# On Exact Pleijel's Constant for Some Domains

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Abstract. We provide an explicit expression for the Pleijel constant for the planar disk and some of its sectors, as well as for $N$-dimensional rectangles. In particular, the Pleijel constant for the disk is equal to $0.4613019\ldots$ Also, we characterize the Pleijel constant for some rings and annular sectors in terms of asymptotic behavior of zeros of certain cross-products of Bessel functions.

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## 1 Introduction and main results

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with the boundary $\partial\Omega$, $N \geq 2$. Consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and denote by $\{\lambda_n\}_{n\in\mathbb{N}}$ the sequence of the corresponding eigenvalues,

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \to \infty \quad \text{as } n \to \infty.$$

For any eigenfunction $\varphi_n$ associated with $\lambda_n$, let $\mu(\varphi_n)$ be a number of nodal domains of $\varphi_n$, that is, a number of connected components of $\Omega \setminus \overline{\{x \in \Omega : \varphi_n(x) = 0\}}$. Courant's nodal domain theorem [11] asserts that $\mu(\varphi_n) \leq n$ for any $n \in \mathbb{N}$. This result was refined by Pleijel for $N = 2$ and then extended by Bérard & Meyer to the general $N \geq 2$ as follows.

Theorem 1.1 ([24, Section 5] and [3, Theorem II.7]). *Let $j_{\frac{N}{2}-1,1}$ be the first zero of the Bessel function $J_{\frac{N}{2}-1}$. Then*

$$Pl(\Omega) := \limsup_{n \to \infty} \frac{\mu(\varphi_n)}{n} \leq \gamma(N) := \frac{2^{N-2} N^2 \Gamma\left(\frac{N}{2}\right)^2}{j_{\frac{N}{2}-1,1}^N} < 1.$$

*In particular, if $N = 2$, then*

$$Pl(\Omega) \leq \frac{4}{j_{0,1}^2} = 0.6916602\ldots \tag{1.2}$$

We will call $Pl(\Omega)$ *the Pleijel constant* for $\Omega$. Note that the function $N \mapsto \gamma(N)$ is strictly decreasing and $\lim_{N \to \infty} \frac{\gamma(N+1)}{\gamma(N)} = \frac{2}{e}$, see Helffer & Sundqvist [17, Theorem 5.1 and Remark 5.4].

In recent years, properties of the Pleijel constant and, more in general, the nodal domains statistics started to be intensively studied with both mathematical and physical accents. For instance, in the work of Blum et al [4], it has been revealed that the distribution of the values $\frac{\mu(\varphi_n)}{n}$ crucially depends on the shape of the domain according to the underlying classical billiard dynamics - with either integrable or chaotic motion, see also Aronovitch et al [1], Gnutzmann & Lois [14], and references therein. Moreover, properties of the support of the limiting distribution function of $\frac{\mu(\varphi_n)}{n}$ are closely related with those of the Pleijel constant.

It was pointed out by Polterovich [25] and then rigorously developed by Bougain [7] and Steinerberger [26] that the upper bound $\frac{4}{j_{0,1}^2}$ for $Pl(\Omega)$ in (1.2) is not sharp. Furthermore, Polterovich conjectured that the optimal upper bound for $N = 2$ should be

$$Pl(\Omega) \leq \frac{2}{\pi} = 0.6366197\ldots$$

If this upper estimate holds true, then its optimality follows from the consideration of any rectangle $\mathcal{R}(a, b) = (0, a) \times (0, b)$ with irrational ratio $\frac{a^2}{b^2}$, see Helffer & Hoffmann-Ostenhof [16, Proposition 5.1].

We start by extending [16, Proposition 5.1] to the general $N$-dimensional case as follows (cf. [14, Section 4.2]).

Proposition 1.2. *Let $\mathcal{R}(a_1, \ldots, a_N) = (0, a_1) \times \cdots \times (0, a_N)$ be a $N$-orthotope such that $\frac{a_i^2}{a_j^2}$ is irrational for any $i \neq j$. Then*

$$Pl(\mathcal{R}(a_1, \ldots, a_N)) = \rho(N) := \frac{2^N \Gamma\left(\frac{N}{2} + 1\right)}{\pi^{\frac{N}{2}} N^{\frac{N}{2}}}.$$

*Moreover, the function $N \mapsto \rho(N)$ is strictly decreasing and $\lim_{N \to \infty} \frac{\rho(N+1)}{\rho(N)} = \sqrt{\frac{2}{\pi e}}$.*

It is clear that $\rho(N) < \gamma(N)$ for any $N \geq 2$ by the same reasoning as in [25, Remark 2.5]. Furthermore, by analogy with the conjecture of Polterovich [25], one can ask whether the estimate $Pl(\Omega) \leq \rho(N)$ is valid for any $\Omega \subset \mathbb{R}^N$ and $N \geq 2$.

Note that the "irrational" rectangles discussed above are the most simple domains whose Pleijel's constant can be worked out explicitly. However, to the best of our knowledge, the exact value of the Pleijel constant has not been known for any other domains[2], and the question of finding of such domains was proposed by Bonnaillie-Noël et al [5, Section 6.1]. Furthermore, numerical experiments (see, e.g., [4, 1, 14]) demonstrate relatively slow convergence of the ratio $\frac{\mu(\varphi_n)}{n}$ as $n \to \infty$. Thus, theoretical results in this direction are needed. The aim of the present note is to obtain the explicit expression for the Pleijel constant for the planar disk. Our main result is the following.

THEOREM 1.3. *Let* $B := \{x \in \mathbb{R}^2 : |x| < 1\}$. *Then*

$$Pl(B) = 8 \sup_{x>0} \left\{ x \left( \cos \theta(x) \right)^2 \right\} = 0.4613019\ldots,$$

*where* $\theta = \theta(x)$ *is the solution of the transcendental equation*

$$\tan \theta - \theta = \pi x, \quad \theta \in \left( 0, \frac{\pi}{2} \right).$$

Note that the value of $Pl(B)$ is in the good correspondence with the numerical simulation of Blum et al [4, Fig. 1]. We also refer the reader to Han et al [15] where the sharp estimates for the length of the nodal set of eigenfunctions of (1.1) on $B$ are obtained.

