# Lattices in Crystalline Representations and Kisin Modules Associated with Iterate Extensions

Yoshiyasu Ozeki

Abstract. Cais and Liu extended the theory of Kisin modules and crystalline representations to allow more general coefficient fields and lifts of Frobenius. Based on their theory, we classify lattices in crystalline representations by Kisin modules with additional structures under a Cais-Liu's setting. Furthermore, we give a geometric interpretation of Kisin modules of height one in terms of Dieudonné crystals of $p$-divisible groups, and show a full faithfulness theorem for a restriction functor on torsion crystalline representations.

2010 Mathematics Subject Classification: 11F85, 11S20.
Keywords and Phrases: crystalline representations, Kisin modules, iterate extensions.

## Contents

## 1   Introduction

Let $K$ be a complete discrete valuation field of mixed characteristics $(0, p)$ with perfect residue field $k$. Let $\overline{K}$ be an algebraic closure of $K$ and $G := \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group of $K$. Let $e$ be the absolute ramification index of $K$ and $r \geq 0$ an integer. It is known that to classify $G$-stable lattices in semi-stable or crystalline representations by some linear data is one of the powerful tools for studies of various interesting problems such as Langlands correspondence. For this, the theory of *Kisin modules*, provided in [Kis], is very useful. Based on Kisin's theory, Liu [Li2] constructed a theory of $(\varphi, \hat{G})$-*modules*, which gives a categorical equivalence between them and a category of $G$-stable lattices in semi-stable representations with certain Hodge-Tate weights. One of the advantages of Liu's theory is that there are no restriction on $e$ and $r$ in his theory. Throughout Kisin and Liu's theory, the non-Galois "Kummer" extension $K_\infty/K$, obtained by adjoining a given compatible system of $p$-power roots of a uniformizer of $K$, plays a central role. Recently, Cais and Liu [CL] generalized Kisin's theory to the setting of many $f$-*iterate extension* $K_{\underline{\pi}}/K$. Here, the $f$-iterate extension $K_{\underline{\pi}}/K$ that we consider is defined as follows. Let $f(u) = u^p + a_{p-1}u^{p-1} + \cdots + a_1 u \in \mathbb{Z}_p[u]$ such that $f(u) \equiv u^p \bmod p\mathbb{Z}_p[u]$. We fix the choice of a uniformizer $\pi_0 = \pi$ of $K$ and $\{\pi_n\}_{n \geq 0}$ such that $f(\pi_{n+1}) = \pi_n$. Then we set $K_{\underline{\pi}} := \bigcup_{n \geq 0} K(\pi_n)$. Thus Kisin's theory is the case where $f(u) = u^p$.

The aim of this paper is to establish the theory of "crystalline" $(\varphi, \hat{G})$-modules under the Cais-Liu's setting, and apply it to a study of torsion crystalline representations. In Section 3.2, following [Li2], we define a notion of $(\varphi, \hat{G})$-modules of height $r$. We show in Theorem 3.7 that, under some mild assumptions, there exists an anti-equivalence between the category of $(\varphi, \hat{G})$-modules of height $r$ (with an additional condition) and the category of $G$-stable lattices in crystalline $\mathbb{Q}_p$-representations with Hodge-Tate weights in $[0, r]$.

As a consequence of our arguments, we can prove a full faithfulness theorem on torsion crystalline representations. Let $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G)$ be the category of torsion crystalline representations of $G$ with Hodge-Tate weights in $[0, r]$. Here, a torsion $\mathbb{Z}_p$-representation $T$ is *torsion crystalline with Hodge-Tate weights in* $[0, r]$ if $T$ is a quotient of lattices in a crystalline $\mathbb{Q}_p$-representation with Hodge-Tate weights in $[0, r]$. It is well-known that the condition that $T$ is torsion crystalline with Hodge-Tate weights in $[0, 1]$ is equivalent to the condition that $T$ is flat in the sense that $T$ is of the form $H(\overline{K})$ where $H$ is a finite flat group

scheme over $\mathcal{O}_{\overline{K}}$ killed by some power of $p$. The theorem below is a torsion analogue of Theorem 1.0.2 of [CL].

THEOREM 1.1 (= Theorems 4.1 and 4.2). *Under some technical assumptions (see Theorems 4.1 and 4.2 for details), the restriction functor $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G) \to \mathrm{Rep}_{\mathrm{tor}}(G_{\underline{\pi}})$ is fully faithful if $e(r-1) < p-1$.*

In the case $f(u) = u^p$, this is Theorem 1.2 of [Oz2]. In this case, previous results have been given by some mathematicians. The theorem was first studied by Breuil for $e = 1$ and $r < p-1$ via the Fontaine-Laffaille theory ([Br1], the proof of Théorèm 5.2). He also proved the theorem for $p > 2$ and $r \leq 1$ as a consequence of a study of the category of finite flat group schemes ([Br2, Theorem 3.4.3]). Later, his result was extended to the case $p = 2$ in [Kim], [La], [Li3] (proved independently). Based on studies of ramification bounds for torsion crystalline representations, Abrashkin proved the theorem in the case $[K : \mathbb{Q}_p] < \infty$, $e = 1$, $p > 2$ and $r < p$ ([Ab, Section 8.3.3]).
On the other hand, our arguments give an affirmative answer to a conjecture suggested in [CL, Remark 5.2.3 and Section 6.3] (in the case where "$F = \mathbb{Q}_p$"). Let $T$ be a $G$-stable lattice in a crystalline $\mathbb{Q}_p$-representation with Hodge-Tate weights in $[0,r]$. Cais-Liu constructed a Kisin module $\mathfrak{M}$ which corresponds to $T|_{G_{\underline{\pi}}}$, where $G_{\underline{\pi}}$ is the absolute Galois group of $K_{\underline{\pi}}$. This Kisin module $\mathfrak{M}$ depends on the choice of $(f(u), (\pi_n)_{n \geq 0})$. If we select another choice of $(f'(u), (\pi'_n)_{n \geq 0})$, then we obtain a different Kisin module $\mathfrak{M}'$. It seems natural to ask for the relationship between $\mathfrak{M}$ and $\mathfrak{M}'$. For this, we show[1]

THEOREM 1.2 (= Corollary 3.22 and Theorem 3.24). *Let the notation be as above. Assume $v_p(a_1) > \max\{r, 1\}$. Furthermore, we assume the condition (P) (cf. Section 3.2) if $r > 1$. Then the Kisin modules $\mathfrak{M}$ and $\mathfrak{M}'$ become isomorphic after base change to $W(R)$.*

Now we consider the case $r = 1$. In this case, Cais-Liu showed in [CL, Theorem 5.0.10] that there exists an anti-equivalence of categories between the category of Kisin modules of height 1 and the category of $p$-divisible groups over the ring of integers $\mathcal{O}_K$ of $K$. On the other hand, in the classical Kisin's setting $f(u) = u^p$, relationships between Kisin modules of height 1 and Dieudonné crystals are well-studied (cf. [Kis]). Combining these facts with the above theorem, we obtain a geometric interpretation of Kisin modules of height 1 for the Cais-Liu's setting.

COROLLARY 1.3 (= Corollary 3.26). *Assume $v_p(a_1) > 1$. Let $H$ be a $p$-divisible group over $\mathcal{O}_K$ and $\mathbb{D}(H)$ be the Dieudonné crystal attached to $H$. Let $\mathfrak{M}$ be the Kisin module attached to $H$. Then there is a functorial isomorphism $A_{\mathrm{cris}} \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M} \simeq \mathbb{D}(H)(A_{\mathrm{cris}})$.*

---

[1]We should note that the anonymous referee pointed out that Theorem 1.2 holds if we replace the assumptions "$v_p(a_1) > \max\{r, 1\}$ and (P)" with only one assumption "$v_p(a_1) > 1$" (see Section 3.7).

Notation : For any topological group $H$, a free $\mathbb{Z}_p$-representation of $H$ (resp. a $\mathbb{Q}_p$-representation of $H$) is a finitely generated free $\mathbb{Z}_p$-module equipped with a continuous $\mathbb{Z}_p$-linear $H$-action (resp. a finite dimensional $\mathbb{Q}_p$-vector space equipped with a continuous $\mathbb{Q}_p$-linear $H$-action). We denote by $\mathrm{Rep}_{\mathbb{Z}_p}(H)$ (resp. $\mathrm{Rep}_{\mathbb{Q}_p}(H)$) the category of them.
For any ring extension $A \subset B$ and any $A$-linear morphism of $A$-modules $f \colon M \to N$, we often abuse notations by writing $f \colon B \otimes_A M \to B \otimes_A N$ for the $B$-linear extension of $f$.

## 2   Preliminary

In this section, we define some basic notation, and we recall some results on iterate extensions given in [CL]. A lot of arguments in this section are deeply depending on [Li1, Sections 2 and 3]. It will be helpful for the reader to refer this reference.

### 2.1   Basic notation

Let $p \geq 2$ be a prime number. Let $K$ be a complete discrete valuation field of mixed characteristics $(0, p)$ with perfect residue field $k$. We denote by $e$ the absolute ramification index of $K$. Let $\overline{K}$ be an algebraic closure of $K$ and $\mathcal{O}_{\overline{K}}$ the integer ring of $\overline{K}$. We denote by $v_p$ the valuation of $\overline{K}$ normalized by $v_p(p) = 1$. We set $G := \mathrm{Gal}(\overline{K}/K)$, the absolute Galois group of $K$. We denote by $K_0$ the field $W(k)[1/p]$, which is the maximal absolutely unramified subfield of $K$.
We fix a uniformizer $\pi$ of $K$ and fix the choice of a system $(\pi_n)_{n \geq 0}$, where $\pi_0 = \pi$ and $f(\pi_{n+1}) = \pi_n$ for any $n \geq 0$. We also fix a polynomial $f(u) = \sum_{i=1}^{p} a_i u^i = u^p + a_{p-1} u^{p-1} + \cdots + a_1 u \in \mathbb{Z}_p[u]$ which satisfies $f(u) \equiv u^p \bmod p$. By an easy computation of the Newton polygon of $f(u) - \pi_{n-1}$, we see that $v_p(\pi_n) = 1/(ep^n)$ for any $n \geq 0$. We denote by $E(u)$ the minimal polynomial of $\pi$ over $K_0$.
Let $R = \varprojlim \mathcal{O}_{\overline{K}}/p$, where the transition maps are given by the $p$-th power map. This is a complete valuation ring with residue field $\overline{k}$. Let $v_R$ be a valuation of $R$ given by $v_R(x) := \lim_{n \to \infty} v_p(\hat{x}_n^{p^n})$ for $x = (x_n)_{n \geq 0} \in R$, where $\hat{x}_n \in \mathcal{O}_{\overline{K}}$ is any lift of $x_n$. Let $\mathfrak{m}_R$ be the maximal ideal of $R$ and set $\mathfrak{m}_R^{\geq c} := \{x \in R \mid v_R(x) \geq c\}$ for any real number $c \geq 0$. We set $\underline{\pi} := (\pi_n \bmod p\mathcal{O}_{\overline{K}})_{n \geq 0} \in R$. Note that $v_R(\underline{\pi}) = 1/e$. By [CL, Lemma 2.2.1], there exists a unique set-theoretic section $\{\cdot\}_f \colon R \to W(R)$ to the reduction modulo $p$ which satisfies $\varphi(\{x\}_f) = f(\{x\}_f)$ for all $x \in R$. The embedding $W(k)[u] \hookrightarrow W(R)$, given by $u \mapsto \{\underline{\pi}\}_f$, extends to a unique $W(k)$-algebra embedding $\mathfrak{S} := W(k)[\![u]\!] \hookrightarrow W(R)$. By this embedding, we identify $\mathfrak{S}$ with a $\varphi$-stable $W(k)$-subalgebra of $W(R)$. Let $\mathcal{O}_{\mathcal{E}}$ be

the $p$-adic completion of $\mathfrak{S}[1/u]$. This is a complete discrete valuation ring with residue field $k((u))$. Note that $p$ is a uniformizer of $\mathcal{O}_{\mathcal{E}}$. Let $\mathcal{E}$ be the fraction field of $\mathcal{O}_{\mathcal{E}}$. Then the embedding $\mathfrak{S} \hookrightarrow W(R)$ extends to $\mathcal{O}_{\mathcal{E}} \hookrightarrow W(\mathrm{Fr}R)$ and $\mathcal{E} \hookrightarrow W(\mathrm{Fr}R)[1/p]$. We denote by $\mathcal{E}^{\mathrm{ur}}$ the $p$-adic completion of the maximal unramified algebraic extension of $\mathcal{E}$, and denote by $\mathcal{O}^{\mathrm{ur}}$ the integer ring of $\mathcal{E}^{\mathrm{ur}}$. We may regard $\mathcal{E}^{\mathrm{ur}}$ and $\mathcal{O}^{\mathrm{ur}}$ as $\varphi$-stable subrings of $W(\mathrm{Fr}R)[1/p]$ and $W(\mathrm{Fr}R)$, respectively. We put $\mathfrak{S}^{\mathrm{ur}} = \mathcal{O}^{\mathrm{ur}} \cap W(R)$.

We set $K_{\underline{\pi}} := \bigcup_{n \geq 0} K(\pi_n)$ and denote by $G_{\underline{\pi}}$ the absolute Galois group of $K_{\underline{\pi}}$. The extension $K_{\underline{\pi}}/K$ is totally wildly ramified. Furthermore, it is shown in [CL, Lemmas 3.1.1 and 3.2.1] that the extension $K_{\underline{\pi}}/K$ is strictly APF in the sense of [Wi], and the $G_{\underline{\pi}}$-action on $R$ induces an isomorphism $G_{\underline{\pi}} \simeq G_{k((\underline{\pi}))} = G_{k((u))}$. Note that $G_{\underline{\pi}}$-action on $W(\mathrm{Fr}R)[1/p]$ preserves $\mathcal{E}^{\mathrm{ur}}$ and $\mathcal{O}^{\mathrm{ur}}$, and $G_{\underline{\pi}}$ acts on $\mathcal{E}$ and $\mathcal{O}_{\mathcal{E}}$ trivial.

Let $\nu \colon W(R) \twoheadrightarrow W(\overline{k})$ be the canonical projection induced by the projection $R \twoheadrightarrow \overline{k}$, which extends to a map $\nu \colon B_{\mathrm{cris}}^+ \to W(\overline{k})[1/p]$. Here, $B_{\mathrm{cris}}^+$ is the usual $p$-adic period ring of Fontaine (see [Fo2] for various $p$-adic period rings). For any subring $A$ of $B_{\mathrm{cris}}^+$, we set $\mathrm{Fil}^i A := A \cap \mathrm{Fil}^i B_{\mathrm{cris}}^+$. We also set

$$I_+ A := A \cap \ker \nu \quad \text{and}$$
$$I^{[1]} A := \{ x \in A \mid \varphi^n(x) \in \mathrm{Fil}^1 A \text{ for any } n \geq 0 \}.$$

Note that we have $I_+ A \supset I^{[1]} A$.

## 2.2 Étale $\varphi$-modules and Kisin modules

Let $\mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}$ (resp. $\mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}, \infty}$) be the category of finite free $\varphi$-modules $M$ over $\mathcal{O}_{\mathcal{E}}$ (resp. of finite type $\varphi$-modules $M$ over $\mathcal{O}_{\mathcal{E}}$ killed by a power of $p$) whose $\mathcal{O}_{\mathcal{E}}$-linearization $1 \otimes \varphi \colon \mathcal{O}_{\mathcal{E}} \otimes_{\varphi, \mathcal{O}_{\mathcal{E}}} M \to M$ is an isomorphism. We call objects of these categories *étale $\varphi$-modules*.

We define a $\mathbb{Z}_p$-representation of $G_{\underline{\pi}}$ for any étale $\varphi$-module $M$ by

$$T_{\mathcal{O}_{\mathcal{E}}}(M) := \begin{cases} \mathrm{Hom}_{\mathcal{O}_{\mathcal{E}}, \varphi}(M, \mathcal{O}^{\mathrm{ur}}) & \text{if } M \in \mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}, \\ \mathrm{Hom}_{\mathcal{O}_{\mathcal{E}}, \varphi}(M, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathcal{O}^{\mathrm{ur}}) & \text{if } M \in \mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}, \infty}. \end{cases}$$

Here, the $G_{\underline{\pi}}$-action on $T_{\mathcal{O}_{\mathcal{E}}}(M)$ is given by $(g.f)(x) := g(f(x))$ for $f \in T_{\mathcal{O}_{\mathcal{E}}}(M)$, $g \in G_{\underline{\pi}}$ and $x \in M$. Then we have a contravariant functor $T_{\mathcal{O}_{\mathcal{E}}} \colon \mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}} \to \mathrm{Rep}_{\mathbb{Z}_p}(G_{\underline{\pi}})$ and $T_{\mathcal{O}_{\mathcal{E}}} \colon \mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}, \infty} \to \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{tor}}(G_{\underline{\pi}})$. By [CL, Corollary 3.2.3], these two functors give equivalences of categories $\mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}} \simeq \mathrm{Rep}_{\mathbb{Z}_p}(G_{\underline{\pi}})$ and $\mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}, \infty} \simeq \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{tor}}(G_{\underline{\pi}})$.

For any integer $r \geq 0$, we denote by $'\mathrm{Mod}_{\mathfrak{S}}^r$ the category of finite type $\varphi$-modules $\mathfrak{M}$ over $\mathfrak{S}$ which are *of height $r$* in the sense that the cokernel of the $\mathfrak{S}$-linearization $1 \otimes \varphi_{\mathfrak{M}} \colon \mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \to \mathfrak{M}$ of $\varphi_{\mathfrak{M}}$ is killed by $E(u)^r$. A $\varphi$-modules $\mathfrak{M}$ is *$p'$-torsion free* if, for any non-zero element $x \in \mathfrak{M}$, $\mathrm{Ann}_{\mathfrak{S}}(x)$ is $0$ or $p^n \mathfrak{S}$ for some $n$. If $\mathfrak{M}$ is killed by some power of $p$, then we can check that $\mathfrak{M}$ is $p'$-torsion free if and only if $\mathfrak{M}$ is $u$-torsion free. We denote by $\mathrm{Mod}_{\mathfrak{S}}^r$ the full

subcategory of $'\mathrm{Mod}_{\mathfrak{S}}^r$ consisting of those objects which are finite and free over
$\mathfrak{S}$. We also denote by $\mathrm{Mod}_{\mathfrak{S}_\infty}^r$ the full subcategory of $'\mathrm{Mod}_{\mathfrak{S}}^r$ consisting of
those objects which are $p'$-torsion and killed by a power of $p$. We call objects
of $\mathrm{Mod}_{\mathfrak{S}}^r$ or $\mathrm{Mod}_{\mathfrak{S}_\infty}^r$ free Kisin modules or torsion Kisin modules, respectively.
If $\mathfrak{M}$ is a Kisin module, then one can check that $\mathcal{O}_\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}$ is an étale $\varphi$-module.
We describe standard linear algebraic properties of Kisin modules.

PROPOSITION 2.1. *Let* $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ *be an exact sequence of*
$\varphi$-*modules over* $\mathfrak{S}$. *If* $\mathfrak{M}', \mathfrak{M}$ *and* $\mathfrak{M}''$ *are of finite type and* $p'$-*torsion free and*
$\mathfrak{M}$ *is of height* $r$, *then* $\mathfrak{M}'$ *and* $\mathfrak{M}$ *are of height* $r$.

*Proof.* See Propositions B. 1.3.3 and B. 1.3.5 of [Fo1].                                    □

PROPOSITION 2.2. *Let* $\mathfrak{M} \in {}'\mathrm{Mod}_{\mathfrak{S}}^r$ *be killed by a power of* $p$. *Then the following*
*are equivalent.*

   (1) $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^r$,

   (2) *the natural map* $\mathfrak{M} \to \mathcal{O}_\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}$ *is injective,*

   (3) *there exists an increasing sequence*

$$0 = \mathfrak{M}_0 \subset \mathfrak{M}_1 \subset \mathfrak{M}_2 \subset \cdots \subset \mathfrak{M}_n = \mathfrak{M}$$

   *of* $\varphi$-*modules over* $\mathfrak{S}$ *such that, for each* $i$, $\mathfrak{M}_i/\mathfrak{M}_{i-1}$ *is finite free over*
   $k[\![u]\!]$ *and* $\mathfrak{M}_i/\mathfrak{M}_{i-1} \in {}'\mathrm{Mod}_{\mathfrak{S}}^r$,

   (4) $\mathfrak{M}$ *is a quotient of two finite free* $\mathfrak{S}$-*modules* $\mathfrak{N}'$ *and* $\mathfrak{N}''$ *with* $\mathfrak{N}', \mathfrak{N}'' \in$
   $\mathrm{Mod}_{\mathfrak{S}}^r$.

*Moreover, if this is the case,* $\mathfrak{M}_i$ *and* $\mathfrak{M}_i/\mathfrak{M}_{i-1}$ *are objects of* $\mathrm{Mod}_{\mathfrak{S}_\infty}^r$ *for each*
*i.*

*Proof.* The same proof as [Li1, Proposition 2.3.2] proceeds.                                    □

COROLLARY 2.3. *Let* $A$ *be a* $p$-*torsion free* $\mathfrak{S}$-*algebra and* $\mathfrak{M}$ *a Kisin module.*
*Then we have* $\mathrm{Tor}_1^{\mathfrak{S}}(\mathfrak{M}, A) = 0$. *In particular, the functor from the category of*
*Kisin modules to the category of* $A$-*modules defined by* $\mathfrak{M} \mapsto A \otimes_{\mathfrak{S}} \mathfrak{M}$ *is exact.*

*Proof.* By Proposition 2.2 and *dévissage* argument, we can reduce a proof to
the case where $\mathfrak{M}$ is killed by a power of $p$. In this case, $\mathfrak{M}$ is a free $k[\![u]\!]$-module
of finite rank. Thus it suffices to show $\mathrm{Tor}_1^{\mathfrak{S}}(k[\![u]\!], A) = 0$. This equality in fact
follows from the assumption that $A$ is $p$-torsion free.                                    □

By this proposition, the following corollaries immediately follow:

COROLLARY 2.4. *Let* $\mathfrak{M}$ *be a Kisin module. Let* $A \subset B$ *be a ring extension of*
$p$-*torsion free* $\mathfrak{S}$-*algebras such that the natural map* $A/pA \to B/pB$ *is injective.*
*Then the natural map* $A \otimes_{\mathfrak{S}} \mathfrak{M} \to B \otimes_{\mathfrak{S}} \mathfrak{M}$ *is injective.*

COROLLARY 2.5. *Let $\mathfrak{M}$ be a Kisin module and $\mathfrak{N}$ a $\varphi$-module over $\mathfrak{S}$ with $\mathfrak{M} \subset \mathfrak{N}$. Let $\mathfrak{S} \subset A \subset W(\mathrm{Fr}R)$ be ring extensions. Suppose that such that the natural map $A/pA \to \mathrm{Fr}R$ is injective.*
*(1) The natural map $A \otimes_{\mathfrak{S}} \mathfrak{M} \to A \otimes_{\mathfrak{S}} \mathfrak{N}$ is injective.*
*(2) If $A$ is $\varphi$-stable, then the natural map $A \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to A \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}$ is injective.*

We define a $\mathbb{Z}_p$-representation of $G_{\underline{\pi}}$ for any Kisin module $\mathfrak{M}$ by

$$T_{\mathfrak{S}}(\mathfrak{M}) := \begin{cases} \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}) & \text{if } \mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}}^r, \\ \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathfrak{S}^{\mathrm{ur}}) & \text{if } \mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^r. \end{cases}$$

Here, the $G_{\underline{\pi}}$-action on $T_{\mathfrak{S}}(\mathfrak{M})$ is given by $(g.f)(x) := g(f(x))$ for $f \in T_{\mathfrak{S}}(\mathfrak{M})$, $g \in G_{\underline{\pi}}$ and $x \in \mathfrak{M}$. If $\mathfrak{M}$ is a Kisin module, then $M := \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ is an étale $\varphi$-module.

