Relative Homological Algebra via Truncations

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Abstract. To do homological algebra with unbounded chain complexes one needs to first find a way of constructing resolutions. Spaltenstein solved this problem for chain complexes of R-modules by truncating further and further to the left, resolving the pieces, and gluing back the partial resolutions. Our aim is to give a homotopy theoretical interpretation of this procedure, which may be extended to a relative setting. We work in an arbitrary abelian category $\mathcal{A}$ and fix a class of "injective objects" $\mathcal{I}$. We show that Spaltenstein's construction can be captured by a pair of adjoint functors between unbounded chain complexes and towers of non-positively graded ones. This pair of adjoint functors forms what we call a Quillen pair and the above process of truncations, partial resolutions, and gluing, gives a meaningful way to resolve complexes in a relative setting *up to a split error term*. In order to do homotopy theory, and in particular to construct a well behaved relative derived category $D(\mathcal{A};\mathcal{I})$, we need more: the split error term must vanish. This is the case when $\mathcal{I}$ is the class of all injective R-modules but not in general, not even for certain classes of injectives modules over a Noetherian ring. The key property is a relative analogue of Roos's AB4*-n axiom for abelian categories. Various concrete examples such as Gorenstein homological algebra and purity are also discussed.

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Introduction

Our aim in this work is to present a framework to do relative homological algebra. If homological algebra is understood as a means to study objects and functors in abelian categories through invariants determined by projective or injective resolutions, then relative homological algebra should give us more flexibility in constructing resolutions, meaning we would like to be allowed to use a priori *any* object as an injective. This idea goes back at least to Adamson [1] for group cohomology and Chevalley-Eilenberg [8] for Lie algebra homology. Both were then subsumed in a general theory by Hochschild [20]. The most complete reference for the classical point of view is Eilenberg–Moore [12].

Analogously, in homotopy theory one would traditionally use spheres to "resolve spaces" by constructing a CW-approximation, but it has become very common nowadays to replace them by some other spaces and do A-homotopy theory, as developed for instance by Farjoun [15]. In fact, homotopical methods have already been applied to do relative homological algebra. Christensen and Hovey [9] show that, in many cases, one can equip the category of unbounded chain complexes with a model category structure where the weak equivalences reflect a choice of new projective objects. It is their work, and the relationship to Spaltenstein's explicit construction of a resolution for unbounded chain complex [34], that motivated us originally. We wish to stress the point that, for us, it is as important to have a constructive method to build relative resolutions as to know that there exists a formal method to invert certain relative quasi-isomorphisms (because there is a relative model structure or a relative derived category for example).

More precisely we fix in an abelian category $\mathcal{A}$ a class $\mathcal{I} \subset \mathcal{A}$ of objects, called the *relative injectives*, that will play the role of usual injectives. This determines in turn two classes of maps: a class of *relative monomorphisms* and a class of *relative quasi-isomorphisms*. If $\mathcal{I}$ is the class of injective objects these reduce to ordinary monomorphisms and ordinary quasi-isomorphisms. Denote by $\mathrm{Ch}(\mathcal{A})$ the category of chain complexes over $\mathcal{A}$ and by $\mathcal{W}_\mathcal{I}$ the class of relative quasi-isomorphisms. Our aim is to construct the localized category $\mathrm{Ch}(\mathcal{A})[\mathcal{W}_\mathcal{I}]^{-1}$, in particular we would like to find a way to resolve chain complexes.

Disregarding set-theoretical problems, one could formally add inverses of the elements in $\mathcal{W}_\mathcal{I}$ to get $D(\mathcal{A};\mathcal{I}) = \mathrm{Ch}(\mathcal{A})[\mathcal{W}_\mathcal{I}]^{-1}$. With a little more care, for instance using the theory of null systems, one can construct $\mathrm{Ch}(\mathcal{A})[\mathcal{W}_\mathcal{I}]^{-1}$ by the calculus of fractions and endow it with a natural triangulated structure; this is done at the end of Section 1. It is unwise though to completely disregard set-theoretic problems and Quillen devised in the late sixties the notion of a model category, see [31], which provides a technique for overcoming this difficulty. On the category of left bounded chain complexes Bousfield [4] showed how to use Quillen's machine to construct the relative derived category $D_{\leq 0}(\mathcal{A};\mathcal{I})$. An elementary exposition of Bousfield's relative model structure, including explicit methods to construct factorizations (and hence resolutions), can be found in Appendix A.

Our objective is to extend this construction and the model structure to unbounded complexes, but this is a more delicate issue, even in the classical setting, see Spaltenstein [34] or Serpé [33]. A relative model structure on $\mathrm{Ch}(\mathcal{A})$ would be nice, but we cannot apply homotopical localization techniques in a straightforward way since there is no obvious *set* of maps to invert. Anyway, we need less. Therefore we introduce a more flexible framework, namely that of a model approximation [7]. Our idea in this work is to approximate a complex by the tower of its truncations, just as Spaltenstein did. For this we observe first in Proposition 4.3 that a relative model structure on left bounded complexes induces a model structure on towers of left bounded complexes. Diagrams of model categories have been studied by Greenlees and Shipley [16] and play an important role in equivariant stable homotopy theory, see for example [2]. Recent work of Harpaz and Prasma [19] proposes another viewpoint on such diagrams and model structures.
Second, we package the relationship between unbounded chain complexes and the category of towers $\mathrm{Tow}(\mathcal{A}, \mathcal{I})$ equipped with the relative model structure into what we call a Quillen pair. It consists of a pair of adjoint functors

$$\mathrm{tow} : \mathrm{Ch}(\mathcal{A}) \rightleftarrows \mathrm{Tow}(\mathcal{A}, \mathcal{I}) : \lim$$

where the "tower functor" associates to a complex the tower given by truncating it further and further to the left, and the limit functor takes limits degreewise, see Proposition 5.5. The left hand side is not a model category but its homotopical features are reflected in the right hand side. To do homotopy theory with unbounded chain complexes we need this Quillen pair to form a *model approximation*, i.e. to verify some extra compatibility condition of the adjoint pair with resolutions, see Definition 3.2. When this is the case resolutions of complexes are provided by an explicit recipe. Thus we need to understand when the Quillen pair is a model approximation. To solve this difficulty we introduce in Section 6 a relative version of Roos axiom AB4*-n, [32]. Our main result is the following.

Theorem 6.4 *Let $\mathcal{I}$ be an injective class and assume that the abelian category $\mathcal{A}$ satisfies axiom* AB4*-$\mathcal{I}$-n. *Then the standard Quillen pair*

$$\mathrm{tow} : \mathrm{Ch}(\mathcal{A}) \rightleftarrows \mathrm{Tow}(\mathcal{A}, \mathcal{I}) : \lim$$

is a model approximation.

This axiom is satisfied in particular for classes of injective modules over a Noetherian ring of finite Krull dimension. Here we only consider injective classes of R-modules that are injective in the classical sense, which means that we do relative homological algebra with less injectives than in the usual sense. Spaltenstein's classical construction also works for this reason, see Corollary 6.5.

Theorem 7.4. *Let R be a Noetherian ring of finite Krull dimension d, and $\mathcal{I}$ an injective class of injective modules. Then the category of towers forms a model approximation for $Ch(R)$ equipped with $\mathcal{I}$-equivalences.*

When the Krull dimension is infinite it depends on the chosen class of injectives whether or not one can resolve unbounded complexes by truncation. For Nagata's ring [27] we show in Subsection 8.2 that some classes $\mathcal{I}$ satisfy axiom AB4*-$\mathcal{I}$-n, but we also construct in Theorem 8.4 an injective class $\mathcal{I}$ which fails to yield a model approximation. Concretely this means that we exhibit an unbounded complex which is not relatively quasi-isomorphic to the limit of the (relative) injective resolutions of its truncations. Our methods rely on local cohomology computations, see [22]. The failure of being a model approximation is nevertheless rather well behaved, as we never lose any information about the original complex:

Proposition 5.7 *Let* $f : \mathrm{tow}(X) \to Y_\bullet$ *be a weak equivalence in* $\mathrm{Tow}(\mathcal{A}, \mathcal{I})$ *and* $g : X \to \lim(Y_\bullet)$ *be its adjoint. Then, for any* $W \in \mathcal{I}$, $\mathcal{A}(g, W)$ *induces a split epimorphism on homology.*

The failure of the standard Quillen pair to be a model approximation is closely related to the "non-left completeness" of the derived category of some abelian categories, observed by Neeman [29].

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1. Chain complexes and relative weak equivalences

In this section we recall briefly the definition of an abelian category, introduce the notion of an injective class, and study the relative weak equivalences that arise in the category of chain complexes in an abelian category once an injective class has been chosen.

1.1. Abelian categories. Throughout the paper we work with an *abelian category* $\mathcal{A}$, for example the category of left modules over a ring. By an abelian catgory we mean a category with the following structure [17]:

- (AB0) Additivity. The category $\mathcal{A}$ is additive: finite products and coproducts exist; there is a zero object (an object which is both initial and terminal); given two objects $X, Y \in \mathcal{A}$, the morphism set $\mathcal{A}(X, Y)$ has an abelian group structure with the zero given by the unique morphism that factors through the zero object; the composition of maps is a bilinear operation.
- (AB1) Kernels and cokernels. Any morphism has a kernel and cokernel as defined in [25].
- (AB2) Every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.
- (AB3) Limits and colimits. Arbitrary limits and colimits exist in $\mathcal{A}$.

At first we do not ask for any further properties of products beyond their existence, although later on we will make a crucial assumption. Grothendieck's axiom, which we will use, is:

(AB4*) A countable product of epimorphisms in $\mathcal{A}$ is an epimorphism.

Let R be a possibly non-commutative unitary ring. The category of left R-modules, which we call simply R-modules and denote by R-Mod, is an abelian category that satisfies axiom AB4*. However, if X is a topological space then the category of sheaves of abelian groups on X, which is also an abelian category, does not satisfy AB4* in general [17, Proposition 3.1.1].

1.2. INJECTIVE CLASSES. Given an abelian category $\mathcal{A}$ we are interested in understanding relative analogues of monomorphisms and injective objects in $\mathcal{A}$.

DEFINITION 1.1. Let $\mathcal{I}$ be a collection of objects in $\mathcal{A}$. A morphism $f : M \to N$ in $\mathcal{A}$ is said to be an *$\mathcal{I}$-monomorphism* if $f^* : \mathcal{A}(N, W) \to \mathcal{A}(M, W)$ is a surjection of sets for any $W \in \mathcal{I}$. We say that $\mathcal{A}$ has enough $\mathcal{I}$-injectives if, for any object M, there is an $\mathcal{I}$-monomorphism $M \to W$ with $W \in \mathcal{I}$.

REMARK 1.2. It is clear that a composite of $\mathcal{I}$-monomorphisms is also an $\mathcal{I}$-monomorphism. We say that a morphism f has a retraction if there exists a morphism r such that $rf = \mathrm{id}$. Any morphism that has a retraction is an $\mathcal{I}$-monomorphism for any collection $\mathcal{I}$. Observe also that $\mathcal{I}$-monomorphisms are preserved under base change: if $f : M \to N$ is an $\mathcal{I}$-monomorphism, then so is its push-out along any morphism $M \to M'$, by the universal property of a push-out. Similarly an arbitrary coproduct of $\mathcal{I}$-monomorphisms is an $\mathcal{I}$-monomorphism. In general however limits and products of $\mathcal{I}$-monomorphisms may fail to be $\mathcal{I}$-monomorphisms.

Given a class of objects $\mathcal{I}$ denote by $\overline{\mathcal{I}}$ the class of retracts of arbitrary products of elements of $\mathcal{I}$. Since a morphism is an $\mathcal{I}$-monomorphism if and only if it is an $\overline{\mathcal{I}}$-monomorphism, without loss of generality we may assume that $\mathcal{I}$ is closed under retracts and products so that $\mathcal{I} = \overline{\mathcal{I}}$.

DEFINITION 1.3. A collection of objects $\mathcal{I}$ in $\mathcal{A}$ is called an *injective class* if $\mathcal{I}$ is closed under retracts and products and if $\mathcal{A}$ has enough $\mathcal{I}$-injectives.

It should be pointed out that general products have considerably more retracts than direct sums.

EXAMPLE 1.4. The largest injective class $\mathcal{I}$ in $\mathcal{A}$ consists of all the objects in $\mathcal{A}$. Here $\mathcal{I}$-monomorphisms are morphisms $f : M \to N$ that have retractions. It is clear that there are enough $\mathcal{I}$-injectives since for any object N the identity $Id_N : N \to N$ is an $\mathcal{I}$-monomorphism.

Recall that an object W in an abelian category $\mathcal{A}$ is called injective if, for any monomorphism f, $\mathcal{A}(f, W)$ is an epimorphism. Assume that any object of $\mathcal{A}$ admits a monomorphism into an injective object, which is the case for example in the category of left R-modules. Then the collection $\mathcal{I}$ of injective objects in $\mathcal{A}$ is an injective class and $\mathcal{I}$-monomorphisms are the ordinary monomorphisms.

The same holds for the category of $\mathcal{O}_X$-modules for a scheme X: any $\mathcal{O}_X$-module is a submodule of an injective $\mathcal{O}_X$-module.

Adjoint functors allow us to construct new injective classes out of old ones, an idea that goes back to Eilenberg-Moore [12, Theorem 2.1].

Proposition 1.5. *Let $l : \mathcal{B} \leftrightarrows \mathcal{A} : r$ be a pair of functors between abelian categories such that l is left adjoint to r. Let $\mathcal{I}$ be a collection of objects in $\mathcal{A}$.*

1. *A morphism f in $\mathcal{B}$ is an $r(\mathcal{I})$-monomorphism if and only if lf is an $\mathcal{I}$-monomorphism in $\mathcal{A}$.*
2. *If $lM \to W$ is an $\mathcal{I}$-monomorphism in $\mathcal{A}$, then its adjoint $M \to rW$ is an $r(\mathcal{I})$-monomorphism in $\mathcal{B}$.*
3. *If there are enough $\mathcal{I}$-injectives in $\mathcal{A}$, then there are enough $r(\mathcal{I})$-injectives in $\mathcal{B}$.*
4. *If $\mathcal{I}$ is an injective class in $\mathcal{A}$, then the collection of retracts of objects of the form $r(W)$, for $W \in \mathcal{I}$, is an injective class in $\mathcal{B}$.*

Example 1.6. Tensor products. Assume now that S is a commutative ring and $S \to R$ is a ring homomorphism whose image lies in the center of R, hence turns R into an S-algebra. The forgetful functor R-Mod $\to$ S-Mod is right adjoint to $R \otimes_S - : S\text{-Mod} \to R\text{-Mod}$. Thus, by Example 1.4 and Proposition 1.5, both the collection of S-linear summands of R-modules and the collection of S-linear summands of all injective R-modules form injective classes of S-modules. A monomorphism relative to the first collection is a homomorphism f for which $f \otimes_S R$ is a split monomorphism. A monomorphism relative to the second collection is an homomorphism f for which $f \otimes_S R$ is a monomorphism.

