# Hermitian Forms and Systems of Quadratic Forms

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Abstract. We associate to every symmetric (antisymmetric) hermitian form a system of quadratic forms over the base field which determines its isotropy and metabolicity behaviour. It is shown that two even hermitian forms are isometric if and only if their associated systems are equivalent. As an application, it is also shown that an anisotropic symmetric hermitian form over a quaternion division algebra in characteristic two remains anisotropic over all odd degree extensions of the ground field.

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## 1 Introduction

The theory of hermitian forms appears as a natural generalization of the theory of bilinear forms, replacing the ground field by an associative ring with involution. In view of this generalization, a natural problem is to compare important properties of quadratic and bilinear forms with their corresponding properties in hermitian forms. A possible approach to this problem is to associate to every hermitian form $h$ a bilinear or quadratic form over the base field sharing some properties with $h$. Among these properties, the isometry and isotropy of hermitian forms are of particular importance.
The first attempt on the aforementioned problem was made by N. Jacobson [4]. He associated a quadratic form to hermitian forms over quadratic separable extensions or quaternion algebras with the canonical involution in characteristic different from two. This quadratic form determines the isometry class and the isotropy behaviour of hermitian forms (see [11, Ch. 10, §1] for more details and [10] for a generalization to arbitrary characteristic). In other words, the

theory of hermitian forms over these division algebras with involution reduces to the theory of quadratic forms.
In this work we generalize the ideas of [4] and associate a system of quadratic forms to every $\pm 1$-hermitian form over a division algebra with involution of the first kind $(D,\theta)$. This correspondence agrees with the Jacobson's one in the case where $D$ is a quaternion algebra endowed with the canonical involution (see Remark 3.3). We start by studying some basic properties of systems of quadratic forms in §2. In §3 the generalized Jacobson's trace map $q_h$ of a hermitian form $h$ is defined and its basic properties are studied. We then study some characterizing properties of $q_h$ in §4. It is easily seen that $q_h$ determines the isotropy behaviour of $h$ (see Proposition 4.1). Further, it is shown in Theorem 4.2 that if $\operatorname{char} F \neq 2$ or $D \neq F$, a regular $\pm 1$-hermitian form $h$ over $(D,\theta)$ is metabolic if and only if $q_h$ is metabolic. It is also shown that for $\lambda = \pm 1$, the generalized Jacobson's trace map classifies, up to isometry, even $\lambda$-hermitian forms over $(D,\theta)$, except for the case where $\operatorname{char} F \neq 2$, $D = F$ and $\lambda = -1$ (see Theorem 4.5). Finally, in §5 we use the system $q_h$ to prove a characteristic two counterpart of a result of Parimala et al [8], which states that an anisotropic hermitian form over a quaternion division algebra remains anisotropic over all odd degree extensions of the ground field (see Theorem 5.3).

## 2 Systems of quadratic forms

Let $V$ be a finite dimensional vector space over a field $F$ of arbitrary characteristic. A *quadratic form* on $V$ is a map $q : V \to F$ for which (i) $q(\alpha v) = \alpha^2 q(v)$ for all $\alpha \in F$ and $v \in V$; (ii) the map $\mathfrak{b}_q : V \times V \to F$ given by $\mathfrak{b}_q(u,v) = q(u+v) - q(u) - q(v)$ is a bilinear form. The map $\mathfrak{b}_q$ is called the *polar form* of $q$. The *orthogonal complement* of a subspace $W$ of $V$ is defined as

$$W^\perp = \{v \in V \mid \mathfrak{b}_q(v,w) = 0 \text{ for all } w \in W\}.$$

As in [9], we say that $q$ is *regular* if $V^\perp = \{0\}$. Note that this definition is different from the one given in [3] (see [3, p. 42]). A nonzero vector $v \in V$ is called *isotropic* if $q(v) = 0$. The form $q$ is called *isotropic* if there is an isotropic vector in $V$ and *anisotropic* otherwise.
By an *$n$-fold system of quadratic forms* on $V$ we mean an $n$-tuple $q = (q_1, \cdots, q_n)$, where every $q_i$ is a quadratic form on $V$. Note that we may identify $q$ with a quadratic map $q : V \to F^n$ (see [9, p. 132]). Then $q$ induces a polar map $\mathfrak{b}_q : V \times V \to F^n$ given by

$$\mathfrak{b}_q(u,v) = q(u+v) - q(u) - q(v).$$

For a subspace $W$ of $V$ we use the notation

$$W^\perp = \{v \in V \mid \mathfrak{b}_q(v,w) = 0 \in F^n \text{ for all } w \in W\}.$$

Clearly, $W^\perp$ is the intersection of the orthogonal complements of $W$ in the quadratic spaces $(V,q_1), \cdots, (V,q_n)$. The system $q$ is called *regular* if $V^\perp = \{0\}$.