Consider now the circular sector with the angle $\alpha \in (0, 2\pi]$ defined as

$$\Sigma^\alpha := \left\{ (\varrho \cos \vartheta, \varrho \sin \vartheta) : \varrho \in (0, 1), \vartheta \in \left( -\frac{\alpha}{2}, \frac{\alpha}{2} \right) \right\}.$$

THEOREM 1.4. *Let* $\alpha$ *be such that any eigenvalue of* (1.1) *on* $\Sigma^\alpha$ *has the multiplicity* 1. *Then* $Pl(\Sigma^\alpha) = Pl(B)$. *In particular, the assumption on* $\alpha$ *is satisfied if* $\alpha = \pi/m$, $m \in \mathbb{N}$.

REMARK 1.5. There exists $\alpha \in (0, 2\pi)$ such that there is an eigenvalue of (1.1) on $\Sigma^\alpha$ whose multiplicity is at least 2, see Bonnaillie-Noël & Léna [6, Section 2.3]. In general, the question about the multiplicity 1 for eigenvalues of (1.1) on $\Sigma^\alpha$ is equivalent to the following: Find assumptions on $\alpha \in (0, 2\pi]$ which guarantee that the Bessel functions $J_{\nu_1 \frac{\pi}{\alpha}}$ and $J_{\nu_2 \frac{\pi}{\alpha}}$ have no common positive zeros for any $\nu_1, \nu_2 \in \mathbb{N}$. This question is reminiscent of Bourget's hypothesis [27, p. 484].

Finally, we characterize the Pleijel constant for rings and annular sectors. First, consider the ring (annulus) $A_r := \{x \in \mathbb{R}^2 : r < |x| < 1\}$, $r \in (0, 1)$.

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[2] For the quantum harmonic oscillator $-\Delta + \sum_{i=1}^N a_i^2 x_i^2$ with rationally independent $a_i$'s, Charron [8] proved that the Pleijel constant is equal to $N!/N^N$, cf. [14, Section 4.1].

Proposition 1.6. *Let $r$ be such that any eigenvalue of* (1.1) *on $A_r$ has the multiplicity at most* 2. *Then*

$$Pl(A_r) = \frac{8}{1-r^2} \sup_{x>0} \left\{ x \limsup_{k\to\infty} \frac{k^2}{a_{kx,k}^2} \right\},$$

*where $a_{kx,k}$ is the $k$-th positive zero of the cross-product of Bessel functions $J_{kx}(rz)Y_{kx}(z) - J_{kx}(z)Y_{kx}(rz)$.*

Consider now the annular sector with the inner radius $r \in (0,1)$ and the angle $\alpha \in (0, 2\pi]$ defined as

$$\Sigma_r^\alpha := \left\{ (\varrho\cos\vartheta, \varrho\sin\vartheta) : \ \varrho \in (r,1), \ \vartheta \in \left(-\frac{\alpha}{2}, \frac{\alpha}{2}\right) \right\}.$$

Proposition 1.7. *Let $r$ and $\alpha$ be such that any eigenvalue of* (1.1) *on $\Sigma_r^\alpha$ has the multiplicity* 1. *Then $Pl(\Sigma_r^\alpha) = Pl(A_r)$.*

Remark 1.8. It seems that, by now, the asymptotic behavior of $a_{kx,x}$ as $k \to \infty$ is not studied as comprehensively as whose of $j_{kx,k}$ ([13, 12]), which obstructs us to obtain an explicit expression for $Pl(A_r)$ in the spirit of Theorem 1.3. We expect that the following asymptotics should be valid:

$$Pl(A_r) \to Pl(B) \quad \text{as } r \to 0; \qquad Pl(A_r) \to \rho(2) = \frac{2}{\pi} \quad \text{as } r \to 1.$$

Let us note that there exist parameters $r$ and $\alpha$ for which the assumptions of Propositions 1.6 and 1.7 do not hold.

Lemma 1.9. *There exists $r$ (and $\alpha$) such that there is an eigenvalue of* (1.1) *on $A_r$ (on $\Sigma_r^\alpha$) whose multiplicity is at least* 3 *(at least* 2*), see Fig. 1.*

But it should be also emphasized that we do not know whether the sets of admissible parameters for Propositions 1.6 and 1.7 are non-empty.

## 2   Proofs

The structure of all the proofs consists of three main steps. First, we characterize eigenvalues and eigenfunctions of (1.1) explicitly as multi-indexed sets, due to the "separable" nature of the considered domains. Then, after reordering the set of eigenvalues into the increasing sequence $\{\lambda_n\}_{n\in\mathbb{N}}$, we apply the Weyl law (see, e.g., Ivrii [18]) to express the labeling $n$ in terms of the corresponding eigenvalue $\lambda_n$ for sufficiently large $n$. Finally, we elaborate the obtained expression for $Pl(\Omega)$ up to the desired form.

## 2.1 Proof of Proposition 1.2

Let $N \geq 2$ and let $\mathcal{R}(a_1, \ldots, a_N) = (0, a_1) \times \cdots \times (0, a_N)$ be a $N$-orthotope such that the ratio $\frac{a_i^2}{a_j^2}$ is irrational for any $i \neq j$. By separation of variables, any eigenvalue of (1.1) on $\mathcal{R}(a_1, \ldots, a_N)$ is given by

$$\lambda_{m_1,\ldots,m_N} = \frac{\pi^2 m_1^2}{a_1^2} + \cdots + \frac{\pi^2 m_N^2}{a_N^2}, \quad m_1, \ldots, m_N \in \mathbb{N},$$

with the associated eigenfunction

$$\varphi_{m_1,\ldots,m_N} = \sin \frac{\pi m_1 x}{a_1} \cdots \sin \frac{\pi m_N x}{a_N}.$$

Evidently, $\mu(\varphi_{m_1,\ldots,m_N}) = m_1 \cdots m_N$.