Example 1.7. Schemes. Let $f : X \to Y$ be a morphism of schemes. The functor $f^* : \mathcal{O}_Y\text{-Mod} \to \mathcal{O}_X\text{-Mod}$ is left adjoint to $f_* : O_X\text{-Mod} \to \mathcal{O}_Y\text{-Mod}$. It follows that the two collections: $\mathcal{O}_Y$-modules which are retracts of $\mathcal{O}_Y$-modules of the form $f_*(N)$, for any $\mathcal{O}_X$-module N, and retracts of $\mathcal{O}_Y$-modules of the same form, but for all injective $\mathcal{O}_X$-module N, are injective classes in $\mathcal{O}_Y$-Mod.

We wish to see to what extent objects in $\mathcal{I}$ behave like usual injective objects, that is when it is possible to do homological algebra relative to the class $\mathcal{I}$. We therefore turn to the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes over $\mathcal{A}$ and to its homotopy category $\mathcal{K}(\mathcal{A})$.

1.3. Relative weak equivalences in $\mathrm{Ch}(\mathcal{A})$. In this work we mostly consider *homological* complexes (i.e. differentials lower degree by one) in $\mathcal{A}$: $X = (\cdots \to X_i \xrightarrow{d_i} X_{i-1} \to \cdots)$. The category of such chain complexes in $\mathcal{A}$ is denoted by $\mathrm{Ch}(\mathcal{A})$. We identify $\mathcal{A}$ with the full subcategory of $\mathrm{Ch}(\mathcal{A})$ of those complexes concentrated in degree 0 and will use the topologist's suspension symbol ΣX for the shifted complex sometimes denoted by $X[1]$.

The only examples of *cohomological* complexes that we consider are complexes of abelian groups of the form $\mathcal{A}(X, W)$ for some $X \in \mathrm{Ch}(\mathcal{A})$ and $W \in \mathcal{A}$. As usual, if X_k is in homological degree $k \in \mathbb{Z}$, we put $\mathcal{A}(X_k, W)$ in cohomological degree $-k$. The key definition for doing relative homological algebra is the following.

DEFINITION 1.8. Let $k \in \mathbb{Z}$ be an integer. A morphism $f : X \to Y$ in $\mathrm{Ch}(\mathcal{A})$ is called a *k-$\mathcal{I}$-weak equivalence* if and only if, for any $W \in \mathcal{I}$, the induced morphism of cochain complexes $\mathcal{A}(f, W) : \mathcal{A}(Y, W) \to \mathcal{A}(X, W)$ induces an isomorphism in cohomology in degrees $n \geq -k$ and a monomorphism in degree $-k-1$. A morphism that is a k-$\mathcal{I}$-weak equivalence for all $k \in \mathbb{Z}$ is called an *$\mathcal{I}$-weak equivalence.*

DEFINITION 1.9. An object X in $\mathrm{Ch}(\mathcal{A})$ is called *$\mathcal{I}$-trivial* when $X \to 0$ is an $\mathcal{I}$-weak equivalence, i.e. when $\mathcal{A}(X, W)$ is an acyclic complex of abelian groups for all $W \in \mathcal{I}$. It is called *k-$\mathcal{I}$-connected* if $X \to 0$ is a k-$\mathcal{I}$-weak equivalence, i.e., when $\mathcal{A}(X, W)$ has trivial cohomology in degrees $n \geq -k$ for all $W \in \mathcal{I}$.

Let us see what these definitions mean for the examples we introduced in the previous subsection.

EXAMPLE 1.10. We study first the case when $\mathcal{I}$ is the injective class of all objects of $\mathcal{A}$. For an object $M \in \mathcal{A}$ and an integer k denote by $D_k(M)$ the "disc" chain complex

$$\cdots 0 \longrightarrow M \xrightarrow{Id_M} M \longrightarrow 0 \longrightarrow \cdots$$

where the two copies of M are in homological degrees k and $k-1$ respectively. Complexes of the form $D_k(M)$ are prototypical examples of contractible complexes.
A morphism of chain complexes $f : X \to Y$ is an $\mathcal{I}$-weak equivalence if and only if it is a homotopy equivalence. A chain complex is $\mathcal{I}$-trivial if and only if it is isomorphic to $\bigoplus_i D_{k_i}(M_i)$ for some sequence of objects $M_i \in \mathcal{A}$ and integers $k_i \in \mathbf{Z}$.

EXAMPLE 1.11. Let us assume that classical injective objects form an injective class, i.e. any object in $\mathcal{A}$ is a subobject of an injective object. As the functors $\mathcal{A}(-, W)$ are exact when W is injective, a morphism of complexes $f : X \to Y$ in $\mathrm{Ch}(\mathcal{A})$ is an $\mathcal{I}$-weak equivalence if and only if it is a quasi-isomorphism. A chain complex is $\mathcal{I}$-trivial if and only if it has trivial homology.

EXAMPLE 1.12. Consider a pair of adjoint functors $l : \mathcal{B} \leftrightarrows \mathcal{A} : r$ between abelian categories and $\mathcal{I}$ an injective class in $\mathcal{A}$. According to Proposition 1.5.(4), the collection $\mathcal{J}$ of retracts of objects of the form $r(W)$, for $W \in \mathcal{I}$, forms an injective class in $\mathcal{B}$. By applying l and r degree-wise, we get an induced pair of adjoint functors, denoted by the same symbols: $l : \mathrm{Ch}(\mathcal{B}) \leftrightarrows \mathrm{Ch}(\mathcal{A}) : r$. A morphism $f : X \to Y$ in $\mathrm{Ch}(\mathcal{B})$ is a $\mathcal{J}$ -weak equivalence if and only if $l(f) : l(X) \to l(Y)$ is an $\mathcal{I}$-weak equivalence in $\mathrm{Ch}(\mathcal{A})$.

Our next example is based on the classification of injective classes of injective objects in a module category given in [6], to which we refer for more details. Let us recall however that given an ideal I in R and an element r outside of I, then $(I\colon r)$ denotes the ideal $\{s \in R \,|\, sr \in I\}$. This example will play an important role in the final sections of this article.

Example 1.13. Let R be a commutative ring and $\mathcal{L}$ be a saturated set of ideals in R. This means that $\mathcal{L}$ is a set of proper ideals of R closed under intersection and the construction $(I\colon r)$; moreover if an ideal J has the property that $(J\colon r)$ is contained in some ideal in $\mathcal{L}$ for any element $r \notin J$, then J itself must belong to $\mathcal{L}$.
Consider the injective class $\mathcal{E}(\mathcal{L})$ that consists of retracts of products of injective envelopes $E(R/I)$ for $I \in \mathcal{L}$. A morphism $f : X \to Y$ in $\mathrm{Ch}(R)$ is an $\mathcal{E}(\mathcal{L})$-weak equivalence if and only if $\mathrm{Hom}(\mathrm{H}_n(f), E(R/I))$ is a bijection for any n and $I \in \mathcal{L}$. This happens if and only if the annihilator of any element in either $\mathrm{Ker}(\mathrm{H}_n(f))$ or $\mathrm{Coker}(\mathrm{H}_n(f))$ is not included in any ideal that belongs to $\mathcal{L}$.

We denote the class of $\mathcal{I}$-weak equivalences by $\mathcal{W}_{\mathcal{I}}$ or simply $\mathcal{W}$ if there is no ambiguity for the choice of the ambient injective class $\mathcal{I}$. Isomorphisms are always $\mathcal{I}$-weak equivalences and $\mathcal{I}$-weak equivalences satisfy the "2 out of 3" property, as the stronger "2 out of 6" property from [10, Definition 4.5] holds.

Lemma 1.14. *The class $\mathcal{W}_{\mathcal{I}}$ of $\mathcal{I}$-weak equivalences satisfies the* 2 *out of* 6 *property: given any three composable maps*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T$$

if vu and wv are in $\mathcal{W}$ then so are u, v, w and wvu.

Proof. Fix an object $W \in \mathcal{I}$. Then $\mathcal{A}(vu, W) = \mathcal{A}(u, W) \circ \mathcal{A}(v, W)$ is a quasi-isomorphism, hence $\mathcal{A}(v, W)$ induces an epimorphism in cohomology. Similarly, from the fact that $\mathcal{A}(wv, W)$ is a quasi-isomorphism we get that $\mathcal{A}(v, W)$ induces a monomorphism in cohomology, hence v belongs to $\mathcal{W}$. Since quasi-isomorphisms satisfy the "2 out of 3" property we get that $\mathcal{A}(u, W)$ and $\mathcal{A}(w, W)$ are quasi-isomorphisms and v, w is in $\mathcal{W}$. By closure under composition so is wvu. □

Here are some elementary properties of $\mathcal{I}$-weak equivalences:

Proposition 1.15. *Let I be an injective class in an abelian category $\mathcal{A}$.*

1. *A chain homotopy equivalence in* $\mathrm{Ch}(\mathcal{A})$ *is an $\mathcal{I}$-weak equivalence.*
2. *A morphism $f : X \to Y$ in* $\mathrm{Ch}(\mathcal{A})$ *is an $\mathcal{I}$-weak equivalence if and only if the cone* $\mathrm{Cone}(f)$ *is $\mathcal{I}$-trivial.*
3. *Coproducts of $\mathcal{I}$-weak equivalences are $\mathcal{I}$-weak equivalences.*
4. *A contractible chain complex in* $\mathrm{Ch}(\mathcal{A})$ *is $\mathcal{I}$-trivial.*
5. *Coproducts of $\mathcal{I}$-trivial complexes are $\mathcal{I}$-trivial.*
6. *A complex X is k-$\mathcal{I}$-connected if and only if, for any $i \leq k$, the morphism d_i :* $\mathrm{Coker}(d_{i+1}) \to X_{i-1}$ *is an $\mathcal{I}$-monomorphism.*
7. *A complex X is $\mathcal{I}$-trivial if and only if d_i :* $\mathrm{Coker}(d_{i+1}) \to X_{i-1}$ *is an $\mathcal{I}$-monomorphism for any i.*
8. *Let X be a complex such that, for all i,* $\mathrm{Coker}(d_{i+1}) \in \mathcal{I}$. *Then X is $\mathcal{I}$-trivial if and only if X is isomorphic to $\bigoplus D^i(W_i)$.*

Proof. Point (1) is a consequence of the fact that $\mathcal{A}(-,W)$ is an additive functor.
(2) The cone of $\mathcal{A}(f,W):\mathcal{A}(Y,W)\to\mathcal{A}(X,W)$ is isomorphic to the shift of the complex $\mathcal{A}(\mathrm{Cone}(f),W)$, for any $W\in\mathcal{A}$. Thus $\mathcal{A}(f,W)$ is a quasi-isomorphism if and only if $\mathcal{A}(\mathrm{Cone}(f),W)$ is acyclic.
Point (3) is a consequence of two facts. First, $\mathcal{A}(-,W)$ takes coproducts in $\mathcal{A}$ into products of abelian groups. Second, products of quasi-isomorphisms of chain complexes of abelian groups are quasi-isomorphisms.
Point (4) is a special instance of Point (1), and given (4), Point (5) is a special case of Point (3).
(6) The kernel of $\mathcal{A}(d_{i+1},W)$ is $\mathcal{A}(\mathrm{Coker}(d_{i+1}),W)$. Thus the $(-i)$-th cohomology group of the complex $\mathcal{A}(X,W)$ is trivial if and only if the morphism $\mathcal{A}(X_{i-1},W)\to\mathcal{A}(\mathrm{Coker}(d_{i+1}),W)$ induced by d_i is an epimorphism. By definition this happens if and only if the morphism $d_i:\mathrm{Coker}(d_{i+1})\to X_{i-1}$ is an $\mathcal{I}$-monomorphism.
(7) This is a consequence of (6).
(8) If X can be expressed as a direct sum $\bigoplus D_i(W_i)$, then X is contractible and according to (4) it is $\mathcal{I}$-trivial. Assume now that X is $\mathcal{I}$-trivial. Define $W_i:=\mathrm{Coker}(d_{i+1})$. According to (6), the morphism $d_i:\mathrm{Coker}(d_{i+1})\to X_{i-1}$ is an I-monomorphism. As $\mathrm{Coker}(d_{i+1})$ is assumed to belong to $\mathcal{I}$, it follows that the morphism $d_i:\mathrm{Coker}(d_{i+1})\to X_{i-1}$ has a retraction. This retraction can be used to define a morphism of chain complexes $X\to D_i(W_i)$. By assembling these morphisms together we get the desired isomorphism $X\to\bigoplus D_i(W_i)$. □

2. The relative derived category as a large category

Doing homological algebra relative to an injective class $\mathcal{I}$ amounts to inverting the morphisms in $\mathcal{W}$ to form the relative derived category $D(\mathcal{A};\mathcal{I})=\mathrm{Ch}(\mathcal{A})[\mathcal{W}_{\mathcal{I}}^{-1}]$. The formalities of inverting a class of morphisms in a category are well understood. But there is a problem that, without some extra structure, the resulting category turns out to be a large category in general, i.e. with classes of morphisms between two objects instead of sets of morphisms. This becomes an issue if one wants to further localize in this category or study its quotients. Let us nevertheless put this set-theoretical issue aside for the moment, and remind the reader of the classical construction of the relative derived category $D(\mathcal{A};\mathcal{I})$. In particular we recall that the classical results endow $D(\mathcal{A};\mathcal{I})$ with a canonical triangulated structure.
As chain homotopy equivalences are in particular $\mathcal{I}$-equivalences the localization functor $\mathrm{Ch}(\mathcal{A})\to\mathrm{Ch}(\mathcal{A})[\mathcal{W}_{\mathcal{I}}^{-1}]$, if it exists, factors through the canonical localization functor $\mathrm{Ch}(\mathcal{A})\to\mathcal{K}(\mathcal{A})$, where $\mathcal{K}(\mathcal{A})$ is the homotopy category of chain complexes. The category $\mathcal{K}(\mathcal{A})$ is a triangulated category, and we exploit this fact and the theory of *null systems*, [23, Section 10.2], to construct the relative derived category.

