Dimension Theory of the Moduli Space of Twisted k-Differentials

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Received: March 3, 2017
Revised: February 2, 2018

Communicated by Gavril Farkas

Abstract. In this note we extend the dimension theory for the spaces $\widetilde{\mathcal{H}}_g^k(\mu)$ of twisted k-differentials defined by Farkas and Pandharipande in [FP18] to the case $k > 1$. In particular, we show that the intersection $\mathcal{H}_g^k(\mu) = \widetilde{\mathcal{H}}_g^k(\mu) \cap \mathcal{M}_{g,n}$ is a union of smooth components of the expected dimensions for all $k \geq 0$. We also extend a conjectural formula from [FP18] for a weighted fundamental class of $\widetilde{\mathcal{H}}_g^k(\mu)$ and provide evidence in low genus. If true, this conjecture gives a recursive way to compute the cycle class $[\overline{\mathcal{H}}_g^k(\mu)]$ of the closure of $\mathcal{H}_g^k(\mu)$ for $k \geq 1, \mu$ arbitrary.

2010 Mathematics Subject Classification: 14H10, 30F30
Keywords and Phrases: Strata of k-differentials, deformation theory, tautological classes, double ramification cycles

1 Introduction

In [FP18], Farkas and Pandharipande define the moduli space $\widetilde{\mathcal{H}}_g^k(\mu)$ of twisted k-differentials associated to an integer partition $\mu = (m_1, \ldots, m_n)$ of $k(2g-2)$. It is a closed substack $\widetilde{\mathcal{H}}_g^k(\mu) \subset \overline{\mathcal{M}}_{g,n}$, whose interior points $\mathcal{H}_g^k(\mu) = \widetilde{\mathcal{H}}_g^k(\mu) \cap \mathcal{M}_{g,n}$ parametrize curves $(C, p_1, \ldots, p_n)$ such that there exists a meromorphic section s_0 of $\omega_C^{\otimes k}$ with zeroes and poles at $p_1, \ldots, p_n$ with orders prescribed by μ. That is $\mathrm{div}(s_0) = \sum_{i=1}^n m_i p_i$ or equivalently

$$\mathcal{O}_C\left(\sum_{i=1}^n m_i p_i\right) = \omega_C^{\otimes k}.$$

[1]Supported by the the grant SNF-200020162928.

On the boundary of $\overline{\mathcal{M}}_{g,n}$, this definition is modified by allowing to introduce twists $I(q,D')=-I(q,D'')\in\mathbb{Z}$ at nodes q of C where two distinct components D',D'' of C meet. These twists need to satisfy certain combinatorial conditions. Let $\nu: C_I\to C$ be the partial normalization of C resolving the collection N_I of all nodes q of C with nonzero twist $I(q,D'_q)$ and let q',q'' be the preimages of q contained in the preimages of D'_q,D''_q. Then $(C,p_1,\ldots,p_n)$ is called *k-twisted canonical* and is contained in $\widetilde{\mathcal{H}}_g^k(\mu)$ iff we have an equality of line bundles

$$\nu^*\mathcal{O}_C\left(\sum_{i=1}^n m_i p_i\right)=\nu^*\omega_C^{\otimes k}\otimes\mathcal{O}_{C_I}\left(\sum_{q\in N_I} I(q,D'_q)q'+I(q,D''_q)q''\right)$$

on C_I. For details see [FP18, Section 0 and Definition 20]. See also [Gué16] for an approach to k-twisted canonical divisors via log geometry.
For $k=1$, the dimension theory of $\widetilde{\mathcal{H}}_g^k(\mu)$ has been studied in [FP18] (see also Remark 1.3).

- In the holomorphic case, i.e. if all $m_i\geq 0$, the components of the closure $\overline{\mathcal{H}}_g^1(\mu)$ of $\mathcal{H}_g^1(\mu)$ have dimension $2g-2+n$ and all other components of $\widetilde{\mathcal{H}}_g^1(\mu)$ have dimension $2g-3+n$.
- In the strictly meromorphic case, i.e. if there exists some $m_i<0$, all components of $\widetilde{\mathcal{H}}_g^1(\mu)$ have dimension $2g-3+n$.

In the case $k=0$, the space $\widetilde{\mathcal{H}}_g^k(\mu)$ has components of various dimensions, all at least $2g-3+n$.
Extending the result for $k=1$ above, we prove the following.

Theorem 1.1. For $k\geq 1$, all components of $\widetilde{\mathcal{H}}_g^k(\mu)$ are of dimension $2g-3+n$, except in the case that all parts of μ are nonnegative and divisible by k. Then the sublocus

$$\overline{\mathcal{H}}_g^1\left(\frac{1}{k}\mu\right)\subset\widetilde{\mathcal{H}}_g^k(\mu)$$

is a union of irreducible components of dimension $2g-2+n$ and all other components of $\widetilde{\mathcal{H}}_g^k(\mu)$ have dimension $2g-3+n$.

This was conjectured in [FP18]. We are going to see that the crucial point in the argument is the dimension theory of the open locus $\mathcal{H}_g^k(\mu)$.

Proposition 1.2. Let $k\geq 0$, $(C,p)\in\mathcal{H}_g^k(\mu)$ and let s_0 be a meromorphic section of $\omega_C^{\otimes k}$ with $\mathrm{div}(s_0)=\sum_{i=1}^n m_i p_i$. Consider the condition

all m_i are nonnegative and divisible by k and there exists a meromorphic section $\tilde{s}_0$ of ω_C such that $s_0=\tilde{s}_0^{\otimes k}$. (*)

Then

$$\dim\ T_{(C,p)}\mathcal{H}_g^k(\mu)=2g-2+n,$$

if (*) is satisfied and

$$\dim\ T_{(C,p)}\mathcal{H}_g^k(\mu) = 2g - 3 + n$$

otherwise. Here we require that if $k = 0$, the partition μ should not be the trivial partition $\mu = (0, \ldots, 0)$ (otherwise $\mathcal{H}_g^0(\mu) = \mathcal{M}_{g,n}$).
In particular, all components of $\mathcal{H}_g^k(\mu)$ are smooth of the dimensions determined above.

Remark 1.3. The dimension of moduli spaces of k-differentials has previously been studied in a number of contexts.
Using methods from analytic and flat geometry, Veech computed it for quadratic differentials with at worst simple poles (including the distinction when the differential is a square of an abelian differential), see [Vee86, equation (0.9) and Remarks 28.16, 28.17]. Veech also has a result for rational partitions μ of $2g-2$, which, by clearing denomiators, can be interpreted as a result for any k and poles of order at most $k-1$, see [Vee93, Theorem 0.3]. The case of strictly meromorphic abelian differentials has been observed by Boissy [Boi15, Lemma 3.5].
On the other hand, using deformation theory the case of holomorphic abelian differentials has been treated by Polishchuk [Pol06], Möller [Mö8, Theorem 2.3] and Mondello [Mon17, Proposition 2.8]. Finally, the case of general k and possibly meromorphic partitions can also be treated using deformation theory. Apart from our proof below, there exists an unpublished argument by G. Mondello (G. Mondello, personal communication with R. Pandharipande, November 2015) and the paper [BCG$^+$b] by Bainbridge, Chen, Gendron, Grushevsky and Möller, which appeared around the same time as the present work. All three proofs first reduce the claim to the computation of the cokernel of a map of first cohomology groups of sheaves on C induced by a sheaf map. While Mondello and the paper [BCG$^+$b] use Serre duality to reduce to a computation in H^0, we analyze kernel and cokernel of the map of sheaves directly and then use long exact sequences. The deformation theoretic part of our argument below follows closely the proof of Polishchuk in [Pol06].
In [BCG$^+$a] the question if a given twisted differential $(C, p_1, \ldots, p_n) \in \widetilde{\mathcal{H}}_g^k(\mu)$ is actually contained in the closure $\overline{\mathcal{H}}_g^k(\mu)$ is answered in terms of orders and residue conditions on the dual graph of C in the case $k = 1$. The corresponding question for $k > 1$ is answered in [BCG$^+$b]. We do not address this question in the following.

In Section 3 we present a generalized version of a conjecture from [FP18], which relates a fundamental class of $\widetilde{\mathcal{H}}_g^k(\mu)$ with explicit weights on the components inside the boundary to a tautological class described by Pixton in [Pix14]. Here we must distinguish cases.

- If μ has an entry which is negative or not divisible by k, all components of $\widetilde{\mathcal{H}}_g^k(\mu)$ are of codimension g and we obtain a straightforward generalization of the conjectural formula in [FP18] (see Conjecture A).

