Motivic Cohomology of Fat Points in Milnor Range

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Abstract. We introduce a new algebraic-cycle model for the motivic cohomology theory of truncated polynomials $k[t]/(t^m)$ in one variable. This approach uses ideas from the deformation theory and non-archimedean analysis, and is distinct from the approaches via cycles with modulus. We prove that the groups in the Milnor range give the Milnor K-groups of $k[t]/(t^m)$, when the base field is of characteristic 0. Its relative part is the sum of the absolute Kähler differential forms.

1 Introduction

The objective of this paper is to present a new algebraic-cycle model for the motivic cohomology theory of some singular k-schemes and to compute its simplest case concretely, to justify the model.

Bloch's higher Chow groups [3] of smooth k-schemes give the correct motivic cohomology groups as shown by Voevodsky [34], but they fail to do so for singular k-schemes. For instance, the motivic cohomology groups are expected to be part of a conjectural Atiyah-Hirzebruch type spectral sequence that converges to higher algebraic K-groups [26] of Quillen. Here the K-groups do detect the difference of a scheme X and and its reduced scheme X_{red} (see e.g. [33]), while the higher Chow groups do not distinguish X from X_{red}.

The additive higher Chow groups, initiated by Bloch-Esnault [4], were in a sense born as a way to complement this issue for non-reduced schemes. This approach developed further by e.g. [18], [19], [21], [24], [25], [27], has had several successful aspects; for instance, they provide understanding of Witt vectors, de Rham-Witt complexes and crystalline cohomology via algebraic cycles based on the moving lemma of [15]. They also spawned a variation, "higher Chow groups with modulus" (see [2]) that rapidly built connections to various subjects of mathematics such as abelianized fundamental groups [17], Somekawa K-groups and reciprocity functors [12], [28], and motives with modulus [14], to name a few. However, our further attempts to understand the conjectural motivic cohomology for singular schemes through the cycles with modulus were bumping into increasingly complex technical and philosophical issues. Some of these hindrances encouraged the authors to return to the starting point, and to look for and develop some fundamentally new approaches. The new approach of this paper may resolve some of the old issues, while it may create a different set of technical problems. For instance, the Milnor range is now represented by higher dimensional cycles, not by 0-cycles, so that harder algebro-geometric challenges await us. Nevertheless, we choose to work with this new model, because this seems to be leading us further as well as opening new avenues to handle algebraic cycles via some new means and ideas such as deformation theory or non-archimedean analysis, that were thought to be distant from the subject until now.

The particular case studied in depth in this paper is the truncated polynomial ring $k_m := k[t]/(t^m)$. We show that the Milnor K-groups $K_n^M(k_m)$ can be expressed in terms of our new cycle groups in the Milnor range. The precursors of these theorems for higher Chow groups were the theorems of [23] and [29], and for additive higher Chow groups, the theorems of [4] and [27]. Our theorem in this paper is a unification of all those precursors in *ibids.* in a sense. We repeat however that unlike those precursors, our cycle representatives in the Milnor range are now 1-cycles. In fact, the 0-cycles do not appear in our groups (see Remark 2.3.6) at all, so our 1-cycles in the Milnor range form yet the simplest part.

We retain the notations of the cubical version of higher Chow groups (see §2.1) for smooth k-schemes. For k_m, we redefine $\mathrm{CH}^q(k_m, n)$ in §2.3, different from the higher Chow groups of [3], but when $m = 1$ so that $k_m = k$, we do have the agreement $\mathrm{CH}^q(k_m, n) = \mathrm{CH}^q(k, m)$. In this new theory, for $m \geq 1$ we can easily define the relative group $\mathrm{CH}^q((k_m, (t)), n)$ (see Definition 2.4.2). The main theorem, following immediately from Theorem 3.0.1, is:

THEOREM 1.0.1. *Let k be a field of characteristic 0 and let $m, n \geq 1$ be integers. Let $k_m := k[t]/(t^m)$. Then the graph homomorphism $K_n^M(k_m) \to \mathrm{CH}^n(k_m, n)$ to the redefined higher Chow group of k_m is an isomorphism. The isomorphism of the relative parts takes the form $(\Omega_{k/\mathbb{Z}}^{n-1})^{\oplus(m-1)} \simeq \mathrm{CH}^n((k_m, (t)), n)$, where the former is isomorphic to the relative Milnor K-group $K_n^M(k_m, (t))$.*

We mention a few further new aspects of our theory. One of them comes from

that the ring k_m has several presentations $k[t]/(t^m)$, $\mathcal{O}_{\mathbb{A}^1_k,0}/(t^m)$ and $k[[t]]/(t^m)$. Usually those who work on algebraic cycles work with the "algebraic" situations, e.g. cycles over either $k[t]$ or its localization $\mathcal{O} := \mathcal{O}_{\mathbb{A}^1_k,0}$. In this paper, we work with cycles over $k[[t]] = \widehat{\mathcal{O}}_{\mathbb{A}^1_k,0}$, which is henselian. This gives more admissible cycles than the "algebraic" situation over $\mathcal{O}$, and generally they have a better rationality property. For instance, $y = \sqrt{1+t}$ is of degree 2 over $\mathcal{O}$, but it is rational over $\widehat{\mathcal{O}}$ because $\sqrt{1+t} = 1 + \frac{t}{2} - \frac{t^2}{8} + \frac{t^3}{16} - \cdots$. The possibility of using Hensel's lemma could also be a technical benefit.

On a pair of integral cycles over $k[[t]]$, we put a "mod t^m equivalence relation" when their pull-backs to $k[[t]]/(t^m)$ are equal (see Definition 2.3.3). This allows us to use intuitions from the deformation theory to study cycles. The other structure that we use in our new model is the non-archimedean t-adic metric on $k((t)) = \mathrm{Frac}(k[[t]])$. This non-archimedean analytic view-point helps us in proving the following Theorem 1.0.2, stated in Theorem 4.3.2. This is a new type moving result up to mod t^m equivalence. This theorem enables us to transport some technical results already known for cycles over $k[t]$ or $\mathcal{O}_{\mathbb{A}^1_k,0}$, to our cycles over $k[[t]]$:

THEOREM 1.0.2. *Let k be a field. Let $\mathcal{O} := \mathcal{O}_{\mathbb{A}^1_k,0}$ and let $\widehat{\mathcal{O}} := \widehat{\mathcal{O}}_{\mathbb{A}^1_k,0}$. For the completion homomorphism $\xi^n : z^n_{\mathfrak{m}}(\mathcal{O}, n)^c \to z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$, the composition $\xi^n_m : z^n_{\mathfrak{m}}(\mathcal{O}, n)^c \to z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c \to z^n(k_m, n) := z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c / \sim_{t^m}$ is surjective.*

Here, the superscripts "c" signify that the groups consist of the cycles proper over $\mathcal{O}$ and $\widehat{\mathcal{O}}$, respectively. See Definition 2.2.5. Although there generally are more cycles over $k[[t]]$, this theorem shows that modulo t^m in the Milnor range, we can still approximate them by those of the algebraic origin. This eventually allows a reduction of the proof of Theorem 1.0.1 to the graph cycles, providing a great technical simplification.

Since this mod t^m moving lemma of Theorem 1.0.2 is a new type of result for the studies algebraic cycles, to give some motivations to the reader, let us quickly sketch the idea. The essential point behind the proof of Theorem 1.0.2 is the notion of coefficient perturbations of Definition 4.1.1: when $W \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ is an integral cycle, we show that it is possible to choose a suitable system of equations, for which perturbations of the coefficients may produce good deformations of W. The base parameter space is the space of all choices of the coefficients. Some property such as non-emptiness of the solutions is an open condition on this base. For some other properties, we need a flat family. For this, we use a trick that is an explicit version of the flattening stratification theorem. Using so-obtained locally closed nonempty base over which we have a flat family, we prove that we can deform geometrically integral W mod t^m with all the desired properties preserved, such that it comes from the "algebraic world" over $\mathcal{O}$. In the process, we need to resort to the non-archimedean t-adic metric topology. The general integral cycle case is reduced to the geometrically integral case by constructing a Nisnevich cover.

Some follow-up works will treat the cases of off-Milnor range of the relative

Chow group of $(k_m, (t))$, with a cycle-theoretic version of the regulator maps on the additive polylogarithmic complex constructed and studied in [30] and [31]. *cf.* [32]. Its comparison with the regulators in [24] and [25] will also be discussed. Other on-going works deal with cycles over Artin local k-algebras with embedding dimensions ≥ 1.

We remark that our cycle complex seems to have a natural generalization that might be a candidate model for the motivic cohomology of any k-scheme with singularities. This also seems to offer a way to define the relative version for any pair (X, Z) of a scheme and its closed subscheme. The verification that this is well-defined and is the correct definition will require nontrivial works. We hope that this paper provides an evidence that either our approach or its variation has a potential to reach the goal of constructing the ultimate motivic cohomology theory for all k-schemes.

Conventions. For a scheme $X \to \mathrm{Spec}\,(R)$ over a discrete valuation ring R, we always denote the special fiber by X_s, and the generic fiber by X_η.

2   Recollections, new definitions and basic results

In this section, we recall and prove some basic definitions and results on higher Chow complexes needed in this paper. A new one over the truncated polynomial rings $k[t]/(t^m)$ will be defined in §2.3, which is the main complex we work with.

2.1   Recollections of higher Chow cycles

Let k be a field. We recall the cubical version of Bloch's higher Chow complexes (*cf.* [3]). Let $\mathbb{P}_k^1 := \mathrm{Proj}(k[u_0, u_1])$, and let $\overline{\square}_k := \mathbb{P}_k^1$, with $y := u_1/u_0$ as the coordinate. Let $\square_k := \overline{\square}_k \setminus \{1\}$. We let $\square_k^0 = \overline{\square}_k^0 := \mathrm{Spec}\,(k)$, and for $n \geq 1$, we let $\square_k^n$ (resp. $\overline{\square}_k^n$) be the n-fold product of $\square_k$ (resp. $\overline{\square}_k$) with itself over k. A *face* F of $\square_k^n$ (resp. $\overline{\square}_k^n$) is defined to be the closed subscheme given by a finite set of equations of the form $\{y_{i_1} = \epsilon_1, \cdots, y_{i_u} = \epsilon_u\}$, for an increasing sequence of indices $1 \leq i_1 < \cdots < i_u \leq n$, and $\epsilon_j \in \{0, \infty\}$. We allow the case of the empty set of equations, i.e. having $F = \square_k^n$. A *codimension 1 face* is given by a single such equation. We often write F_i^ϵ to be the face given by $\{y_i = \epsilon\}$.

For a smooth k-scheme X, we let $\square_X^n := X \times_k \square_k^n$, $\overline{\square}_X^n := X \times_k \overline{\square}_k^n$, and define the face F_X of $\square_X^n$ to be the pull-back $X \times_k F$, of a face F of $\square_k^n$. We drop the subscript X from $F_{i,X}^\epsilon$ when no confusion arises. Let $\underline{z}^q(X, n)$ be the free abelian group on the set of codimension q integral closed subschemes $Z \subseteq \square_X^n$ that intersect each face properly on $\square_X^n$. For each codimension 1 face $F_{i,X}^\epsilon$, with $1 \leq i \leq n$ and $\epsilon \in \{0, \infty\}$, and an irreducible $Z \in \underline{z}^q(X, n)$, we let $\partial_i^\epsilon(Z)$ be the cycle associated to the scheme-theoretic intersection $Z \cap F_{i,X}^\epsilon$. By definition, $\partial_i^\epsilon(Z) \in \underline{z}^q(X, n-1)$. This forms a cubical abelian group $(\underline{n} \mapsto \underline{z}^q(X, n))$, where $\underline{n} = \{0, 1, \cdots, n\}$, in the sense of [22, §1.1]. Let $\partial := \sum_{i=1}^n (-1)^i (\partial_i^\infty - \partial_i^0)$ on $\underline{z}^q(X, n)$. One checks immediately from the formalism of cubical abelian groups that $\partial \circ \partial = 0$ and hence one obtains the associated nondegenerate complex $z^q(X, \bullet) := \underline{z}^q(X, \bullet)/\underline{z}^q(X, \bullet)_{\mathrm{degn}}$, where $\underline{z}^q(X, n)_{\mathrm{degn}}$ is the subgroup of degenerate cycles, i.e. sums of those obtained by pulling back via one of the standard projections $\square_X^n \to \square_X^{n-1}$, for $0 \leq i \leq n$, which omits one of the coordinates on $\square_X^n$. This complex $(z^q(X, \bullet), \partial)$ is called the (cubical) *higher Chow complex* of X and its homology groups $\mathrm{CH}^q(X, n) := \mathrm{H}_n(z^q(X, \bullet))$ are the higher Chow groups of X. It is a theorem of Voevodsky [34] that the higher Chow groups form a universal bigraded ordinary cohomology theory $\mathrm{H}_{\mathcal{M}}^{2q-n}(X, \mathbb{Z}(q)) := \mathrm{CH}^q(X, n)$ on the category of smooth k-varieties X.

2.2 Some subgroups

If we are given an integral closed subscheme $W \subseteq X$, we have a subcomplex $z_W^q(X, \bullet) \subseteq z^q(X, \bullet)$ defined as follows: first, let $\underline{z}_W^q(X, n) \subseteq \underline{z}^q(X, n)$ be the subgroup generated by integral closed subschemes $Z \subseteq \square_X^n$ that intersect each $W \times F$ properly on $\square_X^n$, as well as each $F_X = X \times F$, for every face F of $\square_k^n$. More precisely, we require that the codimension of $Z \cap (W \times F)$ in $W \times F$ is at least q. Modding out by degenerate cycles, we obtain the subcomplex $z_W^q(X, \bullet) \subseteq z^q(X, \bullet)$. In this paper, we are interested only in the cases when (X, W) is $(\mathrm{Spec}\,(\widehat{\mathcal{O}}), \widehat{\mathfrak{m}})$ or $(\mathrm{Spec}\,(\mathcal{O}), \mathfrak{m})$ where:

Definition 2.2.1. Let $\mathcal{O} := \mathcal{O}_{\mathbb{A}_k^1, 0}$ and $\mathfrak{m} := \mathfrak{m}_{|mathbbA_k^1, 0}$. Let $\widehat{\mathcal{O}} := \widehat{\mathcal{O}}_{\mathbb{A}_k^1, 0}$ be the completion of $\mathcal{O}$ at $\mathfrak{m}$, and let $\widehat{\mathfrak{m}}$ be its unique maximal ideal. Here $\widehat{\mathcal{O}} \simeq k[[t]]$. For $m \geq 1$, let $k_m := \widehat{\mathcal{O}}/(t^m) = k[[t]]/(t^m)$. We use these notations throughout this paper.

Remark 2.2.2. We have $z_{\widehat{\mathfrak{m}}}^{n+1}(\widehat{\mathcal{O}}, n) = 0$. To see this, suppose that $z_{\widehat{\mathfrak{m}}}^{n+1}(\widehat{\mathcal{O}}, n) \neq 0$ and let $\mathfrak{p} \in z_{\widehat{\mathfrak{m}}}^{n+1}(\widehat{\mathcal{O}}, n)$ be a closed point on $\square_{\widehat{\mathcal{O}}}^n$. Here, $[k(\mathfrak{p}) : k] < \infty$ so that the image of the composition $\mathfrak{p} \hookrightarrow \square_{\widehat{\mathcal{O}}}^n \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ must be the unique closed point $\widehat{\mathfrak{m}}$ of $\mathrm{Spec}\,(\widehat{\mathcal{O}})$, i.e. $\mathfrak{p}$ lies in the special fiber of the morphism $\square_{\widehat{\mathcal{O}}}^n \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$, contradicting the assumption that $\mathfrak{p} \in z_{\widehat{\mathfrak{m}}}^{n+1}(\widehat{\mathcal{O}}, n)$. Hence $z_{\widehat{\mathfrak{m}}}^{n+1}(\widehat{\mathcal{O}}, n) = 0$.

Remark 2.2.3. We have $z_{\mathfrak{m}}^{n+1}(\mathcal{O}, n) = 0$ as well. The proof is identical to that of Remark 2.2.2 by simply replacing $(\widehat{\mathcal{O}}, \widehat{\mathfrak{m}})$ by $(\mathcal{O}, \mathfrak{m})$. We use Remarks 2.2.2 and 2.2.3 later.

Corollary 2.2.4. *For* $n \geq 1$, *let* $Z \in z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)$ *be an integral cycle. Then for any proper face* $F \subset \square_{\widehat{\mathcal{O}}}^n$, *we have* $Z \cap F = \emptyset$. *In particular, we have* $\partial_i^\epsilon(Z) = 0$ *for any* $1 \leq i \leq n$ *and* $i \in \{0, \infty\}$, *thus* $\partial(Z) = 0$. *A similar result holds for* $Z \in z_{\mathfrak{m}}^n(\mathcal{O}, n)$.

Proof. If $Z \cap F \neq \emptyset$ for a codimension 1 face $F \subset \square_{\widehat{\mathcal{O}}}^n$ given by $\{y_i = \epsilon\}$, then $\partial_i^\epsilon(W) \neq 0$ in $z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n-1)$. But this contradicts Remark 2.2.2 that says $z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n-1) = 0$. So Z does not intersect any codimension 1 face. On the other hand, any proper face is contained in a codimension 1 face, so the corollary follows. $\square$

Definition 2.2.5. Let $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^c \subset z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)$ be the subgroup generated by the integral cycles $Z \in z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)$ that are proper over $\mathrm{Spec}\,(\widehat{\mathcal{O}})$. Similarly, define $z_{\mathfrak{m}}^q(\mathcal{O}, n)^c \subset z_{\mathfrak{m}}^q(\mathcal{O}, n)$.

There are some technical advantages in working with cycles in $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^c$. Firstly we have the following, inspired by the finiteness criterion in [20, §2.3] (Its shortcomings follow soon in Lemma 2.2.8.):

Lemma 2.2.6. *Let* X *be a* k-scheme. *Let* $W \subset \square_X^n$ *be a nonempty closed subscheme and let* $\overline{W} \subset \overline{\square}_X^n$ *be its Zariski closure. Then* $W \to X$ *is proper if and only if* $W = \overline{W}$.

Proof. $(\Rightarrow)$ The structure morphism $W \to X$ factors into the composite $W \hookrightarrow \overline{\square}_X^n \to X$. Since the composite is assumed to be proper and the second morphism is separated by [11, Theorem II-4.9, p.103], the inclusion $W \hookrightarrow \overline{\square}_X^n$ is proper by [11, Corollary II-4.8(e), p.102]. In particular W is closed in $\overline{\square}_X^n$ and $W = \overline{W}$.
$(\Leftarrow)$ If $W = \overline{W}$, then $W \hookrightarrow \overline{\square}_X^n$ is closed, in particular it is a proper morphism by [11, Corollary II-4.9(a), p.102]. Hence composed with the proper projective morphism $\overline{\square}_X^n \to X$, the composite $W \to X$ is proper by [11, Corollary II-4.8(b), p.102]. $\square$

Lemma 2.2.7. *Let* $W \in z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^c$ *be a nonempty integral cycle. Then we have:*

  1. W *is closed in* $\overline{\square}_{\widehat{\mathcal{O}}}^n$, *so that its Zariski closure* $\overline{W}$ *is* W *itself.*

  2. *The structure morphism* $W \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ *is surjective.*

A similar assertion holds for $W \in z_{\mathfrak{m}}^q(\mathcal{O}, n)^c$.

Proof. (1) is a corollary to Lemma 2.2.6.
If $W \to \mathrm{Spec}(\widehat{\mathcal{O}})$ is not dominant, then $W = W_s$, which violates the assumption that $W \in z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)$. Hence $W \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ is dominant. Now, being proper and dominant, it must be surjective, proving (2). $\square$

The above Lemmas 2.2.6 and 2.2.7 give interesting results for nonempty integral cycles in $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^c$. However, this non-emptiness assumption is nontrivial:

LEMMA 2.2.8. *The group $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^c$ is nonzero if and only if $q = n$.*

Proof. If $q \geq n+2$, the group $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^c = 0$ since $\dim(\square_{\widehat{\mathcal{O}}}^n) = n+1$. If $q = n+1$, it is zero by Remark 2.2.2. Let $q < n$ and suppose that $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^c \neq 0$, so there is a nonempty integral closed subscheme $W \subset \square_{\widehat{\mathcal{O}}}^n$ that is proper over $\operatorname{Spec}(\widehat{\mathcal{O}})$. Then $W \to \operatorname{Spec}(\widehat{\mathcal{O}})$ is a proper morphism affine schemes, so that it must be finite. (See [11, Exercise II-4.6, p.106]) This forces $q = n$, a contradiction. On the other hand, when $q = n$, certainly $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^c$ is nonzero, for instance, they contain all the graph cycles given by $\{y_1 = c_1, \cdots, y_n = c_n\}$ for $c_i \in \widehat{\mathcal{O}}^\times$. $\square$

This apparent shortcoming shows that assuming properness over $\operatorname{Spec}(\widehat{\mathcal{O}})$ for all codimension is too restrictive except for the Milnor range. We thus consider the following a bit weaker cycles with "partial compactness" defined inductively:

DEFINITION 2.2.9. For integers $q \leq n$, define the subgroup $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^{pc} \subset z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)$ as follows:

1. If $n \leq q$, we let $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^{pc} := z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^c$.