PROPOSITION 2.6. *(1) Let $\mathfrak{M}$ be a Kisin module and put $M = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$. Then, we have a canonical isomorphism $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T_{\mathcal{O}_{\mathcal{E}}}(M)$ of $\mathbb{Z}_p[G_{\underline{\pi}}]$-modules.*
*(2) Let $\mathfrak{M}$ be a free (resp. torsion) Kisin module. Then the inclusion $\mathfrak{S}^{\mathrm{ur}} \hookrightarrow W(R)$ induces a natural isomorphism $T_{\mathfrak{S}}(\mathfrak{M}) \simeq \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, W(R))$ (resp. $T_{\mathfrak{S}}(\mathfrak{M}) \simeq \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R)))$ of $\mathbb{Z}_p[G_{\underline{\pi}}]$-modules.*
*(3) Assume that $\varphi^n(f(u)/u)$ is not a power of $E(u)$ for any $n \geq 0$. Then the contravariant functor $T_{\mathfrak{S}} : \mathrm{Mod}_{\mathfrak{S}}^r \to \mathrm{Rep}_{\mathbb{Z}_p}(G_{\underline{\pi}})$ is fully faithful.*
*(4) The contravariant functors $T_{\mathfrak{S}} : \mathrm{Mod}_{\mathfrak{S}}^r \to \mathrm{Rep}_{\mathbb{Z}_p}(G_{\underline{\pi}})$ and $T_{\mathfrak{S}} : \mathrm{Mod}_{\mathfrak{S}_\infty}^r \to \mathrm{Rep}_{\mathrm{tor}}(G_{\underline{\pi}})$ are exact and faithful.*
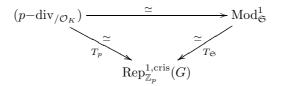
*Proof.* Assertions (1) and (2) for free Kisin modules are [CL, Proposition 3.3.1], and a proof for the torsion case is essentially the same. For this, the proof of [Li1, Corollary 2.2.2] is helpful for the readers. The assertion (3) is [CL, Proposition 3.3.5]. To show (4), it suffices to show that $\mathrm{Mod}_{\mathfrak{S}}^r \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{E}}}$ and $\mathrm{Mod}_{\mathfrak{S}_\infty}^r \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{E}},\infty}$ given by $\mathfrak{M} \to \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ are exact and faithful. The exactness follows from the fact that the inclusion map $\mathfrak{S} \to \mathcal{O}_{\mathcal{E}}$ is flat. The faithfulness follows from Proposition 2.2 (2) or Corollary 2.4. $\square$

Let $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G)$ be the category of torsion crystalline representations of $G$ with Hodge-Tate weights in $[0, r]$. Here, a torsion $\mathbb{Z}_p$-representation $T$ is *torsion crystalline with Hodge-Tate weights in $[0, r]$* if $T$ is a quotient of lattices in a crystalline $\mathbb{Q}_p$-representation with Hodge-Tate weights in $[0, r]$. The following is the main results of Section 5 of [CL].

THEOREM 2.7 ([CL], Theorem 1.0.3). *Assume $v_p(a_1) > 1$. Then there exists an anti-equivalence of categories between the category $\mathrm{Mod}_{\mathfrak{S}}^1$ of free Kisin modules of height 1 and the category $(p-\mathrm{div}_{/\mathcal{O}_K})$ of $p$-divisible groups over the ring of integers $\mathcal{O}_K$ of $K$. If $\mathfrak{M}$ is a free Kisin module of height 1, then the $G_{\underline{\pi}}$-action on $T_{\mathfrak{S}}(\mathfrak{M})$ naturally extends to $G$. This induces an anti-equivalence of categories between $\mathrm{Mod}_{\mathfrak{S}}^1$ and $\mathrm{Rep}_{\mathbb{Z}_p}^{1,\mathrm{cris}}(G)$. Moreover, the following diagram is commutative:*

Assume that $v_p(a_1) > 1$. Let $\mathfrak{S}(1)$ be the free Kisin module of rank 1 corresponding to $\mathbb{Z}_p(1)$ via Theorem 2.7. Let $\mathfrak{e}_{(1)}$ be a generator of $\mathfrak{S}(1)$. By [CL, Lemma 5.2.1 (2)], we have $\varphi(\mathfrak{e}_{(1)}) = \mu_0 E(u)\mathfrak{e}_{(1)}$ for some $\mu_0 \in \mathfrak{S}^{\times}$.

Cartier duality. Here we give a Cartier duality theorem for étale $\varphi$-modules and Kisin modules. Since arguments here are completely the same as [Li1, Section 3.1], we only give a brief sketch here. We fix an integer $r \geq 0$. A lot of notion in this subsection depend on the choice of $r$ but we omit it from subscripts for an abbreviation.

Assume that $v_p(a_1) > 1$. Let $\mu_0 \in \mathfrak{S}^{\times}$ be as in the previous section. Let $\mathfrak{S}^{\vee}$ be the free Kisin module of rank 1 such that $\varphi(\mathfrak{e}) = (\mu_0 E(u))^r \mathfrak{e}$. Here $\mathfrak{e}$ is a generator of $\mathfrak{S}^{\vee}$. (Clearly, we have $\mathfrak{S}^{\vee} = \mathfrak{S}(1)$ if $r = 1$.) We see that $\mathfrak{S}^{\vee}$ is of height $r$. We set $\mathcal{O}_{\mathcal{E}}^{\vee} := \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{S}^{\vee}$, which is an étale $\varphi$-module. Note that we have isomorphisms $T_{\mathcal{O}_{\mathcal{E}}}(\mathcal{O}_{\mathcal{E}}^{\vee}) \simeq T_{\mathfrak{S}}(\mathfrak{S}^{\vee}) \simeq \mathbb{Z}_p(r)$. For any Kisin module $\mathfrak{M}$, we define an $\mathfrak{S}$-module $\mathfrak{M}^{\vee}$ by

$$\mathfrak{M}^{\vee} := \begin{cases} \operatorname{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}) & \text{if } \mathfrak{M} \in \operatorname{Mod}_{\mathfrak{S}}^r, \\ \operatorname{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}_{\infty}) & \text{if } \mathfrak{M} \in \operatorname{Mod}_{\mathfrak{S}_{\infty}}^r. \end{cases}$$

For any étale $\varphi$-module $M$, we define an $\mathcal{O}_{\mathcal{E}}$-module $M^{\vee}$ by

$$M^{\vee} := \begin{cases} \operatorname{Hom}_{\mathcal{O}_{\mathcal{E}}}(M, \mathcal{O}_{\mathcal{E}}) & \text{if } M \in \operatorname{Mod}_{\mathcal{O}_{\mathcal{E}}}, \\ \operatorname{Hom}_{\mathcal{O}_{\mathcal{E}}}(M, \mathcal{O}_{\mathcal{E},\infty}) & \text{if } M \in \operatorname{Mod}_{\mathcal{O}_{\mathcal{E},\infty}}. \end{cases}$$

We then have canonical parings

$$\langle \cdot, \cdot \rangle \colon \mathfrak{M} \times \mathfrak{M}^{\vee} \to \mathfrak{S}^{\vee} \qquad \text{if } \mathfrak{M} \in \operatorname{Mod}_{\mathfrak{S}}^r,$$
$$\langle \cdot, \cdot \rangle \colon \mathfrak{M} \times \mathfrak{M}^{\vee} \to \mathfrak{S}_{\infty}^{\vee} \qquad \text{if } \mathfrak{M} \in \operatorname{Mod}_{\mathfrak{S}_{\infty}}^r$$

and

$$\langle \cdot, \cdot \rangle \colon M \times M^{\vee} \to \mathcal{O}_{\mathcal{E}}^{\vee} \qquad \text{if } M \in \operatorname{Mod}_{\mathcal{O}_{\mathcal{E}}},$$
$$\langle \cdot, \cdot \rangle \colon M \times M^{\vee} \to \mathcal{O}_{\mathcal{E},\infty}^{\vee} \qquad \text{if } M \in \operatorname{Mod}_{\mathcal{O}_{\mathcal{E},\infty}}.$$

Proposition 2.8. *Assume that $v_p(a_1) > 1$.*
*(1) There exist a unique $\varphi$-semi-linear map $\varphi_{M^{\vee}} \colon M^{\vee} \to M^{\vee}$ which satisfies the following:*

 *(a) $(M^{\vee}, \varphi_{M^{\vee}})$ is an étale $\varphi$-module,*

 *(b) $\varphi_{M^{\vee}}$ is compatible with the pairing $\langle \cdot, \cdot \rangle$ for $M$,*

(c) $T_{\mathcal{O}_{\mathcal{E}}}(M^\vee) \simeq T_{\mathcal{O}_{\mathcal{E}}}(M)^\vee(r)$.

(2) *Suppose that* $M = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$. *There exist a unique* $\varphi$-*semi-linear map* $\varphi_{\mathfrak{M}^\vee} : \mathfrak{M}^\vee \to \mathfrak{M}^\vee$ *which satisfies the following:*

(a) $(\mathfrak{M}^\vee, \varphi_{\mathfrak{M}^\vee})$ *is a Kisin module of height* $r$,

(b) $\varphi_{M^\vee} = 1 \otimes \varphi_{\mathfrak{M}^\vee}$. *In particular,* $\varphi_{M^\vee}$ *is compatible with the pairing* $\langle \cdot, \cdot \rangle$ *for* $\mathfrak{M}$,

(c) $T_{\mathfrak{S}}(\mathfrak{M}^\vee) \simeq T_{\mathfrak{S}}(\mathfrak{M})^\vee(r)$.

*Proof.* The same proof as [Li1, Section 3.1] proceeds. $\qquad\square$

Comparison morphism of Kisin modules. We define a comparison morphism between Kisin modules and their representations. Precise arguments are given in [Li1, Section 3.2].
Let $\mathfrak{M}$ be a Kisin module of height $r$. We define a $W(R)$-linear map $\iota_{\mathfrak{S}} : W(R) \otimes_{\mathfrak{S}} \mathfrak{M} \to W(R) \otimes_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathfrak{M})^\vee$ by the composite

$$W(R) \otimes_{\mathfrak{S}} \mathfrak{M} \to \mathrm{Hom}_{\mathbb{Z}_p}(T_{\mathfrak{S}}(\mathfrak{M}), W(R)) \simeq W(R) \otimes_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathfrak{M})^\vee,$$

where the first map is given by $x \mapsto (f \mapsto f(x))$ and the second is the natural map. It is not difficult to check that $\iota_{\mathfrak{S}}$ is $\varphi$-equivalent and $G_{\underline{\pi}}$-equivalent.
Assume that $v_p(a_1) > 1$. Take any generator $f$ of $T_{\mathfrak{S}}(\mathfrak{S}(1))$ and set $\mathfrak{t} := f(\mathfrak{e}_{(1)}) \in W(R)$. Since $f$ is compatible with $\varphi$ and is a generator of $T_{\mathfrak{S}}(\mathfrak{S}(1))$, we see

$$\varphi(\mathfrak{t}) = \mu_0 E(u)\mathfrak{t} \quad \text{and} \quad \mathfrak{t} \in W(R) \smallsetminus pW(R).$$

Such $\mathfrak{t}$ is unique up to multiplication by $\mathbb{Z}_p^\times$ and is independent of the choice of $f$.

PROPOSITION 2.9. *Assume that* $v_p(a_1) > 1$. *There exist natural* $W(R)$-*linear morphisms*

$$\iota_{\mathfrak{S}} : W(R) \otimes_{\mathfrak{S}} \mathfrak{M} \to W(R) \otimes_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathfrak{M})^\vee$$

*and*

$$\iota_{\mathfrak{S}}^\vee : W(R)^\vee \otimes_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathfrak{M})^\vee \to W(R)(-r) \otimes_{\mathfrak{S}} \mathfrak{M}$$

*which satisfy the following:*

(1) $\iota_{\mathfrak{S}}$ *and* $\iota_{\mathfrak{S}}^\vee$ *are* $\varphi$-*equivalent and* $G_{\underline{\pi}}$-*equivalent.*

(2) *If we identify* $W(R)^\vee = W(R)(-r) = W(R)$, *then we have* $\iota_{\mathfrak{S}}^\vee \circ \iota_{\mathfrak{S}} = \mathfrak{t}^r \otimes Id_{\mathfrak{M}}$ *and* $\iota_{\mathfrak{S}} \circ \iota_{\mathfrak{S}}^\vee = \mathfrak{t}^r \otimes Id_{T_{\mathfrak{S}}(\mathfrak{M})^\vee}$.

*Proof.* The proof is completely the same as that of [Li1, Theorem 3.2.2]. $\qquad\square$

COROLLARY 2.10. *Assume that* $v_p(a_1) > 1$. *The maps* $\iota_{\mathfrak{S}}$ *and* $\iota_{\mathfrak{S}}^\vee$ *are injective, and we have* $\mathfrak{t}^r(W(R) \otimes_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathfrak{M})^\vee) \subset \mathrm{Im}(\iota_{\mathfrak{S}})$ *and* $\mathfrak{t}^r(W(R)(-r) \otimes_{\mathfrak{S}} \mathfrak{M}) \subset \mathrm{Im}(\iota_{\mathfrak{S}}^\vee)$.

## 3   Lattices in crystalline representations

In this section, we study Galois actions on Kisin modules which corresponds to crystalline representations. It gives an anti-equivalence between a category of Kisin modules with certain Galois actions and a category of lattices in crystalline representations with some Hodge-Tate weights.

### 3.1   $(\varphi, \hat{G})$-modules

Let $\widehat{K}_{\underline{\pi}}/K$ be the Galois closure of the extension $K_{\underline{\pi}}/K$ and put $\hat{G} = \mathrm{Gal}(\overline{K}/\widehat{K}_{\underline{\pi}})$. Following [CL], we set $\mathfrak{O}_\alpha := \mathfrak{S}[\![\frac{E(u)^p}{p}]\!][1/p] \subset B_{\mathrm{cris}}^+$. It is not difficult to check $I_+ \mathfrak{O}_\alpha = u \mathfrak{O}_\alpha$ and $\mathfrak{O}_\alpha/I_+\mathfrak{O}_\alpha \simeq K_0$. We note that we have $\mathfrak{S}[\![\frac{E(u)^p}{p}]\!] = \mathfrak{S}[\![\frac{u^{ep}}{p}]\!] \subset A_{\mathrm{cris}}$ and $\mathfrak{S}[\![\frac{E(u)^p}{p}]\!]$ is $p$-adically complete and $\varphi$-stable. *In the rest of this paper, we fix the choice of a $K_0$-subalgebra $\mathcal{R}_{K_0}$ of $B_{\mathrm{cris}}^+$* which satisfies the following properties:

- $\mathfrak{O}_\alpha \subset \mathcal{R}_{K_0}$ and $\nu(\mathcal{R}_{K_0}) = K_0$,

- $\mathcal{R}_{K_0} \subset B_{\mathrm{cris}}^+$ is stable under $\varphi$ and $G$-actions, and

- the $G$-action on $\mathcal{R}_{K_0}$ factors through $\hat{G}$.

REMARK 3.1. (1) Such $\mathcal{R}_{K_0}$ exists. In fact, the $K_0$-subalgebra of $B_{\mathrm{cris}}^+$ generated by $\{gx \mid g \in G, x \in \mathfrak{O}_\alpha\}$ satisfies all the desired properties.
(2) In the classical setting $f(u) = u^p$, an explicitly described $\mathcal{R}_{K_0}$ has been considered. For this, see [Li2].

We set $\widehat{\mathcal{R}} := \mathcal{R}_{K_0} \cap W(R)$. By definition, we see that $\widehat{\mathcal{R}} \subset W(R)$ is stable under $\varphi$ and $G$-actions, the $G$-action on $\widehat{\mathcal{R}}$ factors through $\hat{G}$, and the map $\nu$ induces isomorphisms $\mathcal{R}_{K_0}/I_+\mathcal{R}_{K_0} \simeq K_0$ and $\widehat{\mathcal{R}}/I_+\widehat{\mathcal{R}} \simeq W(k)$.

DEFINITION 3.2. A $(\varphi, \hat{G})$-*module* (*of height $r$*) is a triple $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G})$ where

(1) $(\mathfrak{M}, \varphi)$ is a free Kisin module $\mathfrak{M}$ of height $r$,

(2) $\hat{G}$ is an $\widehat{\mathcal{R}}$-semi-linear continuous[2] $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$,

(3) the $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ commutes with $\varphi_{\widehat{\mathcal{R}}} \otimes \varphi_{\mathfrak{M}}$, and

(4) $\varphi^* \mathfrak{M} \subset (\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M})^{G_{\underline{\pi}}}$.

We denote by $\mathrm{Mod}_{\mathfrak{S}}^{r, \hat{G}}$ the category of $(\varphi, \hat{G})$-modules of height $r$.

---

[2]This means that the $G$-action on $W(R) \otimes_{\widehat{\mathcal{R}}} (\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}) = W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ induced by the $\hat{G}$-action on $\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ is continuous with respect to the weak topology of $W(R)$.

We define a $\mathbb{Z}_p$-representation $\hat{T}(\hat{\mathfrak{M}})$ of $G$ for any $(\varphi, \hat{G})$-module $\hat{\mathfrak{M}}$ by

$$\hat{T}(\hat{\mathfrak{M}}) := \mathrm{Hom}_{\widehat{\mathcal{R}}, \varphi}(\widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, W(R)).$$

Here, the $G$-action on $\hat{T}(\hat{\mathfrak{M}})$ is given by $(g.f)(x) := g(f(g^{-1}(x)))$ for $f \in \hat{T}(\hat{\mathfrak{M}})$, $g \in G$ and $x \in \widehat{\mathcal{R}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$. Note that we have a natural isomorphism of $\mathbb{Z}_p[G_{\underline{\pi}}]$-modules

$$\theta \colon T_{\mathfrak{S}}(\mathfrak{M}) \xrightarrow{\sim} \hat{T}(\hat{\mathfrak{M}})$$

given by $\theta(f)(a \otimes x) := a\varphi(f(x))$ for $f \in T_{\mathfrak{S}}(\hat{\mathfrak{M}})$, $a \in \widehat{\mathcal{R}}$ and $x \in \mathfrak{M}$ (see the proof of [Li2, Theorem 2.3.1 (1)]). In particular, $\hat{T}(\hat{\mathfrak{M}})$ is a free $\mathbb{Z}_p$-module of rank $d$, where $d := \mathrm{rank}_{\mathfrak{S}} \mathfrak{M}$. Hence we obtain a contravariant functor

$$\hat{T} \colon \mathrm{Mod}_{\mathfrak{S}}^{r, \hat{G}} \to \mathrm{Rep}_{\mathbb{Z}_p}(G).$$

Note also that we have a canonical isomorphism $\hat{T}(\hat{\mathfrak{M}}) \simeq \mathrm{Hom}_{W(R), \varphi}(W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, W(R))$.

Following [CL], we set $\tilde{B}_\alpha := W(R)\llbracket \frac{E(u)^p}{p} \rrbracket[1/p] = W(R)\llbracket \frac{u^{pe}}{p} \rrbracket[1/p]$, which is a subring of $B_{\mathrm{cris}}^+$, stable under $\varphi$ and $G_K$-actions.

DEFINITION 3.3. (1) We denote by $'\mathrm{Mod}_{\mathfrak{S}}^{r, \hat{G}, \mathrm{cris}}$ the full subcategory of $\mathrm{Mod}_{\mathfrak{S}}^{r, \hat{G}}$ consisting of objects $\hat{\mathfrak{M}}$ which satisfy the following condition: For any $g \in \hat{G}$ and $x \in \mathfrak{M}$, we have

$$g(1 \otimes x) - (1 \otimes x) \in \varphi(gu - u)\tilde{B}_\alpha \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}.$$

(Note that, if this is the case, $g(1 \otimes x) - (1 \otimes x)$ is in fact contained in $\varphi(gu - u)\tilde{B}_\alpha \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \cap I^{[1]}W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ since we have $\varphi(gu - u) \in I^{[1]}W(R)$ by [CL, Lemma 2.3.2].)

(2) We denote by $\mathrm{Mod}_{\mathfrak{S}}^{r, \hat{G}, \mathrm{cris}}$ the full subcategory of $\mathrm{Mod}_{\mathfrak{S}}^{r, \hat{G}}$ consisting of objects $\hat{\mathfrak{M}}$ which satisfy the following condition: For any $g \in \hat{G}$ and $x \in \mathfrak{M}$, we have

$$g(1 \otimes x) - (1 \otimes x) \in \varphi(gu - u)B_{\mathrm{cris}}^+ \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}.$$

(Note that, if this is the case, $g(1 \otimes x) - (1 \otimes x)$ is in fact contained in $\varphi(gu - u)B_{\mathrm{cris}}^+ \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \cap I^{[1]}W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ since we have $\varphi(gu - u) \in I^{[1]}W(R)$ by [CL, Lemma 2.3.2].)

By definition, the category $'\mathrm{Mod}_{\mathfrak{S}}^{r, \hat{G}, \mathrm{cris}}$ is a full subcategory of $\mathrm{Mod}_{\mathfrak{S}}^{r, \hat{G}, \mathrm{cris}}$.

REMARK 3.4. To understand $(\varphi, \hat{G})$-module, it is very important to study the structure of the Galois group $\hat{G}$ and to find a "good choice" of $\mathcal{R}_{K_0}$. In the classical Kisin's setting $f(u) = u^p$, these are well studied. For this, see [Li2]. We should remark that in this classical setting, we may consider $(\varphi, \hat{G})$-modules as "linear data" like $(\varphi, \Gamma)$-modules. In fact, $\hat{G}$ is topologically generated by $\mathrm{Gal}(\widehat{K}_\pi/K_\pi)$ and a (fixed) generator $\tau$ of $\mathrm{Gal}(\widehat{K}_\pi/K(\mu_{p^\infty}))$. Here, $\mu_{p^\infty}$ is the set of $p$-power roots of unity. Hence the $\hat{G}$-action on a $(\varphi, \hat{G})$-module is essentially determined by the $\tau$-action only.

REMARK 3.5. To understand objects of the category $\mathrm{Mod}_{\mathfrak{S}}^{r,\hat{G},\mathrm{cris}}$, studying ideals $I'_g\varphi(gu-u)\tilde{B}_\alpha \cap I^{[1]}W(R)$ or $I_g := \varphi(gu-u)B_{\mathrm{cris}}^+ \cap I^{[1]}W(R)$ of $W(R)$ must be important. However, it is not so easy (at least for the author). Later, we give partial results in Propositions 4.18 and 4.19. Here we describe some known facts about $I_g$ and give some remarks.

(1) Suppose $v_p(a_1) > 1$. Then we can check $I_g \subset I^{[1+]}W(R)$ as follows: Let $\mathfrak{t}$ be as in the previous section. It follows from $\varphi(\mathfrak{t}) = \mu_0 E(u)\mathfrak{t}$ and [CL, Lemma 2.3.1 (2)] that $\mathfrak{t}$ is not in $I^{[1]}W(R)$ and $\varphi(\mathfrak{t})$ is a generator of $I^{[1]}W(R)$. Take $x = \varphi(gu-u)y = \varphi(\mathfrak{t})z$ with $y \in B_{\mathrm{cris}}^+$ and $z \in W(R)$. It suffices to show $z \in I_+W(R)$. By [CL, Lemma 2.3.2] (see also Proposition 3.11), we have $gu - u \in \varphi(\mathfrak{t})I_+W(R)$. This implies $\varphi(gu-u) \in \varphi^2(\mathfrak{t})I_+W(R) = \varphi(E(u))\varphi(\mathfrak{t})I_+W(R) \subset \varphi(\mathfrak{t})I_+W(R)$, and thus we obtain $z = \varphi(gu-u)y/\varphi(\mathfrak{t}) \in W(R) \cap I_+B_{\mathrm{cris}}^+ = I_+W(R)$ as desired.

(2) (Kisin's setting) If $f(u) = u^p$, then we can show that $I_g \subset u^pI^{[1]}W(R)$ as follows: Since $gu - u \in uW(R)$ in this case. it suffices to show $u^pB_{\mathrm{cris}}^+ \cap I^{[1]}W(R) \subset u^pI^{[1]}W(R)$. Take any $x = u^py \in u^pB_{\mathrm{cris}}^+ \cap I^{[1]}W(R)$. By [Li3, Lemma 3.2.2], $u^py \in W(R)$ shows $y \in W(R)$. On the other hand, $u^py \in I^{[1]}W(R)$ and $\varphi^n(u^p) \notin \mathrm{Fil}^1B_{\mathrm{dR}}$ for any $n \geq 0$ implies that $y \in I^{[1]}B_{\mathrm{cris}}^+$. Hence we have $y \in I^{[1]}W(R)$, which induces $x \in u^pI^{[1]}W(R)$ as desired.
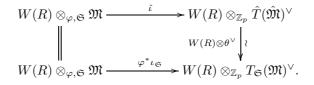
The ideal $u^pI^{[1]}W(R)$ of $W(R)$ plays an important role for studies of $(\varphi, \hat{G})$-modules (cf. [Li2]) which correspond to lattices in crystalline representations. It allows us to study reductions of crystalline representations and also gives interesting applications such as the weight part of Serre's conjecture (cf. [Ga],[GLS1],[GLS2]).

COMPARISON MORPHISM OF $(\varphi, \hat{G})$-MODULES. Let $\hat{\mathfrak{M}}$ be a $(\varphi, \hat{G})$-module of height $r$. We define a $W(R)$-linear map $\hat{\iota}\colon W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to W(R) \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})^\vee$ by the composite

$$W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to \mathrm{Hom}_{\mathbb{Z}_p}(\hat{T}(\hat{\mathfrak{M}}), W(R)) \simeq W(R) \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})^\vee,$$

where the first map is given by $x \mapsto (f \mapsto f(x))$ and the second is the natural map. It is not difficult to check that $\hat{\iota}$ is $\varphi$-equivariant and $G$-equivariant. By the same argument as that in the proof of [Li2, Proposition (2),(3)], we can check the following.