Definition 2.1. Let $\mathcal{T}$ be a triangulated category and $\mathcal{N}$ be a class of objects in $\mathcal{T}$ closed under isomorphisms. Then $\mathcal{N}$ is a *null system* if and only if the following axioms are satisfied:

(N0) The zero object of $\mathcal{T}$ is in $\mathcal{N}$.
(N1) For any $X \in \mathcal{T}$, $X \in \mathcal{N} \Leftrightarrow \Sigma X \in \mathcal{N}$.
(N2) Given a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ in $\mathcal{T}$, if $X, Z \in \mathcal{N}$ then $Y \in \mathcal{N}$.

The main property of null systems is that it allows us to construct the Verdier quotient $\mathcal{T}/\mathcal{N}$ by a simple calculus of fractions (although recall that this quotient may have proper classes of morphisms). For a proof of the following proposition we refer the reader to [23].

Proposition 2.2. *Given a triangulated category $\mathcal{T}$ and a null system $\mathcal{N}$ in $\mathcal{T}$, let $\mathcal{S}(\mathcal{N})$ denote the set of those arrows $f : X \to Y \in \mathcal{T}$ that fit into a triangle:*

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X$$

whith $Z \in \mathcal{N}$. Then $\mathcal{S}(\mathcal{N})$ admits a left and right calculus of fractions. In particular:

(1) The localization $\mathcal{T}/\mathcal{N} := \mathcal{T}[\mathcal{S}(\mathcal{N})^{-1}]$ exists.
(2) Let us declare the isomorphs in $\mathcal{T}/\mathcal{N}$ of images of triangles in $\mathcal{T}$, via the canonical quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{N}$, to be the triangles in $\mathcal{T}/\mathcal{N}$. Then the category $\mathcal{T}/\mathcal{N}$ becomes triangulated and the canonical quotient functor is triangulated.

We apply this to our situation of interest, where we want to invert the relative equivalences, i.e. kill the cones of $\mathcal{W}_\mathcal{I}$-equivalences, which are $\mathcal{I}$-trivial by Proposition 1.15.(2).

Proposition 2.3. *In $\mathcal{K}(\mathcal{A})$, the homotopy category of $\mathcal{A}$ with its standard triangulated structure, the class $\mathcal{WN}$ of $\mathcal{I}$-trivial objects forms a null system.*

Proof. Axioms (N0) and (N1) hold by definition of $\mathcal{I}$-triviality, see Definition 1.9.
(N2) Let W be an object in $\mathcal{I} \subset \mathcal{A}$. Let $X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$ be a triangle in $\mathcal{K}(\mathcal{A})$, with $X, Z \in \mathcal{WN}$. Applying the functor $\mathcal{A}(-, W)$ to the triangle we deduce a triangle, in the homotopy category $\mathcal{K}(\mathrm{Ab})$ where Ab is the abelian category of abelian groups,

$$\mathcal{A}(\Sigma X, W) \longrightarrow \mathcal{A}(Z, W) \longrightarrow \mathcal{A}(Y, W) \longrightarrow \mathcal{A}(X, W)$$

Since $\mathcal{A}(Z, W)$ and $\mathcal{A}(X, W)$ are both acyclic so is $\mathcal{A}(Y, W)$. □

From the general theory it follows that the class $\mathcal{W}_\mathcal{I}$ of $\mathcal{I}$-equivalences admits simple right and left calculuses of fractions. As a consequence we have:

Corollary 2.4. *Let $\mathcal{A}$ be an abelian category, $\mathcal{I}$ a class of injective objects, and $\mathcal{W}_\mathcal{I}$ the associated class of $\mathcal{I}$-weak equivalences.*

(1) *The localization* $\mathrm{Ch}(\mathcal{A})[\mathcal{W}_{\mathcal{I}}^{-1}] =: D(\mathcal{A};\mathcal{I})$ *exists and has a natural triangulated category structure which is functorial with respect to inclusions of classes of relative weak equivalences.*
(2) *The canonical functor* $\mathcal{K}(\mathcal{A}) \to D(\mathcal{A};\mathcal{I})$ *is triangulated.*
(3) *The class* $\mathcal{W}_{\mathcal{I}}$ *is saturated: a map* $f \in \mathcal{K}(\mathcal{A})$ *is an isomorphism in* $D(\mathcal{A};\mathcal{I})$ *if and only if* $f \in \mathcal{W}_{\mathcal{I}}$.

Proof. The only non-immediate consequence from Proposition 2.2 is point (3), which is a consequence of the "2 out of 6" property, Lemma 1.14, see [23, Prop. 7.1.20]. □

3. Model categories and model approximations

We now present our set-up for doing homotopical algebra. In homotopy theory a convenient framework for localizing categories and constructing derived functors is given by Quillen model categories; we use the term model category as defined in [11]. There are however situations in which, either it is very hard to construct a model structure, or one simply does not know whether such a structure does exist. We will explain how to localize and construct right derived functors in a more general context than model categories. We do not to try to impose a model structure on a given category with weak equivalences directly but rather use model categories to *approximate* the given category.
Let $\mathcal{C}$ be a category and $\mathcal{W}$ be a collection of morphisms in $\mathcal{C}$ which contains all isomorphisms and satisfies the"2 out of 3" property: if f and g are composable morphisms in $\mathcal{C}$ and 2 out of $\{f, g, gf\}$ belong to $\mathcal{W}$ then so does the third. We call elements of $\mathcal{W}$ weak equivalences and a pair $(\mathcal{C},\mathcal{W})$ a category with weak equivalences. The following definitions come from [7, 3.12].

DEFINITION 3.1. A *right Quillen pair* for $(\mathcal{C},\mathcal{W})$ is a model category $\mathcal{M}$ and a pair of functors $l : \mathcal{C} \rightleftarrows \mathcal{M} : r$ satisfying the following conditions:

(1) l is left adjoint to r;
(2) if f is a weak equivalence in $\mathcal{C}$, then lf is a weak equivalence in $\mathcal{M}$;
(3) if f is a weak equivalence between fibrant objects in $\mathcal{M}$, then rf is a weak equivalence in $\mathcal{C}$.

DEFINITION 3.2. We say that an object A in $\mathcal{C}$ is *approximated* by a right Quillen pair $l : \mathcal{C} \rightleftarrows \mathcal{M} : r$ if the following condition is satisfied:

(4) if $lA \to X$ is a weak equivalence in $\mathcal{M}$ and X is fibrant, then its adjoint $A \to rX$ is a weak equivalence in $\mathcal{C}$.

If all objects of $\mathcal{C}$ are approximated by $l : \mathcal{C} \rightleftarrows \mathcal{M} : r$, then this Quillen pair is called a *right model approximation* of $\mathcal{C}$.

REMARK 3.3. For an object A to be approximated by a Quillen pair, we only need the existence of *some* fibrant object X in the model category together with a weak equivalence $lA \to X$ and such that the adjoint map is a weak equivalence. Condition (4) is then automatically satisfied for *any* such fibrant object. The reason is that any weak equivalence $lA \to X'$ to a fibrant object X'

factors as an acyclic cofibration followed by an acyclic fibration $lA \overset{\sim}{\hookrightarrow} X'' \overset{\sim}{\twoheadrightarrow} X'$. By the lifting axiom this means that $lA \to X$ factors through X'' so that $A \to rX''$ is a weak equivalence by the "2 out of 3" axiom. Finally so is $A \to rX'$ for the same reason.

Let us fix a right Quillen pair $l : \mathcal{C} \rightleftarrows \mathcal{M} : r$ and choose a full subcategory $\mathcal{D}$ of $\mathcal{C}$ with the following properties: all objects in $\mathcal{D}$ are approximated by the Quillen pair and, for a weak equivalence $f : X \to Y$, if one of X and Y belongs to $\mathcal{D}$ then so does the other ($\mathcal{D}$ is closed under weak equivalences). We are going to think of $\mathcal{D}$ as a category with weak equivalences given by the morphisms in $\mathcal{D}$ that belong to $\mathcal{W}$. Here are some fundamental properties of this category, whose proofs extend those for model approximations in [7, Section 5]:

Proposition 3.4. *(1) A morphism f in $\mathcal{D}$ is a weak equivalence if and only if lf is a weak equivalence in $\mathcal{M}$.*

(2) The localization $\mathrm{Ho}(\mathcal{D})$ *of $\mathcal{D}$ with respect to weak equivalences exists and can be constructed as follows: objects of* $\mathrm{Ho}(\mathcal{D})$ *are the same as objects of $\mathcal{D}$ and* $\mathrm{mor}_{\mathrm{Ho}(\mathcal{D})}(X, Y) = \mathrm{mor}_{\mathrm{Ho}(\mathcal{M})}(lX, lY)$.

(3) A morphism in $\mathcal{D}$ is a weak equivalence if and only if it induces an isomorphism in $\mathrm{Ho}(\mathcal{D})$.

(4) The class of weak equivalences in $\mathcal{D}$ is closed under retracts.

(5) Let $F : \mathcal{C} \to \mathcal{T}$ be a functor. Assume that the composition $Fr : \mathcal{M} \to \mathcal{T}$ takes weak equivalences between fibrant objects in $\mathcal{M}$ to isomorphisms in $\mathcal{T}$. Then the right derived functor of the restriction $F : \mathcal{D} \to \mathcal{T}$ exists and is given by $A \mapsto F(rX)$, where X is a fibrant replacement of lA in $\mathcal{M}$.

Proof. (1) Assume that $lf : lA \to lB$ is a weak equivalence in $\mathcal{M}$. Choose a weak equivalence $lB \to Y$ with fibrant target Y. By taking adjoints we form the following commutative diagram in $\mathcal{D}$:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow & \downarrow \\ & & rY \end{array}$$

Since A and B belong to $\mathcal{D}$, the morphisms $A \to rY$ and $B \to rY$ are weak equivalences, as their adjoints are so. By the "two out of three" property, f is then also a weak equivalence.

(2) Let $\alpha : \mathcal{D} \to \mathcal{T}$ be a functor that sends weak equivalences to isomorphisms. We prove that there is a unique functor $\beta : \mathrm{Ho}(\mathcal{D}) \to \mathcal{T}$ for which the composition $\mathcal{D} \to \mathrm{Ho}(D) \xrightarrow{\beta} \mathcal{T}$ equals α. On objects we have no choice, we define $\beta(A) := \alpha(A)$.

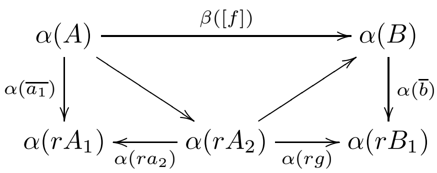
Let A and B be objects in $\mathcal{D}$. Since $\mathrm{mor}_{\mathrm{Ho}(\mathcal{D})}(A, B) = \mathrm{mor}_{\mathrm{Ho}(\mathcal{M})}(lA, lB)$, a morphism $[f] : A \to B$ in $\mathrm{Ho}(\mathcal{D})$ is given by a sequence of morphisms in $\mathcal{M}$:

$$lA \xrightarrow{a_1} A_1 \xleftarrow{a_2} A_2 \xrightarrow{g} B_1 \xleftarrow{b} lB$$

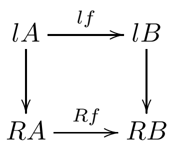
where a_1 is a weak equivalence with fibrant target A_1, a_2 is a weak equivalence with fibrant and cofibrant domain A_2, and b is a weak equivalence with fibrant target B_1. By adjunction we get a sequence of morphisms in $\mathcal{D}$:

$$A \xrightarrow{\overline{a}_1} rA_1 \xleftarrow{ra_2} rA_2 \xrightarrow{rg} rB_1 \xleftarrow{\overline{b}} B$$

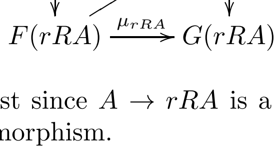
Note that $\overline{a}_1$, ra_2, and $\overline{b}$ are weak equivalences. We define $\beta([f])$ to be the unique morphism in $\mathcal{T}$ for which the following diagram commutes:



Since α takes weak equivalences to isomorphisms such a morphism $\beta([f])$ exists and is unique. One can finally check that this process defines the desired functor $\beta : \mathrm{Ho}(\mathcal{D}) \to \mathcal{T}$.
(3) is a consequence of (1) and (2). Point (4) follows from (3).
(5) For any object $A \in \mathcal{D}$ let us fix a fibrant replacement $lA \to RA$ in $\mathcal{M}$. For any morphism $f : A \to B$ in $\mathcal{D}$ let us fix a morphism $Rf : RA \to RB$ in $\mathcal{M}$ for which the following diagram commutes:



Since Fr takes weak equivalences between fibrant objects to isomorphisms, the association $A \mapsto F(rRA)$ and $f \mapsto F(rRf)$ defines a functor $RF : \mathcal{D} \to \mathcal{T}$. We claim that RF together with the natural transformation given by $F(A \to rRA)$ is the right derived functor of $F : \mathcal{D} \to \mathcal{T}$. It is clear that RF takes weak equivalences to isomorphisms. Let $G : \mathcal{D} \to \mathcal{T}$ be a functor that takes weak equivalences to isomorphisms and let $\mu : F \to G$ be a natural transformation. For any $A \in \mathcal{D}$ define $F(rRA) \to G(A)$ to be the unique morphism that fits into the following commutative diagram in $\mathcal{T}$:



Such a morphism does exist since $A \to rRA$ is a weak equivalence and thus $G(A) \to G(rRA)$ is an isomorphism. □

4. Towers

For a given category with weak equivalences $(\mathcal{C}, \mathcal{W})$ and a full subcategory $\mathcal{D}$ our strategy is to construct a right Quillen pair $l : \mathcal{C} \rightleftarrows \mathcal{M} : r$ which approximates objects of $\mathcal{D}$. We can then use this Quillen pair to localize $\mathcal{D}$ with respect to weak equivalences and construct right derived functors as explained in Proposition 3.4. For this strategy to work we need adequate examples of model categories. The purpose of this section is to show how to assemble model categories together to build new model categories that are suitable to approximate $\mathcal{D}$. Such diagrams of model categories have appeared meanwhile in work of Greenlees and Shipley, [16], see also Bergner's construction of a homotopy limit model category for a diagram of left Quillen functors, [3]. We include the following definitions and results to fix notation and so as to be able to refer to specific constructions in the next sections.
We start with a tower $\mathcal{T}$ of model categories consisting of a sequence of model categories $\{\mathcal{T}_n\}_{n\geq 0}$ and a sequence of Quillen functors $\{l : \mathcal{T}_{n+1} \rightleftarrows \mathcal{T}_n : r\}_{n\geq 0}$: for any n, l is left adjoint to r and r preserves fibrations and acyclic fibrations. The model categories in a tower $\mathcal{T}$ can be assembled to form its category of towers.

