Prolongations of t-Motives and Algebraic Independence of Periods

Andreas Maurischat

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Abstract. In this article we show that the coordinates of a period lattice generator of the n-th tensor power of the Carlitz module are algebraically independent, if n is prime to the characteristic. The main part of the paper, however, is devoted to a general construction for t-motives which we call *prolongation*, and which gives the necessary background for our proof of the algebraic independence. Another ingredient is a theorem which shows hypertranscendence for the Anderson-Thakur function $\omega(t)$, i.e. that $\omega(t)$ and all its hyperderivatives with respect to t are algebraically independent.

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1 Introduction

Periods of t-modules play a central role in number theory in positive characteristic, and questions about their algebraic independence are of major interest. The most prominent period is the Carlitz period

$$\tilde{\pi} = \lambda_\theta \theta \prod_{j\geq 1} (1-\theta^{1-q^j})^{-1} \in K_\infty(\lambda_\theta),$$

where $\lambda_\theta \in \bar{K}$ is a $(q-1)$-th root of $-\theta$. Here, $K = \mathbb{F}_q(\theta)$ is the rational function field over the finite field $\mathbb{F}_q$, $\bar{K}$ its algebraic closure, and K_∞ is the completion of K with respect to the absolute value $|\cdot|_\infty$ given by $|\theta|_\infty = q$. The Carlitz period is the function field analog of the complex number $2\pi i$, and it was already proven by Wade in 1941 that $\tilde{\pi}$ is transcendental over K (see [18]).

For proving algebraic independence of periods (and other "numbers" like zeta values and logarithms) the ABP-criterion (cf. [2, Thm. 3.1.1]) and a consequence of it - which is part of the proof of [16, Thm. 5.2.2] - turned out to be very useful. To state this consequence, let $\mathbb{C}_\infty$ denote the completion of the algebraic closure of K_∞, and $\mathbb{C}_\infty[\![t]\!]$ the power series ring over $\mathbb{C}_\infty$, as well as $\mathbb{T} = \mathbb{C}_\infty\langle t\rangle$ the subring consisting of those power series which converge on the closed unit disc $|t|_\infty \leq 1$. Finally, let $\mathbb{E}$ be the subring of entire functions, i.e. of those power series which converge for all $t \in \mathbb{C}_\infty$ and whose coefficients lie in a finite extension of K_∞. On $\mathbb{T}$ we consider the inverse Frobenius twist σ given by

$$\sigma\Big(\sum_{i=0}^{\infty} x_i t^i\Big) = \sum_{i=0}^{\infty} (x_i)^{1/q} t^i,$$

which will be applied on matrices entry-wise.

Theorem 1.1. *(See proof of [16, Thm. 5.2.2])* [1]
Let $\Phi \in \mathrm{Mat}_{r\times r}(\bar{K}[t])$ be a matrix with determinant $\det(\Phi) = c(t-\theta)^s$ for some $c \in \bar{K}^\times$ and $s \geq 1$. If $\Psi \in \mathrm{GL}_r(\mathbb{T}) \cap \mathrm{Mat}_{r\times r}(\mathbb{E})$ is a matrix such that

$$\sigma(\Psi) = \Psi\Phi,$$

then the transcendence degree of $\bar{K}(t)(\Psi)$ over $\bar{K}(t)$ is the same as the transcendence degree of $\bar{K}(\Psi(\theta))$ over $\bar{K}$.
Here, $\bar{K}(t)(\Psi)$ denotes the field extension of $\bar{K}(t)$ generated by the entries of Ψ, and $\bar{K}(\Psi(\theta))$ denotes the field extension of $\bar{K}$ generated by the entries of $\Psi(\theta)$, the evaluation of the entries of Ψ at $t = \theta$.

[1] Note that the difference equation in [16] is given as $\sigma(\Psi) = \Phi\Psi$ from which our version is obtained by transposing the matrices. We use this transposed version as it fits better to our convention on notation (cf. Sect. 2.2).

Actually, the matrix Φ occurs as a matrix which represents the σ-action on a dual t-motive $\mathfrak{M}$ with respect to some $\bar{K}[t]$-basis of $\mathfrak{M}$, and Ψ is the corresponding rigid analytic trivialization.
Using this statement, one can also reprove the transcendence of $\tilde{\pi}$ by using the power series

$$\Omega(t) = \lambda_\theta^{-q} \prod_{j\geq 1} (1 - \frac{t}{\theta^{q^j}}) \in \mathbb{E}.$$

This power series satisfies the difference equation $\sigma(\Omega) = \Omega\cdot(t-\theta)$ and is indeed the rigid analytic trivialization of the dual Carlitz motive $\mathfrak{C}$. The function Ω is transcendental over $\bar{K}(t)$ - as it has infinitely many zeros - and

$$\Omega(\theta) = \Omega|_{t=\theta} = \lambda_\theta^{-q} \prod_{j\geq 1} (1 - \frac{\theta}{\theta^{q^j}}) = -\frac{1}{\theta\lambda_\theta} \prod_{j\geq 1} (1 - \theta^{1-q^j}) = -\frac{1}{\tilde{\pi}}.$$

Hence by the criterion, $\tilde{\pi}$ is transcendental over $\bar{K}$.
Several proofs on algebraic independence (see e.g. [9],[15],[16]) follow the strategy to construct dual t-motives such that for the rigid analytic trivialization Ψ of this module, the inverse of its specialization $\Psi(\theta)^{-1}$ has the desired values as entries. Then one shows algebraic independence for the corresponding entries of Ψ or Ψ^{-1} using different methods (like the Galois theoretical methods developed in [16]) and deduces algebraic independence of the desired values.

The main theorem in the present paper is about the periods of the n-th tensor power of the Carlitz module. The n-th tensor power $E = C^{\otimes n}$ of the Carlitz module is a uniformizable t-module of dimension n and rank 1. Hence, the period lattice for E is an $\mathbb{F}_q[\theta]$-submodule of $\mathrm{Lie}(E)(\mathbb{C}_\infty) \cong \mathbb{C}_\infty^n$ of rank 1, and we will show the following.

THEOREM 1.2. *(see Thm. 8.1)*
Let $n \in \mathbb{N}$ be prime to q, let $C^{\otimes n}$ be the n-th tensor power of the Carlitz module and let

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{C}_\infty^n$$

be a generator for the period lattice. Then $z_1, z_2, \ldots, z_n$ are algebraically independent over $\bar{K}$.

The first step will be the definition of an appropriate dual t-motive such that the specialization at $t = \theta$ of the inverse of the rigid analytic trivialization contains such coordinates $z_1, \ldots, z_n$. As this is a special case of a general construction of new t-motives from old ones, we present this construction in detail. Actually, the main part of the paper is devoted to this construction which we call *prolongation*, due to its similarities to prolongations in differential geometry.

In Section 3, we start by defining the prolongations of (non-dual) t-motives, since they are often defined over a smaller base field than the dual t-motives, and we show various properties which transfer from the original t-motive to its prolongation. We also give the explicit descriptions with matrices for abelian t-motives. In Section 4, we transfer the definition of prolongation and the explicit description to dual t-motives, and in Section 5, we transfer it to t-modules, too.

For the definition of prolongations, we make use of hyperdifferential operators (also called iterative higher derivations) with respect to the variable t. These are the family of $\mathbb{C}_\infty$-linear maps $(\partial_t^{(n)})_{n\geq 0}$ given by

$$\partial_t^{(n)}\left(\sum_{i=i_0}^{\infty} x_i t^i\right) = \sum_{i=i_0}^{\infty} \binom{i}{n} x_i t^{i-n}$$

for Laurent series $\sum_{i=i_0}^{\infty} x_i t^i \in \mathbb{C}_\infty(\!(t)\!)$, where $\binom{i}{n} \in \mathbb{F}_p \subset \mathbb{F}_q$ is the residue of the usual binomial coefficient. One should think of the n-th hyperdifferential operator $\partial_t^{(n)}$ as $\frac{1}{n!}(d/dt)^n$, although in characteristic p, we can't divide by $n!$, if $n \geq p$. In characteristic zero, however, $\partial_t^{(n)}$ would be exactly $\frac{1}{n!}(d/dt)^n$.
As a warning to the reader, we would like to note that in the literature usually the hyperdifferential operators with respect to $\theta \in K$ are used (e.g. in [5], [6], [7]), and hence the operation on power series is by hyperdifferentiating the coefficients. In this article, we will not use those hyperdifferential operators, but exclusively the hyperdifferentiation by t.

