $\lambda \in \widehat{A}$. Then, since Proposition 3.6 tells us that $f'(\lambda) = \frac{1}{\|f(\lambda)\|} f(\lambda) \otimes \overline{f(\lambda)}$, Proposition 3.8 provides a $\rho_\lambda(M^{(1)})$-invariant vector $v(\lambda) \in \mathcal{H}$, such that $\|v(\lambda) - f(\lambda)\| \leq 7\|f'_M(\lambda) - f'(\lambda)\|$. (Also note that the von Neumann Selection Theorem [1, Thm. 3.4.3, p. 77] implies that we may choose $v(\lambda)$ to be a measurable function of λ.) Then v is $\pi(M^{(1)})$-invariant, and $\|v - f\|_2 \leq 7\|f'_M - f'\|_2 < \delta$. $\qquad \square$

REMARK 4.2. Here are two situations that satisfy Theorem 4.1's assumption that every M-invariant finite measure on $\widehat{A}$ is supported on the set of fixed points of M:

(1) If A is contained in the center of M, then M acts trivially on A (so it also acts trivially on $\widehat{A}$), so every point in $\widehat{A}$ is a fixed point.

(2) If M is a connected, solvable Lie group that is real split, and A is a closed, 1-connected, abelian, normal subgroup (so $\widehat{A} \cong A$), then the desired conclusion is a well known result in the spirit of the Borel Density Theorem (cf. [9, Cor. 1.3]).

REMARK 4.3. If G, H, M, and A are as described in the first two sentences of the statement of Theorem 4.1, then the proof establishes the following quantitative version of the theorem: *Suppose $Q \subseteq H$ and $\delta, \epsilon > 0$, such that, for every (Q, ϵ)-invariant vector ξ' for a unitary representation of G/A, there is an M-invariant vector η', such that $\|\eta' - \xi'\| < \delta/7$. Then, for every $(Q, \epsilon/3)$-invariant vector ξ for a unitary representation of G, there is an $M^{(1)}$-invariant vector η, such that $\|\eta - \xi\| < \delta$.*

REMARK 4.4. Taking $G = H = M$ in Remark 4.3 establishes that if Q is a subset of a locally compact group G, and A is a closed, abelian, normal subgroup of G, such that

(1) the image of Q in G/A is a Kazhdan set for G/A, and

(2) every G-invariant finite measure on $\widehat{A}$ is supported on the set of fixed points of G,

then Q is a Kazhdan set for the pair $(G, G^{(1)})$. (That is, there exists $\epsilon > 0$, such that every unitary representation of G that has (Q, ϵ)-invariant vectors also has $G^{(1)}$-invariant vectors.)

The following theorem removes the phrase "with approximation" from the statement of Theorem 4.1, at the expense of placing restrictions on M and A.

NOTATION 4.5. We use $Z(M)$ to denote the center of a group M.

THEOREM 4.6. *Let H and M be closed subgroups of a locally compact group G, and let A be a closed subgroup of $Z(M) \cap M^{(1)}$ that is normal in G. Assume that HA is closed, that M is nilpotent, and that either M has no closed, proper*

subgroups of finite index, or A is contained in H and is compactly generated. If $(G/A, HA/A, M/A)$ has relative Property (T), then (G, H, M) has relative Property (T).

Proof. Let π be a unitary representation of G, such that $\pi|_H$ has almost-invariant vectors.

Assume, for the moment, that the triple (G, H, A) has relative Property (T). Then the space of A-invariant vectors is nonzero. Since A is a normal subgroup, this space is G-invariant, and therefore yields a representation π^A of G/A. Also (because A is a normal subgroup), Lemma 2.6 tells us that (G, H, A) has relative Property (T) with approximation, so the restriction of π^A to H has almost-invariant vectors. Since $(G/A, HA/A, M/A)$ has relative Property (T), we conclude that π^A (and hence π) has nonzero M-invariant vectors. So (G, H, M) has relative Property (T), as desired.

To complete the proof, we show that the triple (G, H, A) does indeed have relative Property (T). That is, we show that π has nonzero A-invariant vectors. Arguing as in the proof of Theorem 4.1, we see that we may assume that $\pi|_A$ is homogeneous (by replacing π with $\pi \oplus \pi \oplus \cdots$), so Section 3A provides a measure μ on $\widehat{A}$, a Borel cocycle $\alpha \colon G \times \widehat{A} \to \mathcal{U}(\mathcal{H})$, a corresponding realization of π as a representation on the Hilbert space $L^2(\widehat{A}, \mu; \mathcal{H})$, and a unitary representation $\pi' = \pi \otimes_{\widehat{A}} \overline{\pi}$ of G. Also, M acts trivially on $\widehat{A}$, so, for each fixed $\lambda \in \widehat{A}$, the function $\alpha(m, \lambda)$ is a representation ρ_λ of M on $\mathcal{H}$, so M acts on $L^2(\widehat{A}, \mu; \mathcal{H} \otimes \overline{\mathcal{H}})$ by

$$\big(\pi'(m)f'\big)(\lambda) = \big((\rho_\lambda \otimes \overline{\rho_\lambda})(m)\big)f'(\lambda).$$

Also, since $(G/A, HA/A, M/A)$ has relative Property (T), there is a nonzero $\pi'|_M$-invariant vector f'_M in $L^2(\widehat{A}, \mu; \mathcal{H} \otimes \overline{\mathcal{H}})$.

Therefore, $\rho_\lambda \otimes \overline{\rho_\lambda}$ has a nonzero M-invariant vector for all λ in a set E of positive measure. For each $\lambda \in E$, there must be a finite-dimensional $\rho_\lambda(M)$-invariant subspace F_λ of $\mathcal{H}$ [3, Prop. A.1.2, p. 295]. Let M_λ be the closure of $\rho_\lambda(M)|_{F_\lambda}$. This is a closed (hence compact) subgroup of $\mathrm{SU}(n)$, for some $n \in \mathbb{N}$. Every compact, nilpotent Lie group is virtually abelian [11, Cor. 11.2.11, p. 447], so we see that M_λ has a closed, abelian subgroup of finite index.

Case 1. *Assume M has no closed, proper subgroups of finite index.* Then the entire group M_λ must be abelian. This means that $\rho_\lambda(M^{(1)})|_{F_\lambda}$ is trivial. Since $A \subseteq M^{(1)}$, this implies that $\rho_\lambda(A)$ fixes every element of F_λ. So $\lambda(a) = 1$ for all $a \in A$ and all $\lambda \in E$.

This means that $\mu(\{1\}) \neq 0$. Therefore, if we fix any nonzero $\xi_0 \in \mathcal{H}$, then the function

$$f(\lambda) = \begin{cases} \xi_0 & \text{if } \lambda = 1, \\ 0 & \text{otherwise} \end{cases}$$

is nonzero in $L^2(\widehat{A}, \mu; \mathcal{H})$. And it is obviously fixed by A. So π has a nonzero A-invariant vector, as desired.