2. Suppose $n > q$. Inductively, suppose $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, i)^{pc}$ is defined for each $0 \leq i \leq n-1$. Then let $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^{pc}$ be the subgroup of cycles $Z \in z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)$ such that $\partial(Z) \in z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n-1)^{pc}$.

By construction, ∂ maps $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^{pc}$ into $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n-1)^{pc}$, and we have $\partial \circ \partial = 0$ so that $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, \bullet)^{pc}$ is a subcomplex of $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, \bullet)$. We can similarly define $z_{\mathfrak{m}}^q(\mathcal{O}, n)^{pc}$. Define $\operatorname{CH}_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^{pc} := \operatorname{H}_n(z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, \bullet)^{pc})$ and similarly define $\operatorname{CH}_{\mathfrak{m}}^q(\mathcal{O}, n)^{pc}$.

Remark 2.2.10. Our definition does not necessarily imply that for each $1 \leq i \leq n$ and $\epsilon \in \{0, \infty\}$, the individual face operator ∂_i^ϵ maps $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^{pc}$ into $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n-1)^{pc}$, unlike the boundary operator ∂.

One good side of our definition is the following:

COROLLARY 2.2.11. *We have*

$$\partial(z_{\mathfrak{m}}^n(\mathcal{O}, n+1)^{pc}) = \partial(z_{\mathfrak{m}}^n(\mathcal{O}, n+1)) \cap z_{\mathfrak{m}}^n(\mathcal{O}, n)^c,$$

$$\partial(z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n+1)^{pc}) = \partial(z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n+1)) \cap z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^c.$$

So, the maps $\operatorname{CH}_{\mathfrak{m}}^n(\mathcal{O}, n)^{pc} \to \operatorname{CH}_{\mathfrak{m}}^n(\mathcal{O}, n)$ and $\operatorname{CH}_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^{pc} \to \operatorname{CH}_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)$ are injections.

We thank the referee for suggesting Corollaries 2.2.12 and 2.2.13:

Corollary 2.2.12. *For $n = q + 1$, we have*

$$\begin{cases} \ker(\partial : z_{\mathfrak{m}}^q(\mathcal{O}, q+1) \to z_{\mathfrak{m}}^q(\mathcal{O}, q)) \subseteq z_{\mathfrak{m}}^q(\mathcal{O}, q+1)^{pc}, \\ \ker(\partial : z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, q+1) \to z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, q)) \subseteq z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, q+1)^{pc}. \end{cases} \tag{2.2.1}$$

For $n \geq q + 2$, we have

$$z_{\mathfrak{m}}^q(\mathcal{O}, n)^{pc} = z_{\mathfrak{m}}^q(\mathcal{O}, n) \text{ and } z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^{pc} = z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n). \tag{2.2.2}$$

Proof. We do it just for $\mathcal{O}$; the other case is similar. Note that the trivial subgroup of $z_{\mathfrak{m}}^q(\mathcal{O}, q)$ is a subgroup of $z_{\mathfrak{m}}^q(\mathcal{O}, q)^{pc}$ as well so that the boundaries of the members of $\ker(\partial : z_{\mathfrak{m}}^q(\mathcal{O}, q+1) \to z_{\mathfrak{m}}^q(\mathcal{O}, q))$ belong to $z_{\mathfrak{m}}^q(\mathcal{O}, q)^{pc}$. This means $\ker(\partial : z_{\mathfrak{m}}^q(\mathcal{O}, q+1) \to z_{\mathfrak{m}}^q(\mathcal{O}, q)) \subseteq z_{\mathfrak{m}}^q(\mathcal{O}, q+1)^{pc}$, proving the first assertion.

Since $\partial \circ \partial = 0$, we immediately have $z_{\mathfrak{m}}^q(\mathcal{O}, n) \subseteq z_{\mathfrak{m}}^q(\mathcal{O}, n)^{pc}$ for $n \geq q + 2$. This proves the second assertion. $\qquad\square$

Corollary 2.2.13. *For $n \neq q$, we have the equalities*

$$\mathrm{CH}_{\mathfrak{m}}^q(\mathcal{O}, n)^{pc} = \mathrm{CH}_{\mathfrak{m}}^q(\mathcal{O}, n) \text{ and } \mathrm{CH}_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^{pc} = \mathrm{CH}_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n).$$

Proof. We do it just for $\mathcal{O}$; the other case is similar. When $n \leq q - 2$, we have $z_{\mathfrak{m}}^q(\mathcal{O}, n)^{pc} = z_{\mathfrak{m}}^q(\mathcal{O}, n) = 0$, because $\dim(\square_{\mathcal{O}}^n) = n + 1$. When $n = q - 1$, we have $z_{\mathfrak{m}}^q(\mathcal{O}, n)^{pc} = z_{\mathfrak{m}}^q(\mathcal{O}, n) = 0$ by Remark 2.2.3. (See Remark 2.2.2 for the case of $\widehat{\mathcal{O}}$) Hence $\mathrm{CH}_{\mathfrak{m}}^q(\mathcal{O}, n)^{pc} = \mathrm{CH}_{\mathfrak{m}}^q(\mathcal{O}, n) = 0$ for $n < q$. When $n \geq q + 2$, by (2.2.2) of Corollary 2.2.12, the equality $\mathrm{CH}_{\mathfrak{m}}^q(\mathcal{O}, n)^{pc} = \mathrm{CH}_{\mathfrak{m}}^q(\mathcal{O}, n)$ holds. When $n = q + 1$, the injective map $\mathrm{CH}_{\mathfrak{m}}^q(\mathcal{O}, n)^{pc} \to \mathrm{CH}_{\mathfrak{m}}^q(\mathcal{O}, n)$ of Corollary 2.2.11 reads

$$\frac{\ker(\partial : z_{\mathfrak{m}}^q(\mathcal{O}, q+1)^{pc} \to z_{\mathfrak{m}}^q(\mathcal{O}, q)^{pc})}{\partial z_{\mathfrak{m}}^q(\mathcal{O}, q+2)^{pc}} \to \frac{\ker(\partial : z_{\mathfrak{m}}^q(\mathcal{O}, q+1) \to z_{\mathfrak{m}}^q(\mathcal{O}, q))}{\partial z_{\mathfrak{m}}^q(\mathcal{O}, q+2)}. \tag{2.2.3}$$

By (2.2.1) and (2.2.2) of Corollary 2.2.12, both the numerators and the denominators of (2.2.3) are equal. Hence $\mathrm{CH}_{\mathfrak{m}}^q(\mathcal{O}, n)^{pc} \to \mathrm{CH}_{\mathfrak{m}}^q(\mathcal{O}, n)$ is the identity map. $\qquad\square$

Remark 2.2.14. We guess that the equalities of Corollary 2.2.13 extend to the case when $n = q$ as well, but we have only partial results in this direction.

When $n = q = 1$, we have $z_{\mathfrak{m}}^1(\mathcal{O}, 1)^{pc} = z_{\mathfrak{m}}^1(\mathcal{O}, 1)$ and $z_{\widehat{\mathfrak{m}}}^1(\widehat{\mathcal{O}}, 1)^{pc} = z_{\widehat{\mathfrak{m}}}^1(\widehat{\mathcal{O}}, 1)$. We prove it for $\mathcal{O}$; the case of $\widehat{\mathcal{O}}$ is similar.

An integral cycle $Z \in z_{\mathfrak{m}}^1(\mathcal{O}, 1)$ is given by an irreducible polynomial $f \in \mathcal{O}[y_1]$. Since its intersection with the faces $\{y_1 = 0\}$ and $\{y_1 = \infty\}$ are both empty, both the leading coefficient c and the constant term are units in $\mathcal{O}^{\times}$. Replacing f by $c^{-1}f$, we see that f is a monic irreducible polynomial. In particular $Z \to \mathrm{Spec}\,(\mathcal{O})$ is finite, hence proper, so that $Z \in z_{\mathfrak{m}}^1(\mathcal{O}, 1)^{pc}$. Hence we have the equality.

By the definition of the groups with the superscripts "pc", this implies that $z_{\mathfrak{m}}^1(\mathcal{O}, 2)^{pc} = z_{\mathfrak{m}}^1(\mathcal{O}, 2)$ and $z_{\widehat{\mathfrak{m}}}^1(\widehat{\mathcal{O}}, 2)^{pc} = z_{\widehat{\mathfrak{m}}}^1(\widehat{\mathcal{O}}, 2)$. Hence we deduce that $\mathrm{CH}_{\mathfrak{m}}^1(\mathcal{O}, 2)^{pc} = \mathrm{CH}_{\mathfrak{m}}^1(\mathcal{O}, 2)$ and $\mathrm{CH}_{\widehat{\mathfrak{m}}}^1(\widehat{\mathcal{O}}, 2)^{pc} = \mathrm{CH}_{\widehat{\mathfrak{m}}}^1(\widehat{\mathcal{O}}, 2)$.

When $n = q \geq 2$, for $\mathcal{O}$ we have a partial result in Lemma 5.1.2 that when k is infinite the equality holds. However we do not know much about it for $\widehat{\mathcal{O}}$. See Remark 5.5.1 as well.

2.3 Cycles modulo t^m

DEFINITION 2.3.1. Let $m \geq 1$ be an integer. Let X be an integral $\mathrm{Spec}(\widehat{\mathcal{O}})$-scheme and let $Z_1, Z_2 \subset X$ be two *integral* closed subschemes of X. We allow the case when Z_1 or Z_2 is the empty scheme. We say that Z_1 *and* Z_2 *are equivalent mod* t^m, if we have the equality $Z_1 \times_{\mathrm{Spec}(\widehat{\mathcal{O}})} \mathrm{Spec}(\widehat{\mathcal{O}}/(t^m)) =$
$Z_2 \times_{\mathrm{Spec}(\widehat{\mathcal{O}})} \mathrm{Spec}(\widehat{\mathcal{O}}/(t^m))$ as closed subschemes of $X \times_{\mathrm{Spec}(\widehat{\mathcal{O}})} \mathrm{Spec}(\widehat{\mathcal{O}}/(t^m))$. We can extend this notion to algebraic cycles on X by extending $\mathbb{Z}$-linearly.

Remark 2.3.2. It might be tempting to define the mod t^m-equivalence on each pair of closed subschemes Z_1 and Z_2 as long as we have $Z_1 \times_{\mathrm{Spec}(\widehat{\mathcal{O}})}$
$\mathrm{Spec}(\widehat{\mathcal{O}}/(t^m)) = Z_2 \times_{\mathrm{Spec}(\widehat{\mathcal{O}})} \mathrm{Spec}(\widehat{\mathcal{O}}/(t^m))$. But this finer relation may result in some technically very undesirable effects in dealing with algebraic cycles. One of such problems is that this "tempting" definition often identifies an irreducible closed subscheme with possibly reducible ones, and this makes an analysis of the behaviors of algebraic cycles very intractable. We thus put this mod t^m-equivalence only on pairs of integral closed subschemes with the above equality.

DEFINITION 2.3.3. For two integral schemes $Z_1, Z_2 \in z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)$, we say that Z_1 and Z_2 are *naively mod* t^m*-equivalent*, if their Zariski closures $\overline{Z}_1, \overline{Z}_2$ in $\overline{\square}_{\widehat{\mathcal{O}}}^n$ are mod t^m-equivalent in the sense of Definition 2.3.1. Extend this notion $\mathbb{Z}$-linearly to cycles. We say that Z_1 and Z_2 are *mod* t^m*-equivalent as higher Chow cycles* and write $Z_1 \sim_{t^m} Z_2$, if the pair (Z_1, Z_2) and all pairs of faces $(Z_1 \cap F, Z_2 \cap F)$ for each face $F \subset \square_{\widehat{\mathcal{O}}}^n$ are all naively mod t^m-equivalent.

For simplicity, when Z_1, Z_2 are mod t^m-equivalent as higher Chow cycles, we will simply say they are mod t^m-equivalent.

The inductive nature of the definition of mod t^m-equivalence shows:

LEMMA 2.3.4. *The boundary operator ∂ of the complex $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, \bullet)$ induces the boundary operator, also denoted by ∂, on the mod t^m groups $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)/ \sim_{t^m}$, turning them into a complex. Similarly, we obtain the mod t^m complex $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, \bullet)^{pc}/ \sim_{t^m}$.*

To avoid a technical difficulty (see Remark 5.5.1), we will use $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, \bullet)^{pc}/ \sim_{t^m}$:

DEFINITION 2.3.5. Let $m \geq 1$, $q, n \geq 0$ be integers. Define

$$z^q(k_m, n) := z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^{pc}/ \sim_{t^m}, \tag{2.3.1}$$

where $\sim_{t^m}$ is the mod t^m-equivalence in Definition 2.3.3. By Lemma 2.3.4, this $z^q(k_m, \bullet)$ is a complex of abelian groups. We denote its homology by $\mathrm{CH}^q(k_m, n)$.

Remark 2.3.6. The group $z^{n+1}(k_m, n)$ is 0 because $z_{\widehat{\mathfrak{m}}}^{n+1}(\widehat{\mathcal{O}}, n) = 0$ by Remark 2.2.2. Hence the group $z^n(k_m, n)$ is the simplest nontrivial group in our cycle theory.

2.4 Relative mod t^m cycle complex

We have k-algebra homomorphisms $k \xrightarrow{p^\sharp} \widehat{\mathcal{O}} \xrightarrow{s^\sharp} k$, where $p^\sharp$ is the natural k-algebra map and $s^\sharp$ is reduction modulo (t). Their composition is the identity of k.

LEMMA 2.4.1. *The flat structure morphism $p : \mathrm{Spec}\,(\widehat{\mathcal{O}}) \to \mathrm{Spec}\,(k)$ given by $p^\sharp$ induces the flat pull-back $p^* : z^q(k, \bullet) \to z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, \bullet)^{pc}$.*

Proof. A priori, the flat pull-back p^* maps $z^q(k, n)$ into $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)$. It is enough to show that the image lies in $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^{pc}$. If $q > n$, then $z^q(k, n) = 0$ so that there is nothing to prove. If $q = n$, then $z^n(k, n)$ and $z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^{pc} = z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^c$. But every irreducible cycle $Z \in z^n(k, n)$ is proper over $\mathrm{Spec}\,(k)$ so that its pull-back $p^*(Z)$ is proper over $\mathrm{Spec}\,(\widehat{\mathcal{O}})$, thus the assertion holds in this case. For $q < n$, the group $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^{pc}$ is defined inductively via the boundary operator ∂. Suppose the statement of the lemma holds for $n - 1$, i.e. p^* maps $z^q(k, n - 1)$ into $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n - 1)^{pc}$. By [3, Proposition (1.3)], we have a commutative diagram

$$
\begin{array}{ccc}
z^q(k, n) & \xrightarrow{\;\partial\;} & z^q(k, n - 1) \\
\downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} \\
z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n) & \xrightarrow{\;\partial\;} & z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n - 1).
\end{array}
$$

By the induction hypothesis, the right vertical map p^* maps into $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n-1)^{pc}$. Since the above diagram commutes, this means $\partial(p^*(z^q(k, n))) \subset z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n - 1)^{pc}$. Hence, by definition $p^*(z^q(k, n)) \subset z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^{pc}$. Hence by induction, the lemma holds. $\square$

The map $s^\sharp$ induces the closed immersion $s : \mathrm{Spec}\,(k) \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$. This gives the intersection-restriction to the special fiber $s^* : z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, \bullet)^{pc} \to z^q(k, \bullet)$. Since $s^* \circ p^* = \mathrm{Id}$, we can regard $z^q(k, \bullet)$ as a subcomplex of $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^{pc}$ via p^*. Going modulo t^m as in Definition 2.3.5, which does not do anything on $z^q(k, \bullet)$, we obtain $z^q(k, \bullet) \xrightarrow{p^*} z^q(k_m, \bullet) \xrightarrow{s^*} z^q(k, \bullet)$. Here, we still have $s^* \circ p^* = \mathrm{Id}$. This gives a splitting

$$z^q(k_m, \bullet) = z^q(k, \bullet) \oplus \ker s^*, \qquad (2.4.1)$$

of the complex $z^q(k_m, \bullet)$.

Definition 2.4.2. Define the relative mod t^m cycle complex to be $z^q((k_m, (t)), \bullet) := \ker s^*$, and its homology is denoted by $\mathrm{CH}^q((k_m, (t)), n) := \mathrm{H}_n(z^q((k_m, (t)), \bullet))$.

2.5   Schemes of type $X \otimes_k k_m$ and basic functoriality

In §2.3 and §2.4, we defined cycle complexes associated to k_m. Following the referee's suggestion we can attempt to generalize the construction to schemes of type $X \otimes_k k_m = X \times_{\mathrm{Spec}\,(k)} \mathrm{Spec}\,(k_m)$ for any k-scheme X. For simplicity, when X is a k-scheme and $m \geq 1$ is an integer, let $X_m := X \otimes_k k_m$ and $X^\wedge := X \otimes_k \widehat{\mathcal{O}} = X \times_{\mathrm{Spec}\,(k)} \mathrm{Spec}\,(\widehat{\mathcal{O}})$.

In this generality, it is not yet clear which conditions would give us the "ultimate correct" definition of the cycle groups, but we can still try to push this direction as far as we can. In the future the situation will get clearer. Here is the provisional definition that generalizes the notions in §2.3:

Definition 2.5.1. Let X be a k-scheme. Let $\widehat{\mathfrak{m}} \subset \widehat{\mathcal{O}}$ be the maximal ideal and let $X_{\widehat{\mathfrak{m}}} := X \times_k \{\widehat{\mathfrak{m}}\} \subset X^\wedge$ be the closed subscheme. Let $z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)^c \subseteq z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)$ be the subgroup generated by integral closed subschemes $W \subset X^\wedge \times_k \square^n_k$ that are proper over $X^\wedge$. Since the morphism $W \to X^\wedge$ is proper and affine, it must be finite so that only when $q \geq n$ we may have a possibly nontrivial group.

We define $z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc} \subseteq z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)$ inductively, by imitating what we did before. Namely, for $q \geq n$, we define $z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc} := z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)^c$. Suppose that $z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, i)^{pc}$ is defined for all $i \leq n - 1$. Then let $z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc}$ be the subgroup of cycles $Z \in z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)$ such that $\partial(Z) \in z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n - 1)^{pc}$. By definition, this gives a complex with respect to the boundary operator ∂.

As in Definition 2.3.3, for $m \geq 1$, we define the mod t^m-equivalence inductively on integral cycles in $z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc}$ as well using $- \times_{X^\wedge} X_m = - \otimes_{\widehat{\mathcal{O}}} k_m$. Define $z^q(X_m, n) := z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc}/ \sim_{t^m}$. One checks immediately that this gives a complex $z^q(X_m, \bullet)$ as in Lemma 2.3.4. We define $\mathrm{CH}^q(X_m, n) := \mathrm{H}_n(z^q(X_m, \bullet))$.

For the rest of §2.5, we discuss some basic functoriality properties, namely the existence of the push-forward for a proper morphism and the pull-back for a flat morphism. The groups $\mathrm{CH}^q(X_m, n)$ have two types of relations: the first is given by the boundaries of cycles from $z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n + 1)^{pc}$ and the second is given by the mod t^m-equivalence. So, to discuss some basic functoriality properties, we need to show that the pull-backs and push-forwards respect both those relations. Fortunately, for the usual push-forwards and pull-backs in the sense of [6, §1.4, §1.7], it is already known by [3, Proposition (1.3)] that they respect the first type of relations given by the boundaries, if we ignore the superscripts pc.