Note that $q$ could be regular, while none of the forms $q_1, \cdots, q_n$ are regular. We say that $q$ is *strongly regular* if $q_i$ is regular for some $1 \leqslant i \leqslant n$.
The system $q$ is called *isotropic* if $q(v) = 0$ for some nonzero vector $v \in V$. In other words, $q$ is isotropic if the forms $q_1, \cdots, q_n$ have a common isotropic vector. Two systems $q : V \to F^n$ and $q' : V' \to F^n$ are called *equivalent* if there exists an $F$-linear isomorphism $f : V \to V'$ such that $q'(f(v)) = q(v)$ for every $v \in V$. In this case, we write $(V, q) \simeq (V', q')$.

Definition 2.1. A system $(V, q)$ of quadratic forms over $F$ is called *metabolic* if there exists a subspace $L$ of $V$ with $\dim_F L \geqslant \frac{1}{2} \dim_F V$ such that $q|_L = 0$.

We call such a subspace $L$ a *lagrangian* of $(V, q)$. Note that if $q$ is strongly regular and metabolic, then $\dim_F L = \frac{1}{2} \dim_F V$ and $L^\perp = L$.
It is worth noting that there are other possible definitions for the *metabolicity* of a system of quadratic forms. However, Definition 2.1 is the weakest one (see [9, p. 133]).
Given two systems $q : V \to F^n$ and $q' : V' \to F^n$ of quadratic forms, one can consider the *orthogonal sum* $q \perp q' : V \oplus V' \to F^n$ given by

$$(q \perp q')((v, v')) = q(v) + q'(v') \quad \text{for } v \in V \text{ and } v' \in V'.$$

Lemma 2.2. *Let $(V, q)$ be a system of quadratic forms over a field $F$. Then $q \perp (-q)$ is metabolic.*

*Proof.* Let $\{v_1, \cdots, v_n\}$ be a basis of $V$, so that

$$\{(v_1, 0), \cdots, (v_n, 0), (0, v_1), \cdots, (0, v_n)\}$$

is a basis of $V \oplus V$. It readily follows that the subspace $L \subseteq V \oplus V$ spanned by $((v_1, v_1), \cdots, (v_n, v_n))$ is a lagrangian of $q \perp (-q)$. □

Note that there is no Witt group of systems of quadratic forms. Indeed, there exists an equivalence $q \simeq q_1 \perp q_2$ of systems of quadratic forms such that $q$ and $q_1$ are metabolic, but $q_2$ is not metabolic (see [9, pp. 132–133]). However, for strongly regular systems we can prove the following result.

Proposition 2.3. *Let $(V, q) \simeq (U, \rho) \perp (W, \phi)$ be an equivalence of systems of quadratic forms. Suppose that $q$ is strongly regular. If $q$ and $\phi$ are metabolic then $\rho$ is isotropic.*

*Proof.* Suppose that $\rho$ is anisotropic. Considering the isomorphism $V \simeq U \oplus W$, we may identify $U$ and $W$ with subspaces of $V$ in such a way that $V = U + W$ and $U \cap W = \{0\}$. Hence, every element $v \in V$ can be uniquely written as $v = u + w$, where $u \in U$ and $w \in W$. Therefore,

$$q(v) = \rho(u) + \phi(w). \tag{1}$$

Let $L$ be a lagrangian of $(V, q)$ and set $W_1 = L \cap W$. Let $\{v_1, \cdots, v_k\}$ be a basis of $W_1$ and extend it to a basis $\{v_1, \cdots, v_n\}$ of $L$, where $n = \frac{1}{2} \dim_F V$.

For $i = 1, \cdots, n$, write $v_i = u_i + w_i$, where $u_i \in U$ and $w_i \in W$. Then $u_i = 0$ for $i \leqslant k$. Since $q|_L$ is trivial, (1) implies that

$$\rho(u_i) = -\phi(w_i) \quad \text{and} \quad \mathfrak{b}_\rho(u_i, u_j) = -\mathfrak{b}_\phi(w_i, w_j) \quad \text{for } i, j = 1, \cdots, n. \tag{2}$$

We claim that the set $\{w_1, \cdots, w_n\}$ is linearly independent. Suppose that $\sum_{i=1}^n \alpha_i w_i = 0$ for some $\alpha_1, \cdots, \alpha_n \in F$. Then $\phi(\sum_{i=1}^n \alpha_i w_i) = 0$, which implies that $\rho(\sum_{i=1}^n \alpha_i u_i) = 0$, thanks to (2). Since $\rho$ is anisotropic, we obtain $\sum_{i=1}^n \alpha_i u_i = 0$. Hence, $\sum_{i=1}^n \alpha_i v_i = 0$, which yields $\alpha_1 = \cdots = \alpha_n = 0$, because $\{v_1, \cdots, v_n\}$ is a basis of $L$. This proves the claim.

Let $W'$ be the subspace of $W$ spanned by $w_1, \cdots, w_n$. For $i \leqslant k$ and $j = 1, \cdots, n$, the equality $\mathfrak{b}_q(v_i, v_j) = 0$ implies that $\mathfrak{b}_q(w_i, u_j + w_j) = 0$. Since $\mathfrak{b}_q(w_i, u_j) = 0$ for all $i, j$, we have

$$\mathfrak{b}_\phi(w_i, w_j) = \mathfrak{b}_q(w_i, w_j) = 0 \quad \text{for all } i \leqslant k \text{ and } j = 1, \cdots, n.$$

It follows that $W' \subseteq W_1^\perp$ with respect to the polar form of $\phi$.