Since $\frac{a_i^2}{a_j^2}$ is irrational for any $i \neq j$, all eigenvalues are simple (have the multiplicity 1). Therefore, no other eigenfunctions occur, and if we put $\lambda_{m_1,\ldots,m_N}$'s in the increasing order as $\lambda_1 \leq \ldots \lambda_n \leq \ldots$, then for any $n$ there exists a unique $N$-tuple $(m_{1,n}, \ldots, m_{N,n})$ such that $\lambda_n = \lambda_{m_{1,n},\ldots,m_{N,n}}$. Since the explicit relation between $n$ and $(m_{1,n}, \ldots, m_{N,n})$ is not clear, we use the Weyl law which states that

$$n = (2\pi)^{-N} \omega_N |\mathcal{R}(a_1, \ldots, a_N)| \lambda_n^{N/2} + o(\lambda_n^{N/2}),$$

where $\omega_N$ is the volume of the unit ball in $\mathbb{R}^N$ and $|\mathcal{R}(a_1, \ldots, a_N)|$ is the volume of $\mathcal{R}(a_1, \ldots, a_N)$. Thus, we get

$$n = \frac{\omega_N \, a_1 \cdots a_N}{2^N} \left( \frac{m_{1,n}^2}{a_1^2} + \cdots + \frac{m_{N,n}^2}{a_N^2} \right)^{N/2} + o(\lambda_n^{N/2}).$$

Substituting this expression into the definition of the Pleijel constant and omitting, for simplicity, the subindex $n$ in $m_{i,n}$, we obtain

$$Pl(\mathcal{R}(a_1, \ldots, a_N)) = \frac{2^N}{\omega_N} \limsup_{m_1+\cdots+m_N \to \infty} \frac{\frac{m_1}{a_1} \cdots \frac{m_N}{a_N}}{\left( \frac{m_1^2}{a_1^2} + \cdots + \frac{m_N^2}{a_N^2} \right)^{N/2}}.$$

Applying the inequality between arithmetic and geometric means, we deduce that

$$Pl(\mathcal{R}(a_1, \ldots, a_N)) \leq \frac{2^N}{\omega_N N^{\frac{N}{2}}}.$$

On the other hand, taking a sequence of $N$-tuples $\{(m_{1,k}, \ldots, m_{N,k})\}_{k \in \mathbb{N}} \subset \mathbb{N}^N$ such that $\lim_{k \to \infty} \frac{m_{i,k}}{k} = a_i$ for each $i = 1, \ldots, N$, we get

$$Pl(\mathcal{R}(a_1, \ldots, a_N)) \geq \frac{2^N}{\omega_N N^{\frac{N}{2}}},$$

and hence the equality holds. Since $\omega_N = \frac{\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}+1\right)}$, we conclude that

$$Pl(\mathcal{R}(a_1,\ldots,a_N)) = \rho(N) := \frac{2^N \Gamma\left(\frac{N}{2}+1\right)}{\pi^{\frac{N}{2}} N^{\frac{N}{2}}}.$$

Let us show now that $\frac{\rho(N+1)}{\rho(N)} < 1$ for any $N \geq 2$. We have

$$\frac{\rho(N+1)}{\rho(N)} = \frac{2 \, N^{\frac{N}{2}} \Gamma\left(\frac{N}{2}+\frac{1}{2}+1\right)}{\pi^{\frac{1}{2}} (N+1)^{\frac{N+1}{2}} \Gamma\left(\frac{N}{2}+1\right)}. \tag{2.1}$$

Applying Gautschi's-type inequality $\Gamma(x+1) < \left(x+\frac{1}{2}\right)^{\frac{1}{2}} \Gamma\left(x+\frac{1}{2}\right)$ which holds for all $x > 0$ (see Kershaw [19, (1.3)]), we have

$$\Gamma\left(\frac{N}{2}+\frac{1}{2}+1\right) < \left(\frac{N+2}{2}\right)^{1/2} \Gamma\left(\frac{N}{2}+1\right).$$

Substituting this inequality into (2.1) and estimating $N^{\frac{N}{2}} < (N+1)^{\frac{N}{2}}$, we obtain

$$\frac{\rho(N+1)}{\rho(N)} < \left(\frac{2(N+2)}{\pi(N+1)}\right)^{1/2} < 1$$

for any $N \geq 2$, that is, the function $N \mapsto \rho(N)$ is strictly decreasing.

Finally, we show that $\lim\limits_{N \to \infty} \frac{\rho(N+1)}{\rho(N)} = \sqrt{\frac{2}{\pi e}}$. Let us rewrite (2.1) as

$$\frac{\rho(N+1)}{\rho(N)} = \frac{2^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \left(1+\frac{1}{N}\right)^{-\frac{N}{2}} \frac{(N+2)^{\frac{1}{2}}}{(N+1)^{\frac{1}{2}}} \frac{\Gamma\left(\frac{N}{2}+\frac{1}{2}+1\right)}{\Gamma\left(\frac{N}{2}+1\right)\left(\frac{N}{2}+1\right)^{\frac{1}{2}}}.$$

Then, noting that $\lim\limits_{x \to \infty} \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)\left(x+\frac{1}{2}\right)^{\frac{1}{2}}} = 1$ (see, e.g., [19, (2.3)]) and $\lim\limits_{N \to \infty} \left(1+\frac{1}{N}\right)^{-\frac{N}{2}} = e^{-\frac{1}{2}}$, we get the desired result. $\qquad\square$

