

ABSTRACT ℓ -ADIC 1-MOTIVES
AND TATE'S CANONICAL CLASS FOR NUMBER FIELDS

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ABSTRACT. In [GP2] we constructed a new class of Iwasawa modules as ℓ -adic realizations of what we called *abstract ℓ -adic 1-motives* in the number field setting. We proved in loc. cit. that the new Iwasawa modules satisfy an equivariant main conjecture. In this paper we link the new modules to the ℓ -adified Tate canonical class, defined by Tate in 1960 [Ta1] and give an explicit construction of (the minus part of) ℓ -adic Tate sequences for any Galois CM extension K/k of an arbitrary totally real number field k . These explicit constructions are significant and useful in their own right but also due to their applications (via results in [GP2]) to a proof of the minus part of the far reaching Equivariant Tamagawa Number Conjecture for the Artin motive associated to the Galois extension K/k .

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1. SETUP AND PREPARATION

Let K/k be a Galois extension of number fields of Galois group G . Assume that K is a CM field and that k is totally real. We fix an odd prime ℓ and denote by K_∞ and k_∞ the cyclotomic \mathbb{Z}_ℓ -extensions of K and k , respectively. We fix two finite, disjoint, G -invariant sets of primes S and T in K , such that S contains the ramification locus $S_{\text{ram}}(K_\infty/k)$ of K_∞/k (in particular, it contains the set S_ℓ of all ℓ -adic primes and the set S_∞ of all the archimedean primes) and T contains at least two primes of distinct residual characteristics. We assume throughout that the *classical Iwasawa μ -invariant associated to K_∞ and ℓ vanishes*, as conjectured by Iwasawa. In earlier work [GP2] we defined the category of “abstract ℓ -adic 1-motives” (which contains Deligne’s category of Picard 1-motives as a full subcategory) and from the data $(K/k, S, T, \ell)$ as above we constructed a canonical abstract ℓ -adic 1-motive $\mathcal{M} := \mathcal{M}_{S,T}^\ell(K/k)$. Its ℓ -adic realization (Tate module) $T_\ell(\mathcal{M})$, which was defined in loc.cit., is a free \mathbb{Z}_ℓ -module of finite rank which comes endowed with a natural $\mathbb{Z}_\ell[[\mathcal{G}]]$ -module structure, where $\mathcal{G} := \text{Gal}(K_\infty/k)$. In fact, the unique complex conjugation automorphism j of the CM field K_∞ acts upon $T_\ell(\mathcal{M})$ with eigenvalue (-1) , so $T_\ell(\mathcal{M})$ can be naturally viewed as a module over the quotient ring $\mathbb{Z}_\ell[[\mathcal{G}]]^- := \mathbb{Z}_\ell[[\mathcal{G}]]/(1+j)$. The main result in [GP2] states the following.

THEOREM 1.1. *Under the above hypotheses, the following hold.*

- (1) $\text{pd}_{\mathbb{Z}_\ell[[\mathcal{G}]]} T_\ell(\mathcal{M}) = 1$.
- (2) *If G is abelian, then $\text{Fit}_{\mathbb{Z}_\ell[[\mathcal{G}]]^-} T_\ell(\mathcal{M}) = (\Theta_{S,T}^\infty)$.*

Above, “Fit” denotes as usual the initial (0-th) Fitting ideal and $\Theta_{S,T}^\infty$ denotes a certain equivariant ℓ -adic L -function (a distinguished element of $\mathbb{Z}_\ell[[\mathcal{G}]]^-$) defined in loc.cit. Part (2) of the above theorem is what we called an “equivariant main conjecture” and it is a G -equivariant refinement of the classical Iwasawa Main Conjecture for arbitrary totally real number fields and odd primes ℓ proved by Wiles in [Wi]. As shown in [GP2], this refinement implies refined versions of the classical (imprimitive) Brumer-Stark and Coates-Sinnott conjectures. At this point we want to stress that Ritter and Weiss, and Kakde, have also proved Equivariant Main Conjectures which are in a way broader in scope, as they

are valid in the non-abelian setting as well. The relation with our version stated above was very nicely clarified by Nickel in [Ni].

From now on we will assume for simplicity that *the extensions k_∞/k and K/k are linearly disjoint (over k)*. This hypothesis will be removed in Remark 5.11. As a consequence of this hypothesis, Galois restriction induces a group isomorphism $\mathcal{G} \simeq G \times \Gamma$, where $\Gamma := \text{Gal}(K_\infty/K) \simeq \text{Gal}(k_\infty/k)$. Consequently, we have ring isomorphisms $\mathbb{Z}_\ell[[\mathcal{G}]]^- \simeq \mathbb{Z}_\ell[G]^-[[\Gamma]] \simeq \Lambda[G]^-$, where $\Lambda = \mathbb{Z}_\ell[[\Gamma]]$ is the usual Iwasawa algebra. Consequently (see [GP2] and the references therein) part (1) of the theorem above is equivalent to

$$\text{pd}_{\mathbb{Z}_\ell[G]} T_\ell(\mathcal{M}) = 0,$$

i.e. $T_\ell(\mathcal{M})$ is a finitely generated projective module over $\mathbb{Z}_\ell[G]$ (and over $\mathbb{Z}_\ell[G]^-$, obviously.) As a consequence, if we fix a topological generator γ of Γ , we obtain a perfect complex of $\mathbb{Z}_\ell[G]$ -modules

$$C^\bullet = [T_\ell(\mathcal{M}) \xrightarrow{1-\gamma} T_\ell(\mathcal{M})],$$

concentrated in degrees 0 and 1. Of course, the two cohomology groups of C^\bullet are given by the Γ -invariants $T_\ell(\mathcal{M})^\Gamma$ and Γ -coinvariants $T_\ell(\mathcal{M})_\Gamma$ of $T_\ell(\mathcal{M})$, respectively.

The main goal of this paper is twofold: first, fully describe the two cohomology groups of C^\bullet in terms of arithmetic invariants associated to $(K/k, S, T)$ and second, fully describe the class of C^\bullet in the appropriate $\text{Ext}_{\mathbb{Z}_\ell[G]^-}^2(*, *)$. More precisely, the main result of our paper is that C^\bullet gives a T -modified version of Tate's class (defined in [Ta1]) in the appropriate Ext^2 . (See Theorem 5.9 for the precise statement.) In the next remark and the paragraph that follows we explain these goals in more detail.

REMARK 1.2. *In [GP1] we proved the exact analogue of Theorem 1.1 in the case where K/k is a Galois extension of global fields of characteristic $p > 0$ (i.e. function fields) and K_∞ (respectively k_∞) is the maximal constant field extension of K (respectively k .) In that case there exists an actual geometric 1-motive (Deligne's Picard 1-motive) $\mathcal{M}_{S,T}(K/k)$ whose ℓ -primary part $\mathcal{M}_{S,T}(K/k) \otimes \mathbb{Z}_\ell$ gives the abstract ℓ -adic 1-motive $\mathcal{M}_{S,T}^\ell(K/k)$, for all prime numbers ℓ (including $\ell = 2, p$.) In that geometric context there is no analogue of complex conjugation, so taking (-1) -eigenspaces does not make sense. Also, there is no analogue of the sets S_ℓ*

or S_∞ and, most importantly, the extension K_∞/K (which in that case is the maximal constant field extension of K) is unramified. Moreover, in [GPff] we studied the function field analogue of the complex C^\bullet and under a natural largeness hypothesis on the set S emerging from work of Tate (see below for details) we showed that there are $\mathbb{Z}_\ell[G]$ -module isomorphisms

$$(1) \quad H^0(C^\bullet) \simeq U_{S,T} \otimes \mathbb{Z}_\ell, \quad H^1(C^\bullet) \simeq X_S \otimes \mathbb{Z}_\ell,$$

where $U_{S,T}$ is the group of S -units in K which are congruent to 1 modulo all primes in T and X_S is the group of degree 0 divisors in K supported at S . For all G -Galois extensions K/k of global fields and data $(K/k, S, T)$ as above Tate [Ta1, Ta2] defined a canonical class $\tau_{K/k,S} \in \text{Ext}_{\mathbb{Z}_\ell[G]}^2(X_S, U_S)$, for “large” S and independent on T , where U_S is the group of S -units in K and X_S is as above. It turns out that under the “largeness” hypothesis (to be explained below), the $\mathbb{Z}[G]$ -module inclusion $\iota : U_{S,T} \rightarrow U_S$ induces a group isomorphism (more on this below)

$$(2) \quad \iota_* : \text{Ext}_{\mathbb{Z}[G]}^2(X_S, U_{S,T}) \simeq \text{Ext}_{\mathbb{Z}[G]}^2(X_S, U_S).$$

In the function field setting we proved in [GPff] that if $c_{K/k,S,T}^\ell$ is the extension class of C^\bullet and $\tau_{K/k,S}^\ell$ is the ℓ -primary part of Tate’s class, then

$$(3) \quad (\iota_* \otimes \text{id}_{\mathbb{Z}_\ell})(c_{K/k,S,T}^\ell) = \tau_{K/k,S}^\ell,$$

for all primes $\ell \neq p$. The same result should hold for $\ell = p$, but as explained in *loc. cit.* we will address that case in a separate paper as the calculations would be somewhat different in nature, involving crystalline rather than ℓ -adic étale cohomology. This way we obtained in the function field setting a very explicit ℓ -adic realization

$$0 \rightarrow U_{S,T} \otimes \mathbb{Z}_\ell \rightarrow T_\ell(\mathcal{M}) \xrightarrow{1-\gamma} T_\ell(\mathcal{M}) \rightarrow X_S \otimes \mathbb{Z}_\ell \rightarrow 0.$$

of a so-called Tate sequence (meaning that its middle terms are finitely generated, projective $\mathbb{Z}_\ell[G]$ -modules and representing the ℓ -adic Tate class via $\iota_* \otimes \text{id}_{\mathbb{Z}_\ell}$.)

The remark above makes it easier for us to state the goals of this paper more precisely: prove (1) and (3) in the number field setting laid out above, under the assumption that ℓ is an odd prime, with $U_{S,T} \otimes \mathbb{Z}_\ell$, $X_S \otimes \mathbb{Z}_\ell$ and $\tau_{K/k,S}^\ell$ replaced by $(U_{S,T} \otimes \mathbb{Z}_\ell)^-$, $(X_S \otimes$

$\mathbb{Z}_\ell)^-$ and $\tau_{K/k,S,T}^{\ell,-}$, respectively. As in [GPff], we will approach the question of linking C^\bullet to the Tate class from two sides. On one hand we calculate the Γ -invariants and Γ -coinvariants of $T_\ell(\mathcal{M})$ directly, via Iwasawa theoretic methods, in sections 2 and 3. On the other hand, in section 5 (see Theorem 5.9) we establish the desired link between C^\bullet and the Tate class $\tau_{K/k,S}^{\ell,-}$ via calculations in a certain derived category and by relying in an essential way upon deep results of Burns–Flach [BF] and [Bu1]. The reason why we insist on presenting the explicit calculations of the cohomology of C^\bullet is because the proof of Theorem 5.9 relies on less explicit, not so easily transparent derived category arguments. It is satisfactory to see that the results obtained via the two approaches agree at the cohomology level. We must admit that at present the explicit calculation in the coinvariant case is somewhat laborious and not as smooth as the result one extracts from the “identification” with Tate’s canonical class. However, it is definitely much more explicit. To put things into context, it is important to note the following. In his Habilitationsschrift [Wt], Witte has reproved the results of [GP2] in a much wider context, which also covers Nickel’s generalization [Ni]. In the present paper we need a link between torsion of the realization of abstract 1-motives and cohomology. This is established in Section 5. As it turns out, this is also to be found in Chapter 6 of [Wt], see Cor. 6.3.3 in particular. Witte’s language is much more general and abstract. The arguments in Section 5 of the present paper were developed several years ago; we feel that their more pedestrian presentation might be useful, and we would like to let them stand. It was only in the summer of 2017 that the authors became aware of [Wt].