Definition 4.1. The objects $a_\bullet$ of the *category of towers* $\mathrm{Tow}(\mathcal{T})$ are sequences $\{a_n\}_{n\geq 0}$ of objects $a_n \in \mathcal{T}_n$ together with a sequence of *structure morphisms* $\{a_{n+1} \to r(a_n)\}_{n\geq 0}$. The set of morphisms in $\mathrm{Tow}(\mathcal{T})$ between $a_\bullet$ and $b_\bullet$ consists of sequences of morphisms $\{f_n : a_n \to b_n\}_{n\geq 0}$ for which the following squares commute:

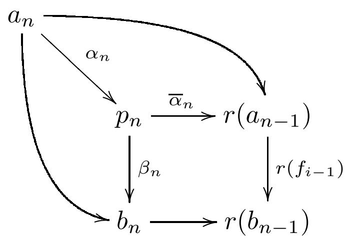
$$\begin{array}{ccc} a_{n+1} & \longrightarrow & r(a_n) \\ {\scriptstyle f_{n+1}}\downarrow & & \downarrow{\scriptstyle r(f_n)} \\ b_{n+1} & \longrightarrow & r(b_n) \end{array}$$

We write $f_\bullet : a_\bullet \to b_\bullet$ to denote the morphism $\{f_n : a_n \to b_n\}_{n\geq 0}$ in $\mathrm{Tow}(\mathcal{T})$.

The following construction will be useful to describe a model structure on $\mathrm{Tow}(\mathcal{T})$. For a morphism $f_\bullet : a_\bullet \to b_\bullet$, define $p_0 := b_0$ and, for $n > 0$, define:

$$p_n := \lim\big(b_n \to r(b_{n-1}) \xleftarrow{r(f_{n-1})} r(a_{n-1})\big)$$

Set $\alpha_0 : a_0 \to p_0$ to be f_0 and $\beta_0 : p_0 \to b_0$ to be the identity. For any $n > 0$, let $\beta_n : p_n \to b_n$ and $\overline{\alpha}_n : p_n \to r(a_{n-1})$ be the projection from the inverse limit onto the components b_n, respectively $r(a_{n-1})$. Finally $\alpha_n : a_n \to p_n$ is the unique morphism for which the following diagram commutes:



The sequence $\{p_n\}_{n\geq 0}$ and the morphisms $\{p_{n+1} \xrightarrow{\overline{\alpha}_{n+1}} r(a_n) \xrightarrow{r(\alpha_n)} r(p_n)\}_{n\geq 0}$ defines an object $p_\bullet$ in $\mathrm{Tow}(\mathcal{T})$. Moreover $\{\alpha_n : a_n \to p_n\}_{n\geq 0}$, respectively $\{\beta_n : p_n \to b_n\}_{n\geq 0}$, define morphisms $\alpha_\bullet : a_\bullet \to p_\bullet$ and $\beta_\bullet : p_\bullet \to b_\bullet$ whose composite is $f_\bullet$. For example, let $*_\bullet$ be given by the sequence consisting of the terminal objects $\{*\}_{n\geq 0}$ in $\mathcal{T}_n$ and $f_\bullet : a_\bullet \to *_\bullet$ be the unique morphism in $\mathrm{Tow}(\mathcal{T})$. Then $p_0 = *$, and, for $n > 0$, $p_n = r(a_{n-1})$. The morphism $\alpha_n : a_n \to p_n = r(a_{n-1})$ is given by the structure morphism of $a_\bullet$.

DEFINITION 4.2. A morphism $\{f_n : a_n \to b_n\}_{n\geq 0}$ in $\mathrm{Tow}(\mathcal{T})$ is a *weak equivalence* (respectively a *cofibration*) if, for any $n \geq 0$, the morphism f_n is a weak equivalence (respectively a cofibration) in $\mathcal{T}_n$. It is a *fibration* if $\alpha_n : a_n \to p_n$ is a fibration in $\mathcal{T}_n$ for any $n \geq 0$.

For example the morphism $a_\bullet \to *_\bullet$ is a fibration if and only if a_0 is fibrant in $\mathcal{T}_0$ and the structure morphisms $a_n \to r(a_{n-1})$ are fibrations in $\mathcal{T}_n$ for all n.
The following result is a particular case of the existence of the injective model structure for diagrams of model categories, [16, Theorem 3.1]. We provide some details of the proof as we will refer to the explicit construction of the factorizations.

PROPOSITION 4.3. *The above choice of weak equivalences, cofibrations, and fibrations equips* $\mathrm{Tow}(\mathcal{T})$ *with a model category structure.*

Proof. First, the category $\mathrm{Tow}(\mathcal{T})$ is bicomplete, as limits and colimits are formed "degree-wise". The structural morphisms of the limit are the limits of the structural morphisms since the functors r, as right adjoints, commute with limits. For colimits, one considers the adjoints $l(a_{n+1}) \to a_n$ of the structural morphisms, and takes colimits $l(\mathrm{colim}(a_{n+1})) \cong \mathrm{colim}\, l(a_{n+1}) \to \mathrm{colim}(a_n)$. The adjoint morphisms $\mathrm{colim}(a_{n+1}) \to r(\mathrm{colim}(a_n))$ are precisely the structural morphisms of the colimit.
The "2 out of 3" property (MC2) for weak equivalences and the fact that retracts of weak equivalences (respectively cofibrations) are weak equivalences (respectively cofibrations) follow immediately from the same properties for the categories $\mathcal{T}_n$. To prove axiom (MC3), notice that if $\{c_n \to d_n\}_{n\geq 0}$ is a retract of a fibration $\{a_n \to b_n\}_{n\geq 0}$, then $c_0 \to d_0$ is a fibration in $\mathcal{T}_0$. Next consider

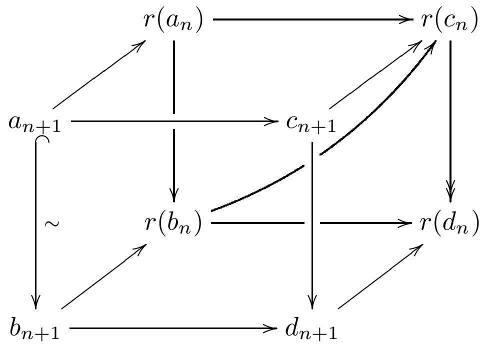
the following commutative diagram for $n > 0$:

$$\begin{array}{ccccccccc}
d_n & \longrightarrow & r(d_{n-1}) & \twoheadleftarrow & r(c_{n-1}) & & q_n & \longleftarrow & c_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
b_n & \longrightarrow & r(b_{n-1}) & \twoheadleftarrow & r(a_{n-1}) & \overset{\lim}{\rightsquigarrow} & p_n & \twoheadleftarrow & a_n \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
d_n & \longrightarrow & r(d_{n-1}) & \twoheadleftarrow & r(c_{n-1}) & & q_n & \longleftarrow & c_n
\end{array}$$

where the penultimate column has been obtained by taking pull-backs. By the retract property in $\mathcal{T}_n$ the morphism $c_n \to q_n$ is fibration, for any $n > 0$, and therefore so is $\{c_n \to d_n\}_{n\geq 0}$ in $\mathrm{Tow}(\mathcal{T})$.
Let us prove now the right and left lifting properties (MC4). Consider a commutative diagram:

$$\begin{array}{ccc}
a_\bullet & \longrightarrow & c_\bullet \\
{\scriptstyle\sim}\big\downarrow & & \big\downarrow \\
b_\bullet & \longrightarrow & d_\bullet
\end{array}$$

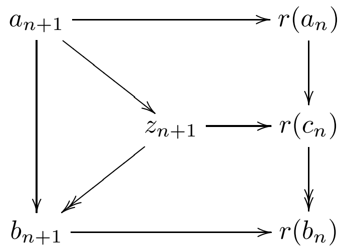
where the indicated arrows are respectively an acyclic cofibration and a fibration. In degree 0, a lift $b_0 \to c_0$ is provided by the model structure on $\mathcal{T}_0$. We construct the lift inductively. Take the solved lifting problem at level n and complete with the structural maps to get the following commutative cube:



As above, denote by q_{n+1} the pull-back of $d_{n+1} \to r(d_n) \leftarrow r(c_n)$. By the universal property of the pull-back there is a morphism $b_{n+1} \to q_{n+1}$ that makes the resulting diagram commutative. Since by definition $c_{n+1} \to q_{n+1}$ is a fibration, the lifting problem

$$\begin{array}{ccc}
a_{n+1} & \longrightarrow & c_{n+1} \\
{\scriptstyle\sim}\big\downarrow & & \big\downarrow \\
b_{n+1} & \longrightarrow & q_{n+1}
\end{array}$$

has a solution, which is the desired morphism. The proof for the right lifting property for acyclic fibrations with respect to cofibrations is analogous.
Finally, to prove the factorization axiom (MC5), consider a morphism $a_\bullet \to b_\bullet$. The morphism $a_0 \to b_0$ can be factored as an acyclic cofibration followed by a fibration (respectively as a cofibration followed by an acyclic fibration) because (MC5) holds in $\mathcal{T}_0$. We construct a factorization $a_{n+1} \hookrightarrow c_{n+1} \twoheadrightarrow b_{n+1}$ by induction on the degree. Consider the following commutative diagram:



where the right column is obtained by applying the functor r to the factorization at level n and the bottom right square is a pull-back. Since both r and cobase-change preserve (acyclic) fibrations, $z_{n+1} \to b_{n+1}$ is an (acyclic) fibration as long as $c_n \to b_n$ is. It is now enough to factor $a_{n+1} \to z_{n+1}$ in $\mathcal{T}_{n+1}$ in the desired way to obtain the factorization of $a_{n+1} \to b_{n+1}$. □

Example 4.4. Let $\mathcal{M}$ be a model category. The constant sequence $\{\mathcal{M}\}_{n\geq 0}$ together with the sequence of identity functors $\{\mathrm{id} : \mathcal{M} \rightleftarrows \mathcal{M} : \mathrm{id}\}_{n\geq 0}$ forms a tower of model categories. Its category of towers can be identified with the category of functors $\mathrm{Fun}(\mathbf{N}^{op}, \mathcal{M})$, where $\mathbf{N}$ is the poset whose objects are natural numbers, $\mathbf{N}(n,l) = \emptyset$ if $n > l$, and $\mathbf{N}(n,l)$ consists of one element if $n \leq l$. The model structure on $\mathrm{Fun}(\mathbf{N}^{op}, \mathcal{M})$, given by Proposition 4.3, coincides with the standard model structure on the functor category $\mathrm{Fun}(\mathbf{N}^{op}, \mathcal{M})$ (see [7]). For example, a functor F in $\mathrm{Fun}(\mathbf{N}^{op}, \mathcal{M})$ is fibrant if the object $F(0)$ is fibrant in $\mathcal{M}$ and, for any $n > 0$, the morphism $F(n) \to F(n-1)$ is a fibration in $\mathcal{M}$. A morphism $\alpha : F \to G$ is a cofibration in $\mathrm{Fun}(\mathbf{N}^{op}, \mathcal{M})$ if it consists levelwise of cofibrations in $\mathcal{M}$.

5. A model approximation for relative homological algebra

In this section we construct a Quillen pair suitable for doing relative homological algebra with unbounded chain complexes. The model category we propose is a tower of categories of *bounded* chain complexes, each equipped with a relative model structure. Therefore we first define a model structure on bounded chain complexes, then introduce the category of towers, and finally study the associated Quillen pair.

5.1. Bounded chain complexes. Let n be an integer. The full subcategory of $\mathrm{Ch}(\mathcal{A})_{\leq n} \subset \mathrm{Ch}(\mathcal{A})$ consists of the chain complexes X such that $X_i = 0$ for $i > n$. The inclusion functor $\mathrm{in} : \mathrm{Ch}(\mathcal{A})_{\leq n} \subset \mathrm{Ch}(\mathcal{A})$ has both a right and a left adjoint. The left adjoint is denoted by $\tau_n : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})_{\leq n}$ and is called *truncation.* Explicitly τ_n assigns to a complex X the truncated complex

$$\tau_n(X) := (\mathrm{Coker}(d_{n+1}) \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \xrightarrow{d_{n-2}} \cdots)$$

where in degree n we have $\tau_n(X)_n = \mathrm{Coker}(d_{n+1})$, and for $i < n$ the formula is $\tau_n(X)_i = X_i$. For a morphism $f : X \to Y$ in $\mathrm{Ch}(\mathcal{A})$ the map $\tau_n(f)_n$ is induced by f_n, while for $i < n$ we have $\tau_n(f)_i = f_i$.
For any $X \in \mathrm{Ch}(\mathcal{A})$, the *truncation morphism* $t_n : X \to \mathrm{in}\tau_n(X)$ is the unit of the adjunction $\tau_n \dashv \mathrm{in}$. Explicitly this morphism we will abusively write as $t_n : X \to \tau_n(X)$ is the following chain map:

$$\begin{array}{ccccccccccc}
X & & \cdots & \xrightarrow{d_{n+2}} & X_{n+1} & \xrightarrow{d_{n+1}} & X_n & \xrightarrow{d_n} & X_{n-1} & \xrightarrow{d_{n-1}} & \cdots \\
\downarrow & & & & \downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\tau_n(X) & & \cdots & \longrightarrow & 0 & \longrightarrow & \mathrm{Coker}(d_{n+1}) & \xrightarrow{d_n} & X_{n-1} & \xrightarrow{d_{n-1}} & \cdots
\end{array}$$

where q denotes the quotient morphism. With respect to the injective classes we introduced in Definition 1.3, the key property of the truncation morphism is the following.