## 2.2 PROOF OF THEOREM 1.3

Consider the unit planar disk $B = \{x \in \mathbb{R}^2 : |x| < 1\}$. It is well-known that, by separation of variables, there exists a basis of eigenfunctions of (1.1) on $B$ expressed (up to rotation) in polar coordinates $(\varrho, \vartheta)$ as

$$\varphi_{\nu,k}(\varrho, \vartheta) = J_\nu(j_{\nu,k}\varrho) \cos(\nu\vartheta), \quad \nu \in \mathbb{N} \cup \{0\}, \ k \in \mathbb{N},$$

and $\lambda_{\nu,k} = j_{\nu,k}^2$ is the eigenvalue associated with $\varphi_{\nu,k}$. Here $j_{\nu,k}$ stands for the $k$-th positive zero of the Bessel function $J_\nu$. Any eigenvalue $\lambda_{0,k}$ has the multiplicity 1 (and $\varphi_{0,k}$ is radial), while any other eigenvalue has the multiplicity 2 (and corresponding eigenfunctions are nonradial, one is a rotation of another). These facts follow from the validity of Bourget's hypothesis, which asserts that

$J_\nu$ and $J_{\nu+m}$ do not have common positive zeros for any natural $m$, see [27, p. 484].

Clearly, $\mu(\varphi_{0,k}) = k$ and $\mu(\varphi_{\nu,k}) = 2\nu k$ for $\nu \in \mathbb{N}$. Below, for brevity, we combine both cases by writing $\mu(\varphi_{\nu,k}) = (2\nu + \sigma(\nu))k$, where $\sigma(0) = 1$ and $\sigma(\nu) = 0$ for $\nu \in \mathbb{N}$.

Let us put $\lambda_{\nu,k}$'s in the increasing order as $\lambda_1 \leq \cdots \leq \lambda_n \leq \ldots$. Since we are interested in the behavior as $n \to \infty$ and the explicit relation between $(\nu, k)$ and $n$ is not known, we use the Weyl law to get

$$n = \lambda_n \frac{|B|^2}{4\pi^2} + o(\lambda_n) = \frac{\lambda_n}{4} + o(\lambda_n).$$

Hence, noting that for any $\lambda_n$ there exists a unique pair $(\nu_n, k_n)$ such that $\lambda_n = \lambda_{\nu_n,k_n} = j_{\nu_n,k_n}^2$, we obtain

$$n = \frac{j_{\nu_n,k_n}^2}{4} + o(j_{\nu_n,k_n}^2).$$

Substituting this relation to the definition of $Pl(B)$, we deduce that

$$Pl(B) = \limsup_{n \to \infty} \frac{4(2\nu_n + \sigma(\nu_n))k_n}{j_{\nu_n,k_n}^2}.$$

Extracting a subsequence which delivers the value $Pl(B)$, omitting (for simplicity) the subindex for $(\nu_n, k_n)$, and noting that $n \to \infty$ iff $\nu + k \to \infty$, we obtain

$$Pl(B) = \lim_{\nu+k \to \infty} \frac{4(2\nu + \sigma(\nu))k}{j_{\nu,k}^2}.$$

All we need now is to study the behavior of $j_{\nu,k}$ as $\nu + k \to \infty$. We will use the inequality of McCann [22, Corollary, p. 102] which states that

$$j_{\nu,k} > \left(\nu^2 + \pi^2 \left(k - \frac{1}{4}\right)^2\right)^{1/2} \qquad \text{for any } \nu \geq 0 \text{ and } k \in \mathbb{N}. \qquad (2.2)$$

Note first that the sequence of $\varphi_{0,k}$'s cannot be a maximizing sequence for $Pl(B)$ since otherwise the inequality (2.2) implies $j_{0,k} > \pi(k-1)$ and hence

$$Pl(B) = \lim_{k \to \infty} \frac{4k}{j_{0,k}^2} \leq \lim_{k \to \infty} \frac{4k}{\pi^2(k-1)^2} = 0, \qquad (2.3)$$

but we will see later that $Pl(B) > 0$. Thus, we may assume that $\nu > 0$, which yields $\mu(\varphi_{\nu,k}) = 2\nu k$ and

$$Pl(B) = \lim_{\nu+k \to \infty} \frac{8\nu k}{j_{\nu,k}^2}. \qquad (2.4)$$

Estimating now (2.2) from below as $j_{\nu,k} > \nu$ and as $j_{\nu,k} > \frac{\pi k}{2}$, we deduce that

$$Pl(B) \le \lim_{\nu+k\to\infty} \min\left\{\frac{8k}{\nu}, \frac{32\nu}{\pi^2 k}.\right\} \tag{2.5}$$

As we already mentioned, it will be shown later that $Pl(B) > 0$. Using this inequality, we conclude from (2.5) that both $\nu$ and $k$ tend to infinity, and there exist $A_1, A_2 > 0$ such that

$$A_1\nu < k < A_2\nu \quad \text{for all sufficiently large } \nu \in \mathbb{N}.$$

Recalling that $(\nu, k)$ is a maximizing subsequence for $Pl(B)$, we can select a sub-subsequence (which is hence also a maximizing subsequence for $Pl(B)$) still denoted by $(\nu, k)$, such that

$$\lim_{\nu\to\infty} \frac{k}{\nu} = x_0 \in [A_1, A_2]. \tag{2.6}$$

That is, we have $k = \nu x_0 + o(\nu)$ for all large $\nu \in \mathbb{N}$.
Let us now use the result of Elbert & Laforgia [13] (see [12, Section 1.5] for the precise statement employed in (2.7)) which states that

$$\lim_{\nu\to\infty} \frac{j_{\nu,\nu x}}{\nu} = \frac{1}{\cos\theta(x)}, \quad x > 0, \tag{2.7}$$

where $\theta = \theta(x)$ is the solution of the (transcendental) equation

$$\tan\theta - \theta = \pi x, \quad \theta \in \left(0, \frac{\pi}{2}\right). \tag{2.8}$$

Combining (2.4), (2.6), and (2.7), we see that $Pl(B) = 8x_0 \left(\cos\theta(x_0)\right)^2$, and $x_0$ have to satisfy

$$Pl(B) = 8x_0 \left(\cos\theta(x_0)\right)^2 = 8 \sup_{x>0}\left\{x\left(\cos\theta(x)\right)^2\right\} > 0. \tag{2.9}$$