CASE 2.   *Assume $A \subseteq H$ and A is compactly generated.*   Since M_λ is virtually abelian, we know there is a finite-index subgroup M'_λ of M, such that $\rho_\lambda(M'_\lambda)|_{F_\lambda}$ is abelian. So $\rho_\lambda|_{F_\lambda}$ is trivial on $(M'_\lambda)^{(1)}$, which is a finite-index subgroup of $M^{(1)}$ (since M is nilpotent and M'_λ has finite index in M). Since $A \subseteq M^{(1)}$, we conclude that $\rho_\lambda|_{F_\lambda}$ is trivial on a finite-index subgroup A_λ of A. Therefore, there is some $m \in \mathbb{N}$, such that ρ_λ is trivial on A^m for all λ in a set E of positive measure (where $A^m = \mathrm{cl}(\{a^m \mid a \in A\})$). This means $\rho_\lambda(A^m)$ fixes every element of F_λ (for all $\lambda \in E$), so $\pi(A^m)$ has a nonzero fixed vector. This implies that (G, H, A^m) has relative Property (T).

Note that A^m is normal in G (because it is characteristic in the normal subgroup A). Also, the quotient A/A^m is compact (because A is compactly generated and abelian). Therefore $(G/A^m, H/A^m, A/A^m)$ obviously has relative Property (T) (because $A \subseteq H$). Combining this with the fact that (G, H, A^m) has relative Property (T) (with approximation, by Lemma 2.6), we conclude that (G, H, A) has relative Property (T), as desired.   □

REMARK 4.7. The proof of Theorem 4.6 applies somewhat more generally than is specified in the statement of the theorem. More precisely, after the assumption that HA is closed, it suffices to make the following two additional assumptions:

(1) For every finite-dimensional, unitary representation ρ of M, the closure of $\rho(M)$ has an abelian subgroup of finite index. (For example, this is true when M is virtually solvable, and also when M is a connected Lie group whose Levi subgroup has no compact factors.)

(2) For every finite-index, closed subgroup M' of M, if m is the index of $(M')^{(1)} \cap A$ in A, then $m < \infty$, and there is a compact subset C of H, such that $A \subseteq C \cdot A^m$.

Also note that every closed subgroup of a compactly generated nilpotent group is compactly generated [21, Thm. 6, p. 38]. Therefore, if M is nilpotent, then it would suffice to assume M is compactly generated, instead of assuming that A is compactly generated.

5. Proofs of results stated in the Introduction

In this section, we prove that all of the results stated in the Introduction are consequences of Theorem 4.1. We first prove Theorem 1.5, Corollary 1.6, Theorem 1.2, and Corollaries 1.9 and 1.12 (while mentioning an additional corollary and remark along the way). These are followed by Corollary 1.4, which is a special case of $(1 \Leftrightarrow 5)$ of Corollary 5.4 below.

Recall that, as stated in Assumption 2.1, all locally compact groups are assumed to be second countable, and therefore σ-compact.

PROOF OF THEOREM 1.5. This is the special case of Theorem 4.1 in which $G = H$.   □

Proof of Corollary 1.6. This is a special case of Theorem 1.5 (see Remark 4.2(1)). □

The following immediate consequence of Corollary 1.6 is a generalization of [3, Cor. 3.5.3, p. 177] (which is the special case where $G = H$).

Corollary 5.1. *Let H be a closed subgroup of a locally compact group G, and let A be a closed subgroup of the center of H, such that A is normal in G. If $(G/A, H/A)$ has relative Property (T), and $H/H^{(1)}$ is compact, then (G, H) has relative Property (T).*

Proof of Theorem 1.2. The direction $(\Rightarrow)$ is obvious. The other direction follows easily from Corollary 1.6 (or, if the reader prefers, from Theorem 1.5, 4.1, or 4.6), by induction on the nilpotence class of N. □

Remark 5.2. In the statement of Theorem 1.2, the assumption that N is normal can be weakened slightly, to the assumption that N has a central series $N = N_0 \rhd N_1 \rhd N_2 \rhd \cdots \rhd N_c = \{e\}$, such that $N_i \lhd G$, for all $i > 0$. (The subgroup $N = N_0$ does not need to be normal in G.)

Proof of Corollary 1.9. Since $(\Rightarrow)$ is obvious, we prove only $(\Leftarrow)$. Let $A = Z(N) \cap N^{(1)}$. By induction on the nilpotence class of N, we may assume that $\bigl(H \ltimes (N/A), H, N/A\bigr)$ has relative Property (T). Then the desired conclusion is immediate from Theorem 4.1, by letting $G = H \ltimes N$ and $M = N$. □

Proof of Corollary 1.12. Let $G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_k = \{e\}$ be:
- the descending central series of G, if G is nilpotent, or
- the derived series of G, if G is a connected, real split, solvable Lie group.

In either case, we have $G_k = \{e\}$ for some k. By induction on k, we may assume the image of Q in G/G_{k-1} is a Kazhdan set for G/G_{k-1}. Applying Remarks 4.2 and 4.4 (with $A = G_{k-1}$) tells us that Q is a Kazhdan set for the pair (G, G_1). By combining this with the fact that the image of Q in $G/G_1 = G^{ab}$ is a Kazhdan set for G/G_1, we conclude that Q is a Kazhdan set for G. □

Our next goal is Corollary 5.4, which is an extension of Corollary 1.4 that also incorporates Corollary 1.9. Its proof uses the following result.

Proposition 5.3 (Raja [24, Lem. 3.1]). *Let H be a locally compact group that acts on a 1-connected, abelian Lie group N. Then $(H \ltimes N, H, N)$ has relative Property (T) if and only if there does not exist a closed, connected, H-invariant, proper subgroup L of N, such that $\mathrm{Int}_{N/L}(H)^{\bullet}$ is amenable.*

Although our main interest is in groups that are locally compact, we state the following result without this assumption on H:

Corollary 5.4. *If a topological group H acts on a 1-connected, nilpotent Lie group N, then the following are equivalent:*

(1) *The pair $(H \ltimes N, N)$ has relative Property (T).*
(2) *The pair $(H \ltimes N^{ab}, N^{ab})$ has relative Property (T).*
(3) *The triple $(H \ltimes N, H, N)$ has relative Property (T).*
(4) *The triple $(H \ltimes N^{ab}, H, N^{ab})$ has relative Property (T).*
(5) *For every closed, connected, H-invariant, proper subgroup L of N that contains $N^{(1)}$, the group $\mathrm{Int}_{N/L}(H)^{\bullet}$ is <u>not</u> amenable.*
(6) *There exists a <u>finite</u> subset Q of H, and $\epsilon > 0$, such that if π is any unitary representation of $H \ltimes N$ that has nonzero (Q, ϵ)-invariant vectors, then π has nonzero N-invariant vectors.*

Proof. It is easy to establish $(3 \Rightarrow 1 \Rightarrow 2)$ and $(6 \Rightarrow 3 \Rightarrow 4 \Rightarrow 2)$. Also, $(2 \Rightarrow 5)$ is well known [7, Prop. 2.2 (ii' $\Rightarrow$ i), p. 391].

Therefore, it suffices to prove $(5 \Rightarrow 6)$. Let $H^{\bullet}$ be the closure of the image of H in $\mathrm{Aut}(N^{ab})$. Now, as $H^{\bullet}$ is a closed subgroup of the Lie group $\mathrm{Aut}(N^{ab})$, it is separable. Therefore, there is a countable subgroup Γ of H whose image is dense in $H^{\bullet}$. Every Γ-invariant, closed subgroup of N^{ab} is also $H^{\bullet}$-invariant, and is therefore H-invariant. If we give Γ the discrete topology, then Γ is locally compact, so we see from Proposition 5.3 that $(\Gamma \ltimes N^{ab}, \Gamma, N^{ab})$ has relative Property (T). Then Corollary 1.9 tells us that $(\Gamma \ltimes N, \Gamma, N)$ has relative Property (T).