PROPOSITION 3.6. (1) We have $\hat{\iota} \simeq W(R) \otimes_{\varphi,W(R)} \iota_{\mathfrak{S}}$, that is, the following diagram is commutative.

$$
\begin{array}{ccc}
W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} & \xrightarrow{\hat{\iota}} & W(R) \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})^\vee \\
\Big\| & & \Big\downarrow{\wr}\ {\scriptstyle W(R)\otimes\theta^\vee} \\
W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} & \xrightarrow{\varphi^*\iota_{\mathfrak{S}}} & W(R) \otimes_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathfrak{M})^\vee.
\end{array}
$$

*Here,* $\varphi^* \iota_{\mathfrak{S}} := W(R) \otimes_{\varphi, W(R)} \iota_{\mathfrak{S}}$.
(2) *Assume that* $v_p(a_1) > 1$. *Then the map* $\hat{\iota}$ *is injective and we have* $\mathfrak{t}_0^r(W(R) \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})^\vee) \subset \mathrm{Im}(\hat{\iota})$. *Here,* $\mathfrak{t}_0$ *is any generator of* $I^{[1]}W(R)$ *(e.g.,* $\mathfrak{t}_0 = \varphi(\mathfrak{t})$ *(cf., [Fo2, Proposition 5.1.3])).*

## 3.2 MAIN RESULTS

We often use the following conditions.

CONDITION (P): $\varphi^n(f(u)/u)$ is not a power of $E(u)$ for any $n \geq 0$.

CONDITION: $v_p(a_1) > \max\{r, 1\}$.

Note that these conditions are satisfied if $a_1 = 0$. We denote by $\mathrm{Rep}_{\mathbb{Z}_p}^{r,\mathrm{cris}}(G)$ the category of $G$-stable $\mathbb{Z}_p$-lattices in crystalline $\mathbb{Q}_p$-representations of $G$ with Hodge-Tate weights in $[0, r]$. Now we state our main theorem of this paper.

THEOREM 3.7. *Assume the conditions* (P) *and* $v_p(a_1) > \max\{r, 1\}$.
(1) *We have* $\mathrm{Mod}_{\mathfrak{S}}^{r,\hat{G},\mathrm{cris}} = {}'\mathrm{Mod}_{\mathfrak{S}}^{r,\hat{G},\mathrm{cris}}$.
(2) *The contravariant functor* $\hat{T}$ *induces an anti-equivalence of categories between* $\mathrm{Mod}_{\mathfrak{S}}^{r,\hat{G},\mathrm{cris}}$ *and* $\mathrm{Rep}_{\mathbb{Z}_p}^{r,\mathrm{cris}}(G)$.

Summary, we have

$$'\mathrm{Mod}_{\mathfrak{S}}^{r,\hat{G},\mathrm{cris}} = \mathrm{Mod}_{\mathfrak{S}}^{r,\hat{G},\mathrm{cris}} \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{Z}_p}^{r,\mathrm{cris}}(G)$$

under the conditions (P) and $v_p(a_1) > \max\{r, 1\}$. The theorem is an easy consequence of the following result, which we show in the rest of this section.

THEOREM 3.8. (1) *Assume the conditions* (P) *and* $v_p(a_1) > 1$. *Then the contravariant functor* $\hat{T}: \mathrm{Mod}_{\mathfrak{S}}^{r,\hat{G}} \to \mathrm{Rep}_{\mathbb{Z}_p}(G)$ *is fully faithful.*
(2) *Assume the condition* $v_p(a_1) > \max\{r, 1\}$. *Then the contravariant functor* $\hat{T}: \mathrm{Mod}_{\mathfrak{S}}^{r,\hat{G},\mathrm{cris}} \to \mathrm{Rep}_{\mathbb{Z}_p}(G)$ *has values in* $\mathrm{Rep}_{\mathbb{Z}_p}^{\infty,\mathrm{cris}}(G)$. *If we furthermore assume the condition* (P), *then it has values in* $\mathrm{Rep}_{\mathbb{Z}_p}^{r,\mathrm{cris}}(G)$.
(3) *Assume the conditions* (P) *and* $v_p(a_1) > \max\{r, 1\}$. *Then the contravariant functor* $\hat{T}: {}'\mathrm{Mod}_{\mathfrak{S}}^{r,\hat{G},\mathrm{cris}} \to \mathrm{Rep}_{\mathbb{Z}_p}^{r,\mathrm{cris}}(G)$ *is essentially surjective.*

The contravariant functor $T_{\mathfrak{S}}: \mathrm{Mod}_{\mathfrak{S}}^r \to \mathrm{Rep}_{\mathbb{Z}_p}(G_{\underline{\pi}})$ is fully faithful under the condition (P). By the condition $v_p(a_1) > 1$, we know the injectivity of comparison morphisms (cf. Corollary 2.10 and Proposition 3.6 (2)). Thus Theorem (1) follows by completely the same way as the last paragraph of [Li2, Section 3.1] and so we leave the proof of (1) for the readers.
In the rest of this section, we show Theorem 3.8 (2) and (3).

REMARK 3.9. In fact, we can remove the assumption (P) from Theorem 3.8 (1). See Section 3.7.

3.3    Some notations and Properties

Before a proof of Theorem 3.8 (2) and (3), we give some notations and their properties.

The map $\xi_\alpha$.  Let $\mathfrak{M} \in \mathrm{Mod}_{\mathfrak{S}}^r$ be a Kisin module of rank $d$ and set $M := \varphi^*\mathfrak{M}/u\varphi^*\mathfrak{M}$.

Lemma 3.10 ([CL], Lemma 4.5.6.).  *Assume that $v_p(a_1) > r$. Then there exists a unique $\varphi$-equivalent $\mathfrak{O}_\alpha$-linear isomorphism*

$$\xi_\alpha \colon \mathfrak{O}_\alpha \otimes_{W(k)} M \xrightarrow{\sim} \mathfrak{O}_\alpha \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M}$$

*whose reduction modulo $u$ is the identity map on $M$.*

We recall how to define $\xi_\alpha$. Let $\mathfrak{e}_1, \ldots, \mathfrak{e}_d$ be a basis of $\mathfrak{M}$ and let $A \in M_d(\mathfrak{S})$ be a matrix such that $\varphi(\mathfrak{e}_1, \ldots, \mathfrak{e}_d) = (\mathfrak{e}_1, \ldots, \mathfrak{e}_d)A$. Put $e_i = 1 \otimes \mathfrak{e}_i \in \varphi^*\mathfrak{M}$ for each $i$. Then $e_1, \ldots, e_d$ is a basis of $\varphi^*\mathfrak{M}$ and $\varphi(e_1, \ldots, e_d) = (e_1, \ldots, e_d)\varphi(A)$. Put $\bar{e}_i := e_i \bmod u\varphi^*\mathfrak{M}$ for each $i$. Then $\bar{e}_1, \ldots, \bar{e}_d$ is a basis of $M$ and $\varphi(\bar{e}_1, \ldots, \bar{e}_d) = (\bar{e}_1, \ldots, \bar{e}_d)\varphi(A_0)$ where $A_0 = A \bmod u\mathfrak{S} \in M_d(W(k))$.
It was shown in the proof of [CL, Lemma 4.5.6] that the matrix

$$\varphi(A) \cdots \varphi^n(A)\varphi^n(A_0^{-1}) \cdots \varphi(A_0^{-1})$$

converges to an element of $GL_d(\mathfrak{O}_\alpha)$. Putting

$$Y := \lim_{n \to \infty} \varphi(A) \cdots \varphi^n(A)\varphi^n(A_0^{-1}) \cdots \varphi(A_0^{-1}),$$

we define $\xi_\alpha \colon \mathfrak{O}_\alpha \otimes_{W(k)} M \xrightarrow{\sim} \mathfrak{O}_\alpha \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M}$ by $\xi_\alpha(\bar{e}_1, \ldots, \bar{e}_d) = (e_1, \ldots, e_d)Y$.

The map $\xi_\alpha'$.  Let $T$ be an object of $\mathrm{Rep}_{\mathbb{Z}_p}^{r,\mathrm{cris}}(G)$ and put $V = T[1/p]$. Let $D = D_{\mathrm{cris}}(V) := (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V^\vee)^G$ be the filtered $\varphi$-module corresponding to $V$. Let $\mathfrak{O}$ be the subring of $K_0(\!(u)\!)$ consisting of those elements which converge for all $x \in \overline{K}$ with $v_p(x) \geq 0$. We equip $\mathfrak{O}$ with a $K_0$-semi-linear Frobenius $\varphi \colon \mathfrak{O} \to \mathfrak{O}$ such that $\varphi(u) = f(u)$. We see that $\mathfrak{O}$ is a $\varphi$-stable subring of $\mathfrak{O}_\alpha$. By [CL, Section 4.2], there exists a $\varphi$-module $\mathcal{M} = \mathcal{M}(D)$ over $\mathfrak{O}$ such that

- $\mathcal{D}_0 \subset \mathcal{M} \subset \lambda^{-r}\mathcal{D}_0$ where $\mathcal{D}_0 := \mathfrak{O} \otimes_{K_0} D$ and $\lambda := \prod_{n=0}^\infty \varphi^n(E(u)/E(0)) \in \mathfrak{O}$.

- $\mathcal{M}$ is of height $r$ in the sense that the cokernel of the $\mathfrak{O}$-linearization $1 \otimes \varphi_{\mathcal{M}} \colon \mathfrak{O} \otimes_{\varphi, \mathfrak{O}} \mathcal{M} \to \mathcal{M}$ of $\varphi_{\mathcal{M}}$ is killed by $E(u)^r$.

- $\mathcal{M}$ is étale in the sense of [CL, Section 4.4].

By Theorem 4.4.1 of *loc. cit.*, there exists a Kisin module $\mathfrak{M} \subset \mathcal{M}$ of height $r$ such that $\mathfrak{O} \otimes_{\mathfrak{S}} \mathfrak{M} = \mathcal{M}$. Now we define an isomorphism $\xi_\alpha' \colon \mathfrak{O}_\alpha \otimes_{K_0} D \xrightarrow{\sim} \mathfrak{O}_\alpha \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M}$ as follows: The isomorphism $1 \otimes \varphi \colon \varphi^*D \xrightarrow{\sim} D$ induces

an isomorphism $1 \otimes \varphi \colon \varphi^* \mathcal{D}_0 \xrightarrow{\sim} \mathcal{D}_0$. Thus we obtain an injection $\xi' \colon \mathcal{D}_0 \xrightarrow{\sim} \varphi^* \mathcal{D}_0 \hookrightarrow \varphi^* \mathcal{M} \simeq \mathfrak{O} \otimes_{\mathfrak{S}} \varphi^* \mathfrak{M}$. Then we define $\xi'_\alpha = \mathfrak{O}_\alpha \otimes_{\mathfrak{O}} \xi'$. It is shown in Lemma 4.2.2 of *loc. cit.* that $\xi'_\alpha$ is an isomorphism.

THE MAP $\iota_0$. Following Proposition 4.5.1 of *loc. cit.*, we define a $G_{\underline{\pi}}$-equivalent injection $\iota_0 \colon T_{\mathfrak{S}}(\mathfrak{M}) \hookrightarrow V$ by the composite

$$
\begin{aligned}
T_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, W(R)) &\hookrightarrow \mathrm{Hom}_{\mathfrak{O},\varphi,\mathrm{Fil}}(\varphi^* \mathcal{M}, B^+_{\mathrm{cris}}) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{O}_\alpha,\varphi,\mathrm{Fil}}(\mathfrak{O}_\alpha \otimes_{\mathfrak{O}} \varphi^* \mathcal{M}, B^+_{\mathrm{cris}}) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{O}_\alpha,\varphi,\mathrm{Fil}}(\mathfrak{O}_\alpha \otimes_{K_0} D, B^+_{\mathrm{cris}}) \\
&\simeq V_{\mathrm{cris}}(D) \simeq V,
\end{aligned}
$$

where the first arrow is given by $f \mapsto (a \otimes x \in \mathfrak{O} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} = \varphi^* \mathcal{M} \mapsto a\varphi(f(x)))$, the second and the fourth arrows are natural isomorphisms, and the third arrow is given by $(f \mapsto f \circ \xi'_\alpha)$. We omit definitions of filtrations of various modules appeared above since precise informations of them are not so important here. We only note that definitions of filtrations are given in [CL, Section 4].

THE $G$-ACTION ON $u$. We consider a difference between $gu$ and $u$ for $g \in G$. We recall that $f(u) = \sum_{i=1}^{p} a_i u^i = u^p + a_{p-1}u^{p-1} + \cdots + a_1 u \in \mathbb{Z}_p[u]$ with the property $f(u) \equiv u^p \bmod p$.

At first, here is a Cais-Liu's observation.

PROPOSITION 3.11 ([CL], Lemma 2.3.2). *Let $g \in G$ be arbitrary.*
(1) *We have $gu - u \in I^{[1]}W(R)$.*
(2) *If $v_p(a_1) > 1$, then we have $gu - u \in I^{[1+]}W(R)$.*

We use the following proposition in the final section.

PROPOSITION 3.12. *Let $j_0$ be the minimum integer $1 \leq j \leq p$ such that $v_p(ja_j) = 1$. Let $g \in G \smallsetminus G_{\underline{\pi}}$ and $N \geq 1$ the integer such that $g\pi_{N-1} = \pi_{N-1}$ and $g\pi_N \neq \pi_N$. We denote by $\bar{u}$ the image of $u$ for the projection $W(R) \twoheadrightarrow R$. Then we have $v_R(g\bar{u} - \bar{u}) = p^N/(p-1) + (j_0 - 1)/(e(p-1))$.*

*Proof.* Since $v_R(g\bar{u} - \bar{u}) = \lim_{n\to\infty} p^n v_p(g\pi_n - \pi_n)$, it suffices to show that $v_p(g\pi_n - \pi_n) = c_n$ for $n \geq N$, where $c_n := p^N/((p-1)p^n) + (j_0 - 1)/(ep^n(p-1))$. We note that we have an equation $\sum_{i=1}^{p} a_i(g\pi_n^i - \pi_n^i) = g\pi_{n-1} - \pi_{n-1}$. Putting $b_n^{(\ell)} = \sum_{j=0}^{p-\ell} a_{\ell+j} \binom{\ell+j}{j} \pi_n^j$, we have

$$
\sum_{i=1}^{p} a_i(g\pi_n^i - \pi_n^i) = \sum_{i=1}^{p}\sum_{j=0}^{i-1} a_i \binom{i}{j}(g\pi_n - \pi_n)^{i-j}\pi_n^j = \sum_{\ell=1}^{p} b_n^{(\ell)}(g\pi_n - \pi_n)^\ell.
$$

Hence we obtain that $g\pi_n - \pi_n$ is a solution of the equation

$$
\sum_{\ell=1}^{p} b_n^{(\ell)} X^\ell - (g\pi_{n-1} - \pi_{n-1}) = 0.
$$

We note that we have $b_n^{(p)} = 1$ and

$$v_p\left(a_{\ell+j}\binom{\ell+j}{j}\pi_n^j\right) = \begin{cases} v_p(a_{\ell+j}) + \frac{j}{ep^n} & \text{for } 0 \le j < p - \ell, \\ 1 + \frac{p-\ell}{ep^n} & \text{for } j = p - \ell \end{cases}$$

if $1 \le \ell \le p - 1$.

THE CASE $j_0 = p$: By the assumption $v_p(a_1), \ldots, v_p(a_{p-1}) > 1$, we have

$$v_p(b_n^{(\ell)}) = 1 + \frac{p-\ell}{ep^n} \tag{3.1}$$

for $1 \le \ell \le p - 1$. Now we show $v_p(g\pi_n - \pi_n) = c_n$ by induction on $n \ge N$. Suppose $n = N$. Then $g\pi_N - \pi_N$ is a solution of the equation

$$\sum_{\ell=0}^{p-1} b_N^{(\ell+1)} X^\ell = 0.$$

Hence it is enough to show that the Newton polygon of the polynomial $\sum_{\ell=0}^{p-1} b_N^{(\ell+1)} X^\ell \in \mathbb{Z}_p[X]$ is the line segment, denoted by $l_N$, connecting $(0, (p-1)c_N)$ to $(p-1, 0)$. This follows immediately by (3.1).
We suppose that the assertion holds for $n$ and consider the case $n + 1$. We recall that $g\pi_{n+1} - \pi_{n+1}$ is a solution of the equation

$$\sum_{\ell=1}^{p} b_{n+1}^{(\ell)} X^\ell - (g\pi_n - \pi_n) = 0.$$

Thus it is enough to show that the Newton polygon of the polynomial $\sum_{\ell=1}^{p} b_{n+1}^{(\ell)} X^\ell - (g\pi_n - \pi_n)$ is the line segment, denoted by $l_{n+1}$, connecting $(0, c_n)$ to $(p, 0)$. This follows immediately by (3.1) again.
THE CASE $j_0 < p$: Let $s$ be the number of integers $j$ such that $1 \le j \le p - 1$ and $v_p(a_j) = 1$. By assumption we have $s > 0$. Let $j_{-1}, j_0, j_1, \ldots, j_{s-1}$ be integers such that $j_{-1} = 0 < j_0 < j_1 < \cdots < j_{s-1} \le p - 1$ and $v_p(a_{j_0}) = \cdots = v_p(a_{j_{s-1}}) = 1$. Then we see

$$v_p(b_n^{(\ell)}) = \begin{cases} 1 + \frac{j_k - \ell}{ep^n} & \text{if } j_{k-1} < \ell \le j_k \text{ for some } 0 \le k \le s - 1, \\ 1 + \frac{p-\ell}{ep^n} & \text{if } j_{s-1} < \ell \le p - 1. \end{cases} \tag{3.2}$$

By a similar strategy to the proof of (1), we can show $v_p(g\pi_n - \pi_n) = c_n$ by induction on $n \ge N$. We leave a proof to the readers. $\square$

We recall that $\mathfrak{D}_\alpha = \mathfrak{S}[\![\frac{E(u)^p}{p}]\!][1/p] \subset B_{\mathrm{cris}}^+$ and $\mathfrak{S}[\![\frac{E(u)^p}{p}]\!] = \mathfrak{S}[\![\frac{u^{ep}}{p}]\!] \subset A_{\mathrm{cris}}$.

LEMMA 3.13. (1) *We have $u^i \in p^{[i/(ep)] + [i/(ep^2)]} A_{\mathrm{cris}}$ for any $i \ge 0$, where $[\cdot]$ is the floor function.*
(2) *We have*

$$\mathfrak{S}[\![\frac{u^{ep}}{p}]\!] = \left\{ \sum_{i=0}^{\infty} a_i p^{-[\frac{i}{ep}]} u^i \mid a_i \in W(k) \right\}.$$

*Proof.* (1) Write $i = epj + h$ with $0 \leq h < ep$ and $j = pk + h'$ with $0 \leq h' < p$. Note that we have $j = [i/(ep)]$ and $k = [i/(ep^2)]$. Since we have $u^{ep}/p \in E(u)^p/p + pA_{\mathrm{cris}}$, we see $(u^{ep}/p)^p \in pA_{\mathrm{cris}}$. Thus we obtain $u^i/p^j = u^h(u^{ep}/p)^{h'}(u^{ep}/p)^{pk} \in p^k A_{\mathrm{cris}}$.

(2) If $x = \sum_{i=0}^{\infty} a_i p^{-[i/(ep)]} u^i$ with $a_i \in W(k)$, then $x = \sum_{j=0}^{\infty} \left\{ \sum_{h=0}^{ep-1} a_{epj+h} u^h \right\} (u^{ep}/p)^j \in \mathfrak{S}[\![\frac{u^{ep}}{p}]\!]$. Conversely, let $x = \sum_{i=0}^{\infty} x_i (u^{ep}/p)^i$ with $x_i \in \mathfrak{S}$. Writing $x_i = \sum_{j=0}^{\infty} a_{ij} u^j$ with $a_{ij} \in W(k)$, then we have $x = \sum_{h=0}^{\infty} \left\{ \sum_{i,j \geq 0, epi+j=h} p^{-i} a_{ij} \right\} u^h$. If we have $i, j \geq 0$ and $epi + j = h$, then we have $i \leq [h/(ep)]$ and hence $p^{-i} a_{ij} \in p^{-[h/(ep)]} W(k)$. □

We recall that $\tilde{B}_\alpha = W(R)[\![\frac{E(u)^p}{p}]\!][1/p] = W(R)[\![\frac{u^{pe}}{p}]\!][1/p] \subset B_{\mathrm{cris}}^+$.

LEMMA 3.14. *For any $g \in G$ and $x \in \mathfrak{D}_\alpha$, we have $gx - x \in (gu - u)\tilde{B}_\alpha$.*

*Proof.* We may suppose $x \in \mathfrak{S}[\![\frac{u^{ep}}{p}]\!]$ and write $x = \sum_{i=0}^{\infty} a_i u^i$ with $a_i \geq -[i/(ep)]$ (see Lemma 3.13). For $i \geq 1$, we have $gu^i - u^i = (gu - u)\sum_{j=0}^{i-1} \binom{i}{j} (gu - u)^{i-1-j} u^j$ . By Proposition 3.11, we have $gu - u \in \mathrm{Fil}^1 W(R) \subset u^e W(R) + pW(R)$. Thus we have $(gu^i - u^i)/(gu - u) = \sum_{j=0}^{i-1} \sum_{h=0}^{i-1-j} c_{ijh} p^h u^{e(i-1-j-h)+j} = \sum_{k=0}^{\infty} \sum_{(j,h) \in S_{ik}} c_{ijh} p^{h+[k/(ep)]} u^k / p^{[k/(ep)]}$ for some $c_{ijh} \in W(R)$. Here, $S_{ik}$ is the set of pairs $(j,h)$ of integers such that $0 \leq j \leq i-1, 0 \leq h \leq i-1-j, e(i-1-j-h)+j = k$. Note that, if $S_{ik}$ is not empty, then $k \leq e(i-1-0-0) + (i-1) = (e+1)(i-1)$. Thus we have

$$\frac{gu^i - u^i}{gu - u} = \sum_{k=0}^{(e+1)(i-1)} \sum_{(j,h) \in S_{ik}} c_{ijh} p^{h+[k/(ep)]} \frac{u^k}{p^{[k/(ep)]}}.$$

For $(j,h) \in S_{ik}$, we have $v_p\left(a_i p^{h+[k/(ep)]}\right) > -i/(ep) + h + k/(ep) - 1 = h(1 - 1/p) + (e-1)(i-j-1)/(ep) - 1 - 1/(ep)$. Since $h, i-j-1 \geq 0$, we have $v_p\left(a_i p^{h+[k/(ep)]}\right) > -1 - 1/(ep)$, that is, $v_p\left(a_i p^{h+[k/(ep)]}\right) \geq -1$. Thus $t_{ik} := \sum_{(j,h) \in S_{ik}} a_i c_{ijh} p^{h+[k/(ep)]}$ is an element of $p^{-1} W(R)$. On the other hand, we have $h = (e(i-1) - (e-1)j - k)/e \geq (e(i-1) - (e-1)(i-1) - k)/e = (i-k-1)/e$ and hence $v_p\left(a_i p^{h+[k/(ep)]}\right) > (i-k-1)/e \cdot (1 - 1/p) + (e-1) \cdot 0/(ep) - 1 - 1/(ep) =: m_k(i)$. Since $m_k(i)$ goes to $\infty$ as $i \to \infty$, we obtain that $t_k := \sum_{i \geq 1 + k/(e+1)} t_{ik}$ converges in $p^{-1} W(R)$ (here, $t_{ik} := 0$ if $S_{ik}$ is empty). Now we claim that the following equality holds;

$$\sum_{i=1}^{\infty} \sum_{k=0}^{(e+1)(i-1)} t_{ik} p^{-[k/(ep)]} u^k = \sum_{k=0}^{\infty} t_k p^{-[k/(ep)]} u^k. \qquad (3.3)$$

Admitting this equality, we complete the proof since we have

$$gx - x = \sum_{i=1}^{\infty} a_i(gu^i - u^i) = (gu - u)\sum_{i=1}^{\infty}\sum_{k=0}^{(e+1)(i-1)} t_{ik}p^{-[k/(ep)]}u^k$$

$$= (gu - u)\sum_{k=0}^{\infty} t_k p^{-[k/(ep)]}u^k \in (gu - u)\tilde{B}_\alpha.$$

Hence it suffices to show (3.3). Denote by $\alpha$ and $\beta$ the left hand side and the right hand side of (3.3), respectively. For simplicity, we put $u_k = p^{-[k/(ep)]}u^k$. Since we have

$$\alpha - \beta = \left(\alpha - \sum_{i=1}^{m}\sum_{k=0}^{(e+1)(i-1)} t_{ik}u_k\right) - \left(\beta - \sum_{k=0}^{(e+1)(m-1)} t_k u_k\right)$$

$$- \left(\sum_{k=0}^{(e+1)(m-1)} t_k u_k - \sum_{i=1}^{m}\sum_{k=0}^{(e+1)(i-1)} t_{ik}u_k\right)$$

for any $m \geq 1$, it suffices to show that $\gamma_m := \sum_{k=0}^{(e+1)(m-1)} t_k u_k - \sum_{i=1}^{m}\sum_{k=0}^{(e+1)(i-1)} t_{ik}u_k$ converges to zero $p$-adically in $B_{\mathrm{cris}}^+$. Note that we see $\gamma_m = \sum_{k=0}^{(e+1)(m-1)}\sum_{i=m+1}^{\infty} t_{ik}u_k$. Let $s > 0$ be any integer. By Lemma 3.13 (1), there exists an integer $k_0$ such that $u_k \in p^{s+1}A_{\mathrm{cris}}$ for any $k > k_0$. Since sequences $\{t_{i0}\}_i, \ldots, \{t_{ik_0}\}_i$ converge to zero, there exists $m_0$ large enough such that $t_{ik} \in p^s W(R)$ for any $0 \leq k \leq k_0$ and $i > m_0$. Therefore, if $m > m_0$, the decomposition

$$\gamma_m = \sum_{k=0}^{k_0}\sum_{i=m+1}^{\infty} t_{ik}u_k + \sum_{k=k_0+1}^{(e+1)(m-1)}\sum_{i=m+1}^{\infty} t_{ik}u_k$$

and the fact that $\sum_{i=m+1}^{\infty} t_{ik} \in p^{-1}W(R)$ implies $\gamma_m \in p^s A_{\mathrm{cris}}$.          $\square$

## 3.4   Essential image of $\hat{T}$

The goal of this subsection is to show Theorem 3.8 (2). We continue to use the same notation as in previous section.