In the proof of the main theorem, the Anderson-Thakur function $\omega(t)$ and its hyperderivatives appear, as ω is related to Ω via

$$\omega = \frac{1}{(t-\theta)\Omega}.$$

In Section 7, we show a property of ω which is of interest on its own, namely we show

THEOREM 1.3. *(see Thm. 7.2) The Anderson-Thakur function $\omega(t)$ is hypertranscendental over $\bar{K}(t)$, i.e. the set $\{\partial_t^{(n)}(\omega) \mid n \geq 0\}$ is algebraically independent over $\bar{K}(t)$.*

This will be deduced from properties of specializations of ω and its hyperderivatives at roots of unity which were investigated in [4] and [14]. This statement has also been given in [17] whose proof uses different methods.

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2 Generalities

2.1 Base rings and operators

Let $\mathbb{F}_q$ be the finite field with q elements, and K a finite extension of the rational function field $\mathbb{F}_q(\theta)$ in the variable θ. We choose an extension to K of the absolute value $|\cdot|_\infty$ which is given on $\mathbb{F}_q(\theta)$ by $|\theta|_\infty = q$. Furthermore, $K_\infty \supseteq \mathbb{F}_q(\!(\frac{1}{\theta})\!)$ denotes the completion of K at this infinite place, and $\mathbb{C}_\infty$ the completion of an algebraic closure of K_∞. Furthermore, let $\bar{K}$ be the algebraic closure of K inside $\mathbb{C}_\infty$.

All the commutative rings occuring will be subrings of the field of Laurent series $\mathbb{C}_\infty(\!(t)\!)$, like the polynomial rings $K[t]$ and $\bar{K}[t]$, the power series ring $\mathbb{C}_\infty[\![t]\!]$ and the Tate algebra $\mathbb{T} = \mathbb{C}_\infty\langle t\rangle$, i.e. the algebra of series which are convergent for $|t|_\infty \leq 1$.

On $\mathbb{C}_\infty(\!(t)\!)$ we have several operations which will induce operations on these subrings.

First of all, there is the twisting $\tau : \mathbb{C}_\infty(\!(t)\!) \to \mathbb{C}_\infty(\!(t)\!)$ given by

$$f^\tau := \sum_{i=i_0}^{\infty} (x_i)^q t^i$$

for $f = \sum_{i=i_0}^{\infty} x_i t^i \in \mathbb{C}_\infty(\!(t)\!)$, and the inverse twisting $\sigma : \mathbb{C}_\infty(\!(t)\!) \to \mathbb{C}_\infty(\!(t)\!)$ given by

$$f^\sigma := \sum_{i=i_0}^{\infty} (x_i)^{1/q} t^i$$

for $f = \sum_{i=i_0}^{\infty} x_i t^i \in \mathbb{C}_\infty(\!(t)\!)$. While the twisting restricts to endomorphisms on all subrings of $\mathbb{C}_\infty(\!(t)\!)$ which occur in this paper, the inverse twisting is only defined for perfect coefficient fields, in particular not on $K[t]$, but on $\bar{K}[t]$.

On the Laurent series ring $\mathbb{C}_\infty(\!(t)\!)$ we furthermore have an action of the hyperdifferential operators with respect to t, i.e. the sequence of $\mathbb{C}_\infty$-linear maps $(\partial_t^{(n)})_{n\geq 0}$ given by

$$\partial_t^{(n)}\biggl(\sum_{i=i_0}^{\infty} x_i t^i\biggr) = \sum_{i=i_0}^{\infty} \binom{i}{n} x_i t^{i-n}.$$

The image $\partial_t^{(n)}(f)$ of some $f \in \mathbb{C}_\infty(\!(t)\!)$ is called the n-th hyperderivative of f. The hyperdifferential operators satisfy $\partial_t^{(0)}(f) = f$ for all $f \in \mathbb{C}_\infty(\!(t)\!)$,

$$\partial_t^{(n)}(fg) = \sum_{i=0}^{n} \partial_t^{(i)}(f)\, \partial_t^{(n-i)}(g) \quad \text{for all } f, g \in \mathbb{C}_\infty(\!(t)\!), n \in \mathbb{N}$$

as well as

$$\partial_t^{(n)}\Bigl(\partial_t^{(m)}(f)\Bigr) = \binom{n+m}{n} \partial_t^{(n+m)}(f) \quad \text{for all } f \in \mathbb{C}_\infty(\!(t)\!), n, m \in \mathbb{N}.$$

It is not hard to verify that the subrings $\mathbb{C}_\infty[\![t]\!]$, $\mathbb{T}$, $L[t]$, and $L(t)$ (for any subfield L of $\mathbb{C}_\infty$) are stable under all the hyperdifferential operators. It is also obvious that the hyperdifferential operators commute with the twistings τ and σ.
Another way to obtain these hyperdifferential operators is to consider the $\mathbb{C}_\infty$-algebra map $\mathcal{D} : \mathbb{C}_\infty[\![t]\!] \to \mathbb{C}_\infty[\![t]\!][\![X]\!]$

$$f(t) \mapsto f(t+X) = \sum_{n\geq 0} f_n(t)X^n$$

given by replacing the variable t in the power series expansion for f by $t+X$, expanding each $(t+X)^n$ using the binomial theorem, and rearranging to obtain a power series in X. Then, one has

$$\partial_t^{(n)}(f) = f_n.$$

Since, $\partial_t^{(0)}(f) = f$ for all $f \in \mathbb{C}_\infty[\![t]\!]$, the homomorphism $\mathcal{D}$ can be extended to a $\mathbb{C}_\infty$-algebra map $\mathcal{D} : \mathbb{C}_\infty(\!(t)\!) \to \mathbb{C}_\infty(\!(t)\!)[\![X]\!]$, and we still have the identity

$$\mathcal{D}(f) = \sum_{n\geq 0} \partial_t^{(n)}(f)\, X^n.$$

For more background on hyperdifferential operators (iterative higher derivations) see for example [13, §27]. [2]

When we apply the twisting operators τ and σ as well as the hyperdifferential operators to matrices it is meant that we apply them coefficient-wise.

We will frequently use the following family (in $n \geq 0$) of homomorphisms of $\mathbb{C}_\infty$-algebras $\rho_{[n]} : \mathbb{C}_\infty(\!(t)\!) \to \mathrm{Mat}_{(n+1)\times(n+1)}(\mathbb{C}_\infty(\!(t)\!))$ defined by

$$\rho_{[n]}(f) := \begin{pmatrix} f & \partial_t^{(1)}(f) & \cdots & \partial_t^{(n)}(f) \\ 0 & f & \ddots & \vdots \\ \vdots & \ddots & \ddots & \partial_t^{(1)}(f) \\ 0 & \cdots & 0 & f \end{pmatrix}, \tag{1}$$

which already appears in [14]. This map arises from the homomorphism $\mathcal{D}$ by evaluation of X at the $(n+1)\times(n+1)$ nilpotent matrix

$$N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

[2] As already mentioned in the introduction, these hyperdifferential operators are not the one commonly used for constructing t-modules.

We will also apply $\rho_{[n]}$ to square matrices $\Theta \in \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty(\!(t)\!))$. In that case, $\rho_{[n]}(\Theta)$ is defined to be the block square matrix

$$\rho_{[n]}(\Theta) := \begin{pmatrix} \Theta & \partial_t^{(1)}(\Theta) & \partial_t^{(2)}(\Theta) & \cdots & \partial_t^{(n)}(\Theta) \\ 0 & \Theta & \partial_t^{(1)}(\Theta) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \partial_t^{(2)}(\Theta) \\ \vdots & & \ddots & \Theta & \partial_t^{(1)}(\Theta) \\ 0 & \cdots & \cdots & 0 & \Theta \end{pmatrix} \tag{2}$$

in the ring of $r(n+1) \times r(n+1)$-matrices. As mentioned before $\partial_t^{(1)}(\Theta)$ etc. is the matrix where we apply the hyperdifferential operators entry-wise. It is not hard to check that $\rho_{[n]} : \mathrm{Mat}_{r\times r}(\mathbb{C}_\infty(\!(t)\!)) \to \mathrm{Mat}_{r(n+1)\times r(n+1)}(\mathbb{C}_\infty(\!(t)\!))$ is a ring homomorphism, too.

As the hyperdifferential operators commute with twisting, $\rho_{[n]}$ also commutes with twisting.