Most likely, (2.8) and hence (2.9) cannot be resolved in closed forms. However, one can convince himself that the left-hand side of (2.8) is strictly increasing in $\left(0, \frac{\pi}{2}\right)$ and the function $x\left(\cos\theta(x)\right)^2$ for $x > 0$ attains its unique maximum, and hence the unique root of (2.8) and the value of $Pl(B)$ can be found with arbitrary precision via the standard numerical methods. In particular, using the build-in methods of *Mathematica*, we obtain

$$Pl(B) = 0.4613019\ldots \quad \text{and} \quad x_0 = \lim_{\nu\to\infty}\frac{k}{\nu} = 0.3710096\ldots \qquad \square$$

REMARK 2.1. *Mathematica*'s code for finding $Pl(B)$ via (2.9) can look like that:

```
T[x_?NumericQ] := y /. FindRoot[Tan[y] - y == Pi*x, {y, Pi/4}];
FindMaximum[8*x*(Cos[T[x]])^2, {x, 0.37}]
```

REMARK 2.2. Consideration of the sequence of radial eigenfunctions of (1.1) on $B$ reveals (see (2.3)) that

$$\liminf_{n \to \infty} \frac{\mu(\varphi_n)}{n} = 0.$$

REMARK 2.3. If we consider the Neumann eigenvalues instead of the Dirichlet ones, then the result of Theorem 1.3 remains valid. Indeed, Neumann eigenfunctions are expressed as

$$\phi_{\nu,k}(\varrho, \vartheta) = J_\nu(j'_{\nu,k}\varrho) \cos(\nu\vartheta), \quad \nu \in \mathbb{N} \cup \{0\}, \ k \in \mathbb{N},$$

where $j'_{\nu,k}$ is the $k$-th positive zero of the derivative $J'_\nu$ of the Bessel function $J_\nu$. Moreover, $\lambda_{\nu,k} = (j'_{\nu,k})^2$ is the associated eigenvalue. (Note that $\lambda = 0$ is an additional eigenvalue corresponding to the constant eigenfunction.) According to the result of Ashu [2, Theorem 3.2] (see also a generalization obtained by Helffer & Sundqvist [17, Lemma 2.5]), $J'_\nu$ and $J'_{\nu+m}$ have no common zeros for any $m \in \mathbb{N}$. Thus, the multiplicity of $\lambda_{0,k}$ is 1, and the multiplicity of any other eigenvalue is 2. Note that the Weyl law is still valid for the Neumann case (see, e.g., Ivrii [18]). Moreover, $j_{0,k} < j'_{0,k} < j_{0,k+1}$ for $k \geq 1$, and $j_{\nu,k-1} < j'_{\nu,k} < j_{\nu,k}$ for $\nu > 0$ and $k \geq 2$. These inequalities imply that $\phi_{0,k}$ ($\phi_{\nu,k}$ for $\nu > 1$) is, in fact, a restriction to $B$ of the Dirichlet eigenfunction $\varphi_{0,k+1}$ ($\varphi_{\nu,k}$ for $\nu > 1$) defined on a slightly bigger ball $B_R$. Hence, $\mu(\phi_{0,k}) = k+1$ and $\mu(\phi_{\nu,k}) = 2\nu k$ for $\nu \in \mathbb{N}$. Arguing now as in the proof of Theorem 1.3 and estimating $j'_{\nu,k}$ from both sides as above, we obtain the desired result.

Note that the Pleijel theorem (Theorem 1.1) for the Neumann eigenvalues was proved by Polterovich [25] for $N = 2$ and piecewise real analytic domains, and by Léna [21] for $N \geq 2$ and $C^{1,1}$-smooth domains.

## 2.3    PROOF OF THEOREM 1.4

Consider the circular sector of a unit disk

$$\Sigma^\alpha = \left\{ (\varrho \cos\vartheta, \varrho \sin\vartheta) : \ \varrho \in (0,1), \ \vartheta \in \left(-\frac{\alpha}{2}, \frac{\alpha}{2}\right) \right\}, \quad \alpha \in (0, 2\pi].$$

By separation of variables, it can be shown (see, e.g., [6, Proposition 2.1]) that there is a basis of eigenfunctions of (1.1) on $\Sigma^\alpha$ of the form

$$\varphi^\alpha_{\nu,k}(\varrho, \vartheta) = J_{\nu\frac{\pi}{\alpha}}\left(j_{\nu\frac{\pi}{\alpha},k}\varrho\right) \sin\left(\nu\pi\left(\frac{\vartheta}{\alpha} + \frac{1}{2}\right)\right), \quad \nu, k \in \mathbb{N},$$

with the associated eigenvalues $\lambda_{\nu,k} = j^2_{\nu\frac{\pi}{\alpha},k}$. Evidently, $\mu(\varphi^\alpha_{\nu,k}) = \nu k$. We assume that $\alpha$ is chosen in such a way that each $\lambda_{\nu,k}$ has the multiplicity 1, that is, no other eigenfunctions occur.