LEMMA 3.15.  *For any $\hat{\mathfrak{M}} \in \mathrm{Mod}_{\mathfrak{S}}^{r,\hat{G},\mathrm{cris}}$, we have $\xi_\alpha(M) \subset (B_{\mathrm{cris}}^+ \otimes_{\varphi,\mathfrak{S}} \mathfrak{M})^G$.*

*Proof.* It suffices to show $g((e_1, \ldots, e_d)Y) = (e_1, \ldots, e_d)Y$ for any $g \in G$. We define $X_g \in GL_d(W(R))$ by

$$g(e_1, \ldots, e_d) = (e_1, \ldots, e_d)X_g.$$

It is enough to show $X_g g(Y) = Y$. To simplify notation, put $u_g = gu - u$. We know $X_g - I_d \in \varphi(u_g)M_d(B_{\mathrm{cris}}^+)$. Hence we have $X_g = I_d + \varphi(u_g)Y_g$ for

some $Y_g \in M_d(B^+_{\mathrm{cris}})$. Furthermore, we have $X_g g \varphi(A) = \varphi(A) \varphi(X_g)$ since $\varphi$ commutes with the $G$-action. Thus we have

$$X_g g(\varphi(A) \cdots \varphi^n(A) \varphi^n(A_0^{-1}) \cdots \varphi(A_0^{-1}))$$
$$= X_g g \varphi(A) g \varphi^2(A) \cdots g \varphi^n(A) \varphi^n(A_0^{-1}) \cdots \varphi(A_0^{-1})$$
$$= \varphi(A) \varphi(X_g) g \varphi^2(A) \cdots g \varphi^n(A) \varphi^n(A_0^{-1}) \cdots \varphi(A_0^{-1})$$
$$= \varphi(A) \varphi^2(A) \varphi^2(X_g) g \varphi^3(A) \cdots g \varphi^n(A) \varphi^n(A_0^{-1}) \cdots \varphi(A_0^{-1})$$
$$= \cdots$$
$$= \varphi(A) \cdots \varphi^n(A) \varphi^n(X_g) \varphi^n(A_0^{-1}) \cdots \varphi(A_0^{-1})$$
$$= \varphi(A) \cdots \varphi^n(A) \varphi^n(A_0^{-1}) \cdots \varphi(A_0^{-1})$$
$$\quad + \varphi^{n+1}(u_g) \varphi(A) \cdots \varphi^n(A) \varphi^n(Y_g) \varphi^n(A_0^{-1}) \cdots \varphi(A_0^{-1}).$$

Hence the proof completes if we show that

$$Z_n := \varphi^{n+1}(u_g) \varphi(A) \cdots \varphi^n(A) \varphi^n(Y_g) \varphi^n(A_0^{-1}) \cdots \varphi(A_0^{-1})$$

converges to zero $p$-adically in $B^+_{\mathrm{cris}}$. Let $\lambda > 0$ be an integer such that $p^\lambda Y_g \in M_d(A_{\mathrm{cris}})$. Since $\mathfrak{M}$ is of height $r$, we see that $Z_n$ is contained in $\varphi^{n+1}(u_g)/p^{nr} \cdot p^{-\lambda} M_d(A_{\mathrm{cris}})$. Since $\varphi^{n+1}(u_g)/p^{nr}$ converges to zero by [CL, Lemma 2.2.2], we obtain the desired result. $\square$
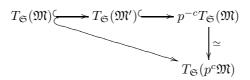
*Proof of Theorem 3.8 (2).* We continue to use the same notation. First we assume the condition $v_p(a_1) > \max\{r, 1\}$. Proposition 3.6 (2) and Lemma 3.15, we have injections

$$M \xrightarrow{\xi_\alpha} (B^+_{\mathrm{cris}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M})^G \xhookrightarrow{\hat{\iota}} (B^+_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})^\vee)^G.$$

Hence we the equality $\dim_{\mathbb{Q}_p}(B^+_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})^\vee)^G = \dim_{\mathbb{Q}_p} \hat{T}(\hat{\mathfrak{M}})^\vee[1/p]$. This implies that $\hat{T}(\hat{\mathfrak{M}})[1/p]$ is a crystalline $\mathbb{Q}_p$-representation with non-negative Hodge-Tate weights. In the rest of this proof, we show that the Hodge-Tate weights of $\hat{T}(\hat{\mathfrak{M}})[1/p]$ are at most $r$.

From now on, we assume the condition (P). Under this assumption, we know that $T_{\mathfrak{S}}$ is fully faithful (cf. Proposition 2.6). Put $V = \hat{T}(\hat{\mathfrak{M}})[1/p]$ and $D = D_{\mathrm{cris}}(V)$. Take an integer $r' > 0$ such that Hodge-Tate weights of $V$ are at most $r'$. Let $\mathcal{M} = \mathcal{M}(D)$ be the $\varphi$-module over $\mathfrak{O}$ corresponding to $D$ and take any free Kisin module $\mathfrak{M}' \subset \mathcal{M}$ of height $r'$ such that $\mathfrak{O} \otimes_{\mathfrak{S}} \mathfrak{M}' = \mathcal{M}$.

We claim that $\mathfrak{M}'$ is of height $r$. Note that $T_{\mathfrak{S}}(\mathfrak{M})$ and $T_{\mathfrak{S}}(\mathfrak{M}')$ are lattices of $V$. By replacing $\mathfrak{M}'$ with some $p^\ell \mathfrak{M}$, we may assume that we have $T_{\mathfrak{S}}(\mathfrak{M}) \subset T_{\mathfrak{S}}(\mathfrak{M}')$. Let $c > 0$ be an integer such that $T_{\mathfrak{S}}(\mathfrak{M}') \subset p^{-c} T_{\mathfrak{S}}(\mathfrak{M})$. We consider the following commutative diagram.

$$
\begin{array}{ccccc}
T_{\mathfrak{S}}(\mathfrak{M}) & \hookrightarrow & T_{\mathfrak{S}}(\mathfrak{M}') & \hookrightarrow & p^{-c} T_{\mathfrak{S}}(\mathfrak{M}) \\
& \searrow & & & \downarrow{\simeq} \\
& & & & T_{\mathfrak{S}}(p^c \mathfrak{M})
\end{array}
$$

Here, $p^{-c}T_{\mathfrak{S}}(\mathfrak{M}) \xrightarrow{\sim} T_{\mathfrak{S}}(p^c\mathfrak{M})$ in the diagram is the map given by $f \mapsto f|_{p^c\mathfrak{M}}$, and the other arrows are natural injections. Since $T_{\mathfrak{S}}$ is fully faithful, we obtain maps $\eta' \colon \mathfrak{M}' \to \mathfrak{M}$ and $\eta \colon p^c\mathfrak{M} \to \mathfrak{M}'$ such that $\eta' \circ \eta$ is the inclusion map $p^c\mathfrak{M} \hookrightarrow \mathfrak{M}$. We see that $\eta$ and $\eta'$ are injective and $p^c\mathfrak{M} \subset \eta'(\mathfrak{M}')$. We regard $\mathfrak{M}'$ as a $\varphi$-stable submodule of $\mathfrak{M}$ by $\eta'$. Since $\mathfrak{M}/\mathfrak{M}'$ is killed by a power of $p$, Proposition 2.2 shows that the natural map $\mathfrak{M}/\mathfrak{M}' \to \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}/\mathfrak{M}'$ is injective. Thus we obtain the fact that $\mathfrak{M}/\mathfrak{M}'$ is $p'$-torsion free in the sense that $\mathrm{Ann}_{\mathfrak{S}}(\mathfrak{M}/\mathfrak{M}')$ is zero or of the form $p^{\ell}\mathfrak{S}$. It follows from [Fo2, Proposition B.1.3.5] that $\mathfrak{M}'$ is of height $r$. In particular, $\mathcal{M}$ is of height $r$.

Note that $\xi'_{\alpha}$ induces an isomorphism $\varphi^*\mathcal{M}/E(u)\varphi^*\mathcal{M} \simeq K \otimes_{K_0} D =: D_K$. If we define a decreasing filtration $\mathrm{Fil}^i\varphi^*\mathcal{M}$ of $\varphi^*\mathcal{M}$ by

$$\mathrm{Fil}^i\varphi^*\mathcal{M} = \{x \in \varphi^*\mathcal{M} \mid (1 \otimes \varphi)(x) \in E(u)^i\mathcal{M}\},$$

then the natural projection

$$\varphi^*\mathcal{M} \twoheadrightarrow \varphi^*\mathcal{M}/E(u)\varphi^*\mathcal{M} \simeq D_K$$

is strict compatible with filtrations (cf. [CL, Corollary 4.2.4]). Since $\mathcal{M}$ is of height $r$, we have $\mathrm{Fil}^{r+1}\varphi^*\mathcal{M} \subset E(u)\varphi^*\mathcal{M}$, which induces the fact $\mathrm{Fil}^{r+1}D_K = 0$ as desired.                                                                                       $\square$

### 3.5   Essential surjectiveness of $\hat{T}$

We show Theorem 3.8 (3). Let $T$ be an object of $\mathrm{Rep}^{r,\mathrm{cris}}_{\mathbb{Z}_p}(G)$ and put $V = T[1/p]$. Let $D = D_{\mathrm{cris}}(V)$ be the filtered $\varphi$-module corresponding to $V$. Throughout this subsection, we identify $V$ with $V_{\mathrm{cris}}(D) = \mathrm{Hom}_{K_0,\varphi}(D, B^+_{\mathrm{cris}}) \cap \mathrm{Hom}_{K,\mathrm{Fil}}(D_K, B^+_{\mathrm{dR}})(\subset \mathrm{Hom}_{K_0}(D, B^+_{\mathrm{cris}}))$. Let $\mathcal{M} = \mathcal{M}(D)$ be the $\varphi$-module over $\mathfrak{O}$ corresponding to $D$. By Theorem 4.4.1 of $loc.\ cit.$, there exists a Kisin module $\mathfrak{M} \subset \mathcal{M}$ of height $r$ such that $\mathfrak{O} \otimes_{\mathfrak{S}} \mathfrak{M} = \mathcal{M}$. In Section 3.3, we defined a $G_{\underline{\pi}}$-equivalent injection $\iota_0 \colon T_{\mathfrak{S}}(\mathfrak{M}) \hookrightarrow V$. The image of $\iota_0$ might not coincide with $T$. However, we have

**Lemma 3.16.** *Assume the condition* $(P)$. *Then we can choose* $\mathfrak{M}$ *so that* $\iota_0(T_{\mathfrak{S}}(\mathfrak{M})) = T$.

*Proof.* We identify $V$ with $\mathrm{Hom}_{\mathfrak{O},\varphi,\mathrm{Fil}}(\varphi^*\mathcal{M}, B^+_{\mathrm{cris}})$ by isomorphisms $\mathrm{Hom}_{\mathfrak{O},\varphi,\mathrm{Fil}}(\varphi^*\mathcal{M}, B^+_{\mathrm{cris}}) \simeq \mathrm{Hom}_{\mathfrak{O}_{\alpha},\varphi,\mathrm{Fil}}(\mathfrak{O}_{\alpha} \otimes_{\mathfrak{O}} \varphi^*\mathcal{M}, B^+_{\mathrm{cris}}) \simeq \mathrm{Hom}_{\mathfrak{O}_{\alpha},\varphi,\mathrm{Fil}}(\mathfrak{O}_{\alpha} \otimes_{K_0} D, B^+_{\mathrm{cris}}) \simeq V_{\mathrm{cris}}(D) = V$ (see the definition of $\iota_0$). Under this identification, $\iota_0$ is the injection

$$\iota_0 \colon T_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},W(R)}(\mathfrak{M}, W(R)) \to \mathrm{Hom}_{\mathfrak{O},\varphi,\mathrm{Fil}}(\varphi^*\mathcal{M}, B^+_{\mathrm{cris}}) = V$$

given by $\iota_0(f)(a \otimes x) = a\varphi(f(x))$ for $f \in T_{\mathfrak{S}}(\mathfrak{M})$, $a \in \mathcal{O}$, $x \in \mathfrak{M}$ (here we identify $\varphi^*\mathcal{M}$ with $\mathcal{O} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$). Put $L = \iota_0(T_{\mathfrak{S}}(\mathfrak{M}))$. For any integer $\ell \geq 0$, we have natural injections $T_{\mathfrak{S}}(p^{-\ell}\mathfrak{M}) \hookrightarrow T_{\mathfrak{S}}(\mathfrak{M}) \hookrightarrow T_{\mathfrak{S}}(p^{\ell}\mathfrak{M})$ induced by embeddings $p^{\ell}\mathfrak{M} \subset \mathfrak{M} \subset p^{-\ell}\mathfrak{M}$. It is not difficult to check the equality

$\iota_0(T_{\mathfrak{S}}(p^{\pm\ell}\mathfrak{M})) = p^{\mp\ell}L$. Thus by replacing $\mathfrak{M}$ with $p^{-\ell}\mathfrak{M}$ for $\ell$ large enough, we may assume that $L$ is a submodule of $T$. Let $N \to M$ be the morphism of free étale $\varphi$-modules which corresponds to the natural injection $L \hookrightarrow T$. This implies that we have the following commutative diagram:

$$
\begin{array}{ccc}
T & \xleftarrow{\ \sim\ } & T_{\mathcal{O}_{\mathcal{E}}}(N) \\
\uparrow & & \uparrow \\
L & \xleftarrow{\ \sim\ } & T_{\mathcal{O}_{\mathcal{E}}}(M)
\end{array}
$$

We denote by $\eta$ the isomorphism $T_{\mathcal{O}_{\mathcal{E}}}(M) \simeq L$ in the diagram. We see that $N \to M$ is injective and $M/N$ is a torsion étale $\varphi$-module killed by $p^c$. Here, $c$ is any integer $c > 0$ such that $p^c$ kills $T/L$. Let $\mathfrak{g}\colon \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\sim} M$ be the morphism of étale $\varphi$-modules which corresponds to the composition $T_{\mathcal{O}_{\mathcal{E}}}(M) \underset{\eta}{\to} L \underset{\iota_0^{-1}}{\to} T_{\mathfrak{S}}(\mathfrak{M}) \xrightarrow{\sim} T_{\mathcal{O}_{\mathcal{E}}}(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M})$. We have the following commutative diagram:

$$
\begin{array}{ccccccc}
T_{\mathcal{O}_{\mathcal{E}}}(M) & \xrightarrow[\eta]{\ \sim\ } & L & \xrightarrow[\iota_0^{-1}]{\ \sim\ } & T_{\mathfrak{S}}(\mathfrak{M}) & \xrightarrow{\ \sim\ } & T_{\mathcal{O}_{\mathcal{E}}}(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}) \\
\| & & & & & & \| \\
T_{\mathcal{O}_{\mathcal{E}}}(M) & \lhook\joinrel\xrightarrow{\hspace{2cm} T_{\mathcal{O}_{\mathcal{E}}}(\mathfrak{g}) \hspace{2cm}} & & & & & T_{\mathcal{O}_{\mathcal{E}}}(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M})
\end{array}
$$

Let $\mathrm{pr}\colon M \to M/N$ be the natural projection. Then $\mathfrak{N}' := \ker(\mathrm{pr} \circ \mathfrak{g}) \subset \mathfrak{M}$ is a $\varphi$-module of height $r$ by [Fo2, Proposition B.1.3.5]. Put $\mathfrak{N} = \mathfrak{N}'[1/p] \cap (\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N}')$. It follows from [CL, Lemma 3.3.4] that $\mathfrak{N}$ is a free Kisin module of height $r$. By the condition (P) and [CL, Proposition 3.3.5], the embedding $\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N} = \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N}' \hookrightarrow \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ induces an embedding $\mathfrak{N} \hookrightarrow \mathfrak{M}$. We see that we have an isomorphism $\mathfrak{g}'\colon \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N} \xrightarrow{\sim} N$ which makes the diagram

$$
\begin{array}{ccc}
N & \xleftarrow[\mathfrak{g}']{\ \sim\ } & \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N} \\
\uparrow\downarrow & & \uparrow \\
M & \xleftarrow[\mathfrak{g}]{\ \sim\ } & \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}
\end{array}
$$

commutative. Here we consider the following commutative diagram.

$$
\begin{array}{ccccccc}
T & \xrightarrow{\ \sim\ } & T_{\mathcal{O}_{\mathcal{E}}}(N) & \xrightarrow[T_{\mathcal{O}_{\mathcal{E}}}(\mathfrak{g}')]{\ \sim\ } & T_{\mathcal{O}_{\mathcal{E}}}(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N}) & \xleftarrow{\ \sim\ } & T_{\mathfrak{S}}(\mathfrak{N}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
L & \xrightarrow[\eta]{\ \sim\ } & T_{\mathcal{O}_{\mathcal{E}}}(M) & \xrightarrow[T_{\mathcal{O}_{\mathcal{E}}}(\mathfrak{g})]{\ \sim\ } & T_{\mathcal{O}_{\mathcal{E}}}(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}) & \xleftarrow{\ \sim\ } & T_{\mathfrak{S}}(\mathfrak{M})
\end{array}
$$

The composite map $L \xrightarrow{\sim} T_{\mathcal{O}_{\mathcal{E}}}(M) \xrightarrow{\sim} T_{\mathcal{O}_{\mathcal{E}}}(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}) \xrightarrow{\sim} T_{\mathfrak{S}}(\mathfrak{M})$ in the diagram is just $\iota_0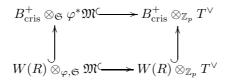^{-1}$. It suffices to show that the inverse $\iota_0'$ of the composite map $T \xrightarrow{\sim} T_{\mathcal{O}_{\mathcal{E}}}(N) \xrightarrow{\sim} T_{\mathcal{O}_{\mathcal{E}}}(\mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{N}) \xrightarrow{\sim} T_{\mathfrak{S}}(\mathfrak{N})$ is just $\iota_0\colon T_{\mathfrak{S}}(\mathfrak{N}) \hookrightarrow V$.

Since $\mathfrak{M}/\mathfrak{N}' \subset M/N$ is killed by $p^c$, we have $p^c\mathfrak{M} \subset \mathfrak{N}' \subset \mathfrak{N} \subset \mathfrak{M}$. Consider the following diagram:
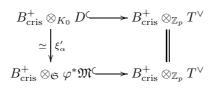
$$
\begin{array}{ccccc}
T_{\mathfrak{S}}(\mathfrak{M}) & \hookrightarrow & T_{\mathfrak{S}}(\mathfrak{N}) & \hookrightarrow & T_{\mathfrak{S}}(p^c\mathfrak{M}) \\
\simeq\downarrow\iota_0 & & \simeq\downarrow\iota_0' & & \simeq\downarrow\iota_0 \\
L & \hookrightarrow & T & \hookrightarrow & p^{-c}L \hookrightarrow V = \mathrm{Hom}_{\mathfrak{O},\varphi,\mathrm{Fil}}(\varphi^*\mathcal{M}, B_{\mathrm{cris}}^+)
\end{array}
$$

The biggest square in the diagram clearly commutes. The left square in the diagram also commutes by definition of $\iota_0'$. Thus we see that the right square commutes. This implies that $\iota_0'$ is the map $\iota_0 \colon T_{\mathfrak{S}}(p^c\mathfrak{M}) \hookrightarrow V$ restricted to $T_{\mathfrak{S}}(\mathfrak{N})$, which must coincide with $\iota_0 \colon T_{\mathfrak{S}}(\mathfrak{N}) \hookrightarrow V$. □

In the rest of this subsection, we always assume the condition (P) and $v_p(a_1) > \max\{r, 1\}$. Let $\mathfrak{M}$ be as in Lemma 3.16. Then $\iota_0 \colon T_{\mathfrak{S}}(\mathfrak{M}) \hookrightarrow V$ induces an isomorphism $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T$. By this isomorphism, we equip $T_{\mathfrak{S}}(\mathfrak{M})$ with a $G$-action. Here, we consider the following diagram:
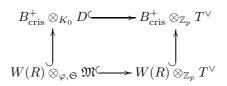
$$
\begin{array}{ccccc}
B_{\mathrm{cris}}^+ \otimes_{K_0} D & \hookrightarrow \mathrm{Hom}_{\mathbb{Q}_p}(V, B_{\mathrm{cris}}^+) & \xrightarrow{\sim} & B_{\mathrm{cris}}^+ \otimes_{\mathbb{Q}_p} V^\vee & == B_{\mathrm{cris}}^+ \otimes_{\mathbb{Z}_p} T^\vee \\
\simeq\downarrow\xi_\alpha' & & & & \| \\
B_{\mathrm{cris}}^+ \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M} & \xrightarrow{\varphi^*\iota_{\mathfrak{S}}} & & B_{\mathrm{cris}}^+ \otimes_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathfrak{M})^\vee \xleftarrow[\iota_0^\vee]{\sim} B_{\mathrm{cris}}^+ \otimes_{\mathbb{Z}_p} T^\vee \\
\cup\uparrow & & & \cup\uparrow \\
W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} & \xrightarrow{\varphi^*\iota_{\mathfrak{S}}} & & W(R) \otimes_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathfrak{M})^\vee \xleftarrow[\iota_0^\vee]{\sim} W(R) \otimes_{\mathbb{Z}_p} T^\vee
\end{array}
$$
$$(3.4)$$

The square

$$
\begin{array}{ccc}
B_{\mathrm{cris}}^+ \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M} & \longrightarrow & B_{\mathrm{cris}}^+ \otimes_{\mathbb{Z}_p} T^\vee \\
\uparrow & & \uparrow \\
W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} & \longrightarrow & W(R) \otimes_{\mathbb{Z}_p} T^\vee
\end{array}
$$

in the above diagram is clearly commutative. Furthermore, by direct computations, we can check that the square

$$
\begin{array}{ccc}
B_{\mathrm{cris}}^+ \otimes_{K_0} D & \hookrightarrow & B_{\mathrm{cris}}^+ \otimes_{\mathbb{Z}_p} T^\vee \\
\simeq\downarrow\xi_\alpha' & & \| \\
B_{\mathrm{cris}}^+ \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M} & \longrightarrow & B_{\mathrm{cris}}^+ \otimes_{\mathbb{Z}_p} T^\vee
\end{array}
$$

in the diagram is also commutative (here we note that $\xi'_\alpha$ appears in the definition of $\iota_0$). Hence, seeing the biggest square in the diagram (3.4), we obtain a commutative diagram

$$
\begin{array}{ccc}
B^+_{\mathrm{cris}} \otimes_{K_0} D & \hookrightarrow & B^+_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} T^\vee \\
\uparrow & & \uparrow \\
W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} & \hookrightarrow & W(R) \otimes_{\mathbb{Z}_p} T^\vee
\end{array}
$$

By this diagram, we regard $B^+_{\mathrm{cris}} \otimes_{K_0} D, W(R) \otimes_{\mathbb{Z}_p} T^\vee$ and $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ as $\varphi$-stable submodules of $B^+_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} T^\vee$. Note that $B^+_{\mathrm{cris}} \otimes_{K_0} D$ and $W(R) \otimes_{\mathbb{Z}_p} T^\vee$ are $G$-stable submodules of $B^+_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} T^\vee$.

LEMMA 3.17. *Let the notation be as above.*
(1) $G_{\underline{\pi}}$ *acts on* $\varphi^*\mathfrak{M}$ *trivial.*
(2) *The $G$-action on $W(R) \otimes_{\mathbb{Z}_p} T^\vee$ preserves $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$.*
(3) *The $G$-action on $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ commutes with $\varphi$.*
(4) $G(\varphi^*\mathfrak{M}) \subset \widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$.