2.2 Convention on notation

In the following sections, we will deal with t-modules, t-motives and dual t-motives. We use the definitions of these terms as given in the survey article [8]. For the convenience of the reader, we repeat these definitions below, but refer the reader to ibid. for more details. For recognizing the objects at first glance, t-modules will be denoted by italic letters, like E, t-motives with serif-less letters, like M, and dual t-motives in Fraktur font, like $\mathfrak{M}$.

Bases of finitely generated free modules (over some ring) will always be written as row vectors $\boldsymbol{e} = (e_1, \ldots, e_r)$ such that one obtains the familiar identification of the module with a module of column vectors by writing an arbitrary element $x = \sum_{i=1}^r x_i e_i$ as

$$\boldsymbol{e} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}.$$

A t-module (E, Φ) (or shortly, E) consists of an algebraic group E over K which is isomorphic to $\mathbb{G}_a^d$ for some $d > 0$, and an $\mathbb{F}_q$-algebra homomorphism

$$\Phi : \mathbb{F}_q[t] \to \mathrm{End}_{\mathrm{grp},\mathbb{F}_q}(E) \cong \mathrm{Mat}_{d\times d}(K\{\tau\}),$$

with the additional property that $\Phi(t) - \theta \cdot \mathrm{id}_E$ induces a nilpotent endomorphism on $\mathrm{Lie}(E)$. In other terms, if one writes

$$\Phi(t) = A_0 + A_1\tau + \ldots + A_s\tau^s \in \mathrm{Mat}_{d\times d}(K\{\tau\})$$

with respect to some isomorphism $\mathrm{End}_{\mathrm{grp},\mathbb{F}_q}(E) \cong \mathrm{Mat}_{d\times d}(K\{\tau\})$, then the matrix $A_0 - \theta \cdot \mathbb{1}_d \in \mathrm{Mat}_{d\times d}(K)$ is nilpotent.

A t-motive M is a left $K[t]\{\tau\}$-module which is free and finitely generated as $K\{\tau\}$-module, and such that

$$(t-\theta)^{\ell}(\mathsf{M}) \subseteq K[t]\cdot \tau(\mathsf{M})$$

for some $\ell \in \mathbb{N}$. A t-motive M is called abelian if it is also finitely generated as $K[t]$-module in which case it is even free as $K[t]$-module. An abelian t-motive M is called pure, if there exists a $K[\![1/t]\!]$-lattice H inside $\mathsf{M}\otimes_{K[t]} K(\!(1/t)\!)$ and $u,v\geq 1$ such that

$$t^u H = K[\![1/t]\!]\cdot \tau^v H.$$

The fraction $w=\frac{u}{v}$ is called the weight of M.
Given an abelian t-motive M with $K[t]$-basis $\boldsymbol{e}=(e_1,\ldots,e_r)$, then there is a matrix $\Theta\in \mathrm{Mat}_{r\times r}(K[t])$ representing the τ-action on M with respect to $\{e_1,\ldots,e_r\}$, i.e.

$$\tau(e_j)=\sum_{h=1}^{r}\Theta_{hj}e_h$$

for all $j=1,\ldots,r$. This will be written in matrix notation as

$$\tau(\boldsymbol{e})=\boldsymbol{e}\cdot\Theta.$$

For an arbitrary element $x=\sum_{i=1}^r x_ie_i$ one therefore has

$$\tau(x)=\boldsymbol{e}\cdot\Theta\cdot\begin{pmatrix}x_1\\ \vdots\\ x_r\end{pmatrix}^{\tau}.$$

Writing the basis as a row vector instead of a column vector, as for example in [16], causes the difference equations for the rigid analytic trivializations to have a different form which we will review now. However, the usual form is obtained by taking transposes of the matrices given here:
Given an abelian t-motive M with $K[t]$-basis $\boldsymbol{e}=(e_1,\ldots,e_r)$ and $\Theta\in \mathrm{Mat}_{r\times r}(K[t])$ such that

$$\tau(\boldsymbol{e})=\boldsymbol{e}\cdot\Theta,$$

a rigid analytic trivialization (if it exists) is a matrix $\Upsilon\in \mathrm{GL}_r(\mathbb{T})$ such that $\tau(\boldsymbol{e}\cdot\Upsilon)=\boldsymbol{e}\cdot\Upsilon$, i.e. such that

$$\Theta\cdot\Upsilon^{\tau}=\Upsilon.$$

If Υ exists, M is called rigid analytically trivial.
In [1], Anderson associated to a t-module E a t-motive $\mathsf{E}:=\mathrm{Hom}_{\mathrm{grp},\mathbb{F}_q}(E,\mathbb{G}_a)$ with t-action given by composition with $\Phi_t\in \mathrm{End}_{\mathrm{grp},\mathbb{F}_q}(E)$ and left-$K\{\tau\}$-action given by composition with elements in $K\{\tau\}\cong \mathrm{End}_{\mathrm{grp},\mathbb{F}_q}(\mathbb{G}_a)$. A t-module is then called abelian if the associated t-motive is abelian, and

Anderson proved (cf. [1, Thm. 1]) that this correspondence induces an anti-equivalence of categories between abelian t-modules and abelian t-motives. However, the proof even shows that it induces an anti-equivalence of categories between t-modules and t-motives.

A DUAL t-MOTIVE $\mathfrak{M}$ is a left $\bar{K}[t]\{\sigma\}$-module that is free and finitely generated as $\bar{K}\{\sigma\}$-module, and such that

$$(t-\theta)^{\ell}(\mathfrak{M}) \subseteq \sigma(\mathfrak{M})$$

for some $\ell \in \mathbb{N}$. A dual t-motive is called t-FINITE if it is also finitely generated as $\bar{K}[t]$-module in which case it is even free as $\bar{K}[t]$-module.
For a t-finite dual t-motive $\mathfrak{M}$ with $K[t]$-basis $\boldsymbol{e} = (e_1, \ldots, e_r)$ and $\tilde{\Theta} \in \mathrm{Mat}_{r\times r}(K[t])$ such that

$$\sigma(\boldsymbol{e}) = \boldsymbol{e} \cdot \tilde{\Theta}$$

a RIGID ANALYTIC TRIVIALIZATION (if it exists) is a matrix $\Psi \in \mathrm{GL}_r(\mathbb{T})$ such that $\sigma(\boldsymbol{e} \cdot \Psi^{-1}) = \boldsymbol{e} \cdot \Psi^{-1}$, i.e. such that

$$\Psi \cdot \tilde{\Theta} = \Psi^{\sigma}.$$

If Ψ exists, $\mathfrak{M}$ is called RIGID ANALYTICALLY TRIVIAL.
Similar, as for t-motives, Anderson associated to a t-module E over $\bar{K}$ a dual t-motive $\mathfrak{E} := \mathrm{Hom}_{\mathrm{grp},\mathbb{F}_q}(\mathbb{G}_a, E)$ with t-action given by composition with $\Phi_t \in \mathrm{End}_{\mathrm{grp},\mathbb{F}_q}(E)$ and left-$K\{\sigma\}$-action given by composition with elements in $K\{\sigma\} \cong K\{\tau\}^{\mathrm{op}} \cong \mathrm{End}_{\mathrm{grp},\mathbb{F}_q}(\mathbb{G}_a)^{\mathrm{op}}$. Anderson showed (cf. [11]) that this induces an equivalence of categories between t-modules over $\bar{K}$ and t-motives.

3 PROLONGATIONS OF t-MOTIVES

In this section, we introduce a construction of new t-motives from old ones which we call *prolongation*. The construction is taken from [12] where prolongations of difference modules are described. We also show (see Theorems 3.4 and 3.6) that the prolongations inherit the properties of abelianness, rigid analytic triviality as well as pureness from the original t-motive.

DEFINITION 3.1. For a $K[t]\{\tau\}$-module M and $k \geq 0$, the k-TH PROLONGATION of M is the $K[t]$-module $\rho_k\mathsf{M}$ which is generated by symbols $D_i m$, for $i = 0, \ldots, k$, $m \in \mathsf{M}$, subject to the relations

1. $D_i(m_1 + m_2) = D_i m_1 + D_i m_2$,
2. $D_i(a \cdot m) = \sum_{i_1+i_2=i} \partial_t^{(i_1)}(a) \cdot D_{i_2} m$,

for all $m, m_1, m_2 \in \mathsf{M}$, $a \in K[t]$ and $i = 0, \ldots, k$. The semi-linear τ-action on $\rho_k\mathsf{M}$ is given by

$$\tau(a \cdot D_i m) = a^{\tau} \cdot D_i(\tau(m)).$$

for $a \in K[t]$, $m \in \mathsf{M}$.