Proposition 5.1. *The truncation morphism $t_n : X \to \tau_n(X)$ is an n-$\mathcal{I}$-weak equivalence for any injective class $\mathcal{I}$.*

Proof. For any $W \in \mathcal{I}$, the morphism $\mathcal{A}(t_n, W)$ is given by the following commutative diagram:

$$\begin{array}{ccccccc}
0 & \longleftarrow & \mathcal{A}(\mathrm{Coker}(d_{n+1}), W) & \longleftarrow & \mathcal{A}(X_{n-1}, W) & \xleftarrow{\mathcal{A}(d_{n-1},W)} & \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\mathcal{A}(X_{n+1}, W) & \xleftarrow{\mathcal{A}(d_{n+1},W)} & \mathcal{A}(X_n, W) & \xleftarrow{\mathcal{A}(d_n,W)} & \mathcal{A}(X_{n-1}, W) & \xleftarrow{\mathcal{A}(d_{n-1},W)} &
\end{array}$$

Clearly $\mathcal{A}(t_n, W)$ induces an isomorphism on cohomology in degrees $> -n$. Since the kernel of $\mathcal{A}(d_{n+1}, W)$ is given by $\mathcal{A}(\mathrm{Coker}(d_{n+1}), W)$, $\mathcal{A}(t_n, W)$ induces also an isomorphism on H^{-n}. As $H^{-n-1}(\mathcal{A}(\tau_n(X), W)) = 0$, $\mathcal{A}(t_n, W)$ induces a monomorphism on H^{-n-1}. $\square$

We begin by recalling a theorem of Bousfield [4, Section 4.4]. A proof may also be found in the appendix, see Theorem A.16 – it is there both for the reader's convenience and because it gives an explicit construction of fibrant replacements.

Theorem 5.2. *Let $\mathcal{I}$ be an injective class. The following choice of weak equivalences, cofibrations and fibrations endows $\mathrm{Ch}(\mathcal{A})_{\leq n}$ with a model category structure:*

- *$f : X \to Y$ is called an $\mathcal{I}$-weak equivalence if $f^* : \mathcal{A}(Y, W) \to \mathcal{A}(X, W)$ is a quasi-isomorphism of complexes of abelian groups for any $W \in \mathcal{I}$.*

- *$f\colon X \to Y$ is called an $\mathcal{I}$-cofibration if $f_i\colon X_i \to Y_i$ is an $\mathcal{I}$-monomorphism for all $i \leq n$.*
- *$f\colon X \to Y$ is called an $\mathcal{I}$-fibration if $f_i\colon X_i \to Y_i$ has a section and its kernel belongs to $\mathcal{I}$ for all $i \leq n$. In particular X is $\mathcal{I}$-fibrant if $X_i \in \mathcal{I}$ for all $i \leq n$.*

Among other things this model structure gives, for an object $A \in \mathcal{A}$ seen as a chain complex concentrated in degree zero, a fibrant replacement $A \to I$. This turns out to be nothing else than a relative injective resolution for A. Here are some basic properties of this model structure on $\mathrm{Ch}(\mathcal{A})_{\leq n}$.

Proposition 5.3. *(1) All objects in $\mathrm{Ch}(\mathcal{A})_{\leq n}$ are $\mathcal{I}$-cofibrant.*

(2) Let $f\colon X \to Y$ be an $\mathcal{I}$-fibration. Then $\mathrm{Ker}(f)$ is fibrant and f is a k-$\mathcal{I}$-weak equivalence if and only if $\mathrm{Ker}(f)$ is k-$\mathcal{I}$-connected.

(3) An $\mathcal{I}$-fibration $f\colon X \to Y$ is an $\mathcal{I}$-weak equivalence if and only if $\mathrm{Ker}(f)$ is $\mathcal{I}$-trivial. Moreover, if f is an acyclic $\mathcal{I}$-fibration, then there is an isomorphism $\alpha\colon Y \oplus \mathrm{Ker}(f) \to X$ for which the following diagram commutes:

$$\begin{array}{ccc} Y \oplus \mathrm{Ker}(f) & \xrightarrow{\quad\alpha\quad} & X \\ & {\scriptstyle pr}\searrow \quad \swarrow{\scriptstyle f} & \\ & Y & \end{array}$$

(4) An $\mathcal{I}$-weak equivalence between $\mathcal{I}$-fibrant chain complexes in $\mathrm{Ch}(\mathcal{A})_{\leq n}$ is a homotopy equivalence.

(5) An $\mathcal{I}$-fibrant object in $\mathrm{Ch}(\mathcal{A})_{\leq n}$ is $\mathcal{I}$-trivial if and only if it is isomorphic to a complex of the form $\bigoplus_{i\leq n} D_i(W_i)$, where disc complexes have been defined in Example 1.10.

(6) Products of $\mathcal{I}$-fibrant and $\mathcal{I}$-trivial complexes are $\mathcal{I}$-trivial.

(7) Assume that the following is a sequence of $\mathcal{I}$-fibrations and $\mathcal{I}$-weak equivalences in $\mathrm{Ch}(\mathcal{A})_{\leq n}$:

$$(\cdots X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0)$$

Then, the projection morphism $\lim_{i\geq 0} X_i \to X_k$ is an $\mathcal{I}$-fibration and an $\mathcal{I}$-weak equivalence for any $k \geq 0$.

Proof. (1) follows from the fact that, for any $W \in \mathcal{A}$, the morphism $0 \to W$ is an $\mathcal{I}$-monomorphism.
(2): For any W, the following is an exact sequence of chain complexes of abelian groups:

$$0 \to \mathcal{A}(Y, W) \xrightarrow{\mathcal{A}(f,W)} \mathcal{A}(X, W) \to \mathcal{A}(\mathrm{Ker}(f), W) \to 0\,.$$

The first part of (3) follows from (2). If $f : X \to Y$ is an acyclic $\mathcal{I}$-fibration, then because all objects in $\mathrm{Ch}(\mathcal{A})_{\leq n}$ are $\mathcal{I}$-cofibrant, there is a morphism $s : Y \to X$ for which $fs = \mathrm{id}_Y$. This implies the second part of (3).
(4): All objects in $\mathrm{Ch}(\mathcal{A})_{\leq n}$ are $\mathcal{I}$-cofibrant, so an $\mathcal{I}$-weak equivalence between $\mathcal{I}$-fibrant objects is a homotopy equivalence in the $\mathcal{I}$-model structure.

But, the standard path object $P(Z)$ (see A.5), is a very good path object for any $\mathcal{I}$-fibrant chain complex $Z \in \mathrm{Ch}(\mathcal{A})_{\leq n}$ (in the terminology used in [11], which means that the factorization $Z \subset P(Z) \xrightarrow{\pi} Z \oplus Z$ consists in an acyclic cofibration followed by a fibration). Hence, a homotopy equivalence in the $\mathcal{I}$-model structure on $\mathrm{Ch}(\mathcal{A})_{\leq n}$ is nothing but a homotopy equivalence.
(5): Assume that X is $\mathcal{I}$-fibrant and $\mathcal{I}$-trivial. According to Proposition 1.15.(8) we need to show that, for all i, $W_i := \mathrm{Coker}(d_{i+1})$ belongs to $\mathcal{I}$. We do it by induction on i. For $i = n$, $\mathrm{Coker}(d_{n+1}) = X_n$ belongs to $\mathcal{I}$ since X is $\mathcal{I}$-fibrant. Assume now that $W_{i+1} \in \mathcal{I}$. As $d_{i+1} : W_{i+1} \to X_i$ is an $\mathcal{I}$-monomorphism, it has a retraction. It follows that $X_i = W_{i+1} \oplus W_i$. Consequently W_i, as a retract of a member of $\mathcal{I}$, also belongs to $\mathcal{I}$.
(6) is a consequence of (5), and (7) follows from (3) and (6). □

We will use the model categories $\mathrm{Ch}(\mathcal{A})_{\leq n}$ with their $\mathcal{I}$-relative model structure to approximate the category of unbounded chain complexes $\mathrm{Ch}(\mathcal{A})$ equipped with the $\mathcal{I}$-relative weak equivalences.

Proposition 5.4.

1. *The following pair of functors is a right Quillen pair:*
$$\tau_n : \mathrm{Ch}(\mathcal{A}) \rightleftarrows \mathrm{Ch}(\mathcal{A})_{\leq n} : \mathrm{in}$$
2. *A chain complex $X \in \mathrm{Ch}(\mathcal{A})$ is approximated by the above right Quillen pair if and only if $\mathcal{A}(X, W)$ has trivial cohomology for $i < -n$ and any $W \in \mathcal{I}$.*

Proof. Both statements follow directly from the definitions and Proposition 5.1. □

Our aim is to find other Quillen pairs for $\mathrm{Ch}(\mathcal{A})$ that approximate more unbounded chain complexes than just those with "bounded $\mathcal{I}$-homology". For that we construct a suitable model category by assembling the categories $\mathrm{Ch}(\mathcal{A})_{\leq n}$ into a tower. This is the content of the next subsection.

5.2. Towers of bounded chain complexes. For $n \geq k$, the restriction of $\tau_k : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})_{\leq k}$ to the subcategory $\mathrm{Ch}(\mathcal{A})_{\leq n} \subset \mathrm{Ch}(\mathcal{A})$ is denoted by the same symbol $\tau_k : \mathrm{Ch}(\mathcal{A})_{\leq n} \to \mathrm{Ch}(\mathcal{A})_{\leq k}$ (and is left adjoint to the inclusion in : $\mathrm{Ch}(\mathcal{A})_{\leq k} \subset \mathrm{Ch}(\mathcal{A})_{\leq n}$). Moreover the canonical morphism $X \to \tau_k(X)$ can be expressed uniquely as the composite $X \to \tau_n(X) \to \tau_k(X)$, of the truncation morphism $X \to \tau_n(X)$ for X and n, and the truncation morphism $\tau_n(X) \to \tau_k(X) = \tau_k(\tau_n(X))$ for $\tau_n(X)$ and k.
Consider now the sequence of model categories $\{\mathrm{Ch}(\mathcal{A})_{\leq n}\}_{n\geq 0}$, with the model structures given by Theorem 5.2. The functor in : $\mathrm{Ch}(\mathcal{A})_{\leq n} \subset \mathrm{Ch}(\mathcal{A})_{\leq n+1}$ takes (acyclic) fibrations to (acyclic) fibrations and hence the following is a sequence of Quillen functors:
$$\{\tau_n : \mathrm{Ch}(\mathcal{A})_{\leq n+1} \rightleftarrows \mathrm{Ch}(\mathcal{A})_{\leq n} : \mathrm{in}\}_{n\geq 0}$$
We will denote this tower of model categories by $\mathcal{T}(\mathcal{A}, \mathcal{I})$ and use the symbol $\mathrm{Tow}(\mathcal{A}, \mathcal{I})$ to denote the category of towers in $\mathcal{T}(\mathcal{A}, \mathcal{I})$.

Let $X_\bullet$ be an object in $\mathrm{Tow}(\mathcal{A},\mathcal{I})$. We can think about this object as a tower of morphisms:

$$\cdots \xrightarrow{t_3} X_2 \xrightarrow{t_2} X_1 \xrightarrow{t_1} X_0$$

in $\mathrm{Ch}(\mathcal{A})$ given by the structure morphisms of $X_\bullet$. Conversely, for any such tower where X_n is a chain complex that belongs to $\mathrm{Ch}(\mathcal{A})_{\leq n}$, we can define an object $X_\bullet$ in $\mathrm{Tow}(\mathcal{A},\mathcal{I})$ given by the sequence $\{X_n\}_{n\geq 0}$ with the morphisms $\{t_{n+1}\}_{n\geq 0}$ as its structure morphisms. In this way we can think about $\mathrm{Tow}(\mathcal{A},\mathcal{I})$ as a full subcategory of the functor category $\mathrm{Fun}(\mathbf{N}^{op},\mathrm{Ch}(\mathcal{A}))$.
To be very explicit, $\mathrm{Tow}(\mathcal{A},\mathcal{I})$ is the category of commutative diagrams in $\mathcal{A}$ of the following form:

$$\begin{array}{ccccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_{2,2} & \xrightarrow{d_{2,2}} & X_{2,1} & \xrightarrow{d_{2,1}} & X_{2,0} & \xrightarrow{d_{2,0}} & X_{2,-1} & \xrightarrow{d_{2,-1}} & X_{2,-2} & \xrightarrow{d_{2,-2}} & \cdots \\
 & & \downarrow{\scriptstyle t_{2,2}} & & \downarrow{\scriptstyle t_{2,1}} & & \downarrow{\scriptstyle t_{2,0}} & & \downarrow{\scriptstyle t_{2,-1}} & & \downarrow{\scriptstyle t_{2,-2}} & & \\
 & & 0 & \longrightarrow & X_{1,1} & \xrightarrow{d_{1,1}} & X_{1,0} & \xrightarrow{d_{1,0}} & X_{1,-1} & \xrightarrow{d_{1,-1}} & X_{1,-2} & \xrightarrow{d_{1,-2}} & \cdots \\
 & & & & \downarrow{\scriptstyle t_{1,1}} & & \downarrow{\scriptstyle t_{1,0}} & & \downarrow{\scriptstyle t_{1,-1}} & & \downarrow{\scriptstyle t_{1,-2}} & & \\
 & & & & 0 & \longrightarrow & X_{0,0} & \xrightarrow{d_{0,0}} & X_{0,-1} & \xrightarrow{d_{0,-1}} & X_{0,-2} & \xrightarrow{d_{0,-2}} & \cdots
\end{array} \tag{1}$$

where, for any $n \geq 0$ and $i \leq n$, $d_{n,i-1}d_{n,i} = 0$, i.e., horizontal lines are chain complexes.
We will always think about $\mathrm{Tow}(\mathcal{A},\mathcal{I})$ as a model category, with the model structure given by Proposition 4.3. For example, if we think about $X_\bullet$ as a tower $(\cdots \xrightarrow{t_3} X_2 \xrightarrow{t_2} X_1 \xrightarrow{t_1} X_0)$, then $X_\bullet$ is fibrant if and only if X_0 is $\mathcal{I}$-fibrant in $\mathrm{Ch}(\mathcal{A})_{\leq 0}$ and, for any $n \geq 0$, $t_{n+1} : X_{n+1} \to X_n$ is an $\mathcal{I}$-fibration in $\mathrm{Ch}(\mathcal{A})_{\leq n+1}$. If we think about $X_\bullet$ as a commutative diagram as above, then $X_\bullet$ is fibrant if, for any $i \leq 0$, the objects $X_{0,i}$ belongs to $\mathcal{I}$, and, for any $n > 0$ and $i \leq n$, $t_{n,i}$ has a section and its kernel belongs to $\mathcal{I}$. Note also that since all objects in $\mathrm{Ch}(\mathcal{A})_{\leq n}$ are cofibrant, then so are all objects in $\mathrm{Tow}(\mathcal{A},\mathcal{I})$.