By a standard argument [15, Thm. 1.2 (a2 $\Rightarrow$ a1)], this implies there is a finite subset Q of Γ, and $\epsilon > 0$, such that if π is any unitary representation of $\Gamma \ltimes N$ that has nonzero (Q, ϵ)-invariant vectors, then π has nonzero N-invariant vectors. Since every unitary representation of $H \ltimes N$ restricts to a (continuous) unitary representation of $\Gamma \ltimes N$, this completes the proof.   □

Remark 5.5. It is obvious that if a triple (G, H, N) has relative Property (T), then the pair (G, N) also has relative Property (T). The converse does not hold in general. (For example, it is easy to see that if N is infinite and discrete, then the triple $(H \ltimes N, H, N)$ never has relative Property (T) [14, Rem. 2.1.8]. But the pair $(H \ltimes N, N)$ may have relative Property (T).) Corollary 5.4 shows that the converse does hold when N is a 1-connected, nilpotent Lie group (and $G = H \ltimes N$).

Furthermore, the equivalence of (1) and (2) in Corollary 5.4 does not require N to be a Lie group:

Corollary 5.6. *If a topological group H acts on a compactly generated, locally compact, nilpotent group N, then the following are equivalent:*

(1) *The pair $(H \ltimes N, N)$ has relative Property (T).*
(2) *The pair $(H \ltimes N^{ab}, N^{ab})$ has relative Property (T).*

Proof. It suffices to prove $(2 \Rightarrow 1)$. Since the maximal compact subgroup of N is unique (see Lemma 6.1 below), it is normal in $H \ltimes N$, so there is no harm in modding it out. Therefore, we may assume that N has no nontrivial compact subgroups, so N is a (nilpotent) Lie group (see Theorem 6.3 below), such that N° is 1-connected (see Corollary 6.4 below) and N/N° is finitely generated

(because N is compactly generated) and torsion-free (see Corollary 6.4 below). Then N can be embedded as a closed, cocompact subgroup of a 1-connected Lie group N_1 [23, Thm. 2.20, p. 42]. Every automorphism of N extends uniquely to an automorphism of N_1 [23, Cor. 1 on p. 34], so we may form the semidirect product $H \ltimes N_1$, which contains $H \ltimes N$ as a closed, cocompact subgroup. Since $(H \ltimes N^{ab}, N^{ab})$ has relative Property (T), and N is cocompact in N_1, it is easy to see that $(H \ltimes N_1^{ab}, N_1^{ab})$ has relative Property (T). Namely, let $\rho \colon H \ltimes N^{ab} \to H \ltimes N_1^{ab}$ be the natural homomorphism, and let $N^\bullet$ be the closure of $\rho(N^{ab})$. Since $(H \ltimes N^{ab}, N^{ab})$ has relative Property (T), we know that $(H \ltimes N_1^{ab}, N^\bullet)$ has relative Property (T). Also, $N_1^{ab}/N^\bullet$ is compact (because N_1/N is compact), so $\left(H \ltimes (N_1^{ab}/N^\bullet), N_1^{ab}/N^\bullet\right)$ has relative Property (T). Therefore $(H \ltimes N_1^{ab}, N_1^{ab})$ has relative Property (T).

Now Corollary 5.4(6) provides a finitely generated subgroup Γ of H, such that $(\Gamma \ltimes N_1, N_1)$ has relative Property (T) (where Γ is given the discrete topology). Since N_1/N is compact, and N_1 is nilpotent, there is a unique N_1-invariant probability measure on N_1/N. The uniqueness implies that the measure is Γ-invariant, so we obtain a $(\Gamma \ltimes N_1)$-invariant probability measure on $(\Gamma \ltimes N_1)/(\Gamma \ltimes N)$. Therefore, since $(\Gamma \ltimes N_1, N_1)$ has relative Property (T), we conclude from [12, Prop. 2.4(1)] that $(\Gamma \ltimes N, N)$ has relative Property (T).

Since every unitary representation of $H \ltimes N$ restricts to a (continuous) unitary representation of $\Gamma \ltimes N$, this implies that $(H \ltimes N, N)$ has relative Property (T). □

6. The largest subgroup with relative Property (T)

It is easy to construct examples in which (G, L_1), (G, L_2), $\ldots$, (G, L_k) have relative Property (T), but if we let L be the subgroup generated by $L_1 \cup L_2 \cup \cdots \cup L_k$, then (G, L) does not have relative Property (T). (For example, let G be a simple Lie group that does not have Property (T), and let $L_1, L_2, \ldots, L_k$ be compact subgroups that generate G.) Corollary 6.7 provides a situation in which this pathology does not arise. The proof does not require the main results proved in Section 5, but it does use several basic facts about locally compact groups and relative Property (T).

Lemma 6.1 (cf. [16, Thm. 2] and [8, Lem. 3.1]). *Every compactly generated, locally compact, nilpotent group N has a unique maximal compact subgroup.*

Notation 6.2. We use G° for the identity component of the topological group G.

Theorem 6.3 ([18, Theorem on p. 175 (and Lem. 2.3.1, p. 54)]). *Let G be a locally compact group. Then some open subgroup H of G has a compact, normal subgroup C, such that H/C is a Lie group.*
Moreover, for an appropriate choice of H, the compact subgroup C can be chosen to be contained in any neighborhood of the identity in H.
Furthermore, if G/G° is compact, then we may take $H = G$.

The following consequence is well known.

Corollary 6.4. *Let G be a locally compact group. If G/G° is compact, and G has no nontrivial, compact subgroups, then G is a 1-connected Lie group.*

Lemma 6.5 (cf. [25, Lem. 2.2]). *Assume $H, H_1, \ldots, H_n$ are closed subgroups of a locally compact group G, and C is a compact subset of G, such that $CH_1H_2\cdots H_n$ contains H. If the pair (G, H_i) has relative Property (T), for each i, then (G, H) has relative Property (T).*

Proof. Let π be a unitary representation of G on a Hilbert space $\mathcal{H}$, such that π has almost-invariant vectors, and let $\delta = 4^{-(n+2)}$. For each i, Theorem 2.3 provides a compact subset Q_i of G and $\epsilon_i > 0$, such that if η is any (Q_i, ϵ_i)-invariant unit vector, then there is an H_i-invariant unit vector η_i, such that $\|\eta - \eta_i\| < \delta/2$. Now, let η be a (Q, ϵ)-invariant unit vector, where $Q = C \cup \bigcup_{i=1}^n Q_i$ and $\epsilon = \min(\delta, \epsilon_1, \ldots, \epsilon_n)$. Then

$$\|\pi(g)\eta - \eta\| < \delta, \text{ for all } g \in H_1 \cup \cdots \cup H_n \cup C.$$

This implies $\|\pi(h)\eta - \eta\| < 1/2$ for all $h \in H$. So π has a nonzero H-invariant vector [15, Lem. 2.2]. □

Lemma 6.6. *Let H be a compactly generated, locally compact, nilpotent group, and let $\mathcal{L}$ be a collection of closed subgroups of H that generates a dense subgroup of H. Then, for some n, there exist $L_1, \ldots, L_n \in \mathcal{L}$, and a compact subset C of H, such that $CL_1L_2\cdots L_n = H$.*