Let $L'$ be a lagrangian of $(W, \phi)$. We claim that $W_1 \cap L' = W' \cap L'$. Since $w_i = v_i$ for $i \leqslant k$, we have $W_1 \subseteq W'$, hence $W_1 \cap L' \subseteq W' \cap L'$. Conversely, let $w \in W' \cap L'$. Write $w = \sum_{i=1}^n \beta_i w_i$ for some $\beta_1, \cdots, \beta_n \in F$. Since $\phi(w) = 0$, we obtain $\rho(\sum_{i=1}^n \beta_i u_i) = 0$ by (2). Hence, $\sum_{i=1}^n \beta_i u_i = 0$, because $\rho$ is anisotropic. It follows that

$$w = \sum_{i=1}^n \beta_i w_i = \sum_{i=1}^n \beta_i w_i + \sum_{i=1}^n \beta_i u_i = \sum_{i=1}^n \beta_i v_i \in L.$$

On the other hand, we have $W' \subseteq W$, hence $w \in W \cap L = W_1$, proving the converse inclusion.

Set $X = W_1 \cap L' = W' \cap L'$ and let $l = \dim_F X$. The inclusion $X \subseteq W_1$ shows that $W_1^\perp \subseteq X^\perp$, hence $W' \subseteq X^\perp$ (with respect to the polar form of $\phi$). Similarly, the inclusion $X \subseteq L'$ implies that $L' = L'^\perp \subseteq X^\perp$. Let $\dim_F W = 2s$. Since $\phi$ is strongly regular, $X^\perp$ is a subspace of $W$ of dimension at most $2s - l$. Also, $W'$ is an $n$-dimensional subspace of $X^\perp$ and $L'$ is an $s$-dimensional subspace of $X^\perp$ with $\dim_F W' \cap L' = l$. Hence, $n + s - l \leqslant 2s - l$, which implies that $n = s$, because $s \leqslant n$. But this means that the form $(U, \rho)$ is trivial, a contradiction. □

## 3 The generalized Jacobson's trace map

Let $A$ be a central simple algebra over a field $F$. An *involution* on $A$ is a map $\sigma : A \to A$ satisfying $\sigma(x + y) = \sigma(x) + \sigma(y)$, $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma^2(x) = x$ for $x, y \in A$. An involution $\sigma$ on $A$ is said to be of *the first kind* if it restricts to the identity on $F$. Otherwise, it is said to be of the *second kind*. For an algebra with involution $(A, \sigma)$ and $\lambda = \pm 1$ we use the notation

$$\mathrm{Sym}_\lambda(A, \sigma) = \{x \in A \mid \sigma(x) = \lambda x\},$$
$$\mathrm{Symd}_\lambda(A, \sigma) = \{x + \lambda\sigma(x) \mid x \in A\}.$$

Note that if $\mathrm{char}\, F \neq 2$ then $\mathrm{Sym}_\lambda(A,\sigma) = \mathrm{Symd}_\lambda(A,\sigma)$ (see [6, p. 14]). We will simply denote $\mathrm{Sym}_1(A,\sigma)$ by $\mathrm{Sym}(A,\sigma)$ and $\mathrm{Symd}_1(A,\sigma)$ by $\mathrm{Symd}(A,\sigma)$. From now on, we fix $(D,\theta)$ as a finite dimensional division algebra with involution of the first kind over a field $F$. We also fix the element $\lambda = \pm 1$. A *$\lambda$-hermitian space* over $(D,\theta)$ is a pair $(V,h)$, where $V$ is a finite dimensional right vector space over $D$ and $h : V \times V \to D$ is a bi-additive map satisfying $h(ud, vd') = \theta(d)h(u,v)d'$ and $h(v,u) = \lambda\theta(h(u,v))$ for all $u,v \in V$ and $d,d' \in D$. It follows immediately that $h(v,v) \in \mathrm{Sym}_\lambda(D,\theta)$ for every $v \in V$.
A $\lambda$-hermitian space $(V,h)$ is called *even* if $h(v,v) \in \mathrm{Symd}_\lambda(D,\theta)$ for all $v \in V$. Note that if $\mathrm{char}\, F \neq 2$ then all $\lambda$-hermitian forms are even, because $\mathrm{Sym}_\lambda(D,\theta) = \mathrm{Symd}_\lambda(D,\theta)$. A $\lambda$-hermitian space $(V,h)$ is called *regular* if for every nonzero vector $u \in V$ there exists a vector $v \in V$ such that $h(u,v) \neq 0$.