Proposition 2.5.2. *Let $f : X \to Y$ be a proper morphism of k-schemes and let $f : X^\wedge \to Y^\wedge$ also denote the induced proper morphism. Then the push-forward $f_* : z^q_{\{X_{\hat{\mathfrak{m}}}\}}(X^\wedge, n) \to z^q_{\{Y_{\hat{\mathfrak{m}}}\}}(Y^\wedge, n)$ satisfies the following properties:*

1. *f_* sends $z^q_{\{X_{\hat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc}$ into $z^q_{\{Y_{\hat{\mathfrak{m}}}\}}(Y^\wedge, n)^{pc}$, and $f_*\partial = \partial f_*$ so that f_* is a morphism of complexes:*

$$
\begin{array}{ccc}
z^q_{\{X_{\hat{\mathfrak{m}}}\}}(X^\wedge, n+1)^{pc} & \xrightarrow{\ \partial\ } & z^q_{\{X_{\hat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc} \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
z^q_{\{Y_{\hat{\mathfrak{m}}}\}}(Y^\wedge, n+1)^{pc} & \xrightarrow{\ \partial\ } & z^q_{\{Y_{\hat{\mathfrak{m}}}\}}(Y^\wedge, n)^{pc}.
\end{array}
$$

2. *f_* respects the mod t^m-equivalence, i.e. it induces the right vertical arrow of the following diagram, that makes the diagram commutes:*

$$
\begin{array}{ccc}
z^q_{\{X_{\hat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc} & \longrightarrow & z^q_{\{X_{\hat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc}/ \sim_{t^m} \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
z^q_{\{Y_{\hat{\mathfrak{m}}}\}}(Y^\wedge, n)^{pc} & \longrightarrow & z^q_{\{Y_{\hat{\mathfrak{m}}}\}}(Y^\wedge, n)^{pc}/ \sim_{t^m} .
\end{array}
$$

Proof. Without the superscripts pc, we already know by [3, Proposition (1.3)] (which uses [6, Theorem 6.2(a), p.98]) that the push-forwards f_* are compatible with the boundary maps ∂. For (1), we need to check that this still holds after putting the superscripts pc.

For $q \geq n$, an integral cycle $Z \in z^q_{\{X_{\hat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc}$ is proper over $X^\wedge$ by definition. Hence its image under the proper morphism $f : X^\wedge \times_k \square^n_k \to Y^\wedge \times_k \square^n_k$ is again proper over $Y^\wedge$. In particular, $f_*(Z) \in z^q_{\{Y_{\hat{\mathfrak{m}}}\}}(Y^\wedge, n)^{pc}$.

For $q < n$, since the groups $z^q_{\{X_{\hat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc}$ and $z^q_{\{Y_{\hat{\mathfrak{m}}}\}}(Y^\wedge, n)^{pc}$ are defined inductively in such a way that their images under ∂ lie in the previous steps $z^q_{\{X_{\hat{\mathfrak{m}}}\}}(X^\wedge, n-1)^{pc}$ and $z^q_{\{Y_{\hat{\mathfrak{m}}}\}}(Y^\wedge, n-1)^{pc}$, the known compability of f_* and ∂ for the cycle groups without the superscripts pc and the case of $q \geq n$ imply that f_* maps $z^q_{\{X_{\hat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc}$ into $z^q_{\{Y_{\hat{\mathfrak{m}}}\}}(Y^\wedge, n)^{pc}$ and $f_*\partial = \partial f_*$ by induction. This proves (1).

The part (2) is an easy application of the following projection formula: *suppose we have a Cartesian diagram of k-schemes*

$$
\begin{array}{ccc}
D_P & \overset{i_P}{\hookrightarrow} & P \\
\downarrow{\scriptstyle p|} & & \downarrow{\scriptstyle p} \\
D_Q & \overset{i_Q}{\hookrightarrow} & Q,
\end{array}
$$

where p is a proper morphism, D_P, D_Q are effective divisors such that $D_P = p^*(D_Q)$, and $A \subset P$ is a closed subscheme that intersects D_P properly so that $D_P \cdot A = i_P^*(A)$ is well-defined. Then $p_*(A) \cdot D_Q = p_*(A \cdot p^*(D_Q)) = p_*(A \cdot D_P)$.

Its proof is given in [6, Proposition 2.3-(c), p.34]. The statement in *loc.cit* is given in the cycle group modulo rational equivalence, but in our case we already suppose the proper intersection condition so that the equality of our cycles holds on the level of cycles.

We apply the above formula: for two integral closed cycles $Z_1, Z_2 \in z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc}$ such that $Z_1 \sim_{t^m} Z_2$, we take $P = X^\wedge \times_k \overline{\square}^n_k$, $Q = Y^\wedge \times_k \overline{\square}^n_k$, $A_i = \overline{Z}_i$, D_Q = the divisor of t^m in Q, and D_P = the divisor of t^m in P. The proper map $p = f$. Hence we have $f_*(A_i) \cdot D_Q = f_*(A_i \cdot f^*(D_Q)) = f_*(A_i \cdot D_P)$. That $Z_1 \sim_{t^m} Z_2$ means $A_1 \cdot D_P = A_2 \cdot D_P$. Hence we deduce that $f_*(A_1) \cdot D_Q = f_*(A_2) \cdot D_Q$, i.e. $f_*(Z_1) \sim_{t^m} f_*(Z_2)$. This proves (2). $\qquad\square$

PROPOSITION 2.5.3. *Let $f : X \to Y$ be a flat morphism of k-schemes and let $f : X^\wedge \to Y^\wedge$ also denote the induced flat morphism. Then the pull-back $f^* : z^q_{\{Y_{\widehat{\mathfrak{m}}}\}}(Y^\wedge, n) \to z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)$ satisfies the following properties:*

1. *f^* sends $z^q_{\{Y_{\widehat{\mathfrak{m}}}\}}(Y^\wedge, n)^{pc}$ into $z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc}$, and $f^*\partial = \partial f^*$, so that f^* is a morphism of complexes:*

$$
\begin{array}{ccc}
z^q_{\{Y_{\widehat{\mathfrak{m}}}\}}(Y^\wedge, n+1)^{pc} & \xrightarrow{\ \partial\ } & z^q_{\{Y_{\widehat{\mathfrak{m}}}\}}(Y^\wedge, n)^{pc} \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n+1)^{pc} & \xrightarrow{\ \partial\ } & z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc}.
\end{array}
$$

2. *f^* respects the $\bmod\, t^m$-equivalence, i.e. it induces the right vertical arrow of the following diagram, that makes the diagram commutes:*

$$
\begin{array}{ccc}
z^q_{\{Y_{\widehat{\mathfrak{m}}}\}}(Y^\wedge, n)^{pc} & \longrightarrow & z^q_{\{Y_{\widehat{\mathfrak{m}}}\}}(Y^\wedge, n)^{pc}/\sim_{t^m} \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc} & \longrightarrow & z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc}/\sim_{t^m}.
\end{array}
$$

Proof. Without the superscripts *pc*, we already know by [3, Proposition (1.3)] which uses [6, Proposition 1.7, p.18] that the pull-backs f^* are compatible with the boundary maps ∂. For (2), we need to check that this still holds after putting the superscripts *pc*.

For $q \geq n$, an integral cycle $Z \in z^q_{\{Y_{\widehat{\mathfrak{m}}}\}}(Y^\wedge, n)^{pc}$ is proper over $Y^\wedge$ by definition. On the other hand, since the pull-back of a proper morphism is again proper, this time over $X^\wedge$, we immediately have that the flat pull-back $f^*(Z) \in z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc}$.

For $q < n$, since the groups $z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc}$ and $z^q_{\{Y_{\widehat{\mathfrak{m}}}\}}(Y^\wedge, n)^{pc}$ are defined inductively in such a way that their images under ∂ lie in the previous steps $z^q_{\{X_{\widehat{\mathfrak{m}}}\}}(X^\wedge, n-1)^{pc}$ and $z^q_{\{Y_{\widehat{\mathfrak{m}}}\}}(Y^\wedge, n-1)^{pc}$, the known compability of f^* and ∂ for the cycle groups without the superscripts *pc* and the case of $q \geq n$ imply

that f^* maps $z^q_{\{Y_{\hat{\mathfrak{m}}}\}}(Y^\wedge, n)^{pc}$ into $z^q_{\{X_{\hat{\mathfrak{m}}}\}}(X^\wedge, n)^{pc}$ and $f^*\partial = \partial f^*$ by induction. This proves (1).

The part (2) is an easy application of the following fact: *suppose we have a Cartesian diagram of k-schemes*

$$
\begin{array}{ccc}
D_P & \overset{i_P}{\hookrightarrow} & P \\
\downarrow{\scriptstyle f|} & & \downarrow{\scriptstyle f} \\
D_Q & \overset{i_Q}{\hookrightarrow} & Q,
\end{array}
$$

where f is a flat morphism, D_P, D_Q are effective divisors such that $D_P = f^*(D_Q)$, and $A \subset Q$ is a closed subscheme that intersects D_Q properly so that $D_Q \cdot A = i_Q^*(A)$ is well-defined. Then $f^*(A) \cdot f^*(D_Q) = f^*(A \cdot D_Q)$.

Its proof is given in [6, Proposition 2.3-(d), p.34]. The statement in *loc.cit* is given in the cycle group modulo rational equivalence, but in our case we already suppose the proper intersection condition so that the equality of our cycles holds on the level of cycles.

We apply the above formula: for two integral closed cycles $Z_1, Z_2 \in z^q_{\{Y_{\hat{\mathfrak{m}}}\}}(Y^\wedge, n)^{pc}$ such that $Z_1 \sim_{t^m} Z_2$, we take $P = X^\wedge \times_k \overline{\square}^n_k$, $Q = Y^\wedge \times_k \overline{\square}^n_k$, $A_i = \overline{Z}_i$, $D_Q =$ the divisor of t^m in Q, and $D_P =$ the divisor of t^m in P. Hence we have $f^*(A_i) \cdot D_P = f^*(A_i \cdot D_Q)$. That $Z_1 \sim_{t^m} Z_2$ means $A_1 \cdot D_Q = A_2 \cdot D_Q$. Hence $f^*(A_1) \cdot D_P = f^*(A_2) \cdot D_P$, i.e. $f^*(Z_1) \sim_{t^m} f^*(Z_2)$. This proves (2). $\square$

2.6   The non-archimedean norm

We recall some facts on the non-archimedean t-adic metric topology on the local field $k((t))$, needed in §4. Recall that the field $k((t))$ has a natural discrete valuation $v : k((t)) \to \mathbb{Z}$ given by the order of vanishing function $v = \mathrm{ord}_t$ with $v(0) := \infty$. Its ring of integers $\mathcal{O}_{k((t))} = \widehat{\mathcal{O}} = k[[t]]$ is simply $\{f \in k((t)) \mid v(f) \geq 0\}$. We have a norm $|-|_v : k((t)) \to \mathbb{R}$ given by $|f|_v := e^{-v(f)}$. For any integer $M > 0$, we have the supremum norm on the vector space $k((t))^M$ given by $|(f_1, \cdots, f_M)|_v := \sup_{1 \leq i \leq M} |f_i|_v$. This gives the non-archimedean t-adic metric topology, which is finer than the Zariski topology on $k((t))^M = \mathbb{A}^M(k((t)))$. For any $\alpha_0 \in k((t))^M$, we let $\mathcal{B}_N(\alpha_0)$ be the open ball around α_0 of radius e^{-N}. Here $k[[t]]^M \subset k((t))^M$ is open, while $k[t]^M \subset k[[t]]^M$ is dense.

2.7   Milnor K-groups

Let R be a commutative local ring with unity. Recall that the Milnor K-ring $K_*^M(R)$ of R is the graded tensor algebra $T_{\mathbb{Z}}(R^\times)$ of $R^\times$ over $\mathbb{Z}$ modulo the two-sided ideal generated by the elements of the form $\{a \otimes (1-a) \mid a, 1-a \in R^\times\}$. Its degree n part is the n-th Milnor K-group $K_n^M(R)$.

3 Milnor range I: reciprocity

The goal of the paper is to prove the following Theorem 3.0.1. In the case of additive higher Chow groups over fields, similar results were obtained by Bloch-Esnault [4] and Rülling [27].

THEOREM 3.0.1. *Let k be a field of characteristic 0 and let $m, n \geq 1$ be integers. Then we have* $\mathrm{CH}^n((k_m, (t)), n) \simeq (\Omega^{n-1}_{k/\mathbb{Z}})^{\oplus(m-1)}$.

The proof of Theorem 3.0.1 is largely broken into two parts: the first is to define regulator maps on cycles and to prove that they vanish on the boundaries, as done in Proposition 3.0.2 below. The second part, done later in §4 and §5, is to show that the regulator maps respect the mod t^m-equivalence. Here, we emphasize that although we are in the Milnor range, our representatives are 1-cycles, unlike the additive Chow group versions of [4] or [27] that used 0-cycles. In our discussion, the argument of the first part follows a path similar to one paved in [25]:

PROPOSITION 3.0.2. *Let k be a field of characteristic 0. For each $1 \leq i \leq m-1$, define $\Upsilon_i : z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n) \to \Omega^{n-1}_{k/\mathbb{Z}}$ as follows. Consider the rational form $\gamma_{i,n} := \frac{1}{t^i} d\log y_1 \wedge \cdots \wedge d\log y_n \in \Omega^n_{\overline{\square}^n_{\widehat{\mathcal{O}}}/\mathbb{Z}}(*\{t=0\})(\log F)$. For each integral 1-cycle $Z \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)$, let $\nu : \widetilde{Z} \to \overline{Z} \hookrightarrow \overline{\square}^n_{\widehat{\mathcal{O}}}$ be a normalization of the closure $\overline{Z}$ of Z in $\overline{\square}^n_{\widehat{\mathcal{O}}}$. Define $\Upsilon_i(Z) := \sum_{p \in \widetilde{Z}_s} \mathrm{Tr}_{k(p)/k} \mathrm{res}_p \nu^* \gamma_{i,n} \in \Omega^{n-1}_{k/\mathbb{Z}}$, and $\mathbb{Z}$-linearly extend Υ_i to all of $z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)$. Then $\Upsilon_i(\partial W) = 0$ for $W \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n+1)$.*

Proof. It is enough to prove the statement for any integral $W \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n+1)$. Let $\overline{W} \subset \overline{\square}^{n+1}_{\widehat{\mathcal{O}}}$ be the Zariski closure of W, which is also integral. For each $1 \leq \ell \leq n+1$ and $\epsilon \in \{0, \infty\}$, via the codimension 1 face map $\iota_{\ell,\epsilon} : \overline{\square}^n_{\widehat{\mathcal{O}}} \hookrightarrow \overline{\square}^{n+1}_{\widehat{\mathcal{O}}}$, identify the Zariski closure of $\partial^\epsilon_\ell(W)$ in $\overline{\square}^n_{\widehat{\mathcal{O}}}$ with its image $\overline{\partial^\epsilon_\ell(W)}$ in $\overline{\square}^{n+1}_{\widehat{\mathcal{O}}}$. Consider the divisor $D := \sum_{\ell,\epsilon} \{y_\ell = \epsilon\}$ on $\overline{\square}^{n+1}_{\widehat{\mathcal{O}}}$. We omit the proof of the following claim, which is easily deduced by a standard argument using a finite sequence of point blow-ups and [11, Exercise II-7.12, p.171]:

CLAIM: *There exists a sequence of blow-ups $\widetilde{\phi} : \widetilde{\overline{\square}^{n+1}_{\widehat{\mathcal{O}}}} \to \overline{\square}^{n+1}_{\widehat{\mathcal{O}}}$ centered at points, such that for the strict transform $\widetilde{W} \subset \widetilde{\overline{\square}^{n+1}_{\widehat{\mathcal{O}}}}$ of $\overline{W}$ and the restriction $\phi := \widetilde{\phi}|_{\widetilde{W}} : \widetilde{W} \to \overline{W} \hookrightarrow \overline{\square}^{n+1}_{\widehat{\mathcal{O}}}$, we have the following properties: (1) each irreducible component of the strict transform $\phi^!(\overline{\partial^\epsilon_\ell(W)})$ of the 1-cycle $\overline{\partial^\epsilon_\ell(W)}$ is regular. We let $\phi^!(D) := \sum_{\ell,\epsilon} \phi^!(\overline{\partial^\epsilon_\ell(W)})$, the strict transform of $\overline{D \cap W}$; (2) each closed point $p \in \widetilde{W}_s = \phi^*\{t=0\}$ satisfies exactly one of the following three possibilities:*

(2-i) p belongs to a unique irreducible component of $\widetilde{W}_s$, but does not meet $\phi^!(D)$.

(2-ii) *p belongs to a unique irreducible component of* $\widetilde{W}_s$, *and belongs to precisely one irreducible component of* $\phi^!(D)$.

(2-iii) *p belongs to exactly two irreducible components of* $\widetilde{W}_s$, *but does not meet* $\phi^!(D)$.

Going back to the proof of the proposition, notice that the irreducible components of $\phi^!(\overline{\partial_\ell^\epsilon(W)})$ are all regular, and are in one-to-one correspondence with the irreducible components of $\overline{\partial_\ell^\epsilon(W)}$ via ϕ. Hence each component of $\phi^!(\overline{\partial_\ell^\epsilon(W)})$ gives a normalization of the Zariski closure in $\overline{\square}_{\hat{\mathcal{O}}}^n$ of the corresponding component of $\partial_\ell^\epsilon(W)$. Express the special fiber $\widetilde{W}_s$ as the union of (not necessarily regular) irreducible projective curves $C_1, \cdots, C_M$.

We use the theory of Parshin-Lomadze residues associated to pseudo-coefficient fields (see [36, Definitions 4.1.1, 4.1.3]). For each generic point of C_j seen as a point of the scheme $\widetilde{W}$, choose a pseudo-coefficient field σ_j. Consider the Parshin-Lomadze residue $\Xi_{\sigma_j} := \mathrm{res}_{(\widetilde{W}, C_j), \sigma_j} \phi^*(\gamma_{i,n+1})$ along the chain $(\widetilde{W}, C_j)$ for the choice of σ_j. For each $1 \leq j \leq M$, this Ξ_{σ_j} is a rational absolute Kähler n-form on C_j.

Let $p \in \widetilde{W}_s$. By our construction of $\widetilde{W}$ in the above claim, for the point p, exactly one of (2-i), (2-ii), and (2-iii) holds.

If (2-i) holds for p, then let C_j be the unique component of $\widetilde{W}_s$ with $p \in C_j$. Since p does not lie over any face $\overline{\partial_\ell^\epsilon(W)}$ for $1 \leq \ell \leq n+1$, $\epsilon \in \{0, \infty\}$, from the shape of $\gamma_{i,n+1}$, the form $\Xi_{\sigma_j} = \mathrm{res}_{(\widetilde{W}, C_j), \sigma_j}(\phi^* \gamma_{i,n+1})$ is regular at p so that we have $\mathrm{res}_{p \in C_j}(\Xi_{\sigma_j}) = 0$.

If (2-iii) holds for p, then let $C_j, C_{j'}$ with $j \neq j'$ be the two distinct components of $\widetilde{W}_s$ such that $p \in C_j \cap C_{j'}$. Here, again p does not lie over any face $\overline{\partial_\ell^\epsilon(W)}$ for $1 \leq \ell \leq n+1$, $\epsilon \in \{0, \infty\}$, therefore by [36, Theorem 4.2.15-(a)], we have $\mathrm{res}_{p \in C_j}(\Xi_{\sigma_j}) + \mathrm{res}_{p \in C_{j'}}(\Xi_{\sigma_{j'}}) = 0$.

Now suppose (2-ii) holds for p. Thus there exist (i) a unique index $1 \leq j(p) \leq M$ with $p \in C_{j(p)}$, (ii) a unique pair (ℓ_0, ϵ_0) with $1 \leq \ell_0 \leq n+1$, $\epsilon_0 \in \{0, \infty\}$, and a unique irreducible component $G \subset \phi^!(\overline{\partial_{\ell_0}^{\epsilon_0}(W)})$ such that $p \in G$.

From the shape of $\gamma_{i,n+1}$, the form $\phi^* \gamma_{i,n+1}$ on $\widetilde{W}$ has a simple (or logarithmic) pole (see [36, Definition 4.2.10]) along G, so that the residue of $\phi^* \gamma_{i,n+1}$ along the chain $(\widetilde{W}, G)$ is independent of the choice of a pseudo-coefficient field for G by [36, Corollary 4.2.13]. On the other hand, by [36, Theorem 4.2.15-(a)], we have

$$\mathrm{res}_{p \in C_{j(p)}}\big(\mathrm{res}_{(\widetilde{W}, C_{j(p)}), \sigma_{j(p)}}(\phi^* \gamma_{i,n+1})\big) = -\mathrm{res}_{p \in G}\big(\mathrm{res}_{(\widetilde{W}, G)}(\phi^* \gamma_{i,n+1})\big). \quad (3.0.1)$$

From the shape of $\gamma_{i,n+1} = \frac{1}{t^i} \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{n+1}}{y_{n+1}}$ again, since $G \subset \phi^!(\overline{\partial_{\ell_0}^{\epsilon_0}(W)})$, we have

$$\mathrm{res}_{(\widetilde{W}, G)}(\phi^* \gamma_{i,n+1}) = (-1)^{\ell_0} \cdot \iota(G; \ell_0, \epsilon_0) \cdot \mathrm{sgn}(\epsilon_0) \phi^*(\gamma_{i,n+1}^{\ell_0})|_G, \quad (3.0.2)$$

where $\iota(G; \ell_0, \epsilon_0)$ is the intersection multiplicity of G in $\partial_{\ell_0}^{\epsilon_0}(W)$, $\mathrm{sgn}(0) :=$ $1, \mathrm{sgn}(\infty) := -1$, and $\gamma_{i,n+1}^{\ell_0} := \frac{1}{t^i} \frac{dy_1}{y_1} \wedge \cdots \widehat{\frac{dy_{\ell_0}}{y_{\ell_0}}} \wedge \cdots \wedge \frac{dy_{n+1}}{y_{n+1}}$.