*Proof.* (1) is trivial. If we admit (2), the statement (3) follows from the fact that $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ is a $\varphi$-stable submodule of $W(R) \otimes_{\mathbb{Z}_p} T^\vee$ and the $G$-action on $W(R) \otimes_{\mathbb{Z}_p} T^\vee$ commutes with $\varphi$. Hence it suffices to show (2) and (4).
We show (2). Take any $g \in G$. Let $e_1, \ldots, e_d$ be a basis of $\varphi^*\mathfrak{M}$. Note that this is also a basis of $B^+_{\mathrm{cris}} \otimes_{K_0} D$. Hence we have $g(e_1, \ldots, e_d) = (e_1, \ldots, e_d)X_g$ for some $X_g \in GL_d(B^+_{\mathrm{cris}})$. By Proposition 3.6 (2), $\varphi(\mathfrak{t})^r g(e_1, \ldots, e_d) = (e_1, \ldots, e_d)A_g$ for some $A_g \in M_d(W(R))$. Hence we have $\varphi(\mathfrak{t})^r X_g = A_g \in M_d(W(R))$. Note that $\varphi(\mathfrak{t})$ is a generator of $I^{[1]}W(R)$ by [Fo1, Proposition 5.1.3]. Hence we have $X_g \in M_d(W(R))$ by [Li4, Lemma 3.1.3].
Finally we show (4). By (2), it suffices to show that $X_g$ has coefficients in $\mathcal{R}_{K_0}$. Put $M := \varphi^*\mathfrak{M}/u\varphi^*\mathfrak{M}$. Let $\xi_\alpha \colon \mathfrak{D}_\alpha \otimes_{W(k)} M \xrightarrow{\sim} \mathfrak{D}_\alpha \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M}$, $\xi'_\alpha \colon \mathfrak{D}_\alpha \otimes_{K_0} D \xrightarrow{\sim} \mathfrak{D}_\alpha \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M}$ and $Y$ be as in Section 3.3. By [CL, Corollary 4.5.7], we have an equality $\xi_\alpha(M[1/p]) = \xi'_\alpha(D)$. By definition of the $G$-action on $\mathfrak{M}$, we know that $B^+_{\mathrm{cris}} \otimes_{\mathfrak{D}_\alpha} \xi'_\alpha$ is $G$-equivalent and thus $G$ acts on $\xi_\alpha(M)$ trivial. This implies $g((e_1, \ldots, e_d)Y) = (e_1, \ldots, e_d)Y$. Thus we have $X_g = Yg(Y)^{-1}$, which is an element of $GL_d(\mathcal{R}_{K_0})$. $\qquad\square$

By the above lemma, we have a natural $\widehat{\mathcal{R}}$-semi-linear $G$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$, which commutes with $\varphi$. Since $\mathrm{Gal}(\overline{K}/\widehat{K}_{\underline{\pi}})$ acts on $\widehat{\mathcal{R}}$ and $\varphi^*\mathfrak{M}$ trivial, the $G$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ factors through $\hat{G}$. Hence $\mathfrak{M}$ has a structure of an object of $\mathrm{Mod}_{\mathfrak{S}}^{r,\hat{G}}$, which we denote by $\hat{\mathfrak{M}}$.

LEMMA 3.18. *Let the notation be as above. Then we have a natural isomorphism $\hat{T}(\hat{\mathfrak{M}}) \simeq T$ of $\mathbb{Z}_p[G]$-modules.*

*Proof.* We follow the method of [Li2, Section 3.2]. First we recall that we defined a $G$-action on $T_{\mathfrak{S}}(\mathfrak{M})$ by the isomorphism $\iota_0 \colon T_{\mathfrak{S}}(\mathfrak{M}) \simeq T$, and also recall that the injection $\varphi^* \iota_{\mathfrak{S}} \colon W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \hookrightarrow W(R) \otimes_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathfrak{M})^\vee$ is $G$-equivariant by definition of the $G$-action on $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. We consider the following commutative diagram:

$$
\begin{array}{ccccc}
W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} & \xrightarrow{\;\varphi^* \iota_{\mathfrak{S}}\;} & W(R) \otimes_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathfrak{M})^\vee & \xleftarrow[\iota_0^\vee]{\;\sim\;} & W(R) \otimes_{\mathbb{Z}_p} T^\vee \\[4pt]
\Big\| & & {\scriptstyle \simeq}\Big\uparrow{\scriptstyle \eta} & & \\[4pt]
W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} & \xrightarrow{\;\hat{\iota}\;} & W(R) \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})^\vee & &
\end{array}
$$

Here, $\eta := W(R) \otimes \theta^\vee$. It suffices to show that $\eta$ is $G$-equivariant. Note that all arrows in the diagram except $\eta$ are known to be $G$-equivariant and $\varphi(\mathfrak{t})W(R) = I^{[1]}W(R)$ is stable under the $G$-action on $W(R)$. By Corollary 2.10 and Proposition 3.6 (2), we can regard $\varphi(\mathfrak{t})W(R) \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})^\vee$ and $\varphi(\mathfrak{t})W(R) \otimes_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathfrak{M})^\vee$ as $G$-stable submodules of $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$, and thus $\eta$ restricted to $\varphi(\mathfrak{t})W(R) \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})^\vee$ induces an $G$-equivariant isomorphism $\varphi(\mathfrak{t})W(R) \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})^\vee \simeq \varphi(\mathfrak{t})W(R) \otimes_{\mathbb{Z}_p} T_{\mathfrak{S}}(\mathfrak{M})^\vee$. It follows from this that $\eta$ is $G$-equivariant. $\qquad\square$

Finally, we show the following, which completes a proof of Theorem 3.8 (3).

Lemma 3.19. *Let the notation be as above. Then $\hat{\mathfrak{M}}$ is an object of $'\mathrm{Mod}_{\mathfrak{S}}^{r,\hat{G},\mathrm{cris}}$.*

*Proof.* Let $\mathfrak{e}_1, \ldots, \mathfrak{e}_d$ be a basis of $\mathfrak{M}$ and let $A \in M_d(\mathfrak{S})$ be a matrix such that $\varphi(\mathfrak{e}_1, \ldots, \mathfrak{e}_d) = (\mathfrak{e}_1, \ldots, \mathfrak{e}_d)A$. Put $e_i = 1 \otimes \mathfrak{e}_i \in \varphi^*\mathfrak{M}$ for each $i$. Then $e_1, \ldots, e_d$ is a basis of $\varphi^*\mathfrak{M}$ and $\varphi(e_1, \ldots, e_d) = (e_1, \ldots, e_d)\varphi(A)$. Put $M := \varphi^*\mathfrak{M}/u\varphi^*\mathfrak{M}$ and $\bar{e}_i = e_i \bmod u\varphi^*\mathfrak{M}$ for each $i$. Then $\bar{e}_1, \ldots, \bar{e}_d$ is a basis of $M$ and $\varphi(\bar{e}_1, \ldots, \bar{e}_d) = (\bar{e}_1, \ldots, \bar{e}_d)\varphi(A_0)$ where $A_0 = A \bmod u\mathfrak{S} \in M_d(W(k))$. Take any $g \in G$ and put $u_g = gu - u$. Let $X_g \in GL_d(\widehat{\mathcal{R}})$ be a matrix given by

$$g(e_1, \ldots, e_d) = (e_1, \ldots, e_d)X_g.$$

Let $Y$ be as in Section 3.3. Then we have $X_g = Yg(Y)^{-1}$ (see the proof of Lemma 3.17 (4)). First we show $X_g - I_d \in \varphi(u_g)M_d(\tilde{B}_\alpha)$. We claim $Y \in \varphi(M_d(\mathfrak{O}_\alpha))$. To check this, we use almost the same method as the proof of [CL, Lemma 4.5.6]. Since $\mathfrak{M}$ is of height $r$, there exists a matrix $B \in M_d(\mathfrak{S})$ such that $AB = E(u)^r I_d$. We denote by $A_0$ and $B_0$ the image of $A$ and $B$ for $\nu \colon M_d(\mathfrak{S}) \to M_d(W(k))$. To simplify notation, we assume $E(0) = p$. We write $A = A_0 + uC$ by some $C \in M_d(\mathfrak{S})$. Put

$$Y_n = \varphi(A) \cdots \varphi^n(A)\varphi^n(A_0^{-1}) \cdots \varphi(A_0^{-1}).$$

Then $Y_n$ converges to $Y$ and we have

$$Y_{n+1} - Y_n = \varphi\left(\varphi^n(u)/p^{r(n+1)} \cdot Z_n\right)$$

where $Z_n := A \cdots \varphi^{n-1}(A)\varphi^n(CB_0)\varphi^{n-1}(B_0)\cdots B_0 \in M_d(\mathfrak{S})$ . Since $\varphi^n(u)/p^{r(n+1)}$ converges to zero $p$-adically in $\mathfrak{O}_\alpha$, we have $\sum_{n=1}^\infty (Y_{n+1} - Y_n) \in \varphi(M_d(\mathfrak{O}_\alpha))$. Therefore, we have $Y = \sum_{n=1}^\infty (Y_{n+1} - Y_n) + Y_1 \in \varphi(M_d(\mathfrak{O}_\alpha))$. Thus we can write $Y = I_d + \varphi(Z)$ by some $Z \in M_d(\mathfrak{O}_\alpha)$ and then we have

$$X_g = (I_d + \varphi(Z))g(Y^{-1}) = ((I_d + \varphi(g(Z)) - \varphi(g(Z) - Z))g(Y^{-1})$$
$$= (g(Y) - \varphi(g(Z) - Z))g(Y^{-1}) = I_d - \varphi(g(Z) - Z)g(Y^{-1}).$$

By Lemma 3.14, the matrix $\varphi(g(Z) - Z)$ has coefficients in $\varphi(u_g)\tilde{B}_\alpha$. This shows $X_g - I_d \in \varphi(u_g)M_d(\tilde{B}_\alpha)$ as desired.

Now we are ready to finish the proof of Lemma 3.19. Take any $g \in G$ and $x \in \mathfrak{M}$. We want to show $g(1\otimes x)-(1\otimes x) \in \varphi(u_g)\tilde{B}_\alpha \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. Let $\mathbf{x} \in M_{d,1}(\varphi(\mathfrak{S}))$ be a matrix such that $1 \otimes x = (e_1, \ldots e_d)\mathbf{x}$. Then we have $g(1 \otimes x) - (1 \otimes x) = (e_1, \ldots e_d)(X_g g\mathbf{x} - \mathbf{x})$. Since we can write $X_g = I_d + \varphi(u_g)X_g'$ by some matrix $X_g' \in M_d(\tilde{B}_\alpha)$, we have $X_g g\mathbf{x} - \mathbf{x} = \varphi(u_g)X_g' g\mathbf{x} + (g\mathbf{x} - \mathbf{x})$. Since we have $g\mathbf{x} - \mathbf{x} \in \varphi(u_g)M_{d,1}(W(R))$, we finish the proof. $\square$

### 3.6 COMPATIBILITY OF DIFFERENT UNIFORMIZERS, AND DIEUDONNÉ CRYSTALS

Suppose the conditions (P) and $v_p(a_1) > \max\{r, 1\}$. Let $T$ be an object of $\mathrm{Rep}_{\mathbb{Z}_p}^{r,\mathrm{cris}}(G)$. Then there exists a $(\varphi, \hat{G})$-module $\hat{\mathfrak{M}}$ such that $\hat{T}(\hat{\mathfrak{M}}) \simeq T$. Note that our arguments depends on the choice of a uniformizer $\pi$ of $K$, a polynomial $f(u)$ and a system $(\pi_n)_{n\geq0}$.

If we select a different choice of a uniformizer $\pi'$ of $K$, a polynomial $f'(u)$ and a system $(\pi_n')_{n\geq0}$, then we get another $(\varphi, \hat{G}')$-module $\hat{\mathfrak{M}}'$.

QUESTION 3.20. What is the relationship between $\hat{\mathfrak{M}}$ and $\hat{\mathfrak{M}}'$ ?

We denote by $\mathfrak{S}_{\underline{\pi}}$ (resp. $\mathfrak{S}_{\underline{\pi}'}$) the image of the injection $W(k)[\![u]\!] \to W(R)$ given by $u \mapsto \{\underline{\pi}\}_f$ (resp. $u \mapsto \{\underline{\pi}'\}_{f'}$). We may regard $\mathfrak{M}$ (resp. $\mathfrak{M}'$) as a $\varphi$-module over $\mathfrak{S}_{\underline{\pi}}$ (resp. $\mathfrak{S}_{\underline{\pi}'}$). Write $\mathfrak{S} := \mathfrak{S}_{\underline{\pi}}$ (resp. $\mathfrak{S}' := \mathfrak{S}_{\underline{\pi}'}$). We have comparison morphisms

$$\hat{\iota}\colon W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \hookrightarrow W(R) \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}})^\vee \simeq W(R) \otimes_{\mathbb{Z}_p} T^\vee$$

and

$$\hat{\iota}'\colon W(R) \otimes_{\varphi,\mathfrak{S}'} \mathfrak{M}' \hookrightarrow W(R) \otimes_{\mathbb{Z}_p} \hat{T}(\hat{\mathfrak{M}}')^\vee \simeq W(R) \otimes_{\mathbb{Z}_p} T^\vee$$

THEOREM 3.21. *Assume the conditions (P) and $v_p(a_1) > \max\{r, 1\}$. Let the notation be as above. Then we have $\hat{\iota}(W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}) = \hat{\iota}(W(R) \otimes_{\varphi,\mathfrak{S}'} \mathfrak{M}')$. In particular, we have a functorial isomorphism $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \simeq W(R) \otimes_{\varphi,\mathfrak{S}'} \mathfrak{M}'$ which commutes with $\varphi$ and $G$-actions.*

*Proof.* Let $d$ be the $\mathbb{Z}_p$-rank of $T$. Put $M = \varphi^*\mathfrak{M}/u\varphi^*\mathfrak{M}$. We have $G$-equivariant injections $B_{\mathrm{cris}}^+ \otimes_{W(k)} M \overset{\xi_\alpha}{\simeq} B_{\mathrm{cris}}^+ \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M} \overset{\hat{\iota}}{\hookrightarrow} B_{\mathrm{cris}}^+ \otimes_{\mathbb{Z}_p} T^\vee$. By Lemma

3.15, we have $\xi_\alpha(M) \subset (B_{\mathrm{cris}}^+ \otimes_{\mathfrak{S}} \varphi^* \mathfrak{M})^G \overset{\hat{\iota}}{\hookrightarrow} (B_{\mathrm{cris}}^+ \otimes_{\mathbb{Z}_p} T^\vee)^G \subset D_{\mathrm{cris}}(V)$. Since the $W(k)$-rank of $M$ is $d$, we have isomorphisms

$$M[1/p] \overset{\xi_\alpha}{\simeq} (B_{\mathrm{cris}}^+ \otimes_{\mathfrak{S}} \varphi^* \mathfrak{M})^G \overset{\hat{\iota}}{\simeq} (B_{\mathrm{cris}}^+ \otimes_{\mathbb{Z}_p} T^\vee)^G = D_{\mathrm{cris}}(V). \qquad (3.5)$$

Therefore, we obtain the following diagram:



Here, two vertical arrows in the diagram are natural maps. We see that the left vertical arrow is isomorphism by the commutativity of the diagram.

Let $e_1, \ldots, e_d$ be a basis of $\varphi^* \mathfrak{M}$ and $e_1', \ldots, e_d'$ be a basis of $\varphi^* \mathfrak{M}'$. Seeing the above diagram, we obtain the fact that $\hat{\iota}(e_1), \ldots, \hat{\iota}(e_d)$ is a basis of $B_{\mathrm{cris}}^+ \otimes_{K_0} D_{\mathrm{cris}}(V)$. Similarly, $\hat{\iota}'(e_1'), \ldots, \hat{\iota}'(e_d')$ is also. Hence there exist a matrix $X \in GL_d(B_{\mathrm{cris}}^+)$ such that $\hat{\iota}(e_1, \ldots, e_d) = \hat{\iota}'(e_1', \ldots, e_d')X$. On the other hand, if we take any generator $\mathfrak{t}_0$ of $I^{[1]}W(R)$, we have $\mathfrak{t}_0^r \hat{\iota}'(W(R) \otimes_{\mathfrak{S}} \varphi^* \mathfrak{M}') \subset \mathfrak{t}_0^r(W(R) \otimes_{\mathbb{Z}_p} T^\vee) \subset \hat{\iota}(W(R) \otimes_{\varphi, \mathfrak{S}} \varphi^* \mathfrak{M})$. Thus we obtain $\mathfrak{t}_0^r X \in M_d(W(R))$. By [Li4, Lemma 3.1.3], $X \in M_d(W(R))$. By the similar manner we can check $X^{-1} \in M_d(W(R))$. This finishes the proof. (The assertion for the functoriality follows immediately by construction.) $\qquad\square$

The following statements gives an affirmative answer of [CL, Section 6.3].

COROLLARY 3.22. *Assume the conditions* $(P)$ *and* $v_p(a_1) > \max\{r, 1\}$. *Let* $T$ *be an object of* $\mathrm{Rep}_{\mathbb{Z}_p}^{r,\mathrm{cris}}(G)$. *Let* $\mathfrak{M}$ (*resp.* $\mathfrak{M}'$) *be a Kisin module with respect to the choice of* $(f(u), (\pi_n)_{n \geq 0})$ (*resp.* $(f'(u), (\pi_n')_{n \geq 0})$) *such that* $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T$ (*resp.* $T_{\mathfrak{S}'}(\mathfrak{M}') \simeq T$). *Then we have a functorial isomorphism* $W(R) \otimes_{\mathfrak{S}} \mathfrak{M} \simeq W(R) \otimes_{\mathfrak{S}'} \mathfrak{M}'$ *of* $\varphi$-*modules over* $W(R)$.

*Proof.* Let $\hat{\mathfrak{N}}$ (resp. $\hat{\mathfrak{N}}'$) be a $(\varphi, \hat{G})$-module with respect to the choice of $(f(u), (\pi_n)_{n \geq 0})$ (resp. $(f'(u), (\pi_n')_{n \geq 0})$) corresponding to $T$. By Theorem 3.21, we have an isomorphism $W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{N} \simeq W(R) \otimes_{\varphi, \mathfrak{S}'} \mathfrak{N}'$. Taking $W(R) \otimes_{\varphi^{-1}, W(R)}$, we obtain an isomorphism $W(R) \otimes_{\mathfrak{S}} \mathfrak{N} \simeq W(R) \otimes_{\mathfrak{S}'} \mathfrak{N}'$. On the other hand, we have isomorphisms $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T|_{G_\pi} \simeq \hat{T}(\hat{\mathfrak{N}})|_{G_\pi} \simeq T_{\mathfrak{S}}(\mathfrak{N})$. Similarly, we also have $T_{\mathfrak{S}'}(\mathfrak{M}') \simeq T_{\mathfrak{S}'}(\mathfrak{N}')$. By the condition (P) and Proposition 2.6, we have isomorphisms $\mathfrak{M} \simeq \mathfrak{N}$ and $\mathfrak{M}' \simeq \mathfrak{N}'$. Thus the result follows. $\qquad\square$

REMARK 3.23. In fact, we can replace the conditions "$(P)$ and $v_p(a_1) > \max\{r, 1\}$" in Theorem 3.21 and Corollary 3.22 with "$v_p(a_1) > 1$". See Section 3.7.

THE CASE $r \leq 1$. In the case $r \leq 1$, we can omit the assumption (P) from Theorem 3.21 and Corollary 3.22.

THEOREM 3.24. *Assume $v_p(a_1) > 1$. Let $T$ be an object of $\mathrm{Rep}_{\mathbb{Z}_p}^{1,\mathrm{cris}}(G)$. Let $\mathfrak{M}$ (resp. $\mathfrak{M}'$) be the Kisin module with respect to the choice of $(f(u), (\pi_n)_{n \geq 0})$ (resp. $(f'(u), (\pi'_n)_{n \geq 0})$) corresponding to $T$ via Theorem 2.7. Then we have a functorial isomorphism $W(R) \otimes_{\mathfrak{S}} \mathfrak{M} \simeq W(R) \otimes_{\mathfrak{S}'} \mathfrak{M}'$ of $\varphi$-modules over $W(R)$.*

*Proof.* At first, in the proof of Theorem 3.21, we used the assumption (P) to apply Lemma 3.15 and to obtain (3.5). Following the arguments of [CL, Section 5], we can obtain the same result without (P) in the case $r = 1$, as follows. By the arguments of [CL, Section 5], we can equip $W(R) \otimes_{\mathfrak{S}} \mathfrak{M}$ with a (unique) $G$-action which satisfies the following:

- $G_{\underline{\pi}}$ acts on $\mathfrak{M}$ trivial, and

- $g(1 \otimes x) - 1 \otimes x \in \mathfrak{t}M_d(I_+W(R))$ for any $g \in G$ and $x \in \mathfrak{M}$.

(Note that their arguments do not work for $r > 1$.) Moreover, if we equip $T_{\mathfrak{S}}(\mathfrak{M})$ with a $G$-action by the isomorphism $T_{\mathfrak{S}}(\mathfrak{M}) \simeq \mathrm{Hom}_{W(R),\varphi}(W(R) \otimes_{\mathfrak{S}} \mathfrak{M}, W(R))$, then we have an isomorphism $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T$ of $\mathbb{Z}_p[G]$-modules. Now we recall how to define a $G$-action on $W(R) \otimes_{\mathfrak{S}} \mathfrak{M}$. Let $\mathfrak{e}_1, \ldots, \mathfrak{e}_d$ be a basis of $\mathfrak{M}$ and let $A \in M_d(\mathfrak{S})$ be the matrix given by $\varphi(\mathfrak{e}_1, \ldots, \mathfrak{e}_d) = (\mathfrak{e}_1, \ldots, \mathfrak{e}_d)A$. Set $X'_g := \lim_{n \to \infty} A\varphi(A) \cdots \varphi^n(A)g\varphi^n(A)^{-1} \cdots g\varphi(A)^{-1}gA^{-1}$, which is an element of $GL_d(W(R))$. We put $X_g = \varphi(X'_g)$. Then we have $X_g = Yg(Y)^{-1}$ where $Y$ is the matrix defined in Section 3.3. Hence we see that the composite $B_{\mathrm{cris}}^+ \otimes_{W(k)} M \overset{\xi_\alpha}{\simeq} B_{\mathrm{cris}}^+ \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M} \overset{\hat{\iota}}{\hookrightarrow} B_{\mathrm{cris}}^+ \otimes_{\mathbb{Z}_p} T^\vee$ induces $\xi_\alpha(M) \subset (B_{\mathrm{cris}}^+ \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M})^G \overset{\hat{\iota}}{\hookrightarrow} (B_{\mathrm{cris}}^+ \otimes_{\mathbb{Z}_p} T^\vee)^G$, which gives $M[1/p] \overset{\xi_\alpha}{\simeq} (B_{\mathrm{cris}}^+ \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M})^G \overset{\hat{\iota}}{\simeq} (B_{\mathrm{cris}}^+ \otimes_{\mathbb{Z}_p} T^\vee)^G = D_{\mathrm{cris}}(V)$ as (3.5). Then the same arguments as Theorem 3.21 proceeds. □

COMPARISON WITH DIEUDONNÉ CRYSTALS. In this section, we give a geometric interpretation of Kisin modules in terms of Dieudonné crystals of $p$-divisible groups under our $K_{\underline{\pi}}/K$-setting, which is well-known in the Kisin's setting $f(u) = u^p$. We recall that (cf. Theorem 2.7), under the assumption $v_p(a_1) > 1$, there exists an anti-equivalence of categories between the category $\mathrm{Mod}_{\mathfrak{S}}^1$ of free Kisin modules of height 1 and the category of $p$-divisible groups over the ring of integers $\mathcal{O}_K$ of $K$.

REMARK 3.25. Consider the Kisin's setting $f(u) = u^p$. In this case Theorem 2.7 is well-studied. Let $S$ be the $p$-adic completion of the divided power envelope of the surjection $W[\![u]\!] \twoheadrightarrow \mathcal{O}_K$ given by $u \mapsto \pi$. Let $H$ be a $p$-divisible group over $\mathcal{O}_K$ and $\mathfrak{M}$ the free Kisin module attached to $H$. Then it is known that we have a functorial isomorphism $S \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M} \simeq \mathbb{D}(H)(S)$. For this, see [Kis, Theorem 2.2.7 and Proposition A.6] for $p > 2$ and [Kim, Proposition 4.2] for $p = 2$.

Combining Theorems 2.7, 3.24 and Remark 3.25, the result below follows immediately.