Lemma 3.1. *Let $(V,h)$ be a $\lambda$-hermitian space over $(D,\theta)$ and let $\pi : \mathrm{Sym}_\lambda(D,\theta) \to F$ be an $F$-linear map. Considering $V$ as a vector space over $F$, the map $q : V \to F$ defined by $q(v) = \pi(h(v,v))$ is a quadratic form with the polar form*

$$\mathfrak{b}_q(u,v) = \pi(h(u,v) + h(v,u)) \quad \text{for } u,v \in V. \tag{3}$$

*Proof.* For $\alpha \in F$ and $v \in V$ we have

$$q(\alpha v) = \pi(h(\alpha v, \alpha v)) = \pi(\alpha^2 h(v,v)) = \alpha^2 \pi(h(v,v)) = \alpha^2 q(v).$$

Consider the map $\mathfrak{b}_q : V \times V \to F$ given by $\mathfrak{b}_q(u,v) = q(u+v) - q(u) - q(v)$. Then the relation (3) follows from the equality

$$h(u+v, u+v) - h(u,u) - h(v,v) = h(u,v) + h(v,u).$$

It readily follows that $\mathfrak{b}_q$ is a symmetric bilinear form on $V$, i.e., $q$ is a quadratic form. □

We now fix a basis $\mathcal{B} = \{u_1, \cdots, u_n\}$ of $\mathrm{Sym}_\lambda(D,\theta)$ over $F$ and denote by $\{\pi_1, \cdots, \pi_n\}$ its dual basis of $\mathrm{Hom}(\mathrm{Sym}_\lambda(D,\theta), F)$. By Lemma 3.1, the map $q^{u_i}_{h,\mathcal{B}} : V \to F$ given by

$$q^{u_i}_{h,\mathcal{B}}(v) = \pi_i(h(v,v))$$

is a quadratic form with the polar form

$$\mathfrak{b}_{q^{u_i}_{h,\mathcal{B}}}(u,v) = \pi_i(h(u,v) + h(v,u)).$$

Note that

$$h(v,v) = \sum_{i=1}^{n} q^{u_i}_{h,\mathcal{B}}(v) u_i \quad \text{for all } v \in V.$$

Let $q_{h,\mathcal{B}} = (q^{u_1}_{h,\mathcal{B}}, \cdots, q^{u_n}_{h,\mathcal{B}})$. Then $q_{h,\mathcal{B}} : V \to F^n$ is a system of quadratic forms over $F$. We will simply denote $q^{u_i}_{h,\mathcal{B}}$ by $q_i$ and $q_{h,\mathcal{B}}$ by $q_h$ if no confusion arises. Note that if $(V,h)$ and $(V',h')$ are two hermitian spaces over $(D,\theta)$ then $q_{h \perp h', \mathcal{B}} = q_{h,\mathcal{B}} \perp q_{h',\mathcal{B}}$.

Remark 3.2. Suppose that $(D,\theta) = (F, \mathrm{id})$. If $\lambda = 1$ then $h$ is a symmetric bilinear form and $\mathrm{Sym}_\lambda(D,\theta) = F$. Taking $\mathcal{B} = \{1\}$, the form $q_h$ is just the quadratic form associated to the bilinear form $h$ given by $q_h(v) = h(v,v)$. If $\lambda = -1$ and $\mathrm{char}\, F \neq 2$ then $h$ is an *alternating* bilinear form (i.e., $h(v,v) = 0$ for all $v \in V$) and $\mathrm{Sym}_\lambda(D,\theta) = \{0\}$. In this case, $q_h$ is trivial.

Remark 3.3. Suppose that $\mathrm{char}\, F \neq 2$, $\lambda = 1$ and $D$ is a quaternion algebra. Let $\theta$ be the *canonical* involution of $D$, i.e., $\theta(x) = \mathrm{Trd}_D(x) - x$ for $x \in D$, where $\mathrm{Trd}_D(x)$ is the reduced trace of $x$ in $D$. Then $\mathrm{Sym}_\lambda(D,\theta) = F$ and one can choose $\mathcal{B} = \{1\}$ (see [6, p. 26]). In this case, the system $q_{h,\mathcal{B}}$ is just a quadratic form, known as the *Jacobson's trace form*. This form was introduced first in [4] (see [10] for a characteristic two counterpart).

In view of Remark 3.3, we call $q_h$ the *generalized Jacobson's trace map* of $h$. The next result shows that $q_h$ is an invariant of the isometry class of $h$.

Proposition 3.4. *Let $(V,h)$ and $(V',h')$ be two $\lambda$-hermitian spaces over $(D,\theta)$. If $(V,h) \simeq (V',h')$ then $q_{h,\mathcal{B}} \simeq q_{h',\mathcal{B}}$.*

*Proof.* Let $\phi : (V,h) \simeq (V',h')$ be an isometry. Considering $\phi$ as an isomorphism of $F$-linear spaces, we have

$$q^{u_i}_{h',\mathcal{B}}(\phi(v)) = \pi_i(h'(\phi(v),\phi(v))) = \pi_i(h(v,v)) = q^{u_i}_{h,\mathcal{B}}(v),$$

for $i = 1, \cdots, n$ and $v \in V$. It follows that $q_{h',\mathcal{B}}(\phi(v)) = q_{h,\mathcal{B}}(v)$ for all $v \in V$, i.e., $q_{h,\mathcal{B}} \simeq q_{h',\mathcal{B}}$. □