Now, by the definition of Υ_i, we have

$$(-1)^{\ell_0} \mathrm{sgn}(\epsilon_0) \Upsilon_i(\partial_{\ell_0}^{\epsilon_0}(W))$$

$$= (-1)^{\ell_0} \mathrm{sgn}(\epsilon_0) \sum_G \iota(G; \ell_0, \epsilon_0) \sum_{p \in G_s} \mathrm{Tr}_{k(p)/k} \mathrm{res}_{p \in G} \phi^*(\gamma_{i,n+1}^{\ell_0})|_G$$

$$=^\dagger \sum_G \sum_{p \in G_s} \mathrm{Tr}_{k(p)/k} \mathrm{res}_{p \in G} (\mathrm{res}_{(\widetilde{W}, G)}(\phi^* \gamma_{i,n+1}))$$

$$=^\ddagger - \sum_G \sum_{p \in G_s} \mathrm{Tr}_{k(p)/k} \mathrm{res}_{p \in C_{j(p)}} (\mathrm{res}_{(\widetilde{W}, C_{j(p)}), \sigma_{j(p)}}(\phi^* \gamma_{i,n+1}))$$

$$=^1 - \sum_G \sum_{p \in G_s} \mathrm{Tr}_{k(p)/k} \mathrm{res}_{p \in C_{j(p)}} (\Xi_{\sigma_{j(p)}}), \qquad (3.0.3)$$

where $\sum_G$ is the sum over all irreducible components of $\phi^!(\overline{\partial_{\ell_0}^{\epsilon_0}(W)})$, $\dagger$ holds by $(3.0.2)$, $\ddagger$ holds by $(3.0.1)$, and $=^1$ holds by definition. Note that the set of all points $p \in G_s$ over all irreducible components G of $\phi^!(D)$ is precisely equal to the set $\widetilde{W}_s^{(2-\mathrm{ii})}$ of all points of $\widetilde{W}_s$ of type (2-ii) in the claim. Hence, taking the sum of $(3.0.3)$ over all $1 \le \ell_0 \le n+1$ and $\epsilon_0 \in \{0, \infty\}$, we obtain

$$\Upsilon_i(\partial W) = - \sum_{\ell=1}^{n+1} \sum_{\epsilon \in \{0, \infty\}} (-1)^\ell \mathrm{sgn}(\epsilon) \Upsilon_i(\partial_\ell^\epsilon(W))$$

$$= \sum_{p \in \widetilde{W}_s^{(2-\mathrm{ii})}} \mathrm{Tr}_{k(p)/k} \mathrm{res}_{p \in C_{j(p)}} (\Xi_{\sigma_{j(p)}}). \qquad (3.0.4)$$

On the other hand, for the points of $\widetilde{W}_s$ of type (2-i) and (2-iii), we saw previously that there is no contribution of residues from them. Hence continuing $(3.0.4)$, we have

$$\Upsilon_i(\partial W) = \sum_{p \in \widetilde{W}_s^{(2-\mathrm{ii})}} \mathrm{Tr}_{k(p)/k} \mathrm{res}_{p \in C_{j(p)}} (\Xi_{\sigma_{j(p)}})$$

$$= \sum_{j=1}^{M} \sum_{p \in C_j} \mathrm{Tr}_{k(p)/k} \mathrm{res}_{p \in C_j} (\Xi_{\sigma_j}) =^\dagger 0,$$

where $\dagger$ holds by the residue theorem (see [36, Theorem 4.2.15-(b)]), i.e. the sum of all residues of a form over a projective curve $\widetilde{W}_s$ is 0. This shows $\Upsilon_i(\partial W) = 0$ as desired. $\qquad \square$

The remaining part of the proof of Theorem 3.0.1 is to check that the regulator maps in Proposition 3.0.2 restricted to $z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^c$ respect the mod t^m-equivalence. This requires further discussions, and the rest of the paper deals with it.

4 Some perturbation lemmas and the mod t^m moving lemma

In working with cycles over the complete local ring $\widehat{\mathcal{O}}$, it is maybe convenient if one can transfer some known results for cycles over $\mathcal{O}$ to cycles over $\widehat{\mathcal{O}}$. The completion ring homomorphism $\mathcal{O} \to \widehat{\mathcal{O}}$ induces a natural flat pull-back homomorphism $\xi^n : z_{\mathfrak{m}}^q(\mathcal{O}, n)^? \to z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}}, n)^?$, for $? = pc, \emptyset$, given by $[Z] \mapsto [\widehat{Z} := \mathrm{Spec}\,(\widehat{\mathcal{O}}) \times_{\mathrm{Spec}\,(\mathcal{O})} Z]$, but in general, ξ^n is not surjective. The goal of §4 is to prove the "mod t^m moving lemma" in Theorem 4.3.2, which states that this natural homomorphism induces a surjection modulo t^m in the Milnor range with $? = pc$. In this case $z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^c = z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^{pc}$.

In this section we suppose k is any field unless specified otherwise. In §4.1, we discuss some preparatory results needed in what follows. In §4.2, we discuss various general position results as in Lemmas 4.2.2, 4.2.5, 4.2.7, and 4.2.10, needed in the proof of the mod t^m moving lemma in §4.3. These results might appear to be related to the Artin approximation theorem [1], but they do not follow from it. The results are stated in terms of schemes over $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_{\mathbb{A}_k^1, 0}$, but some of them might work for more general integral k-schemes with the methods presented here. We leave such generalizations to the reader.

In what follows in §4, to ease the proof, via the automorphism $y \mapsto 1/(1-y)$ of $\mathbb{P}^1$, we identify $(\square, \{\infty, 0\})$ with $(\mathbb{A}^1, \{0, 1\})$ so that $\square^n \simeq \mathbb{A}^n$, and the faces of $\square^n$ under this identification are given by a finite set of equations of the form $y_j = \epsilon_j$ with $\epsilon_j \in \{0, 1\}$.

4.1 Some preparatory lemmas

We are interested in understanding "small changes" of a given integral closed subscheme $W \subseteq \square_{\widehat{\mathcal{O}}}^n$ when we "perturb" the coefficients of a generating set of the ideal of W. So, we introduce:

DEFINITION 4.1.1. For a closed subscheme $W \subseteq \square_{\widehat{\mathcal{O}}}^n$, let $\{f_1, \cdots, f_r\} \subset \widehat{\mathcal{O}}[y_1, \cdots, y_n]$ be a set of generators of the ideal of W.

The *coefficient perturbation* of the set $\{f_1, \cdots, f_r\}$ is the set $\{F_1, \cdots, F_r\}$ of polynomials obtained as follows: for each nonzero monomial term of each of f_j over $1 \le j \le r$, consider an indeterminate and a copy of $\mathbb{A}_{\widehat{\mathcal{O}}}^1$. Replace each nonzero coefficient by the corresponding indeterminate. Let M be the total number of them and let $F_1, \cdots, F_r \in \widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, \cdots, y_n]$ be the so-obtained polynomials from $f_1, \cdots, f_r$, respectively. Let $V \subset \mathbb{A}_{\widehat{\mathcal{O}}}^M \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^n$ be the closed subscheme defined by the ideal $(F_1, \cdots, F_r)$. We may also say V is the *coefficient perturbation of W with respect to the generators* $\{f_1, \cdots, f_r\}$.

For each $\alpha \in \mathbb{A}_{\widehat{\mathcal{O}}}^M$, we let V_α be the fiber over α. If $\alpha_0 \in \widehat{\mathcal{O}}^M$ is the original sequence of coefficients of $\{f_1, \cdots, f_r\}$, we have $V_{\alpha_0} = W$.

Example 4.1.2. For $n = 2$, consider $\{f_1, f_2\} = \{3y_1 y_2^2 + y_1 + 2y_2 + 1, -y_1^2 y_2 - 5y_1 + 3\}$. Then the corresponding coefficient perturbation is given by $\{F_1, F_2\} = \{x_1 y_1 y_2^2 + x_2 y_1 + x_3 y_2 + x_4, x_5 y_1^2 y_2 + x_6 y_1 + x_7\}$.

The coefficient perturbation depends on the choice of a generating set $\{f_1, \cdots, f_r\}$. If we make a "bad" choice, then we might end up having undesirable phenomena:

Example 4.1.3. For $n = 2$, consider $W \subseteq \square_{\widehat{\mathcal{O}}}^2$ defined by $f_1 := y_1 + 1, f_2 := y_2 + 1$. The ideal of W also contains $f_3 := f_1 f_2 = y_1 y_2 + y_1 + y_2 + 1$.

If we take the coefficient perturbation with respect to just $\{f_1, f_2\}$, then we have $F_1 = x_1 y_1 + x_2$, $F_2 = x_3 y_3 + x_4$. So, if $\alpha = (x_1, x_2, x_3, x_4)$ is in the open subset of $\mathbb{A}_{\widehat{\mathcal{O}}}^4$ given by $x_1 \neq 0$ and $x_3 \neq 0$, then V_α is given by $(y_1, y_2) = (-x_2/x_1, -x_4/x_3)$. In particular, $V_\alpha \neq \emptyset$.

However, this time with respect to $\{f_1, f_2, f_3\}$, with a redundant generator f_3, the corresponding coefficient perturbation is given by $F_1 = x_1 y_1 + x_2$, $F_2 = x_3 y_2 + x_4$, $F_3 = x_5 y_1 y_2 + x_6 y_1 + x_7 y_2 + x_8$. Unfortunately, for V_α to be nonempty, we need a necessary condition. Suppose for a choice $\alpha = (x_1, \cdots, x_8)$, we have $V_\alpha \neq \emptyset$. Then $F_1 = 0$ and $F_2 = 0$ give $y_1 = -x_2/x_1, y_2 = -x_4/x_3$ so that by plugging them into F_3, we obtain $x_2 x_4 x_5/(x_1 x_3) - x_2 x_6/x_1 - x_4 x_7/x_3 + x_8 = 0$, i.e. we have an algebraic dependence $x_2 x_4 x_5 - x_2 x_3 x_6 - x_1 x_4 x_7 + x_1 x_3 x_8 = 0$ for $x_1, \cdots, x_8$. Hence we can expect to have a nonempty fiber V_α only over this proper closed subset of $\mathbb{A}_{\widehat{\mathcal{O}}}^8$. This is not desirable for our purposes.

An aim of §4.1 is to show that when $W \in z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^c$ is integral of relative dimension 0 in the Milnor range, it is possible to choose a "nice" generating set. We will make this precise in what follows.

LEMMA 4.1.4. *Let* $W \in z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^c$ *be a nonempty integral cycle. Then* (1) *the structure morphism* $f : W \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ *is surjective, flat, and finite, and* (2) *the generic fiber* W_η *is the singleton given by the generic point* η_W *of* W.

Proof. The surjectivity of f was proven in Lemma 2.2.7. The morphism f is flat by [10, Proposition (14.5.6), p.217] (or [11, Proposition III-9.7, p.257]) because $\mathrm{Spec}\,(\widehat{\mathcal{O}})$ is a regular scheme of dimension 1. The morphism f is finite because it is a proper morphism of affine schemes (see [11, Exercise II-4.6, p.106]). This proves (1).

Since dim $W = 1$, the integral scheme W is the union of the generic point η_W of W and its closed points. Here, all the closed points map to the unique closed point of $\mathrm{Spec}\,(\widehat{\mathcal{O}})$, while η_W cannot map to the closed point of $\mathrm{Spec}\,(\widehat{\mathcal{O}})$ for otherwise f would not be surjective, contradicting Lemma 2.2.7. Hence η_W is the unique point of the generic fiber W_η. This proves (2). $\square$

PROPOSITION 4.1.5. *Let* $W \in z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^c$ *be a nonempty integral cycle. Then* W *is a complete intersection in* $\square_{\widehat{\mathcal{O}}}^n$ *defined by a subset* $\{f_1, \cdots, f_n\} \subseteq \widehat{\mathcal{O}}[y_1, \cdots, y_n]$ *of precisely n polynomials of the triangular form*

$$
\begin{cases}
f_1(y_1), \\
f_2(y_1, y_2), \\
\quad \vdots \\
f_n(y_1, \cdots, y_n),
\end{cases}
\tag{4.1.1}
$$

such that (1) $f_i(y_1, \cdots, y_i)$ has y_i-degree ≥ 1 for each $1 \leq i \leq n$, (2) the highest y_i-degree term of f_i does not involve any variable other than y_i, and (3) the constant term of each f_i is 1.

Proof. This is inspired by [29, Lemma 2], but we need some modifications as our base ring is $\widehat{\mathcal{O}}$, not a field. For each $1 \leq i \leq n$, let $W^{(i)} \subset \square_{\widehat{\mathcal{O}}}^i$ be the image of the projection $\square_{\widehat{\mathcal{O}}}^n \to \square_{\widehat{\mathcal{O}}}^i$, $(y_1, \cdots, y_n) \mapsto (y_1, \cdots, y_i)$. Let $W^{(0)} := \mathrm{Spec}\,(\widehat{\mathcal{O}})$. Since the map $W = W^{(n)} \to W^{(0)} = \mathrm{Spec}\,(\widehat{\mathcal{O}})$ is finite and surjective by Lemma 4.1.4, we deduce that $W^{(i)} \to W^{(j)}$ is finite and surjective for each pair $0 \leq j < i \leq n$ of indices, and each $W^{(i)} \in z_{\widehat{\mathcal{O}}}^i(\widehat{\mathcal{O}}, i)^c$ is a nonempty integral cycle for each $1 \leq i \leq n$.
We prove the proposition by induction. Since $W^{(1)} \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ is finite and surjective, there exists a monic irreducible polynomial $f_1(y_1) \in \widehat{\mathcal{O}}[y_1]$ of y_1-degree ≥ 1 that defines $W^{(1)}$. Since its intersection with the face $\{y_1 = 0\}$ is empty, the constant term of $f_1(y_1)$ is a unit c in $\widehat{\mathcal{O}}^\times$. Replacing f_1 by $c^{-1} f_1$, we may assume that the constant term of $f_1(y_1)$ is 1. This shows (1), (2), (3) for $i = 1$.
Let $n \geq 2$ and let $1 \leq i < n$. Suppose we constructed $f_1(y_1), f_2(y_1, y_2), \cdots, f_i(y_1, \cdots, y_i)$ that define $W^{(i)} \in z_{\widehat{\mathfrak{m}}}^i(\widehat{\mathcal{O}}, i)^c$, with the conditions (1), (2), (3). In particular, $\widehat{\mathcal{O}} \to \widehat{\mathcal{O}}[y_1, \cdots, y_i]/(f_1, \cdots, f_i)$ is a finite ring extension. Since $W^{(i+1)} \to W^{(i)}$ is finite surjective, there exists a monic irreducible polynomial in the ring $(\widehat{\mathcal{O}}[y_1, \cdots, y_i]/(f_1, \cdots, f_i))[y_{i+1}]$ in y_{i+1} of y_{i+1}-degree ≥ 1 that defines $W^{(i+1)}$. Choose any lifting of this polynomial in $\widehat{\mathcal{O}}[y_1, \cdots, y_{i+1}]$ such that the coefficient of the highest y_{i+1}-degree term does not involve any variable other than y_{i+1}. Call it $f_{i+1}(y_1, \cdots, y_{i+1})$. This thus satisfies (1) and (2) by construction.
If the constant term of $f_{i+1}(y_1, \cdots, y_{i+1})$ is a unit c in $\widehat{\mathcal{O}}^\times$, then replace f_{i+1} by $c^{-1} f_{i+1}$. If the constant term of $f_{i+1}(y_1, \cdots, y_{i+1})$ is not a unit in $\widehat{\mathcal{O}}^\times$, then it is divisible by $t \in \widehat{\mathcal{O}}$. Then first replace f_i by $f_i + f_1$. This procedure does not disturb the triangular shape of (4.1.1), nor (1) or (2), and now the constant term of the new f_i is a unit c in $\widehat{\mathcal{O}}^\times$, because any element of the form $1 + th \in \widehat{\mathcal{O}}$ is a unit. Replacing f_{i+1} by $c^{-1} f_{i+1}$, we thus make it satisfies (3). Hence by induction, we have the triangular shaped generators $f_1, \cdots, f_n$ as in (4.1.1) satisfying (1), (2) and (3). $\square$

COROLLARY 4.1.6. *Let $W \in z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^c$ be a nonempty integral cycle. For a defining set $\{f_1, \cdots, f_n\} \subseteq \widehat{\mathcal{O}}[y_1, \cdots, y_n]$ of W in Proposition 4.1.5, consider the corresponding coefficient perturbation $V \subset \mathbb{A}_{\widehat{\mathcal{O}}}^M \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^n$ given by $\{F_1, \cdots, F_n\}$ of $\{f_1, \cdots, f_n\}$ as in Definition 4.1.1. Then the codimension of V in $\mathbb{A}_{\widehat{\mathcal{O}}}^M \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^n$ is n.*

Proof. For $1 \leq i \leq n$, let V_i be the closed subscheme of $\mathbb{A}_{\widehat{\mathcal{O}}}^M \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^i$ given by $(F_1, \cdots, F_i)$. We prove that the codimension of V_i in $B_i := \mathbb{A}_{\widehat{\mathcal{O}}}^M \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^i$ is i by induction on i.

When $i = 1$, this is obvious because V_1 is given by a single polynomial $F_1(y_1)$ and $\deg_{y_1} F_1(y_1) \geq 1$, so that $F_1(y_1) \neq 0$. Suppose the statement holds for $i \geq 1$. Then $V_i \subset B_i$ has codimension i so that $V_i \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^1 \subset B_i \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^1 = B_{i+1}$ has codimension i. On the other hand, V_{i+1} is given in $V_i \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^1$ by $F_{i+1}(y_1, \cdots, y_{i+1})$, and $\deg_{y_{i+1}} F_{i+1} \geq 1$, so that the codimension of V_{i+1} in $V_i \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^1$ is 1. Hence the codimension of V_{i+1} in B_{i+1} is $i+1$, thus by induction the statement holds for all $1 \leq i \leq n$. $\qquad\square$

4.2 Perturbation lemmas

We discuss several perturbation lemmas that play essential roles in the proof of the mod t^m-moving lemma in §4.3.

4.2.1 Non-emptiness of fibers

Here is the basic set-up we consider:

> SITUATION ($\star$): Let $W \in z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^c$ be a nonempty integral cycle and choose a triangular generating set $\{f_1, \cdots, f_n\} \subset \widehat{\mathcal{O}}[y_1, \cdots, y_n]$ of the form (4.1.1) using Proposition 4.1.5. Consider the coefficient perturbation V of W with respect to $\{f_1, \cdots, f_n\}$ given by
>
> $$(F_1, \cdots, F_n) \subset \widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, \cdots, y_n]$$
>
> as in Definition 4.1.1. Let $\alpha_0 \in \widehat{\mathcal{O}}^M$ be the coefficient vector corresponding to the generating set $\{f_1, \cdots, f_n\}$ of W. By Lemma 2.2.7, W is closed in $\overline{\square}_{\widehat{\mathcal{O}}}^n$. We regard $y_i = Y_{i1}/Y_{i0}$ and use $((Y_{10}; Y_{11}), \cdots, (Y_{n0}; Y_{n1})) \in \overline{\square}_{\widehat{\mathcal{O}}}^n$ as the projective coordinates. By homogenizing each f_j, we obtain $\bar{f}_j \in \widehat{\mathcal{O}}[\{Y_{10}, Y_{11}\}, \cdots, \{Y_{n0}, Y_{n1}\}]$. Here $\overline{W} = W$ in $\overline{\square}_{\widehat{\mathcal{O}}}^n$ is given by the ideal $(\bar{f}_1, \cdots, \bar{f}_n)$. Similarly, the homogenization $(\bar{F}_1, \cdots, \bar{F}_n)$ of $(F_1, \cdots, F_n)$ defines the Zariski closure $\overline{V} \subseteq \mathbb{A}_{\widehat{\mathcal{O}}}^M \times_{\widehat{\mathcal{O}}} \overline{\square}_{\widehat{\mathcal{O}}}^n$ of V.

> Let $\overline{pr} : \overline{V} \to \mathbb{A}_{\widehat{\mathcal{O}}}^M$ and $pr : V \to \mathbb{A}_{\widehat{\mathcal{O}}}^M$ be the restrictions of the projections $\mathbb{A}_{\widehat{\mathcal{O}}}^M \times \overline{\square}_{\widehat{\mathcal{O}}}^n \to \mathbb{A}_{\widehat{\mathcal{O}}}^M$ and $\mathbb{A}_{\widehat{\mathcal{O}}}^M \times \square_{\widehat{\mathcal{O}}}^n \to \mathbb{A}_{\widehat{\mathcal{O}}}^M$, respectively. For each $\alpha \in \mathbb{A}_{\widehat{\mathcal{O}}}^M$, let $\overline{V}_\alpha := \overline{pr}^{-1}(\alpha)$. We have $V_\alpha = \overline{V}_\alpha \cap \square_{\widehat{\mathcal{O}}}^n = pr^{-1}(\alpha)$, while $\overline{V}_{\alpha_0} = \overline{W} = W = V_{\alpha_0}$.