The rest of this section reviews additional notation and presents some preparations. For the construction of \mathcal{M} and its ℓ -adic realization $T_\ell(\mathcal{M})$, the reader should consult [GP2]. For the definition of Tate’s class $\tau_{K/k,S}$ the reader should consult [Ta1, Ta2]. In order to simplify notation we will let $\mathcal{K} := K_\infty$. For any algebraic field extension N/K , $S(N)$ denotes the set of places of N above places in S , but often we will be sloppy in context, just writing S instead of $S(N)$. In the particular case $N = \mathcal{K}$ we write \mathcal{S} for $S(\mathcal{K})$. The same notational convention should be used for the set T , but for simplicity we will use T for $T(N)$ and for \mathcal{T} most of

the time. No confusion will ensue. The superscript minus always means the (-1) -eigenspace under the unique complex conjugation of \mathcal{K} , as customary. As in Remark 1.2 above, U_S denotes the group of S -units in K and $U_{S,T}$ denotes its subgroup consisting of those S -units which are congruent to 1 modulo every prime in T . For an algebraic extension N/K , $U_S(N)$ and $U_{S,T}(N)$ have similar meanings. If X is a set and O is a commutative ring, then $O[X]$ will denote the free O -module of basis X . If X happens to be a group (or a set endowed with an action by a group H), then $O[X]$ is viewed with its additional group-ring structure (or $O[H]$ -module structure). Note that since K and \mathcal{K} are CM, we have

$$(X_S \otimes \mathbb{Z}_\ell)^- = \mathbb{Z}_\ell[S]^- = \mathbb{Z}_\ell[S \setminus S_\infty]^-, \mathbb{Z}_\ell[\mathcal{S} \setminus \mathcal{S}_\ell]^- = \mathbb{Z}_\ell[\mathcal{S} \setminus (\mathcal{S}_\ell \cup \mathcal{S}_\infty)]^-.$$

For an algebraic extension N/K , the group $cl_T(N)$ denotes the ray class group of N with conductor equal to the product of the prime ideals belonging to places in $T(N)$. In less elaborate language, this is the group of all fractional ideals coprime to $T(N)$ modulo all principal ideals admitting a generator u which is congruent to 1 modulo all $v \in T(N)$. We let $cl(N)$ denote the usual class-group of N . For simplicity, we let $C^T(N) := (cl_T(N) \otimes \mathbb{Z}_\ell)^-$ and $C_\infty^T := C^T(\mathcal{K})$. We give similar meanings to $C(N)$ and C_∞ .

DEFINITION 1.3. *The set S is called large (respectively ℓ -large) if $cl_T(K)$ (respectively $C^T(K)$) is generated by ideal classes supported at primes in S .*

Note that Tate's definition of "large" involves the usual class-group $cl(K)$ instead of the ray-class group $cl_T(K)$. However, the existence of a canonical surjective group morphism $cl_T(K) \twoheadrightarrow cl(K)$ shows that "large" in the sense of the definition above implies "large" in Tate's sense. Also, there is a well known canonical exact sequence of $\mathbb{Z}[G]$ -modules

$$(4) \quad 0 \rightarrow U_{S,T} \xrightarrow{\iota} U_S \rightarrow \kappa(T) \rightarrow cl_T(K)_S \rightarrow cl(K)_S \rightarrow 0$$

where $\kappa(T) = \bigoplus_{v \in T} \kappa(v)^\times$ (here $\kappa(v)$ is the residue field at v) and $cl_T(K)_S$ and $cl(K)_S$ are the quotients of the corresponding ideal-class groups by the subgroups of S -ideal classes. It is well known (see [GP2], for example) that $\text{pd}_{\mathbb{Z}[G]} \kappa(T) = 1$. Consequently, if S is large then $cl_T(K)_S = cl(K)_S = 0$ and ι induces the isomorphism

ι_* mentioned in (2) above. Under the weaker “ ℓ -largeness” hypothesis this line of arguments yields the isomorphism $(\iota_* \otimes \text{id}_{\mathbb{Z}_\ell})^-$, which is in fact all that is needed for our goals. We repeat our first goal: compute the modules $T_\ell(\mathcal{M})^\Gamma$ and $T_\ell(\mathcal{M})_\Gamma$ directly. The main problems we are going to encounter are caused by the set \mathcal{S}_ℓ of ℓ -adic places, which have no analog in the function field case. To guide us in our task, we recall from [GP2] that there is a canonical short exact sequence of $\mathbb{Z}_\ell[[\mathcal{G}]]$ -modules

$$0 \rightarrow T_\ell(C_\infty^T) \rightarrow T_\ell(\mathcal{M}) \rightarrow \mathbb{Z}_\ell[\mathcal{S} \setminus \mathcal{S}_\ell]^- \rightarrow 0,$$

and we rely on the following largely self-explanatory diagram arising from that s.e.s; the two dotted arrows indicate the snake map. The resulting 6-term exact sequence of Γ -invariants and Γ -coinvariants, connected by the snake map in the middle, is well visible in this diagram and will be used later on. Here γ is a fixed generator of Γ .

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & T_\ell(C_\infty^T)^\Gamma & \longrightarrow & T_\ell(\mathcal{M})^\Gamma & \longrightarrow & \mathbb{Z}_\ell[\mathcal{S} \setminus \mathcal{S}_\ell]^{-,\Gamma} \dashrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & T_\ell(C_\infty^T) & \longrightarrow & T_\ell(\mathcal{M}) & \longrightarrow & \mathbb{Z}_\ell[\mathcal{S} \setminus \mathcal{S}_\ell]^- \longrightarrow 0 \\
 & & \downarrow 1-\gamma & & \downarrow 1-\gamma & & \downarrow 1-\gamma \\
 0 & \longrightarrow & T_\ell(C_\infty^T) & \longrightarrow & T_\ell(\mathcal{M}) & \longrightarrow & \mathbb{Z}_\ell[\mathcal{S} \setminus \mathcal{S}_\ell]^- \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dashrightarrow & & T_\ell(C_\infty^T)_\Gamma & \longrightarrow & T_\ell(\mathcal{M})_\Gamma & \longrightarrow & \mathbb{Z}_\ell[\mathcal{S} \setminus \mathcal{S}_\ell]^-_\Gamma \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

2. INVARIANTS

We begin by dealing with the Γ -invariants. This is a relatively easy task in light of a very concrete interpretation given to $T_\ell(\mathcal{M})$ in [GP2], §3. In this section and the next, we make two blanket assumptions:

- (1) S is ℓ -large, i.e. $C^T(K)$ is generated by the classes of primes in S .
- (2) All primes in S_ℓ are totally ramified in K_∞/K .

The second assumption will be eliminated in section 4 below.

PROPOSITION 2.1. *There is an isomorphism*

$$\varphi_\infty : T_\ell(\mathcal{M})^\Gamma \cong (\mathbb{Z}_\ell \otimes_{\mathbb{Z}} U_{S,T})^-.$$

PROOF: Recall from §3 of [GP2] that $T_\ell(\mathcal{M}) \cong \varprojlim_{\nu} \mathcal{M}[\ell^\nu]$ and that there are canonical module isomorphisms

$$\mathcal{M}[\ell^\nu] \cong \left(\mathcal{K}_{S,T}^{(\ell^\nu)} / \mathcal{K}_T^{\times \ell^\nu} \right)^-.$$

For simplicity, fix ν , denote $m := \ell^\nu$ and let $E_m := \mathcal{K}_{S,T}^{(m)} / \mathcal{K}_T^{\times m}$. Recall that

$$\mathcal{K}_T^\times := \{x \in \mathcal{K}^\times \mid x \equiv 1 \pmod{v}, \forall v \in \mathcal{T}\},$$

$$\mathcal{K}_{S,T}^{(m)} := \{x \in \mathcal{K}_T^\times \mid \operatorname{div}_{\mathcal{K}}(x) = mD + D'\},$$

where $\operatorname{div}_{\mathcal{K}}(x)$ denotes the non-archimedean \mathcal{K} -divisor of x and D' is a divisor supported at \mathcal{S} . In plainer terms $\mathcal{K}_{S,T}^{(m)}$ consists of those elements of \mathcal{K}_T^\times whose divisors are multiples of m away from \mathcal{S} .

(1) We claim that $E_m^\Gamma \cong (\mathcal{K}_{S,T}^{(m)})^\Gamma / (\mathcal{K}_T^{\times m})^\Gamma$, and that the denominator is simply $K_T^{\times m}$, where K_T^\times is defined as above, but at the K -level. Indeed, the second statement is clear (raising to the power m induces an isomorphism $\mathcal{K}_T^\times \cong \mathcal{K}_T^{\times m}$, just as in loc.cit., since there are no nontrivial ℓ -power roots of unity in \mathcal{K}_T^\times , due to our assumptions on T). For the first statement, we need the vanishing of $H^1(\Gamma, \mathcal{K}_T^{\times m})$. Again the exponent m can be omitted, due to the isomorphism above. The vanishing follows, very similarly as in loc.cit., from Hilbert 90 and weak approximation. The ingredient which makes this work is the fact that T is unramified in the extension \mathcal{K}/K .

(2) By the previous step we have $E_m^\Gamma \cong (\mathcal{K}_{S,T}^{(m)})^\Gamma / K_T^{\times m}$. Now, we establish a canonical isomorphism

$$\pi_m : (\mathcal{K}_{S,T}^{(m)})^\Gamma / K_T^{\times m} \cong U_{S,T} / U_{S,T}^m.$$

Take an element $x \in (\mathcal{K}_{S,T}^{(m)})^\Gamma \subseteq K_T^\times$. We have a unique writing $\text{div}_{\mathcal{K}}(x) = m\mathcal{D} + \mathcal{D}'$ where \mathcal{D} and \mathcal{D}' are \mathcal{K} -divisors with \mathcal{D}' supported on \mathcal{S} and \mathcal{D} supported away from \mathcal{S} . Since \mathcal{K}/K is unramified away from S and $x \in K_T^\times$, we also have $\text{div}_K(x) = mD + D'$ with K -divisors D and D' supported away from and on S , respectively. Using the first of our blanket hypotheses we get that $D = \text{div}_K(y) + D''$ with $y \in K_T^\times$ and D'' supported on S . Hence $\text{div}_K(xy^{-m}) = mD'' + D'$ is supported on S , and therefore $xy^{-m} \in U_{S,T}$. We let $\tilde{\pi}_m(x) := \widehat{xy^{-m}}$ (the class of xy^{-m} modulo $U_{S,T}^m$). It is easy to see that $\tilde{\pi}_m : (\mathcal{K}_{S,T}^{(m)})^\Gamma \rightarrow U_{S,T}/U_{S,T}^m$ is well defined and onto, and also easily checked that the kernel is exactly $K_T^{\times m}$. Therefore $\tilde{\pi}_m$ induces the desired isomorphism π_m .