Proposition 3.5. *Let $(V,h)$ be a regular $\lambda$-hermitian space over $(D,\theta)$ and let $u \in \mathrm{Symd}_\lambda(D,\theta)$. Then for every nonzero vector $v \in V$ there exists $w \in V$ such that $h(v,w) + h(w,v) = u$. In particular, if $u_i \in \mathrm{Symd}_\lambda(D,\theta)$ for some basis element $u_i \in \mathcal{B}$, where $1 \leqslant i \leqslant n$, then $q_i$ is regular.*

*Proof.* Let $v \in V$ be a nonzero vector. Since $h$ is regular, there exists $v' \in V$ for which $h(v,v') = 1$. Write $u = d + \lambda\theta(d)$, where $d \in D$ and set $w = v'd$. Then

$$h(v,w) + h(w,v) = h(v,v')d + \theta(d)h(v',v) = d + \lambda\theta(d) = u.$$

If $u_i \in \mathrm{Symd}_\lambda(D,\theta)$ for some $i = 1, \cdots, n$, then the above argument shows that for every nonzero vector $v \in V$ there exists $w \in V$ such that $h(v,w) + h(w,v) = u_i$. Hence, $\mathfrak{b}_{q_i}(v,w) = 1$, i.e., $q_i$ is regular. □

Remark 3.6. The last statement of Proposition 3.5 is not necessarily true if $u_i \notin \mathrm{Symd}_\lambda(D,\theta)$. Indeed, let $\mathrm{char}\, F = 2$ and suppose that the basis $\mathcal{B}$ is chosen with the additional property that $\{u_1, \cdots, u_r\}$ is a basis of $\mathrm{Symd}(D,\theta)$ for some nonnegative integer $r < n$. Since for all $v, w \in V$ we have $h(v,w) + h(w,v) \in \mathrm{Symd}(D,\theta)$, the polar form of $q_i$ is zero for all $i > r$. In particular, if $D = F$ then $h$ is a bilinear form and $q_h$ is a quadratic form, whose polar form is zero (note that in this case, $\mathrm{Sym}(D,\theta) = F$ and $\mathrm{Symd}(D,\theta) = \{0\}$).

Corollary 3.7. *Let $(V, h)$ be a $\lambda$-hermitian space over $(D, \theta)$. Suppose that either $D \neq F$ or $\lambda \neq -1$. If $h$ is regular then $q_h$ is strongly regular.*

*Proof.* The assumption $D \neq F$ or $\lambda \neq -1$ implies that $\mathrm{Symd}_\lambda(D, \theta) \neq \{0\}$ (see [6, (2.6)]). Let $u \in \mathrm{Symd}_\lambda(D, \theta)$ be a nonzero element. Write $u = \sum_{i=1}^n a_i u_i$ for some $a_1, \cdots, a_n \in F$. By re-indexing if necessary, we may assume that $a_1 \cdots a_r \neq 0$, where $r \leqslant n$ is a positive integer. Let $v \in V$ be an arbitrary nonzero vector. By Proposition 3.5 there exists $w \in V$ for which $h(v, w) + h(w, v) = u$. It follows that $\mathfrak{b}_{q_i}(v, w) = a_i \neq 0$ for $i \leqslant r$. Hence, $q_i$ is regular for $i = 1, \cdots, r$. □

Remark 3.8. Corollary 3.7 does not hold in the case where $D = F$ and $\lambda = -1$. Indeed, if $\mathrm{char}\, F \neq 2$ then as observed in Remark 3.2, $q_h$ is trivial. Also, if $\mathrm{char}\, F = 2$ then the polar form of $q_h$ is zero (see Remark 3.6).

## 4 Classification of hermitian forms

In this section we state some characterizing properties of the generalized Jacobson's trace map. We first show that the system $(V, q_h)$ completely determines the isotropy behaviour of the $\lambda$-hermitian space $(V, h)$. Recall that $h$ is called *isotropic* if $h(v, v) = 0$ for some nonzero vector $v \in V$. Let $W$ be a subspace of $V$. The *orthogonal complement* of $W$ is defined as

$$W^{\perp_h} = \{v \in V \mid h(v, w) = 0 \quad \text{for all } w \in W\}.$$

The form $h$ is called *metabolic* if there exists a subspace $L \subseteq V$ such that $L = L^{\perp_h}$.