- If $\mu = k\mu'$ for μ' nonnegative, we replace the codimension $g-1$ contribution $\overline{\mathcal{H}}_g^1(\mu')$ of this formula by a codimension g virtual fundamental class $[\overline{\mathcal{H}}_g^1(\mu')]^{\mathrm{vir}}$. This virtual cycle is obtained by inserting the parameters $k=1$ and μ' in the formula of Conjecture A (see Conjecture A′).

We check both conjectures for genera $g = 0, 1$ and for each conjecture two nontrivial cases in genus $g = 2$.

Assuming Conjecture A, we show how to determine the class $[\overline{\mathcal{H}}_g^k(\mu)]$ of the closure of $\mathcal{H}_g^k(\mu)$ for $k > 1$ in the case where $\mathcal{H}_g^k(\mu)$ has pure codimension g, i.e. for $\mu \neq k\mu'$ with μ' nonnegative. On the other hand, if μ is nonnegative and divisible by k, the fundamental class $[\overline{\mathcal{H}}_g^k(\mu)]$ has a codimension $g-1$ part given by $[\overline{\mathcal{H}}_g^1(\mu/k)]$ (determined by the original Conjecture A as described in [FP18]) and Conjecture A′ allows to compute its codimension g part.

In the appendix of the paper, we give an elementary argument, explained to us by Dimitri Zvonkine, showing that an untwisted node of a curve $(C, p) \in \widetilde{\mathcal{H}}_g^k(\mu)$ can be smoothed inside $\widetilde{\mathcal{H}}_g^k(\mu)$. This technical result is needed to identify the boundary components of $\widetilde{\mathcal{H}}_g^k(\mu)$ in terms of their dual graphs.

Acknowledgements

I would like to thank my advisor Rahul Pandharipande for introducing me to the topic of twisted differentials and for his guidance throughout the writing of this paper. I am grateful to Dimitri Zvonkine for explaining the proof in the appendix and to Aaron Pixton for verifying the tautological relations appearing in the examples in Section 3.3.2. I also want to thank Felix Janda for very helpful discussions and advice and for his comments on a preliminary version of the paper.

Finally I want to thank the anonymous referee for their very detailed and helpful comments and for pointing out many important references in the literature.

2 Dimension estimates

Let $g, n, k \geq 0$ and assume $2g - 2 + n > 0$. Let $\mu = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ be a partition of $k(2g-2)$.

For the proof of Theorem 1.1, many of the dimension estimates for the case $k = 1$ in [FP18] carry over verbatim to the case of general k. By [FP18, Theorem 21] we know that all components of $\widetilde{\mathcal{H}}_g^k(\mu)$ have dimension at least $2g - 3 + n$.

Now assuming Proposition 1.2, we know that the dimension of $\mathcal{H}_g^k(\mu)$ is bounded above by $2g-2+n$ and in the meromorphic case it is even bounded by $2g - 3 + n$. Using this, the proof of [FP18, Proposition 7] shows the following.

Corollary 2.1. For $k \geq 1$, every irreducible component Z of $\widetilde{\mathcal{H}}_g^k(\mu)$ supported in the boundary has dimension at most $2g-3+n$. Here equality can be achieved only if the dual graph of a generic element $(C,p) \in Z$ is a star-graph (see Section 3.1).

Remark 2.2. Note that for $k = 0$, the proof of [FP18, Proposition 7] breaks down because for $\mu = (0,\dots,0)$ the codimension of $\mathcal{H}_g^0(\mu)$ is 0 and not g.

We see that Corollary 2.1 covers all components of $\widetilde{\mathcal{H}}_g^k(\mu)$ supported in the boundary and Proposition 1.2 treats the components in the interior $\mathcal{M}_{g,n}$. Here we note that the $(C,p) \in \mathcal{H}_g^1(\frac{1}{k}\mu)$ are exactly the points of $\mathcal{H}_g^k(\mu)$ such that the corresponding meromorphic section s_0 has a k-th root in ω_C. This finishes the proof of Theorem 1.1, assuming Proposition 1.2.

To show the Proposition, we adapt the original dimension estimate from [Pol06] by Polishchuk. Let $\mathcal{C} \to \mathcal{M}_{g,n}$ be the universal curve over $\mathcal{M}_{g,n}$ and let $\mathcal{J}^d$ be the relative Jacobian of degree d over $\mathcal{M}_{g,n}$ for $d \in \mathbb{Z}$. Then $\mathcal{H}_g^k(\mu)$ can be defined as a fibre product involving the following morphisms:

- $\sigma_k^\mu : \mathcal{M}_{g,n} \to \mathcal{J}^{k(2g-2)}$ sending (C,p) to $(C,p,\mathcal{O}_C(\sum_{i=1}^n m_i p_i))$,
- $c : \mathcal{M}_{g,n} \to \mathcal{J}^{2g-2}$ sending (C,p) to (C,p,ω_C),
- $\psi_k : \mathcal{J}^{2g-2} \to \mathcal{J}^{k(2g-2)}$ sending (C,p,L) to $(C,p,L^{\otimes k})$.

Then indeed the diagram

$$\begin{array}{ccc} \mathcal{H}_g^k(\mu) & \longrightarrow & \mathcal{M}_{g,n} \\ \downarrow & & \downarrow \sigma_k^\mu \\ \mathcal{M}_{g,n} \xrightarrow{c} \mathcal{J}^{2g-2} & \xrightarrow{\psi_k} & \mathcal{J}^{k(2g-2)} \end{array} \tag{1}$$

is cartesian.

Let $(C,p) \in \mathcal{H}_g^k(\mu)$ and let s_0 be a meromorphic section of $\omega_C^{\otimes k}$ with $\operatorname{div}(s_0) = \sum_{i=1}^n m_i p_i$. The cartesian diagram above implies that the tangent space of $\mathcal{H}_g^k(\mu)$ at (C,p) is the kernel of the map

$$T_{(C,p)}\mathcal{M}_{g,n} \oplus T_{(C,p)}\mathcal{M}_{g,n} \xrightarrow{(d\sigma_k^\mu, -d(\psi_k \circ c))} T_{(C,p,\omega_C^{\otimes k})}\mathcal{J}^{k(2g-2)}. \tag{2}$$

Let K be the dimension of the cokernel of this map, then

$$\begin{aligned} \dim\ T_{(C,p)}\mathcal{H}_g^k(\mu) &= 2(3g-3+n) - (3g-3+n+g) + K \\ &= 2g-3+n+K. \end{aligned}$$

Thus we need to show that $K = 1$ if (*) holds and $K = 0$ otherwise.

As in Polishchuk's paper, we can explicitly identify the above tangent spaces and the morphisms between them. We have

$$\begin{aligned} T_{(C,p)}\mathcal{M}_{g,n} &= H^1(C, \mathcal{T}_C(-p_1 - \ldots - p_n)), \\ T_{(C,p,L)}\mathcal{J}^d &= H^1(C, A_{L,p}), \end{aligned}$$

for $d \in \mathbb{Z}$ and L a line bundle on C of degree d. Here $A_{L,p}$ is the sheaf of differential operators $L \to L$ of order ≤ 1 with vanishing principal symbol at $p_1, \ldots, p_n$. In the following, let $E = p_1 + \ldots + p_n$.

Lemma 2.3. The tangent map of the morphism

$$\mathcal{J}^0 \to \mathcal{J}^{k(2g-2)}, \quad (C, p, L) \mapsto \left(C, p, L\left(\sum_i m_i p_i\right)\right)$$

at $(C, p, \mathcal{O}_C)$ is the map $H^1(C, A_{\mathcal{O}_C,p}) \to H^1(C, A_{\omega_C^{\otimes k},p})$ induced by the isomorphism of sheaves

$$A_{\mathcal{O}_C,p} \cong A_{\omega_C^{\otimes k},p}$$

sending an operator $\partial' : \mathcal{O}_C \to \mathcal{O}_C$ to the operator

$$\partial : \omega_C^{\otimes k} \to \omega_C^{\otimes k}, \quad s \mapsto s_0 \partial'\left(\frac{s}{s_0}\right).$$

Proof. See the proof of [Pol06, Lemma 2.2]. We note that for a pole p_i of s_0 of order $|m_i|$ (and s regular around p_i), we have $\mathrm{ord}_{p_i} \partial'(\frac{s}{s_0}) \geq |m_i|$ as the symbol of ∂' vanishes at p_i. Thus the expression $s_0\partial'(\frac{s}{s_0})$ is again regular around p_i. □