5.3. Alternative description. Let us briefly outline another way of describing the category $\mathrm{Tow}(\mathcal{A},\mathcal{I})$. Consider the constant sequence $\{\mathrm{Ch}(\mathcal{A})_{\leq 0}\}_{n\geq 0}$ equipped with the model structure given by Theorem 5.2 and the sequence of adjoint functors $\{\tau : \mathrm{Ch}(\mathcal{A})_{\leq 0} \rightleftarrows \mathrm{Ch}(\mathcal{A})_{\leq 0} : \Sigma^{-1}\}_{n\geq 0}$, where Σ^{-1} is the shift functor. It is clear that Σ^{-1} takes (acyclic) $\mathcal{I}$-fibrations in $\mathrm{Ch}(\mathcal{A})_{\leq 0}$ into (acyclic) $\mathcal{I}$-fibrations in $\mathrm{Ch}(\mathcal{A})_{\leq 0}$. Let us denote this tower of model categories by $\mathcal{T}$.
Let $X_\bullet$ be an object in $\mathrm{Tow}(\mathcal{T})$. The structure morphisms of $X_\bullet$ and the differentials of the chain complexes X_i can be assembled to form a commutative diagram in $\mathcal{A}$ as in (1). This defines an isomorphism between the category of

such commutative diagrams and the category of towers $\mathrm{Tow}(\mathcal{T})$. It then follows that $\mathrm{Tow}(\mathcal{T})$ is also isomorphic to $\mathrm{Tow}(\mathcal{A}, \mathcal{I})$.

5.4. A right Quillen pair for $\mathrm{Ch}(\mathcal{A})$. In this subsection we use the model category $\mathrm{Tow}(\mathcal{A}, \mathcal{I})$ described above to define a right Quillen pair for $\mathrm{Ch}(\mathcal{A})$ that has potential to approximate more than complexes with bounded $\mathcal{I}$-homology (see Proposition 5.4). We define first a pair of adjoint functors $\mathrm{tow} : \mathrm{Ch}(\mathcal{A}) \rightleftarrows \mathrm{Tow}(\mathcal{A}, \mathcal{I}) : \lim$.

Let X be an object in $\mathrm{Ch}(\mathcal{A})$. Define $\mathrm{tow}(X)$ to be the object in $\mathrm{Tow}(\mathcal{A}, \mathcal{I})$ given by the sequence $\{\tau_n(X)\}_{n\geq 0}$ where the structural morphisms are the truncation morphisms $\{t_{n+1} : \tau_{n+1}(X) \to \tau_n(X)\}_{n\geq 0}$. Explicitly, $\mathrm{tow}(X)$ is represented by the following commutative diagram in $\mathcal{A}$:

(2)

$$\begin{array}{ccccccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\tau_2(X) & & 0 & \to & \mathrm{Coker}(d_3) & \xrightarrow{d_2} & X_1 & \xrightarrow{d_1} & X_0 & \xrightarrow{d_0} & X_{-1} & \xrightarrow{d_{-1}} & \cdots \\
\downarrow & & & & \downarrow & & \downarrow q & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \\
\tau_1(X) & & & & 0 & \to & \mathrm{Coker}(d_2) & \xrightarrow{d_1} & X_0 & \xrightarrow{d_0} & X_{-1} & \xrightarrow{d_{-1}} & \cdots \\
\downarrow & & & & & & \downarrow & & \downarrow q & & \downarrow \mathrm{id} & & \\
\tau_0(X) & & & & & & 0 & \to & \mathrm{Coker}(d_1) & \xrightarrow{d_0} & X_{-1} & \xrightarrow{d_{-1}} & \cdots
\end{array}$$

where all q's denote quotient morphisms. For a chain map $f : X \to Y$, the morphism $\mathrm{tow}(f)$ is given by the sequence of morphisms $\{\tau_n(f)\}_{n\geq 0}$.

We define next the limit functor $\lim : \mathrm{Tow}(\mathcal{A}, \mathcal{I}) \to \mathrm{Ch}(\mathcal{A})$ to be the restriction of the standard limit functor defined on $\mathrm{Fun}(\mathbf{N}^{op}, \mathrm{Ch}(\mathcal{A}))$ to the full subcategory $\mathrm{Tow}(\mathcal{A}, \mathcal{I})$. Explicitly, let $X_\bullet$ be an object in $\mathrm{Tow}(\mathcal{A}, \mathcal{I})$ described by a diagram of the form (1). Then $\lim(X_\bullet)$ is the chain complex obtained by taking inverse limits in the vertical direction:

$$\lim(X_\bullet)_i := \lim(\cdots \xrightarrow{t_{3,i}} X_{2,i} \xrightarrow{t_{2,i}} X_{1,i} \xrightarrow{t_{1,i}} X_{0,i})$$

and the differential $d_i : \lim(X_\bullet)_i \to \lim(X_\bullet)_{i-1}$ is given by $\lim_n(d_{n,i})$. On morphisms, the functor $\lim : \mathrm{Tow}(\mathcal{A}, \mathcal{I}) \to \mathrm{Ch}(\mathcal{A})$ is defined in the analogous way by taking the inverse limits in the vertical direction.

Proposition 5.5. *The functors* $\mathrm{tow} : \mathrm{Ch}(\mathcal{A}) \rightleftarrows \mathrm{Tow}(\mathcal{A}, \mathcal{I}) : \lim$ *form a right Quillen pair for* $\mathrm{Ch}(\mathcal{A})$ *with $\mathcal{I}$-weak equivalences as weak equivalences.*

Proof. We need to verify that the three conditions in Definition 3.1 are fulfilled.
(1) We must show that the tower functor $\mathrm{tow} : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Tow}(\mathcal{A}, \mathcal{I})$ is left adjoint to the limit functor $\lim : \mathrm{Tow}(\mathcal{A}, \mathcal{I}) \to \mathrm{Ch}(\mathcal{A})$.

Let Y be a chain complex in $\mathrm{Ch}(\mathcal{A})$ and $X_\bullet$ be an object in $\mathrm{Tow}(\mathcal{A}, \mathcal{I})$ given by the tower $(\cdots X_2 \xrightarrow{t_2} X_1 \xrightarrow{t_1} X_0)$ of morphisms in $\mathrm{Ch}(\mathcal{A})$ with $X_n \in \mathrm{Ch}(\mathcal{A})_{\leq n}$.

Consider a morphism of chain complexes $f : Y \to \lim(X_\bullet)$. Since $\lim(X_\bullet)$ is the inverse limit of the tower $X_\bullet$, the morphism f corresponds to a sequence of morphisms $\{f_n : Y \to X_n\}_{n\geq 0}$ which are compatible with the structural morphisms t_n.
Since the chain complex X_n belongs to $\mathrm{Ch}(\mathcal{A})_{\leq n}$, the morphism $f_n : Y \to X_n$ can be expressed in a unique way as a composition $Y \to \tau_n(Y) \to X_n$ where $Y \to \tau_n(Y)$ is the truncation morphism. The sequence $\{\tau_n(Y) \to X_n\}_{n\geq 0}$ describes a morphism $\mathrm{tow}(Y) \to X_\bullet$ in $\mathrm{Tow}(\mathcal{A}, \mathcal{I})$. It is straightforward to check that this procedure defines a natural bijection from the set of morphisms between Y and $\lim(X_\bullet)$ in $\mathrm{Ch}(\mathcal{A})$ onto the set of morphisms between $\mathrm{tow}(Y)$ and $X_\bullet$ in $\mathrm{Tow}(\mathcal{A}, \mathcal{I})$.

Condition (2) is a consequence of Proposition 5.4: If $f : X \to Y$ is an $\mathcal{I}$-weak equivalence in $\mathrm{Ch}(\mathcal{A})$, then $\mathrm{tow}(f)$ is a weak equivalence in $\mathrm{Tow}(\mathcal{A}, \mathcal{I})$.

To prepare the proof of the third and last condition we show the following.

(2.5) Let $K_\bullet \in \mathrm{Tow}(\mathcal{A}, \mathcal{I})$ be a fibrant object such that K_n is $\mathcal{I}$-trivial in $\mathrm{Ch}(\mathcal{A})_{\leq n}$ for any $n \geq 0$. Then $\lim(K_\bullet)$ is $\mathcal{I}$-trivial in $\mathrm{Ch}(\mathcal{A})$.
Since $K_\bullet$ is fibrant in $\mathrm{Tow}(\mathcal{A}, \mathcal{I})$, K_0 is $\mathcal{I}$-fibrant in $\mathrm{Ch}(\mathcal{A})_{\leq 0}$ and, for $n > 0$, the structure morphism $t_n : K_n \to K_{n-1}$ is an $\mathcal{I}$-fibration in $\mathrm{Ch}(\mathcal{A})_{\leq n}$. As all K_n's are assumed to be $\mathcal{I}$-trivial, the $\mathcal{I}$-fibrations t_n are also $\mathcal{I}$-weak equivalences. It then follows from Proposition 5.3.(3) that $K_\bullet$ is isomorphic to the following tower of chain complexes:

$$\cdots \to M_0 \oplus M_1 \oplus M_2 \oplus M_3 \xrightarrow{\mathrm{pr}} M_0 \oplus M_1 \oplus M_2 \xrightarrow{\mathrm{pr}} M_0 \oplus M_1 \xrightarrow{\mathrm{pr}} M_0$$

where $M_0 := K_0$ and, for $n > 0$, $M_n := \mathrm{Ker}\, t_n$. Thus $\lim(K_\bullet) \cong \prod_{n\geq 0} M_n$. Because M_n is $\mathcal{I}$-trivial and $\mathcal{I}$-fibrant in $\mathrm{Ch}(\mathcal{A})_{\leq n}$, Proposition 5.3.(4) implies that M_n is isomorphic to $\bigoplus_{i\leq n} D_i(W_{n,i})$ for some sequence $\{W_{n,i}\}_{i\leq n}$ of objects in $\mathcal{I}$. Substituting this to the above product describing $\lim(K_\bullet)$ we get the following isomorphisms:

$$\begin{aligned}\lim(K_\bullet) &\cong \prod_{n\geq 0} M_n = \prod_{n\geq 0} \bigoplus_{i\leq n} D_i(W_{n,i}) = \prod_{n\geq 0} \prod_{i\leq n} D_i(W_{n,i}) \cong \\ &\cong \prod_{i} \prod_{i\leq n} D_i(W_{n,i}) \cong \prod_{i} D_i(\prod_{i\leq n} W_{n,i}) \cong \bigoplus_{i} D_i(\prod_{i\leq n} W_{n,i})\end{aligned}$$

It is now clear that $\lim(K_\bullet)$ is $\mathcal{I}$-trivial. In fact $\lim(K_\bullet)$ is even homotopy equivalent to the zero chain complex.

(3) Let $f_\bullet : X_\bullet \to Y_\bullet$ be a weak equivalence in $\mathrm{Tow}(\mathcal{A}, \mathcal{I})$ between fibrant objects. Then $\lim(f_\bullet)$ is an $\mathcal{I}$-weak equivalence in $\mathrm{Ch}(\mathcal{A})$.
By Ken Brown's Lemma (see [5, Factorization Lemma], or [11, Lemma 9.9] for a more explicit treatment), it is enough to show the statement under the additional assumption that $f_\bullet : X_\bullet \to Y_\bullet$ is an $\mathcal{I}$-fibration. Let us define $K_\bullet$ to be an object in $\mathrm{Tow}(\mathcal{A}, \mathcal{I})$ given by the sequence $\{\mathrm{Ker}(f_n)\}_{n\geq 0}$ with the structure morphisms being the restrictions of the structure morphisms of $X_\bullet$. Since all

objects in $\mathrm{Ch}(\mathcal{A})_{\leq n}$ are $\mathcal{I}$-cofibrant, then so are all objects in $\mathrm{Tow}(\mathcal{A},\mathcal{I})$. It follows that there is $s_\bullet : Y_\bullet \to X_\bullet$ for which $f_\bullet s_\bullet = \mathrm{id}$. By applying the functor lim, we then get the following split exact sequence in $\mathrm{Ch}(\mathcal{A})$:

$$0 \to \lim(K_\bullet) \to \lim(X_\bullet) \xrightarrow{\lim(f_\bullet)} \lim(Y_\bullet) \to 0$$

Since $X_\bullet$ is isomorphic to $K_\bullet \oplus Y_\bullet$, as a retract of a fibrant object $X_\bullet$, the object $K_\bullet$ is then also fibrant. Moreover, as f_n is an $\mathcal{I}$-equivalence in $\mathrm{Ch}(\mathcal{A})_{n\geq}$, the complex K_n is $\mathcal{I}$-trivial in $\mathrm{Ch}(\mathcal{A})_{\leq n}$ for any $n \geq 0$. We can then apply statement (2.5) to conclude that $\lim(K_\bullet)$ is an $\mathcal{I}$-trivial chain complex in $\mathrm{Ch}(\mathcal{A})$. The morphism $\lim(f_\bullet) : \lim(X_\bullet) \to \lim(Y_\bullet)$ must be then an $\mathcal{I}$-weak equivalence (which is in fact a homotopy equivalence). □

Definition 5.6. The right Quillen pair $\mathrm{tow} : \mathrm{Ch}(\mathcal{A}) \rightleftarrows \mathrm{Tow}(\mathcal{A},\mathcal{I}) : \lim$ is called the *standard Quillen pair* for $\mathrm{Ch}(\mathcal{A})$.

5.5. Complexes approximated by the standard Quillen pair. The key task now is to find out which chain complexes are approximated by the standard Quillen pair, i.e. we need to understand which complexes X have the following property:

- If $f : \mathrm{tow}(X) \to Y_\bullet$ is a weak equivalence in $\mathrm{Tow}(\mathcal{A},\mathcal{I})$, with fibrant target $Y_\bullet$, then its adjoint $g : X \to \lim(Y_\bullet)$ is an $\mathcal{I}$-weak equivalence in $\mathrm{Ch}(\mathcal{A})$.