The proof of the claim follows the same lines as the proof of Theorem 1.3. Let us present the arguments sketchily. Since $|\Sigma^\alpha| = \frac{\alpha}{2}$, the Weyl law implies

$$n = \lambda_n \frac{|\Sigma^\alpha||B|}{4\pi^2} + o(\lambda_n) = \frac{\lambda_n \alpha}{8\pi} + o(\lambda_n) = \frac{j^2_{\nu_n \frac{\pi}{\alpha}, k_n} \alpha}{8\pi} + o(j^2_{\nu_n \frac{\pi}{\alpha}, k_n}),$$

and hence, for a maximizing subsequence,

$$Pl(\Sigma^\alpha) = \lim_{\nu+k\to\infty} \frac{8\pi\nu k}{j^2_{\nu \frac{\pi}{\alpha}, k} \alpha} = \lim_{\tau+k\to\infty} \frac{8\tau k}{j^2_{\tau, k}},$$

where $\tau = \nu \frac{\pi}{\alpha}$. Applying the inequality (2.2), we deduce that, under the assumption $Pl(\Sigma^\alpha) > 0$, it holds $k = \tau x_0 + o(\tau)$ for some $x_0 > 0$ and all sufficiently large $\tau \in \mathbb{N}$. Using now (2.7), we finally obtain

$$Pl(\Sigma^\alpha) = 8x_0 (\cos\theta(x_0))^2 = 8 \sup_{x>0} \left\{ x (\cos\theta(x))^2 \right\} > 0.$$

Thus, we see that $Pl(\Sigma^\alpha) = Pl(B)$ for any admissible $\alpha \in (0, 2\pi]$. However, the value of $\lim_{\nu\to\infty} \frac{k}{\nu}$ depends on $\alpha$ as follows:

$$\lim_{\nu\to\infty} \frac{k}{\nu} = \frac{\pi}{\alpha} \lim_{\nu\to\infty} \frac{k}{\tau} = \frac{\pi x_0}{\alpha} = \frac{1.165561\ldots}{\alpha}.$$

Finally, let us note that if $\alpha = \pi/m$ for $m \in \mathbb{N}$, then any eigenvalue of (1.1) on $\Sigma^\alpha$ has the multiplicity 1. This fact trivially follows from the validity of Bourget's hypothesis [27, p. 484], since $J_{\nu \frac{\pi}{\alpha}} = J_{\nu m}$, that is, we have the Bessel function of an integer order for any $\nu \in \mathbb{N}$. $\qquad\square$

## 2.4 Proof of Proposition 1.6

Consider the ring $A_r = \{x \in \mathbb{R}^2 : r < |x| < 1\}$. Separating the variables, it is not hard to find a set of eigenfunctions (up to rotation) of (1.1) on $A_r$ in the form

$$\psi_{\nu,k}(\varrho, \vartheta) = (J_\nu(a_{\nu,k}\varrho)Y_\nu(a_{\nu,k}) - J_\nu(a_{\nu,k})Y_\nu(a_{\nu,k}\varrho))\cos(\nu\vartheta), \qquad (2.10)$$

where $\nu \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{N}$, and $\lambda_{\nu,k} = a^2_{\nu,k}$ is the eigenvalue associated with $\psi_{\nu,k}$. Here $a_{\nu,k}$ is the $k$-th positive zero of the following cross-product of Bessel functions of the first and second kind:

$$J_\nu(rz)Y_\nu(z) - J_\nu(z)Y_\nu(rz). \qquad (2.11)$$

That is,

$$J_\nu(ra_{\nu,k})Y_\nu(a_{\nu,k}) - J_\nu(a_{\nu,k})Y_\nu(ra_{\nu,k}) = 0.$$

It is possible to show that the set of $\psi_{\nu,k}$'s forms an orthogonal basis of $L^2(A_r)$. Any eigenvalue $\lambda_{0,k}$ has the multiplicity *at least* 1 and any other eigenvalue has

the multiplicity *at least* 2. Moreover, $\mu(\psi_{0,k}) = k$ and $\mu(\psi_{\nu,k}) = 2\nu k$ for $\nu \in \mathbb{N}$. As in the proof of Theorem 1.3, we will write $\mu(\psi_{\nu,k}) = (2\nu + \sigma(\nu))k$, where $\sigma(0) = 1$ and $\sigma(\nu) = 0$ for $\nu \in \mathbb{N}$.

Assume that the multiplicity of any eigenvalue is at most 2. As a consequence, we see that any eigenfunction of (1.1) on $A_r$ is of the form (2.10), up to scaling. Putting $\lambda_{\nu,k}$ in the increasing order as $\lambda_1 \le \ldots \lambda_n \le \ldots$, noting that $|A_r| = \pi(1 - r^2)$, and applying the Weyl law, we get

$$n = \lambda_n \frac{|B||A_r|}{4\pi^2} + o(\lambda_n) = \lambda_n \frac{1 - r^2}{4} + o(\lambda_n) = a_{\nu_n,k_n}^2 \frac{1 - r^2}{4} + o(a_{\nu_n,k_n}^2).$$

Therefore, passing to a maximizing subsequence and omitting the subindex for $(\nu_n, k_n)$, we deduce that

$$Pl(A_r) = \lim_{\nu+k\to\infty} \frac{4(2\nu + \sigma(\nu))k}{a_{\nu,k}^2(1 - r^2)}.$$

Let us note that the inequality (2.2) from [22, Corollary, p. 102] used in the proof of Theorem 1.3 is, in fact, a consequence of the inequality [22, (10), p. 102] for $a_{\nu,k}$:

$$\left(\frac{a_{\nu,k}}{\nu}\right)^2 \ge \left(\frac{a_{0,k}}{\nu}\right)^2 + 1.$$

If, for the maximizing sequence of $Pl(A_r)$, $k$ is bounded, then we can find $C > 0$ such that $a_{0,k} > Ck$ for all $k$. Otherwise, if $k \to \infty$, then we use the following approximation of $a_{0,k}$ of McMahon [23, (24), p. 29] (see also Cochran [9, Theorem, p. 583]):

$$a_{0,k} = \frac{\pi k}{1 - r} + O\left(\frac{1}{k}\right). \tag{2.12}$$

Therefore, in both cases, we have the existence of $C > 0$ such that

$$a_{\nu,k}^2 \ge Ck^2 + \nu^2 \quad \text{for all admissible } \nu, k \in \mathbb{N}. \tag{2.13}$$

Assume first that $Pl(A_r) > 0$. Then the case $\nu = 0$ can be discarded via (2.12), which implies that $\nu > 0$ and $\mu(\psi_{\nu,k}) = 2\nu k$. Estimating (2.13) as $a_{\nu,k}^2 \ge Ck^2$ and as $a_{\nu,k}^2 \ge \nu^2$, we obtain

$$Pl(A_r) \le \frac{8}{1 - r^2} \lim_{\nu+k\to\infty} \min\left\{\frac{k}{\nu}, \frac{\nu}{Ck}\right\}.$$