Lemma 2.4. For L a line bundle on C of degree d, the tangent map

$$d\psi_k : H^1(C, A_{L,p}) \to H^1(C, A_{L^{\otimes k},p})$$

of $\psi_k : \mathcal{J}^d \to \mathcal{J}^{dk}$ at (C, p, L) is induced by the sheaf map $\Psi : A_{L,p} \to A_{L^{\otimes k},p}$ sending an operator $\partial : L \to L$ to

$$\partial \otimes \mathrm{id}^{\otimes k-1} + \mathrm{id} \otimes \partial \otimes \mathrm{id}^{\otimes k-2} + \ldots + \mathrm{id}^{\otimes k-1} \otimes \partial : L^{\otimes k} \to L^{\otimes k}.$$

Proof. Assume we are given a first order deformation

$$\begin{array}{ccc} C & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2)) \end{array}$$

of (C, p) and a line bundle $\mathcal{L}$ on $\mathcal{C}$ deforming L. Then for an affine cover $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ of C trivializing $\mathcal{L}$ and $U_{\alpha\beta} = U_\alpha \cap U_\beta$ we have a cycle $(d_{\alpha\beta} \in \mathcal{Z}^1(U_{\alpha\beta}, \mathcal{T}_C(-E)))$ describing the deformation of (C, p). If $f_{\alpha\beta}$ are the transition functions of L, the transition functions of $\mathcal{L}$ have the form $F_{\alpha\beta} = f_{\alpha\beta} + \epsilon g_{\alpha\beta}$. Then under the identification of first-order deformations of (C, p, L) with $H^1(C, A_{L,p})$ this data corresponds to the 1-cycle

$$\left(\partial_{\alpha\beta} = d_{\alpha\beta} + \frac{g_{\alpha\beta}}{f_{\alpha\beta}} \in A_{L,p}(U_{\alpha\beta})\right).$$

Under the map ψ_k, the data $d_{\alpha\beta}$ of the deformation of (C,p) remains unchanged, but the transition functions of $\mathcal{L}^{\otimes k}$ are now

$$F_{\alpha\beta}^k = f_{\alpha\beta}^k + k\epsilon g_{\alpha\beta} f_{\alpha\beta}^{k-1}.$$

On the other hand, the sheaf map Ψ above gives us a 1-cycle $(\Psi(\partial_{\alpha\beta}) \in A_{L^{\otimes k},p}(U_{\alpha\beta}))$ and for a section $s \otimes 1 \otimes \ldots \otimes 1$ of $L^{\otimes k}$ on $U_{\alpha\beta}$ we have

$$\begin{aligned} &\Psi(\partial_{\alpha\beta}) s \otimes 1 \otimes \ldots \otimes 1 \\ =&(d_{\alpha\beta}s + \frac{g_{\alpha\beta}}{f_{\alpha\beta}}s) \otimes 1 \otimes \ldots \otimes 1 + s \otimes (\underbrace{d_{\alpha\beta}1}_{=0} + \frac{g_{\alpha\beta}}{f_{\alpha\beta}}1) \otimes 1 \otimes \ldots \otimes 1 + \ldots \\ =&(d_{\alpha\beta}s + s\frac{kg_{\alpha\beta}}{f_{\alpha\beta}}) \otimes 1 \otimes \ldots \otimes 1. \end{aligned}$$

Indeed, this data corresponds to the transition functions

$$f_{\alpha\beta}^k + \epsilon f_{\alpha\beta}^k \frac{kg_{\alpha\beta}}{f_{\alpha\beta}} = F_{\alpha\beta}^k$$

as claimed. □

Corollary 2.5. The tangent map

$$d(\psi_k \circ c) : H^1(C, \mathcal{T}_C(-E)) \to H^1(C, A_{\omega_C^{\otimes k},p})$$

to the morphism $\psi_k \circ c : \mathcal{M}_{g,n} \to \mathcal{J}^{k(2g-2)}$ at (C,p) is induced by the map

$$\mathcal{T}_C(-E) \to A_{\omega_C^{\otimes k},p}, \quad v \mapsto L_v,$$

where L_v is the Lie-derivative along the tangent field v.

Proof. The case $k=1$ (and thus $\psi_k = \mathrm{id}$) was shown in [Pol06, Lemma 2.3]. For general k we know $d(\psi_k \circ c) = d\psi_k \circ dc$ and dc is induced by $v \mapsto (L_v : \omega_C \to \omega_C)$. Thus by Lemma 2.4, $d(\psi_k \circ c)$ is induced by

$$v \mapsto L_v \otimes \mathrm{id}^{\otimes k-1} + \ldots + \mathrm{id}^{\otimes k-1} \otimes L_v = L_v : \omega^{\otimes k} \to \omega^{\otimes k}.$$

Here we use that $L_v(S \otimes T) = (L_v S) \otimes T + S \otimes (L_v T)$ for tensor fields S, T. □

Lemma 2.6. The cokernel of the map (2) is isomorphic to the cokernel of the map

$$H^1(C, \mathcal{T}_C(-E)) \xrightarrow{H^1(\varphi)} H^1(C, \mathcal{O}_C)$$

induced by the sheaf map

$$\varphi : \mathcal{T}_C(-E) \to \mathcal{O}_C, \quad v \mapsto \frac{1}{s_0} L_v(s_0).$$

Proof. This can be seen exactly as in the proof of [Pol06, Proposition 2.4], where we use Lemma 2.3, Lemma 2.4 and Corollary 2.5 instead of the corresponding results in [Pol06]. □

To prove Proposition 1.2 we must show that the cokernel of $H^1(\varphi)$ has dimension 1 if (*) is satisfied and dimension 0 otherwise.
Our strategy for computing the cokernel above is to explicitly identify the kernel $\mathcal{K}$ and the cokernel $\mathcal{N}$ of the sheaf map $\varphi : \mathcal{T}_C(-E) \to \mathcal{O}_C$ (in the analytic category) and then to use long exact sequences in cohomology induced by the exact sequence

$$0 \to \mathcal{K} \to \mathcal{T}_C(-E) \xrightarrow{\varphi} \mathcal{O}_C \to \mathcal{N} \to 0.$$

We will first treat the case $k \geq 1$.

PROPOSITION 2.7. Let $k \geq 1$. The subsheaf $\mathcal{K}$ of $\mathcal{T}_C(-E)$ associates to an open set $U \subset C$ the vector fields v on U vanishing at all p_i such that the natural pairing $\langle v^{\otimes k}, s_0 \rangle$ induced from the contraction $\mathcal{T}_C^{\otimes k} \otimes (\mathcal{T}_C^\vee)^{\otimes k} \to \mathbb{C}$ is locally constant. The sheaf $\mathcal{K}$ can be expressed as $\mathcal{K} = \iota_! \widetilde{\mathcal{K}}$, where

$$\iota : \tilde{C} = C \setminus \{p_i : m_i \geq 0 \text{ or } k \text{ does not divide } m_i\} \hookrightarrow C$$

is the inclusion of the complement of some of the p_i. The sheaf $\widetilde{\mathcal{K}}$ is a local system of rank 1 on $\tilde{C}$.
The cokernel $\mathcal{N}$ of φ is isomorphic to $j_* \underline{\mathbb{C}}$, where

$$j : P = \{p_i : m_i < 0 \text{ and } k \text{ divides } m_i\} \hookrightarrow C.$$

Proof. Fix an open set $U \subset C$ with local coordinate z and write $s_0 = g(dz)^k$, where g is a meromorphic function on U. Given a vector field $v = f \frac{d}{dz}$ on U, we have

$$\begin{aligned} \varphi(U)(v) &= \frac{1}{s_0} L_v(s_0) = \frac{1}{g(dz)^k} L_v(g(dz)^k) \\ &= \frac{f \frac{dg}{dz} + kg \frac{df}{dz}}{g} = f \frac{d \log(g)}{dz} + k \frac{df}{dz}. \end{aligned}$$

This is zero iff $\frac{d\log(g)}{dz} + k \frac{d\log(f)}{dz} = 0$, i.e. $f^k g = \langle v^{\otimes k}, s_0 \rangle$ is constant. If this has a nonzero solution f then all solutions are scalar multiples of f (as g only has isolated zeroes).
Concerning the (local) existence of a solution around a point p, note that if g has an isolated zero at p, this equation has no nonzero holomorphic solution f and such zeroes p of g occur exactly on the p_i with $m_i > 0$. If g is regular but nonzero at p, we can choose a local logarithm and thus a solution f exists and $f(p) \neq 0$. However, if $p = p_i$ with $m_i = 0$, the definition of $\mathcal{T}_C(-E)$ requires f to have a zero at p_i, so again at these points p_i there is no solution. Finally if g has a pole at p, i.e. we have $p = p_i$ with $m_i < 0$, a solution f exists iff the

order m_i of the pole is divisible by k. This finishes the proof that the kernel $\mathcal{K}$ is the exceptional pushforward of a local system $\widetilde{\mathcal{K}}$ on the set $\tilde{C}$ above.
To compute the cokernel, let $p \in U$ (which we identify with $0 \in \mathbb{C}$ below), let h be a function on U, then we try to solve the differential equation

$$\frac{f(z)}{g(z)}\frac{dg}{dz} + k\frac{df}{dz} = h(z)$$

on U. For this we go to the branched covering $z = u^k$. Let $b = \mathrm{ord}_p g$ and write $g(z) = z^b\tilde{g}(z)$. Let g_2 be a local k-th root of $\tilde{g}$, then for $g_1(u) = u^b g_2(u^k)$ we have $g_1(u)^k = g(u^k)$. With these preparations, our differential equation in coordinate u has the form

$$\frac{f(u^k)}{g_1(u)^k}\frac{dg_1}{du}g_1(u)^{k-1}\frac{1}{u^{k-1}} + \frac{df(u^k)}{du}\frac{1}{u^{k-1}} = h(u^k).$$