(3) After a compatibility check for the π_m 's and passing to the projective limit, $\pi_\infty^- = \varprojlim_\nu \pi_{\ell^\nu}^-$ gives the desired isomorphism

$$T_\ell(\mathcal{M}) \cong \varprojlim_\nu E_{\ell^\nu}^- \cong (\mathbb{Z}_\ell \otimes_{\mathbb{Z}} U_{S,T})^-.$$

We may safely leave these details to the reader. Q.E.D.

3. COINVARIANTS

Now, we turn to the calculation of Γ -coinvariants of $T_\ell(\mathcal{M})$. We remind the reader that the assumptions (1) and (2), see beginning of Section 2, are in force. The desired isomorphism $T_\ell(\mathcal{M})_\Gamma \cong \mathbb{Z}_\ell[S]^-$ will result via a simple homological algebra lemma (Lemma 3.8) from Thm. 3.4 (ii) below, which yields a short exact sequence

$$0 \rightarrow \mathbb{Z}_\ell[S_\ell]^- \rightarrow T_\ell(\mathcal{M})_\Gamma \rightarrow \mathbb{Z}_\ell[S \setminus S_\ell]^- \rightarrow 0.$$

Unfortunately there does not seem to be a simple proof of the existence of this sequence. We begin with some notation and some fairly easy auxiliary results. Then we present the calculation of the coinvariants modulo three lemmas (one of which is highly technical), and finally we proceed to prove the lemmas.

Let K_n be the unique intermediate field of \mathcal{K}/K with $[K_n : K] = \ell^n$. Let $\Gamma_n = \text{Gal}(K_n/K)$ and let $\gamma_n \in \Gamma_n$ be the image of the generator γ of Γ via Galois restriction. Let d be the \mathbb{Z}_ℓ -rank of $\mathbb{Z}_\ell[S_\ell]^-$ (note that this is unchanged if S_ℓ is replaced by $S_\ell(K_n)$ or \mathcal{S}_ℓ due to our blanket hypotheses). We remind the reader that

$C^T(N)$ (respectively $C(N)$) is shorthand for the minus part of the ℓ -part of the ray class group $cl_T(N)$ (respectively class group $cl(N)$), for any appropriate field N . (Usually N is one of the fields K_n .) It is well known (see [GP2], for example) that the canonical maps $C^T(K_n) \rightarrow C^T(K_{n+1})$ and $C(K_n) \rightarrow C(K_{n+1})$ are injective and that $C_\infty^T = \bigcup_n C_T(K_n)$ and $C_\infty = \bigcup_n C(K_n)$.

For an appropriate field N , let $D^T(N) \subset C^T(N)$ be the subgroup generated by the classes of the prime ideals in N dividing ℓ . It is easy to see that

$$|\operatorname{Im}(D^T(K_n) \rightarrow C^T(K_n)/C^T(K))| \leq \ell^{nd}.$$

Note that it is legitimate to consider $C^T(K)$ as a subgroup of $C^T(K_n)$.

LEMMA 3.1. *The preceding inequality is an equality, that is:*

$$|\operatorname{Im}(D^T(K_n) \rightarrow C^T(K_n)/C^T(K))| = \ell^{nd}, \quad \text{for all } n.$$

PROOF: Let b_1, \dots, b_d be a \mathbb{Z}_ℓ -basis of $\mathbb{Z}_\ell[S_\ell(K_n)]^-$ where each b_i has the form $(1-j)\mathfrak{p}$ for some prime $\mathfrak{p}|\ell$ in K_n . (The letter j means complex conjugation of course; we have to take exactly those \mathfrak{p} that split from K^+ to K .) There is a map

$$\varphi_n^T : (\mathbb{Z}/\ell^n)^d \rightarrow C^T(K_n)/C^T(K)$$

sending the i -th basis vector e_i of the left-hand module to the class of b_i . It is well-defined since $\ell^n b_i$ comes from an ideal of K . The image of φ_n^T is equal to the image of $D^T(K_n)$ in $C^T(K_n)/C^T(K)$. We claim that φ_n^T is injective. For this it clearly suffices to show the injectivity of the analogously defined map

$$\varphi_n : (\mathbb{Z}/\ell^n)^d \rightarrow C(K_n)/C(K),$$

as φ_n factors through φ_n^T . Let $(m_1, \dots, m_d) \in \mathbb{Z}^d$ and assume that the class $[\prod_i b_i^{m_i}]$ in $C(K_n)$ is equal to $[\mathfrak{c}]$ where \mathfrak{c} is a fractional ideal in K . This means that there exists $x \in (K_n^\times \otimes \mathbb{Z}_\ell)^-$ such that $\operatorname{div}_{K_n}(x) = -\mathfrak{c} + \sum_i m_i \cdot b_i$. Then the divisor on the right is Γ_n -invariant, hence $x^{\gamma_n-1} \in (O_{K_n}^\times \otimes \mathbb{Z}_\ell)^- = \mu(K_n) \otimes \mathbb{Z}_\ell$. Since the module of roots of unity $\mu(K_n)$ is Γ_n -cohomologically trivial (well known fact), we may arrange that $x^{\gamma_n-1} = 1$, that is x is already in $K^\times \otimes \mathbb{Z}_\ell$. Then the divisor $\sum_i m_i \cdot b_i$ also comes from K , and this is only possible if all m_i are divisible by ℓ^n (remember that all

primes above ℓ in K_n are totally ramified in K_n/K). This shows that φ_n is injective as claimed. Q.E.D.

Recall that $C_\infty^T = \bigcup_n C^T(K_n)$. Define $D_\infty^T := \bigcup_n D^T(K_n)$.

LEMMA 3.2. (i) $(C_\infty^T)^\Gamma = C^T(K) \cdot D_\infty^T$, and D_∞^T is divisible. (ii) D_∞^T is the divisible part of $(C_\infty^T)^\Gamma$. (iii) We have $D_\infty^T \cap C^T(K) = D^T(K)$.

PROOF: (i) We start with the ‘‘ambiguous class number formula’’, both for K_n/K and for K_n^+/K^+ , see Lemma 13.4.1 in [La]. If we divide the former by the latter and note that the second factor in the denominator in loc.cit. just goes away in the minus part (again, cohomological triviality of roots of unity), we end up, after some comparison of notation, with the following:

$$|C(K_n)^{\Gamma_n}| = |C(K)| \cdot \ell^{nd}.$$

It is straightforward to deduce from this the following T -variant:

$$|C^T(K_n)^{\Gamma_n}| = |C^T(K)| \cdot \ell^{nd}.$$

Indeed, it suffices to prove that $\ker(C^T(K_n) \rightarrow C(K_n))$ is Γ_n -cohomologically trivial, and the Γ_n -invariants of this kernel identify naturally with $\ker(C^T(K) \rightarrow C(K))$. By taking the minus part of the exact sequence (4) tensored with \mathbb{Z}_ℓ (in the case $S = S_\infty$), one can see that

$$\begin{aligned} \ker(C^T(K_n) \rightarrow C(K_n)) &= \left(\frac{\kappa(T_{K_n})}{\mu_{K_n}} \otimes \mathbb{Z}_\ell\right)^-, \\ \ker(C^T(K) \rightarrow C(K)) &= \left(\frac{\kappa(T)}{\mu_K} \otimes \mathbb{Z}_\ell\right)^-. \end{aligned}$$

Since $\kappa(T_{K_n})$ is Γ_n -cohomologically trivial (apply Shapiro’s Lemma or see [GP2]) and so is $(\mu_{K_n} \otimes \mathbb{Z}_\ell)^-$, the above result follows immediately.

When combined with Lemma 3.1 (and its proof) the above equality implies that the natural map $D_T(K_n) \rightarrow C^T(K_n)^{\Gamma_n}/C^T(K)$ is bijective. Therefore we obtain

$$C^T(K_n)^{\Gamma_n} = C^T(K) \cdot D^T(K_n).$$

By passing to the inductive limit, we obtain

$$(C_\infty^T)^\Gamma = C^T(K) \cdot D_\infty^T.$$

This proves the equality in (i). Now, D_∞^T is divisible since all ℓ -adic primes are infinitely ramified in K_∞/K . Since $C^T(K)$ is finite, we

get (ii) at once. Part (iii) is proved using the method of proof of Lemma 3.1: any element of D_∞^T fixed by $\Gamma = \Gamma_0$ has to come from an ideal of K supported above ℓ . Q.E.D.

We now present the initial step towards calculating the coinvariants. We need one more object. Let $B^T(K)$ denote the quotient of $C^T(K)$ by the subgroup $D^T(K)$.

PROPOSITION 3.3. *There is an exact sequence*

$$0 \rightarrow B^T(K) \rightarrow T_\ell(C_\infty^T)_\Gamma \rightarrow T_\ell(\mathcal{M})_\Gamma \rightarrow \mathbb{Z}_\ell[S \setminus S_\ell]^- \rightarrow 0.$$

PROOF: We extract the following sequence from the diagram at the end of Section 1 (the second arrow is the snake map):

$$T_\ell(\mathcal{M})^\Gamma \rightarrow \mathbb{Z}_\ell[S \setminus S_\ell]^{-,\Gamma} \rightarrow T_\ell(C_\infty^T)_\Gamma \rightarrow T_\ell(\mathcal{M})_\Gamma \rightarrow \mathbb{Z}_\ell[S \setminus S_\ell]^- \rightarrow 0.$$

The second and last nontrivial terms are isomorphic to $\mathbb{Z}_\ell[S \setminus S_\ell]^-$ (as primes in $S \setminus S_\ell$ are not ramified in K_∞/K .) Going back to the proof of Proposition 2.1, one may verify the following: if we identify $T_\ell(\mathcal{M})^\Gamma$ with $(\mathbb{Z}_\ell \otimes U_{S,T})^-$ as in loc.cit, then the first arrow $T_\ell(\mathcal{M})^\Gamma \rightarrow \mathbb{Z}_\ell[S \setminus S_\ell]^-$ corresponds to the S_ℓ -forgetful divisor map $\text{div}_{K,S \setminus S_\ell}$ from $(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} U_{S,T})^-$ to $\mathbb{Z}_\ell[S \setminus S_\ell]^-$. Hence the cokernel of the first arrow of the above sequence agrees with the cokernel of $\text{div}_{K,S \setminus S_\ell}$; this gives exactly $B^T(K)$, by definition, because of our assumption that $C^T(K)$ is generated by S -ideal classes. The following commutative diagram, with surjective second column of vertical arrows captures what is going on.

$$\begin{array}{ccccccc} 0 & \longrightarrow & (U_{S_\ell,T} \otimes \mathbb{Z}_\ell)^- & \longrightarrow & (U_{S,T} \otimes \mathbb{Z}_\ell)^- & \longrightarrow & (U_{S,T}/U_{S_\ell,T} \otimes \mathbb{Z}_\ell)^- \longrightarrow 0 \\ & & \downarrow & & \downarrow & \searrow^{\text{div}_{K,S \setminus S_\ell}} & \downarrow \\ 0 & \longrightarrow & \mathbb{Z}_\ell[S_\ell]^- & \longrightarrow & \mathbb{Z}_\ell[S]^- & \longrightarrow & \mathbb{Z}_\ell[S \setminus S_\ell]^- \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & D^T(K) & \longrightarrow & C^T(K) & \longrightarrow & B^T(K) \longrightarrow 0 \end{array}$$

This produces the exact sequence in the statement of part (ii) of the lemma. Q.E.D.