Proof. We may assume that H is a Lie group with no nontrivial compact subgroups (by modding out the maximal compact subgroup (see Lemma 6.1 and Theorem 6.3)). Let $H^{(k)}$ be the closure of the last nontrivial term of the descending central series of H. The desired conclusion is easy if H is abelian (and therefore isomorphic to $\mathbb{R}^m \times \mathbb{Z}^n$ for some m and n), so we may assume $H^{(k)} \neq H$. By induction on the nilpotence class of H, we may assume that there is a finite product $X = L_1 \cdots L_n$ of subgroups in $\mathcal{L}$, and a compact subset C of H, such that $CH^{(k)}X = H$. Note that, for each $g \in H^{(k-1)}$, the map $x \mapsto [x, g]$ is a homomorphism from $H/H^{(k)}$ to $H^{(k)}$. Since $\dim H^{(k)} + \operatorname{rank} H^{(k)}/(H^{(k)})^\circ$ is finite, there is a finite subset $\{g_1, \ldots, g_m\}$ of $H^{(k-1)}$, such that $\prod_{i=1}^m [H, g_i]$ is dense in a cocompact subgroup of the abelian group $H^{(k)}$. So there is a compact subset C_0 of $H^{(k)}$, such that $C_0 \cdot \prod_{i=1}^m [H, g_i] = H^{(k)}$. Since $H^{(k)}$ is abelian, this implies that $C_0 \cdot \prod_{i=1}^m [C, g_i] \cdot \prod_{i=1}^m [X, g_i] = H^{(k)}$. Then

$$C \cdot C_0 \cdot \prod_{i=1}^m [C, g_i] \cdot \prod_{i=1}^m (X \cdot g_i X g_i^{-1}) \cdot X = CH^{(k)}X = H. \qquad \square$$

Corollary 6.7. *Let N be a closed, compactly generated, nilpotent, normal subgroup of a locally compact group G, and let $\mathcal{T}$ be the collection of all subgroups L of N, such that (G, L) has relative Property (T). Then $\mathcal{T}$ has a unique largest element $L^\dagger$, and $L^\dagger$ is a closed, normal subgroup of G.*

Proof. Let $L^\dagger$ be the closure of the subgroup generated by the subgroups in $\mathcal{T}$. Lemma 6.6 tells us there is a product $L_1L_2\cdots L_n$ of finitely many elements

of $\mathcal{T}$, and a compact set C, such that $CL_1L_2\cdots L_n = L^\dagger$. Since (G, L_i) has relative Property (T), for each i, it follows from Lemma 6.5 that $(G, L^\dagger)$ has relative Property (T). So $L^\dagger \in \mathcal{T}$. By definition, $L^\dagger$ contains every element of $\mathcal{T}$, so this implies that $L^\dagger$ is the unique largest element of $\mathcal{T}$.

Furthermore, $L^\dagger$ is closed by definition. Also, if L is any conjugate of $L^\dagger$, then (G, L) has relative Property (T) (because $(G, L^\dagger)$ has relative Property (T)), so $L \subseteq L^\dagger$. Therefore, $L^\dagger$ is normal. $\qquad\square$

We also have the following weaker conclusion without the assumption that G is locally compact.

COROLLARY 6.8. *Let N be a closed, locally compact, compactly generated, nilpotent, normal subgroup of a topological group G, and let $\mathcal{T}_\triangleleft$ be the collection of all subgroups L of N, such that (G, L) has relative Property (T) and $L \triangleleft G$. Then $\mathcal{T}_\triangleleft$ has a unique largest element $L_\triangleleft^\dagger$, and $L_\triangleleft^\dagger$ is a closed, normal subgroup of G.*

Proof. Nothing in the proof of Corollary 6.7 relies on the assumption that G is locally compact, other than the application of Theorem 2.3 in the proof of Lemma 6.5. Although Theorem 2.3 may not be true for general topological groups, its conclusion holds when $H \triangleleft G$, by the same standard argument that is used in the proof of Lemma 2.6. $\qquad\square$

7. RELATIVE PROPERTY (T) AND AMENABILITY

The main result in this section is Corollary 7.2, which provides additional information about the subgroup $L^\dagger$ of Corollary 6.7 (under a connectivity assumption on N). This implies Corollary 7.9, which is a generalization of Corollary 1.4 that does not require the subgroup N to be a Lie group. The statements of these results require the following extension of Notation 1.3 to this setting:

NOTATION 7.1. Suppose N and L are closed, normal subgroups of a locally compact group G, such that $L \subseteq N$, and N/L is a Lie group.

(1) $\mathrm{Int}_{N/L}\colon G \to \mathrm{Aut}(N/L)$ is the natural map defined by the action of G on N/L by conjugation.
(2) We use $\mathrm{Int}_{N/L}(G)^\bullet$ to denote the *closure* of the image of this homomorphism.

COROLLARY 7.2. *Let N be a closed, locally compact, nilpotent, normal subgroup of a topological group G, such that N/N° is compact. Then N has a unique largest subgroup $L^\dagger$, such that $(G, L^\dagger)$ has relative Property (T). Furthermore, $L^\dagger$ is a closed, normal subgroup of G, the quotient $N/L^\dagger$ is a 1-connected Lie group, and $\mathrm{Int}_{N/L^\dagger}(G)^\bullet$ is amenable.*

The following consequence of $(5 \Rightarrow 2)$ of Corollary 5.4 is essentially the special case of Corollary 7.2 in which N is a connected, abelian Lie group. It will be the basis of a proof by induction.

Corollary 7.3 (cf. [7, Prop. 2.2 (i $\Rightarrow$ ii$'$)] and [24, Cor. 3.2]). *Assume a topological group H acts on a connected, abelian Lie group N. Then N contains a closed, connected, H-invariant, normal subgroup L, such that $(H \ltimes N, L)$ has relative Property (T), and $\mathrm{Int}_{N/L}(H)^\bullet$ is amenable.*

The following elementary observation can reduce problems about arbitrary normal subgroups to the easier case of semidirect products.

Lemma 7.4. *Let N be a closed, normal subgroup of a topological group G, and let L be a subgroup of N. Form the semidirect product $G \ltimes N$, where G acts on N by conjugation. If $(G \ltimes N, L)$ has relative Property (T), then (G, L) also has relative Property (T).*

Proof. For any unitary representation π of G, there is a unitary representation π' of $G \ltimes N$ that is defined by $\pi'(g, n) = \pi(g)\,\pi(n)$, for $g \in G$ and $n \in N$. If π has almost-invariant vectors, then so does π'. $\qquad\square$

Remark 7.5. The converse of Lemma 7.4 is not true. For example, let G be a Lie group with Kazhdan's Property (T), such that $Z(G)$ is not compact, and let $N = L = Z(G)$. (In particular, G could be the universal cover of $\mathrm{Sp}(4, \mathbb{R})$ [3, Example 1.7.13(ii), p. 67].) Then (G, N) has relative Property (T) (in fact, (G, G) has relative Property (T)), but $G \ltimes N \cong G \times N$, and $(G \times N, N)$ does not have relative Property (T) (because N is a noncompact, abelian Lie group, and therefore does not have Kazhdan's Property (T)).