Multiplying by $u^{k-1}g_1(u) = u^{k+b-1}g_2(u^k)$ we obtain

$$\frac{d(f(u^k)g_1(u))}{du} = u^{k+b-1}(g_2 \cdot h)(u^k).$$

The right hand side is the derivative of a function G (with respect to u) iff its residue at 0 vanishes. If this is the case, choose G with constant term 0 in its Laurent series around $u = 0$, then the function $G(u)/g_1(u)$ has only terms $u^{kj}, j \in \mathbb{Z}_{\geq 1}$ appearing in its power series, so we find a solution $f(z)$ vanishing at 0.
The condition on the residue is automatic if b is not negative and divisible by k (that is $p \notin P$) and otherwise it is an obstruction with values in $\mathbb{C}$, so the cokernel of φ has the form claimed above. □

In the case $k = 0$, with notation as in the proof above, the function $\varphi(v)$ is locally given by $f\frac{d\log(g)}{dz}$. Thus if the partition μ is not the trivial partition $\mu = (0, \ldots, 0)$, the meromorphic function g is nonconstant and hence $f\frac{d\log(g)}{dz} = 0$ iff $f = 0$. This shows that the kernel $\mathcal{K}$ is trivial. On the other hand, for a function h the equation

$$f\frac{d\log(g)}{dz} = h$$

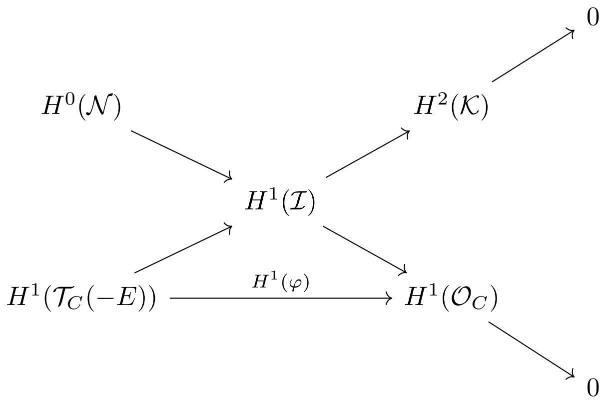
has the unique solution $f = h/\frac{d\log(g)}{dz}$ away from the zeroes of the derivative of g, so the cokernel $\mathcal{N}$ is again supported on a finite set in C.
We return now to the general case of $k \geq 0$. Let $\mathcal{I} = \mathrm{im}(\varphi) \hookrightarrow \mathcal{O}_C$, then we have two short exact sequences of sheaves on C:

$$0 \to \mathcal{K} \to \mathcal{T}_C(-E) \xrightarrow{\varphi} \mathcal{I} \to 0,$$
$$0 \to \mathcal{I} \to \mathcal{O}_C \to \mathcal{N} \to 0.$$

These induce long exact sequences in cohomology groups on C, given by

$$\cdots \to H^1(\mathcal{K}) \to H^1(\mathcal{T}_C(-E)) \to H^1(\mathcal{I}) \to H^2(\mathcal{K}) \to 0,$$
$$\cdots \to H^0(\mathcal{N}) \to H^1(\mathcal{I}) \to H^1(\mathcal{O}_C) \to 0.$$

Here we use that the higher cohomologies of sheaves supported on isolated points vanish. Thus we have the following diagram



(3)

where the diagonal sequences are exact. From this we see immediately that $H^2(\mathcal{K}) = 0$ implies that $H^1(\varphi)$ is surjective, hence has trivial cokernel. This already finishes the proof of the case $k = 0$. Fortunately, for $k \geq 1$ we can compute $H^2(\mathcal{K})$ easily.

LEMMA 2.8. For $k \geq 1$ we have

$$\dim\ H^2(\mathcal{K}) = \begin{cases} 1 & \text{if there is a merom. section } \tilde{s}_0 \text{ of } \omega_C \text{ with } \tilde{s}_0^{\otimes k} = s_0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For the composition

$$\tilde{C} \xrightarrow{\iota} C \xrightarrow{p} \{pt\},$$

the Leray spectral sequence gives us

$$R^i p_! R^j \iota_! \widetilde{\mathcal{K}} \implies R^{i+j}(p \circ \iota)_! \widetilde{\mathcal{K}}.$$

But note that the fibres of ι are either empty or single points and thus $R^j \iota_! \widetilde{\mathcal{K}} = 0$ for $j > 0$. Recall now that for $f : X \to Y$ a continuous map of locally compact spaces and $\mathcal{F}$ a sheaf on X we have $(R^i f_! \mathcal{F})_y = H^i_c(X_y, \mathcal{F}_y)$. Thus we can conclude

$$H^i(C, \iota_! \widetilde{\mathcal{K}}) = H^i_c(C, \iota_! \widetilde{\mathcal{K}}) = H^i_c(\tilde{C}, \widetilde{\mathcal{K}}).$$

Now for $i = 2$ by [Dim04, Corollary 3.3.12] we have an isomorphism

$$H^2_c(\tilde{C}, \widetilde{\mathcal{K}}) \cong H^0(\tilde{C}, \widetilde{\mathcal{K}}^\vee)^\vee.$$

Recall that $\widetilde{\mathcal{K}}$ parametrizes tangent fields v of C with $\langle v^{\otimes k}, s_0\rangle = \text{const}$. From this we see that the dual local system $\widetilde{\mathcal{K}}^\vee$ parametrizes (meromorphic) sections $\tilde{s}_0$ of ω_C such that $\tilde{s}_0^{\otimes k}$ is a locally constant multiple of s_0. Such a section exists on all of $\tilde{C}$ iff we are in the first case of the Lemma above and then it is unique up to a constant, so indeed $H^2(C, \mathcal{K})$ is one-dimensional. □

The Lemma above is already sufficient to prove parts of Proposition 1.2. Clearly condition (*) implies on the one hand that dim $H^2(\mathcal{K}) = 1$ and as there are no poles, we also have $\mathcal{N} = 0$. Looking at the diagram (3), we see $H^1(\mathcal{I}) \cong H^1(\mathcal{O}_C)$ and thus $\text{coker}H^1(\varphi) \cong H^2(\mathcal{K})$ is one-dimensional. This finishes the proof in this case.
In general, if there exists no meromorphic section $\tilde{s}_0$ of ω_C with $\tilde{s}_0^{\otimes k} = s_0$ our proof is also done, as in this case the cokernel of $H^1(\varphi)$ is trivial.
Thus we may from now on assume that we are in the strictly meromorphic case and that a section $\tilde{s}_0$ as above exists. Then from our description of $\widetilde{\mathcal{K}}^\vee$ it is obvious that it is the trivial local system $\underline{\mathbb{C}}$ on $\tilde{C}$ and thus the same is true for $\widetilde{\mathcal{K}}$. Moreover we must have that all weights m_i in μ are divisible by k (as s_0 has a k-th root), so there exists at least one marking p_i with strictly negative weight m_i divisible by k.
With these assumptions, we claim that to finish the proof of Proposition 1.2, it suffices to show the following Lemma.

Lemma 2.9. The composition of morphisms

$$H^0(\mathcal{N}) \xrightarrow{\delta} H^1(\mathcal{I}) \xrightarrow{\delta'} H^2(\mathcal{K})$$

in the diagram (3) above is nonzero and hence surjective.

Indeed this implies that while the map $H^1(\mathcal{T}_C(-E)) \to H^1(\mathcal{I})$ is not surjective, its image together with the kernel of $H^1(\mathcal{I}) \to H^1(\mathcal{O}_C)$ generate the space $H^1(\mathcal{I})$ and therefore $H^1(\varphi)$ is surjective.