THEOREM 3.26. *Assume $v_p(a_1) > 1$. Let $H$ be a p-divisible group over $\mathcal{O}_K$ and $\mathbb{D}(H)$ be the Dieudonné crystal attached to $H$. Let $\mathfrak{M}$ be the Kisin module attached to $H$. Then there exists a functorial isomorphism $A_{\mathrm{cris}} \otimes_{\mathfrak{S}} \varphi^*\mathfrak{M} \simeq \mathbb{D}(H)(A_{\mathrm{cris}})$.*

### 3.7 Appendix

I leave here some comments from the anonymous referee, which refines some results given in this section. His/Her idea is based on the theory of Shtuka and Breuil-Kisin-Fargues modules. It is helpful for the reader to refer Section 4 of [BMS]. We follow notions in *loc. cit.*

ON THEOREM 3.21 AND COROLLARY 3.22. We can replace the assumptions (P) and $v_p(a_1) > \max\{r, 1\}$ in these results with only one assumption $v_p(a_1) > 1$. The proof is as follows.
By [BMS, Theorem 4.28], there exists an equivariant covariant functor $\mathfrak{F}$ from the category of finite free Breuil-Kisin-Fargues modules $\tilde{\mathfrak{M}}$ over $W(R)$ to the category of pairs $(T, \Xi)$, where $T$ is a finite free $\mathbb{Z}_p$-module and $\Xi$ is a $B_{\mathrm{dR}}^+$-lattice of $B_{\mathrm{dR}} \otimes_{\mathbb{Z}_p} T$. Explicitly, $\tilde{\mathfrak{M}}$ corresponds to the pair $(T, \Xi)$ where

$$T = (W(\mathrm{Fr}\ R) \otimes_{W(R)} \tilde{\mathfrak{M}})^{\varphi=1} \quad \text{and} \quad \Xi = B_{\mathrm{dR}}^+ \otimes_{W(R)} \tilde{\mathfrak{M}}.$$

Now let $T$ be an object of $\mathrm{Rep}_{\mathbb{Z}_p}^{r,\mathrm{cris}}(G)$ and $\mathfrak{M}$ a Kisin module such that $T_{\mathfrak{S}}(\mathfrak{M}) \simeq T|_{G_\infty}$. Since $\mathfrak{M}$ is of finite $E(u)$-height, we see that $\tilde{\mathfrak{M}} := W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ is a finite free Breuil-Kisin-Fargues modules over $W(R)$. Let $(T', \Xi')$ be the pair corresponding to $\tilde{\mathfrak{M}}$. Since we have $v_p(a_1) > 1$ and $\varphi(\mathfrak{t})$ is a unit of $W(\mathrm{Fr}\ R)$, we have an isomorphism $W(\mathrm{Fr}\ R) \otimes_{W(R)} \tilde{\mathfrak{M}} \xrightarrow{\sim} W(\mathrm{Fr}\ R) \otimes_{\mathbb{Z}_p} T^\vee$ by Corollary 2.10. This gives $f: T' \xrightarrow{\sim} T^\vee$. On the other hand, we see that the map $B_{\mathrm{dR}} \otimes f: B_{\mathrm{dR}} \otimes_{\mathbb{Z}_p} T' \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{\mathbb{Z}_p} T^\vee$ induces $\Xi' = B_{\mathrm{dR}}^+ \otimes_{W(R)} \tilde{\mathfrak{M}} \xrightarrow{\sim} B_{\mathrm{dR}}^+ \otimes_{K_0} D_{\mathrm{cris}}(T[1/p])$ (in fact, it is not difficult to check that this map coincides with the inverse of $B_{\mathrm{dR}}^+ \otimes \xi'_\alpha: B_{\mathrm{dR}}^+ \otimes_{K_0} D_{\mathrm{cris}}(T[1/p]) \xrightarrow{\sim} B_{\mathrm{dR}}^+ \otimes_{W(R)} \tilde{\mathfrak{M}}$). Therefore, $\tilde{\mathfrak{M}}$ corresponds to the pair $(T^\vee, B_{\mathrm{dR}}^+ \otimes_{K_0} D_{\mathrm{cris}}(T[1/p]))$ via $\mathfrak{F}$, which does not depend on the choice of $(f(u), (\pi_n)_{n\geq 0})$. Thus we obtain the desired result.

ON THEOREM 3.8 (1). We can remove the assumption (P) from the statement of the theorem.
Let $\hat{\mathfrak{M}}$ and $\hat{\mathfrak{N}}$ be objects of $\mathrm{Mod}_{\mathfrak{S}}^{r,\hat{G}}$. Set $T := \hat{T}(\hat{\mathfrak{M}})$ and $T' := \hat{T}(\hat{\mathfrak{M}}')$, and let $f: T' \to T$ be a $G$-equivariant morphism. Since $f$ induces a morphism from $(T^\vee, B_{\mathrm{dR}}^+ \otimes_{K_0} D_{\mathrm{cris}}(T[1/p]))$ to $(T'^\vee, B_{\mathrm{dR}}^+ \otimes_{K_0} D_{\mathrm{cris}}(T'[1/p]))$, we obtain $\mathfrak{f}: W(R) \otimes_{\mathfrak{S}} \mathfrak{M} \to W(R) \otimes_{\mathfrak{S}} \mathfrak{N}$ which commutes with $\varphi$. On the other hand, we have a morphism of étale $\varphi$-modules $M \to M'$ which corresponds to $f$. Since

we have $\mathfrak{M} = M \cap (W(R) \otimes_{\mathfrak{S}} \mathfrak{M})$ and $\mathfrak{M}' = M' \cap (W(R) \otimes_{\mathfrak{S}} \mathfrak{M}')$, we obtain a map $\mathfrak{f} \colon \mathfrak{M} \to \mathfrak{M}'$ of Kisin modules. To check that this induces a desired morphism $\mathfrak{f} \colon \hat{\mathfrak{M}} \to \hat{\mathfrak{M}}'$ of $(\varphi, \hat{G})$-modules is the same as our method (see just after Theorem 3.8).

## 4   Torsion representations and full faithfulness theorem

In this section, we study torsion Kisin modules and show a full faithfulness theorem for a restriction functor on a category of torsion crystalline representations.

### 4.1   Statements of full faithfulness theorems

We state main results of this section. Let $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G)$ be the category of torsion crystalline representations of $G$ with Hodge-Tate weights in $[0, r]$. Here, a torsion $\mathbb{Z}_p$-representation $T$ of $G$ is *torsion crystalline with Hodge-Tate weights in $[0, r]$* if $T$ is a quotient of lattices in a crystalline $\mathbb{Q}_p$-representation of $G$ with Hodge-Tate weights in $[0, r]$. For example, it is well-known that the category $\mathrm{Rep}_{\mathrm{tor}}^{1,\mathrm{cris}}(G)$ coincides with the category of flat representations of $G$. Here, a torsion $\mathbb{Z}_p$-representation $T$ of $G$ is *flat* if it is of the form $H(\overline{K})$ with some finite flat group scheme $H$ over the integer ring of $K$ killed by a power of $p$. In the case where $r = 1$, we have

THEOREM 4.1.   *Assume the condition (P) and $v_p(a_i) > 1$ for any $1 \le i \le p-1$. Then the restriction functor $\mathrm{Rep}_{\mathrm{tor}}^{1,\mathrm{cris}}(G) \to \mathrm{Rep}_{\mathrm{tor}}(G_{\underline{\pi}})$ is fully faithful.*

We recall that the condition (P) is that $\varphi^n(f(u)/u)$ is not a power of $E(u)$ for any $n \ge 0$. For general $r$, we need some more technical assumptions.

THEOREM 4.2.   *Assume the following conditions.*

(i)  *$gu \in uW(R)$ for any $g \in G$.*

(ii)  *$f^{(n)}(\pi) \ne 0$ for any $n \ge 1$.*

(iii)  *$v_p(a_1) > r$.*

*Then the restriction functor $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G) \to \mathrm{Rep}_{\mathrm{tor}}(G_{\underline{\pi}})$ is fully faithful if $e(r-1) < p-1$.*

REMARK 4.3 (This is pointed out by the anonymous referee).   The conditions (i) and (ii) in the theorem just above imply $v_p(a_i) > 1$ for any $1 \le i \le p-1$,. This can be checked as follows. Let $j_0$ be the minimum integer $1 \le j \le p$ such that $v_p(ja_j) = 1$. It suffices to show $j_0 = p$. Take any $g \in G$ such that $g(\pi_1) \ne \pi_1$ and let $\bar{u}$ be the image of $u$ for the projection $W(R) \twoheadrightarrow R$. By Proposition 3.12, we have $v_R(g\bar{u} - \bar{u}) = p/(p-1) + (j_0 - 1)/(e(p-1))$. On the other hand, it follows from Proposition 4.19 (1) (given later) that we have $v_R(g\bar{u} - \bar{u}) \ge p/(p-1) + 1/e$. Hence we obtain $j_0 = p$ as desired.

### 4.2 Some remarks

We give some remarks about the statement of Theorem 4.2.

THE CONDITION $e(r-1) < p-1$. We mention the condition $e(r-1) < p-1$ in the theorem. If we remove this condition, the full faithfulness property in the theorem does not hold as explained in [Oz2] even for the classical case $f(u) = u^p$; moreover, the condition $e(r-1) < p-1$ is (almost) optimal in this case. However, we expect that such full faithfulness should be weaker for general $f(u)$. For example, if $f(u)$ is chosen for cyclotomic extension (cf. [CL, Example 2.1.2]), it is not difficult to check that the full faithfulness holds only for $r = 0$. Motivated by Propositions 4.18 and 4.19, the invariant $j_0$ (defined in Proposition 4.18) might say something on such difference.

THE CONDITIONS (I) AND (II). Next we give some remarks about the conditions (i) and (ii) in Theorem 4.2. Here are some examples of iterated extensions which satisfy the condition (i).

- If $f(u) = u^p$, it is clear that the conditions (i), (ii) and (iii) above are satisfied.

- If $p$ is odd, $K$ is a finite extension of $\mathbb{Q}_p$ and $K_\pi/K$ is Galois (in this case this is abelian (cf. Remark 7.16 of [CD])), then the condition (i) is satisfied. In fact, the $G$-action on $W(R)$ preserves $\mathfrak{S}$ if $K_\pi/K$ is Galois and hence we have $gu \in I_+W(R) \cap \mathfrak{S} = u\mathfrak{S} \subset uW(R)$.

We give two remarks for the condition (ii). First, it is not difficult to check that the condition (ii) implies the condition (P). Next, for a fixed $f(u)$, the condition (ii) is satisfied except only finitely many choice of uniformizers $\pi$ of $K$. Moreover, we have the following. (We recall that $i_0$ is the integer defined by $f(u) = \sum_{i=i_0}^{p} a_i u^i$ with $a_{i_0} \neq 0$.)

PROPOSITION 4.4. *Put*

$$n_0 = \begin{cases} ev_p(a_1) & \text{if } i_0 = 1, \\ \max\{n \in \mathbb{Z} \mid i_0^n \leq e(i_0 - 1)v_p(a_{i_0}) + 1\} & \text{if } i_0 \neq 1. \end{cases}$$

*Then the following are equivalent.*

(ii) $f^{(n)}(\pi) \neq 0$ *for any* $n \geq 1$.

(ii$'$) $f^{(n)}(\pi) \neq 0$ *for any* $1 \leq n \leq n_0$.

*Proof.* Assume that there exists an integer $n \geq 1$ such that $f^{(i)}(\pi) \neq 0$ for any $0 \leq i \leq n-1$ and $f^{(n)}(\pi) = 0$. (In particular, we have $f(u) \neq u^p$.) It suffices to show $n \leq n_0$. Put $c_i = v_p(f^{(i)}(\pi))$ for $0 \leq i \leq n-1$. We have $c_0 = 1/e$ by definition. Note that $f^{(n-1)}(\pi)$ is a root of $X^{p-i_0} + \sum_{i=i_0}^{p-1} a_i X^{i-i_0}$. Seeing the Newton polygon of this polynomial, it is not difficult to check that

the inequality $c_{n-1} \leq v_p(a_{i_0})$ holds. On the other hand, we claim that the inequality

$$c_j \geq \frac{1}{e} \sum_{k=0}^{j} i_0^k \tag{4.1}$$

holds for any $0 \leq j \leq n-1$. We show this claim by induction on $j$. The case $j = 0$ is clear. Assume that (4.1) holds for $j = m-1$ and consider the case where $j = m$. It follows from the equation $f^{(m)}(\pi) = f^{(m-1)}(\pi)^p + \sum_{i=i_0}^{p-1} a_i f^{(m-1)}(\pi)^i$ that we have

$$c_m \geq \min\{pc_{m-1}, v_p(a_i) + ic_{m-1} \mid i = i_0, \ldots, p-1\} \geq \min\{pc_{m-1}, 1 + i_0 c_{m-1}\}$$

$$\geq \min\left\{\frac{p}{e} \sum_{k=0}^{m-1} i_0^k, \frac{1}{e} + \frac{i_0}{e} \sum_{k=0}^{m-1} i_0^k\right\} = \min\left\{\frac{p}{e} \sum_{k=0}^{m-1} i_0^k, \frac{1}{e} \sum_{k=0}^{m} i_0^k\right\}.$$

Since we have $p \sum_{k=0}^{m-1} i_0^k - \sum_{k=0}^{m} i_0^k \geq (1+i_0) \sum_{k=0}^{m-1} i_0^k - \sum_{k=0}^{m} i_0^k = \sum_{k=0}^{m-1} i_0^k - 1 \geq 0$, we obtain $c_m \geq e^{-1} \sum_{k=0}^{m} i_0^k$ as desired. Therefore, we obtain

$$\frac{1}{e} \sum_{k=0}^{n-1} i_0^k \leq c_{n-1} \leq v_p(a_{i_0}).$$

The desired result immediately follows from this. $\qquad\square$

## 4.3 MAXIMAL OBJECTS

We recall that the contravariant functor $T_{\mathfrak{S}} \colon \mathrm{Mod}_{\mathfrak{S}_\infty}^r \to \mathrm{Rep}_{\mathrm{tor}}(G_{\underline{\pi}})$ is exact and faithful (cf. Proposition 2.6). However, this is not full in general. In this section, following [CL1], we first define a notion of *maximal* Kisin modules[34]. Almost the arguments given in [CL1] carry over to the present situation. In particular, we can check that a category of maximal Kisin modules is abelian and the functor $T_{\mathfrak{S}}$ restricted to a category of maximal Kisin modules is fully faithful. These play an important role in the proof of Theorems 4.1 and 4.2.

Let $M$ be an étale $\varphi$-module over $\mathcal{O}_{\mathcal{E}}$ which is killed by a power of $p$. Let $F_{\mathfrak{S}}^r(M)$ be the set of torsion Kisin modules $\mathfrak{M}$ over $\mathfrak{S}$ of height $r$ such that $\mathfrak{M} \subset M$ and $\mathfrak{M}[1/u] = M$. The set $F_{\mathfrak{S}}^r(M)$ is an partially ordered set by inclusion.

LEMMA 4.5. *If $\mathfrak{M}, \mathfrak{M}' \in F_{\mathfrak{S}}^r(M)$, then we have $\mathfrak{M} + \mathfrak{M}', \mathfrak{M} \cap \mathfrak{M}' \in F_{\mathfrak{S}}^r(M)$.*

*Proof.* See the proof of Proposition 3.2.3 of [CL1]. $\qquad\square$

---

[3]We can also study the theory of *minimal* Kisin modules by similar arguments to [CL1]. However, we do not consider it in this paper since we do not need it for our purpose.

[4]As well as [CL1], results in this section can be applied also for the case "$r = \infty$" with suitable (minor) modifications.

LEMMA 4.6. *Let $\mathfrak{M}$ be a torsion Kisin module $\mathfrak{M}$ over $\mathfrak{S}$ of height $r$ and put $M = \mathfrak{M}[1/u]$. If $\mathfrak{M}' \in F_{\mathfrak{S}}^r(M)$ and $\mathfrak{M} \subset \mathfrak{M}'$, then we have*

$$\mathrm{length}_{\mathfrak{S}}(\mathfrak{M}'/\mathfrak{M}) \leq \left[\frac{er}{p-1}\right] \cdot \mathrm{length}_{\mathcal{O}_{\mathcal{E}}} M.$$

*Here, $[x]$ denotes the integer part of $x$.*

*Proof.* See the proof of Lemma 3.2.4 of [CL1]. □

By the above lemmas, we immediately obtain

COROLLARY 4.7. *Let $M \in \mathrm{Mod}_{\mathcal{O}_{\mathcal{E}},\infty}$ and suppose that $F_{\mathfrak{S}}^r(M) \neq \emptyset$.*
(1) *The set $F_{\mathfrak{S}}^r(M)$ has a greatest element and a smallest element.*
(2) *If $er < p - 1$, then $F_{\mathfrak{S}}^r(M)$ contains only one element.*

DEFINITION 4.8. *Let $\mathfrak{M}$ be a torsion Kisin module over $\mathfrak{S}$ of height $r$. We denote by $\mathrm{Max}^r(\mathfrak{M})$ the greatest element of $F_{\mathfrak{S}}^r(\mathfrak{M}[1/u])$. We say that $\mathfrak{M}$ is maximal (of height $r$) if $\mathfrak{M} = \mathrm{Max}^r(\mathfrak{M})$.*

We denote by $\mathrm{Max}_{\mathfrak{S}_\infty}^r$ the full subcategory of $\mathrm{Mod}_{\mathfrak{S}_\infty}^r$ consisting of maximal Kisin modules. By Corollary 4.7, we have $\mathrm{Mod}_{\mathfrak{S}_\infty}^r = \mathrm{Max}_{\mathfrak{S}_\infty}^r$ if $er < p - 1$.
We can check that all the properties given in Section 3.3 in [CL1] holds also for the present situation by the same arguments given in *loc. cit.* Here we describe only a part of properties on maximal Kisin modules that we need later.

THEOREM 4.9. (1) *The implication $\mathfrak{M} \mapsto \mathrm{Max}^r(\mathfrak{M})$ defines a covariant functor $\mathrm{Max}^r\colon \mathrm{Mod}_{\mathfrak{S}_\infty}^r \to \mathrm{Mod}_{\mathfrak{S}_\infty}^r$. Furthermore, this is left exact and $\mathrm{Max}^r \circ \mathrm{Max}^r = \mathrm{Max}^r$.*
(2) *The category $\mathrm{Max}_{\mathfrak{S}_\infty}^r$ is abelian. Moreover, for any morphism $f\colon \mathfrak{M} \to \mathfrak{M}'$ in $\mathrm{Max}_{\mathfrak{S}_\infty}^r$, we have the following.*

  (i) *The kernel $\ker(f)$ of $f$ in the usual sense is an object of $\mathrm{Max}_{\mathfrak{S}_\infty}^r$. Furthermore, it is the kernel of $f$ in the abelian category $\mathrm{Max}_{\mathfrak{S}_\infty}^r$.*

  (ii) *The cokernel $\mathrm{coker}(f)$ in the usual sense is of height $r$ and $\mathrm{coker}(f)/(u\text{-tors})$ is a Kisin module of height $r$. Moreover, $\mathrm{Max}^r(\mathrm{coker}(f)/(u\text{-tors}))$ is the cokernel of $f$ in the abelian category $\mathrm{Max}_{\mathfrak{S}_\infty}^r$. If $f$ is injective, then $\mathrm{coker}(f)$ is $u$-torsion free.*

  (iii) *The image $\mathrm{im}(f)$ (resp. the coimage $\mathrm{coim}(f)$) of $f$ in the usual sense is a Kisin module of height $r$. Moreover, $\mathrm{Max}^r(\mathrm{im}(f))$ (resp. $\mathrm{Max}^r(\mathrm{coim}(f))$) is the image (resp. the coimage) of $f$ in the abelian category $\mathrm{Max}_{\mathfrak{S}_\infty}^r$.*

(3) *Let $0 \to \mathfrak{M}' \xrightarrow{\alpha} \mathfrak{M} \xrightarrow{\beta} \mathfrak{M}'' \to 0$ be a sequence in $\mathrm{Max}_{\mathfrak{S}_\infty}^r$ such that $\beta \circ \alpha = 0$. Then this sequence is exact in the abelian category $\mathrm{Max}_{\mathfrak{S}_\infty}^r$ if and only if $0 \to \mathfrak{M}'[1/u] \xrightarrow{\alpha[1/u]} \mathfrak{M}[1/u] \xrightarrow{\beta[1/u]} \mathfrak{M}''[1/u] \to 0$ is exact as $\mathcal{O}_{\mathcal{E}}$-modules.*
(4) *The functor $\mathrm{Max}_{\mathfrak{S}_\infty}^r \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{E}},\infty}$ given by $\mathfrak{M} \mapsto \mathcal{O}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ is exact and fully faithful.*
(5) *The functor $T_{\mathfrak{S}}\colon \mathrm{Max}_{\mathfrak{S}_\infty}^r \to \mathrm{Rep}_{\mathrm{tor}}(G_{\underline{\pi}})$ is exact and fully faithful.*

*Proof.* (1) : See the proof of Propositions 3.3.2 to 3.3.4 of [CL1].
(2) : See the proof of Theorem 3.3.8 [CL1].
(3) and (4) : See the proof of Lemma 3.3.9 of [CL1].
(5) : This follows from (4) immediately.

<div align="right">□</div>

Let us consider simple objects in the abelian category $\mathrm{Max}^r_{\mathfrak{S}_\infty}$. Let $\mathcal{S}$ be the set of sequences $\mathfrak{n} = (n_i)_{i \in \mathbb{Z}/d\mathbb{Z}}$ of integers $0 \leq n_i \leq er$ with smallest period $d$ for some integer $d > 0$.

DEFINITION 4.10. Let $\mathfrak{n} = (n_i)_{i \in \mathbb{Z}/d\mathbb{Z}} \in \mathcal{S}$ be a sequence with smallest period $d$. We define a torsion Kisin module $\mathfrak{M}(\mathfrak{n})$ of height $r$, killed by $p$, as follows:

- as a $k[\![u]\!]$-module, $\mathfrak{M}(\mathfrak{n}) = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} k[\![u]\!] e_i$;

- for all $i \in \mathbb{Z}/d\mathbb{Z}$, $\varphi(e_i) = u^{n_i} e_{i+1}$.

We denote by $\mathcal{S}^r_{\max}$ the set of sequences $\mathfrak{n} = (n_i)_{i \in \mathbb{Z}/d\mathbb{Z}}$ of integers $0 \leq n_i \leq \min\{er, p-1\}$ with smallest period $d$ for some integer $d$ except the constant sequence with value $p-1$ (if necessary).

PROPOSITION 4.11. *Assume that $k$ is algebraically closed. Then all simple objects in the abelian category $\mathrm{Max}^r_{\mathfrak{S}_\infty}$ are of the form $\mathfrak{M}(\mathfrak{n})$ with some $\mathfrak{n} \in \mathcal{S}^r_{\max}$.*

*Proof.* This is a part of Propositions 3.6.8 and 3.6.12 in [CL1]. 

<div align="right">□</div>

### 4.4 $(\varphi, G)$-MODULES

DEFINITION 4.12. A *free (resp. torsion) $(\varphi, G)$-module (of height $r$)* is a triple $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, G)$ where

(1) $(\mathfrak{M}, \varphi)$ is a free (resp. torsion) Kisin module $\mathfrak{M}$ of height $r$,

(2) $G$ is a $W(R)$-semi-linear continuous $G$-action on $W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$,

(3) the $G$-action on $W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}$ commutes with $\varphi_{W(R)} \otimes \varphi_\mathfrak{M}$, and

(4) $\varphi^* \mathfrak{M} \subset (W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M})^{G_\pi}$.

We denote by $\mathrm{Mod}^{r,G}_{\mathfrak{S}}$ (resp. $\mathrm{Mod}^{r,G}_{\mathfrak{S}_\infty}$) the category of free (resp. torsion) $(\varphi, G)$-modules of height $r$.

We define a $\mathbb{Z}_p$-representation $\hat{T}(\hat{\mathfrak{M}})$ of $G$ for any $(\varphi, G)$-module $\hat{\mathfrak{M}}$ by

$$\hat{T}(\hat{\mathfrak{M}}) := \begin{cases} \mathrm{Hom}_{W(R),\varphi}(W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, W(R)). & \text{if } \hat{\mathfrak{M}} \in \mathrm{Mod}^{r,G}_{\mathfrak{S}}, \\ \mathrm{Hom}_{W(R),\varphi}(W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}, W(R) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p). & \text{if } \hat{\mathfrak{M}} \in \mathrm{Mod}^{r,G}_{\mathfrak{S}_\infty}. \end{cases}$$

Here, the $G$-action on $\hat{T}(\hat{\mathfrak{M}})$ is given by $(g.f)(x) := g(f(g^{-1}(x)))$ for $f \in \hat{T}(\hat{\mathfrak{M}})$, $g \in G$ and $x \in W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. Note that we have a natural isomorphism of $\mathbb{Z}_p[G_{\underline{\pi}}]$-modules

$$\theta \colon T_{\mathfrak{S}}(\mathfrak{M}) \xrightarrow{\sim} \hat{T}(\hat{\mathfrak{M}})$$

given by $\theta(f)(a \otimes x) := a\varphi(f(x))$ for $f \in T_{\mathfrak{S}}(\hat{\mathfrak{M}})$, $a \in W(R)$ and $x \in \mathfrak{M}$ (see the proof of [CL2, Theorem 3.1.3 (1)]). In particular, if $\hat{\mathfrak{M}}$ is free, then $\hat{T}(\hat{\mathfrak{M}})$ is a free $\mathbb{Z}_p$-module of rank $d$, where $d := \mathrm{rank}_{\mathfrak{S}} \mathfrak{M}$. Hence we obtain a contravariant functor

$$\hat{T} \colon \mathrm{Mod}_{\mathfrak{S}}^{r,G} \to \mathrm{Rep}_{\mathbb{Z}_p}(G) \quad \text{and} \quad \hat{T} \colon \mathrm{Mod}_{\mathfrak{S}_\infty}^{r,G} \to \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{tor}}(G).$$

For a $(\varphi,\hat{G})$-module $\hat{\mathfrak{M}}$, by extending the $G$-action on $\widehat{\mathcal{R}} \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ (naturally obtained by the $\hat{G}$-action on this module) to $W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$ by $W(R)$-semi-linearity, we obtain a $(\varphi,G)$-module; we abuse notation by writing $\hat{\mathfrak{M}}$ for it.

DEFINITION 4.13. Let $\alpha \in W(R) \smallsetminus pW(R)$. We define a full subcategory $\mathrm{Mod}_{\mathfrak{S}}^{r,G}(\alpha)$ (resp. $\mathrm{Mod}_{\mathfrak{S}_\infty}^{r,G}(\alpha)$) of $\mathrm{Mod}_{\mathfrak{S}}^{r,G}$ (resp. $\mathrm{Mod}_{\mathfrak{S}_\infty}^{r,G}$) consisting of objects $\hat{\mathfrak{M}}$ with the condition that

$$g(1 \otimes x) - (1 \otimes x) \in \alpha I^{[1]} W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$$

for any $g \in G$ and $x \in \mathfrak{M}$. We put $\bar{\alpha} = \alpha \bmod pW(R) \in R$.