Proposition 4.1. *Let $(V, h)$ be a $\lambda$-hermitian space over $(D, \theta)$. Then $h$ is isotropic if and only if $q_h$ is isotropic.*

*Proof.* The result follows from the equality

$$h(v, v) = q_1(v)u_1 + \cdots + q_n(v)u_n \quad \text{for } v \in V,$$

together with the linear independence of $\{u_1, \cdots, u_n\}$. □

Theorem 4.2. *Let $(V, h)$ be a $\lambda$-hermitian space over $(D, \theta)$. If $h$ is metabolic then $q_h$ is metabolic. The converse is also true if $h$ is regular and either $\mathrm{char}\, F \neq 2$ or $D \neq F$.*

*Proof.* If there exists a subspace $L \subseteq V$ such that $L = L^{\perp_h}$, then $\dim_D L \geqslant \frac{1}{2} \dim_D V$ and $h|_{L \times L} = 0$. Hence, $\dim_F L \geqslant \frac{1}{2} \dim_F V$ and $q_h|_L = 0$, i.e., $q_h$ is metabolic.
Suppose now that $q_h$ is metabolic, $h$ is regular and either $D \neq F$ or $\mathrm{char}\, F \neq 2$. In the case where $D = F$ and $\lambda = -1$ we have $\mathrm{char}\, F \neq 2$. Hence, $h$ is an alternating bilinear form, which is metabolic by [3, (1.8)]. Otherwise, the hypotheses of Corollary 3.7 are satisfied, hence $q_h$ is strongly regular. By [5,

Ch. I, (6.1.1)], one can write $h \simeq h_{\rm an} \perp h_{\rm met}$, where $h_{\rm an}$ is anisotropic and $h_{\rm met}$ is metabolic. Hence, $q_h \simeq q_{h_{\rm an}} \perp q_{h_{\rm met}}$ by Proposition 3.4. If $h_{\rm an}$ is nontrivial, then $q_{h_{\rm an}}$ is anisotropic by Proposition 4.1. However, the above argument shows that $q_{h_{\rm met}}$ is metabolic. This contradicts Proposition 2.3. □

REMARK 4.3. The converse of Theorem 4.2 does not necessarily hold if either $h$ is not regular or $\operatorname{char} F = 2$ and $D = F$. For the first case, let $h = \langle 1, 0\rangle_{(D,\theta)}$ be the diagonal form $h((x_1, x_2), (y_1, y_2)) = \theta(x_1)y_1$. Then $h$ is not metabolic, but $q_h$ is metabolic. For the second case, let $\mathfrak{b}$ be a two-dimensional anisotropic symmetric bilinear form and set $h = \mathfrak{b} \perp \mathbb{H}$, where $\mathbb{H}$ is the hyperbolic plane (note that in this case $h$ is a bilinear form and $q_h$ is a quadratic form). Then $h$ is a regular form which is not metabolic, but $q_h$ is metabolic.

LEMMA 4.4. *Let $f : (V, q) \perp (W, \rho) \simeq (V', q') \perp (W', \rho')$ be an equivalence of systems of quadratic forms. If $\rho$ and $\rho'$ are zero forms and $q$ and $q'$ are regular, then $(V, q) \simeq (V', q')$.*

*Proof.* Since $(V \oplus W)^\perp = W$ and $(V' \oplus W')^\perp = W'$, we have $\dim_F W = \dim_F W'$, hence $\dim_F V = \dim_F V'$. Let $p_1 : V' \oplus W' \to V'$ be the natural projection $(v', w') \mapsto v'$. Consider the map $g : V \to V'$ defined by $g(v) = p_1 \circ f(v, 0)$. It is easily seen that $g$ is an injective map satisfying $q'(g(v)) = q(v)$ for all $v \in V$. Dimension count shows that $g : (V, q) \simeq (V', q')$ an equivalence. □

We now consider the converse of Proposition 3.4 for even $\lambda$-hermitian forms over $(D, \theta)$.

THEOREM 4.5. *Let $(V, h)$ and $(V', h')$ be two even $\lambda$-hermitian spaces over $(D, \theta)$. If $q_{h,\mathcal{B}} \simeq q_{h',\mathcal{B}}$ then $(V, h) \simeq (V', h')$, except for the case where* $\operatorname{char} F \neq 2$*, $D = F$ and $\lambda = -1$.*

*Proof.* Suppose that $\operatorname{char} F = 2$ or $D \neq F$ or $\lambda \neq -1$. If $D = F$ and $\lambda = -1$, then $\operatorname{char} F = 2$ and $\operatorname{Symd}(F, \mathrm{id}) = \{0\}$. Hence, $h$ and $h'$ are zero forms and the result holds by dimension count. Otherwise, we have $D \neq F$ or $\lambda \neq -1$. Write $h \simeq h_1 \perp h_2$ and $h' \simeq h_1' \perp h_2'$, where $h_1$ and $h_1'$ are regular and $h_2$ and $h_2'$ are zero forms. Then the equivalence $q_{h,\mathcal{B}} \simeq q_{h',\mathcal{B}}$ implies that

$$q_{h_1,\mathcal{B}} \perp q_{h_2,\mathcal{B}} \simeq q_{h_1',\mathcal{B}} \perp q_{h_2',\mathcal{B}}.$$