Recall that if, for a chain complex X, the above statement is true for some fibrant $Y_\bullet$, then it is true for any other.
Assume that $f : \mathrm{tow}(X) \to Y_\bullet$ is a weak equivalence in $\mathrm{Tow}(\mathcal{A},\mathcal{I})$ (for now even without the fibrancy assumption on $Y_\bullet$) and let $g : X \to \lim(Y_\bullet)$ be its adjoint. Fix an integer $k \geq 0$ and consider the following commutative diagram in $\mathrm{Ch}(\mathcal{A})$:

$$\begin{array}{ccc} X & \xrightarrow{g} & \lim(Y_\bullet) \\ \downarrow & & \downarrow \\ \tau_k(X) & \xrightarrow{f_k} & Y_k \end{array}$$

where $\lim(Y_\bullet) \to Y_k$ is the projection and $X \to \tau_k(X)$ is the truncation morphism, which, according to Proposition 5.1, is a k-$\mathcal{I}$-weak equivalence. By assumption f_k is an $\mathcal{I}$-weak equivalence so that the composite of g with the projection $\lim(Y_\bullet) \to Y_k$ is a k-$\mathcal{I}$-weak equivalence.
As a consequence, the error in approximating a complex is always of a somewhat tame nature. For any $W \in \mathcal{I}$, $\mathcal{A}(g, W) : \mathcal{A}(\lim(Y_\bullet), W) \to \mathcal{A}(X, W)$ must induce a split epimorphism in cohomology in degrees $i \geq -k$ and this happens for all k's.

Proposition 5.7. *Let* $f : \mathrm{tow}(X) \to Y_\bullet$ *be a weak equivalence in* $\mathrm{Tow}(\mathcal{A},\mathcal{I})$ *and* $g : X \to \lim(Y_\bullet)$ *be its adjoint. Then, for any* $W \in \mathcal{I}$, $\mathcal{A}(g, W)$ *induces a split epimorphism on homology.* □

6. A relative version of Roos' axiom AB4*-n

In this section we show that under Roos' axiom AB4*-n, see [32], every complex is approximated by the standard Quillen pair. In fact we introduce a relative version of this axiom and extend this result to provide a construction of relative resolutions for unbounded chain complexes via towers of truncations.

Definition 6.1. Let $\mathcal{A}$ be an abelian category, $\mathcal{I}$ an injective class and $n \geq 0$ an integer. We say that the category $\mathcal{A}$ satisfies axiom AB4*-$\mathcal{I}$-n if and only if, for any countable family of objects $(A_j)_{j\in J}$ and any choice of relative resolutions $A_j \to I_j$, with $I_j \in \mathrm{Ch}(\mathcal{A})_{\leq 0}$, the product complex $\prod_{j\in J} I_j$ is $(-n-1)$-$\mathcal{I}$-connected.

Roos' axiom AB4*-n is stated in terms of the derived functors of products, namely that all infinite derived product functors $\Pi^{(i)} C_\alpha$ vanish for $i > n$. Our axiom involves countable products because we only need towers indexed by the natural integers. Except for this our axioms are closely related.

Proposition 6.2. *An abelian category $\mathcal{A}$ satisfies axiom* AB4*-$\mathcal{I}$-n *for the class $\mathcal{I}$ of all injective objects if and only if all derived countable product functors $\prod^{(i)} A_j$ vanish for $i > n$.*

Proof. Given a countable family of objects $(A_j)_{j\in J}$, let us choose injective resolutions $A_j \to I_j$, and form the product complex $\prod_{j\in J} I_j$. This complex is $(-n-1)$-$\mathcal{I}$-connected if and only if it is $(-n-1)$-connected since we deal here with the class of all injectives. The higher homology of this complex computes the derived functors of the countable product $\prod^{(i)} A_j$. They vanish for $i > n$ precisely when the complex is $(-n-1)$-connected. □

Proposition 6.3. *Let $\mathcal{I}$ be an injective class and assume that the abelian category $\mathcal{A}$ satisfies axiom* AB4*-$\mathcal{I}$-n. *Let $K_\bullet$ be a fibrant tower in $Tow(\mathcal{A}, \mathcal{I})$ such that, for any n, K_n is k-$\mathcal{I}$-connected. Then the limit complex $\lim K_\bullet$ is $(k-n-1)$-$\mathcal{I}$-connected.*

Proof. The kernel of the "one minus shift" map $1-t : \prod K_n \to \prod K_n$, defined by $(1-t)(x_n) = (x_n - t_{n+1}(x_{n+1}))$, is $\lim K_\bullet$. Since $K_\bullet$ is fibrant, the vertical structure maps t_n are degreewise split epimorphisms and we may choose, in each degree, a splitting $\sigma : K_n \to K_{n+1}$. We define then maps $\prod K_n \to K_{m+1}$ for all $m \geq 0$ by the formula

$$(x_0, x_1, x_2, \cdots) \mapsto -\sum_{j=0}^{m} \sigma^{m+1-j}(x_j)$$

which assemble to form a degreewise splitting $s : \prod K_n \to \prod K_n$ of $1-t$. This proves first that the sequence

$$0 \longrightarrow \lim K_\bullet \longrightarrow \Pi_n K_n \xrightarrow{1-t} \Pi_n K_\bullet \longrightarrow 0$$

is exact, and second, that applying $\mathcal{A}(-,W)$ for any $W \in \mathcal{I}$ to the previous sequence gives an exact sequence of complexes, which is also split in each degree:

$$0 \longleftarrow \mathcal{A}(\lim K_\bullet, W) \longleftarrow \mathcal{A}(\prod_n K_n, W) \underset{1-t}{\longleftarrow} \mathcal{A}(\prod_n K_n, W) \longleftarrow 0$$

Therefore, any bound on the connectivity of $\mathcal{A}(\prod_n K_n, W)$ is a bound on the connectivity of $\mathcal{A}(\lim K_\bullet, W)$. We will conclude the proof by showing that $k-n-1$ is such a bound. Observe now that, because each complex K_m is k-$\mathcal{I}$-connected, the following sequence is exact:

$$\mathcal{A}((K_m)_{k+1}, W) \longleftarrow \mathcal{A}((K_m)_k, W) \longleftarrow \mathcal{A}((K_m)_{k-1}, W) \longleftarrow \dots$$

Left exactness of the functor $\mathcal{A}(-,W)$ shows that the kernel of the leftmost arrow above is $\mathcal{A}((K_m)_k/(K_m)_{k+1}, W)$. In particular, as the complex K_m is fibrant and k-connected, the truncated complex $\tau_k(K_m)$ yields a (shifted) relative $\mathcal{I}$-resolution of the object $(K_m)_k/(K_m)_{k+1}$.
Hence, in computing in degree $q < k-n-1$ the cohomology of the complex $\mathcal{A}(\prod K_n, W)$ we are computing, under the axiom AB4*-$\mathcal{I}$-n, the cohomology in degree $< -n$ of an acyclic complex. □

And finally we get our expected approximation:

Theorem 6.4. *Let $\mathcal{I}$ be an injective class and assume that the abelian category $\mathcal{A}$ satisfies axiom* AB4*-$\mathcal{I}$-n. *Then the standard Quillen pair*

$$\text{tow} : \text{Ch}(\mathcal{A}) \rightleftarrows \text{Tow}(\mathcal{A}, \mathcal{I}) : \lim$$

is a model approximation.

Proof. In view of Proposition 5.5 it remains to show that to any fibrant replacement $f : \text{tow}(X) \to Y_\bullet$ corresponds an adjoint $X \to \lim Y_\bullet$ that is an $\mathcal{I}$-weak equivalence. Let $\{t_{n+1} : Y_{n+1} \to Y_n\}_{n\geq 0}$ be the structure morphisms of $Y_\bullet$ and, for any $n \geq k$, let $c_{n-k} : Y_n \to Y_k$ denote the composite

$$Y_n \xrightarrow{t_{n-k}} Y_{n-1} \xrightarrow{t_{n-1}} \cdots \xrightarrow{t_{k+1}} Y_k$$

These morphisms fit into the following commutative square in $\text{Ch}(A)_{\leq n}$ where the top horizontal morphism is a truncation morphism:

$$\begin{array}{ccc} \tau_n(X) & \longrightarrow & \tau_k(X) \\ \downarrow f_n & & \downarrow f_k \\ Y_n & \xrightarrow{c_{n-k}} & Y_k \end{array}$$

By assumption, f_n and f_k are $\mathcal{I}$-weak equivalences, and the top horizontal arrow is a k-$\mathcal{I}$-weak equivalence according to Proposition 5.1. It follows that c_{n-k} is a k-$\mathcal{I}$-weak equivalence. Fibrancy of $Y_\bullet$ implies that c_{n-k} is also an $\mathcal{I}$-fibration in $\text{Ch}(A)_{\leq n}$. In particular, for $n \geq k$, $K_n := \text{Ker}(c_{n-k} : Y_n \to Y_k)$ is k-$\mathcal{I}$-connected (see Proposition 5.3.(2)). Set $K_n := 0$ for $n < k$, and define $t_{n+1} : K_{n+1} \to K_n$ to be the restriction of the structure morphism $t_{n+1} : Y_{n+1} \to Y_n$, if $n \geq k$, and the zero morphism, if $n < k$. In this way we have defined a fibrant object $K_\bullet$ in $\mathrm{Tow}(\mathcal{A}, \mathcal{I})$. We have moreover a degreewise split exact sequence in $\mathrm{Ch}(\mathcal{A})$:

$$0 \longrightarrow \lim(K_\bullet) \longrightarrow \lim(Y_\bullet) \longrightarrow Y_k \longrightarrow 0 \tag{3}$$

By Proposition 6.3, $\lim(K_\bullet)$ is $(k-n-1)$-$\mathcal{I}$-connected, hence $\lim(Y_\bullet) \to Y_k$ is a $(k-n-1)$-$\mathcal{I}$-equivalence. But the composite $X = \lim(\tau_n(X)) \to \lim(Y_\bullet) \to Y_k$ is a k-$\mathcal{I}$-weak equivalence, and it follows by the "2 out of 3" property that $X \to \lim(Y_\bullet)$ is a $(k-n-1)$-$\mathcal{I}$-weak equivalence. This is so for any value of k, which concludes the proof. □

This explains also why Spaltenstein's construction of resolutions via truncations works in the absolute setting.

Corollary 6.5. *Let R be a ring and $\mathcal{I}$ be the class of all injective R-modules. The category of R-modules satisfies axiom* AB4*-$\mathcal{I}$-0. *In particular the standard Quillen pair*

$$\mathrm{tow} : \mathrm{Ch}(R) \rightleftarrows \mathrm{Tow}(R, \mathcal{I}) : \lim$$

is a model approximation.

Proof. Relative connectivity for the class of all injective modules is connectivity and the category of R-modules satisfies axiom AB4*, which is AB4*-0 as stated in [32, Remark 1.2]. □

7. Example: Noetherian rings with finite Krull dimension

In this section R is a Noetherian ring and we focus on injective classes of injectives, which were classified in [6]: they are in one-to-one correspondence with the generization closed subsets of $\mathrm{Spec}R$. We show that, under the additional assumption that R is of finite Krull dimension, the standard Quillen pair is a model approximation for all injective classes $\mathcal{I}$ of injectives.
We need some preparation before proving this theorem, and we refer to Appendix B for elementary facts about local cohomology. The key ingredient is the vanishing of the homology of an $\mathcal{I}$-relative resolution above the Krull dimension of the ring.

Lemma 7.1. *Let R be a Noetherian ring, $\mathfrak{p} \subset R$ a prime ideal of height d, and $I \in Ch(R)_{\leq 0}$ an injective resolution of a module M. The complex $I(\mathfrak{p})$, obtained from I by keeping only the direct summands isomorphic to $E(R/\mathfrak{p})$, has no homology in degrees $< -d$.*

Proof. First form $I \otimes R_\mathfrak{p}$, that is localize I at $\mathfrak{p}$ to kill all the summands of I isomorphic to $E(R/\mathfrak{q})$ with $\mathfrak{q} \not\subset \mathfrak{p}$, see Lemma B.2. The subcomplex $\Gamma_\mathfrak{p}(I \otimes R_\mathfrak{p})$ is precisely what we obtain from $I \otimes R_\mathfrak{p}$ by excising summands isomorphic to $E(R/\mathfrak{q})$ for $\mathfrak{q} \subsetneq \mathfrak{p}$, see Lemma B.5. Thus $\Gamma_\mathfrak{p}(I \otimes R_\mathfrak{p}) = I(\mathfrak{p})$, in the notation of the current lemma. The ring $R_\mathfrak{p}$ is flat over R, hence $I \otimes R_\mathfrak{p}$ is an injective resolution over $R_\mathfrak{p}$ of the module $M \otimes R_\mathfrak{p}$, and the cohomology of $I(\mathfrak{p}) = \Gamma_\mathfrak{p}(I \otimes R_\mathfrak{p})$ is the local cohomology of $M \otimes R_\mathfrak{p}$ at the maximal ideal $\mathfrak{p}R_\mathfrak{p} \subset R_\mathfrak{p}$. The vanishing follows from Proposition B.7 and Remark B.8 because the Krull dimension of $R_\mathfrak{p}$ is d. □

Proposition 7.2. *Let R be a Noetherian ring of finite Krull dimension d, and $\mathcal{I}$ an injective class of injective modules. For any module M and an $\mathcal{I}$-relative resolution $I \in Ch(R)_{\leq 0}$, we have $H_k(I) = 0$ if $k < -d - 1$.*

Proof. The injective class $\mathcal{I}$ corresponds to a generization closed subset S of $Spec(R)$ by [6, Corollary 3.1]. Let a be the length of the maximal chain of prime ideals in the complement of S. If $a = 0$ then $\mathcal{I}$ consists of all injective modules, so that $H_k(I) = 0$ for all $k < 0$.
Assume now that $a \geq 1$ and we prove the proposition by induction on a. Consider the set S'' of minimal ideals $\mathfrak{p}_i$ in the complement of S; we know the result is true for the injective class $\mathcal{I}'$ corresponding to the set $S' = S \cup S''$. We denote by I' the $\mathcal{I}'$-relative resolution of M. Replacing I by a homotopy equivalent complex if necessary, we obtain a degree-wise split short exact sequence of chain complexes $0 \to I'' \to I' \to I \to 0$, where I'' is a direct sum of $E(R/\mathfrak{p}_i)$ with $\mathfrak{p}_i \in S''$. But there are no inclusions among the primes $\mathfrak{p}_i \in S' - S$, hence I'' is the direct sum of the complexes $I(\mathfrak{p}_i)$ that we introduced in Lemma 7.1. The proposition now follows from Lemma 7.1 and the long exact sequence in homology induced by the short exact sequence of complexes $0 \to I'' \to I' \to I \to 0$. □

Next comes the last proposition we will use in the proof of our main theorem, it measures the difference between the resolutions of a bounded complex and of a truncation. Recall that $I(X)$ denotes the fibrant replacement of the bounded complex X in the $\mathcal{I}$-relative model structure described in Theorem 5.2, i.e. an $\mathcal{I}$-relative injective resolution of X.