Let $\alpha : B^T(K) \rightarrow T_\ell(C_\infty^T)_\Gamma$ denote the first map in the statement of the preceding proposition. We will determine the cokernel of this

map, and this will give the desired coinvariants. Let us state the result:

THEOREM 3.4. *The following hold true. (i) The cokernel of $\alpha : B^T(K) \rightarrow T_\ell(C_\infty^T)_\Gamma$ is isomorphic to $\mathbb{Z}_\ell[S_\ell]^-$. (ii) We have a short exact sequence*

$$0 \rightarrow \mathbb{Z}_\ell[S_\ell]^- \rightarrow T_\ell(\mathcal{M})_\Gamma \rightarrow \mathbb{Z}_\ell[S \setminus S_\ell]^- \rightarrow 0.$$

(iii) We have a $\mathbb{Z}_\ell[G]$ -module isomorphism $T_\ell(\mathcal{M})_\Gamma \cong \mathbb{Z}_\ell[S]^-$.

PROOF: Part (ii) is a direct consequence of Part (i) and Proposition 3.3. To prove part (i), we will need several lemmas. For simplicity, from this point on we will let $C := C_\infty^T$. We will state the lemmas, explain why they suffice to prove (ii) of the theorem, and then give the proofs of the lemmas. Then we will prove part (iii).

LEMMA 3.5. *There is an exact sequence*

$$0 \rightarrow C^\Gamma / (C^\Gamma)_{div} \rightarrow T_\ell(C)_\Gamma \rightarrow T_\ell(C_\Gamma) \rightarrow 0.$$

LEMMA 3.6. *The left-hand term $C^\Gamma / (C^\Gamma)_{div}$ in Lemma 3.5 is isomorphic to $B^T(K)$.*

LEMMA 3.7. *The right-hand term $T_\ell(C_\Gamma)$ in Lemma 3.5 is isomorphic to $\mathbb{Z}_\ell[S_\ell]^-$. In particular, it is torsion-free as a \mathbb{Z}_ℓ -module.*

Proof of Thm. 3.4(ii): From Lemmas 3.5 and 3.7 we see that the torsion part of $T_\ell(C)_\Gamma$ is exactly the image of the arrow $C^\Gamma / (C^\Gamma)_{div} \rightarrow T_\ell(C)_\Gamma$. Hence by Lemma 3.6 we infer that the torsion part of $T_\ell(C)_\Gamma$ is isomorphic to $B^T(K)$. Now this is exactly the domain of definition of the map α . Even if we do not know the (injective) map α , we thus obtain that its cokernel identifies with the quotient of $T_\ell(C)_\Gamma$ modulo its torsion. Using the isomorphism of Lemma 3.7, we may conclude that the cokernel of α is isomorphic to $\mathbb{Z}_\ell[S]^-$ as claimed. This concludes the proof of Thm. 3.4(i) and (ii) pending the proofs of the lemmas. Q.E.D.

We now give the proof of the three lemmas in turn, the third one being by far the most complex one. We tried to find a simpler argument, without success.

PROOF of Lemma 3.5: Recall that our hypothesis that Iwasawa μ -invariant associated to K and ℓ vanishes implies that C is divisible. (See [GP2] for details.) Therefore the short exact sequence of divisible groups

$$0 \rightarrow C/C^\Gamma \xrightarrow{1-\gamma} C \rightarrow C_\Gamma \rightarrow 0$$

produces a short exact sequence of ℓ -adic Tate modules

$$0 \rightarrow T_\ell(C/C^\Gamma) \xrightarrow{1-\gamma} T_\ell(C) \rightarrow T_\ell(C_\Gamma) \rightarrow 0.$$

Furthermore, noting that $T_\ell(C)^\Gamma = T_\ell(C^\Gamma)$, there is a canonical s.e.s.

$$0 \rightarrow T_\ell(C)/T_\ell(C)^\Gamma \rightarrow T_\ell(C/C^\Gamma) \rightarrow C^\Gamma/(C^\Gamma)_{div} \rightarrow 0.$$

A diagram chase based on the two s.e.s.'s above then produces the desired s.e.s.

$$0 \rightarrow C^\Gamma/(C^\Gamma)_{div} \rightarrow T_\ell(C)_\Gamma \rightarrow T_\ell(C_\Gamma) \rightarrow 0.$$

Q.E.D.

PROOF of Lemma 3.6: We need to calculate the quotient of C^Γ by its maximal divisible subgroup. The latter is, by Lemma 3.2 (ii), equal to D_∞^T . Hence

$$\begin{aligned} C^\Gamma/(C^\Gamma)_{div} &= C^T(K)D_\infty^T/D_\infty^T \cong C^T(K)/C^T(K) \cap D_\infty^T = \\ &= C^T(K)/D^T(K) = B^T(K). \end{aligned}$$

We used Lemma 3.2 (i) and (iii). Q.E.D.

PROOF of Lemma 3.7: We have to calculate the module $T_\ell(C_\Gamma)$. As already mentioned, this is the most delicate part. We rely on Kurihara's paper [Ku], in particular on its Prop. 5.2, which is proved using Lemma 5.1 of that paper. We apply this to the Γ_n -extension K_n/K , and we note that we may omit the μ -term at the left of the sequence in Prop. 5.2. Kurihara's notation for the field extension is L/K ; and we may omit the μ -term since it comes from a H^1 of the (-1) -eigenspace of global units (first term in second line of the long sequence in Lemma 5.1), so we may invoke cohomological triviality of roots of unity again. Since all inertia groups of K_n/K at primes

$v|\ell$ are the whole of Γ_n , the mentioned Proposition of [Ku] gives the s.e.s.

$$0 \rightarrow \left(\bigoplus_{v \in S_\ell} \Gamma_n\right)^- \rightarrow C(K_n)_{\Gamma_n} \rightarrow C(K) \rightarrow 0.$$

We will now establish that the following variant also holds:

$$0 \rightarrow \left(\bigoplus_{v \in S_\ell} \Gamma_n\right)^- \rightarrow C^T(K_n)_{\Gamma_n} \rightarrow C^T(K) \rightarrow 0.$$

where the surjection is induced by the norm map at the level of ray class groups. Indeed, since this norm map is onto, its kernel is isomorphic to $\widehat{H}^{-1}(\Gamma_n, C^T(K_n))$; on the other hand the term $\Gamma_n \cong \widehat{H}^0(\Gamma_n, O_{K_{n,v}}^\times) \cong \widehat{H}^0(\Gamma_n, K_{n,v}^\times)$, via the local Artin map. (Keep in mind that $K_{n,v}/K_v$ is totally ramified.) Consequently, we obtain an isomorphism

$$(5) \quad \widehat{H}^{-1}(\Gamma_n, C^T(K_n)) \cong \left(\bigoplus_{v \in S_\ell} \widehat{H}^0(\Gamma_n, K_{n,v}^\times)\right)^-$$

(see Kurihara's argument). This isomorphism will be needed below. We denote the norm map from $C^T(K_n)$ to $C^T(K)$ by π_n . Now, we need to pass to an inductive limit. To this end, we look at the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widehat{H}^{-1}(\Gamma_n, C^T(K_n)) & \longrightarrow & C^T(K_n)_\Gamma & \xrightarrow{\pi_n} & C^T(K) \\ & & \downarrow j_{n,n+1} & & \downarrow & & \downarrow \cdot \ell \\ 0 & \longrightarrow & \widehat{H}^{-1}(\Gamma_{n+1}, C^T(K_{n+1})) & \longrightarrow & C^T(K_{n+1})_\Gamma & \xrightarrow{\pi_{n+1}} & C^T(K). \end{array}$$

Here the transition map $j_{n,n+1}$ has a direct and simple definition: it is induced by the inclusion map $C^T(K_n) \rightarrow C^T(K_{n+1})$ and the usual description of \widehat{H}^{-1} as the kernel of the norm modulo the multiples of $(1 - \sigma)$, with σ a generator of the cyclic group in question. As $C^T(K)$ is finite, the inductive limit gives an isomorphism

$$C_\Gamma \cong \varinjlim \widehat{H}^{-1}(\Gamma_n, C^T(K_n)),$$

where the limit is taken along the maps $j_{n,n+1}$. Now (5) leads to an isomorphism

$$C_\Gamma \cong \varinjlim \widehat{H}^0(\Gamma_n, K_{n,v}^\times)$$

where the inductive limit is taken along certain canonical maps

$$i_{n,n+1} : \widehat{H}^0(\Gamma_n, K_{n,v}^\times) \rightarrow \widehat{H}^0(\Gamma_{n+1}, K_{n+1,v}^\times).$$

An easy direct calculation reveals that $i_{n,n+1}$ is given by multiplication with the relative norm element $\nu_{n+1,n} := N_{G(K_{n+1}/K_n)}$. But in our case the action of this element is the same as multiplication (or more properly, exponentiation) by ℓ . Therefore we have a commutative diagram:

$$\begin{array}{ccc} \widehat{H}^0(\Gamma_n, K_{n,v}^\times) & \longrightarrow & \Gamma_n \\ \nu_{n+1,n} = \ell \downarrow & & \downarrow \ell \\ \widehat{H}^0(\Gamma_{n+1}, K_{n+1,v}^\times) & \longrightarrow & \Gamma_{n+1} \end{array}$$

where the horizontal maps are local Artin maps. So we find that

$$C_\Gamma \cong \varinjlim_{v \in S_\ell} \left(\bigoplus \Gamma_n \right)^-,$$

where the transition maps are multiplication by ℓ . The choice of a generator for Γ identifies the above injective limit with $(\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell[S_\ell]^-$. This proves, by applying the functor T_ℓ , that

$$T_\ell(C_\Gamma) \cong T_\ell((\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell[S_\ell]^-) \cong \mathbb{Z}_\ell[S_\ell]^-,$$

which concludes the proof of Lemma 3.7. Q.E.D. As mentioned earlier, the preceding series of arguments finishes the proof of Theorem 3.4 parts (i) and (ii).