Proof of Lemma 2.9. Note that by Lemma 2.8, the space $H^2(\mathcal{K})$ is one-dimensional, so indeed the composition above is surjective iff it is nonzero. We will start with a suitable nonzero element in $H^0(\mathcal{N})$ and show that its image under $\delta' \circ \delta$ is nonzero.
For this observe that $H^0(\mathcal{N}) = \bigoplus_{p_i \in P} \mathbb{C}$, i.e. we have a direct summand $\mathbb{C}$ for every element of $P = \{p_i : m_i < 0 \text{ and } k \text{ divides } m_i\}$. We show that every one of these summands maps surjectively to $H^2(\mathcal{K})$. Let $p_i \in P$, then δ is a boundary map for the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_C \to \mathcal{N} \to 0.$$

Cover C by a small open disc D around p_i (such that $D \cap P = \{p_i\}$) and $U = C \setminus \{p_i\}$, then we can find a function $h \in \mathcal{O}_C(D)$ mapping to some nonzero $\lambda \in H^0(D, \mathcal{N}) = \mathbb{C}$. That is, there exists no solution f of the equation $k\frac{df}{dz} + f\frac{d\log(g)}{dz} = h$ around p_i. Then the functions h on D and 0 on U map to the restrictions of the global section $\lambda \in H^0(D,\mathcal{N}) \subset H^0(C,\mathcal{N})$ to the cover D, U of C.

Thus the Čech 1-cycle

$$(U \cap D, h|_{U \cap D}) \in H^1(\mathcal{I})$$

is the image of λ under δ.

For the map δ' coming from the exact sequence

$$0 \to \mathcal{K} \to \mathcal{T}_C(-E) \xrightarrow{\varphi} \mathcal{I} \to 0$$

we need to refine our cover of C: take still D the open disc around p_i and let $U^+ = C \setminus s^+$, $U^- = C \setminus s^-$, where s^+, s^- are short path segments, starting in p_i such that $D^+ = D \cap U^+, D^- = D \cap U^-$ are simply connected (see also Figure 1). Thus on D^+, D^- we find vector fields $v^+ = f^+ \frac{d}{dz}$, $v^- = f^- \frac{d}{dz}$ both mapping via φ to the restrictions of h to D^+, D^-. That is, the functions $f^\pm$ satisfy

$$k \frac{df^\pm}{dz} + f^\pm \frac{d\log(g)}{dz} = h. \tag{4}$$

This follows from the proof of Proposition 2.7 as $D^\pm$ are simply connected. Thus the difference $f^+ - f^-$ on $D \cap U^+ \cap U^-$ solves the corresponding homogeneous equation describing the kernel $\mathcal{K}$ of φ . By our assumptions, this kernel has a section $v_0 = f_0 \frac{d}{dz}$ on all of $D \setminus \{p_i\}$. Thus $f^+ - f^-$ restricts to af_0, bf_0 on the two components of $D \cap U^+ \cap U^-$ for some $a, b \in \mathbb{C}$.

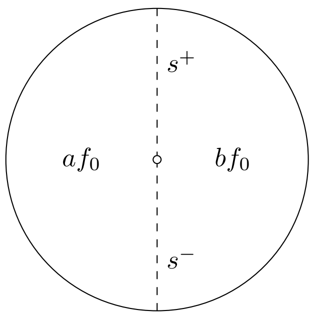


Figure 1: The restriction of $f^+ - f^-$ to $D \cap U^+ \cap U^-$

Then $\delta'(\delta(\lambda))$ is given by $(D \cap U^+ \cap U^-, (f^+ - f^-)\frac{d}{dz}) \in H^2(C, \mathcal{K})$.

To show that this cycle class is nonzero, we first show that $a \neq b$. For this, we will go back to the computation of $\delta(\lambda)$ and adapt our choice of h. Consider the (multivalued) function $F(z) = f_0(z)\log(z)$ on D. It satisfies

$$\begin{aligned} k\frac{dF}{dz} + F(z)\log(g(z)) &= \frac{kf_0(z)}{z} + \underbrace{\left(k\frac{df_0}{dz} + f_0(z)\log(g(z))\right)}_{=0}\log(z) \\ &= \frac{kf_0(z)}{z} = H. \end{aligned}$$

Then H is single-valued and as f_0 (coming from $\mathcal{T}_C(-E)$) vanishes at $z = 0$, it is holomorphic. Moreover, the solution space of the equation (4) with $h = H$ is given by $F + cf_0$ for $c \in \mathbb{C}$. Thus there exists no holomorphic solution around p_i and H maps to a nonzero element $\lambda' \in H^0(\mathcal{N})$. But then for $\delta'(\delta(\lambda'))$ we have $b - a = 2\pi i$, so in particular $a \neq b$.
Note that by assumption $\widetilde{\mathcal{K}} = \underline{\mathbb{C}}$ on $\tilde{C}$. Thus the injection $\iota_!\widetilde{\mathcal{K}} \to \underline{\mathbb{C}}$ induces a map of the second cohomology groups and the image of the two-cycle above in $H^2(C, \mathbb{C})$ is nonzero (see [Cle03, Chapter IV, Section 4.3]). This finishes the proof. □

3 Conjectural relation to Pixton's formula

As in the appendix of [FP18] we want to state for any $k \geq 1$ a conjectural relation between a weighted fundamental class of $\widetilde{\mathcal{H}}_g^k(\mu)$ and an explicit element of the tautological ring $R(\overline{M}_{g,n})$ described by Pixton in [Pix14] and [JPPZ17].

3.1 The weighted fundamental class of $\widetilde{\mathcal{H}}_g^k(\mu)$

As a first step, we identify the components Z of $\widetilde{\mathcal{H}}_g^k(\mu)$ with dimension $2g-3+n$. Let Γ be the generic dual graph of such a component, then the proof of [FP18, Proposition 7] shows that

(a) there exists a unique center vertex v_0 in Γ such that all edges of Γ connect v_0 to one of the outlying vertices $v_1, \ldots, v_r$ or they are self-loops at v_0,

(b) the vertices $v_1, \ldots, v_r$ can only have markings p_i with nonnegative weights m_i divisible by k.

Concerning the self-loops, Corollary A.2 shows that for $\mathcal{M}_{g,n}^{\mathrm{irr}} \subset \overline{\mathcal{M}}_{g,n}$ the locus of irreducible curves, we have

$$\overline{\mathcal{H}}_g^k(\mu) \cap \mathcal{M}_{g,n}^{\mathrm{irr}} = \widetilde{\mathcal{H}}_g^k(\mu) \cap \mathcal{M}_{g,n}^{\mathrm{irr}}.$$

Hence there is no component of $\widetilde{\mathcal{H}}_g^k(\mu)$ whose general element is an irreducible curve with at least one (self)node. This implies the additional restriction that

(c) at the vertex v_0 of Γ there is no self-loop.

For later use, recall that there is a gluing morphism

$$\xi_\Gamma : \prod_{v \in V(\Gamma)} \overline{\mathcal{M}}_{g(v),n(v)} \to \overline{\mathcal{M}}_{g,n},$$

where $g(v)$ is the genus of a vertex and $n(v)$ is the total number of markings and edges connected to the vertex v.

Let $S^k_{g,\mu}$ be the set of dual graphs satisfying the conditions (a), (b) and (c) above, called *star graphs*. For $\Gamma \in S^k_{g,\mu}$ let $V_{\mathrm{out}}(\Gamma)$ be the set of outlying vertices. A *twist* for Γ is a function

$$I : E(\Gamma) \to k\mathbb{Z}_{>0}$$

such that

- for the center vertex v_0 we have
$$k(2g(v_0)-2) + \sum_{e \mapsto v_0} (I(e)+k) = \sum_{i \mapsto v_0} m_i,$$
- for each $v_j \in V_{\mathrm{out}}(\Gamma)$ we have
$$k(2g(v_j)-2) + \sum_{e \mapsto v_j} (-I(e)+k) = \sum_{i \mapsto v_j} m_i.$$

We denote by $\mathrm{Tw}(\Gamma)$ the set of possible twists. For a vertex v of Γ let $\mu[v]$ be the vector of the weights of markings p_i mapping to v and let $I[v]-k$ and $-I[v]-k$ be the vectors of numbers $I(e)-k$ and $-I(e)-k$ for edges e adjacent to v.
Now if $Z \subset \widetilde{\mathcal{H}}^k_g(\mu)$ is a component of dimension $2g-3+n$ we must have that the generic dual graph Γ is in $S^k_{g,\mu}$ and that there exists a twist $I \in \mathrm{Tw}(\Gamma)$ such that Z is a component of the closed set

$$\xi_\Gamma\left(\overline{\mathcal{H}}^k_g(\mu[v_0], -I[v_0]-k) \times \prod_{v \in V_{\mathrm{out}}(\Gamma)} \overline{\mathcal{H}}^1_g\left(\frac{\mu[v]}{k}, \frac{I[v]-k}{k}\right)\right).$$