As the final preparation for the proof of Corollary 7.2, we establish one more lemma:

Lemma 7.6 (cf. [7, Lem. 2.3(i)]). *Suppose N and L are closed, normal subgroups of a locally compact group G, such that $L \subseteq N$. Assume that L is connected, and that N is a 1-connected, nilpotent Lie group. If $\mathrm{Int}_{N/L}(G)^\bullet$ and $\mathrm{Int}_L(G)^\bullet$ are amenable, then $\mathrm{Int}_N(G)^\bullet$ is amenable.*

Proof. Let $\mathfrak{N}$ and $\mathfrak{L}$ be the Lie algebras of N and L, respectively, and let

$$P = \{\, T \in \mathrm{GL}(\mathfrak{N}) \mid T(\mathfrak{L}) = \mathfrak{L} \,\}.$$

Since N is a 1-connected, nilpotent Lie group, we can identify $\mathrm{Aut}(N)$ with $\mathrm{Aut}(\mathfrak{N})$, which is a closed subgroup of P. It is well known that, by choosing a complement W to $\mathfrak{L}$ in $\mathfrak{N}$, we have $P = \big(\mathrm{GL}(W) \times \mathrm{GL}(\mathfrak{L})\big) \ltimes K$, where K is the kernel of the natural map $P \to \mathrm{GL}(\mathfrak{N}/\mathfrak{L}) \times \mathrm{GL}(\mathfrak{L})$. Therefore,

$$\mathrm{Int}_N(G)^\bullet \subseteq \big(\mathrm{Int}_{N/L}(G)^\bullet \times \mathrm{Int}_L(G)^\bullet\big) \ltimes K.$$

Since K is abelian (and hence amenable), we conclude that $\mathrm{Int}_N(G)^\bullet$ is a closed subgroup of an amenable group, and is therefore amenable. $\qquad\square$

Proof of Corollary 7.2. By modding out $L_\lhd^\dagger$ (see Corollary 6.8), there is no harm in assuming that it is trivial, which means:

$(*)$ N does not contain any nontrivial normal subgroup L of G,
           such that (G, L) has relative Property (T).

Since N is a compactly generated, nilpotent group, it has a unique maximal compact subgroup K (see Lemma 6.1). Then (G, K) has relative Property (T) (since K is compact), and $K \lhd G$ (because of the uniqueness), so $(*)$ implies that K is trivial. This means that N has no nontrivial compact subgroups. So N is a 1-connected Lie group (see Corollary 6.4) (and, by assumption, N is nilpotent).

All that remains is to show that $\mathrm{Int}_N(G)^\bullet$ is amenable. By Lemma 7.4, we may assume that G is a semidirect product $H \ltimes N$. Let $A = N^{(1)} \cap Z(N)$. (Note that A is a closed, connected, abelian, normal subgroup of G.) By induction on the rank of N (and Corollary 7.3 for the base case where A is trivial, so N is abelian), we may assume that N/A contains a closed, normal subgroup L'/A of G/A, such that $(G/A, L'/A)$ has relative Property (T), N/L' is 1-connected, and $\mathrm{Int}_{N/L'}(G)^\bullet$ is amenable. From Corollary 1.6, we conclude that $\big(G, (L')^{(1)}\big)$ has relative Property (T). So $(*)$ tells us that $(L')^{(1)}$ is trivial, which means L' is abelian. Also, since N is connected and N/L' is 1-connected, we know that L' is connected. Therefore, we may apply Corollary 7.3 to the semidirect product $G \ltimes L'$ (and compare with $(*)$), to conclude that $\mathrm{Int}_{L'}(G)^\bullet$ is amenable. We now know that $\mathrm{Int}_{N/L'}(G)^\bullet$ and $\mathrm{Int}_{L'}(G)^\bullet$ are amenable, so Lemma 7.6 tells us that $\mathrm{Int}_N(G)^\bullet$ is amenable, as desired. $\qquad\square$

Remark 7.7. Assume G, N, and $L^\dagger$ are as in Corollary 7.2.

(1) If N is connected, then $L^\dagger$ is also connected. To see this from the proof of Corollary 7.2, it suffices to note that the maximal subgroup K must be connected, since N is homeomorphic to $K \times \mathbb{R}^n$. (This is well known for connected Lie groups and, in fact, was proved by Iwasawa [13, Thm. 13, p. 549] for connected groups that are approximated by Lie groups. Theorem 6.3 implies that every connected, locally compact group can be so approximated.)

(2) If G is a semidirect product $H \ltimes N$, then the subgroup $L^\dagger$ either is compact, or projects nontrivially into $N/N^{(1)}$. To establish this, assume, without loss of generality, that N is a 1-connected Lie group (by modding out the maximal compact subgroup). If $\mathrm{Int}_N(H)^\bullet$ is amenable, then $L^\dagger$ is trivial. (This is an easy generalization of $(2 \Rightarrow 5)$ of Corollary 5.4.) If not, then the Zariski closure of $\mathrm{Int}_N(H)^\bullet$ has a noncompact, semisimple subgroup M. Since M acts nontrivially on the 1-connected, nilpotent group N, it is well known that M must act nontrivially on $N/N^{(1)}$. So $\mathrm{Int}_{N/N^{(1)}}(G)^\bullet$ is not amenable. Therefore, Corollary 7.2 tells us that $L^\dagger$ is not contained in $N^{(1)}$.

Remark 7.8. The assumption that N/N° is compact cannot be deleted from the statement of Corollary 7.2. (So a connectivity assumption is also necessary in Corollary 7.3.) Here is a counterexample that is adapted from the proof of [7, Prop. 2.2 (ii' $\Rightarrow$ i)].

Let $\alpha = \sqrt{2}$, $\mathcal{O} = \mathbb{Z}[\alpha]$, $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + \alpha x_3^2$, $\Gamma = \mathrm{SO}_3(Q; \mathcal{O})$, $N = \mathcal{O}^3 \cong \mathbb{Z}^6$, $G = \Gamma \ltimes N$, and $H = \mathrm{SO}(3) \ltimes \mathbb{R}^3$, and let π be the unitary

representation of G obtained by composing the homomorphism

$$G = \Gamma \ltimes N \hookrightarrow \mathrm{SO}(3) \ltimes \mathbb{R}^3 = H$$

with the left-regular representation of H. Since H is amenable, π has almost-invariant vectors. However, if L is any nontrivial subgroup of N, then the image of L in $\mathbb{R}^3$ is noncompact, so π has no nonzero L-invariant vector. Therefore, L must be trivial if (G, L) has relative Property (T).

However, Restriction of Scalars embeds G as a lattice in $\big(\mathrm{SO}(3) \ltimes \mathbb{R}^3\big) \times \big(\mathrm{SO}(2,1) \ltimes \mathbb{R}^3\big)$ (cf. [19, §5.5]), so it is clear that $\mathrm{Int}_N(G)^\bullet$ is not amenable.

Corollary 7.2 implies the following generalization of $(1 \Leftrightarrow 5)$ of Corollary 5.4:

COROLLARY 7.9. *Assume a topological group H acts on a locally compact, nilpotent group N, such that N/N° is compact. Then $(H \ltimes N, N)$ has relative Property (T) if and only if there does <u>not</u> exist a closed, H-invariant, normal subgroup L of N, such that N/L is a nontrivial, 1-connected Lie group, and $\mathrm{Int}_{N/L}(H)^\bullet$ is amenable.*

REMARK 7.10. Corollary 7.9 can be used to determine whether $(H \ltimes N, N)$ has relative Property (T), even if we replace the assumption that N/N° is compact with the weaker assumption that N is compactly generated. This is because the argument in the first paragraph of the proof of Corollary 5.6 constructs a 1-connected, nilpotent Lie group N_1, such that $(H \ltimes N, N)$ has relative Property (T) if and only if $(H \ltimes N_1, N_1)$ has relative Property (T).