PROOF of Thm. 3.4 part (iii): This will follow immediately if we can prove that the short exact sequence in Thm. 3.4(ii) is split. Luckily, this is indeed the case, as shown by the next lemma.

LEMMA 3.8. *Let G be any finite group and U and V two subgroups of G . Then the Ext group $\text{Ext}_{\mathbb{Z}[G]}^1(\mathbb{Z}[G/U], \mathbb{Z}[G/V])$ vanishes. (Our modules are left modules, so G/U denotes the set of left cosets xU , with the obvious G -action.) Consequently $\text{Ext}_{\mathbb{Z}[G]}^1(M, N)$ vanishes for any two permutation G -modules M and N , and this holds as well if the base ring \mathbb{Z} is replaced by \mathbb{Z}_ℓ .*

PROOF: First step: note that for any subgroup $W < G$ we have $H^1(G, \mathbb{Z}[G/W]) = 0$. This is a well-known consequence of Shapiro's Lemma and the fact $H^1(W, \mathbb{Z}) = 0$. Consequently, we also have $H^1(G, P) = 0$ for every permutation module P over G .

Second step: By a standard homological algebra argument, the Ext group in question is isomorphic to $H^1(G, \text{Hom}(\mathbb{Z}[G/U], \mathbb{Z}[G/V]))$. Note that Hom is taken over \mathbb{Z} , and G acts on it in the usual way. Finally, we claim that $P(U, V) := \text{Hom}(\mathbb{Z}[G/U], \mathbb{Z}[G/V])$ is a permutation module over G . Then by the two previous steps, we will be done. There is a canonical \mathbb{Z} -basis of $P(U, V)$, consisting of homomorphisms $e(x, y)$, defined as follows. For any choice of $x \in G/U$ and $y \in G/V$, declare that $e(x, y)$ maps x to y , and maps every other $x' \in G/U$ to zero. It is now easy to check that the G -action permutes this basis. Actually one can easily determine the stabilizer of each $e(x, y)$, but this does not matter; it suffices to know that $P(U, V)$ is a permutation module. Q.E.D.

The preceding Lemma applies in particular to the permutation modules $N = \mathbb{Z}_\ell[S_\ell]^-$ and $M = \mathbb{Z}_\ell[S \setminus S_\ell]^-$: the exact sequence in Thm. 3.4 (ii) is split, and the module in the middle is therefore isomorphic to $\mathbb{Z}_\ell[S]^-$. Q.E.D.

REMARK 3.9. *Let us remark that we do not quite get an explicit isomorphism between $T_\ell(\mathcal{M})_\Gamma$ and $\mathbb{Z}_\ell[S]^-$. It is explicit up to a splitting of an exact sequence, which exists but is not unique. Unfortunately, although the final theorem in the next section does also imply, as a corollary, that $T_\ell(\mathcal{M})_\Gamma$ is indeed isomorphic to $\mathbb{Z}_\ell[S]^-$, since the ℓ -adified Tate sequence in the minus part has exactly $\mathbb{Z}_\ell[S]^-$ on the right, that isomorphism is much less explicit.*

4. REMOVING A TECHNICAL ASSUMPTION

In this short section we explain how to eliminate condition (2) (see Section 2) in the end results of the preceding two sections. (Condition (1) is built into the theory of Tate sequences and therefore indispensable.) The idea is the same for invariants and for coinvariants. One chooses n_0 large enough so that condition (2) holds for K_∞/K_{n_0} and puts $\Gamma_0 = \text{Gal}(K_\infty/K_{n_0})$. If we replace K by K_{n_0} in the results Prop. 2.1 and Thm. 3.4 (ii) (jointly with Lemma 3.8, we obtain descriptions of $T_\ell(\mathcal{M}^{\Gamma_0})$ and $T_\ell(M_{\Gamma_0})$; the isomorphisms in these descriptions are invariant under $G' := G \times (\Gamma/\Gamma_0)$. We then

perform a final (co)descent, taking invariants (resp. coinvariants) under the action Γ/Γ_0 . For the invariants everything is clear: the Γ/Γ_0 -invariants of $(U_{S,T}(K_{n_0}) \otimes \mathbb{Z}_\ell)^-$ coincide with $(U_{S,T}(K) \otimes \mathbb{Z}_\ell)^-$. For the coinvariants, it is also easy to check that $\mathbb{Z}_\ell[S(K_{n_0})]_{\Gamma/\Gamma_0}^-$ is isomorphic to $\mathbb{Z}_\ell[S(K)]^-$. The resulting isomorphism at level K is, of course, not quite explicit, since the isomorphism at level K_{n_0} , coming about through Lemma 3.8, was not totally explicit.

5. THE LINK WITH TATE’S CANONICAL CLASS

We now consider the Tate canonical class $\tau := \tau_{K/k,S} \in \text{Ext}_{\mathbb{Z}[G]}^2(X_S, U_S)$ introduced in Remark 1.2. We retain all our working hypotheses as well as notations introduced in Section 1. In particular, S is assumed large which means that the S -classes generate $cl_T(K)$, and consequently $cl(K)$. Tate proved (see [Ta2], Ch. 5, §2) that there exists a Yoneda 2-extension of $\mathbb{Z}[G]$ -modules (not unique and not canonical)

$$(6) \quad 0 \rightarrow U_S \rightarrow A \rightarrow B \rightarrow X_S \rightarrow 0$$

which represents τ and such that A and B are finitely generated and of finite projective dimension over $\mathbb{Z}[G]$ (i.e. cohomologically trivial or c.t. over G .) Such a Yoneda extension is called a Tate sequence. As mentioned before, we do not review the defining properties of τ here. The reader can consult [Ta1] and [Ta2] for details. It is our goal now to link τ with $T_\ell(\mathcal{M})$. For this, one has to ℓ -adify, T -modify and take the minus part of τ , as explained in Section 1. Next, we follow [BF] and [Bu1] and interpret the ℓ -adification $(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \tau)$ of τ as the isomorphism class (in a sense to be made precise below) of the complex $[A_\ell \rightarrow B_\ell]$ in the derived category $D^{\text{perf}}(\mathbb{Z}_\ell[G])$ of perfect cochain complexes of $\mathbb{Z}_\ell[G]$ -modules, where $A_\ell := A \otimes \mathbb{Z}_\ell$ and $B_\ell := B \otimes \mathbb{Z}_\ell$ are viewed in degrees 0 and 1 respectively.

Let C^\bullet be a complex in the derived category $D(\mathbb{Z}_\ell[G])$ (or $D(\mathbb{Z}[G])$) with differential maps $(\partial^s)_{s \in \mathbb{Z}}$ and some $i \in \mathbb{Z}$ such that

$$(7) \quad H^j(C^\bullet) = 0, \text{ for all } j \neq i, i + 1.$$

Then one can associate to C^\bullet the (correctly) truncated complex

$$\tau_{\geq i}(\tau_{\leq i+1} C^\bullet) : [C^i / \text{im } \partial^{i-1} \xrightarrow{\partial^i} \ker \partial^{i+1}]$$

concentrated in degrees i and $i + 1$, with the same cohomology as C^\bullet . This truncated complex leads to the canonical exact sequence

$$0 \rightarrow H^i(C^\bullet) \rightarrow (\tau_{\geq i}(\tau_{\leq i+1}C^\bullet))^i \rightarrow (\tau_{\geq i}(\tau_{\leq i+1}C^\bullet))^{i+1} \rightarrow H^{i+1}(C^\bullet) \rightarrow 0,$$

which determines a Yoneda extension class

$$e(C^\bullet) \in \text{Ext}_{\mathbb{Z}_\ell[G]}^2(H^{i+1}(C^\bullet), H^i(C^\bullet))$$

(or in $\text{Ext}_{\mathbb{Z}[G]}^2$) canonically associated to C^\bullet .

LEMMA 5.1 (Burns-Flach, [BF]). *Let $i \in \mathbb{Z}$ and C^\bullet and D^\bullet complexes in $D(\mathbb{Z}_\ell[G])$ satisfying (7). Assume that we are given isomorphisms at the level of cohomology*

$$\alpha_i : H^i(C^\bullet) \xrightarrow{\sim} H^i(D^\bullet), \quad \alpha_{i+1} : H^{i+1}(C^\bullet) \xrightarrow{\sim} H^{i+1}(D^\bullet).$$

Then there exists an isomorphism $\alpha : C^\bullet \cong D^\bullet$ in $D(\mathbb{Z}_\ell[G])$ such that $H^i(\alpha) = \alpha_i$ and $H^{i+1}(\alpha) = \alpha_{i+1}$ if and only if

$$(\alpha_{i+1}^{-1})^* \circ (\alpha_i)_*(e(C^\bullet)) = e(D^\bullet),$$

where $(\alpha_{i+1}^{-1})^ \circ (\alpha_i)_*$ is the canonical isomorphism*

$$\text{Ext}_{\mathbb{Z}_\ell[G]}^2(H^{i+1}(C^\bullet), H^i(C^\bullet)) \xrightarrow{\sim} \text{Ext}_{\mathbb{Z}_\ell[G]}^2(H^{i+1}(D^\bullet), H^i(D^\bullet))$$

induced by α_i and α_{i+1} .

PROOF: See [BF], page 1353 or work out your own proof from the definitions. Q.E.D.

REMARK 5.2. *Note that for any complex C^\bullet in $D(\mathbb{Z}_\ell[G])$ satisfying (7) for some $i \in \mathbb{Z}$ there exists an isomorphism in $D(\mathbb{Z}_\ell[G])$*

$$C^\bullet \cong \tau_{\geq i}(\tau_{\leq i+1}C^\bullet),$$

inducing the identity maps at the level of cohomology. So $e(C^\bullet) = e(\tau_{\geq i}(\tau_{\leq i+1}C^\bullet))$. Most importantly, note that, by definition, any two Tate sequences

$$0 \rightarrow U_S \xrightarrow{u} A \xrightarrow{f} B \xrightarrow{x} X_S \rightarrow 0,$$

$$0 \rightarrow U_S \xrightarrow{u'} A' \xrightarrow{f'} B' \xrightarrow{x'} X_S \rightarrow 0$$

give perfect complexes in $D^{perf}(\mathbb{Z}_\ell[G])$ in levels 0 and 1

$$C^\bullet : [A_\ell \xrightarrow{f} B_\ell], \quad C'^\bullet : [A'_\ell \xrightarrow{f'} B'_\ell]$$

and isomorphisms at the level of cohomology (induced by u, u' and x, x' , respectively)

$$H^0(C^\bullet) \cong U_S \otimes \mathbb{Z}_\ell \cong H^0(C'^\bullet), \quad H^1(C^\bullet) \cong U_S \otimes \mathbb{Z}_\ell \cong H^1(C'^\bullet)$$

which map the class $e(C^\bullet)$ to $e(C'^\bullet)$. Therefore, we have an isomorphism $C^\bullet \cong C'^\bullet$ in $D^{perf}(\mathbb{Z}_\ell[G])$ which induces the above isomorphisms at the level of cohomology.