One should think of $D_i m$ as being the formal i-th hyperderivative of the element m.

Remark 3.2. It is not difficult to verify that the definition of the τ-action is well-defined. Hence, the k-th prolongation $\rho_k\mathsf{M}$ is again a $K[t]\{\tau\}$-module.
Furthermore, $\rho_0\mathsf{M}$ is naturally isomorphic to M (via $D_0 m \mapsto m$), and for $0 \leq l < k$ the l-th prolongation $\rho_l\mathsf{M}$ naturally is a $K[t]\{\tau\}$-submodule of $\rho_k\mathsf{M}$. For $0 \leq l < k$, we even obtain a short exact sequence of $K[t]\{\tau\}$-modules

$$0 \longrightarrow \rho_l\mathsf{M} \longrightarrow \rho_k\mathsf{M} \xrightarrow{\mathrm{pr}} \rho_{k-l-1}\mathsf{M} \to 0$$

where $\mathrm{pr}(D_i m) := D_{i-l-1}m$ for $i > l$ and all $m \in \mathsf{M}$, as well as $\mathrm{pr}(D_i m) := 0$ for $i \leq l$ and all $m \in \mathsf{M}$. In particular, taking $l = k-1$ and using the identification $\rho_0\mathsf{M} \cong \mathsf{M}$, we obtain the short exact sequence

$$0 \longrightarrow \rho_{k-1}\mathsf{M} \longrightarrow \rho_k\mathsf{M} \longrightarrow \mathsf{M} \to 0. \qquad (*)$$

Inductively, we see that $\rho_k\mathsf{M}$ is a $(k+1)$-fold extension of M with itself.
From this description as a $(k+1)$-fold extension of M with itself, we will be able to transfer several additional properties of M to the prolongation $\rho_k\mathsf{M}$ (see Theorem 3.4 and Theorem 3.6).

Lemma 3.3. *As a K-vector space, the k-th prolongation $\rho_k\mathsf{M}$ is generated by the symbols $D_i m$, for $i = 0, \ldots, k$, $m \in \mathsf{M}$, subject to the relations*

$$D_i(x_1 m_1 + x_2 m_2) = x_1 \cdot D_i m_1 + x_2 \cdot D_i m_2$$

for all $m_1, m_2 \in \mathsf{M}$, $x_1, x_2 \in K$ and $i = 0, \ldots, k$. The actions of t and τ are described by

$$\begin{aligned} t \cdot D_i m &= D_i(tm) - D_{i-1}m \\ \tau(D_i m) &= D_i(\tau(m)) \end{aligned}$$

for $m \in \mathsf{M}$, $i = 0, \ldots, k$ where we set $D_{-1}m := 0$.

Proof. Applying relation (2) above to $a = t$, leads to

$$D_i(tm) = t \cdot D_i m + 1 \cdot D_{i-1}m$$

for all $m \in \mathsf{M}$. Hence, $t \cdot D_i m = D_i(tm) - D_{i-1}m$.
This shows that $K[t]$-multiples of the $D_i m$ are in the K-span of all $D_i(m')$, and therefore $\rho_k\mathsf{M}$ is generated by all $D_i m$ as a K-vector space.
Restricting relation (2) to $a \in K$, we obtain $D_i(a \cdot m) = a \cdot D_i m$ for all $a \in K$ and $m \in \mathsf{M}$. Hence, the relations above reduce to

$$D_i(x_1 m_1 + x_2 m_2) = x_1 \cdot D_i m_1 + x_2 \cdot D_i m_2$$

for all $m_1, m_2 \in \mathsf{M}$, $x_1, x_2 \in K$ and $i = 0, \ldots, k$.
The given actions are clear from the equation above and the definition of $\rho_k\mathsf{M}$. □

THEOREM 3.4. *Let M be a t-motive. Then the k-th prolongation $\rho_k\mathsf{M}$ is a t-motive for all $k \geq 0$.*
If M is abelian, then so is $\rho_k\mathsf{M}$.

Proof. By Remark 3.2, we have an exact sequence of $K[t]\{\tau\}$-modules

$$0 \longrightarrow \rho_{k-1}\mathsf{M} \longrightarrow \rho_k\mathsf{M} \longrightarrow \mathsf{M} \to 0$$

using the identification $\rho_0\mathsf{M} \cong \mathsf{M}$ (see Equation (*)). Hence, it follows by induction on k that $\rho_k\mathsf{M}$ is free and finitely generated as $K\{\tau\}$-module if M is. Furthermore, if $\ell \in \mathbb{N}$ is such that

$$(t-\theta)^{\ell}(\mathsf{M}) \subseteq K[t] \cdot \tau(\mathsf{M}),$$

we obtain

$$(t-\theta)^{\ell}(\rho_k\mathsf{M}) \subseteq K[t] \cdot \tau(\rho_k\mathsf{M}) + \rho_{k-1}\mathsf{M},$$

and hence, inductively,

$$(t-\theta)^{\ell\cdot(k+1)}(\rho_k\mathsf{M}) \subseteq K[t] \cdot \tau(\rho_k\mathsf{M}).$$

Therefore, $\rho_k\mathsf{M}$ is a t-motive.
If M is abelian, i.e. free and finitely generated as a $K[t]$-module, then $\rho_k\mathsf{M}$ is free and finitely generated as a $K[t]$-module, since it is a $(k+1)$-fold extension of copies of M. □

LEMMA 3.5. *Let M be a t-motive, and $\boldsymbol{b} = (b_1, \dots, b_d)$ be a $K\{\tau\}$-basis of M. Then a $K\{\tau\}$-basis of $\rho_k\mathsf{M}$ is given by*

$$\boldsymbol{Db} = (D_0b_1, \dots, D_0b_d, D_1b_1, \dots, D_1b_d, \dots, \dots, D_kb_1, \dots, D_kb_d).$$

Proof. From the short exact sequence (*) we see that a $K\{\tau\}$-basis of $\rho_k\mathsf{M}$ is given by the join of a $K\{\tau\}$-basis of $\rho_{k-1}\mathsf{M}$ and the preimage of a basis of M. As such a preimage is given by $(D_kb_1, \dots, D_kb_d)$ the proof follows by induction. □

We are now going to explicitly describe the t-motive $\rho_k\mathsf{M}$ as $K[t]$-module with τ-action in the abelian case, i.e. we give a basis as $K[t]$-module as well as a matrix representation of the τ-action with respect to this $K[t]$-basis.

Assume that M is an abelian t-motive, and let $\boldsymbol{e} = (e_1, \dots, e_r)$ be a $K[t]$-basis of M. As in the previous lemma, from the short exact sequence (*) in Remark 3.2 we obtain that $\boldsymbol{De} = (D_0e_1, \dots, D_0e_r, D_1e_1, \dots, D_1e_r, \dots, \dots, D_ke_1, \dots, D_ke_r)$ is a $K[t]$-basis of $\rho_k\mathsf{M}$.

Let $\Theta \in \mathrm{Mat}_{r\times r}(K[t])$ be the matrix representing the τ-action on M with respect to $\boldsymbol{e} = (e_1, \dots, e_r)$, i.e.

$$\tau(e_j) = \sum_{h=1}^{r} \Theta_{hj} e_h$$

for all $j = 1, \dots, r$, or in matrix notation

$$\tau(\boldsymbol{e}) = \boldsymbol{e} \cdot \Theta.$$

Then τ acts on $D_i e_j \in \rho_k \mathsf{M}$ as

$$\tau(D_i e_j) = D_i(\tau(e_j)) = D_i(\sum_{h=1}^{r} \Theta_{hj} e_h) = \sum_{h=1}^{r} \sum_{i_1+i_2=i} \partial_t^{(i_1)}(\Theta_{hj}) \cdot D_{i_2} e_h.$$

In block matrix notation this reads as

$$\tau(\boldsymbol{De}) = \boldsymbol{De} \cdot \begin{pmatrix} \Theta & \partial_t^{(1)}(\Theta) & \partial_t^{(2)}(\Theta) & \cdots & \partial_t^{(k)}(\Theta) \\ 0 & \Theta & \partial_t^{(1)}(\Theta) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \partial_t^{(2)}(\Theta) \\ \vdots & & \ddots & \Theta & \partial_t^{(1)}(\Theta) \\ 0 & \cdots & \cdots & 0 & \Theta \end{pmatrix} = \boldsymbol{De} \cdot \rho_{[k]}(\Theta),$$

where we use the homomorphism $\rho_{[k]}$ defined in Equation (2).