8. Relative Property (T) for subsets

As was mentioned in Remark 1.13, Y. Cornulier [5, p. 302] has generalized the notion of relative Property (T) to pairs (G, H) in which H is a sub*set* of G, rather than a sub*group*. We propose the following natural analogue for triples (G, H, M) in which M is a subset:

DEFINITION 8.1. Assume H is a closed subgroup of a topological group G, and M is a subset of G.

(1) To say that the triple (G, H, M) has *relative Property (T)* means that for every $\epsilon > 0$, there exist a compact subset Q of H and $\delta > 0$, such that every unitary representation of G with nonzero (Q, δ)-invariant vectors also has nonzero (M, ϵ)-invariant vectors.

(2) To say that the triple (G, H, M) has *relative Property (T) with approximation* means that for every $\epsilon > 0$, there exist a compact subset Q of H and $\delta > 0$, such that every (Q, δ)-invariant vector for any unitary representation of G is also (M, ϵ)-invariant.

REMARK 8.2. If $H = G$, and G is locally compact, then it follows from the proof of [5, Thm. 2.2.3 $(1 \Leftrightarrow 2)$] that (1) and (2) of Definition 8.1 are equivalent to each other, and to Cornulier's definition of relative Property (T) for the pair (G, M). Also, it is easy to see that Definition 8.1 is consistent with Definitions 1.8 and 2.5 (because being (M, ϵ)-invariant is equivalent to being

close to an M-invariant vector in the situation where M is a closed subgroup of G [15, Lem. 2.2]).

NOTATION 8.3. For elements m_1 and m_2 of any group, we let

$$[m_1, m_2] = m_1^{-1} m_2^{-1} m_1 m_2.$$

In our discussion of relative Property (T) for subsets, the following trivial observation replaces Proposition 3.8 as a way to obtain almost-invariant vectors for π from almost-invariant vectors for $\pi \otimes_{\widehat{A}} \overline{\pi}$.

LEMMA 8.4. *Assume ρ is a unitary representation of a topological group G on a Hilbert space $\mathcal{H}$, $m_1, m_2 \in G$, $\xi \in \mathcal{H}$, $\xi' = \xi \otimes \overline{\xi} \in \mathcal{H} \otimes \overline{\mathcal{H}}$, and $\rho' = \rho \otimes \overline{\rho}$. Then*

$$\|\rho([m_1, m_2])\xi - \xi\| \leq 2\big(\|\rho'(m_1)\xi' - \xi'\| + \|\rho'(m_2)\xi' - \xi'\|\big).$$

Proof. For convenience, let $\epsilon_i = \|\rho'(m_i)\xi' - \xi'\|$ (for $i = 1, 2$), and assume, without loss of generality, that $\|\xi\| = 1$. For $i = 1, 2$, there exists a unique $\lambda_i \in \mathbb{C}$ (with $|\lambda_i| \leq 1$), such that

$$\rho(m_i)\xi - \lambda_i \xi \ \perp \ \xi.$$

To avoid some uncomfortably long expressions in the following sentence, let $v_i = \rho(m_i)\xi - \lambda_i \xi$, so $v_i \perp \xi$. Then the three vectors

$$v_i \otimes \overline{\rho(m_i)\xi}, \quad \lambda_i \xi \otimes \overline{v_i}, \quad \text{and} \quad \xi'$$

are pairwise orthogonal, so the Pythagorean Theorem tells us

$$\begin{aligned}
\big\|\rho'(m_i)\xi' - \xi'\big\|^2 &= \big\|v_i \otimes \overline{\rho(m_i)\xi}\big\|^2 + \big\|\lambda_i \xi \otimes \overline{v_i}\big\|^2 + \big\|(|\lambda_i|^2 - 1)\xi'\big\|^2 \\
&\geq \big\|v_i \otimes \overline{\rho(m_i)\xi}\big\|^2 \\
&= \|v_i\|^2,
\end{aligned}$$

which means

$$\epsilon_i \geq \|\rho(m_i)\xi - \lambda_i \xi\|.$$

Therefore

$$\begin{aligned}
\|\rho(m_1 m_2)\xi - \lambda_1 \lambda_2 \xi\| &\leq \big\|\rho(m_1)\big(\rho(m_2)\xi - \lambda_2 \xi\big)\big\| + \|\rho(m_1)(\lambda_2 \xi) - \lambda_1 \lambda_2 \xi\| \\
&= \|\rho(m_2)\xi - \lambda_2 \xi\| + |\lambda_2| \cdot \|\rho(m_1)\xi - \lambda_1 \xi\| \\
&\leq \|\rho(m_2)\xi - \lambda_2 \xi\| + \|\rho(m_1)\xi - \lambda_1 \xi\| \\
&\leq \epsilon_2 + \epsilon_1.
\end{aligned}$$

Since the same is true after interchanging the subscripts 1 and 2 (and $\lambda_1 \lambda_2 = \lambda_2 \lambda_1$), we conclude that

$$\begin{aligned}
\big\|\rho([m_1, m_2])\xi - \xi\big\| &= \|\rho(m_1 m_2)\xi - \rho(m_2 m_1)\xi\| \\
&\leq \|\rho(m_1 m_2)\xi - \lambda_1 \lambda_2 \xi\| + \|\rho(m_2 m_1)\xi - \lambda_1 \lambda_2 \xi\| \\
&\leq (\epsilon_2 + \epsilon_1) + (\epsilon_1 + \epsilon_2) \\
&= 2(\epsilon_1 + \epsilon_2). \qquad \square
\end{aligned}$$

The following theorem is an analogue of Theorem 4.1 that does not require the set M to be a subgroup.

Definition 8.5.

> (1) A *probability measure* is a finite real positive Borel regular measure that has been normalized to have total mass 1.
> (2) We use the total variation norm $\|\cdot\|$ to provide a metric on the space of probability measures (on any topological space).
> (3) For a subset M of a group G, we let $[\![M, M]\!] = \{\, [m_1, m_2] \mid m_1, m_2 \in M \,\}$.

Theorem 8.6. *Let H be a closed subgroup of a locally compact group G, let A be a closed, abelian, normal subgroup of G, and let M be a subset of G. For every $\epsilon' > 0$, assume there exists $\delta' > 0$, such that if μ is any (M, δ')-invariant probability measure on $\widehat{A}$ that is quasi-invariant for the action of G, then $\mu(\widehat{A}^M) \geq 1 - \epsilon'$, where $\widehat{A}^M$ is the set of fixed points of M. If HA is closed and $(G/A, HA/A, MA/A)$ has relative Property (T) with approximation, then $\big(G, H, [\![M, M]\!]\big)$ has relative Property (T) with approximation.*

Proof. This is adapted from the proof of Theorem 4.1. Given an arbitrary $\epsilon > 0$, choose $\epsilon' > 0$ small enough that if ξ is any ϵ'-invariant unit vector, and $\|\xi - \eta\|^2 < \epsilon'$, then η is $\epsilon/4$-invariant. Also, let δ' be a value that corresponds to this value of ϵ' in the assumption in the statement of the theorem (and assume $\delta' < \epsilon'/2$). Since $(G/A, HA/A, M/A)$ has relative Property (T) with approximation (and HA is closed), there exist a compact subset Q of H and $\delta > 0$, such that if ξ is any (Q, δ)-invariant vector for any unitary representation of G/A, then ξ is $(M, \delta'/2)$-invariant.