We define a weighted fundamental class $H^k_{g,\mu}$ of $\widetilde{\mathcal{H}}^k_g(\mu)$ by the formula

$$H^k_{g,\mu} = \sum_{\Gamma \in S^k_{g,\mu}} \sum_{I \in \mathrm{Tw}(\Gamma)} \frac{\prod_{e \in E(\Gamma)} I(e)}{|\mathrm{Aut}(\Gamma)| \cdot k^{|V_{\mathrm{out}}(\Gamma)|}} (\xi_\Gamma)_* \Big[\left[\overline{\mathcal{H}}^k_{g(v_0)}(\mu[v_0], -I[v_0]-k)\right] \cdot \prod_{v \in V_{\mathrm{out}}(\Gamma)} \left[\overline{\mathcal{H}}^1_{g(v)}\left(\frac{\mu[v]}{k}, \frac{I[v]-k}{k}\right)\right] \Big]. \tag{5}$$

3.2 Pixton's tautological class

Consider now the shifted vector

$$\tilde{\mu} = (\tilde{m}_1, \ldots, \tilde{m}_n), \quad \tilde{m}_i = m_i + k.$$

Then in [JPPZ17, Section 1.1] a tautological class

$$P^{g,k}_{g,\mu} = P^{g,k}_g(\tilde{\mu}) \in R^g(\overline{\mathcal{M}}_{g,n})$$

is defined. We make the following conjecture, extending the case $k = 1$ from [FP18].

Conjecture A. For $k \geq 1$ and μ not of the form $\mu = k\mu'$ for a nonnegative partition μ' of $2g-2$, we have

$$H^k_{g,\mu} = 2^{-g} P^{g,k}_{g,\mu}.$$

With Theorem 1.1 we see that the condition on μ above exactly ensures that all components of $\widetilde{\mathcal{H}}^k_g(\mu)$ are of codimension g. However, we also want to propose a conjecture in the case $\mu = k\mu'$ with μ' holomorphic. Here we have noted that the codimension $g-1$ part of $\widetilde{\mathcal{H}}^k_g(\mu)$ is exactly given by

$$\overline{\mathcal{H}}^1_g(\mu') \subset \widetilde{\mathcal{H}}^k_g(\mu).$$

Now the idea is to use the formula of Conjecture A for $k = 1$ and μ' to assign to this locus a virtual fundamental class of codimension g. Denote by $\mathrm{Cont}^k_{g,\mu}(\Gamma, I)$ the contribution of graph Γ and twist I to $H^k_{g,\mu}$ in (5). Then this virtual fundamental class is defined by the formula

$$\left[\overline{\mathcal{H}}^1_g(\mu')\right]^{\mathrm{vir}} + \sum_{\substack{\Gamma \in S^1_{g,\mu'} \\ \Gamma \text{ nontrivial}}} \sum_{I \in \mathrm{Tw}(\Gamma)} \mathrm{Cont}^1_{g,\mu'}(\Gamma, I) = 2^{-g} P^{g,1}_{g,\mu'}.$$

The conjecture we want to propose is that when we write down the formula of Conjecture A for k and $\mu = k\mu'$ and replace the codimension $g-1$ part $\left[\overline{\mathcal{H}}^1_g(\mu')\right]$ of $\left[\overline{\mathcal{H}}^k_g(\mu)\right]$ by $\left[\overline{\mathcal{H}}^1_g(\mu')\right]^{\mathrm{vir}}$, we obtain a true equality of codimension g cycle classes. To make this precise, let

$$\mathcal{H}^k_g(\mu)' = \mathcal{H}^k_g(\mu) \setminus \mathcal{H}^1_g(\mu'). \tag{6}$$

Then we propose the following.

Conjecture A′. For $k \geq 1$ and $\mu = k\mu'$ for a nonnegative partition μ' of $2g-2$, we have

$$\left[\overline{\mathcal{H}}^1_g(\mu')\right]^{\mathrm{vir}} + \left[\overline{\mathcal{H}}^k_g(\mu)'\right] + \sum_{\substack{\Gamma \in S^k_{g,\mu} \\ \Gamma \text{ nontrivial}}} \sum_{I \in \mathrm{Tw}(\Gamma)} \mathrm{Cont}^k_{g,\mu}(\Gamma, I) = 2^{-g} P^{g,k}_{g,\mu}.$$

3.3 Examples

For a list of examples where the case $k = 1$ of Conjecture A has been verified see [FP18, Section A.5].

3.3.1 Genus at most 1

For $g = 0$, Conjecture A is trivial and true for all $k \geq 1$ and Conjecture A′ is empty, as there is no nonnegative partition of -2.

For $g = 1$ and a partition μ of $k(2g-2) = 0$ we can first look at $H^k_{1,\mu}$. As expected we have a contribution of $[\overline{\mathcal{H}}^k_1(\mu)]$ from the trivial star graph. But $\mathcal{H}^k_1(\mu)$ parametrizes points $(C,p) \in \overline{\mathcal{M}}_{1,n}$ with $\mathcal{O}_C(\sum_i m_i p_i) = \mathcal{O}_C$, so it is independent of k and we have

$$[\overline{\mathcal{H}}^k_1(\mu)] = [\overline{\mathcal{H}}^1_1(\mu)].$$

As outlying vertices must have genus at least 1, we only have one more class of star graphs Γ contributing to $H^k_{1,\mu}$, namely those with

- exactly one edge e between a genus 0 vertex v_0 and a genus 1 vertex v_1,
- all markings going to v_1 having weight 0,
- the unique twist $I(e) = k$.

For $I \subset \{1, \ldots, n\}$ with $|I| \leq n-2$ denote by δ_I the divisor in $\overline{\mathcal{M}}_{1,n}$ of curves with two components of genera 0 and 1 where the genus 1 component carries the markings in I. Then we have

$$H^k_{1,\mu} = [\overline{\mathcal{H}}^1_1(\mu)] + \sum_{I \subset \{i : m_i = 0\}, |I| \leq n-2} \delta_I.$$

In particular this is independent of k, so $H^k_{1,\mu} = H^1_{1,\mu}$.
On the other hand we compute directly from the definition in [JPPZ17] that

$$P^{1,k}_{1,\mu} = -k^2\kappa_1 + \sum_{i=1}^n (m_i + k)^2 \psi_i - \frac{1}{12}\delta_{\mathrm{irr}} - \sum_{|I| \leq n-2} (k - \sum_{i \in I} m_i)^2 \delta_I.$$

Here $\kappa_1 = \pi_*(\psi_{n+1}^2)$ is the first kappa-class (with $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ the forgetful map) and

$$\delta_{\mathrm{irr}} = \frac{1}{2}\xi_* \left[\overline{\mathcal{M}}_{0,n+2}\right], \quad \xi : \overline{\mathcal{M}}_{0,n+2} \to \overline{\mathcal{M}}_{1,n}$$

is the divisor class of curves with a non-separating node in $\overline{\mathcal{M}}_{1,n}$.
At first this seems to depend on k. However, using the relations

$$\kappa_1 = \sum_{i=1}^n \psi_i - \sum_{|I| \leq n-2} \delta_I, \quad \psi_i = \frac{1}{12}\delta_{\mathrm{irr}} + \sum_{I \not\ni i, |I| \leq n-2} \delta_I$$

a straightforward computation shows that

$$P^{1,k}_{1,\mu} = \sum_{i=1}^n m_i^2 \psi_i - \frac{1}{6}\delta_{\mathrm{irr}} - \sum_{|I| \leq n-2} (\sum_{i \in I} m_i)^2 \delta_I.$$

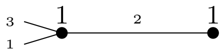
This expression is now also independent of k so $P^{1,k}_{1,\mu} = P^{1,1}_{1,\mu}$. Hence Conjecture A for $k \geq 1$ follows from the case $k = 1$, which was already shown in [FP18]. On the other hand Conjecture A$'$ is also true, as all terms for $k = 1$ and $\mu' = (0, \ldots, 0)$ exactly cancel the terms for k and μ.

3.3.2 Genus 2

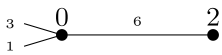
For genus $g=2$ and $k=2$ below we check Conjectures A and A′ in two cases each. Here the first such test is presented in considerable detail, listing all star graphs and their contributions. For the remaining three, we only give indications how the classes $[\overline{\mathcal{H}}_g^k(\mu)], [\overline{\mathcal{H}}_g^k(\mu)']$ can be determined, as the remaining contributions are straightforward to compute.