PROPOSITION 4.2.1. *Under the Situation ($\star$), there exists a nonempty open neighborhood $U_{\mathrm{ne}} \subset \mathbb{A}_{\widehat{\mathcal{O}}}^M$ of α_0 such that for each $\alpha \in U_{\mathrm{ne}}$, the fiber V_α is nonempty. Furthermore, this open set contains $\mathbb{G}_{m,\widehat{\mathcal{O}}}^M$.*

Proof. By Proposition 4.1.5, the coefficient perturbation V is given by polynomials

$$
\begin{cases}
F_1(y_1) \in \widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1], \\
F_2(y_1, y_2) \in \widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, y_2], \\
\quad \vdots \\
F_n(y_1, \cdots, y_n) \in \widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, \cdots, y_n].
\end{cases}
$$

Let $K = \mathrm{Frac}(\widehat{\mathcal{O}}) = k((t))$. Here $\deg_{y_1} F_1 \geq 1$ and the coefficient in $\widehat{\mathcal{O}}[x_1, \cdots, x_M]$ of the highest y_1-degree term is a variable x_{ℓ_1} for some $1 \leq \ell_1 \leq M$. For the open subset $U_1 \subseteq \mathbb{A}_{\widehat{\mathcal{O}}}^M$ given by $x_{\ell_1} \neq 0$, y_1 is algebraic over $K(x_1, \cdots, x_M)$, and there is a solution y_1 in an algebraic extension of $K(x_1, \cdots, x_M)$. Plug this solution y_1 into the second equation. Since $\deg_{y_2} F_2 \geq 1$, and the coefficient of the highest y_2-degree term is x_{ℓ_2} for some $1 \leq \ell_2 \leq M$ with $\ell_2 \neq \ell_1$. Thus, for the open set $U_2 \subseteq \mathbb{A}_{\widehat{\mathcal{O}}}^M$ given by $x_{\ell_1} \neq 0$ and $x_{\ell_2} \neq 0$, y_2 is algebraic over $K(x_1, \cdots, x_M)$, and in particular there is a solution y_2 in an algebraic extension of $K(x_1, \cdots, x_M)$. Continuing this way, the coefficient of the highest y_n-degree term of F_n is x_{ℓ_n} for some $1 \leq \ell_n \leq M$ with $\ell_n \neq \ell_1, \cdots, \ell_{n-1}$, and for the open set $U_n \subseteq \mathbb{A}_{\widehat{\mathcal{O}}}^M$ given by $\{x_{\ell_1} \neq 0, \cdots, x_{\ell_n} \neq 0\}$ we have a system of solutions $y_1, \cdots, y_n$ in an algebraic extension of $K(x_1, \cdots, x_M)$. In other words, for each $\alpha \in U_{\mathrm{ne}} := U_n$, the fiber V_α is nonempty. By construction U_{ne} is given by the product of $\mathbb{A}_{\widehat{\mathcal{O}}}^1$ for each x_i with $i \notin \{\ell_1, \cdots, \ell_n\}$ and $\mathbb{G}_{m, \widehat{\mathcal{O}}}$ for each x_i with $i \in \{\ell_1, \cdots, \ell_n\}$, so that the second statement follows. That $\alpha_0 \in U_{\mathrm{ne}}$ follows immediately. □

4.2.2 Properness over $\widehat{\mathcal{O}}$

Lemma 4.2.2. *We are under the Situation* $(\star)$. *Then there exists an open neighborhood* $U_{\mathrm{pr}} \subseteq \mathbb{A}_{\widehat{\mathcal{O}}}^M$ *of* α_0 *such that* V_α *is a proper scheme over* $\mathrm{Spec}\,(\widehat{\mathcal{O}})$ *for each* $\alpha \in U_{\mathrm{pr}}$.

Proof. Let F^∞ be the divisor associated to $\overline{\square}_{\widehat{\mathcal{O}}}^n \setminus \square_{\widehat{\mathcal{O}}}^n$. By Lemma 2.2.6, to make V_α proper over $\mathrm{Spec}\,(\widehat{\mathcal{O}})$, it is enough to require that $\overline{V}_\alpha \cap F^\infty = \emptyset$. Here $F^\infty = \sum_{i=1}^n \{y_i = \infty\} = \sum_{i=1}^n \{Y_{i0} = 0\}$ so that $\overline{V}_\alpha \cap F^\infty = \emptyset$ if and only if $\overline{V}_\alpha \cap \{Y_{i0} = 0\} = \emptyset$, for all $1 \leq i \leq n$. Recall we have $y_i = Y_{i1}/Y_{i0}$ for the projective coordinate $(Y_{i0}; Y_{i1}) \in \overline{\square}_{\widehat{\mathcal{O}}} = \mathbb{P}_{\widehat{\mathcal{O}}}^1$.

To see which open subset of $\mathbb{A}_{\widehat{\mathcal{O}}}^M$ would do this job, note that the scheme $\overline{V}_\alpha$ *does* intersect $\{Y_{i0} = 0\}$ if and only if the scheme given by $\{\bar{F}_1, \cdots, \bar{F}_n, Y_{i0}\}$ has a point over α. The system $\{\bar{F}_1, \cdots, \bar{F}_n, Y_{i0}\}$ defines a closed subscheme of $\mathbb{A}_{\widehat{\mathcal{O}}}^M \times_{\widehat{\mathcal{O}}} \overline{\square}_{\widehat{\mathcal{O}}}^n$ of dimension $\leq M + n + 1 - (n + 1) = M$ by Corollary 4.1.6. Thus its image C_i under the projective morphism $\mathbb{A}_{\widehat{\mathcal{O}}}^M \times_{\widehat{\mathcal{O}}} \overline{\square}_{\widehat{\mathcal{O}}}^n \to \mathbb{A}_{\widehat{\mathcal{O}}}^M$ is a closed subscheme of dimension $\leq M$, thus $C_i \subsetneq \mathbb{A}_{\widehat{\mathcal{O}}}^M$ is a proper closed subscheme. Hence $\overline{V}_\alpha$ *does not* intersect with F^∞ if and only if $\alpha \in U_{\mathrm{pr}} := \bigcap_{i=1}^n (\mathbb{A}_{\widehat{\mathcal{O}}}^M \setminus C_i)$. By construction, we have $\alpha_0 \in U_{\mathrm{pr}}$. This proves the lemma. □

Corollary 4.2.3. *Under the Situation $(\star)$, for every sufficiently large integer $N > 0$, the open ball $\mathcal{B}_N(\alpha_0) \subseteq k[[t]]^M$ in the non-archimedean t-adic sup-norm satisfies (1) $\mathcal{B}_N(\alpha_0) \cap (k[t]^M)$ is nonempty, (2) for every $\alpha \in \mathcal{B}_N(\alpha_0) \cap (k[t]^M)$, the closed subscheme V_α is proper over $Spec(\widehat{\mathcal{O}})$, and (3) these so obtained polynomials $f_{1,\alpha}, \cdots, f_{n,\alpha} \in k[t][y_1, \cdots, y_n] \subseteq \mathcal{O}[y_1, \cdots, y_n]$ of V_α satisfy $f_{j,\alpha} \equiv f_j \mod t^m$, for each $1 \leq j \leq n$.*

Proof. Since the induced non-archimedean t-adic topology is finer than the Zariski topology on $\mathbb{A}_{\widehat{\mathcal{O}}}^M$ and $\alpha_0 \in \widehat{\mathcal{O}}^M = k[[t]]^M$, for every sufficiently large integer $N > 0$, the open ball $\mathcal{B}_N(\alpha_0) \subseteq k[[t]]^M$ of radius e^{-N} centered at α_0 is contained in the open subset $U_{\mathrm{pr}} \subset \mathbb{A}_{\widehat{\mathcal{O}}}^{M-r}$ of Lemma 4.2.2. We may assume $N > m$. But $k[t]^M \subseteq k[[t]]^M$ is dense in the non-archimedean topology, so $\mathcal{B}_N(\alpha_0) \cap (k[t]^M) \neq \emptyset$, proving (1). Since $\mathcal{B}_N(\alpha_0) \subseteq U_{\mathrm{pr}}$, we have (2). On the other hand, $\alpha \in \mathcal{B}_N(\alpha_0) \Leftrightarrow |\alpha - \alpha_0| < e^{-N} \Leftrightarrow$ for each $1 \leq j \leq n$, we have $f_{j,\alpha} \equiv f_j \mod t^N$. In particular, since $N > m$, $f_{j,\alpha} \equiv f_j \mod t^m$, proving (3). □

4.2.3 Flat stratum

Let $pr : V \to \mathbb{A}_{\widehat{\mathcal{O}}}^M$ be the restriction of the projection $\mathbb{A}_{\widehat{\mathcal{O}}}^M \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^n$. By Proposition 4.2.1, we know that the restriction $pr_{U_{\mathrm{ne}}} : pr^{-1}(U_{\mathrm{ne}}) \to U_{\mathrm{ne}}$ is surjective, but we do not know whether this is flat. By the generic flatness theorem of [9, Théorème (6.9.1), p.153], there is a nonempty open subset of U_{ne} over which $pr_{U_{\mathrm{ne}}}$ is flat, but this theorem does not tell us whether this open set contains α_0. This causes an inconvenience. On the other hand, by the flattening stratification theorem of [9, Corollaire (6.9.3), p.154], we do know that there is a stratification partition $\{S_i\}$ of U_{ne} by locally closed subsets such the restriction of pr over the inverse image of each S_i is flat, and some stratum S_{i_0} must contain α_0. We will construct explicitly in Lemma 4.2.4 a locally closed subset of U_{ne} containing α_0 over which a more general collection of coherent sheaves are flat. This result will be used in §4.2.4 and §4.2.5.

Here is the set-up updated from Situation $(\star)$:

> Situation $(\star')$: Let $W \in z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^c$ be a nonempty integral cycle, and choose a triangular generating set $\{f_1, \cdots, f_n\} \subset \widehat{\mathcal{O}}[y_1, \cdots, y_n]$ of the form (4.1.1) using Proposition 4.1.5. Let V be the coefficient perturbation of W given by $\{F_1, \cdots, F_n\} \subset \widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, \cdots, y_n]$ as in Situation $(\star)$. By renaming the variables x_i, we may assume that $x_{M-n+1}, \cdots, x_M$ corresponds to the constant terms (= 1) of $f_1, \cdots, f_n$. By Lemma 2.2.7, W is closed in $\overline{\square}_{\widehat{\mathcal{O}}}^n$ and it is given by $(\bar{f}_1, \cdots, \bar{f}_n)$ as in Situation $(\star)$, with its coefficient perturbation $\overline{V} \subseteq \mathbb{A}_{\widehat{\mathcal{O}}}^M \times_{\widehat{\mathcal{O}}} \overline{\square}_{\widehat{\mathcal{O}}}^n$ given by $(\bar{F}_1, \cdots, \bar{F}_n)$.
>
> Let $B := \mathbb{A}_{\widehat{\mathcal{O}}}^{M-n} \times \mathbf{1} \subset \mathbb{A}_{\widehat{\mathcal{O}}}^M$, where $\mathbf{1} \subset \mathbb{A}_{\widehat{\mathcal{O}}}^n$ is the closed subscheme defined by $x_j = 1$ for $M - n + 1 \leq j \leq M$, and $\overline{pr}_B : \overline{pr}^{-1}(B) \to B$

and $pr_B : pr^{-1}(B) \to B$ are the restrictions of $\overline{pr}$ and pr, respectively. Here, $\alpha_0 = \beta_0 \times \mathbf{1} \in B$.

LEMMA 4.2.4. *Under the Situation* $(\star')$, *denote* $\mathbb{A}_{\widehat{\mathcal{O}}}^M \times_{\widehat{\mathcal{O}}} \overline{\square}_{\widehat{\mathcal{O}}}^n$ *by* X. *For each face* $\overline{F} \subseteq \overline{\square}_{\widehat{\mathcal{O}}}^n$, *including the case* $\overline{F} = \overline{\square}_{\widehat{\mathcal{O}}}^n$, *consider the coherent sheaf* $\mathcal{F}_{\overline{F}} := \mathcal{O}_X/(\mathcal{I}_{\overline{V}} + \mathcal{I}_{\overline{F}})$, *where* $\mathcal{I}_{\overline{V}}$ *is the ideal sheaf of* $\overline{V} \subseteq X$ *and* $\mathcal{I}_{\overline{F}}$ *is the pull-back to* X *of the ideal sheaf of* $\overline{F}$. *Then* $\mathcal{F}_{\overline{F}}$ *restricted to* $\overline{pr}^{-1}(B)$ *is* $\overline{pr}_B$-*flat. In particular, its restriction to* $pr^{-1}(B)$ *is* pr_B-*flat as well.*

Proof. Fix a face $\overline{F}$, and denote $\mathcal{F}_{\overline{F}}$ by $\mathcal{F}$. Let $X' := B \times_{\widehat{\mathcal{O}}} \overline{\square}_{\widehat{\mathcal{O}}}^n = \overline{pr}^{-1}(B)$, which is closed in X. Let $\mathcal{F}'$ be the restriction of $\mathcal{F}$ to X'. For each open chart $U' \subseteq X'$ from an affine cover of X' and each $x \in U'$, we need to show that the stalk $\mathcal{F}'_x$ is a flat $\mathcal{O}_{B,pr_B(x)}$-module. We prove it for the chart $U' := B \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^n$ of X', which is obtained from the open chart $U := \mathbb{A}_{\widehat{\mathcal{O}}}^M \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^n$ of X via $U' = U \cap X'$.

Now, $\mathcal{F}|_U = \mathcal{O}_U/(\mathcal{I}_V + \mathcal{I}_F)$ is given by the quotient of $\widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, \cdots, y_n]$ by $(F_1, \cdots, F_n) + (\{y_{i_1} = \epsilon_1, \cdots, y_{i_s} = \epsilon_s\})$, where $\{y_{i_1} = \epsilon_1, \cdots, y_{i_s} = \epsilon_s\}$ for some $\epsilon_j \in \{0, 1\}$, is the set of equations of the face $F = \overline{F} \cap \square_{\widehat{\mathcal{O}}}^n$.

Recall the constant term of each of $f_1, \cdots, f_n$ is 1. By the labeling convention of the Situation $(\star')$, x_{M-n+j} is the variable corresponding to the nonzero constant term of f_j for $1 \leq j \leq n$. So, we have $F_j = x_{M-n+j} + G_j$ for some $G_j \in \widehat{\mathcal{O}}[x_1, \cdots, x_{M-n}][y_1, \cdots, y_n]$. Hence, the sections $(\mathcal{O}_U/\mathcal{I}_V)(U) = \widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, \cdots, y_n]/(F_1, \cdots, F_n)$ can be obtained from $\widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, \cdots, y_n]$ by replacing each x_{M-n+j} by $-G_j$ for $1 \leq j \leq n$. Here each G_j does not involve any of the variables $x_{M-n+1}, \cdots, x_M$. Thus, $(\mathcal{O}_U/\mathcal{I}_V)(U) \simeq$

$$\widehat{\mathcal{O}}[x_1, \cdots, x_{M-n}, -G_1, \cdots, -G_n][y_1, \cdots, y_n] = \widehat{\mathcal{O}}[x_1, \cdots, x_{M-n}][y_1, \cdots, y_n],$$

which is a polynomial ring over $\widehat{\mathcal{O}}$ with the variables $\{x_1, \cdots, x_{M-n}\} \cup \{y_1, \cdots, y_n\}$. Now the further quotient

$$R_F := \widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, \cdots, y_n]/((F_1, \cdots, F_n) + (\{y_{i_1} = \epsilon_1, \cdots, y_{i_s} = \epsilon_s\}))$$

can be obtained from $(\mathcal{O}_U/\mathcal{I}_V)(U) \simeq \widehat{\mathcal{O}}[x_1, \cdots, x_{M-n}][y_1, \cdots, y_n]$ by replacing each variable y_{i_u} by ϵ_u for $1 \leq u \leq s$, i.e.

$$R_F \simeq \widehat{\mathcal{O}}[x_1, \cdots, x_{M-n}][y_1, \cdots, y_n]/(\{y_{i_1} = \epsilon_1, \cdots, y_{i_s} = \epsilon_s\})$$

$$\simeq \widehat{\mathcal{O}}[x_1, \cdots, x_{M-n}][\{y_\ell \mid 1 \leq \ell \leq n, \ell \neq i_1, \cdots, i_s\}],$$

which is again a polynomial ring over $\widehat{\mathcal{O}}$ with the variables $\{x_1 \cdots, x_{M-n}\} \cup \{y_\ell \mid 1 \leq \ell \leq n, \ell \neq i_1, \cdots, i_s\}$. In particular, the natural map $\widehat{\mathcal{O}}[x_1, \cdots, x_{M-n}] \to R_F$ induced by the projection $\overline{pr}$ is injective and it is flat. Here, we have $\mathrm{Spec}\,(\widehat{\mathcal{O}}[x_1, \cdots, x_{M-n}]) = \mathbb{A}_{\widehat{\mathcal{O}}}^{M-n} \simeq \mathbb{A}_{\widehat{\mathcal{O}}}^{M-n} \times \mathbf{1} = B$. Hence in particular, $\mathcal{F}' = \mathcal{F}|_{U'}$ is flat. The proof for other charts of X' is similar, so we omit it. $\square$

4.2.4   Dominance

LEMMA 4.2.5. *Under the Situation $(\star')$, recall that $W \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ is dominant. Then there is an open neighborhood $U_{\mathrm{dom}} \subseteq \mathbb{A}_{\widehat{\mathcal{O}}}^{M-r}$ of β_0 such that for every $\beta \in U_{\mathrm{dom}}$, the associated closed subscheme $V_\alpha \subseteq \square_{\widehat{\mathcal{O}}}^n$ with $\alpha := \beta \times \mathbf{1}$, is dominant over $\mathrm{Spec}\,(\widehat{\mathcal{O}})$ as well.*

Proof. Let $K := \mathrm{Frac}(\widehat{\mathcal{O}}) = k((t))$. Note that a morphism $Z \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ is dominant if and only if the base change $Z_K \to \mathrm{Spec}\,(K)$ is a nonempty K-scheme. So, we consider the situation after the base change via $\mathrm{Spec}\,(K) \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$.

By Lemma 4.2.4, the morphism $pr_B : pr^{-1}(B) \to B$ is flat. For the open set U_{ne} of Proposition 4.2.1, we have $\alpha_0 \in B \cap U_{\mathrm{ne}}$ so that $B \cap U_{\mathrm{ne}} \neq \emptyset$, and this proposition shows that the restriction $pr_{B \cap U_{\mathrm{ne}}} : pr^{-1}(B \cap U_{\mathrm{ne}}) \to B \cap U_{\mathrm{ne}}$ is flat and surjective. Since U_{ne} contains $\mathbb{G}_m^M$, there exists a nonempty open neighborhood $U' \subset \mathbb{A}_{\widehat{\mathcal{O}}}^{M-n}$ of β_0 such that $U' \times \mathbf{1} \subseteq B \cap U_{\mathrm{ne}}$. Hence $pr_{U'} : pr^{-1}(U' \times \mathbf{1}) \to U' \times \mathbf{1}$ is flat and surjective. So, after base change via $\mathrm{Spec}\,(K) \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$, the new morphism $pr_{U'_K} : pr^{-1}(U' \times \mathbf{1})_K \to (U' \times \mathbf{1})_K$ is flat and surjective. We implicitly used [9, Proposition (2.1.4), p.6] several times. For this flat family, the dimensions of the fibers are all equal (see [9, Corollaire (6.1.2), p.135], or [11, Corollary III-9.6, p.257]). In particular, for every $\beta \in \mathbb{A}_K^{M-n} \cap U'$, we have $0 \leq \dim(V_{\alpha_0}) = \dim(V_\alpha)$ with $\alpha = \beta \times \mathbf{1}$. In particular $V_\alpha \neq \emptyset$. But, $\mathbb{A}_K^{M-n}$ is a nonempty open subset of $\mathbb{A}_{\widehat{\mathcal{O}}}^{M-n}$, so that we can take $U_{\mathrm{dom}} := \mathbb{A}_K^{M-n} \cap U'$ to finish the proof of the lemma. $\square$

COROLLARY 4.2.6. *Under the assumptions of Lemma 4.2.5, for every sufficiently large integer $N > 0$, the open ball $\mathcal{B}_N(\beta_0) \subseteq k[[t]]^{M-n}$ in the non-archimedean t-adic sup-norm satisfies (1) $\mathcal{B}_N(\beta_0) \cap (k[t]^{M-n})$ is nonempty, (2) for every $\beta \in \mathcal{B}_N(\beta_0) \cap (k[t]^{M-n})$, the closed subscheme V_α for $\alpha = \beta \times \mathbf{1}$, is dominant over $\mathrm{Spec}\,(\widehat{\mathcal{O}})$, and (3) these so obtained polynomials $f_{1,\alpha}, \cdots, f_{n,\alpha} \in k[t][y_1, \cdots, y_n] \subseteq \mathcal{O}[y_1, \cdots, y_n]$ of V_α satisfy $f_{j,\alpha} \equiv f_j \mod t^m$ for each $1 \leq j \leq n$.*

Proof. The proof is almost identical to that of Corollary 4.2.3, where we use Lemma 4.2.5 instead of Lemma 4.2.2, so we omit it. $\square$

4.2.5   Geometric integrality

Although we began with an integral scheme W, this integrality may not necessarily be preserved under "small" perturbations of the coefficients. However, we will show that the geometrical integrality over k in the sense of [9, Définition (4.6.2), p.68] is better behaved. Later in Case 2 of the proof of Theorem 4.3.2, we will reduce the general integral situation to the geometrically integral situation.