From now on we will denote by $(\tau \otimes \mathbb{Z}_\ell)$ (respectively $(\tau \otimes \mathbb{Z}_\ell)^-$) the complex $C^\bullet : [A_\ell \xrightarrow{f} B_\ell]$ (respectively $(C^\bullet)^- : [A_\ell^- \xrightarrow{f} B_\ell^-]$) associated to a Tate sequence (6) as in the above remark. According to the above remark these complexes are unique up to isomorphisms in $D(\mathbb{Z}_\ell[G])$.

We will consider the affine schemes

$$X := \mathrm{Spec}(O_K) \setminus S = \mathrm{Spec}(O_{K,S}),$$

$$\mathcal{X} := \mathrm{Spec}(O_{\mathcal{K}}) \setminus \mathcal{S} = \mathrm{Spec}(O_{\mathcal{K},\mathcal{S}}).$$

We will let $j : \mathcal{T} \rightarrow \mathcal{X}$ and $i : \mathcal{X} \setminus \mathcal{T} \rightarrow \mathcal{X}$ be the usual closed and open immersion, respectively. When confusion is unlikely, we will use the same notation $j : T \rightarrow X$ and $i : X \setminus T \rightarrow X$ for the corresponding immersions at the finite level. From now on all cohomology is viewed in the étale sense, so in particular $R\Gamma(X, *) := R\Gamma(X_{et}, *)$, $R\Gamma_c(X, *) := R\Gamma_c(X_{et}, *)$ and similarly for the scheme \mathcal{X} .

PROPOSITION 5.3. *There is an isomorphism in $D^{perf}(\mathbb{Z}_\ell[G])$*

$$(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \tau)[-1] \cong R\Gamma(X, \mathbb{Z}_\ell(1)).$$

(The $[-1]$ -shift on the left produces a complex with cohomology concentrated in degrees 1 and 2.)

PROOF: This is a fairly short argument. All the same, it is not very direct, since it uses the full strength of the key paper [BF]. Unexplained notation is taken literally from there; all references in the present proof are to this paper, if not said otherwise. According to the last line of p.1383, the complex Ψ_S represents Tate's class τ (see the definition of K_S , p.1351 and p.1353). By Prop. 3.3, we have an isomorphism in the derived category (of course one also has

to check, using the explicit information given in loc.cit. eqn.(69) that it gives the canonical maps on cohomology):

$$\mathbb{Z}_\ell \otimes \tau \cong R\Gamma_c(X, \mathbb{Z}_\ell)^*[-2].$$

The superscript star stands for $R\text{Hom}(-, \mathbb{Z}_\ell)$ (a functor of the derived category to itself). Now we invoke Lemma 16(b), which gives

$$R\Gamma_c(X, \mathbb{Z}_\ell)^* \cong R\Gamma_c(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)^\vee,$$

where the superscript \vee is $R\text{Hom}(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$. In contrast to the functor $()^*$, the functor \vee can be evaluated on *any* complex in a quasi-isomorphism class, termwise, since $\text{Hom}(-, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$ is exact. As a third and last ingredient, we invoke Artin-Verdier duality:

$$R\Gamma_c(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell)^\vee[-3] \cong R\Gamma(X, \mathbb{Z}_\ell(1)).$$

Again one has to make sure that the two preceding isomorphisms are canonical on cohomology level. Putting the three displayed isomorphisms together (the first shifted by -1 , and the second by -3), we obtain the formula of the proposition. Note: We have been following the sign conventions of [BF]. It appears that in the terminology of [Bu1], a minus sign would come up. Q. E. D.

Another important step is a description of $T_\ell(\mathcal{M})$ in terms of étale cohomology. The following result will be important in the sequel; we would like to mention that it was proved independently in Malte Witte's habilitation thesis, see [Wt] Prop. 6.3.2.

THEOREM 5.4. *We have a canonical isomorphism*

$$T_\ell(\mathcal{M}) \cong H^1(\mathcal{X}, j_!\mathbb{Z}_\ell(1))^-.$$

PROOF: We will actually prove $\mathcal{M}[m] \cong H^1(\mathcal{X}, j_!\mathbb{Z}/m(1))^-$ for $m := \ell^\nu$ and all $\nu \geq 1$. The isomorphisms will be compatible and produce the desired result in the projective limit. We again resort to the description given in [GP2]:

$$\mathcal{M}[m] \cong (\mathcal{K}_{S,T}^{(m)}/\mathcal{K}_T^{\times m})^-.$$

(See the proof of Proposition 2.1 and the notations therein.)

PROPOSITION 5.5. *For any fixed m as above, there is a natural isomorphism*

$$\varphi = \varphi_m : \mathcal{K}_{S,T}^{(m)}/\mathcal{K}_T^{\times m} \xrightarrow{\sim} H^1(\mathcal{X}, j_!\mathbb{Z}/m(1)).$$

PROOF: This will take several steps. Most of the underlying ideas are from [De], see Section 10.3.6 in particular, but the mathematical language in loc.cit. is so different that we prefer to give a reasonably self-contained argument. To make the main points more clearly visible, we will first prove a simplified version: replace T by the empty set. (In particular, $j_! \mathbb{Z}/m(1)$ just becomes $\mathbb{Z}/m(1)$.) Then there is an explicit geometric interpretation of $H^1(\mathcal{X}, \mathbb{Z}/m(1))$: it is canonically isomorphic to the group D_m of equivalence classes of pairs (\mathcal{L}, α) , where \mathcal{L} is a projective rank one module over $O_{\mathcal{K}, \mathcal{S}}$ (in other words a line bundle over \mathcal{X}), and

$$\alpha : \mathcal{L}^{\otimes m} \rightarrow O_{\mathcal{X}}$$

is an isomorphism. The equivalence relation is as expected: $(\mathcal{L}, \alpha) \sim (\mathcal{L}', \alpha')$ iff there is an isomorphism $h : \mathcal{L} \rightarrow \mathcal{L}'$ with $\alpha' \circ h^{\otimes m} = \alpha$. The group structure is obvious. The relation between D_m and $H^1(\mathcal{X}, \mathbb{Z}/m(1))$ can be easily seen in the light of Grothendieck's descent theory; the automorphism group of the trivial element $(O_{\mathcal{X}}, 1)$ of D_m is $\mathbb{Z}/m(1)$, just as the automorphism group of the trivial (or any) line bundle is \mathbb{G}_m . The isomorphism φ may now be constructed directly. Given $f \in \mathcal{K}_{\mathcal{S}}^{(m)}$, we know that the principal $O_{\mathcal{K}}$ -ideal generated by f is an m -th power away from \mathcal{S} , so the sheaf $fO_{\mathcal{X}}$ is the m -th power of a unique ideal sheaf \mathcal{I} . We let $\varphi(\widehat{f})$ be the class of the pair (\mathcal{I}, f^{-1}) in D_m . There are two things to check: The kernel of φ is precisely $\mathcal{K}^{\times m}$, and φ is surjective. Both are straightforward. This settles the case where T is replaced by the empty set.

Now we put T and \mathcal{T} back in. (This is the part where our terminology and that in [De] differ the most.) We define a modified group $D_m^{\mathcal{T}}$. Its elements are equivalence classes of triples $(\mathcal{L}, \alpha, \beta)$, where \mathcal{L} and α are as before and β is defined as follows. We let $\kappa(\mathcal{T}) := \bigoplus_{v \in \mathcal{T}} \kappa(v)$, where $\kappa(v)$ is the residue field at v , as usual. Now β , a so-called trivialization at \mathcal{T} , is an isomorphism

$$\beta : \kappa(\mathcal{T}) \otimes_{O_{\mathcal{X}}} \mathcal{L} \xrightarrow{\sim} \kappa(\mathcal{T}),$$

which has to be compatible with α in the obvious way:

$$id_{\kappa(\mathcal{T})} \otimes \alpha = \beta^{\otimes m}.$$

Two triples $(\mathcal{L}, \alpha, \beta)$ and $(\mathcal{L}', \alpha', \beta')$ as above are equivalent if there is an isomorphism $h : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ such that

$$\alpha' \circ h^{\otimes m} = \alpha, \quad \beta' = \beta \circ (id_{\kappa(\mathcal{T})} \otimes_{O_{\mathcal{X}}} h).$$

The above argument carries over directly to produce a canonical isomorphism between the groups $\mathcal{K}_{S,T}^{(m)}/\mathcal{K}_T^{\times m}$ and $D_m^{\mathcal{T}}$. It remains to identify $D_m^{\mathcal{T}}$ with étale cohomology. We feel this should be known, and it certainly can be extracted from [De] with some effort. Let us give a direct argument anyway, via Čech cohomology. Using that $m = \ell^\nu$ is invertible in $O_{\mathcal{X}}$, one easily obtains that every element of $D_m^{\mathcal{T}}$ is trivialized by some étale covering $(U_i)_i$ of \mathcal{X} . We may suppose that all U_i connected. The resulting transition maps over $U_i \cap U_j$ are on the one hand sections of $\mathbb{Z}/m(1)$ (as we said, this is the automorphism sheaf of the trivial element of D_m), but because of the trivialisation at \mathcal{T} they are all trivial whenever $U_i \cap U_j$ has a point above \mathcal{T} . This produces therefore a 1-cocycle over the sheaf $j_! \mathbb{Z}/m(1)$ relative to the covering, and hence a canonical map $\delta^{\mathcal{T}}$ from $D_m^{\mathcal{T}}$ to the first Čech cohomology of that sheaf. Since Čech cohomology embeds into étale cohomology, $\delta^{\mathcal{T}}$ gives a morphism $D_m^{\mathcal{T}} \rightarrow H^1(\mathcal{X}, j_! \mathbb{Z}/m(1))$. The analogous map with \mathcal{T} empty is an isomorphism. One has a commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \frac{\mathbb{Z}/m(1)(\kappa(\mathcal{T}))}{\mathbb{Z}/m(1)(\mathcal{K})} & \longrightarrow & D_m^{\mathcal{T}} & \longrightarrow & D_m \longrightarrow 1 \\ & & \downarrow & & \downarrow \delta^{\mathcal{T}} & & \downarrow \delta \\ 1 & \longrightarrow & \frac{\mathbb{Z}/m(1)(\kappa(\mathcal{T}))}{\mathbb{Z}/m(1)(\mathcal{K})} & \longrightarrow & H^1(\mathcal{X}, j_! \mathbb{Z}/m(1)) & \longrightarrow & H^1(\mathcal{X}, \mathbb{Z}/m(1)) \longrightarrow 1. \end{array}$$

The top sequence comes from a standard s.e.s., cf. [GPff]. One can check directly that the leftmost vertical map is the identity. Since δ is an isomorphism, $\delta^{\mathcal{T}}$ is an isomorphism as well. This proves the proposition. Q.E.D.

Now, the proposition above together with the above mentioned identification of $\mathcal{M}[m]$ with $\left(\mathcal{K}_{S,T}^{(m)}/\mathcal{K}_T^{\times m}\right)^-$ and a passage to the projective limit, gives a proof of Theorem 5.4. Q.E.D.