THEOREM 4.14. *Let $r, r' \geq 0$, $\hat{\mathfrak{M}} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{r,G}(\alpha)$ and $\hat{\mathfrak{N}} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{r',G}(\alpha)$. Then we have $\mathrm{Hom}(\hat{\mathfrak{M}}, \hat{\mathfrak{N}}) = \mathrm{Hom}(\mathfrak{M}, \mathfrak{N})$ if $v_R(\bar{\alpha}) > p(r-1)/(p-1)$.*
*In particular, the forgetful functor $\mathrm{Mod}_{\mathfrak{S}_\infty}^{r,G}(\alpha) \to \mathrm{Mod}_{\mathfrak{S}_\infty}^{r}$ is fully faithful if $v_R(\bar{\alpha}) > p(r-1)/(p-1)$.*

*Proof.* The proof is the same as that of Proposition 4.2 in [Oz2]. Here we only explain why we need the condition $v_R(\bar{\alpha}) > p(r-1)/(p-1)$. Assume $p\mathfrak{N} = 0$ for simplicity. Let $g \in G$ and $f \colon \mathfrak{M} \to \mathfrak{N}$ be a morphism of Kisin modules. We also denote by $f \colon W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M} \to W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}$ the $W(R)$-linear extension of $f$. Then it follows from the argument of the proof of Proposition 4.2 of *loc. cit.* that we have $f \circ g(x) - g \circ f(x) \in \mathfrak{m}_R^{\geq c(s)} \otimes_{\varphi,\mathfrak{S}} \mathfrak{N}$ for any $s \geq 0$ and $x \in W(R) \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$. Here, $c(s)$ is defined by $c(0) = v_R(\bar{\alpha}) + p/(p-1)$ and $c(s+1) = pc(s) - pr$, that is, $c(s) = (v_R(\bar{\alpha}) - p(r-1)/(p-1))p^s + pr/(p-1)$. By the assumption $v_R(\bar{\alpha}) > p(r-1)/(p-1)$, we have $\lim_{s \to \infty} c(s) = \infty$ and hence $f$ commutes with $g$. $\square$

## 4.5   A $G$-action on $\mathfrak{M}(\mathfrak{n})$

In this section, we equip a $(\varphi, G)$-module structure on $\mathfrak{M}(\mathfrak{n})$. In the classical setting $f(u) = u^p$, this has been already studied in Section 4.3 of [Oz2] by using the fact that the $G$-action on $u$ is explicitly calculated. In the present setting, the $G$-action on $u$ is not so easy to understand, and so we need more delicate arguments.

THEOREM 4.15. *Assume that $v_p(a_i) > 1$ for any $1 \le i \le p - 1$. Let $\mathfrak{n} = (n_i)_{i \in \mathbb{Z}/d\mathbb{Z}} \in \mathcal{S}$ be a sequence with smallest period $d$. Let $\mathfrak{M}(\mathfrak{n})$ be the Kisin module of height $r$ defined in Definition 4.10. Then there exists a $W(R)$-semi-linear $G$-action on $W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(\mathfrak{n})$ which satisfies the following properties:*
(1) *The $G$-action on $W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(\mathfrak{n})$ commutes with $\varphi_{W(R)} \otimes \varphi_{\mathfrak{M}(\mathfrak{n})}$.*
(2) *$G_{\underline{\pi}}$ acts on $\mathfrak{S} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(\mathfrak{n})$ trivial.*
(3) *For any $g \in G$ and $x \in \mathfrak{M}$, we have $g(1 \otimes x) - (1 \otimes x) \in \mathfrak{m}_R^{\ge p^2/(p-1)} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(\mathfrak{n})$. Moreover, such a $G$-action is uniquely determined if $0 \le n_i \le \min\{er, p - 1\}$ for any $i$.*

REMARK 4.16. For the uniqueness assertion above, we do not need (2).

*Proof of Theorem 4.15.* Take any $(p^d - 1)$-st root $\pi_{(0)}$ of $\pi = \pi_0$. We define $\pi_{(n)}$ inductively by the formula $\pi_{(n)} = \pi_{(n-1)}^{p^{d-1}} \pi_n^{-1}$ for $n \ge 1$. We see $v_p(\pi_{(n)}) = 1/(ep^n(p^d - 1))$ and thus we have $\pi_{(n)} \in \mathcal{O}_{\overline{K}}$. Now we claim the following.

$$\pi_{(n)}^p \equiv \pi_{(n-1)} \bmod p\pi_{(n-1)}\mathcal{O}_{\overline{K}} \quad \text{and} \quad \pi_{(n)}^{p^d-1} \equiv \pi_n \bmod p\pi_n\mathcal{O}_{\overline{K}}. \qquad (4.2)$$

We proceed a proof of this claim by induction on $n$.

Consider the case $n = 1$. We have $\pi_{(1)}^p = \pi_{(0)}^{p^d} \pi_1^{-p} = \pi_{(0)} \pi_{(0)}^{p^d-1} \pi_1^{-p} = \pi_{(0)} \cdot \pi\pi_1^{-p}$ and $\pi = \pi_1^p + \sum_{i=1}^{p-1} a_i \pi_1^i$. Hence we obtain $\pi_{(1)}^p = \pi_{(0)} + \pi_{(0)} \sum_{i=1}^{p-1} a_i \pi_1^{-(p-i)}$. By the assumption $v_p(a_i) > 1$ for any $i$, we obtain $\pi_{(1)}^p \equiv \pi_{(0)} \bmod p\pi_{(0)}\mathcal{O}_{\overline{K}}$. On the other hand, we have $\pi_{(1)}^{p^d-1} = \pi_{(0)}^{p^{d-1}(p^d-1)} \pi_1^{-(p^d-1)} = \pi_1(\pi\pi_1^{-p})^{p^{d-1}} = \pi_1(1 + \sum_{i=1}^{p-1} a_i \pi_1^{-(p-i)})^{p^{d-1}}$. By the assumption $v_p(a_i) > 1$ for any $i$, we have $(1 + \sum_{i=1}^{p-1} a_i \pi_1^{-(p-i)})^{p^{d-1}} \in 1 + p\mathcal{O}_{\overline{K}}$. Hence we have $\pi_{(1)}^{p^d-1} \equiv \pi_1 \bmod p\pi_1\mathcal{O}_{\overline{K}}$ as desired.

Next we assume that (4.2) holds for $n = m - 1$ and consider the case $n = m$. By induction hypothesis, we have $\pi_{(m-1)}^{p^d-1} = \pi_{m-1} + p\pi_{m-1}x$ for some $x \in \mathcal{O}_{\overline{K}}$. Thus we have $\pi_{(m)}^p = \pi_{(m-1)}^{p^d} \pi_m^{-p} = \pi_{(m-1)} \pi_{(m-1)}^{p^d-1} \pi_m^{-p} = \pi_{(m-1)}(\pi_{m-1} + p\pi_{m-1}x)\pi_m^{-p} = \pi_{(m-1)}(1 + \sum_{i=1}^{p-1} a_i \pi_m^{-(p-i)} + p\pi_{m-1}\pi_m^{-p}x)$. By the assumption $v_p(a_i) > 1$ for any $i$, we obtain $\pi_{(m)}^p \equiv \pi_{(m-1)} \bmod p\pi_{(m-1)}\mathcal{O}_{\overline{K}}$. On the other hand, we have $\pi_{(m)}^{p^d-1} = (\pi_{(m-1)}^{p^{d-1}} \pi_m^{-1})^{p^d-1} = (\pi_{m-1} + p\pi_{m-1}x)^{p^{d-1}} \pi_m^{-p^d} \pi_m = \pi_m((\pi_{m-1} + p\pi_{m-1}x)\pi_m^{-p})^{p^{d-1}} = \pi_m(1 + \sum_{i=1}^{p-1} a_i \pi_m^{-(p-i)} + p\pi_{m-1}\pi_m^{-p}x)^{p^{d-1}}$. By the assumption $v_p(a_i) > 1$ for any $i$, we have $(1 + \sum_{i=1}^{p-1} a_i \pi_m^{-(p-i)} + p\pi_{m-1}\pi_m^{-p}x)^{p^{d-1}} \in 1 + p\mathcal{O}_{\overline{K}}$. Therefore, we obtain $\pi_{(m)}^{p^d-1} \equiv \pi_m \bmod p\pi_m\mathcal{O}_{\overline{K}}$. This finishes the proof of (4.2).

By (4.2), we can define an element $\underline{\pi}_d$ of $R$ by $\underline{\pi}_d := (\pi_{(n)} \bmod p\mathcal{O}_{\overline{K}})_{n \ge 0}$. By definition we have $\underline{\pi}_d^{p^d-1} = \underline{\pi}$. On the other hand, for any $g \in G$, there exists a unique $a_g \in \mathbb{F}_{p^d}^{\times}$ such that $g\pi_{(0)}\pi_{(0)}^{-1} = [a_g]$. Here, $[\cdot]$ stands for the Teichmüller lift. We note that we have a cocycle condition $a_{gh} = a_g \cdot ga_h$ for any $g, h \in G$.

Put $x_g = a_g^{-1} g \underline{\pi}_d \underline{\pi}_d^{-1} \in R^\times$. By the cocycle condition above, we can define an $R$-semi-linear $G$-action on $R \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(\mathfrak{n}) = \oplus_{i \in \mathbb{Z}/d\mathbb{Z}} R(1 \otimes e_i)$ by

$$g(1 \otimes e_i) = x_g^{m_i}(1 \otimes e_i)$$

for any $g \in G$ and $i \in \mathbb{Z}/d\mathbb{Z}$. Here, $m_i = \sum_{j=1}^d p^j n_{i-j}$. In the rest of this proof, we show that this $G$-action satisfies the assertions (1), (2) and (3) in the statement of this lemma. The assertion (1) can be checked by a direct computation without difficulty. We check (2) and (3) below.

We show (2). Let $g \in G_{\underline{\pi}}$. It suffices to show that $g \underline{\pi}_d \underline{\pi}_d^{-1}$ coincides with $a_g$. The case $(p, d) = (2, 1)$ is clear. Thus we may assume $(p, d) \neq (2, 1)$. Put $b_g := g \underline{\pi}_d \underline{\pi}_d^{-1}$, which is an element of $\mathbb{F}_{p^d}^\times$. Seeing the 0-th components of both sides of $g \underline{\pi}_d = b_g \underline{\pi}_d$, we have $g \pi_{(0)} \equiv [b_g] \pi_{(0)} \mod p \mathcal{O}_{\overline{K}}$. Thus we have $[a_g] \pi_{(0)} \equiv [b_g] \pi_{(0)} \mod p \mathcal{O}_{\overline{K}}$, and this induces $[a_g] - [b_g] \in p \pi_{(0)}^{-1} \mathcal{O}_{\overline{K}}$. By the assumption $(p, d) \neq (2, 1)$, we have $v_p(p \pi_{(0)}^{-1}) = 1 - 1/(e(p^d - 1)) > 0$, and hence we obtain $[a_g] - [b_g] \in p W(\overline{k})$. Therefore, we have $a_g = b_g$. This shows (2).

We show (3). We may assume $g \notin G_{\underline{\pi}}$. At first we show

$$g(1 \otimes e_i) - (1 \otimes e_i) \in \mathfrak{m}_R^{\geq \frac{p^2}{p-1}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(\mathfrak{n}) \tag{4.3}$$

for any $g \in G$ and $i \in \mathbb{Z}/d\mathbb{Z}$. Since $m_i$ is divided by $p$, it suffices to show $x_g - 1 \in \mathfrak{m}_R^{\geq p/(p-1)} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(\mathfrak{n})$. Note that the $n$-th component of $a_g^{-1} g \underline{\pi}_d - \underline{\pi}_d$ is $[a_g^{-p^{-n}}] g \pi_{(n)} - \pi_{(n)} \mod p \mathcal{O}_{\overline{K}}$. Hence we have

$$
\begin{aligned}
v_R(x_g - 1) &= v_R(a_g^{-1} g \underline{\pi}_d - \underline{\pi}_d) - v_R(\underline{\pi}_d) \\
&= \lim_{n \to \infty} p^n v_p([a_g^{-p^{-n}}] g \pi_{(n)} - \pi_{(n)}) - \frac{1}{e(p^d - 1)} \\
&= \lim_{n \to \infty} p^n (v_p([a_g^{-p^{-n}}] g \pi_{(n)} - \pi_{(n)}) - v_p(\pi_{(n)})) \\
&= \lim_{n \to \infty} p^n v_p \left( [a_g^{-p^{-n}}] \frac{g \pi_{(n)}}{\pi_{(n)}} - 1 \right).
\end{aligned}
$$

Hence it is enough to show that $v_p \left( [a_g^{-p^{-n}}] g \pi_{(n)} \pi_{(n)}^{-1} - 1 \right) \geq p/(p^n(p-1))$ for $n$ large enough. More precisely, we claim the following: Let $N \geq 1$ be the integer such that $g \pi_{N-1} = \pi_{N-1}$ and $g \pi_N \neq \pi_N$. (Such $N$ exists by the assumption $g \notin G_{\underline{\pi}}$.) Then we have

$$v_p \left( [a_g^{-p^{-n}}] \frac{g \pi_{(n)}}{\pi_{(n)}} - 1 \right) \geq \frac{p^N}{p^n(p-1)} \tag{4.4}$$

for $n$ large enough. We show this inequality. Put $c_n = [a_g^{-p^{-n}}] g \pi_{(n)} \pi_{(n)}^{-1}$ for

$n \geq 0$. Since $a_g^{p^d} = a_g$, we have

$$c_n - 1 = [a_g^{-p^{-n}}]^{p^d} \frac{g\pi_{(n-1)}^{p^{d-1}} g\pi_n^{-1}}{\pi_{(n-1)}^{p^{d-1}} \pi_n^{-1}} - 1 = \left( [a_g^{-p^{-(n-1)}}] \frac{g\pi_{(n-1)}}{\pi_{(n-1)}} \right)^{p^{d-1}} \left( \frac{g\pi_n}{\pi_n} \right)^{-1} - 1$$

$$= c_{n-1}^{p^{d-1}} \left( \frac{g\pi_n}{\pi_n} \right)^{-1} - 1 = (c_{n-1}^{p^{d-1}} - 1) \left( \frac{g\pi_n}{\pi_n} \right)^{-1} + \left( \frac{g\pi_n}{\pi_n} \right)^{-1} - 1.$$

In particular, we have $v_p(c_n - 1) \geq \min\{v_p(c_{n-1} - 1), v_p(g\pi_n\pi_n^{-1} - 1)\}$. Repeating this argument, we obtain $v_p(c_n - 1) \geq \min\{v_p(c_0 - 1), v_p(g\pi_1\pi_1^{-1} - 1), \ldots, v_p(g\pi_n\pi_n^{-1} - 1)\}$. Since $c_0 - 1 = 0$, we have $v_p(c_n - 1) \geq \min\{v_p(g\pi_1\pi_1^{-1} - 1), \ldots, v_p(g\pi_n\pi_n^{-1} - 1)\}$. On the other hand, we know $v_R(g\pi_n\pi_n^{-1} - 1) = p^N/(p^n(p-1))$ for any $n \geq N$ by the proof of Proposition 3.12. Hence, to show (4.4), it suffices to show $v_p(g\pi_n\pi_n^{-1} - 1) > 0$ for any $n \geq 1$. More precisely, we show

$$v_p \left( \frac{g\pi_n}{\pi_n} - 1 \right) > \frac{p}{p^n(p-1)} \tag{4.5}$$

for any $n \geq 1$. We note that $x_n := g\pi_n\pi_n^{-1} - 1$ is a root of $\sum_{i=1}^p a_i \pi_n^{-(p-i)}(X + 1)^i - g\pi_{n-1}\pi_n^{-p}$. Put $b_j = \sum_{i=j}^p \binom{i}{j} a_i \pi_n^{-(p-i)} \in p\mathcal{O}_{\overline{K}}$ for any $1 \leq j \leq p - 1$. Then we see the equality $\sum_{i=1}^p a_i \pi_n^{-(p-i)}(X+1)^i = X^p + \sum_{j=1}^{p-1} b_j X^j + \pi_{n-1}\pi_n^{-p}$. Hence $x_n$ for $n \geq 2$ (resp. $n = 1$) is a root of $X^p + \sum_{j=1}^{p-1} b_j X^j + (g\pi_{n-1} - \pi_{n-1})\pi_n^{-p}$ (resp. $X^{p-1} + \sum_{j=1}^{p-1} b_j X^{j-1}$). Now (4.5)) follows by induction on $n$ and arguments of Newton polygons. Consequently we finish the proof of (4.3). To finish the proof of (3), we need to show

$$g(1 \otimes x) - (1 \otimes x) \in \mathfrak{m}_R^{\geq \frac{p^2}{p-1}} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(\mathfrak{n}) \tag{4.6}$$

for any $x \in \mathfrak{M}(\mathfrak{n})$. Writing $x = \sum_{i=1}^d a_i e_i$ with some $a_i \in k[\![u]\!]$, we have $g(1 \otimes x) - (1 \otimes x) = \sum_{i=1}^d (g(1 \otimes a_i e_i) - (1 \otimes a_i e_i)) = \sum_{i=1}^d ((ga_i - a_i)^p g(1 \otimes e_i)) + a_i^p(g(1 \otimes e_i) - (1 \otimes e_i)))$. By (4.3), it suffices to show $ga_i - a_i \in \mathfrak{m}_R^{\geq p/(p-1)}$ but this immediately follows from Proposition 3.12. Consequently, we obtained the proof of (3).

Finally, we show that an $R$-semi-linear $G$-action on $R \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(\mathfrak{n})$ satisfying (1) and (3) is uniquely determined when $0 \leq n_i \leq \min\{er, p-1\}$ for any $i$. Assume that two $G$-actions $\rho_1, \rho_2 \colon G \to \mathrm{End}_R(R \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(\mathfrak{n}))$ on $R \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(\mathfrak{n})$ satisfy (1) and (3), and put $g_*(x) = \rho_1(g)(x)$ and $g_\sharp(x) = \rho_2(g)(x)$ for any $g \in G$ and $x \in R \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(\mathfrak{n})$. By (3), we have $g_*(1 \otimes e_i) - g_\sharp(1 \otimes e_i) \in \mathfrak{m}_R^{\geq c(0)} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(\mathfrak{n})$ where $c(0) = p^2/(p-1)$. Thus, by (1), we obtain

$$gu^{pn_i}(g_*(1 \otimes e_{i+1}) - g_\sharp(1 \otimes e_{i+1})) = \varphi(g_*(1 \otimes e_i) - g_\sharp(1 \otimes e_i)) \in \mathfrak{m}_R^{\geq pc(0)} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(\mathfrak{n}).$$

Furthermore, we have $pc(0) - pn_i/e \geq pc(0) - p(p-1)$ by the assumption $0 \leq n_i \leq \min\{er, p-1\}$. Hence we obtain $g_*(1 \otimes e_{i+1}) - g_\sharp(1 \otimes e_{i+1}) \in$

$\mathfrak{m}_R^{\geq c(1)} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(\mathfrak{n})$ where $c(1) = pc(0) - p(p-1)$. Repeating this argument, we obtain $g_*(1 \otimes e_{i+s}) - g_\sharp(1 \otimes e_{i+s}) \in \mathfrak{m}_R^{\geq c(s)} \otimes_{\varphi, \mathfrak{S}} \mathfrak{M}(\mathfrak{n})$ for any $s \geq 0$ where $c(s) = pc(s-1) - p(p-1) = p^{s+1}/(p-1) + p$. Since $\lim_{s \to \infty} c(s) = \infty$, we obtain $g_*(1 \otimes e_i) = g_\sharp(1 \otimes e_i)$ for any $i$ as desired. □

### 4.6  Proofs of Theorems 4.1 and 4.2

In this section we prove Theorems 4.1 and 4.2. We put $\bar{a} = a \bmod pW(R)$ for any $a \in W(R)$. It is known (cf. Example 3.3.2 of [CL]) that there exists $\mathfrak{t}' \in W(R) \smallsetminus pW(R)$ such that $\varphi(\mathfrak{t}') = E(u)\mathfrak{t}'$. By Lemma 2.3.1 of *loc. cit.*, $\varphi(\mathfrak{t}')$ is a generator of $I^{[1]}W(R)$.

REMARK 4.17. Under the condition $v_p(a_1) > 1$, we defined $\mathfrak{t} \in W(R) \smallsetminus pW(R)$ in Section 2.2 such that $\varphi(\mathfrak{t}) = \mu_0 E(u)\mathfrak{t}$ with some $\mu_0 \in \mathfrak{S}^\times$. Then we have $\mathfrak{t}/\mathfrak{t}' \in W(R)^\times$ since both $\varphi(\mathfrak{t})$ and $\varphi(\mathfrak{t}')$ are generators of a principal ideal $I^{[1]}W(R)$.

We start with two estimations of the ideal $\varphi(gu - u)B_{\mathrm{cris}}^+ \cap W(R)$ of $W(R)$ for $g \in G$ to study its reduction modulo $p$. The first proposition gives a "weak" estimation, however, it does not need any assumption. The second one gives a "strong" estimation although we need some technical assumptions.

PROPOSITION 4.18. *Let $j_0$ be the minimum integer $1 \leq j \leq p$ such that $v_p(ja_j) = 1$. Put $h = 0$ (resp. $h = 1$) if $e < j_0 - 1$ (resp. $e \geq j_0 - 1$).*
*(1) Let $g \in G \smallsetminus G_{\underline{\pi}}$ and $N \geq 1$ the integer such that $g\pi_{N-1} = \pi_{N-1}$ and $g\pi_N \neq \pi_N$. Then*

(i) $gu - u = \varphi^N(\mathfrak{t}')v_g$ *for some $v_g \in W(R)$.*

(ii) $\varphi(v_g) = v_g w_g$ *for some $w_g \in W(R)$.*

(iii) $\varphi(gu - u)B_{\mathrm{cris}}^+ \cap W(R) \subset v_g w_g^h I^{[1]}W(R)$.

*(2) The image of $\varphi(gu - u)B_{\mathrm{cris}}^+ \cap W(R)$ under the projection $W(R) \to R$ is contained in $\mathfrak{m}_R^{\geq c}$ for any $g \in G$. Here,*

$$c = \frac{p}{p-1} + \frac{j_0 - 1}{e(p-1)}p^h.$$

PROPOSITION 4.19. *Assume the following conditions.*

(i) $gu \in uW(R)$ *for any $g \in G$.*

(ii) $f^{(n)}(\pi) \neq 0$ *for any $n \geq 1$.*

*Then we have the following.*
*(1) $gu - u \in uI^{[1]}W(R)$ for any $g \in G$.*
*(2) $\varphi(gu - u)\tilde{B}_\alpha \cap W(R) \subset \varphi(u)I^{[1]}W(R)$ for any $g \in G$.*

(3) *The image of $\varphi(gu-u)\tilde{B}_\alpha \cap W(R)$ under the projection $W(R) \to R$ is contained in $\mathfrak{m}_R^{\geq c}$ for any $g \in G$. Here,*

$$c = \frac{p}{p-1} + \frac{p}{e}.$$

For proofs of these propositions, we use

LEMMA 4.20. (1) *Let $v \in W(R)$ such that $v_R(\bar{v}) \leq 1$. If $x \in B_{\mathrm{cris}}^+$ satisfies $vx \in W(R)$, then we have $x \in W(R)$.*
(2) *Assume that $v_p(a_i) > 1$ for any $1 \leq i \leq p-1$. Put $f_0(u) = f(u)/u$. If $x \in \tilde{B}_\alpha$ satisfies $f_0(u)x \in W(R)$, then we have $x \in W(R)$.*

*Proof.* (1) This is a generalization of Lemma 3.2.2 of [Li3] but almost the same proof can be applied to our setting. We only give one remark that $E(u)$ is contained in $vW(R) + pW(R)$ by the condition $v_R(\bar{v}) \leq 1 = v_R(E(\bar{u}))$, and thus we can write $E(u)^{i+1} = p^{i+1}b_i + vw_i$ by some $b_i, w_i \in W(R)$.
(2) The proof here is given by the anonymous referee. Again we modify the proof of Lemma 3.2.2 of [Li3]. We may assume $x = \sum_{j=0}^{\infty} a_j (E(u)^p/p)^j$ with $a_j \in W(R)$. Put $y = f_0(u)x \in W(R)$. For any $i \geq 0$, we have $p^i y = f_0(u)\tilde{x}_i + \tilde{z}_i$ where $\tilde{x}_i := \sum_{j=0}^{i} a_j p^{i-j} E(u)^{pj}$ and $\tilde{z}_i := p^i f_0(u) \sum_{j=i+1}^{\infty} a_j (E(u)^p/p)^j$. Note that we have $\tilde{x}_i \in W(R)$ and $\tilde{z}_i \in \mathrm{Fil}^{p(i+1)}W(R)$. Since $\mathrm{Fil}^{p(i+1)}W(R)$ is generated by $E(u)^{p(i+1)}$, we have $\tilde{z}_i = E(u)^{p(i+1)}\beta_i$ for some $\beta_i \in W(R)$. By the assumption $v_p(a_i) > 1$ for any $1 \leq i \leq p-1$, we have $E(u)^p \equiv f_0(u)u^{ep-p+1} \bmod p^2$, which implies $E(u)^{p(i+1)} \in p^{2(i+1)}\mathfrak{S} + f_0(u)\mathfrak{S}$. Hence we have $p^i y = f_0(u)x_i' + p^{2(i+1)}w_i'$ for some $x_i', y_i' \in W(R)$. Since $p$ does not divide $f_0(u)$, we see that $p^i$ divides $x_i'$ and thus we obtain $y = f_0(u)x_{(i)} + p^{i+2}w_{(i)}$ for some $x_{(i)}, y_{(i)} \in W(R)$. This gives $y \in f_0(u)W(R)$, which shows $x \in W(R)$ as desired. $\square$

*Proof of Proposition 4.18.* (1) By definition of $N$, we have $\varphi^{-(N-1)}(gu-u) \in I^{[1]}W(R)$ and $\varphi^{-N}(gu-u) \notin \mathrm{Fil}^1 W(R)$ (cf. Lemma 2.1.3 of [CL]). By the condition $\varphi^{-(N-1)}(gu-u) \in I^{[1]}W(R)$ and the fact that $\varphi(\mathfrak{t}')$ is a generator of $I^{[1]}W(R)$, we have $gu-u = \varphi^N(\mathfrak{t}')v_g$ for some $v_g \in W(R)$, which shows (1)-(i). Taking $\varphi$ to both sides of this equality, we have

$$\varphi(gu-u) = \varphi^{N+1}(\mathfrak{t}')\varphi(v_g) = \varphi^N(\mathfrak{t}')\varphi^N(E(u))\varphi(v_g). \tag{4.7}$$

On the other hand, the equation $\varphi(u) = f(u)$ implies

$$\varphi(gu-u) = (gu-u)\tilde{w}_g = \varphi^N(\mathfrak{t}')v_g\tilde{w}_g \tag{4.8}$$

where $\tilde{w}_g = \sum_{i=1}^{p} a_i(gu^i - u^i)/(gu-u) \in W(R)$. By (4.7) and (4.8), we obtain

$$v_g\tilde{w}_g = \varphi^N(E(u))\varphi(v_g). \tag{4.9}$$

Hence we have $\varphi^{-N}(v_g)\varphi^{-N}(\tilde{w}_g) \in \mathrm{Fil}^1 W(R)$. Here we note that $\varphi^{-N}(v_g)$ is not contained in $\mathrm{Fil}^1 W(R)$ since $\mathfrak{t}'\varphi^{-N}(v_g) = \varphi^{-N}(gu-u) \notin \mathrm{Fil}^1 W(R)$. Thus

we obtain $\varphi^{-N}(\tilde{w}_g) \in \mathrm{Fil}^1 W(R)$. Since $E(u)$ is a generator of $\mathrm{Fil}^1 W(R)$ (cf. Lemma 2.1.3 of [CL]), we obtain $\tilde{w}_g = \varphi^N(E(u))w_g$ for some $w_g \in W(R)$. By (4.9), we obtain $\varphi(v_g) = v_g w_g$, which shows (1)-(ii).