As a first example, we look at the partition $\mu=(3,1)$. What makes this case easy to check is that $\overline{\mathcal{H}}_2^2(3,1)=\emptyset$ (see [MS93, Theorem 2]), so the trivial graph does not give a contribution to $H^2_{2,(3,1)}$. The terms coming from nontrivial graphs are listed below, where vertices are labelled with their genus and edges with their twist. Here the central vertex is always the vertex on the left.

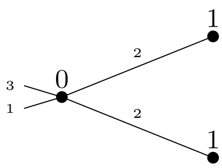
Graph Γ — Contribution



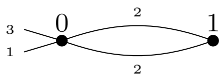
$(\xi_\Gamma)_*\left[\left[\overline{\mathcal{H}}_1^2(3,1,-4)\right]\cdot\left[\overline{\mathcal{M}}_{1,1}\right]\right]$



$3\ (\xi_\Gamma)_*\left[\left[\overline{\mathcal{M}}_{0,3}\right]\cdot\left[\overline{\mathcal{H}}_2^1(2)\right]\right]$



$\frac{1}{2}(\xi_\Gamma)_*\left[\left[\overline{\mathcal{M}}_{0,4}\right]\cdot\left[\overline{\mathcal{M}}_{1,1}\right]\cdot\left[\overline{\mathcal{M}}_{1,1}\right]\right]$



$(\xi_\Gamma)_*\left[\left[\overline{\mathcal{M}}_{0,4}\right]\cdot\left[\overline{\mathcal{M}}_{1,2}\right]\right]$

Here the class $[\overline{\mathcal{H}}_1^2(3,1,-4)]$ is obtained from the case $g=1$ of Conjecture A verified above whereas the class $[\overline{\mathcal{H}}_2^1(2)]$ has been computed in [EH87]. Summing up the contributions above we obtain a tautological cycle which exactly equals $2^{-2}P^{2,2}_{2,(3,1)}$. To verify this equality one uses known relations in the tautological ring.

Another check for Conjecture A is possible for $g=2, k=2$ and $\mu=(2,1,1)$. Here, by [Lan08, Theorem 1.2] (or an elementary argument involving the Residue theorem), we have

$$\mathcal{H}_2^2(2,1,1)=\left\{(C,q,p_1,p_2):\begin{array}{l} q \text{ Weierstrass point,} \\ p_1,p_2 \text{ hyperelliptic conjugate}\end{array}\right\}.$$

The class of the closure of this locus has been computed in [BP00, Lemma 3]. As for the other terms appearing in $H^2_{2,(2,1,1)}$, all of them are obtained from the genus 1 case of Conjecture A or in [BP00]. Again the statement of Conjecture A for $\mu=(2,1,1)$ is true.

Concerning Conjecture A′, we are able to verify it in two cases, namely $g=2, k=2$ and $\mu=(4)$ or $\mu=(2,2)$. For the partition $\mu=(4)$, it follows from [Lan08, Theorem 1.2] that we have

$$\mathcal{H}_2^2(4)=\mathcal{H}_2^1(2),$$

so $\mathcal{H}_2^2(4)' = 0$. All contributions from nontrivial graphs are easily computed and the conjecture holds in this case.
For $\mu = (2,2)$ we find that $(C,p,q) \in \mathcal{H}_2^2(2,2)$ iff either p,q are hyperelliptic conjugate (i.e. $(C,p,q) \in \mathcal{H}_2^1(1,1)$) or if both p,q are Weierstrass points. Thus we have

$$\mathcal{H}_2^2(2,2)' = \{(C,p,q) : p,q \text{ Weierstrass points}\}$$

and the class of the closure of this locus has been computed by Tarasca in [Tar15, Theorem 0.1] (under the name $[\overline{\mathcal{DR}}_2(2)]$). Again all other terms are computed via the genus 1 case of Conjecture A and the claimed relation holds.
All four equalities in this section were checked by Aaron Pixton, using a computer program to expand the cycles $2^{-g}P_{g,\mu}^{g,k}$ in terms of tautological classes and then using known relations ([PPZ15]) in the tautological ring to show the equality to $H_{g,\mu}^k$. We gratefully acknowledge his help.

3.4 Recursions for $\overline{\mathcal{H}}_g^k(\mu)$

Let $k \geq 1$ and let μ be a partition of $k(2g-2)$ not of the form $\mu = k\mu'$ for a nonnegative partition μ' of $2g-2$. Assuming Conjecture A above, we can determine an expression for $\overline{\mathcal{H}}_g^k(\mu)$ in terms of tautological classes. Indeed, in the formula for $H_{g,\mu}^k$, the term $[\overline{\mathcal{H}}_g^k(\mu)]$ appears with a coefficient 1. All other terms are composed from cycles

- $[\overline{\mathcal{H}}_{g'}^k(\mu')]$ for $g' < g$ (as outlying vertices must have genus at least 1),
- $[\overline{\mathcal{H}}_{g''}^1(\mu'')]$, which were determined in [FP18] assuming Conjecture A for $k=1$.

Hence in the equation $H_{g,\mu}^k = 2^{-g}P_{g,\mu}^{g,k}$, we can solve for $[\overline{\mathcal{H}}_g^k(\mu)]$ and determine these cycles by induction on g.
On the other hand, if the partition μ is nonnegative and divisible by k, the closed set $\overline{\mathcal{H}}_g^k(\mu)$ is a union of components of codimension $g-1$ (coming from $\overline{\mathcal{H}}_g^1(\mu/k)$) and components of codimension g (given by the closure of $\overline{\mathcal{H}}_g^k(\mu)'$ as defined in (6)). Again the class $[\overline{\mathcal{H}}_g^1(\mu/k)]$ is determined by Conjecture A for $k=1$. Solving the formula of Conjecture A$'$ for $[\overline{\mathcal{H}}_g^k(\mu)']$, we can express it in terms of classes determined by Conjecture A as described above.

A Smoothing of untwisted nodes for k-differentials

In this section, for the convenience of the reader, we give an elementary argument showing that an untwisted node in a curve $(C,p_1,\ldots,p_n) \in \widetilde{\mathcal{H}}_g^k(\mu)$ can be smoothed in a holomorphic 1-parameter family inside $\widetilde{\mathcal{H}}_g^k(\mu)$. Such a smoothing is known as a plumbing construction in complex analytic geometry

(see [Wol13, Section 6.3] and [Gen, Lemma 3.13] on the plumbing of abelian differentials). Essentially, we verify that the same ideas work for arbitrary k. The proof below was explained to us by Dimitri Zvonkine.
Let $\mathbb{D} = \{z : |z| < 1\} \subset \mathbb{C}$ be the unit disc.

Proposition A.1. Let $g, n, k \geq 0$ with $2g - 2 + n > 0$ and let μ be a partition of $k(2g - 2)$. Assume we have

$$(C, p_1, \ldots, p_n) \in \widetilde{\mathcal{H}}_g^k(\mu) \subset \overline{\mathcal{M}}_{g,n}$$

where C has a node q at the intersection of two components D, D' (where possibly $D = D'$) such that there exists a twist I for the divisor $\sum_i m_i p_i$ with $I(q, D) = I(q, D') = 0$.
Then there exists a holomorphic 1-parameter family $\pi : \mathcal{C} \to \mathbb{D}$ and sections $\sigma_1, \ldots, \sigma_n : \mathbb{D} \to \mathcal{C}$ with central fibre

$$(\mathcal{C}_0, \sigma_1(0), \ldots, \sigma_n(0)) \cong (C, p_1, \ldots, p_n),$$

smoothing the node q such that the induced analytic map $\varphi : \mathbb{D} \to \overline{\mathcal{M}}_{g,n}$ has image in $\widetilde{\mathcal{H}}_g^k(\mu)$.