THEOREM 3.6. *Let* M *be an abelian t-motive,* $k \geq 0$ *and* $\rho_k \mathsf{M}$ *the k-th prolongation of* M.

1. *If* M *is rigid analytically trivial, then* $\rho_k \mathsf{M}$ *is rigid analytically trivial.*
2. *If* M *is pure of weight* w*, then* $\rho_k \mathsf{M}$ *is pure of weight* w.

Proof. Let M be given with respect to a basis $\boldsymbol{e} = (e_1, \dots, e_r)$ by the τ-action

$$\tau(\boldsymbol{e}) = \boldsymbol{e} \cdot \Theta$$

for some $\Theta \in \mathrm{Mat}_{r \times r}(K[t])$. Assume that M is rigid analytically trivial, and that $\Upsilon \in \mathrm{GL}_r(\mathbb{T})$ is a rigid analytic trivialization of M, i.e. Υ satisfies the difference equation

$$\Upsilon = \Theta \Upsilon^{\tau}.$$

Since twisting commutes with $\rho_{[k]}$ and $\rho_{[k]}$ is a ring homomorphism, we have

$$\rho_{[k]}(\Theta) \left(\rho_{[k]}(\Upsilon)\right)^{\tau} = \rho_{[k]}(\Theta \Upsilon^{\tau}) = \rho_{[k]}(\Upsilon).$$

Since the τ-action on $\rho_k \mathsf{M}$ with respect to $\boldsymbol{De}$ from above is given by $\tau(\boldsymbol{De}) = \boldsymbol{De} \cdot \rho_{[k]}(\Theta)$, this just means that $\rho_{[k]}(\Upsilon) \in \mathrm{GL}_{r(k+1)}(\mathbb{T})$ is a rigid analytic trivialization of $\rho_k \mathsf{M}$.
Assume that M is pure of weight w, and let H be a $K[\![1/t]\!]$-lattice inside $\mathsf{M} \otimes_{K[t]} K(\!(1/t)\!)$ such that

$$t^u H = K[\![1/t]\!] \cdot \tau^v H$$

for appropriate $u, v \geq 1$.

After choosing a $K[\![1/t]\!]$-basis $\boldsymbol{b} = (b_1, \ldots, b_r)$ of H, we have

$$\tau^v(\boldsymbol{b}) = \boldsymbol{b} \cdot t^u A$$

for some $A \in \mathrm{GL}_r(K[\![1/t]\!])$. By the explicit description of the τ-action on $\rho_k\mathsf{M}$, we therefore get

$$\begin{aligned}\tau^v(\boldsymbol{Db}) &= \boldsymbol{Db} \cdot \rho_{[k]}(t^u A)\\ &= \boldsymbol{Db} \cdot t^u \begin{pmatrix} A & t^{-u}\partial_t^{(1)}(t^u A) & t^{-u}\partial_t^{(2)}(t^u A) & \cdots & t^{-u}\partial_t^{(k)}(t^u A) \\ 0 & A & t^{-u}\partial_t^{(1)}(t^u A) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & t^{-u}\partial_t^{(2)}(t^u A) \\ \vdots & & \ddots & A & t^{-u}\partial_t^{(1)}(t^u A) \\ 0 & \cdots & \cdots & 0 & A \end{pmatrix}.\end{aligned}$$

For Laurent series $f = \sum_{j=j_0}^{\infty} x_j t^{-j}$ in $1/t$ we have

$$\partial_t^{(n)}(f) = \sum_{j=j_0}^{\infty} \binom{-j}{n} x_j t^{-j-n}.$$

In particular, for any power series $f = \sum_{j=0}^{\infty} x_j t^{-j} \in K[\![1/t]\!]$ and $u \in \mathbb{Z}$,

$$\begin{aligned}t^{-u} \cdot \partial_t^{(n)}(t^u f) &= t^{-u} \cdot \partial_t^{(n)}\left(\sum_{j=0}^{\infty} x_j t^{-j+u}\right) = t^{-u} \cdot \sum_{j=0}^{\infty} \binom{-j+u}{n} x_j t^{-j+u-n}\\ &= \sum_{j=0}^{\infty} \binom{-j+u}{n} x_j t^{-j-n} \in t^{-n} K[\![1/t]\!] \subseteq K[\![1/t]\!].\end{aligned}$$

Hence, the block upper triangular matrix above has entries in $K[\![1/t]\!]$, and is moreover invertible over $K[\![1/t]\!]$, as A is invertible. Hence, by choosing $\rho_k H$ to be the $K[\![1/t]\!]$-lattice inside $\rho_k\mathsf{M} \otimes_{K[t]} K(\!(1/t)\!)$ generated by $\boldsymbol{Db}$ we obtain

$$K[\![1/t]\!] \cdot \tau^v(\rho_k H) = t^u \rho_k H.$$

Hence, $\rho_k\mathsf{M}$ is pure of weight $\frac{u}{v} = w$. □

Remark 3.7. Starting with a Drinfeld module, the associated t-motive is abelian, pure and rigid analytically trivial. Hence, by taking its prolongations we obtain new abelian, pure and rigid analytically trivial t-motives of arbitrary dimension.

4 Prolongations of dual t-motives

Since we will use the dual t-motives in the proof in Section 8, we review the construction and explicit descriptions in this case.

For the definition of a prolongation of a dual t-motive $\mathfrak{M}$ we just transfer the definition for the t-motives above.

Definition 4.1. For a dual t-motive $\mathfrak{M}$ over $\bar{K}[t]$ and $k \geq 0$, the k-th prolongation of $\mathfrak{M}$ is the $\bar{K}[t]$-module $\rho_k\mathfrak{M}$ which is generated by symbols $D_i m$, for $i = 0, \dots, k$, $m \in \mathfrak{M}$, subject to the relations

1. $D_i(m_1 + m_2) = D_i m_1 + D_i m_2$,
2. $D_i(a \cdot m) = \sum_{i_1+i_2=i} \partial_t^{(i_1)}(a) \cdot D_{i_2} m$,

for all $m, m_1, m_2 \in \mathfrak{M}$, $a \in \bar{K}[t]$ and $i = 0, \dots, k$. The semi-linear σ-action on $\rho_k\mathsf{M}$ is given by

$$\sigma(a \cdot D_i m) = a^\sigma \cdot D_i(\sigma(m)).$$

for $a \in \bar{K}[t]$, $m \in \mathfrak{M}$.

We obtain similar explicit descriptions as for abelian t-motives.

Proposition 4.2. *Let $\mathfrak{M}$ be a t-finite dual t-motive with $\bar{K}[t]$-basis $\boldsymbol{e} = (e_1, \dots, e_r)$ and $\tilde{\Theta} \in \mathrm{Mat}_{r\times r}(\bar{K}[t])$ the matrix such that*

$$\sigma(\boldsymbol{e}) = \boldsymbol{e} \cdot \tilde{\Theta}.$$

Then $\boldsymbol{De} = (D_0e_1, \dots, D_0e_r, D_1e_1, \dots, D_1e_r, \dots, \dots, D_ke_1, \dots, D_ke_r)$ is a $\bar{K}[t]$-basis of $\rho_k\mathfrak{M}$ and

$$\sigma(\boldsymbol{De}) = \boldsymbol{De} \cdot \rho_{[k]}(\tilde{\Theta}).$$

If $\mathfrak{M}$ is rigid analytically trivial with rigid analytic trivialization Ψ, i.e. $\Psi^\sigma = \Psi \cdot \tilde{\Theta}$, then $\rho_k\mathfrak{M}$ is rigid analytically trivial and $\rho_{[k]}(\Psi)$ is a rigid analytic trivialization with respect to $\boldsymbol{De}$.

Proof. The proof is along the same lines as for t-motives. □

5 Prolongations of t-modules

Definition 5.1. Let (E, Φ) be a t-module, and E the corresponding t-motive. Then we define the k-th prolongation $(\rho_k E, \rho_k \Phi)$ of (E, Φ) to be the t-module associated to $\rho_k\mathsf{E}$.