Finally we show (1)-(iii). Take any $x = \varphi(gu-u)y \in \varphi(gu-u)B^+_{\mathrm{cris}} \cap W(R)$. We have

$$
\begin{aligned}
x = \varphi(gu-u)y &= \varphi^{N+1}(\mathfrak{t}')\varphi(v_g)y = \varphi^{N+1}(\mathfrak{t}')v_g w_g y \\
&= \varphi^N(E(u))\varphi^N(\mathfrak{t}')v_g w_g y = \varphi^N(E(u))\varphi^{N-1}(E(u))\varphi^{N-1}(\mathfrak{t}')v_g w_g y \\
&= \cdots = \varphi^N(E(u)) \cdots \varphi(E(u)) \cdot E(u)\mathfrak{t}' v_g w_g y \\
&= E(u)\mathfrak{t}' v_g w_g^h z
\end{aligned}
$$

where $z := \varphi^N(E(u)) \cdots \varphi(E(u)) w_g^{1-h} y \in B^+_{\mathrm{cris}}$. Note that we have $v_R(\overline{E(u)}) = ev_R(\bar{u}) = 1$. By the equality $\varphi(\mathfrak{t}') = E(u)\mathfrak{t}'$, we have $v_R(\bar{\mathfrak{t}}') = 1/(p-1) \le 1$. It follows from Proposition 3.12 and the equality $gu-u = \varphi^N(\mathfrak{t}')v_g$ that we have $v_R(\bar{v}_g) = (j_0-1)/(e(p-1)) \le 1$. Furthermore, by the equality $\varphi(v_g) = v_g w_g$, we also see $v_R(\bar{w}_g^h) = h(j_0-1)/e \le 1$. Hence it follows from Lemma 4.20 and $E(u)\mathfrak{t}' v_g w_g^h z = x \in W(R)$ that we have $z \in W(R)$. Therefore, we obtain $x = \varphi(\mathfrak{t}')v_g w_g^h z \in v_g w_g^h I^{[1]}W(R)$ as desired.

(2) Since $v_R(\bar{\mathfrak{t}}') = 1/(p-1)$, $v_R(\bar{v}_g) = (j_0-1)/(e(p-1))$ and $v_R(\bar{w}_g^h) = h(j_0-1)/e$, the result follows from (1)-(iii) immediately. □

*Proof of Proposition 4.19.* The assertion (3) follows from (2) immediately, and thus it suffices to show (1) and (2). By the assumption (i), we have $gu-u = uv_g$ for some $v_g \in W(R)$. By Lemma 2.3.2 of [CL] and the assumption (ii), we see $v_g \in I^{[1]}W(R)$, which shows (1). Take any $x = \varphi(gu-u)y \in \varphi(gu-u)\tilde{B}_\alpha \cap W(R)$. Writing $v_g = \varphi(\mathfrak{t}')v'_g$ with some $v'_g \in W(R)$, we have

$$
x = \varphi(uv_g)y = \varphi(u)\varphi^2(\mathfrak{t}')\varphi(v'_g)y = f_0(u) \cdot u \cdot E(u) \cdot \mathfrak{t}' \cdot z
$$

where $z = \varphi(E(u))\varphi(v'_g)y$, which is an element of $\tilde{B}_\alpha$. Note that we have $v_R(\bar{u}) = 1/e \le 1$, $v_R(\bar{\mathfrak{t}}') = 1/(p-1) \le 1$. Hence it follows from Lemma 4.20 that we have $z \in W(R)$. Therefore, we obtain $x = \varphi(u)\varphi(\mathfrak{t}')z \in \varphi(u)I^{[1]}W(R)$. □

The above propositions allow us to show the existence of "good" $(\varphi, G)$-modules which correspond to objects of $\mathrm{Rep}^{r,\mathrm{cris}}_{\mathrm{tor}}(G)$. For the case $r = 1$, we have

COROLLARY 4.21. *Assume $v_p(a_1) > 1$ and the condition $(P)$. Let $j_0$ be the minimum integer $1 \le j \le p$ such that $v_p(ja_j) = 1$. Put $h = 0$ (resp. $h = 1$) if $e < j_0 - 1$ (resp. $e \ge j_0 - 1$). Let $\alpha \in W(R) \smallsetminus pW(R)$ such that $v_R(\bar{\alpha}) \le (j_0-1)p^h/(e(p-1))$. Let $T$ be an object of $\mathrm{Rep}^{1,\mathrm{cris}}_{\mathrm{tor}}(G)$ such that $pT = 0$. Then there exists a $(\varphi, G)$-module $\hat{\mathfrak{M}} \in \mathrm{Mod}^{1,G}_{\mathfrak{S}_\infty}(\alpha)$ killed by $p$ such that $T \simeq \hat{T}(\hat{\mathfrak{M}})$.*

*Proof.* Take an exact sequence $0 \to L_1 \to L_2 \to T \to 0$ of representations of $G$, where $L_1 \subset L_2$ are $G$-stable $\mathbb{Z}_p$-lattices in a crystalline $\mathbb{Q}_p$-representation of $G$ with Hodge-Tate weights in $[0,1]$. Take a morphism $i: \hat{\mathfrak{L}}_2 \to \hat{\mathfrak{L}}_1$ in $\mathrm{Mod}^{1,\hat{G},\mathrm{cris}}_{\tilde{\mathfrak{S}}}$

which corresponds to the injection $L_1 \hookrightarrow L_2$ via Theorem 3.7. We regard $\hat{\mathfrak{L}}_1$ and $\hat{\mathfrak{L}}_2$ as $(\varphi, G)$-modules by a canonical way. It is not difficult to check that the map $\mathfrak{L}_2 \to \mathfrak{L}_1$ of underlying Kisin modules of $i$ is injective, and thus we may regard $\hat{\mathfrak{L}}_2$ as a sub $(\varphi, G)$-module of $\hat{\mathfrak{L}}_1$. Put $\mathfrak{M} = \mathfrak{L}_1/\mathfrak{L}_2$. It follows from Proposition 2.2 that $\mathfrak{M}$ is an object of $\mathrm{Mod}_{\mathfrak{S}_\infty}^1$. Furthermore, we can naturally equip $\mathfrak{M}$ with a $(\varphi, G)$-module structure; we denote it by $\hat{\mathfrak{M}}$. By construction, we have an exact sequence $0 \to \hat{\mathfrak{L}}_2 \to \hat{\mathfrak{L}}_1 \to \hat{\mathfrak{M}} \to 0$ of $(\varphi, G)$-modules. It follows from (the proof of) Lemma 3.1.4 of [CL2] that this exact sequence induces $0 \to L_1 \to L_2 \to T \to 0$. We note that $\mathfrak{M}[1/u]$ is an étale $\varphi$-module corresponding to $T|_{G_\pi}$, and thus $\mathfrak{M}[1/u]$ is killed by $p$ (see the isomorphism (3.2.1) of [CL]). In particular, $\mathfrak{M}$ is killed by $p$. Combining this with the fact that $\hat{\mathfrak{L}}_1$ and $\hat{\mathfrak{L}}_2$ are objects of $\mathrm{Mod}_{\mathfrak{S}}^{1,\hat{G},\mathrm{cris}}$, it follows from Proposition 4.18 that $\hat{\mathfrak{M}}$ is an object of $\hat{\mathfrak{M}} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{1,G}(\alpha)$. $\square$

Next we consider general $r$.

COROLLARY 4.22. *Assume the following conditions.*

(i) $gu \in uW(R)$ *for any* $g \in G$.

(ii) $f^{(n)}(\pi) \neq 0$ *for any* $n \geq 1$.

(iii) $v_p(a_1) > \max\{r, 1\}$.

*Let* $T$ *be an object of* $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G)$. *Then there exists a* $(\varphi, G)$-*module* $\hat{\mathfrak{M}} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{r,G}(\varphi(u))$ *such that* $T \simeq \hat{T}(\hat{\mathfrak{M}})$.
*Moreover, we have the following: Suppose that we have an exact sequence*

$$(\#) \quad 0 \to L_1 \to L_2 \to T \to 0$$

*of representations of* $G$, *where* $L_1 \subset L_2$ *are* $G$-*stable* $\mathbb{Z}_p$-*lattices in a crystalline* $\mathbb{Q}_p$-*representation of* $G$ *with Hodge-Tate weights in* $[0, r]$. *Then there exist* $\hat{\mathfrak{L}}_1, \hat{\mathfrak{L}}_2 \in \mathrm{Mod}_{\mathfrak{S}}^{r,G}(\varphi(u))$, $\hat{\mathfrak{M}} \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{r,G}(\varphi(u))$ *and an exact sequence*

$$(*) \quad 0 \to \hat{\mathfrak{L}}_2 \to \hat{\mathfrak{L}}_1 \to \hat{\mathfrak{M}} \to 0$$

*of* $(\varphi, G)$-*modules which induces* $(\#)$.

*Proof.* The proof is almost the same as that of Corollary 4.21. We only give a remark that $\hat{\mathfrak{L}}_1$ and $\hat{\mathfrak{L}}_2$ in the present situation are objects of $\mathrm{Mod}_{\mathfrak{S}}^{r,G}(\varphi(u))$ by Proposition 4.19 (2), and thus $\hat{\mathfrak{M}}$ is an object of $\mathrm{Mod}_{\mathfrak{S}_\infty}^{r,G}(\varphi(u))$. $\square$

Now we are ready to prove Theorems 4.1 and 4.2. We essentially follow the method of [Oz2].

*Proof of Theorem 4.2.* The goal is to show the equality

$$\operatorname{Hom}_G(T, T') = \operatorname{Hom}_{G_{\underline{\pi}}}(T, T') \qquad (4.10)$$

for any $T,\ T' \in \operatorname{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G)$.

<u>STEP 1.</u>  We reduce a proof to the case where $k = \overline{k}$. Assume that the theorem holds when $k = \overline{k}$ and consider general cases. We denote by $L$ and $H$ the completion of the maximal unramified extension of $K$ and the absolute Galois group of $L$, respectively. We identify the inertia subgroup $I$ of $G$ with $H$. We set $L_{\underline{\pi}} := \bigcup_{n \geq 0} L(\pi_n)$ and denote by $H_{\underline{\pi}}$ the absolute Galois group of $L_{\underline{\pi}}$. We remark that $\bar{L}_{\underline{\pi}}$ is an $f$-iterate extension of $L$ since $\pi$ is a uniformizer of $L$.
Let $f \colon T \to T'$ be a $G_{\underline{\pi}}$-equivalent homomorphism. Since $T|_H$ and $T'|_H$ are objects of $\operatorname{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(H)$ and $f$ commutes with $H_{\underline{\pi}}$, the assumption above implies that $f$ is $H$-equivalent. Since the extension $K_{\underline{\pi}}/K$ is a totally ramified pro-$p$-extension, we know that $H$ and $G_{\underline{\pi}}$ topologically generates $G$. Hence $f$ commutes with $G$.

<u>STEP 2.</u>  We reduce a proof to the case where $T$ is irreducible. Assume that the equality (4.10) holds when $T$ is irreducible and consider general cases. Since the category $\operatorname{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G)$ is stable under subquotients and direct sums in $\operatorname{Rep}_{\mathrm{tor}}(G)$ (cf. Lemma 4.19 of [Oz2]), it is an exact category in the sense of Quillen [Qu, Section 2]. Hence short exact sequences in $\operatorname{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G)$ give rise to exact sequences of Hom's and Ext[1]'s in the usual way. Thus a standard *dévissage* argument (with respect to a Jordan-Hölder sequence of $T$) reduces a proof to the case where $T$ is irreducible.

<u>STEP 3.</u>  By Steps 1 and 2, it suffices to show the equality (4.10) under the conditions that $k = \overline{k}$ and $T$ is irreducible. Now we assume these conditions.
First we claim that $T|_{G_{\underline{\pi}}}$ is irreducible. Let $W$ be a $G_{\underline{\pi}}$-stable submodule of $T$. Since $T$ is irreducible, the wild inertia subgroup $I^{\mathrm{w}}$ of $G$ acts on $T$ trivial. In particular, the $I^{\mathrm{w}}$-action on $T$ preserves $W$. Since $G_{\underline{\pi}}$ and $I^{\mathrm{w}}$ topologically generates $G$, the irreducibility of $T$ implies that $W$ is $0$ or $T$. Thus the claim follows.
By Corollary 4.22, there exist $(\varphi, G)$-modules $\hat{\mathfrak{M}}, \hat{\mathfrak{M}}' \in \operatorname{Mod}_{\mathfrak{S}_{\infty}}^{r,G}(\varphi(u))$ such that $T \simeq \hat{T}(\hat{\mathfrak{M}})$ and $T' \simeq \hat{T}(\hat{\mathfrak{M}}')$. Then we have $T|_{G_{\underline{\pi}}} \simeq T_{\mathfrak{S}}(\mathfrak{M}) \simeq T_{\mathfrak{S}}(\operatorname{Max}^r(\mathfrak{M}))$. By Theorem 4.9 (5) and the condition that $T|_{G_{\underline{\pi}}}$ is irreducible, we know that $\operatorname{Max}^r(\mathfrak{M})$ is a simple object in the abelian category $\operatorname{Max}_{\mathfrak{S}_{\infty}}^r$. By Proposition 4.11 and the assumption $k = \overline{k}$, there exists an sequence $\mathfrak{n} \in \mathcal{S}_{\max}^r$ such that $\mathfrak{M}(\mathfrak{n}) \simeq \operatorname{Max}^r(\mathfrak{M})$. We note that the ideal $\varphi(u) I^{[1]} W(R)$ of $W(R)$ is generated by $\varphi(u)\varphi(\mathfrak{t})$ and $v_R(\varphi(u)\varphi(\mathfrak{t}) \bmod p) = p/e + p/(p-1) \leq p + p/(p-1) = p^2/(p-1)$. It follows from Theorem 4.15 that there exists a (unique) $(\varphi, G)$-module $\hat{\mathfrak{M}}(\mathfrak{n}) \in \operatorname{Mod}_{\mathfrak{S}_{\infty}}^{r,G}(\varphi(u))$ with underlying Kisin module $\mathfrak{M}(\mathfrak{n})$. Then we have an isomorphism $T|_{G_{\underline{\pi}}} \simeq \hat{T}(\hat{\mathfrak{M}}(\mathfrak{n}))|_{G_{\underline{\pi}}}$. By this isomorphism, we know

that $\hat{T}(\hat{\mathfrak{M}}(\mathfrak{n}))|_{G_{\underline{\pi}}}$ is irreducible since $T|_{G_{\underline{\pi}}}$ is irreducible. Hence $\hat{T}(\hat{\mathfrak{M}}(\mathfrak{n}))$ is irreducible as a representation of $G$. In particular, $T$ and $\hat{T}(\hat{\mathfrak{M}}(\mathfrak{n}))$ are tame. Since $G_{\underline{\pi}}$ and $I^{\mathrm{w}}$ topologically generates $G$, the isomorphism $T|_{G_{\underline{\pi}}} \simeq \hat{T}(\hat{\mathfrak{M}}(\mathfrak{n}))|_{G_{\underline{\pi}}}$ is in fact $G$-equivalent. We consider the following commutative diagram.

$$
\begin{array}{ccc}
\mathrm{Hom}_G(T, T') & \hookrightarrow & \mathrm{Hom}_{G_{\underline{\pi}}}(T, T') \\
\hat{T} \uparrow & & \uparrow T_{\mathfrak{S}} \\
\mathrm{Hom}(\hat{\mathfrak{M}}', \hat{\mathfrak{M}}(\mathfrak{n})) \longrightarrow \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}', \mathfrak{M}(\mathfrak{n})) & \xrightarrow{\mathrm{Max}^r} & \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathrm{Max}^r(\mathfrak{M}'), \mathfrak{M}(\mathfrak{n}))
\end{array}
$$

Here, we recall that we have $v_R(\varphi(\bar{u})) = p/e > p(r-1)/(p-1)$. Hence the first arrow in the bottom line, obtained by forgetting $G$-actions, is bijective by Theorem 4.14. Since $\mathfrak{M}(\mathfrak{n})$ is maximal, it is not difficult to check that the second arrow in the bottom line is also bijective. Furthermore, the right vertical arrow is also bijective by Theorem 4.9 (5). Therefore, the top horizontal arrow must be bijective as desired. This is the end of the proof of Theorem 4.2. $\square$

*Proof of Theorem 4.1.* The goal is to show the equality

$$\mathrm{Hom}_G(T, T') = \mathrm{Hom}_{G_{\underline{\pi}}}(T, T') \tag{4.11}$$

for any $T$, $T' \in \mathrm{Rep}_{\mathrm{tor}}^{1,\mathrm{cris}}(G)$. The arguments in Steps 1 and 2 just above proceed also for the present situation. Thus it suffices to show the equality (4.11) under the conditions that $k = \bar{k}$ and $T$ is irreducible. Put $T'' = \ker(T' \to T'; x \mapsto px)$. This is an object of $\mathrm{Rep}_{\mathrm{tor}}^{1,\mathrm{cris}}(G)$ by Lemma 4.19 of [Oz2]. Since $pT = 0$, we know that any homomorphism $T \to T'$ of $\mathbb{Z}_p$-modules have values in $T''$. Thus, by replacing $T'$ with $T''$, we may assume $pT' = 0$.

Take any $\alpha \in W(R) \smallsetminus pW(R)$ such that $0 < v_R(\bar{\alpha}) \leq (j_0 - 1)/(e(p-1))$. Since $T$ and $T'$ are killed by $p$, there exist $(\varphi, G)$-modules $\hat{\mathfrak{M}}, \hat{\mathfrak{M}}' \in \mathrm{Mod}_{\mathfrak{S}_\infty}^{1,G}(\alpha)$ killed by $p$ such that $T \simeq \hat{T}(\hat{\mathfrak{M}})$ and $T' \simeq \hat{T}(\hat{\mathfrak{M}}')$ by Corollary 4.21. Now we can use the same arguments of the third paragraph of Step 3. $\square$

## References

[Ab] Victor Abrashkin, *Group schemes of period $p > 2$*, Proc. Lond. Math. Soc. (3), 101 (2010), 207–259.

[Be] Laurent Berger, *Iterated extensions and relative Lubin-Tate groups*, Ann. Math. Québec 40 (2016), no. 1, 17–28.

[BMS] Bhargav Bhatt, Matthew Morrow and Peter Scholze, *Integral p-adic Hodge Theory*, Preprint, arXiv:1602.03148

[Br1] Christophe Breuil, *Une application du corps des normes*, Compos. Math. 117 (1999) 189–203.

[Br2]   Christophe Breuil, *Integral p-adic Hodge theory*, in Algebraic geometry 2000, Azumino (Hotaka), Adv. Stud. Pure Math., vol. 36, Math. Soc. Japan, 2002, 51–80.

[CD]    Bryden Cais and Christopher Davis, *Canonical cohen rings for norm fields*, Int. Math. Res. Not. IMRN (2014), 5473–5517.

[CL]    Bryden Cais and Tong Liu, *On F-crystalline representation*, Doc. Math., 21 (2016), 223–270.

[CL1]   Xavier Caruso and Tong Liu, *Quasi-semi-stable representations*, Bull. Soc. Math. France, 137 (2009), no. 2, 185–223.

[CL2]   Xavier Caruso and Tong Liu, *Some bounds for ramification of $p^n$-torsion semi-stable representations*. J. Algebra, 325 (2011), 70–96.

[Fo1]   Jean-Marc Fontaine, *Représentations p-adiques des corps locaux. I*, The Grothendieck Festschrift, Vol. II, Progr. Math., 87, Birkhäuser Boston, Boston, MA (1990), 249–309.

[Fo2]   Jean-Marc Fontaine, *Le corps des périods p-adiques*, Astérisque, no. 223 (1994), 59–111.

[Ga]    Hui Gao, *Crystalline liftings and weight part of Serre's conjecture*, Israel J. Math. 221 (2017), 117–164.

[GLS1]  Toby Gee, Tong Liu and David Savitt, *The Buzzard-Diamond-Jarvis conjecture for unitary groups*, J. Amer. Math. Soc. 27 (2014), 389–435.

[GLS2]  Toby Gee, Tong Liu and David Savitt, *The weight part of Serre's conjecture for $GL(2)$*, Forum of Mathematics, $\pi$ 3 (2015), e2 (52 pages).

[Kim]   Wansu Kim, *The classification of p-divisible groups over 2-adic discrete valuation rings*, Math. Res. Lett. 19 (2012), no. 1, 121–141.

[Kis]   Mark Kisin, *Crystalline representations and F-crystals*, Algebraic geometry and number theory, Progr. Math. 253, Birkhäuser Boston, Boston, MA (2006), 459–496.

[La]    Eike Lau, *A relation between Dieudonné displays and crystalline Dieudonné theory*, Algebra Number Theory 8 (2014), 2201-2262.

[Li1]   Tong Liu, *Torsion p-adic Galois representations and a conjecture of Fontaine*, Ann. Sci. École Norm. Sup. (4) 40 (2007), no. 4, 633–674.

[Li2]   Tong Liu, *A note on lattices in semi-stable representations*, Math. Ann. 346 (2010), 117–138.

[Li3]   Tong Liu, *The correspondence between Barsotti-Tate groups and Kisin modules when $p = 2$*, Journal de Théroie des Nombres de Bordeaux, 25, no. 3 (2013), 661–676.

[Li4]   Tong Liu, *Compatibility of Kisin modules for different uniformizers*, appear at Journal für die reine und angewandte Mathematik.

[Oz1]   Yoshiyasu Ozeki, *Torsion representations arising from $(\varphi, \hat{G})$-modules*, J. Number Theory, No. 133 (2013), 3810-3861.

[Oz2]   Yoshiyasu Ozeki, *On Galois equivariance of homomorphisms between torsion crystalline representations*, Nagoya Math. J. 229 (2018), 169–214.

[Qu]    Daniel Quillen, *Higher algebraic K-theory: I*, in Algebraic $K$-theory, I: Higher $K$-theories (Seattle, 1972), Lecture Notes in Math. 341, Springer-Verlag, New York, 1973, 85–147.

[Wi]    Jean-Pierre Wintenberger, *Le corps des normes de certaines extensions infinies de corps locaux; applications*, Ann. Sci. École Norm. Sup. (4) 16 (1983), 59–89.

Yoshiyasu Ozeki
Dept. of Mathematics and Physics
Kanagawa University
Kanagawa 259-1293
Japan
ft101992yp@kanagawa-u.ac.jp

542