Proof. Let $U \subset C$ be an analytic neighbourhood of q with an isomorphism $\psi : U \xrightarrow{\sim} V(xy) \subset \mathbb{C}^2$ and not containing any of the points $p_1, \ldots, p_n$. Let

$$V = C \setminus \psi^{-1}(V(xy) \cap \{(x, y) : |x| \leq 1, |y| \leq 1\}).$$

Then U, V form an open cover of C and their intersection is a union of two annuli

$$U \cap V \cong \underbrace{\{(x, 0) : |x| > 1\}}_{=:A_x} \amalg \underbrace{\{(0, y) : |y| > 1\}}_{=:A_y}.$$

We construct the family $\pi : \mathcal{C} \to \mathbb{D}$ by gluing the trivial family $\pi_V : V \times \mathbb{D} \to \mathbb{D}$ with constant sections $\sigma_i(t) = (p_i, t)$ to a family $\pi_U : \mathcal{U} \to \mathbb{D}$, which smoothes the node of the central fibre $\mathcal{U}_0 = U$. The gluing happens along the subset $(A_x \amalg A_y) \times \mathbb{D}$ of $V \times \mathbb{D}$.

$$\begin{array}{ccccc} \mathcal{U} & \xleftarrow{i_{\mathcal{U}}} & (A_x \amalg A_y) \times \mathbb{D} & \hookrightarrow & V \times \mathbb{D} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{D} & = & \mathbb{D} & = & \mathbb{D} \end{array}$$

After choosing suitable coordinates on U (i.e. modifying the isomorphism $\psi : U \to V(xy)$), the family $\mathcal{U}$ is easy to write down:

$$\mathcal{U} = \{(x, y, t) \in \mathbb{C}^2 \times \mathbb{D} : xy - t = 0\} \xrightarrow{(x,y,t) \mapsto t} \mathbb{D}$$

and the map $i_{\mathcal{U}}$ sends $(x, t) \in A_x \times \mathbb{D}$ to $(x, t/x, t)$ and $(y, t) \in A_y \times \mathbb{D}$ to $(t/y, y, t)$.

It is easy to check that the above data defines a holomorphic family π smoothing the node q of the central fibre $(C, p_1, \ldots, p_n)$. The crucial point however is to ensure that the corresponding twisted differential deforms in this family, i.e. that the map $\mathbb{D} \to \overline{\mathcal{M}}_{g,n}$ has image in $\widetilde{\mathcal{H}}_g^k(\mu)$.
By assumption, for the partial normalization $\nu : C_I \to C$ resolving all nodes p of C with nonzero twist I we have a section η of $\nu^*\omega_C$ having zeroes and poles at the points $\nu^{-1}(p_i)$ with order m_i and at the preimages of p', p'' of the nodes p with orders determined by the twist I. As the twist I vanishes at the node q, it is not normalized by η, so over $U \subset C$ the map η is an isomorphism. Using this, we obtain a section $\eta_U \in \Gamma(U, \omega_U^{\otimes k})$.
On the horizontal and vertical components $V(y), V(x)$ of U, we can express the restrictions η_x, η_y of η_U in terms of the coordinate x on $V(y)$ and y on $V(x)$ as

$$\eta_x = a(1 + b_{k-1}x + b_{k-2}x^2 + \ldots) \left(\frac{dx}{x}\right)^k,$$
$$\eta_y = (-1)^k a(1 + c_{k-1}y + c_{k-2}y^2 + \ldots) \left(\frac{dy}{y}\right)^k.$$

Here the number $a \neq 0$ is invariant under coordinate changes. The fact that the order $(-k)$-coefficients of η_x, η_y agree (up to a sign $(-1)^k$) follows from the usual fact that for $k = 1$ the residues of the differentials on both components have to sum up to zero. Then for any local section s of ω_C not vanishing around q, s^k is a local section of ω^k around q and all such sections are multiples of s^k.
In the case $k \geq 1$, we claim that after a holomorphic change of coordinates $x = x(w)$ with $x(0) = 0$ we can assume that η_x is of the form $\eta_x = a\left(\frac{dw}{w}\right)^k$. This normal form was derived for quadratic differentials ($k = 2$) in [Str84, Theorem 6.3]. The same argument works in general, and for the convenience of the reader we include it below.
For $f(x) = 1 + b_{k-1}x + b_{k-2}x^2 + \ldots$ as above choose a k-th root $g(x)$ of f around $x = 0$ with $g(0) = 1$. Then for a change of coordinates $x(w)$ we compute

$$\eta_x = af(x(w)) \left(\frac{dx(w)}{x(w)}\right)^k = af(x(w)) \left(\frac{dx(w)}{dw}\frac{w}{x(w)}\right)^k \left(\frac{dw}{w}\right)^k.$$

Thus we want to find a solution of the differential equation

$$f(x(w)) \left(\frac{dx(w)}{dw}\frac{w}{x(w)}\right)^k = 1. \tag{7}$$

Dividing by $f(x(w))$ and taking a k-th rooth, the equation is in particular satisfied if we have

$$\frac{dx(w)}{dw}\frac{w}{x(w)} = g(x(w))^{-1}.$$

We solve this equation by separation of variables. Define $R(z) = g(z)/z$ observing $R(z) = 1/z + \tilde{R}(z)$ with $\tilde{R}$ holomorphic around $z = 0$. Then we write the equation above as

$$\frac{dx(w)}{dw}R(x(w)) = w^{-1}.$$

Define the (multivalued) function

$$S(u) = \int_{z_0}^{u} R(z)dz = (\log(u) - \log(z_0)) + \underbrace{\int_{z_0}^{u} \tilde{R}(z)dz}_{=:\tilde{S}(u)}$$

for a fixed z_0 close to 0. Note that $\tilde{S}$ is well-defined and the logarithm terms are defined up to $2\pi i\mathbb{Z}$. Then the above equation has the form

$$\frac{d}{dw}S(x(w)) = w^{-1}.$$

Integrating and then taking the exponential on both sides, we obtain

$$\frac{x(w)}{z_0}\exp(\tilde{S}(x(w))) = Aw$$

for some $A \in \mathbb{C}^*$. But the function $u \mapsto u\exp(\tilde{S}(u))$ has nonzero derivative at $u = 0$, so we can locally find a holomorphic inverse function T. We conclude

$$x(w) = T(Az_0 w).$$

Going backwards through the argument above one checks that this choice of $x(w)$ then satisfies the original equation (7).

Thus after a change of coordinates and possibly shrinking U, we can assume that η_x, η_y are given by

$$\eta_x = a\left(\frac{dx}{x}\right)^k, \quad \eta_y = (-1)^k a\left(\frac{dy}{y}\right)^k.$$

On the fibre $\mathcal{U}_t = V(xy-t) \subset \mathbb{C}^2$ for t fixed we have $0 = d(xy-t) = xdy+ydx$, so $dx/x = -dy/y$ if $x \neq 0, y \neq 0$. Using this, we see that the meromorphic differentials $a\left(\frac{dx}{x}\right)^k, (-1)^k a\left(\frac{dy}{y}\right)^k$ on $\mathbb{C}^2 \times \mathbb{D}$ restrict to the same relative k-differential on $\mathcal{U} \setminus \{(0,0,0)\} \to \mathbb{D}$. As $\mathcal{U}$ is isomorphic to an open subset of $\mathbb{C}^2$ and as $(0,0,0)$ has codimension 2, this extends to a unique relative differential $\eta_{\mathcal{U}}$ extending η_U on the central fibre by the second Riemann extension theorem. Note that in the case $k = 0$, which was omitted so far, the argument is even easier: for η_x, η_y the restrictions of η_U to the x and y-axis as above, we note that the function $(x,y,t) \mapsto \eta_x(x) + \eta_y(y) - a$ on $\mathbb{C}^2 \times \mathbb{D}$ restricts to a function $\eta_{\mathcal{U}}$ on $\mathcal{U}$ extending η_U on the central fibre.

We can now verify that the pullback of $\eta_{\mathcal{U}}$ by $i_{\mathcal{U}} : (A_x \amalg A_y) \times \mathbb{D} \to \mathcal{U}$ is independent of the point $t \in \mathbb{D}$. This shows that we can glue this k-differential in the family $\mathcal{C}$ and that for all $t \in \mathbb{D}$ we obtain a differential as desired on the partial normalization of the fibres $\mathcal{C}_t$. But for instance for the restriction of $i_{\mathcal{U}}$ to $A_x \times \mathbb{D}$ given by $(x,t) \mapsto (x, t/x, t)$ we have that

$$i_{\mathcal{U}}^* \eta_{\mathcal{U}} = i_{\mathcal{U}}^* a \left(\frac{dx}{x}\right)^k = a \left(\frac{dx}{x}\right)^k$$

is indeed independent of t (and similarly for $A_y \times \mathbb{D}$). Again the case $k = 0$ is even more obvious.

□

COROLLARY A.2. Let $g, n, k \geq 0$ with $2g - 2 + n > 0$, μ a partition of $k(2g-2)$ and $(C, p_1, \ldots, p_n) \in \widetilde{\mathcal{H}}_g^k(\mu)$. Then if C is irreducible, we have $(C, p_1, \ldots, p_n) \in \overline{\mathcal{H}}_g^k(\mu)$.

Proof. We do an induction on the number δ of nodes of C, which are always self-nodes, hence untwisted. The case $\delta = 0$ is clear and the case $\delta \geq 1$ follows from the statement for $\delta - 1$ as we can smooth C in a family with general element having $\delta - 1$ nodes. □

The case $k = 1$ and μ holomorphic of this result was proved in [FP18, Lemma 12].

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