Theorem 5.2. *Let (E, Φ) be a t-module of dimension d, and*

$$\Phi_t = A_0 + A_1\tau + \ldots + A_s\tau^s \in \mathrm{Mat}_{d\times d}(K\{\tau\})$$

with repect to some isomorphism $E \cong \mathbb{G}_a^d$.
Then the k-th prolongation $(\rho_k E, \rho_k\Phi)$ of (E, Φ) is of dimension $d(k+1)$ and $(\rho_k\Phi)_t$ is given in block diagonal form as

$$(\rho_k\Phi)_t = \begin{pmatrix} A_0 & 0 & \cdots & \cdots & 0 \\ -\mathbb{1}_d & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\mathbb{1}_d & A_0 \end{pmatrix} + \mathrm{diag}(A_1)\tau + \ldots + \mathrm{diag}(A_s)\tau^s,$$

where $\mathbb{1}_d$ *is the* $(d\times d)$*-identity matrix, and* $\operatorname{diag}(A_i)$ *is the block diagonal matrix with diagonal entries all equal to* A_i *for* $i=1,\dots,s$.

Proof. Let $\boldsymbol{e}=(e_1,\dots,e_d)$ be the basis of E corresponding to the isomorphism $E\cong\mathbb{G}_a^d$, and hence the t-action is given by

$$t(\boldsymbol{e})=\boldsymbol{e}\cdot\Phi_t.$$

Then a $K\{\tau\}$-basis for the t-motive E is given by the dual basis $\boldsymbol{e}^\vee=(e_1^\vee,\dots,e_d^\vee)$ and the t-action on E is given by

$$t(\boldsymbol{e}^\vee)=\boldsymbol{e}^\vee\cdot\Phi_t^{\mathrm{tr}}.$$

By Lemma 3.5, a $K\{\tau\}$-basis of $\rho_k\mathsf{E}$ is given by

$$\boldsymbol{De}^\vee=(D_0e_1^\vee,\dots,D_0e_d^\vee,D_1e_1^\vee,\dots,D_1e_d^\vee,\dots,\dots,D_ke_1^\vee,\dots,D_ke_d^\vee),$$

and we have

$$t(D_ie_j^\vee)=D_i(te_j^\vee)-D_{i-1}e_j^\vee$$

for $i=0,\dots,k$ and $j=1,\dots,d$, where we set $D_{-1}e_j^\vee=0$. In block matrix notation this is just

$$t(\boldsymbol{De}^\vee)=\boldsymbol{De}^\vee\cdot\begin{pmatrix}\Phi_t^{\mathrm{tr}} & -\mathbb{1}_d & 0 & \cdots & 0\\ 0 & \Phi_t^{\mathrm{tr}} & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ \vdots & & \ddots & \ddots & -\mathbb{1}_d\\ 0 & \cdots & \cdots & 0 & \Phi_t^{\mathrm{tr}}\end{pmatrix}.$$

This finally shows that ρ_kE is isomorphic to $\mathbb{G}_a^{d(k+1)}$ with basis $\boldsymbol{De}$, the dual basis of $\boldsymbol{De}^\vee$, and the t-action is given by

$$t(\boldsymbol{De})=\boldsymbol{De}\cdot\begin{pmatrix}\Phi_t & 0 & \cdots & \cdots & 0\\ -\mathbb{1}_d & \Phi_t & \ddots & & \vdots\\ 0 & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & -\mathbb{1}_d & \Phi_t\end{pmatrix}.$$

Hence,

$$(\rho_k\Phi)_t=\begin{pmatrix}A_0 & 0 & \cdots & \cdots & 0\\ -\mathbb{1}_d & \ddots & \ddots & & \vdots\\ 0 & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & -\mathbb{1}_d & A_0\end{pmatrix}+\operatorname{diag}(A_1)\tau+\ldots+\operatorname{diag}(A_s)\tau^s.$$

□

6 Prolongations of tensor powers of the Carlitz motive

In this section, we apply the constructions of prolongations to the tensor powers of the Carlitz module, the Carlitz motive, as well as the dual Carlitz motive. Let us first recall the (dual) Carlitz motive and its tensor powers. The Carlitz module (C,ϕ) is given by $C \cong \mathbb{G}_a$ and

$$\phi : A \to \mathrm{End}(\mathbb{G}_{a,K}) = K\{\tau\}, f \mapsto \phi_f$$

given by $\phi_t = \theta + \tau$. The Carlitz motive $\mathsf{C} = \mathrm{Hom}_K(C, \mathbb{G}_a) \cong K\{\tau\}$ is also free of rank 1 as $K[t]$-module, and with respect to the basis element $e = 1 \in K\{\tau\} \cong \mathsf{C}$ the τ-action is given by $\tau(e) = e \cdot (t-\theta)$.
The n-th tensor power of the Carlitz motive C is the $K[t]$-module

$$\mathsf{C}^{\otimes n} = \underbrace{\mathsf{C} \otimes_{K[t]} \ldots \otimes_{K[t]} \mathsf{C}}_{n-\text{times}}$$

with diagonal τ-action. I.e. on the canonical basis element $e_{\otimes n}$, we have

$$\tau(e_{\otimes n}) = e_{\otimes n} \cdot (t-\theta)^n.$$

Let $\omega \in \mathbb{T}$ be the Anderson-Thakur function. Then a rigid analytic trivialization for C is given by $\frac{1}{\omega}$, since ω satisfies the difference equation $\omega^\tau = (t-\theta)\omega$. Hence, a rigid analytic trivialization for $\mathsf{C}^{\otimes n}$ is given by ω^{-n}.

The dual Carlitz motive $\mathfrak{C}$ is the $\bar{K}[t]$-module of rank 1 with σ-action given by

$$\sigma(e) = e \cdot (t-\theta),$$

with respect to some basis element $e \in \mathfrak{C}$, and its n-th tensor power $\mathfrak{C}^{\otimes n}$ has σ-action given by

$$\sigma(e_{\otimes n}) = e_{\otimes n} \cdot (t-\theta)^n.$$

The entire function $\Omega(t) := \frac{1}{(t-\theta)\omega(t)}$ is a rigid analytic trivialization of the Carlitz dual t-motive $\mathfrak{C}$, since

$$\Omega^\sigma = \left(\frac{1}{\omega^\tau}\right)^\sigma = \frac{1}{\omega} = (t-\theta)\Omega.$$

Therefore, $\Omega(t)^n$ is a rigid analytic trivialization for the n-th tensor power $\mathfrak{C}^{\otimes n}$.

Proposition 6.1. *The k-th prolongation of the motive $\mathsf{C}^{\otimes n}$ is the $K[t]$-module $\rho_k(\mathsf{C}^{\otimes n}) := K[t]^{k+1}$ with τ-action given by*

$$\tau\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_k \end{pmatrix} = \begin{pmatrix} t-\theta & 1 & 0 & \cdots & 0 \\ 0 & t-\theta & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & t-\theta \end{pmatrix}^n \cdot \begin{pmatrix} f_0^\tau \\ f_1^\tau \\ \vdots \\ f_k^\tau \end{pmatrix}.$$

Its rigid analytic trivialization is given by

$$\Upsilon=\rho_{[k]}(\omega^{-n})=\begin{pmatrix}\omega & \partial_t^{(1)}(\omega) & \cdots & \partial_t^{(k)}(\omega)\\ 0 & \omega & \ddots & \vdots\\ \vdots & \ddots & \ddots & \partial_t^{(1)}(\omega)\\ 0 & \cdots & 0 & \omega\end{pmatrix}^{-n}.$$

Proof. This follows from the general description in Section 3. One just has to recognize that $\rho_{[k]}(t-\theta)$ is just the matrix

$$\begin{pmatrix}t-\theta & 1 & 0 & \cdots & 0\\ 0 & t-\theta & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ \vdots & & \ddots & \ddots & 1\\ 0 & \cdots & \cdots & 0 & t-\theta\end{pmatrix}.$$

□

Proposition 6.2. *The k-th prolongation of the dual motive $\mathfrak{C}^{\otimes n}$ is the $\bar{K}[t]$-module $\rho_k(\mathfrak{C}^{\otimes n}) := \bar{K}[t]^{k+1}$ with σ-action given by*

$$\sigma\begin{pmatrix}f_0\\ f_1\\ \vdots\\ f_k\end{pmatrix}=\begin{pmatrix}t-\theta & 1 & 0 & \cdots & 0\\ 0 & t-\theta & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ \vdots & & \ddots & \ddots & 1\\ 0 & \cdots & \cdots & 0 & t-\theta\end{pmatrix}^n\cdot\begin{pmatrix}f_0^\sigma\\ f_1^\sigma\\ \vdots\\ f_k^\sigma\end{pmatrix}.$$

Its rigid analytic trivialization is given by

$$\Psi=\rho_{[k]}(\Omega^n)=\begin{pmatrix}\Omega^n & \partial_t^{(1)}(\Omega^n) & \partial_t^{(2)}(\Omega^n) & \cdots & \partial_t^{(k)}(\Omega^n)\\ 0 & \Omega^n & \partial_t^{(1)}(\Omega^n) & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \partial_t^{(2)}(\Omega^n)\\ \vdots & & \ddots & \Omega^n & \partial_t^{(1)}(\Omega^n)\\ 0 & \cdots & \cdots & 0 & \Omega^n\end{pmatrix}.$$

For the description of the corresponding t-modules we restrict to the prolongations of the Carlitz module, and let the descriptions for the tensor powers as an exercise for the reader.