Since $Pl(A_r) > 0$, we conclude that both $\nu$ and $k$ tend to infinity, and there exist $A_1, A_2 > 0$ such that

$$A_1 k < \nu < A_2 k \quad \text{for all sufficiently large } k \in \mathbb{N}.$$

Therefore, up to a subsequence, we have $\nu = k x_0 + o(k)$ for some $x_0 > 0$ and all large $k \in \mathbb{N}$. Thus,

$$Pl(A_r) = \frac{8}{1 - r^2} \sup_{x>0}\left\{x \limsup_{k\to\infty} \frac{k^2}{a_{kx,k}^2}\right\}. \tag{2.14}$$

Assume now that $Pl(A_r) = 0$. Then the expression on the right-hand side of (2.14) provides a lower bound for $Pl(A_r)$, which again implies that the equality (2.14) is satisfied. □

Corollary 2.4. *Combining the upper estimate* (1.2) *(which is strict by [7, 26]) with the expression* (2.14)*, we obtain the following lower bound for the k-th positive zero of the cross-product of Bessel functions* (2.11) *of order kx, x > 0:*

$$a_{kx,k} > \frac{\sqrt{2}j_{0,1}k}{\sqrt{1-r^2}} + o(k) > \frac{3.4\,k}{\sqrt{1-r^2}} + o(k)$$

## 2.5  Proof of Proposition 1.7

Consider the annular sector

$$\Sigma_r^\alpha = \left\{ (\varrho\cos\vartheta, \varrho\sin\vartheta): \ \varrho \in (r,1), \ \vartheta \in \left(-\frac{\alpha}{2}, \frac{\alpha}{2}\right) \right\}, \quad r \in (0,1), \ \alpha \in (0, 2\pi].$$

By separation of variables, it can be derived that there is a basis of eigenfunctions of (1.1) on $\Sigma_r^\alpha$ of the form

$$\psi_{\nu,k}^\alpha(\varrho, \vartheta) = \left(J_{\nu\frac{\pi}{\alpha}}(a_{\nu\frac{\pi}{\alpha},k}\varrho)Y_{\nu\frac{\pi}{\alpha}}(a_{\nu\frac{\pi}{\alpha},k}) - J_{\nu\frac{\pi}{\alpha}}(a_{\nu\frac{\pi}{\alpha},k})Y_{\nu\frac{\pi}{\alpha}}(a_{\nu\frac{\pi}{\alpha},k}\varrho)\right)$$
$$\cdot \sin\left(\nu\pi\left(\frac{\vartheta}{\alpha} + \frac{1}{2}\right)\right)$$

for $\nu, k \in \mathbb{N}$, with the associated eigenvalues $\lambda_{\nu,k} = a_{\nu\frac{\pi}{\alpha},k}^2$. Here $a_{\nu\frac{\pi}{\alpha},k}$ is the k-th positive zero of the cross-product of Bessel functions (2.11) of order $\nu\frac{\pi}{\alpha}$. It is not hard to see that $\mu(\psi_{\nu,k}^\alpha) = \nu k$, and any eigenvalue $\lambda_{\nu,k}$ has the multiplicity at least 1.
Assuming that each $\lambda_{\nu,k}$ has the multiplicity exactly 1, we argue in much the same way as in the proof of Theorem 1.6 (and Theorem 1.4) to conclude that

$$Pl(\Sigma_r^\alpha) = Pl(A_r) = \frac{8}{1-r^2} \sup_{x>0}\left\{ x \limsup_{k\to\infty} \frac{k^2}{a_{kx,k}^2} \right\}. \qquad \square$$

## 2.6  Proof of Lemma 1.9

Note that any zero $a_{\nu,k}(r)$ of the cross-product of Bessel functions (2.11) of order $\nu$ is a continuous (in fact, analytic) function with respect to $r > 0$, see Cochran [10, Theorem 1]. Here $r > 0$ stands for the inner radius of the ring $A_r$. Then, the numerical computations show that $a_{0,2}(0.01) \approx 6.0109$ and $a_{3,1}(0.01) \approx 6.3801$, while $a_{0,2}(0.1) \approx 6.8575$ and $a_{3,1}(0.1) \approx 6.3804$. Therefore, by continuity, there exists $r_0 \in (0.01, 0.1)$ such that $a_{3,1}(r_0) = a_{0,2}(r_0)$, which implies that the eigenvalue $\lambda = \lambda_{3,1} = \lambda_{0,2}$ has the multiplicity at least 3. (According to the numerical computations, $r_0 \approx 0.044951$ and $\lambda \approx 40.7064$; see Fig. 1.) Actually, this fact was observed already by Kline [20, Fig. 1]. The claim for annular sectors can be obtained either by the same method as above or by the arguments of [6, Section 2.3]. □
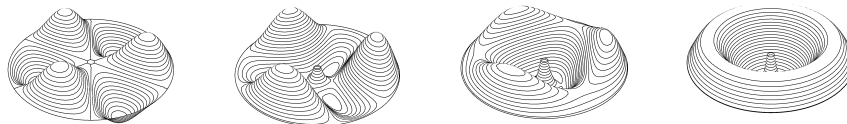
$$C_1 = 1,\ C_2 = 0 \qquad C_1 = 1,\ C_2 = 0.5 \qquad C_1 = 0.5,\ C_2 = 1 \qquad C_1 = 0,\ C_2 = 1$$

Figure 1: Eigenfunctions $C_1 \psi_{3,1} + C_2 \psi_{0,2}$ of (1.1) on $A_{r_0}$ with $r_0 \approx 0.044951$ associated with $\lambda = \lambda_{3,1} = \lambda_{0,2}$.