In order to use the results 5.4 and 5.3 towards our goal of identifying the Tate class in terms of $T_\ell(\mathcal{M})$ we need some intermediate lemmas. All previous notation remains in place.

LEMMA 5.6. *The sheaf $j_! \mathbb{Z}_\ell(1)$ on \mathcal{X} has cohomology concentrated in degree 1.*

PROOF: To show this, one first looks at the cohomology of the ℓ -adic étale sheaf $\mathbb{Z}_\ell(1)$ on \mathcal{X} .

(1) $H^0(\mathcal{X}, \mathbb{Z}_\ell(1)) = \varprojlim \mu_{\ell^n}(\mathcal{K}) \cong \mathbb{Z}_\ell(1)$ or 0 if $\mu_\ell \subseteq \mathcal{K}^\times$ or not. (2) $H^2(\mathcal{X}, \mathbb{Z}_\ell(1)) = 0$. Indeed, if one writes the cohomology sequence attached to the Kummer sequence of étale sheaves on \mathcal{X}

$$0 \rightarrow \mathbb{Z}/\ell^m(1) \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 0,$$

and takes into account that $H^1(\mathcal{X}, \mathbb{G}_m)_\ell = \text{Pic}(O_{\mathcal{K}, \mathcal{S}})_\ell$ which is divisible under our working hypothesis that the μ -invariant of K and ℓ vanishes, one concludes that

$$H^2(\mathcal{X}, \mathbb{Z}/\ell^m(1)) \cong \text{Br}(\mathcal{X})[\ell^m],$$

for every m . (Cautionary remark about notation: The cohomology on the left is not meant as a projective limit of the analogous cohomology groups over $\text{Spec}(O_{K_n})$, as done by a few authors, but literally, taking \mathcal{X} as a scheme in its own right.) However, $\text{Br}(\mathcal{X})[\ell^m] = 0$ for all m : if A is a central simple algebra over \mathcal{K} split outside \mathcal{S} and killed by ℓ^m (i.e. a representative of an element in $\text{Br}(\mathcal{X})[\ell^m]$) then it is defined over some K_n and therefore split by $K_{n'}$, for n' sufficiently large. (If n' is sufficiently large, the extension $K_{n'}/K_n$ has local degree divisible by ℓ^m at all primes in $S(K_n) \setminus S_\infty$ and therefore the algebra A is split by $K_{n'}$ locally everywhere and therefore splits globally.) Passing to the limit gives the claimed vanishing. Now we use the closed immersion $i : \mathcal{T} \rightarrow \mathcal{X}$ and the open immersion $j : \mathcal{X} \setminus \mathcal{T} \rightarrow \mathcal{X}$ and look at the standard exact sequence of sheaves on \mathcal{X}

$$0 \rightarrow j_! \mathbb{Z}_\ell(1) \rightarrow \mathbb{Z}_\ell(1) \rightarrow i_* \mathbb{Z}_\ell(1) \rightarrow 0.$$

The long exact sequence in cohomology reads as follows.

$$\begin{aligned} 0 \rightarrow H^0(\mathcal{X}, j_! \mathbb{Z}_\ell(1)) \rightarrow H^0(\mathcal{X}, \mathbb{Z}_\ell(1)) \xrightarrow{\rho} H^0(\mathcal{X}, i_* \mathbb{Z}_\ell(1)) \rightarrow \\ \rightarrow H^1(\mathcal{X}, j_! \mathbb{Z}_\ell(1)) \rightarrow H^1(\mathcal{X}, \mathbb{Z}_\ell(1)) \rightarrow H^1(\mathcal{X}, i_* \mathbb{Z}_\ell(1)) \rightarrow \\ \rightarrow H^2(\mathcal{X}, j_! \mathbb{Z}_\ell(1)) \rightarrow H^2(\mathcal{X}, \mathbb{Z}_\ell(1)) = 0. \end{aligned}$$

Now, the map ρ is a diagonal embedding and therefore injective, as

$$H^0(\mathcal{X}, i_* \mathbb{Z}_\ell(1)) \cong \bigoplus_{v \in \mathcal{T}} H^0(\kappa(v), \mathbb{Z}_\ell(1)) \cong \bigoplus_{v \in \mathcal{T}} \varprojlim \mu_{\ell^n}(\kappa(v))$$

and (under our working assumption on T) no roots of unity in \mathcal{K} are congruent to 1 mod v for all $v \in \mathcal{T}$. Consequently, $H^0(\mathcal{X}, j_! \mathbb{Z}_\ell(1)) = 0$. Recall that \mathcal{T} is a finite set of closed points on \mathcal{X} , so the natural map

$$H^1(\mathcal{X}, i_* \mathbb{Z}_\ell(1)) \rightarrow H^1(\mathcal{T}, \mathbb{Z}_\ell(1))$$

is an isomorphism. Since \mathcal{T} is a finite union of spectra of fields of char. $\neq \ell$ without algebraic extensions of ℓ -power degree, $H^1(\mathcal{T}, \mathbb{Z}_\ell(1)) = 0$. This implies (via the long exact sequence above) that $H^2(\mathcal{X}, j_! \mathbb{Z}_\ell(1)) = 0$, which concludes the proof. Q. E. D.

For the purpose of the next results, we remind the reader that we are working under the hypothesis that S is large (i.e. $cl_T(K)$ is generated by S -ideal classes.) It is likewise important to point out that the T -modified Tate sequence as in part (2) below was also obtained with different methods by Burns-Kurihara-Sano for abelian extensions of number fields, see [BKS]. However, for CM fields our approach on the minus part appears to be more explicit.

LEMMA 5.7. (1) *The inclusion $\iota : U_{S,T} \rightarrow U_S$ induces a canonical isomorphism*

$$\iota_* : \text{Ext}_{\mathbb{Z}[G]}^2(X_S, U_{S,T}) \cong \text{Ext}_{\mathbb{Z}[G]}^2(X_S, U_S).$$

(2) *The unique class $\tau' := \tau_{K/k,S,T}$ in $\text{Ext}_{\mathbb{Z}[G]}^2(X_S, U_{S,T})$ satisfying $\iota_*(\tau') = \tau$ admits a representative*

$$0 \rightarrow U_{S,T} \rightarrow A' \rightarrow B' \rightarrow X_S \rightarrow 0$$

with A' and B' finitely generated and c.t. over G . (Any such representative will be called a T -modified Tate sequence.)

(3) *The pushout along ι of any T -modified Tate sequence is a Tate sequence.*

PROOF: (1) Recall the exact sequence (4) in Section 1 and let $Z := U_S/U_{S,T}$. Since S is large, $Z \cong \kappa(T)$ as $\mathbb{Z}[G]$ -modules. It is easily seen (see [GP2]) that $\text{pd}_{\mathbb{Z}[G]}(\kappa(T)) = 1$. Therefore Z is c.t. over G . By a routine argument, we get that $\text{Ext}_{\mathbb{Z}[G]}^i(N, Z) = 0$ for all $i > 0$, all G -modules N without \mathbb{Z} -torsion, and all Z that are c.t. over G . This shows that the inclusion $\iota : U_{S,T} \rightarrow U_S$ induces an isomorphism

$$\iota_* : \text{Ext}_{\mathbb{Z}[G]}^2(X_S, U_{S,T}) \cong \text{Ext}_{\mathbb{Z}[G]}^2(X_S, U_S).$$

(2) Now, since X_S is free of \mathbb{Z} -torsion, there is a canonical commutative diagram

$$\begin{CD} H^2(G, \text{Hom}(X_S, U_{S,T})) @>\sim>> \widehat{\text{Ext}}_{\mathbb{Z}[G]}^2(X_S, U_{S,T}) \\ @V \iota_* VV @VV \iota_* V \\ H^2(G, \text{Hom}(X_S, U_S)) @>\sim>> \widehat{\text{Ext}}_{\mathbb{Z}[G]}^2(X_S, U_S) \end{CD}$$

Let $\alpha \in H^2(G, \text{Hom}(X_S, U_S))$ be the preimage of τ via the bottom isomorphism. Then Tate showed (see [Ta2], Ch. II, §5) that the cup product with α induces isomorphisms $\widehat{H}^i(G, X_S) \cong \widehat{H}^{i+2}(G, U_S)$, for all i . Consequently, the cup product with α' (the preimage of α under ι_*) induces similar isomorphisms $\widehat{H}^i(G, X_S) \cong \widehat{H}^{i+2}(G, U_{S,T})$. Now, this is sufficient for the argument in [Ta2] pp. 56-57 (right before Remark 5.3 in loc.cit.) to produce a representative for τ' as required in part (2) of the Lemma. One important note here is that since $U_{S,T}$ has no \mathbb{Z} -torsion (unlike U_S), A' and B' can be picked to be projective, finitely generated $\mathbb{Z}[G]$ -modules. (3) By definition, the push-out along ι of a T -modified Tate sequence as in (2) is a representative of τ . It is of the form $0 \rightarrow U_S \rightarrow A \rightarrow B \rightarrow X_S \rightarrow 0$ with $B' = B$, hence f.g. and c.t. over G and A' part of an exact sequence $0 \rightarrow A' \rightarrow A \rightarrow \kappa(T) \rightarrow 0$, hence c.t. and f.g. over G . We obtain this way a Tate sequence. Q. E. D.

LEMMA 5.8. *We have the following variant of Prop. 5.3:*

$$(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \tau')[-1] \cong R\Gamma(X, j_! \mathbb{Z}_\ell(1)).$$

Here we have abusively used j to indicate the open immersion $j : X \setminus T \rightarrow X$ at the finite level as well.

PROOF: Let $\xi : j_! \mathbb{Z}_\ell(1) \rightarrow \mathbb{Z}_\ell(1)$ denote the canonical inclusion of sheaves. Using the arguments in lemmas 5.6 and 5.7 one checks easily that $H^1(X, \xi)$ is injective with cokernel $U_S/U_{S,T} \cong H^1(X, i^* \mathbb{Z}_\ell(1))$, and $H^2(X, \xi)$ is an isomorphism. It is then clear that the ℓ -adic étale sheaf $j_! \mathbb{Z}_\ell(1)$ of X has cohomology concentrated in degrees 1 and 2 as well, so we can think of $R\Gamma(X, j_! \mathbb{Z}_\ell(1))$ in terms of Yoneda 2-extensions. Let C^\bullet be a complex concentrated in degrees 1 and 2 isomorphic in $D(\mathbb{Z}_\ell[G])$ to $R\Gamma(X, j_! \mathbb{Z}_\ell(1))$. (Take for example the correct truncation of the latter complex.) There

is a map f of complexes from C^\bullet to some complex D^\bullet which represents $R\Gamma(X, \mathbb{Z}_\ell(1))$, such that f induces $H^\bullet(X, \iota)$ on cohomology. In particular it gives the inclusion $U_{S,T} \rightarrow U_S$ on H^1 , and an isomorphism on H^2 . Let $\iota_* C^\bullet = C'^\bullet$ be the complex given by pushing out:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}_\ell \otimes U_{S,T} & \longrightarrow & C^1 & \longrightarrow & C^2 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathbb{Z}_\ell \otimes U_S & \longrightarrow & (C')^1 & \longrightarrow & (C')^2. \end{array}$$

Note that $C'^2 = C^2$. Then f extends to a map of complexes f' from C'^\bullet to D^\bullet , just by the universal property of the pushout. One verifies that f' is now identity on H^1 , and nothing has changed on H^2 , so f' is an isomorphism and actually induces an equivalence. (See reminder before 5.3.) Hence $\iota_* C^\bullet$ represents $R\Gamma(X, \mathbb{Z}_\ell(1))$, and this agrees with $\mathbb{Z}_\ell \otimes \tau$ by Prop. 5.3. Since τ' is the inverse image of τ under ι_* , we conclude by Lemma 5.7 that C^\bullet agrees with $\mathbb{Z}_\ell \otimes \tau'$. Q.E.D.