LEMMA 4.2.7. *Under the Situation $(\star')$, suppose further that W is geometrically integral over k. Then there exists an open neighborhood $U_{\mathrm{gi}} \subseteq \mathbb{A}_{\widehat{\mathcal{O}}}^{M-n}$ of β_0 such that for each $\beta \in U_{\mathrm{gi}}$, the fiber V_α with $\alpha = \beta \times \mathbf{1}$, is geometrically integral over k.*

Proof. Note that V_α is geometrically integral over if and only if so is its Zariski closure $\overline{V}_\alpha$ in $\overline{\square}_{\widehat{\mathcal{O}}}^n$. Now, by Lemma 4.2.4 with $F = \overline{\square}_{\widehat{\mathcal{O}}}^n$, the morphism $\overline{pr}_B : \overline{pr}^{-1}(B) \to B = \mathbb{A}_{\widehat{\mathcal{O}}}^{M-n} \times \mathbf{1}$ is proper and flat. Hence by [10, Théorème (12.2.4)-(viii), p.183], the set $U_{\mathrm{gi}} := \{\beta \in \mathbb{A}_{\widehat{\mathcal{O}}}^{M-n} \mid \overline{V}_\alpha$ with $\alpha = \beta \times \mathbf{1}$, is geometrically integral$\}$ is open in $\mathbb{A}_{\widehat{\mathcal{O}}}^{M-n}$. This U_{gi} is nonempty because $\beta_0 \in U_{\mathrm{gi}}$. But again, for each $\beta \in U_{\mathrm{gi}}$ with $\alpha = \beta \times \mathbf{1}$, we have that $\overline{V}_\alpha$ is geometrically integral if and only if so is V_α. This proves the lemma.   □

COROLLARY 4.2.8. *Under the assumptions of Lemma 4.2.7, the restriction V_B is geometrically integral of codimension n in $B \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^n$, and it intersects each codimension 1 face of $B \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^n$ properly.*

Proof. For the generic point $\eta \in \mathbb{A}_{\widehat{\mathcal{O}}}^{M-n} \simeq B$, we have $\eta \times \mathbf{1} \in U_{\mathrm{gi}} \times \mathbf{1}$ for the nonempty open subset $U_{\mathrm{gi}} \subset \mathbb{A}_{\widehat{\mathcal{O}}}^{M-n}$ of Lemma 4.2.7. Hence $V_{\eta \times \mathbf{1}}$ is geometrically integral. But this is equivalent to that its Zariski closure V_B in $B \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^n$ is geometrically integral. That it has codimension n in $B \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^n$ follows by the same argument as in the proof of Corollary 4.1.6.

Finally, let $F \subset B \times_{\widehat{\mathcal{O}}} \square_{\widehat{\mathcal{O}}}^n$ be a codimension 1 face given by $\{y_i = \epsilon\}$ for some $1 \leq i \leq n$ and $\epsilon \in \{0, 1\}$. Since this is a divisor and since V_B is geometrically integral (in particular irreducible) by the previous paragraph, we just need to show that $V \not\subseteq \mathbb{A}_{\widehat{\mathcal{O}}}^{M-n} \times_{\widehat{\mathcal{O}}} F$. Suppose not, i.e. $V_B \subseteq \mathbb{A}_{\widehat{\mathcal{O}}}^{M-n} \times_{\widehat{\mathcal{O}}} F$. Then specializing at $\beta_0 \times \mathbf{1} \in B$, which corresponds to the given W, we have $W \subseteq F$. But this is impossible because W intersects F properly, in fact $W \cap F = \emptyset$ by Corollary 2.2.4. This is a contradiction, so V_B intersects each codimension 1 face properly.   □

COROLLARY 4.2.9. *Under the assumptions of Lemma 4.2.7, for every sufficiently large integer $N > 0$, the open ball $\mathcal{B}_N(\beta_0) \subseteq k[[t]]^{M-n}$ in the non-archimedean t-adic sup-norm satisfies (1) $\mathcal{B}_N(\beta_0) \cap (k[t]^{M-n})$ is nonempty, (2) for every $\beta \in \mathcal{B}_N(\beta_0) \cap (k[t]^{M-n})$ and $\alpha := \beta \times \mathbf{1}$, the closed subscheme V_α is geometrically integral, and (3) these so obtained polynomials $f_{1,\alpha}, \cdots, f_{n,\alpha} \in k[t][y_1, \cdots, y_n] \subseteq \mathcal{O}[y_1, \cdots, y_n]$ of V_α satisfy $f_{j,\alpha} \equiv f_j \mod t^m$ for each $1 \leq j \leq n$.*

Proof. The proof is almost identical to that of Corollary 4.2.3, where we use Lemma 4.2.7 instead of Lemma 4.2.2, so we omit it.   □

4.2.6   EMPTY INTERSECTION WITH FACES

Recall from Corollary 2.2.4 that for any proper face $F \subsetneq \square_{\widehat{\mathcal{O}}}^n$, we had $W \cap F = \emptyset$, which is stronger than having proper intersection with the face. We assert that

this is an open condition inside a suitable locally closed base in the following sense:

LEMMA 4.2.10. *Under the assumptions of Lemma 4.2.7, for each proper face $F \subsetneq \square_{\widehat{\mathcal{O}}}^n$, there exists an open neighborhood $U_F \subseteq \mathbb{A}_{\widehat{\mathcal{O}}}^{M-n}$ of β_0 such that for each $\beta \in U_F$, we have $V_\alpha \cap F = \emptyset$ with $\alpha = \beta \times \mathbf{1}$. In particular, for each $\beta \in U_{\mathrm{pi}} := \bigcap_F U_F \subseteq \mathbb{A}_{\widehat{\mathcal{O}}}^M$, where the intersection is taken over all proper faces F, the closed subscheme V_α intersects with no proper face at all.*

Proof. By Lemma 2.2.6, we know that $W \cap F = \emptyset$ if and only if $\overline{W} \cap \overline{F} = \emptyset$. So, we want to achieve the stronger assertion that $\overline{V}_\alpha \cap \overline{F} = \emptyset$ for each $\alpha = \beta \times \mathbf{1}$, where β is in an open neighborhood of β_0 in B. We use the projectivized system $\{\bar{F}_1, \cdots, \bar{F}_n\}$ of equations for $\overline{V}$ so that $\{\bar{F}_1, \cdots, \bar{F}_n, x_{M-n+1} - 1, \cdots, x_M - 1\}$ is the system for $\overline{V}_B$.

When $\overline{F}$ is a codimension 1 face of $\overline{\square}_{\widehat{\mathcal{O}}}^n$, it is given by $\{y_{i_1} = \epsilon_1\} = \{Y_{i_1,1} = \epsilon_1 Y_{i_1,0}\}$ for some $1 \leq i_1 \leq n$ and $\epsilon_1 \in \{0, 1\}$. Here, the scheme $\overline{V}_\alpha$ *does* intersect with the face $\overline{F}$ if and only if the scheme given by $\{\bar{F}_1, \cdots, \bar{F}_n, x_{M-n+1} - 1, \cdots, x_M - 1, Y_{i_1,1} - \epsilon_1 Y_{i_1,0}\}$ has a point lying over α. Here, the system $\{\bar{F}_1, \cdots, \bar{F}_n, x_{M-n+1} - 1, \cdots, x_M - 1, Y_{i_1,1} - \epsilon_1 Y_{i_1,0}\}$ defines a closed subscheme of $\mathbb{A}_{\widehat{\mathcal{O}}}^{M-n} \times_{\widehat{\mathcal{O}}} \overline{\square}_{\widehat{\mathcal{O}}}^n$ of dimension $\leq \dim (\mathbb{A}_{\widehat{\mathcal{O}}}^{M-n} \times_{\widehat{\mathcal{O}}} \overline{\square}_{\widehat{\mathcal{O}}}^n) - (n+1) = (M-n) + n + 1 - (n+1) = M - n$ by Corollary 4.2.8. Thus its image C_F under the projective morphism $\mathbb{A}_{\widehat{\mathcal{O}}}^{M-n} \times_{\widehat{\mathcal{O}}} \overline{\square}_{\widehat{\mathcal{O}}}^n \to \mathbb{A}_{\widehat{\mathcal{O}}}^{M-n}$ is a closed subscheme of dimension $\leq M - n$. In particular, $C_F \subsetneq \mathbb{A}_{\widehat{\mathcal{O}}}^{M-n}$ is a proper closed subscheme since $\dim (\mathbb{A}_{\widehat{\mathcal{O}}}^{M-n}) = M - n + 1$. Hence $\overline{V}_\alpha$ *does not* intersect with $\overline{F}$ if and only if $\alpha \in U_F := \mathbb{A}_{\widehat{\mathcal{O}}}^{M-n} \setminus C_F$. By construction we have $\alpha_0 \in U_F$. Here, $\overline{V}_\alpha \cap \overline{F} = \emptyset$ implies that $V_\alpha \cap F = \emptyset$.

Since every proper face is contained in some codimension 1 face, this proves the lemma. $\qquad\square$

COROLLARY 4.2.11. *Under the assumptions of Lemma 4.2.7, for every sufficiently large integer $N > 0$, the open ball $\mathcal{B}_N(\alpha_0) \subseteq k[[t]]^M$ in the non-archimedean t-adic sup-norm satisfies (1) $\mathcal{B}_N(\alpha_0) \cap (k[t]^M)$ is nonempty, (2) for every $\alpha \in \mathcal{B}_N(\alpha_0) \cap (k[t]^M)$, the closed subscheme V_α does not intersect any face $F \subsetneq \overline{\square}_{\widehat{\mathcal{O}}}^n$ at all, and (3) these so obtained polynomials $f_{1,\alpha}, \cdots, f_{n,\alpha} \in k[t][y_1, \cdots, y_n] \subseteq \mathcal{O}[y_1, \cdots, y_n]$ of V_α satisfy $f_{j,\alpha} \equiv f_j \mod t^m$, for each $1 \leq j \leq n$.*

Proof. The proof is almost identical to that of Corollary 4.2.3, where we use Lemma 4.2.10 instead of Lemma 4.2.2, so we omit it. $\qquad\square$

4.3 THE MOD t^m MOVING LEMMAS

First observe:

LEMMA 4.3.1. *Let T be a $\mathrm{Spec}(\widehat{\mathcal{O}})$-scheme of finite type. Let $W_1, W_2 \subseteq T$ be two integral closed subschemes, both surjective over $\mathrm{Spec}(\widehat{\mathcal{O}})$, such that we have the equality $W_{1,s} = W_{2,s}$ of the special fibers. Then $\dim W_1 = \dim W_2$.*

Proof. Let $d_i := \dim W_i$. The morphisms $W_i \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ for $i = 1, 2$ are flat (of relative dimension $d_i - 1$) because they are surjective and $\mathrm{Spec}\,(\widehat{\mathcal{O}})$ is a regular scheme of dimension 1 (see [9, Corollaire (6.1.2), p.135] or [11, Proposition III-9.7, p.257]). Since $W_{1,s} = W_{2,s}$, we have $d_1 - 1 = d_2 - 1$. Hence $d_1 = d_2$. □

We now prove the main result of §4:

THEOREM 4.3.2. *For the completion homomorphism* $\xi^n : z_{\mathfrak{m}}^n(\mathcal{O}, n)^c \to z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^c$, *the composition* $\xi_m^n : z_{\mathfrak{m}}^n(\mathcal{O}, n)^c \xrightarrow{\xi^n} z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^c \to z^n(k_m, n)$ *is a surjection.*

Proof. Let $W \in z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^c$ be a nonempty integral closed subscheme of $\square_{\widehat{\mathcal{O}}}^n$. By Lemma 2.2.7, the structure morphism $W \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ is surjective.

CASE 1: First consider the case when W is geometrically integral over k. Take the generators of the ideal of W given by $f_1, \cdots, f_n \in \widehat{\mathcal{O}}[y_1, \cdots, y_n]$ satisfying the Situation $(\star')$, i.e. of the form in (4.1.1) in Proposition 4.1.5.

By Corollaries 4.2.3, 4.2.6, 4.2.9 and 4.2.11, there exists a sufficiently large integer $N > m$ such that for every $\beta \in \mathcal{B}_N(\beta_0) \cap (k[t])^{M-n}$ with $\alpha := \beta \times \mathbf{1}$, the corresponding cycle $V_\alpha \subseteq \square_{\widehat{\mathcal{O}}}^n$ is proper and dominant (in particular surjective) over $\mathrm{Spec}\,(\widehat{\mathcal{O}})$, is geometrically integral, and has empty intersection with all proper faces of $\square_{\widehat{\mathcal{O}}}^n$ (in particular, the intersections with all faces are proper), and furthermore the defining ideal of V_α in $\widehat{\mathcal{O}}[y_1, \cdots, y_n]$ is given by polynomials $f_{j,\alpha} \in k[t][y_1, \cdots, y_n]$ satisfying $f_j \equiv f_{j,\alpha} \mod t^m$ for all $1 \leq j \leq n$. Since both W and V_α are geometrically integral over k, they are integral, thus we have $W \sim_{t^m} V_\alpha$. By Lemma 4.3.1, this implies $\dim W = \dim V_\alpha$. Furthermore, for each proper face $F \subsetneq \square_{\widehat{\mathcal{O}}}^n$, we have $V_\alpha \cap F_s = \emptyset$ so that $\mathrm{codim}_F(V_\alpha \cap F_s) \geq n$, while for $F = \square_{\widehat{\mathcal{O}}}^n$ we have $\mathrm{codim}_F(V_\alpha \cap F_s) = \mathrm{codim}_F((V_\alpha)_s) = \mathrm{codim}_F(W_s) \geq n$, so that $V_\alpha \in z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^c$.

Note that V_α is given by the ideal generated by $\{f_{1,\alpha}, \cdots, f_{n,\alpha}\}$ in $\widehat{\mathcal{O}}[y_1, \cdots, y_n]$ with $f_{j,\alpha} \in k[t][y_1, \cdots, y_n] \subseteq \mathcal{O}[y_1, \cdots, y_n]$. So, if we let $Z \subseteq \square_{\mathcal{O}}^n$ be the closed subscheme given by the ideal generated by the same set $\{f_{1,\alpha}, \cdots, f_{n,\alpha}\}$, this time in $\mathcal{O}[y_1, \cdots, y_n]$, then we have $\widehat{Z} := Z \times_{\mathcal{O}} \widehat{\mathcal{O}} = V_\alpha$ by definition.
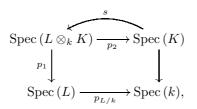
We claim that $Z \in z_{\mathfrak{m}}^n(\mathcal{O}, n)^c$. Here for each face $F_{\mathcal{O}} \subseteq \square_{\mathcal{O}}^n$, we have $\dim (Z \cap F_{\mathcal{O}}) = \dim (\widehat{Z} \cap F_{\widehat{\mathcal{O}}})$, where $F_{\widehat{\mathcal{O}}}$ is the base change of $F_{\mathcal{O}}$. In particular, when $F_{\mathcal{O}} = \square_{\mathcal{O}}^n$, we have $\dim Z = \dim V_\alpha$, while when $F \subsetneq \square_{\mathcal{O}}^n$ is a proper face, Z intersects with F properly. Furthermore via the identification $\mathcal{O}/\mathfrak{m} = \widehat{\mathcal{O}}/\widehat{\mathfrak{m}}$, we have $Z_s = (V_\alpha)_s = W_s$ so that $\mathrm{codim}_{F_{\mathcal{O}}}(Z \cap F_{\mathcal{O},s}) \geq n$ for each face $F_{\mathcal{O}} \subseteq \square_{\mathcal{O}}^n$. Hence $Z \in z_{\mathfrak{m}}^n(\mathcal{O}, n)$. The structure morphism $Z \to \mathrm{Spec}\,(\mathcal{O})$ is proper by [9, Proposition (2.7.1)-(vii), p.29], because its base change $Z \times_{\mathcal{O}} \widehat{\mathcal{O}} = V_\alpha \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ via the faithfully flat morphism $\mathrm{Spec}\,(\widehat{\mathcal{O}}) \to \mathrm{Spec}\,(\mathcal{O})$, is proper. Hence $Z \in z_{\mathfrak{m}}^n(\mathcal{O}, n)^c$ and $V_\alpha = \xi^n(Z)$. Combined with that $W \sim_{t^m} V_\alpha$, we thus have $W \in \mathrm{im}(\xi_m^n)$.

CASE 2: Now we suppose that W is integral, but not geometrically integral over k. Recall from [9, (4.3.1), p.58] that a field extension $k \subset K$ is a *primary extension* if the biggest algebraic separable extension of k in K is k itself. In other words, we say that k is separably closed in K, or that the separable closure of k in K is itself. Here we have:

CLAIM: *Let $w \in W$ be the generic point and take $K := k(w)$. Let L be the algebraic separable closure of k in K. Then we have the Cartesian diagram with a section s of p_2:*

$$
\begin{array}{ccc}
\operatorname{Spec}(L \otimes_k K) & \xrightarrow{\ \ p_2\ \ } & \operatorname{Spec}(K) \\
{\scriptstyle p_1}\Big\downarrow & & \Big\downarrow \\
\operatorname{Spec}(L) & \xrightarrow{\ p_{L/k}\ } & \operatorname{Spec}(k),
\end{array}
$$

for the étale base change map $p_{L/k}$. In other words, p_2 is a Nisnevich cover.

Since $k \subset L$ is separable, i.e. étale, the base change p_2 is étale as well. So, we just need to prove the existence of the section s. Recall that p_2 is given by the k-algebra homomorphism $p_2^\sharp : K \to L \otimes_k K$ given by $a \mapsto 1 \otimes a$. Note that we have the multiplication map $\Delta^\sharp : K \otimes_k K \to K$. Let $s^\sharp := \Delta^\sharp| : L \otimes_k K \to K$ be the restriction of $\Delta^\sharp$ on the subalgebra $L \otimes_k K \subset K \otimes_k K$. Then the composite

$$s^\sharp \circ p_2^\sharp : K \to L \otimes_k K \to K, \quad a \mapsto 1 \otimes a \mapsto a$$

is indeed the identity of K so that we have $p_2 \circ s = \operatorname{Id}_{\operatorname{Spec}(K)}$. This proves the Claim.

Going back to the proof of the Theorem in Case 2, we continue to use the notations of Claim and let $x := s(w)$. Note we have $k(x) \simeq k(w)$. The flat pull-back $p_{L/k}^* : z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^c \to z_{\widehat{\mathfrak{m}}'}^n(\widehat{\mathcal{O}}_L, n)^c$ of Proposition 2.5.3 gives $p_{L/k}^*(W) = \sum_i m_i W_i'$ for some integers $m_i \geq 1$ and integral closed subschemes $W_i' \subset \square_{\widehat{\mathcal{O}}_L}^n$. Here, $\widehat{\mathcal{O}}_L := \widehat{\mathcal{O}}_{\mathbb{A}_L^1, 0} \simeq L[[t]]$ and $\widehat{\mathfrak{m}}' := (t) \subset L[[t]]$.

By the Claim, one of W_i' has $x = s(w)$ as its generic point, say W_{i_0}'. Hence $k(W_{i_0}') = k(x) = K$. On the other hand, this field is a primary extension of L by the choice of L that it is the algebraic separable closure of k in K. Now, [9, Proposition (4.5.9), p.62] says that this is equivalent to that W_{i_0}' is geometrically irreducible in the sense of [9, Définition (4.5.2), p.61].