## References

[1] Aronovitch, A., Band, R., Fajman, D., & Gnutzmann, S. (2012). Nodal domains of a non-separable problem — the right-angled isosceles triangle. Journal of Physics A: Mathematical and Theoretical, 45(8), 085209. DOI:10.1088/1751-8113/45/8/085209 arXiv:1110.1521

[2] Ashu, A. M. (2013). Some properties of Bessel functions with applications to Neumann eigenvalues in the unit disc. Bachelor's thesis (E. Wahlén advisor), Lund University. http://lup.lub.lu.se/student-papers/record/7370411

[3] Bérard, P., & Meyer, D. (1982). Inégalités isopérimétriques et applications. Annales scientifiques de l'École Normale Supérieure, 15(3), 513-541. https://eudml.org/doc/urn:eudml:doc:82104

[4] Blum, G., Gnutzmann, S., & Smilansky, U. (2002). Nodal domains statistics: A criterion for quantum chaos. Physical Review Letters, 88(11), 114101. DOI:10.1103/PhysRevLett.88.114101 arXiv:0109029

[5] Bonnaillie-Noël, V., Helffer, B., & Hoffmann-Ostenhof, T. (2017). Nodal domains, spectral minimal partitions, and their relation to Aharonov-Bohm operators. IAMP News Bulletin, October 2017, 3-28. arXiv:1711.01174 http://www.iamp.org/bulletins/old-bulletins/Bulletin-October2017-print.pdf

[6] Bonnaillie-Noël, V., & Léna, C. (2014). Spectral minimal partitions of a sector. Discrete and Continuous Dynamical Systems-Series B, 19(1), 27-53. DOI:10.3934/dcdsb.2014.19.27

[7] Bourgain, J. (2015). On Pleijel's nodal domain theorem. International Mathematics Research Notices, 2015(6), 16011612. DOI:10.1093/imrn/rnt241 arXiv:1308.4422

[8] Charron, P. (2018). A Pleijel-type theorem for the quantum harmonic oscillator. Journal of Spectral Theory, 8(2), 715-732. DOI:10.4171/JST/211 arXiv:1512.07880

[9] Cochran, J. A. (1964). Remarks on the zeros of cross-product Bessel functions. Journal of the Society for Industrial and Applied Mathematics, 12(3), 580-587. DOI:10.1137/0112049

[10] Cochran, J. A. (1966). The analyticity of cross-product Bessel function zeros. Proceedings of the Cambridge Philosophical Society, 62(2), 215-226. DOI:10.1017/S0305004100039785

[11] Courant, R. (1923). Ein allgemeiner Satz zur Theorie der Eigenfunktionen selbstadjungierter Differentialausdrücke. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, 81-84. http://www.digizeitschriften.de/dms/resolveppn/?PID=GDZPPN002506300

[12] Elbert, Á. (2001). Some recent results on the zeros of Bessel functions and orthogonal polynomials. Journal of computational and applied mathematics, 133(1-2), 65-83. DOI:10.1016/S0377-0427(00)00635-X

[13] Elbert, Á., & Laforgia, A. (1994). A lower bound for the zeros of the Bessel functions. In Inequalities And Applications (pp. 179-185). DOI:10.1142/9789812798879_0015

[14] Gnutzmann, S., & Lois, S. (2013). On the nodal count statistics for separable systems in any dimension. Journal of Physics A: Mathematical and Theoretical, 46(4), 045201. DOI:10.1088/1751-8113/46/4/045201 arXiv:1208.2120

[15] Han, X., Murray, M., & Tran, C. (2017). Nodal lengths of eigenfunctions in the disc. arXiv:1708.08112

[16] Helffer, B., & Hoffmann-Ostenhof, T. (2015). A review on large $k$ minimal spectral $k$-partitions and Pleijel's Theorem. Spectral theory and partial differential equations, 3957, Contemp. Math., 640, Amer. Math. Soc., Providence, RI, 2015. DOI:10.1090/conm/640/12841 arXiv:1509.04501

[17] Helffer, B., & Sundqvist, M. (2016). On nodal domains in Euclidean balls. Proceedings of the American Mathematical Society, 144(11), 4777-4791. DOI:10.1090/proc/13098 arXiv:1506.04033

[18] Ivrii, V. Y. (1980). Second term of the spectral asymptotic expansion of the Laplace-Beltrami operator on manifolds with boundary. Functional Analysis and Its Applications, 14(2), 98-106. DOI:10.1007/BF01086550

[19] Kershaw, D. (1983). Some extensions of W. Gautschi's inequalities for the gamma function. Mathematics of Computation, 607-611. DOI:10.2307/2007697

[20] Kline, M. (1948). Some Bessel equations and their applications to guide and cavity theory. Studies in Applied Mathematics, 27(1-4), 37-48. DOI:10.1002/sapm194827137

[21] Léna, C. (2016). Pleijel's nodal domain theorem for Neumann eigenfunctions. arXiv:1609.02331

[22] McCann, R. C. (1977). Lower bounds for the zeros of Bessel functions. Proceedings of the American Mathematical Society, 64(1), 101-103. DOI:10.1090/S0002-9939-1977-0442316-6

[23] McMahon, J. (1894). On the roots of the Bessel and certain related functions. The Annals of Mathematics, 9(1/6), 23-30. DOI:10.2307/1967501

[24] Pleijel, A. (1956). Remarks on Courant's nodal line theorem. Communications on pure and applied mathematics, 9(3), 543-550. DOI:10.1002/cpa.3160090324

[25] Polterovich, I. (2009). Pleijel's nodal domain theorem for free membranes. Proceedings of the American Mathematical Society, 137(3), 1021-1024. DOI:10.1090/S0002-9939-08-09596-8 arXiv:0805.1553

[26] Steinerberger, S. (2014). A Geometric Uncertainty Principle with an Application to Pleijel's Estimate. Annales Henri Poincaré, 15(12), 22992319. DOI:10.1007/s00023-013-0310-4 arXiv:1306.3103

[27] Watson, G. N. (1944). A treatise on the theory of Bessel functions. Cambridge: The University Press.

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