Now, suppose π is a unitary representation of G, such that π has a nonzero $(Q, \delta/3)$-invariant vector f. (We wish to show that f is $([M, M], \epsilon)$-invariant.) By replacing π with the direct sum $\pi \oplus \pi \oplus \cdots$ of infinitely many copies of itself, we may assume that all irreducible representations appearing in the direct integral decomposition of $\pi|_A$ have the same multiplicity (namely, ∞). By definition, this means that $\pi|_A$ is homogeneous, so Section 3A provides a quasi-invariant probability measure μ on $\widehat{A}$, a Borel cocycle $\alpha \colon G \times \widehat{A} \to \mathcal{U}(\mathcal{H})$, a corresponding realization of π as a representation on the Hilbert space $L^2(\widehat{A}, \mu; \mathcal{H})$, and a unitary representation $\pi' = \pi \otimes_{\widehat{A}} \overline{\pi}$ of G.

Since π has been realized as a representation on $L^2(\widehat{A}, \mu; \mathcal{H})$, we know that $f \in L^2(\widehat{A}, \mu; \mathcal{H})$. Then Proposition 3.6 provides a (Q, δ)-invariant unit vector f' for π'. By the choice of Q and δ (and Remark 3.5), we know that

$$f' \text{ is } (M, \delta'/2)\text{-invariant.}$$

Then it is straightforward to check that $\|f'\|^2 \mu$ is an (M, δ')-invariant measure on $\widehat{A}$. Also, by perturbing f slightly, we could assume that it is nonzero almost everywhere, so f' is also nonzero almost everywhere. Then $\|f'\|^2 \mu$ is quasi-invariant for the G-action (because μ is quasi-invariant). By the choice of δ',

this implies that $\int_{\widehat{A}^M} \|f'\|^2 \, d\mu > 1 - \epsilon'$ (assuming that f' has been normalized to be a unit vector in L^2). Proposition 3.6 tells us

$$f'(\lambda) = \frac{1}{\|f(\lambda)\|} \, f(\lambda) \otimes \overline{f(\lambda)},$$

so $\|f'(\lambda)\| = \|f(\lambda)\|$ for all $\lambda \in \widehat{A}$. This implies

$$\int_{\widehat{A}^M} \|f\|^2 \, d\mu = \int_{\widehat{A}^M} \|f'\|^2 \, d\mu > 1 - \epsilon',$$

so $\left\| f|_{\widehat{A}^M} - f \right\| < \epsilon'$. This means that f is well approximated by a function that is supported on $\widehat{A}^M$, so, to simplify the argument, we will assume that f itself is supported on $\widehat{A}^M$.

For each fixed $\lambda \in \widehat{A}^M$, the function $\alpha(m, \lambda)$ is a representation ρ_λ of M on $\mathcal{H}$, so M acts on $L^2(\widehat{A}^M, \mu; \mathcal{H} \otimes \overline{\mathcal{H}})$ by

$$\big(\pi'(m)f'\big)(\lambda) = (\rho_\lambda \otimes \overline{\rho_\lambda})(m)f'(\lambda).$$

Therefore, we see from Lemma 8.4 that

$$\big\|\pi\big([m_1, m_2]\big)f(\lambda) - f(\lambda)\big\| \leq 2\big(\|\pi'(m_1)f'(\lambda) - f'(\lambda)\| + \|\pi'(m_2)f'(\lambda) - f'(\lambda)\|\big).$$

Since this is true for all λ, we conclude that

$$\begin{aligned}
\big\|\pi\big([m_1, m_2]\big)f - f\big\|_2 &\leq 2\big(\|\pi'(m_1)f' - f'\|_2 + \|\pi'(m_2)f' - f'\|_2\big) \\
&\leq 2\left(\frac{\delta'}{2} + \frac{\delta'}{2}\right) < \epsilon. \qquad \square
\end{aligned}$$

Theorem 8.6 immediately implies the following natural generalization of Corollary 1.6, in which H is a subset, rather than a subgroup. (We do not need to mention approximation, because Jolissaint's proof of Theorem 2.3 shows that relative Property (T) is equivalent to relative Property (T) with approximation for all pairs (G, H), even if H is only a subset, not a subgroup.)

COROLLARY 8.7. *Let A be a closed, abelian, normal subgroup of a locally compact group G, and H be a subset of G that centralizes A. If $(G/A, HA/A)$ has relative Property (T), then $\big(G, [\![H, H]\!]\big)$ has relative Property (T).*

9. Other observations about relative Property (T)

We close the paper with some tangential observations about relative Property (T).

9A. Relative Property (T) for connected, normal, Lie subgroups.

If N is a connected Lie group, then, since the group U/U_S in the following proposition is a connected, nilpotent Lie group, Corollary 7.9 determines whether or not $(H \ltimes N, N)$ has relative Property (T), without the need to assume N is nilpotent.

Proposition 9.1. *Suppose a locally compact group H acts on a connected Lie group N. Let S be the closure of the product of the noncompact simple factors of a Levi subgroup of N, let U be the nilradical of N, and let*

$$U_S = \mathrm{cl}\big([S, U] \cdot (S \cap U)\big).$$

Then $(H \ltimes N, N)$ has relative Property (T) if and only if:

(1) *S has Kazhdan's Property (T),*
(2) *$N/\mathrm{cl}(SU)$ is compact, and*
(3) *$\big(H \ltimes (U/U_S), U/U_S\big)$ has relative Property (T).*

Proof. ($\Rightarrow$) The adjoint group $\mathrm{Ad}\,S$ is a quotient of N, so $(H \ltimes \mathrm{Ad}\,S, \mathrm{Ad}\,S)$ has relative Property (T). However, $\mathrm{Ad}\,S$ is a connected, semisimple Lie group, so its outer automorphism group is finite. Therefore, we may assume that H acts on $\mathrm{Ad}\,S$ by inner automorphisms (after replacing H with a finite-index subgroup). This implies that

$$H \ltimes \mathrm{Ad}\,S = C_{H \ltimes \mathrm{Ad}\,S}(S) \cdot \mathrm{Ad}\,S \cong H \times \mathrm{Ad}\,S.$$

So we now know that $(H \times \mathrm{Ad}\,S, \mathrm{Ad}\,S)$ has relative Property (T). This implies that $\mathrm{Ad}\,S$ has Kazhdan's Property (T). Then (1) follows from Corollary 1.6 (or the special case proved by J.–P. Serre that is mentioned in Remark 1.7).

By definition, S is contained in the closure of some Levi subgroup S^+ of N. Since the pair $\big(H \ltimes N/(S^+U), N/(S^+U)\big)$ has relative Property (T), and the structure theory of Lie groups tells us that $\mathrm{Aut}(N)$ acts on $N/(S^+U)$ via a finite group, we see that $N/(S^+U)$ is compact. Since S^+/S is compact (by the definition of S), this implies (2).

Since $(H \ltimes N, N)$ has relative Property (T), (2) implies that $\big(H \ltimes SU, SU\big)$ has relative Property (T) (see [15, Cor. 4.1(2)]). Passing to a quotient yields (3).

($\Leftarrow$) Suppose π is a unitary representation of $H \ltimes N$ that has almost-invariant vectors. We wish to show that N has invariant vectors. By induction on $\dim N$ (and Theorem 2.3), we may assume that no nontrivial, connected, H-invariant, normal subgroup of N has nonzero invariant vectors.