By Corollary 3.7, $q_{h_1,\mathcal{B}}$ and $q_{h_1',\mathcal{B}}$ are strongly regular. Since $q_{h_2,\mathcal{B}}$ and $q_{h_2',\mathcal{B}}$ are zero forms, Lemma 4.4 implies that $q_{h_1,\mathcal{B}} \simeq q_{h_1',\mathcal{B}}$. It follows from Lemma 2.2 that $q_{h_1 \perp (-h_1'),\mathcal{B}} \simeq q_{h_1,\mathcal{B}} \perp (-q_{h_1',\mathcal{B}})$ is metabolic. The assumption $D \neq F$ or $\lambda \neq -1$ implies that either $D \neq F$ or $\operatorname{char} F \neq 2$, and thus it follows that $h_1 \perp (-h_1')$ is metabolic by Theorem 4.2. Since $h_1$ and $h_1'$ are even, [5, Ch. I, (6.4.5)] implies that $h_1 \simeq h_1'$. Dimension count now shows that $h \simeq h'$. □

REMARK 4.6. If $\operatorname{char} F \neq 2$, $D = F$ and $\lambda = -1$ in Theorem 4.5 then $q_h$ and $q_{h'}$ are trivial (see Remark 3.2). Hence, this result is not necessarily true in this exceptional case. It is also worth noting that Theorem 4.5 does not hold for an arbitrary regular hermitian form. Indeed, let $D = F$ be a field of characteristic two with $F \neq F^2$ and choose an element $a \in F \setminus F^2$. The bilinear forms $h_1 = \langle 1, a\rangle$ and $h_2 = \langle 1, a+1\rangle$ are not equivalent because $a(a+1) \notin F^2$. However, the quadratic forms $q_{h_1} = \langle 1, a\rangle$ and $q_{h_2} = \langle 1, a+1\rangle$, whose corresponding symmetric bilinear forms are trivial, are equivalent because $\{1, a\}$ and $\{1, a+1\}$ generate the same subspace of $F$ over $F^2$.

## 5 An application in characteristic two

Let $K/F$ be a finite extension such that $D_K := D \otimes K$ is a division algebra. If $(V, h)$ is a $\lambda$-hermitian space over $(D, \theta)$, then there exists a $\lambda$-hermitian form $(V_K, h_K)$ over $(D, \theta)_K := (D_K, \theta \otimes \mathrm{id})$, where $h_K : V_K \times V_K \to D_K$ is induced by $h_K(x \otimes \alpha, y \otimes \beta) = h(x, y) \otimes \alpha\beta$. Since $\mathrm{Sym}_\lambda((D, \theta)_K) = \mathrm{Sym}_\lambda(D, \theta) \otimes K$, the set $\mathcal{B}' := \{u_1 \otimes 1, \cdots, u_n \otimes 1\}$ is a $K$-basis of $\mathrm{Sym}_\lambda((D, \theta)_K)$. For $i = 1, \cdots, n$, by identifying $F \otimes K = K$, we obtain a quadratic form $q^{u_i \otimes 1}_{h_K, \mathcal{B}'} : V_K \to K$ satisfying

$$q^{u_i \otimes 1}_{h_K, \mathcal{B}'}(v \otimes \alpha) = \pi_i(h(v, v))\alpha^2 = q^{u_i}_{h, \mathcal{B}}(v)\alpha^2 \quad \text{for all } v \in V \text{ and } \alpha \in K.$$

It readily follows that the definition of $q_{h,\mathcal{B}}$ is functorial, i.e.,

$$q_{h_K, \mathcal{B}'} = (q_{h,\mathcal{B}})_K.$$

The following result is based on Springer's theorem [3, (18.5)] and a theorem of Amer-Brumer (see [1], [2] and [7]).

Theorem 5.1. *Let $K/F$ be a field extension of odd degree and let $q = (q_1, q_2)$ be a 2-fold system of quadratic forms over $F$. If $q$ is anisotropic, then $q_K$ is also anisotropic.*

*Proof.* See [9, Ch. 9, (1.11)]. □

Lemma 5.2. *Let $(A, \sigma)$ be a central simple algebra over $F$. If $x \in \mathrm{Symd}_\lambda(A, \sigma)$ then $\sigma(y)xy \in \mathrm{Symd}_\lambda(A, \sigma)$ for every $y \in A$.*

*Proof.* Write $x = z + \lambda\sigma(z)$ for some $z \in A$. Then

$$\sigma(y)xy = \sigma(y)(z + \lambda\sigma(z))y = \sigma(y)zy + \lambda\sigma(\sigma(y)zy) \in \mathrm{Symd}_\lambda(A, \sigma). \quad \square$$

We conclude by proving the following analogue of [8, (3.5)].