Proposition 6.3. *The k-th prolongation $(\rho_k C, \rho_k\phi)$ of the Carlitz module is the t-module of dimension $k+1$ with*

$$\rho_k\phi:\mathbb{F}_q[t]\to \mathrm{Mat}_{(k+1)\times(k+1)}(K)\{\tau\}$$

given by

$$(\rho_k\phi)_t = \begin{pmatrix} \theta & 0 & \cdots & \cdots & 0 \\ -1 & \theta & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & \theta \end{pmatrix} + \mathbb{1}_{k+1}\cdot\tau.$$

Proof. This follows from the general description in Section 5. □

7 Hypertranscendence of the Anderson-Thakur function

In this section, we show that the Anderson-Thakur function ω is hypertranscendental, i.e. that ω and all its hyperderivatives $\partial_t^{(n)}(\omega)$ $(n>0)$ are algebraically independent over the field $\bar{K}(t)$. This fact is also given by F. Pellarin in [17, Prop. 27] by different methods.
We first recall a fact about the evaluations of the Anderson-Thakur function ω and its hyperderivatives at roots of unity given in [4] and [14]. The evaluation of $\partial_t^{(n)}(\omega)$ at $t=\zeta$ will be shortly denoted by $\partial_t^{(n)}(\omega)\,(\zeta)$.
Moreover, in this section, K will denote the field $\mathbb{F}_q(\theta)$.

Theorem 7.1. *Let $\zeta\in\bar{\mathbb{F}}_q$, let $\mathfrak{p}\in\mathbb{F}_q[t]$ be the minimal polynomial of ζ, and let $d=\deg(\mathfrak{p})$ be its degree.*
For $n\geq 0$, the Carlitz $\mathfrak{p}^{n+1}$-torsion extension of $K(\zeta)$ is generated by $\partial_t^{(n)}(\omega)\,(\zeta)$, i.e.

$$K(\zeta)(C[\mathfrak{p}^{n+1}]) = K(\zeta, \partial_t^{(n)}(\omega)\,(\zeta)).$$

The minimal polynomial of $\omega(\zeta)$ over $K(\zeta)$ is given by

$$X^{q^d-1}-\beta(\zeta)\in K(\zeta)[X],$$

where $\beta(t)=\prod_{h=0}^{d-1}(t-\theta^{q^h})\in K[t]\subseteq\mathbb{T}$.
For $n\geq 1$, the minimal polynomial of $\partial_t^{(n)}(\omega)\,(\zeta)$ over $K(\zeta)(C[\mathfrak{p}^n])$ is given by

$$X^{q^d}-\beta(\zeta)X-\xi_n(\zeta)\in K(\zeta)(C[\mathfrak{p}^n])[X],$$

where

$$\xi_n(t)=\sum_{l=1}^{n}\partial_t^{(l)}(\beta)\cdot\partial_t^{(n-l)}(\omega)\in\mathbb{T}.$$

Proof. The first part is shown in [4, Thm. 3.3] where also the minimal polynomials occur. The minimality of these polynomials, however, is shown in [14, Thm. 3.8 & Rem. 3.9]. □

Theorem 7.2. *The Anderson-Thakur function $\omega(t)$ is hypertranscendental over $\bar{K}(t)$, i.e. the set $\{\partial_t^{(n)}(\omega)\mid n\geq 0\}$ is algebraically independent over $\bar{K}(t)$.*

Proof. Since $\bar{K}(t)$ is algebraic over $K(t)$, it suffices to show algebraic independence over $K(t)$. Now, assume for the contrary, that ω and its hyperderivatives satisfy some algebraic relation. Choose n minimal such that $\omega, \partial_t^{(1)}(\omega), \dots, \partial_t^{(n)}(\omega)$ are algebraically dependent, and choose a polynomial $0 \neq F(X_0, \dots, X_n) \in K(t)[X_0, \dots, X_n]$ such that $F(\omega, \partial_t^{(1)}(\omega), \dots, \partial_t^{(n)}(\omega)) = 0$. Write $F = \sum_{j=0}^{k} f_j X_n^j$ with $f_j \in K(t)[X_0, \dots, X_{n-1}]$ and $f_k \neq 0$. After rescaling we can even assume that the coefficients of the f_j are polynomials in t, i.e. $f_j \in K[t][X_0, \dots, X_{n-1}]$.
As we have chosen n to be minimal, and as $f_k \neq 0$, we also have

$$f_k(\omega, \partial_t^{(1)}(\omega), \dots, \partial_t^{(n-1)}(\omega)) \neq 0 \in \mathbb{T}.$$

Since every nonzero element of $\mathbb{T}$ has only finitely many zeros in the closed unit disc, for almost all $\zeta \in \bar{\mathbb{F}}_q^\times$ we have: $f_k(\omega, \partial_t^{(1)}(\omega), \dots, \partial_t^{(n-1)}(\omega))|_{t=\zeta} \neq 0 \in \mathbb{C}_\infty$. Hence, for such ζ, $\partial_t^{(n)}(\omega)(\zeta)$ is a root of the nonzero polynomial

$$\sum_{j=0}^{k} f_j(\omega, \partial_t^{(1)}(\omega), \dots, \partial_t^{(n-1)}(\omega))|_{t=\zeta} X_n^j \in \mathbb{C}_\infty[X_n]$$

of degree k.
By construction, the coefficients lie in $K(C[\mathfrak{p}^n])(\zeta)$ where $\mathfrak{p} \in \mathbb{F}_q[t]$ is the minimal polynomial of ζ over $\mathbb{F}_q$. By the theorem above, the minimal polynomial of $\partial_t^{(n)}(\omega)(\zeta)$ over $K(C[\mathfrak{p}^n])(\zeta)$ has degree $q^{\deg(\mathfrak{p})} = \#\mathbb{F}_q(\zeta)$ (resp. degree $q^{\deg(\mathfrak{p})} - 1$ if $n = 0$).
Therefore, if we choose ζ such that $\#\mathbb{F}_q(\zeta) - 1 > k$, this leads to a contradiction. □

8 Algebraic independence of periods

In this section, we prove our main theorem on the algebraic independence of the periods.

Theorem 8.1. *Let $n \in \mathbb{N}$ be prime to q, let $C^{\otimes n}$ be the n-th tensor power of the Carlitz module and let*

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{C}_\infty^n$$

be a generator for the period lattice. Then $z_1, z_2, \dots, z_n$ are algebraically independent over $\bar{K}$.

Remark 8.2. As already noted in [3], if n is a power of the characteristic $p = \mathrm{char}(\mathbb{F}_q)$, then all but the last coordinate are 0. We will make a precise statement in the case that p divides n at the end of this section.

For proving the theorem, we first give a formula for these coordinates using evaluations of hyperderivatives.

Lemma 8.3. *Let the generator above be chosen such that $z_n = \tilde{\pi}^n$. Then the coordinates $z_1, z_2, \ldots, z_n$ fulfill the equalities*

$$z_i = (-1)^n \partial_t^{(n-i)}\Big((t-\theta)^n \omega(t)^n \Big) \,|_{t=\theta},$$

i.e. z_i is the $(n-i)$-th hyperderivative of the function $(\theta - t)^n \omega(t)^n$ evaluated at $t = \theta$.