Therefore, Corollary 6.7 implies that there is no nontrivial, connected subgroup U_0 of U, such that $(H \ltimes N, U_0)$ has relative Property (T). So U has no nontrivial, compact subgroups, and is therefore 1-connected (see Corollary 6.4). Then Corollary 7.2 implies that S centralizes U. So $S \lhd H \ltimes N$. However, we see from (1) that $(H \ltimes N, S)$ has relative Property (T). So the assumption of the previous paragraph implies that S is trivial. Then U_S is obviously also trivial. So $U/U_S = U = SU$. Therefore, (3) tells us that $\big(H \ltimes (SU), SU\big)$ has relative Property (T). Then the assumption of the previous paragraph implies that SU is trivial. Therefore, (2) tells us that N is compact, so $(H \ltimes N, N)$ has relative Property (T), as desired. $\square$

Remark 9.2. The proof of Proposition 9.1 yields the following general result (which is slightly weaker than Proposition 9.1 in the special case where G is a semidirect product):

Suppose N is a closed, normal subgroup of a locally compact group G, such that N is a connected Lie group. Define S, U, and U_S as in Proposition 9.1. Then (G, N) has relative Property (T) if and only if:

      (1) *S has Kazhdan's Property (T),*
      (2) *$N/\mathrm{cl}(SU)$ is compact, and*
      (3) *$(G/U_S, U/U_S)$ has relative Property (T).*

Stronger results were already known in the special case where G is a connected Lie group. (See, for example, [5, Cor. 3.3.2].)

9B. Relative Property (T) for solvable subgroups. In the statement of Theorem 1.2, the assumption that N is nilpotent cannot be replaced with the weaker assumption that N is solvable. (For example, let $G = N$ be a noncompact, solvable group, such that $G/G^{(1)}$ is compact.) However, it would suffice to assume that N is a connected, real split, solvable Lie group (see Remark 4.2(2)). Also, we have the following easy consequence of Theorem 1.2 that applies to some other solvable groups.

Notation 9.3. If N is a locally compact group, then

$$N^{(2)} = (N^{(1)})^{(1)} = \mathrm{cl}\big([[N, N], [N, N]]\big)$$

is the closure of the second derived group of N.

Corollary 9.4. *Let N be a closed, normal subgroup of a locally compact group G, such that $N^{(1)}$ is nilpotent. Then (G, N) has relative Property (T) if and only if $(G/N^{(2)}, N/N^{(2)})$ has relative Property (T).*

For example, every virtually polycyclic group has a (characteristic) finite-index subgroup whose commutator subgroup is nilpotent (see [23, Cor. 4.11, p. 59]). Here is another example:

Corollary 9.5. *Let N be a connected, closed, solvable, normal subgroup of a locally compact group G. Then (G, N) has relative Property (T) if and only if $(G/N^{(2)}, N/N^{(2)})$ has relative Property (T).*

Proof. It is well known that the assumptions on N imply that $N^{(1)}$ has a unique maximal compact subgroup C_1, and that $N^{(1)}/C_1$ is nilpotent. So the desired conclusion is obtained by applying Corollary 9.4 to the pair $(G/C_1, N/C_1)$. $\square$

9C. Homomorphisms with a dense image. The following corollary generalizes a result of Y. Cornulier and R. Tessera [6, Cor. 2].

Corollary 9.6. *Let H, N, and H_1 be locally compact groups. Assume N is nilpotent, H acts on N, and we are given a homomorphism $H_1 \to H$ with dense image. Then $(H \ltimes N, N)$ has relative Property (T) if and only if $(H_1 \ltimes N, N)$ has relative Property (T).*
Moreover, if $(H \ltimes N, N)$ has relative Property (T), then there is a finitely generated group Γ and a homomorphism $\Gamma \to H$, such that $(\Gamma \ltimes N, N)$ has relative Property (T).

Proof. In the case where N is abelian, this is [6, Cor. 2]. The general case follows from this by applying Theorem 1.2.   □

If we assume that N is a connected Lie group, then the assumption that N is nilpotent can be eliminated:

COROLLARY 9.7. *Let H and H_1 be locally compact groups, and let N be a connected Lie group. Assume H acts on N, and we are given a homomorphism $H_1 \to H$ with dense image. Then $(H \ltimes N, N)$ has relative Property (T) if and only if $(H_1 \ltimes N, N)$ has relative Property (T).*
Moreover, if $(H \ltimes N, N)$ has relative Property (T), then there is a finitely generated group Γ and a homomorphism $\Gamma \to H$, such that $(\Gamma \ltimes N, N)$ has relative Property (T).

Proof. Proposition 9.1 reduces the problem to the case where N is nilpotent, which is handled by Corollary 9.6.   □

If we assume that N is a 1-connected Lie group, then Corollary 9.6 can be extended to triples, and does not require H or H_1 to be locally compact:

COROLLARY 9.8. *Let H and H_1 be topological groups, and let N be a 1-connected, nilpotent Lie group. Assume H acts on N, and we are given a homomorphism $H_1 \to H$ with dense image. Then the following are equivalent:*

  (1) $(H \ltimes N, N)$ *has relative Property (T).*
  (2) $(H_1 \ltimes N, N)$ *has relative Property (T).*
  (3) $(H \ltimes N, H, N)$ *has relative Property (T).*
  (4) $(H_1 \ltimes N, H_1, N)$ *has relative Property (T).*

Moreover, if these conditions hold, then there is a finitely generated group Γ and a homomorphism $\Gamma \to H$, such that $(\Gamma \ltimes N, N)$ and $(\Gamma \ltimes N, \Gamma, N)$ have relative Property (T).

Proof. A closed subgroup of N is H-invariant if and only if it is H_1-invariant, so the equivalence of the four conditions follows from Corollary 5.4. The final conclusion follows from Corollary 5.4(6).   □

9D. An observation on the center. Although we are mostly interested in the abelianization of a nilpotent subgroup H, we also record the following observation regarding the opposite end of a central series of H.

COROLLARY 9.9. *Let H be a nilpotent subgroup of a locally compact group G. If there is a nontrivial subgroup L of H, such that (G, L) has relative Property (T), then there is a nontrivial subgroup Z of the center of H, such that (G, Z) has relative Property (T).*

Proof. Consider the ascending central series of H:

$$\{e\} = Z_0 \subset Z_1 \subset \cdots \subset Z_c = H.$$

Let k be minimal, such that (G, L) has relative Property (T), for some closed, nontrivial subgroup L of Z_k. We may assume $L \nsubseteq Z(H)$, so there is some i

(which we choose to be minimal), such that $[L, Z_i] \neq \{e\}$. Choose $h \in Z_i$, such that $[L, h] \neq \{e\}$. Then $[L, h] \subseteq [H, Z_i] \subseteq Z_{i-1}$, so the minimality of i implies that $[L, h]$ centralizes L. Therefore $[\ell_1 \ell_2, h] = [\ell_1, h][\ell_2, h]$ for $\ell_1, \ell_2 \in L$, so $[L, h]$ is a subgroup. Also, $[L, h] \subseteq L \cdot hLh^{-1}$ has relative Property (T) by Lemma 6.5. Since $\{e\} \neq [L, h] \subseteq [Z_k, H] \subseteq Z_{k-1}$, this contradicts the minimality of k. □

REMARK 9.10. The proof of Corollary 9.9 establishes the following general fact about subgroups of a nilpotent group: If L is a nontrivial subgroup of a nilpotent group H, then there exist finitely many conjugates of L, such that the product of these conjugates contains a nontrivial subgroup of the center of H.

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