With these preparations, we can state and prove the main result of this section. Recall that $M := T_\ell(\mathcal{M})$ and let C^\bullet be the complex $[M \xrightarrow{1-\gamma} M]$ concentrated in degrees 0 and 1. From this point on, for all $i \in \mathbb{Z}$ we let $M[i]$ denote the complex having terms M in degree $(-i)$, 0 in all other degrees, and (obviously) all differentials zero.

THEOREM 5.9. *If the assumption at the beginning of Lemma 5.7 is satisfied, then there is a canonical isomorphism in $D(\mathbb{Z}_\ell[G])$*

$$(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \tau')^- \cong C^\bullet.$$

PROOF: By Theorem 5.4 and Lemma 5.6 we have a canonical (therefore Γ -equivariant) isomorphism in the derived category $D(\mathbb{Z}_\ell[G])$

$$M[-1] \cong R\Gamma(\mathcal{X}, j_! \mathbb{Z}_\ell(1))^-.$$

We now descend from \mathcal{X} to X . From [Bu1], diagram (8) on p.371 plus comment (see also definition of $C(\theta)^\bullet$ on p.366 of loc. cit.) we get a canonical isomorphism in the derived category $D(\mathbb{Z}_\ell[G])$

$$(8) \quad R\Gamma(X, j_! \mathbb{Z}_\ell(1))^- \cong C^\bullet[-1].$$

Three comments are necessary in order to derive this isomorphism from loc. cit. (1) To link up with the notation in [Bu1], note that the (-1) shift of the mapping cone of the map of complexes $1 - \gamma : M[-1] \rightarrow M[-1]$ (which is the precise definition of Burns' $C(\theta)^\bullet$ in our context) is exactly the complex $C^\bullet[-1]$. (2) We also remark that [Bu1] is concerned with the function field case where there is a canonical choice for γ , to wit Frobenius. But actually the isomorphism class (in the derived sense) of the complex $C^\bullet : [M \xrightarrow{1-\gamma} M]$ does not change when γ is replaced by any other generator of Γ , so the lack of a canonical generator of Γ is not an issue. (3) The rest of the argument taken from [Bu1] is entirely cohomological algebra, so there is no difference between the function field and number field cases in this respect.

Finally, we combine (8) with Lemma 5.8 to get the isomorphism in the statement of the above theorem. Q. E. D.

REMARK 5.10. (1) If one weakens the assumption at the beginning of Lemma 5.7 to say that just $C^T(K)$ (the minus- ℓ -part of $cl(K)$) is S -generated (which is even closer to Hypothesis (1) in Section 2), then an ℓ -adic version of that lemma remains correct, as well as a version of the preceding theorem, in which the modified Tate class τ' only exists as an ℓ -adic object. (2) With notations as in the proof of Lemma 5.8, one also has $(\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \tau)^- \cong (C'^\bullet)^-$. This can be seen as an “explicit Tate sequence”. The complex C'^\bullet arises from M by a very simple and explicit construction involving pushout along ι . For further reference, here is the relevant diagram (basically taken from the proof of Lemma 5.8; we also put in the cokernels on the right for clarity); M' is defined as the pushout, and $(C'^\bullet)^-$ is simply the complex $[M' \rightarrow M]$ that shows up in the lower row.

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & (\mathbb{Z}_\ell \otimes U_{S,T})^- & \longrightarrow & M & \xrightarrow{1-\gamma} & M & \longrightarrow & \mathbb{Z}_\ell[S]^- & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \parallel & & \parallel & & \\
 0 & \longrightarrow & (\mathbb{Z}_\ell \otimes U_S)^- & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & \mathbb{Z}_\ell[S]^- & \longrightarrow & 0.
 \end{array}$$

Note that in order to really work with $[M' \rightarrow M]$, one needs a good grasp on the maps $(\mathbb{Z}_\ell \otimes U_{S,T})^- \rightarrow M$ and $M \rightarrow \mathbb{Z}_\ell[S]^-$. This is another justification, apart from their intrinsic interest, for the explicit calculations in Sections 2 and 3.

REMARK 5.11. *Finally, we would like to indicate briefly how the linear disjointness condition $k_\infty \cap K = k$ can be removed in all of the above considerations. In the case where this condition is not satisfied, $T_\ell(\mathcal{M}_{S,T}(K_\infty))$ does not have a natural $\mathbb{Z}_\ell[G]$ -module structure. Indeed, in this case $G(K_\infty/k) \cong H \rtimes \Gamma$, where $H := G(K/k_\infty \cap K)$ and $\Gamma := G(k_\infty/k)$, so $T_\ell(\mathcal{M}_{S,T}(K_\infty))$ is naturally endowed with a $\mathbb{Z}_\ell[H]$ -module structure only, and it is projective over this ring (see [GP2]). Consequently $T_\ell(\mathcal{M}_{S,T}(K_\infty)) \otimes_{\mathbb{Z}_\ell[H]} \mathbb{Z}_\ell[G]$ is a projective $\mathbb{Z}_\ell[G]$ -module. It is easily seen (see [GP2]) that this is in fact isomorphic to the ℓ -adic realization of the abstract ℓ -adic 1-motive associated to the semisimple k -algebra $K \otimes_k k_\infty$ and the sets S and T , i.e. we have a natural isomorphism of $\mathbb{Z}_\ell[[G \rtimes \Gamma]]$ -modules*

$$T_\ell(\mathcal{M}_{S,T}(K \otimes_k k_\infty)) \cong T_\ell(\mathcal{M}_{S,T}(K_\infty)) \otimes_{\mathbb{Z}_\ell[H]} \mathbb{Z}_\ell[G].$$

All of the above considerations can be easily generalized to show that the complex

$$[T_\ell(\mathcal{M}_{S,T}(K \otimes_k k_\infty)) \xrightarrow{1-\gamma} T_\ell(\mathcal{M}_{S,T}(K \otimes_k k_\infty))]$$

concentrated at levels 0 and 1 represents the (minus ℓ -adic) Tate class and gives an explicit (minus ℓ -adic) Tate sequence just as above.

6. EXAMPLES

We would like to end by giving a couple of concrete examples, illustrating the fact that, unlike the newly constructed Iwasawa modules $M := T_\ell(\mathcal{M})$, the classical Iwasawa modules $M_0 := T_\ell(C_\infty)$ are not $\mathbb{Z}_\ell[G]$ -projective in general and therefore a similar construction of Tate sequences in terms of the latter is hopeless.

More concretely, let us start with the following observation: if G' is an abelian ℓ -group, then the \mathbb{Z}_ℓ -rank of any projective (and therefore free) module over $\mathbb{Z}_\ell[G']$ is a multiple of the order of G' . Since ℓ is odd, the classical λ -invariant $\lambda_{\ell,K}$ of the CM field K splits as $\lambda_{\ell,K} = \lambda_{\ell,K}^- + \lambda_{\ell,K}^+$, where $\lambda_{\ell,K}^- := \text{rk}_{\mathbb{Z}_\ell} M_0$.

We choose two settings which are as simple as possible and compute $\text{rk}_{\mathbb{Z}_\ell} M_0$, showing that the results preclude the possibility of $\mathbb{Z}_\ell[G]$ -projectivity, based on the above observation. For illustrative purposes, we also compute $\text{rk}_{\mathbb{Z}_\ell}(M)$ as well as $\text{rk}_{\mathbb{Z}_\ell}(M_\Gamma)$ and $\text{rk}_{\mathbb{Z}_\ell}(M^\Gamma)$ in both settings. In what follows $\text{rk}(\ast) = \text{rk}_{\mathbb{Z}_\ell}(\ast)$.

Let $k = \mathbb{Q}$ and let K^+ be the cubic field of conductor 7. We take $\ell = 3$. For K , we will look at two choices: $K = K^+L$ where the imaginary quadratic field L is either $\mathbb{Q}(\sqrt{-5})$ or $\mathbb{Q}(\sqrt{-37})$. In both cases K is CM, and of course k is totally real. For S we consistently take the set of ramified primes in K/k together with the 3-adic primes; the contribution of the places over 5 (resp. 37) and of the infinite places disappears in the minus part. For T we take the set of primes in K sitting over a completely split prime in K/k . In both cases, 7 is split in L and ramified in K/L . Moreover $\zeta_3 := e^{2\pi i/3}$ is not contained in K_∞ . Hence the “toric part”, that is, the kernel of $T_\ell(C_\infty^T) \rightarrow T_\ell(C_\infty)$, is a copy of the free module $\mathbb{Z}_\ell[G']$, where $G' = \text{Gal}(K/L) = \text{Gal}(K^+/\mathbb{Q})$. The rank of the lattice $\mathbb{Z}_\ell[S \setminus S_\ell]^-$ is 1 in both cases. In both cases we have:

$$\text{rk}(M) = \lambda_{3,K}^- + 4,$$

where the constant 4 comes about as $3 + 1$ (3 for the toric part and 1 for the lattice part.) Since M is free over $\mathbb{Z}_\ell[G']$ (by Theorem 1.1 (1)), this already tells us that $\lambda_{3,K}^- \equiv 2$ modulo 3. Of course this can also be seen from the Kida’s formula, which says in this case that

$$\lambda_{3,K}^- = 3\lambda_{3,L} + 2.$$

First case: $K = K^+(\sqrt{-5})$. Here 3 is split in L , and by [DFKS], $\lambda_{3,L} = \lambda_{3,L}^- = 1$. Hence $\lambda_{3,K}^- = 5$ and the $\text{rk}(M) = 9$. Both the Γ -invariants and coinvariants of M give a rank 2 module with trivial G' -action. The conclusion is that M_0 is not projective over $\mathbb{Z}_\ell[G']$. Second case: $K = K^+(\sqrt{-37})$. Here 3 is inert in L , and therefore $\lambda_{3,L} = \lambda_{3,L}^- = 0$. Hence $\lambda_{3,K}^- = 2$ and $\text{rk}(M) = 6$. Both the Γ -invariants and coinvariants of M give a rank 1 module with trivial G' -action. Once again, M_0 fails to be projective over $\mathbb{Z}_\ell[G']$.

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