Theorem 5.3. *Suppose that $\mathrm{char}\, F = 2$ and $D$ is a quaternion division $F$-algebra. If $K/F$ is a finite extension of odd degree, then every anisotropic hermitian space over $(D, \theta)$ remains anisotropic over $(D, \theta)_K$.*

*Proof.* Suppose that there exists an anisotropic hermitian space $(V, h)$ over $(D, \theta)$ for which $(V_K, h_K)$ is isotropic. Choose such a hermitian space with $m := \dim_D V$ minimal. Clearly, we have $m > 1$. We claim that there exists

an orthogonal basis $\{v_1, \cdots, v_m\}$ of $(V, h)$ satisfying $h(v_i, v_i) \in \mathrm{Symd}(D, \theta)$ for $i = 1, \cdots, m$.
By [6, (2.6 (2))] we have $\dim_F \mathrm{Symd}(D, \theta) = 1$ and $\dim_F \mathrm{Sym}(D, \theta) = 3$. Let $u_1 \in \mathrm{Symd}(D, \theta)$ be a unit, so that $\mathrm{Symd}(D, \theta) = Fu_1$. Extend $\{u_1\}$ to a basis $\{u_1, u_2, u_3\}$ of $\mathrm{Sym}(D, \theta)$. We construct inductively the required set $\{v_1, \cdots, v_m\}$. First, note that since $h_K$ is isotropic, the system

$$(q_1, q_2, q_3)_K = (q_h)_K = q_{h_K}$$

is isotropic by Proposition 4.1. Applying Theorem 5.1 to the system $(q_2, q_3)$, one can find a nonzero vector $v_1 \in V$ such that $q_2(v_1) = q_3(v_1) = 0$. Hence,

$$h(v_1, v_1) = a_1 u_1 \in \mathrm{Symd}(D, \theta),$$

where $a_1 = q_1(v_1) \in F$. Suppose now that there exists a linearly independent set $\{v_1, \cdots, v_r\} \subset V$ such that

$$h(v_i, v_i) \in \mathrm{Symd}(D, \theta) \quad \text{for } i = 1, \cdots, r, \tag{4}$$

and $h(v_i, v_j) = 0$ for $1 \leqslant i \neq j \leqslant r$, where $1 \leqslant r < m$. Let $W = v_1 D + \cdots + v_r D$ and $S = W^{\perp_h}$. Then $(V, h) \simeq (W, h|_{W\times W}) \perp (S, h|_{S\times S})$ by [5, Ch. I, (3.6.2)]. The minimality of $m$ implies that $(h|_{W\times W})_K$ and $(h|_{S\times S})_K$ are anisotropic. Since $h_K$ is isotropic there exist nonzero vectors $w \in W_K$ and $w' \in S_K$ such that $h_K(w + w', w + w') = 0$. Write $w = v_1 d_1 + \cdots + v_r d_r$ for some $d_1, \cdots, d_r \in D_K$ and set $h' = h|_{S\times S}$. By Lemma 5.2 and (4) we have

$$\begin{aligned} h'_K(w', w') &= h_K(w', w') = -h_K(w, w) \\ &= -\sum_{i=1}^{r} \theta_K(d_i) h(v_i, v_i) d_i \in \mathrm{Symd}(D, \theta) \otimes K. \end{aligned}$$

Hence, $(q^{u_2}_{h',\mathcal{B}})_K(w') = (q^{u_3}_{h',\mathcal{B}})_K(w') = 0$. By Theorem 5.1 there exists a nonzero vector $v_{r+1} \in S$ such that $q^{u_2}_{h',\mathcal{B}}(v_{r+1}) = q^{u_3}_{h',\mathcal{B}}(v_{r+1}) = 0$. Hence,

$$h(v_{r+1}, v_{r+1}) \in \mathrm{Symd}(D, \theta).$$

So we have extended the set $\{v_1, \cdots, v_r\}$ to $\{v_1, \cdots, v_{r+1}\}$ with the required properties. The claim therefore follows from induction.
Now, let $v \in V$ be an arbitrary vector and write $v = v_1 d'_1 + \cdots v_m d'_m$ for some $d'_1, \cdots, d'_m \in D$. Then Lemma 5.2 implies that

$$h(v, v) = \sum_{i=1}^{m} \theta(d'_i) h(v_i, v_i) d'_i \in \mathrm{Symd}(D, \theta),$$

i.e., $h$ is an even hermitian form. It follows that the forms $q_2$ and $q_3$ are trivial. In other words, the system $q_h$ reduces to the quadratic form $q_1$. Since $h_K$ is isotropic, using Proposition 4.1 and Springer's theorem [3, (18.5)], one concludes that $h$ is isotropic, contradicting the assumption. □

Remark 5.4. The proof of Theorem 5.3 is really specific to the characteristic 2 case. Indeed, this proof relies on the fact that $\mathrm{Symd}(D,\theta)$ is a one-dimensional subspace of $\mathrm{Sym}(D,\theta)$ satisfying $\theta(d)\cdot\mathrm{Symd}(D,\theta)\cdot d\subseteq\mathrm{Symd}(D,\theta)$ for all $d\in D$. However, if $\operatorname{char} F\neq 2$, $\dim_F \mathrm{Sym}_\lambda(D,\theta)=3$ and $h$ is a $\lambda$-hermitian form over $(D,\theta)$, then one can easily show that there is no 1-dimensional subspace $S$ of $\mathrm{Sym}_\lambda(D,\theta)$ for which $\theta(d)\cdot S\cdot d\subseteq S$ for all $d\in D$.

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