Proof. As ω has a pole of order 1 at $t = \theta$, ω^n has a pole of order n. Building on work of Anderson and Thakur [3, §2.5], we write ω^n as a Laurent series in $(t - \theta)$,

$$\omega^n = \sum_{j=-n}^{\infty} c_j (t-\theta)^j \in \mathbb{C}_\infty(\!(t-\theta)\!).$$

Then the coordinates are explicitly given by

$$z_i = (-1)^n c_{-i}$$

for $i = 1, \ldots, n$ (see [3, Cor. 2.5.8], and be aware that $\bar{\pi}$ ibid. equals $-\tilde{\pi}$).
On the other hand, for any $0 \le k \le n$:

$$\begin{aligned}
\partial_t^{(k)}\Big((t-\theta)^n \omega(t)^n \Big) &= \partial_t^{(k)}\left(\sum_{j=-n}^{\infty} c_j (t-\theta)^{j+n} \right) \\
&= \sum_{j=k-n}^{\infty} c_j \binom{j+n}{k} (t-\theta)^{j+n-k}.
\end{aligned}$$

Hence for $i = 1, \ldots, n$:

$$\begin{aligned}
(-1)^n \partial_t^{(n-i)}\Big((t-\theta)^n \omega(t)^n \Big) \,|_{t=\theta} &= (-1)^n \sum_{j=-i}^{\infty} c_j \binom{j+n}{n-i} (t-\theta)^{j+i}|_{t=\theta} \\
&= (-1)^n c_{-i} = z_i.
\end{aligned}$$

□

A second ingredient is a relation between hyperderivatives of functions and hyperderivatives of powers of that function.

Lemma 8.4. *Let $0 \neq f \in \mathbb{C}_\infty(\!(t)\!)$, $k \ge 0$ and let E be the field extension of $K(t)$ generated by the entries of $\rho_{[k]}(f)$, i.e. generated by $f, \partial_t^{(1)}(f), \ldots, \partial_t^{(k)}(f)$. For $n \in \mathbb{N}$ prime to q, let F be the field extension of $K(t)$ generated by the entries of $\rho_{[k]}(f^n)$, i.e. generated by $f^n, \partial_t^{(1)}(f^n), \ldots, \partial_t^{(k)}(f^n)$. Then E is generated over F by f, and in particular, E is finite algebraic over F.*

Proof. We only have to show that $\partial_t^{(j)}(f) \in F(f)$ for $1 \leq j \leq k$. Let $p = \mathrm{char}(\mathbb{F}_q)$, and $s \in \mathbb{N}$ such that $p^s > k$. Then for all j not divisible by p^s, one has $\partial_t^{(j)}(f^{p^s}) = 0$, since

$$\mathcal{D}(f^{p^s}) = \mathcal{D}(f)^{p^s}$$

is a power series in X^{p^s}. In particular, we have $\partial_t^{(j)}(f^{p^s}) = 0$ for all $1 \leq j \leq k$, and therefore $\rho_{[k]}(f^{p^s})$ is the scalar matrix with diagonal entries equal to f^{p^s}. As n was prime to q, and hence prime to p, there are $a, b \in \mathbb{Z}$ such that $ap^s + bn = 1$. Therefore,

$$\rho_{[k]}(f) = \rho_{[k]}(f^{ap^s+bn}) = \rho_{[k]}(f^{p^s})^a \cdot \rho_{[k]}(f^n)^b = (f^{p^s})^a \cdot \rho_{[k]}(f^n)^b$$

has entries in $F(f)$. □

Proof of Thm. 8.1. By Prop. 6.2, a rigid analytic trivialization of $\rho_{n-1}(\mathfrak{C}^{\otimes n})$, the $(n-1)$-th prolongation of $\mathfrak{C}^{\otimes n}$, is given by the matrix

$$\rho_{[n-1]}(\Omega^n) = \begin{pmatrix} \Omega^n & \partial_t^{(1)}(\Omega^n) & \partial_t^{(2)}(\Omega^n) & \cdots & \partial_t^{(n-1)}(\Omega^n) \\ 0 & \Omega^n & \partial_t^{(1)}(\Omega^n) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \partial_t^{(2)}(\Omega^n) \\ \vdots & & \ddots & \Omega^n & \partial_t^{(1)}(\Omega^n) \\ 0 & \cdots & \cdots & 0 & \Omega^n \end{pmatrix}.$$

Let F be the field generated by the entries of $\rho_{[n-1]}(\Omega^n)$ over $\bar{K}(t)$. As,

$$\rho_{[n-1]}(\Omega^n) = \rho_{[n-1]}\left((t-\theta)^{-n}\omega(t)^{-n}\right) = \left(\rho_{[n-1]}(t-\theta)\right)^{-n} \cdot \left(\rho_{[n-1]}(\omega^n)\right)^{-1}$$

and $\rho_{[n-1]}(t-\theta) \in \mathrm{GL}_n(K(t))$, the field F is also generated over $\bar{K}(t)$ by the entries of $\rho_{[n-1]}(\omega^n)$, and in particular is a subfield of finite index of the field generated by the entries of $\rho_{[n-1]}(\omega)$, as shown in Lemma 8.4. Since ω is hypertranscendental (see Theorem 7.2), the latter has transcendence degree over $\bar{K}(t)$ equal to n. Hence, the field F has transcendence degree n over $\bar{K}(t)$. Let L be the field extension of $\bar{K}$ generated by the entries of $\rho_{[n-1]}(\Omega^n)|_{t=\theta}$. Then by the proof of [16, Thm. 5.2.2] (see Thm. 1.1), the transcendence degree of $L/\bar{K}$ is the same as the transcendence degree of $F/\bar{K}(t)$, i.e. equals n. On the other hand, L is also generated as a field by the entries of the inverse of $\rho_{[n-1]}(\Omega^n)|_{t=\theta}$, and using Lemma 8.3, we get

$$\begin{aligned} \left(\rho_{[n-1]}(\Omega^n)|_{t=\theta}\right)^{-1} &= \left(\rho_{[n-1]}(\Omega^n)^{-1}\right)|_{t=\theta} = \rho_{[n-1]}(\Omega^{-n})|_{t=\theta} \\ &= \rho_{[n-1]}((t-\theta)^n\omega^n)|_{t=\theta} \\ &= (-1)^n \cdot \begin{pmatrix} z_n & z_{n-1} & z_{n-2} & \cdots & z_1 \\ 0 & z_n & z_{n-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & z_{n-2} \\ \vdots & & \ddots & z_n & z_{n-1} \\ 0 & \cdots & \cdots & 0 & z_n \end{pmatrix}. \end{aligned}$$

Hence, $z_1, \ldots, z_n$ are algebraically independent over $\bar{K}$. □

In the case that the characteristic p divides n, we can also make a precise statement on the algebraic independence.

Corollary 8.5. *Let $n \in \mathbb{N}$ be arbitrary, let $C^{\otimes n}$ be the n-th tensor power of the Carlitz module and let*

$$\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \in \mathbb{C}_\infty^n$$

be the generator for the period lattice with $z_n = \tilde{\pi}^n$. If p^s is the exact power of p dividing n, then $z_i \neq 0$ precisely, when p^s divides i, and all nonzero coordinates are algebraically independent over $\bar{K}$.

Proof. The hyperdifferential operators on $\mathbb{C}_\infty(\!(t)\!)$ satisfy

$$\partial_t^{(i)}\Big(f^{p^s}\Big) = \begin{cases} 0 & \text{if } p^s \text{ does not divide } i \\ \Big(\partial_t^{(i/p^s)}(f)\Big)^{p^s} & \text{if } p^s \text{ divides } i, \end{cases}$$

for all $f \in \mathbb{C}_\infty(\!(t)\!)$, as one readily sees by using the homomorphism $\mathcal{D}$.
Applying this to $f = \Omega^{n/p^s}$, we see that the nonzero entries in $\rho_{[n-1]}(\Omega^n)$ are the $\partial_t^{(i)}(\Omega^n)$ with p^s divides i and those are equal to $\Big(\partial_t^{(i/p^s)}\big(\Omega^{n/p^s}\big)\Big)^{p^s}$.
By specializing the inverse of $\rho_{[n-1]}(\Omega^n)$ to $t = \theta$ as in the proof of Theorem 8.1, we see that the coordinates z_i where p^s does not divide i are equal to zero, and that the other coordinates are just the p^s-powers of the coordinates of a period lattice generator for the n/p^s-th tensor power of the Carlitz module. Hence, by Theorem 8.1, they are algebraically independent over $\bar{K}$. □

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Andreas Maurischat
Lehrstuhl A für Mathematik
RWTH Aachen University
Aachen
Germany
andreas.maurischat@matha.rwth-aachen.de