If k is perfect (thus so is L), then geometric reducedness over L (see [9, Définition (4.6.2), p.68]) is equivalent to reducedness by [9, Corollaire (4.6.11), p.70], so that this W_{i_0}' is actually geometrically integral over L. In case k is not perfect, then by [9, Proposition (4.6.6), p.69], there exists a finite radical (i.e. purely inseparable) extension $L \subset L'$ such that for the further base change of W_{i_0}' to $(W_{i_0}')_{L'}$ from L to L', the scheme $((W_{i_0}')_{L'})_{\mathrm{red}}$ is geometrically reduced over L'. Under this procedure, $((W_{i_0}')_{L'})_{\mathrm{red}}$ is still geometrically irreducible

over L' by definition. If x' is its unique generic point, then $k(x') \simeq k(x) \simeq k(w)$ under this extension. Hence under the further base change $\mathrm{Spec}\,(L') \to \mathrm{Spec}\,(k)$ (which is no longer étale, so the map $\mathrm{Spec}\,(L' \otimes_k K) \to \mathrm{Spec}\,(K)$ may no longer be a Nisnevich cover), but it still has a section $s' : \mathrm{Spec}\,(K) \to \mathrm{Spec}\,(L' \otimes_k K)$ such that the irreducible component of $x' = s'(w)$ is geometrically integral over L', and $k(s'(w)) \simeq k(w)$, i.e. its degree to W is 1. In this case, we replace L by L' and W'_{i_0} by $((W'_{i_0})_{L'})_{\mathrm{red}}$.

Hence for any base field k, we have a geometrically irreducible L-scheme $W'_{i_0} \in z^n_{\widehat{\mathfrak{m}}'}(\widehat{\mathcal{O}}_L, n)^c$ such that $[k(W'_{i_0}) : k(W)] = [K : K] = 1$. In particular, the push-forward $p_{L/k,*}(W'_{i_0}) = W$.

We have the following commutative diagram:

$$
\begin{array}{ccccc}
z^n_{\mathfrak{m}'}(\mathcal{O}_L, n)^c & \xrightarrow{\;\xi^n_L\;} & z^n_{\widehat{\mathfrak{m}}'}(\widehat{\mathcal{O}}_L, n)^c & \longrightarrow & z^n(L_m, n) \\[2pt]
\Big\downarrow{\scriptstyle p_{L/k,*}} & & \Big\downarrow{\scriptstyle p_{L/k,*}} & & \Big\downarrow{\scriptstyle p_{L/k,*}} \\[2pt]
z^n_{\mathfrak{m}}(\mathcal{O}, n)^c & \xrightarrow{\;\xi^n\;} & z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c & \longrightarrow & z^n(k_m, n),
\end{array}
\tag{4.3.1}
$$
$$
\underbrace{\hspace{7cm}}_{\xi^n_m}
$$

where the left square is commutative by [6, Proposition 1.7], while the right square is well-defined and commutative by Proposition 2.5.2-(2) applied to the proper morphism $p_{L/k} : \mathrm{Spec}\,(L) \to \mathrm{Spec}\,(k)$.

Since W'_{i_0} is geometrically integral over L, by Case 1, there exists some $Z' \in z^n_{\mathfrak{m}'}(\mathcal{O}_L, n)^c$ such that $\xi^n_L(Z') = \widehat{Z}' \sim_{t^m} W'_{i_0}$. Hence we have $W = p_{L/k,*}(W'_{i_0}) \sim^\dagger_{t^m} p_{L/k,*}\xi^n_L(Z') =^\ddagger \xi^n p_{L/k,*}(Z')$, where $\dagger$ and $\ddagger$ hold by the commutativity of the right and the left squares of the diagram (4.3.1), respectively. This shows that W lies in the image of ξ^n_m. This finishes the proof that ξ^n_m is surjective. $\square$

COROLLARY 4.3.3. *The morphism $\xi^n_m : z^n_{\mathfrak{m}}(\mathcal{O}, \bullet)^{pc} \to z^n(k_m, \bullet)$ of complexes induces a surjective group homomorphism $\mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)^{pc} \to \mathrm{CH}^n(k_m, n)$.*

Proof. Let $K_\bullet := \ker(\xi^n_m)$ and $I_\bullet := \mathrm{im}(\xi^n_m)$ so that we have a short exact sequence $0 \to K_\bullet \to z^n_{\mathfrak{m}}(\mathcal{O}, \bullet)^{pc} \to I_\bullet \to 0$ of homological complexes. From the morphisms $z^n_{\mathfrak{m}}(\mathcal{O}, \bullet)^{pc} \to I_\bullet \hookrightarrow z^n(k_m, \bullet)$ of complexes, we have homomorphisms

$$
\mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)^{pc} \to \mathrm{H}_n(I_\bullet) \to \mathrm{CH}^n(k_m, n).
\tag{4.3.2}
$$

Here, by Remark 2.2.3, we have $z^n_{\mathfrak{m}}(\mathcal{O}, n-1)^{pc} = 0$ so that $K_{n-1} = 0$, while we have $K_j = 0$ for all $j \leq n-1$ due to the dimension reason. In particular, $\mathrm{H}_{n-1}(K_\bullet) = 0$ and we have part of the associated long exact sequence $\cdots \to \mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)^{pc} \to \mathrm{H}_n(I_\bullet) \to \mathrm{H}_{n-1}(K_\bullet) = 0$ so that the first map $\mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)^{pc} \to \mathrm{H}_n(I_\bullet)$ of (4.3.2) is surjective.

On the other hand, by Theorem 4.3.2, we have $I_n = z^n(k_m, n)$, while $I_j = 0$ for all $j \leq n - 1$ by Remark 2.3.6 and the dimension reason. Hence

$$\mathrm{H}_n(I_\bullet) = \frac{z^n(k_m, n)}{\partial(\xi_m^n(z_{\mathfrak{m}}^n(\mathcal{O}, n+1)^{pc}))}, \quad \mathrm{CH}^n(k_m, n) = \frac{z^n(k_m, n)}{\partial(z^n(k_m, n+1))}$$

with $\partial(\xi_m^n(z_{\mathfrak{m}}^n(\mathcal{O}, n+1)^{pc})) \subseteq \partial(z^n(k_m, n+1))$ in $z^n(k_m, n)$, so that the second map $\mathrm{H}_n(I_\bullet) \to \mathrm{CH}^n(k_m, n)$ in (4.3.2) is the surjective quotient map. Hence the composite in (4.3.2) is surjective, as desired. □

Remark 4.3.4. One may wonder whether Theorem 4.3.2 extends beyond the Milnor range, i.e. when $q < n$, whether the composite $z_{\mathfrak{m}}^q(\mathcal{O}, n)^{pc} \to z_{\hat{\mathfrak{m}}}^q(\hat{\mathcal{O}}, n)^{pc} \to z^q(k_m, n)$ is surjective. To test if this question is affirmatively answerable, concentrate only on the subset of integral effective cycles. Since the cycles considered are flat over $\mathrm{Spec}(\hat{\mathcal{O}})$, such effective cycles may be, under mild additional assumptions, represented by (a locally closed subset of) a Hilbert scheme H, and there exists a (non-constant) morphism $\mathrm{Spec}(\hat{\mathcal{O}}) \to H$ of schemes. On the other hand, if the surjectivity assertion mod t^m would hold for those integral effective cycles, then it implies that for the fpqc cover $\mathrm{Spec}(\hat{\mathcal{O}}) \to \mathrm{Spec}(\mathcal{O})$, the morphism $\mathrm{Spec}(\hat{\mathcal{O}}) \to H$ should give an fpqc descent to a morphism $\mathrm{Spec}(\mathcal{O}) \to H$. However, this means that there exists a non-constant rational map $\mathbb{A}^1 \dashrightarrow H$, which imposes a restrictive condition on H. Thus, we do not expect an extension of Theorem 4.3.2 to cycles of arbitrary dimension.

5 MILNOR RANGE II: MOD t^m-EQUIVALENCE AND CONCLUSION

In §5.1, we use the mod t^m moving lemma of Theorem 4.3.2 to transport the main theorem of [5, Theorem 3.4] (or equivalently, [7], [16]) to our situation of cycles over $\hat{\mathcal{O}}$ modulo t^m. This allows a significant simplification of the generators of our relative cycle group $\mathrm{CH}^n((k_m, (t)), n)$, and helps in finally proving in §5.2 that the regulators Υ_i defined in Proposition 3.0.2 of §3 respect the mod t^m-equivalence. Using this, the proof of Theorem 3.0.1 is finished in §5.3.

5.1 THE GRAPH CYCLES

Recall that for each integral k-domain R of finite Krull dimension, and a sequence $a_1, \cdots, a_n \in R^\times$ of units, we have its associated closed subscheme $\Gamma_{(a_1, \cdots, a_n)} \subset \square_R^n$ given by the set of equations $\{y_1 = a_1, \cdots, y_n = a_n\}$. This is called the *graph cycle* of the sequence, and this is geometrically integral over k. In case R is local with the maximal ideal $\mathfrak{m}$, actually $\Gamma_{(a_1, \cdots, a_n)} \in z_{\mathfrak{m}}^n(R, n)$, and we get the graph homomorphism $gr : K_n^M(R) \to \mathrm{CH}_{\mathfrak{m}}^n(R, n)$. This was proven in [5, Lemma 2.1] for a ring R essentially of finite type over k, but exactly the same argument proves it for the general case. By construction, the Zariski closure $\overline{\Gamma}$ of Γ in $\overline{\square}_R^n$ is equal to Γ, so that in particular Γ is closed in $\overline{\square}_R^n$ as

well. Furthermore, one sees immediately that $\partial_i^\epsilon(\Gamma) = 0$ for each $1 \leq i \leq n$ and $\epsilon \in \{0, \infty\}$. We improve *loc.cit.* a bit as follows:

LEMMA 5.1.1. *The graph homomorphisms* $gr_{\mathcal{O}} : K_n^M(\mathcal{O}) \to \mathrm{CH}_{\mathfrak{m}}^n(\mathcal{O}, n)$ *and* $gr_{\widehat{\mathcal{O}}} : K_n^M(\widehat{\mathcal{O}}) \to \mathrm{CH}_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)$ *of [5, Lemma 2.1] actually map into* $\mathrm{CH}_{\mathfrak{m}}^n(\widehat{\mathcal{O}}, n)^{pc}$ *and* $\mathrm{CH}_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}}, n)^{pc}$, *respectively.*

Proof. We give the proof for $\mathcal{O}$ only. The proof for $\widehat{\mathcal{O}}$ is identical. First note that when $a_1, \cdots, a_n \in \mathcal{O}^\times$, the graph $\Gamma_{(a_1, \cdots, a_n)}$ is already proper over $\mathrm{Spec}\,(\mathcal{O})$ with no intersection with the faces, and the proper intersection condition with respect to the special fiber. Thus we have the set map $gr_{\mathcal{O}} : (\mathcal{O}^\times)^n \to z_{\mathfrak{m}}^n(\mathcal{O}, n)^c$.

The point of the rest of the proof is to repeat part of the argument of [5, Lemma 2.1] and check that the relevant cycles used in *loc.cit.* that put various relations on $z_{\mathfrak{m}}^n(\mathcal{O}, n)^c$ are actually lying in the group $z_{\mathfrak{m}}^n(\mathcal{O}, n+1)^{pc}$.

1. Let $a, 1-a, a_i \in \mathcal{O}^\times$ for $3 \leq i \leq n$. Consider the parametrized cycle

$$W : \square_{\mathcal{O}}^1 \dashrightarrow \square_{\mathcal{O}}^{n+1}, \quad x \mapsto \left(x, 1-x, \frac{a-x}{1-x}, a_3, \cdots, a_n \right).$$

   To check $W \in z_{\mathfrak{m}}^n(\mathcal{O}, n+1)^{pc}$, we need to look at its faces. By a direct calculation, one checks that the only nonzero face is $\partial_3^0 W = (a, 1-a, a_3, \cdots, a_n)$, which is in $z_{\mathfrak{m}}^n(\mathcal{O}, n)^c$, so $\partial W \in z_{\mathfrak{m}}^n(\mathcal{O}, n)^c$. Hence we have $W \in z_{\mathfrak{m}}^n(\mathcal{O}, n+1)^{pc}$ by definition. This also shows that $\{a, 1-a, a_3, \cdots, a_n\} \mapsto 0$ in $\mathrm{CH}_{\mathfrak{m}}^n(\mathcal{O}, n)^{pc}$.

2. Let $a, b, a_i \in \mathcal{O}^\times$ for $2 \leq i \leq n$. Consider the parametrized cycle

$$W : \square_{\mathcal{O}}^1 \dashrightarrow \square_{\mathcal{O}}^{n+1}, \quad x \mapsto \left(x, \frac{ax-ab}{x-ab}, a_2, \cdots, a_n \right).$$

   By direct calculations, its only nontrivial faces are $\partial_1^\infty W = (a, a_2, \cdots, a_n)$, $\partial_2^0 W = (b, a_2, \cdots, a_n)$, $\partial_2^\infty W = (ab, a_2, \cdots, a_n)$, all of which are in $z_{\mathfrak{m}}^n(\mathcal{O}, n)^c$. Hence $\partial W \in z_{\mathfrak{m}}^n(\mathcal{O}, n)^c$ so that $W \in z_{\mathfrak{m}}^n(\mathcal{O}, n+1)^{pc}$ by definition. This also gives the relation $(ab, a_2, \cdots, a_n) \equiv (a, a_2, \cdots, a_n) + (b, a_2, \cdots, a_n)$ in $\mathrm{CH}_{\mathfrak{m}}^n(\mathcal{O}, n)^{pc}$.

We may permute the above cycles to give the induced homomorphism $gr_{\mathcal{O}} : \bigotimes_{i=1}^n \mathcal{O}^\times \to \mathrm{CH}_{\mathfrak{m}}^n(\mathcal{O}, n)^{pc}$ by (2), which descends to $gr_{\mathcal{O}} : K_n^M(\mathcal{O}) \to \mathrm{CH}_{\mathfrak{m}}^n(\mathcal{O}, n)^{pc}$ by (1). This completes the proof. $\square$

LEMMA 5.1.2. *Let* k *be an infinite field. Then the map* $gr_{\mathcal{O}} : K_n^M(\mathcal{O}) \to \mathrm{CH}_{\mathfrak{m}}^n(\mathcal{O}, n)^{pc}$ *of Lemma 5.1.1 is surjective.*

Proof. We first claim that $\mathrm{CH}_{\mathfrak{m}}^n(\mathcal{O}, n)^{pc} = \mathrm{CH}^n(\mathcal{O}, n)$. By [5, Lemma 3.11], every cycle class in $\mathrm{CH}^n(\mathcal{O}, n)$ is represented by a cycle in the group called $\mathrm{CH}_{\mathrm{sfs}}^n(\mathcal{O}, n)$, where each irreducible component is finite (in particular, proper)

surjective over $\mathrm{Spec}\,(\mathcal{O})$ with the proper intersection condition with $\mathfrak{m}$. (See [5] for its precise definition.) This means that the composite inclusions $\mathrm{CH}^n_{\mathrm{sfs}}(\mathcal{O}, n) \hookrightarrow \mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)^{pc} \hookrightarrow \mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)$ is an isomorphism, where the second arrow is injective by Corollary 2.2.11. Hence $\mathrm{CH}^n_{\mathrm{sfs}}(\mathcal{O}, n) = \mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)^{pc} = \mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)$. Now the easy moving lemma of smooth affine k-schemes of higher Chow groups shows that $\mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n) = \mathrm{CH}^n(\mathcal{O}, n)$. This proves the claim.

Now the graph map $K^M_n(\mathcal{O}) \to \mathrm{CH}^n(\mathcal{O}, n)$ is surjective by [5, Theorem 3.4] (which uses that k is infinite), and it factors through $\mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)^{pc}$ by Lemma 5.1.1, so via the equality $\mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)^{pc} = \mathrm{CH}^n(\mathcal{O}, n)$, the map $gr_{\mathcal{O}}$ of Lemma 5.1.1 is surjective. $\square$

DEFINITION 5.1.3. For $R = \mathcal{O}$ or $\widehat{\mathcal{O}}$, let $z^n_{gr}(R, n)$ be the subgroup generated by the images of the graph cycles $\Gamma_{(a_1, \cdots, a_n)}$ over all sequences $a_1, \cdots, a_n \in R^\times$. For the well-defined homomorphism $z^n_{gr}(\widehat{\mathcal{O}}, n) \to \mathrm{CH}^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc} \to \mathrm{CH}^n(k_m, n)$, define $\mathrm{CH}^n_{gr}(k_m, n)$ to be the image of $z^n_{gr}(\widehat{\mathcal{O}}, n)$ in $\mathrm{CH}^n(k_m, n)$.

LEMMA 5.1.4. *Let k be an infinite field. The composite $K^M_n(\widehat{\mathcal{O}}) \overset{gr_{\widehat{\mathcal{O}}}}{\to} \mathrm{CH}^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc} \to \mathrm{CH}^n(k_m, n)$ is surjective, where $gr_{\widehat{\mathcal{O}}}$ is as in Lemma 5.1.1. In particular, the group $\mathrm{CH}^n(k_m, n)$ is generated by the graph cycles $\Gamma_{(a_1, \cdots, a_n)}$ for sequences $a_1, \cdots, a_n \in \widehat{\mathcal{O}}^\times$, and the natural homomorphism $\mathrm{CH}^n_{gr}(k_m, n) \to \mathrm{CH}^n(k_m, n)$ is an isomorphism.*

Proof. We have a commutative diagram

$$
\begin{array}{ccc}
K^M_n(\mathcal{O}) & \overset{gr_{\mathcal{O}}}{\longrightarrow} & \mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)^{pc} \\
\downarrow & & \downarrow{\scriptstyle \xi} \quad\; {}^{*}\!\!\searrow \\
K^M_n(\widehat{\mathcal{O}}) & \overset{gr_{\widehat{\mathcal{O}}}}{\longrightarrow} & \mathrm{CH}^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc} \longrightarrow \mathrm{CH}^n(k_m, n),
\end{array}
$$

where $gr_{\mathcal{O}}$ and $gr_{\widehat{\mathcal{O}}}$ map into their respective target groups of the diagram by Lemma 5.1.1, the left square commutes by [5, Proposition 2.3]. The map $gr_{\mathcal{O}}$ is surjective by Lemma 5.1.2. The sloped map $*$ is surjective by Corollary 4.3.3. By diagram chasing, the map $K^M_n(\widehat{\mathcal{O}}) \to \mathrm{CH}^n(k_m, n)$ is surjective. The second assertion follows immediately from the first one. $\square$

LEMMA 5.1.5. *Let k be an infinite field. The surjection $K^M_n(\widehat{\mathcal{O}}) \to \mathrm{CH}^n(k_m, n)$ of Lemma 5.1.4 induces a surjection $K^M_n(k_m) \to \mathrm{CH}^n(k_m, n)$.*

Proof. There is a natural surjection $K^M_n(\widehat{\mathcal{O}}) \to K^M_n(\widehat{\mathcal{O}}/(t^m)) = K^M_n(k_m)$. So, for any Milnor symbol $\{a_1, \cdots, a_n\} \in K^M_n(k_m)$ with $a_i \in k_m^\times$, we choose any liftings $\tilde{a}_1, \cdots, \tilde{a}_n \in \widehat{\mathcal{O}}^\times = k[[t]]^\times$ and send the symbol $\{\tilde{a}_1, \cdots, \tilde{a}_n\} \in K^M_n(\widehat{\mathcal{O}})$ to the cycle class in $\mathrm{CH}^n(k_m, n)$ of the graph cycle $\Gamma_{(\tilde{a}_1, \cdots, \tilde{a}_n)} \subset \square^n_{\widehat{\mathcal{O}}}$. To prove that this map is well-defined, choose another sequence of liftings $\tilde{a}'_1, \cdots, \tilde{a}'_n \in \widehat{\mathcal{O}}^\times$ of the sequence $a_1, \cdots, a_n \in k_m^\times$, and here $\tilde{a}_i - \tilde{a}'_i \in t^m k[[t]]$. By definition, we have $\Gamma_{(\tilde{a}_1, \cdots, \tilde{a}_n)} \sim_{t^m} \Gamma_{(\tilde{a}'_1, \cdots, \tilde{a}'_n)}$, so that the map $K^M_n(k_m) \to \mathrm{CH}^n(k_m, n)$ is

well-defined. The surjectivity of this map now follows from the surjectivity of $K_n^M(\widehat{\mathcal{O}}) \to \mathrm{CH}^n(k_m, n)$ of Lemma 5.1.4. $\qquad\square$

5.2 The graph cycles over $\widehat{\mathcal{O}}$ mod t^m

For graph cycles, it is easy to describe mod t^m equivalence:

Lemma 5.2.1. *Let $Z_1, Z_2 \in z_{gr}^n(\widehat{\mathcal{O}}, n)^c$ be two integral graph cycles, represented by*

$$Z_1 : \{y_1 = a_1, \cdots, y_n = a_n\}, \quad Z_2 : \{y_1 = b_1, \cdots, y_n = b_n\}, \qquad (5.2.1)$$

where $a_j, b_j \in \widehat{\mathcal{O}}^\times$ for $1 \le j \le n$. Then the following are equivalent:

1. *$Z_1 \sim_{t^m} Z_2$*

2. *For each $1 \le j \le n$, we have $a_j \equiv b_j$ in $\widehat{\mathcal{O}}/(t^m)$.*

3. *For each $1 \le j \le n$, there exists $c_j \in \widehat{\mathcal{O}}$ such that $a_j = b_j(1 + c_j t^m)$ in $\widehat{\mathcal{O}}$.*

Proof. The equivalence $(1) \Leftrightarrow (2)$ and the implication $(3) \Rightarrow (2)$ are obvious. For the implication $(2) \Rightarrow (3)$, note that $a_j \equiv b_j$ in $\widehat{\mathcal{O}}/(t^m)$ implies that $a_j b_j^{-1} \equiv 1$ in $\widehat{\mathcal{O}}/(t^m)$ so that $a_j b_j^{-1} = 1 + c_j t^m$ in $\widehat{\mathcal{O}}$ for some $c_j \in \widehat{\mathcal{O}}$. This proves (3). $\qquad\square$

Proposition 5.2.2. *Let k be a field of characteristic 0. Let $Z_1, Z_2 \in z_{gr}^n(\widehat{\mathcal{O}}, n)^c$ be two integral graph cycles such that $Z_1 \sim_{t^m} Z_2$. Then $\Upsilon_i(Z_1) = \Upsilon_i(Z_2)$ for each $1 \le i \le m - 1$.*

Proof. For Z_1 and Z_2, express them by the equations as in (5.2.1). By Lemma 5.2.1, the assumption that $Z_1 \sim_{t^m} Z_2$ implies that we have $a_j = b_j(1 + c_j t^m)$ for some $c_j \in \widehat{\mathcal{O}}$ for each $1 \le j \le n$. Notice that the common special fiber $(Z_1)_s = (Z_2)_s$ is given by a single closed point $\mathfrak{p}$ whose coordinates are $\bar{a}_1 = \bar{b}_1, \cdots, \bar{a}_n = \bar{b}_n \in k$, where the bars denote the images in the residue field $k = \widehat{\mathcal{O}}/(t)$. For each j, we have

$$d\log a_j - d\log b_j = d\log(1 + c_j t^m) = \frac{t^m dc_j + c_j m t^{m-1} dt}{1 + c_j t^m} \in t^{m-1}\Omega^1_{\widehat{\mathcal{O}}/\mathbb{Z}}.$$

Hence by expanding out $d\log y_1 \wedge \cdots \wedge d\log y_n|_{Z_1} - d\log y_1 \wedge \cdots \wedge d\log y_n|_{Z_2} = d\log a_1 \wedge \cdots \wedge d\log a_n - d\log b_1 \wedge \cdots \wedge d\log b_n$, we directly check that it is in $t^{m-1}\Omega^n_{\widehat{\mathcal{O}}/\mathbb{Z}}$. Thus for each $1 \le i \le m - 1$, we have $\frac{1}{t^i}d\log a_1 \wedge \cdots \wedge d\log a_n - \frac{1}{t^i}d\log b_1 \wedge \cdots \wedge d\log b_n \in t^{m-1-i}\Omega^n_{\widehat{\mathcal{O}}/\mathbb{Z}} \subset \Omega^n_{\widehat{\mathcal{O}}/\mathbb{Z}}$ so that the residue at $t = 0$ (which is the residue at the unique closed point $\mathfrak{p}$ of the common special fiber) of the difference vanishes. In other words, $\Upsilon_i(Z_1) = \Upsilon_i(Z_2)$ for $1 \le i \le n$. $\qquad\square$

Corollary 5.2.3. *Let k be a field of characteristic 0. For $1 \le i \le m - 1$, the map Υ_i of Proposition 3.0.2 induces a homomorphism $\Upsilon_i : \mathrm{CH}^n((k_m, (t)), n) \to \Omega^{n-1}_{k/\mathbb{Z}}$.*

Proof. By Proposition 3.0.2, the map Υ_i descends to $\Upsilon_i : \mathrm{CH}^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc} \to \Omega^{n-1}_{k/\mathbb{Z}}$. Since $\mathrm{CH}^n_{gr}(k_m, n) = \mathrm{CH}^n(k_m, n)$ by Lemma 5.1.4, we may consider only the graph cycles. For all the pairs of mod t^m-equivalent integral graph cycles, by Proposition 5.2.2, the maps Υ_i respect the mod t^m-equivalence, so that we have the induced map $\Upsilon_i : \mathrm{CH}^n(k_m, n) \to \Omega^{n-1}_{k/\mathbb{Z}}$. Now, since $\mathrm{CH}^n(k_m, n) = \mathrm{CH}^n((k_m, (t)), n) \oplus \mathrm{CH}^n(k, n)$, by restriction we have the desired homomorphism. $\qquad\square$

Remark 5.2.4. In fact, $\Upsilon_i|_{\mathrm{CH}^n(k,n)} = 0$ for $1 \leq i \leq m-1$. Indeed, by the theorem of Nesterenko-Suslin [23] and Totaro [29], we have an isomorphism $K^M_n(k) \simeq \mathrm{CH}^n(k, n)$ so that it is enough to check that for the graph cycles Γ given by the equations of the form $\{y_1 = a_1, \cdots, y_n = a_n\}$, with $a_1, \cdots, a_n \in k^\times$, we have $\Upsilon_i(\Gamma) = 0$. The form is $\frac{1}{t^i} d\log y_1 \wedge \cdots \wedge d\log y_n|_\Gamma = \frac{1}{t^i} d\log a_1 \wedge \cdots \wedge d\log a_n$ with each $a_j \in k^\times$ so that there is no term with dt anywhere in the form. Thus its residue along $t = 0$ is 0, i.e. $\Upsilon_i(\Gamma) = 0$.

5.3 Proof of Theorem 3.0.1

Finally, we prove the main theorem of the paper. We show that $\bigoplus_{i=1}^{m-1} \Upsilon_i : \mathrm{CH}^n((k_m, (t)), n) \to \bigoplus_{i=1}^{m-1} \Omega^{n-1}_{k/\mathbb{Z}}$ is an isomorphism. Recall from Lemma 5.1.5 that we had a surjection $K^M_n(k_m) \to \mathrm{CH}^n(k_m, n)$. This induces a surjective map $K^M_n(k_m, (t)) \to \mathrm{CH}^n((k_m, (t)), n)$, where $K^M_n(k_m, (t)) := \ker(K^M_n(k_m) \overset{\mathrm{ev}_{t=0}}{\to} K^M_n(k))$. We know from Proposition 5.4.2 in the appendix §5.4 below that we have an isomorphism

$$K^M_n(k_m, (t)) \overset{\sim}{\to} \Omega^{n-1}_{k_m,(t)/\mathbb{Z}}/d\Omega^{n-2}_{k_m,(t)/\mathbb{Z}} \overset{\sim}{\leftarrow} \bigoplus_{1 \leq i \leq m-1} t^i \Omega^{n-1}_{k/\mathbb{Z}},$$

given by $\{a_1, \cdots, a_n\} \mapsto \log(a_1)d\log(a_2) \wedge \cdots \wedge d\log(a_n)$, where $a_1 \in 1 + tk_m$ and $\Omega^i_{k_m,(t)/\mathbb{Z}} := \ker(\Omega^i_{k_m/\mathbb{Z}} \overset{\mathrm{ev}_{t=0}}{\to} \Omega^i_{k/\mathbb{Z}})$. Then, looking at the $k^\times$-weight i parts, we obtain the maps

$$\Omega^{n-1}_{k/\mathbb{Z}} \overset{\sim}{\to} t^i \Omega^{n-1}_{k/\mathbb{Z}} \hookrightarrow K^M_n(k_m, (t)) \twoheadrightarrow \mathrm{CH}^n((k_m, (t)), n) \overset{\Upsilon_i}{\to} \Omega^{n-1}_{k/\mathbb{Z}}, \qquad (5.3.1)$$

where $r_1 dr_2 \wedge \cdots \wedge dr_n \in \Omega^{n-1}_{k/\mathbb{Z}}$ is mapped to $\{e^{rt^i}, r_2, \cdots, r_n\} \in K^M_n(k_m, (t))$, where $r := r_1 \cdots r_n$. Let $\Gamma \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ denote the graph of this Milnor element. The composition (5.3.1) then sends $r_1 dr_2 \wedge \cdots \wedge dr_n$ to $\Upsilon_i(\Gamma) = ir_1 dr_2 \wedge \cdots \wedge dr_n$ by a straightforward calculation. Since $i \neq 0$, the composition (5.3.1) is an isomorphism. In particular, the composite

$$\bigoplus_{i=1}^{m-1} \Omega^{n-1}_{k/\mathbb{Z}} \simeq K^M_n(k_m, (t)) \twoheadrightarrow \mathrm{CH}^n((k_m, (t)), n) \overset{\oplus_i \Upsilon_i}{\to} \bigoplus_{i=1}^{m-1} \Omega^{n-1}_{k/\mathbb{Z}} \qquad (5.3.2)$$

is an isomorphism. Therefore, the above map $K^M_n(k_m, (t)) \to \mathrm{CH}^n((k_m, (t)), n)$ is injective, hence an isomorphism. Since the composite (5.3.2) is an isomorphism, this implies that $\bigoplus_i \Upsilon_i$ is an isomorphism, as desired. $\qquad\square$

5.4   Appendix

In the middle of the proof of Theorem 3.0.1 in §5.3, we used the following Proposition 5.4.2. This is probably well-known to the experts, and with some effort it should follow from e.g. [8]. However, since the Milnor K-groups are given by the concrete Milnor symbols we sketch a direct argument as follows, partly due to the fact that the authors could not find a suitable reference. We first have:

Lemma 5.4.1. *Let k be a field. Then $K_n^M(k_m, (t))$ is generated by the Milnor symbols $\{a_1, \cdots, a_n\}$ with $a_1 \in 1 + tk_m$ and $a_2, \cdots, a_n \in k_m^\times$.*

Proof. Let $G \subseteq K_n^M(k_m)$ be the subgroup generated by the Milnor symbols $\{a_1, \cdots, a_n\}$ with $a_1 \in 1 + tk_m$ (which is contained in $k_m^\times$) and $a_2, \cdots, a_n \in k_m^\times$. Certainly under the evaluation map $\mathrm{ev}_{t=0} : K_n^M(k_m) \to K_n^M(k)$, we have $\{a_1|_{t=0}, a_2|_{t=0}, \cdots, a_n|_{t=0}\} = \{1, a_2|_{t=0}, \cdots, a_n|_{t=0}\} = 0$ in $K_n^M(k)$. Hence each such generator $\{a_1, \cdots, a_n\}$ with $a_1 \in 1 + tk_m$ is contained in $\ker(\mathrm{ev}_{t=0}) = K_n^M(k_m, (t))$, thus $G \subseteq K_n^M(k_m, (t))$.

Every $a \in k_m^\times$ can be written as the product $a = c \cdot b$ with $c \in k^\times$ and $b \in 1 + tk_m$. Hence by the multi-linearity and the anti-commutativity of $K_n^M(k_m)$, every symbol $\{a_1, \cdots, a_n\}$ with $a_i \in k_m^\times$ can be written as a sum of symbols in G (type I) and symbols $\{c_1, \cdots, c_n\}$ such that $c_i \in k^\times$ (type II). Here the splitting ring homomorphisms $k \to k_m \overset{\mathrm{ev}_{t=0}}{\to} k$ induce the splitting $K_n^M(k_m) = K_n^M(k) \oplus K_n^M(k_m, (t))$. The type II symbols are definitely in $K_n^M(k)$, while the symbols of type I generate G. Hence $K_n^M(k_m, (t)) = G$.   □

Proposition 5.4.2. *Let k be a field of characteristic 0 and let $m \geq 2$ be an integer. Then we have an isomorphism $\phi_n : K_n^M(k_m, (t)) \simeq \Omega_{k_m,(t)/\mathbb{Z}}^{n-1}/d\Omega_{k_m,(t)/\mathbb{Z}}^{n-2}$ given by $\{a_1, \cdots, a_n\} \mapsto \log(a_1)d\log(a_2) \wedge \cdots \wedge d\log(a_n)$, where $a_1 \in 1 + tk_m$, where $\log(a_1)$ makes sense in k_m. The isomorphism can be rewritten as $K_n^M(k_m, (t)) \simeq \bigoplus_{i=1}^{m-1} t^i \Omega_{k/\mathbb{Z}}^{n-1}$, where the map $t^i \Omega_{k/\mathbb{Z}}^{n-1} \to K_n^M(k_m, (t))$ is given by sending $r_1 dr_2 \wedge \cdots dr_n$ to $\{e^{r_1 r_2 \cdots r_n t^i}, r_2, \cdots, r_n\} \in K_n^M(k_m, (t))$.*

Proof. By Lemma 5.4.1, $K_n^M(k_m, (t))$ is generated by $\{a_1, \cdots, a_n\}$ with $a_1 \in 1 + tk_m$ and $a_2, \cdots, a_n \in k_m^\times$. We define $\psi_n : \Omega_{k_m,(t)/\mathbb{Z}}^{n-1}/d\Omega_{k_m,(t)/\mathbb{Z}}^{n-2} \to K_n^M(k_m, (t))$ by sending $r_1 dr_2 \wedge \cdots \wedge dr_n$, where $r_1, \cdots, r_\ell \in (t)$ and $r_{\ell+1}, \cdots, r_n \in k_m^\times$, to $\{e^{r_1 r_{\ell+1} \cdots r_n}, e^{r_2}, \cdots, e^{r_\ell}, r_{\ell+1}, \cdots, r_n\}$ in $K_n^M(k_m, (t))$.

One can check by induction that ϕ_n and ψ_n are well-defined group homomorphisms. We omit the proof as they follow from elementary but tedious arguments. Let's check that ϕ_n and ψ_n are inverse to each other. Indeed, for $x = r_1 dr_2 \wedge \cdots \wedge dr_n \in \Omega_{k_m,(t)/\mathbb{Z}}^{n-1}/d\Omega_{k_m,(t)/\mathbb{Z}}^{n-2}$ with $r_1, \cdots, r_\ell \in (t)$ and $r_{\ell+1}, \cdots, r_n \in k_m^\times$, we have

$$(\phi_n \circ \psi_n)(x) = \phi_n\{e^{r_1 r_{\ell+1} \cdots r_n}, e^{r_2}, \cdots, e^{r_\ell}, r_{\ell+1}, \cdots, r_n\}$$

$$= \log(e^{r_1 r_{\ell+1} \cdots r_n})d\log(e^{r_2}) \wedge \cdots \wedge d\log(e^{r_\ell}) \wedge d\log(r_{\ell+1}) \wedge \cdots \wedge d\log(r_n)$$

$$= r_1 r_{\ell+1} \cdots r_n dr_2 \wedge \cdots \wedge dr_\ell \wedge \frac{dr_{\ell+1}}{r_{\ell+1}} \wedge \cdots \wedge \frac{dr_n}{r_n} = r_1 dr_2 \wedge \cdots \wedge dr_n = x,$$

so that $\phi_n \circ \psi_n = \mathrm{Id}$. On the other hand, for $y = \{a_1, a_2, \cdots, a_n\}$ with $a_1 \in 1 + tk_m$ and $a_i \in k_m^\times$ for $2 \le i \le n$, we have $(\psi_n \circ \phi_n)(y) =$

$$\psi_n(\log(a_1) d\log(a_2) \wedge \cdots \wedge d\log(a_n)) = \psi_n \left(\frac{\log(a_1)}{a_2 \cdots a_n} da_2 \wedge \cdots \wedge da_n \right). \quad (5.4.1)$$

Here $a_1 \in 1 + tk_m$ so that $\log(a_1) \in (t)$, hence $\log(a_1)/(a_2 \cdots a_n) \in (t)$. Hence (5.4.1) equals to $\{e^{\frac{\log(a_1)}{a_2 \cdots a_n} \cdot a_2 \cdots a_n}, a_2, \cdots, a_n\} = \{a_1, \cdots, a_n\} = y$, i.e. $\psi_n \circ \phi_n = \mathrm{Id}$. The second statement follows from Lemma 5.4.3 below. $\square$

We used the following elementary lemma in the middle of the proof of Proposition 5.4.2, which we learned from the proof of [13, Lemma 6.2]:

LEMMA 5.4.3. *Let k be a field of characteristic 0. Let $m \ge 2$ be an integer. Then for $n \ge 2$, we have $\Omega^{n-1}_{k_m,(t)/\mathbb{Z}}/d\Omega^{n-2}_{k_m,(t)/\mathbb{Z}} \simeq d\Omega^{n-1}_{k_m/\mathbb{Z}}/d\Omega^{n-1}_{k/\mathbb{Z}} \xleftarrow[d]{\simeq} tk_m \otimes_k \Omega^{n-1}_{k/\mathbb{Z}} = \bigoplus_{i=1}^{m-1} t^i \Omega^{n-1}_{k/\mathbb{Z}}$.*

Proof. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^{n-1}(\Omega^\bullet_{k_m/\mathbb{Z}}) & \longrightarrow & \Omega^{n-1}_{k_m/\mathbb{Z}}/d\Omega^{n-2}_{k_m/\mathbb{Z}} & \xrightarrow{d} & d\Omega^{n-1}_{k_m/\mathbb{Z}} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \mathrm{ev}_{t=0}} & & \downarrow{\scriptstyle \mathrm{ev}_{t=0}} & & \downarrow{\scriptstyle \mathrm{ev}_{t=0}} & & \\
0 & \longrightarrow & \mathrm{H}^{n-1}(\Omega^\bullet_{k/\mathbb{Z}}) & \longrightarrow & \Omega^{n-1}_k/d\Omega^{n-2}_{k/\mathbb{Z}} & \xrightarrow{d} & d\Omega^{n-1}_{k/\mathbb{Z}} & \longrightarrow & 0,
\end{array}$$

where the vertical maps are all split surjections. Furthermore, by the Poincaré lemma in [35, Corollary 9.9.3], we have $\mathrm{H}^{n-1}(\Omega^\bullet_{k_m,(t)/\mathbb{Z}}) = 0$ so that the left vertical map is actually an isomorphism. Hence the snake lemma gives an isomorphism $\Omega^{n-1}_{k_m,(t)/\mathbb{Z}}/d\Omega^{n-2}_{k_m,(t)/\mathbb{Z}} \simeq d\Omega^{n-1}_{k_m/\mathbb{Z}}/d\Omega^{n-1}_{k/\mathbb{Z}}$. The second isomorphism $d\Omega^{n-1}_{k_m/\mathbb{Z}}/d\Omega^{n-1}_{k/\mathbb{Z}} \xleftarrow[d]{\simeq} tk_m \otimes_k \Omega^{n-1}_{k/\mathbb{Z}} = \bigoplus_{i=1}^{m-1} t^i \Omega^{n-1}_{k/\mathbb{Z}}$ is obvious. $\square$

5.5 FINAL REMARKS

We have two remarks on strengthening Theorem 3.0.1.

Remark 5.5.1. We could have defined $z^q(k_m, n)$ in Definition 2.3.5 as $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)/ \sim_{t^m}$ using the complex $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)$, but then part of the perturbation results in §4 may not be easy to establish. If one can prove an analogue of Corollary 2.2.13 for $n = q$ and $\widehat{\mathcal{O}}$, i.e. the guess that "*every integral cycle $Z \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)$ is equivalent to a cycle in $z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc} = z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ modulo the boundary of a cycle in $z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n+1)$*", then we can still prove a stronger version of Theorem 3.0.1 for the cycles using $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)/ \sim_{t^m}$. It is true when $n = 1$. See Remark 2.2.14. However for $n \ge 2$, we could yet find neither a proof nor a counterexample to the guess, so we gave this version of the definition in Definition 2.3.5.

Remark 5.5.2. Reflecting on the main theorem of [27], it is desirable to remove the assumption that the base field k is of characteristic 0 in Theorem 3.0.1. The right hand side $(\Omega_{k/\mathbb{Z}}^{n-1})^{\oplus(m-1)}$ of the isomorphism of Theorem 3.0.1 should be replaced by the big de Rham-Witt forms $\mathbb{W}_{m-1}\Omega_k^{n-1}$ for a general base field k. To proceed further, we need to understand whether there exists a Parshin-Lomadze residue for the big de Rham-Witt complexes when the base field is of positive characteristic, and especially when it is imperfect. This is not trivial and may require serious works. A more minor problem is to give an explicit description of the relative Milnor K-groups of the ring of truncated polynomials over a field of positive characteristic in terms of the big de Rham-Witt complexes. This would improve Proposition 5.4.2 for a field of positive characteristic. We leave these as future tasks to finish.

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