Proposition 7.3. *Let R be a Noetherian ring of finite Krull dimension d, and $\mathcal{I}$ an injective class of injective modules. Let $X \in Ch(R)_{\leq 0}$ be a bounded complex and $\tau_1 X$ its first truncation. Then the canonical morphism $X \to \tau_1 X$ induces isomorphisms in homology $H_k(I(X)) \to H_k(I(\tau_1 X))$ for any integer $k < -d - 1$.*

Proof. Let us replace $X \to \tau_1 X$ by an $\mathcal{I}$-fibration $I(X) \to I(\tau_1 X)$ between $\mathcal{I}$-fibrant objects. The kernel K is a chain complex made of injective modules in $\mathcal{I}$, and forms therefore an $\mathcal{I}$-fibrant replacement for $H_0(X)$, the kernel of the canonical morphism.
From the previous proposition we know that $H_k(K) = 0$ if $k < -d - 1$. The long exact sequence in homology finishes the proof. □

Theorem 7.4. *Let R be a Noetherian ring with finite Krull dimension d, and $\mathcal{I}$ an injective class of injective modules. Then the category of towers forms a model approximation for $Ch(R)$ equipped with $\mathcal{I}$-equivalences.*

Proof. To show that the Quillen pair is in fact a model approximation, we must check that Condition (4) of Definition 3.1 holds, or equivalently that the canonical morphism $\lim I(\text{tow}X) \to X$ is an $\mathcal{I}$-equivalence for any unbounded chain complex X. We have learned from Proposition 7.3 that the homology of $I(\tau_n X)$ and $I(\tau_{n-1} X)$ only differ in degrees lying between n and $n-d-1$. This means that the homology of the $\mathcal{I}$-fibrant replacement of the tower $\text{tow}(X)$ stabilizes. Therefore $H_k(\lim, I(\text{tow}X)) \cong H_k(I(\tau_{k+d+1}X))$. □

Remark 7.5. The above argument actually shows that the category of R-modules satisfies axiom AB4*-$\mathcal{I}$-$(d+1)$ when R has finite Krull dimension d, and $\mathcal{I}$ is an injective class of injective R-modules. A product of relative injective resolutions of certain R-modules is a special case of an inverse limit of a fibrant tower as above.

8. Example: Nagata's "bad Noetherian ring"

The objective of this section is to show that, even under the Noetherian assumption, towers do not always approximate unbounded chain complexes. We have seen in the previous section that no problems arise when the Krull dimension is finite. However, delicate and interesting issues arise when the Krull dimension is infinite. We first recall an example of Noetherian ring with infinite Krull dimension, constructed by Nagata in the appendix of [27].

Example 8.1. Let k be a field and consider the polynomial ring on countably many variables $A = k[x_1, x_2, \dots]$. Choose the following sequence of prime ideals $\mathfrak{p}_2 = (x_1, x_2)$, $\mathfrak{p}_3 = (x_3, x_4, x_5)$, $\mathfrak{p}_4 = (x_6, x_7, x_8)$, etc. where the depth of $\mathfrak{p}_i$ is precisely i. Take S to be the multiplicative set consisting of elements of A which are not in any of the $\mathfrak{p}_i$'s. The localized ring $R = S^{-1}A$ is Noetherian, but of infinite Krull dimension. In fact its maximal ideals are $\mathfrak{m}_i = S^{-1}\mathfrak{p}_i$, a sequence of ideals of strictly increasing height.

8.1. A problematic class of injectives.

In this subsection we choose the specialization closed subset C of $\text{Spec}(R)$ to consist of all the maximal ideals $\mathfrak{m}_i$. We will do relative homological algebra with respect to the injective class $\mathcal{I}$ of injective R-modules, generated by the injective envelopes $E(R/\mathfrak{p})$ for all prime ideals $\mathfrak{p} \notin C$. We noticed earlier that the class of $\mathcal{I}$-acyclic chain complexes is a localizing subcategory of $D(R)$. As it contains $R/\mathfrak{m}_i$ but not any other $R/\mathfrak{p}$, we know from Neeman's classification [28] that this localizing subcategory is generated by all $R/\mathfrak{m}_i$.

Lemma 8.2. *Let $I(R)$ be an $\mathcal{I}$-injective resolution of R. Then $H_0(I(R)) \cong R$ and $H_{1-i}(I(R)) \cong E(R/\mathfrak{m}_i)$ for any $i > 1$.*

Proof. Consider a minimal injective resolution $R \hookrightarrow I_0 = E(R) \to I_{-1} \to \dots$. By the description Matlis [26] gave of injective modules, each I_n is a direct sum of modules of the form $E(R/\mathfrak{p})$ where $\mathfrak{p}$ runs over prime ideals of R.
By Lemma B.3 we see that there is a subcomplex K of I made of all the copies of $E(R/\mathfrak{m}_i)$, and we take $I(R) = I/K$. This is a fibrant replacement for R in the relative model structure described in Theorem 5.2. Since the homology of I is concentrated in degree 0, we see from the long exact sequence

in homology for the short exact sequence of complexes $K \to I \to I(R)$ that the lower homology modules of $I(R)$ are isomorphic to those of K up to a shift: $H_{1-k}(I(R)) \cong H_{-k}(K)$ for $k > 1$. But K splits as a direct sum $\oplus_i \Gamma_{\mathfrak{m}_i}(I)$ by Lemma B.3 and Definition B.4. Therefore

$$H_{-k}(K) \cong \oplus H^k_{\mathfrak{m}_i}(R) \cong \oplus_i H^k_{\mathfrak{m}_i}(R_{\mathfrak{m}_i})$$

where the second isomorphism comes from Lemma B.5. The local ring $R_{\mathfrak{m}_i}$ is regular, hence Gorenstein, of dimension i. Therefore the computation done in [22, Theorem 11.26] yields that $H_{1-i}(I(R)) \cong E(R/\mathfrak{m}_i)$. It also shows here that $H_0(I(R)) \cong H_0(I) \cong R$ since all local cohomology modules are zero in degree zero. □

Now we consider the unbounded chain complex X with $X_n = R$ for all n and zero differential. The zeroth truncation of X is the non-positively graded complex with zero differential and where every module is R, in other words this complex is $\oplus_{i\leq 0}\Sigma^i R$. We know how to construct explicitly an $\mathcal{I}$-relative resolution for this bounded complex by the previous lemma: it is a direct sum $\oplus_{i\leq 0}\Sigma^i I(R)$.

Lemma 8.3. *Let X be the unbounded complex $\oplus_i \Sigma^i R$, let $\tau_0 X$ be its zeroth truncation, and let $I(\tau_0 X)$ denote the $\mathcal{I}$-relative resolution of the latter. We have then $H_{1-i}(I(\tau_0 X)) \cong R \oplus \oplus_{2\leq j\leq i} E(R/\mathfrak{m}_j)$ for any $i \geq 1$.*

Proof. This is a direct consequence of the previous lemma. □

The unbounded complex X is the key player in our main counterexample.

Theorem 8.4. *For Nagata's ring R and the injective class $\mathcal{I}$ above, the category of towers* $\mathrm{Tow}(R,\mathcal{I})$ *does not form a model approximation for* $\mathrm{Ch}(R)$. *More precisely there exists a complex X which is not $\mathcal{I}$-weakly equivalent to the limit of the fibrant replacement of its truncation tower.*

Proof. The complex X is the one we have constructed above, namely $\oplus_{i\in\mathbf{Z}}\Sigma^i R$. Let us consider its tower approximation, which is, by definition, the limit Y of the tower given by the $\mathcal{I}$-relative resolution of the successive truncations of X. From the previous lemma the nth level of this tower is $\oplus_{i\leq n}\Sigma^i I(R)$ and the structure maps are the projections. Therefore the limit is the product $\prod_i \Sigma^i I(R)$. In particular we identify for any i

$$H_{1-i}(Y) \cong R \times \prod_{j\geq 2} E(R/\mathfrak{m}_j).$$

The homotopy fiber of the natural map $X \to Y$ is thus an unbounded complex whose homology is $\prod_{j\geq 2} E(R/\mathfrak{m}_j)$ in each degree. This complex cannot be $\mathcal{I}$-acyclic since the annihilator of the image of 1 via the (diagonal) composite map

$$R \to \prod_j R \to \prod_j R/\mathfrak{m}_j \to \prod_j E(R/\mathfrak{m}_j)$$

is zero and this contradicts the description of $\mathcal{I}$-acyclic complexes given in Example 1.13. □

8.2. Well behaved classes of injectives. Nagata's ring, or other Noetherian rings of infinite Krull dimension, also have well behaved classes of injective modules. Let us fix for example a maximal ideal $\mathfrak{m}$ of height n. Since the set of primes strictly contained in $\mathfrak{m}$ is saturated by [6] we may consider the injective class $\mathcal{I}_{\mathfrak{m}}$ generated by $\{E(R/\mathfrak{p}) \mid \mathfrak{p} \subsetneq \mathfrak{m}\}$.

Theorem 8.5. *The category R-Mod satisfies axiom* AB4*-$\mathcal{I}_{\mathfrak{m}}$-$(n+1)$*, where n is height $\mathfrak{m}$. In particular the category of towers* $\mathrm{Tow}(R, \mathcal{I}_{\mathfrak{m}})$ *is a model approximation for* $\mathrm{Ch}(R)$.

Proof. Let X be an object in $\mathrm{Ch}(\mathcal{A})_{\leq 0}$, let I be an injective resolution for X, and let $I(X)$ be the $\mathcal{I}$–fibrant replacement of X obtained by excising all the summands of I isomorphic to $E(R/\mathfrak{q})$ for $\mathfrak{q}$ not strictly contained in $\mathfrak{m}$. We have a short exact sequence of chain complexes $0 \to K \to I \to I(X) \to 0$, with K a complex of injectives all of which are direct sums of $E(R/\mathfrak{q})$ for $\mathfrak{q}$ not strictly contained in $\mathfrak{m}$. Since $I(X)$ is a complex of $\mathfrak{m}$–local modules, tensoring with $R_{\mathfrak{m}}$ gives the exact sequence

$$0 \longrightarrow K \otimes R_{\mathfrak{m}} \longrightarrow I \otimes R_{\mathfrak{m}} \longrightarrow I(X) \longrightarrow 0.$$

The first complex is a complex of injectives, each of which is a direct sum of injectives of the form $E(R/\mathfrak{m})$. Thus over the ring $R_{\mathfrak{m}}$, the complex $I(X)$ can be viewed as the fibrant replacement of $I \otimes R_{\mathfrak{m}}$ with respect to the injective class of injectives $\mathcal{I}' = \mathcal{I} \cap \mathrm{Spec}(R_{\mathfrak{m}})$. But this reduces us to the case of the noetherian local ring $R_{\mathfrak{m}}$ which is of finite Krull dimension. Theorem 7.4 finishes the proof. □

9. Further examples

In this section we gather some other examples of relative homological algebra settings that may be found across the literature and show how they tie back to our framework.

9.1. Some Grothendieck categories studied by Roos. The original work of Roos is precisely about finding a way to deal with the failure of axiom AB4*. He provides a nice and elementary example of a Grothendieck category that satisfies axiom AB4*-n but not AB4*-$(n-1)$. This example is very close in spirit to our study of injective classes of injectives for the category of modules over a ring of finite Krull dimension in Section 7.

Proposition 9.1. [32, Theorem 1.15] *The Grothendieck category Qcoh of quasicoherent sheaves on the complement of the maximal ideal $\mathfrak{m}$ of the spectrum of a local Noetherian ring R satisfies condition* AB4*-n *where* $n = \max(\dim(R) - 1, 0)$*, and no lower value of n is possible.*

It turns out that injective classes of injectives on Grothendieck categories correspond to the so called hereditary torsion theories. Building on this observation, Virili recently investigated whether Roos' axiom AB4*-n holds in localizations of Grothendieck categories with respect to these hereditary torsion theories.

The answer depends then on the Gabriel dimension of the localized category, a generalization of the Krull dimension to Grothendieck categories due to Gabriel. We refer to Virili's paper [35] for the precise statements.

9.2. Pure injective classes. Purity is a vast subject, of which we will only present the (very) thin part that is directly related to our framework. As a general reference one could consult Prest [30], but let us recall the basic definitions.
Let R be a ring, a morphism of R-modules $f : M \to N$ is said to be *pure* if and only if for any R-module L, $f \otimes id_L : M \otimes L \to N \otimes R$ is injective. Then a *pure-injective module* (a.k.a. algebraically compact) is an R-module W such that for any pure homomorphism f, the induced map $\mathrm{Hom}(f, W)$ is surjective. A product of pure-injectives is again pure-injective and module categories have enough pure-injectives [30]. Thus, pure-injective modules form an injective class as defined in Definition 1.3.
The following theorem shows that rings of small cardinality satisfy a very strong version of the relative AB4* axiom with respect to the injective class of pure-injectives: all objects are of finite pure-injective dimension.

Theorem 9.2 (Kielpinski-Simson[24], Gruson-Jensen[18]). *Let R be a ring of cardinality $\aleph_t$, with $t \in \mathbb{N}$. Then the pure global dimension of R is $\leq t+1$.*

Applying this to our framework we obtain immediately the analogous result to Theorem 7.4.

Corollary 9.3. *Let R be a ring of cardinality $\aleph_t$ with $t \in \mathbb{N}$, and let $\mathcal{PI}$ denote the class of pure-injective modules. Then the standard Quillen pair*

$$\mathrm{tow} : \mathrm{Ch}(R) \rightleftarrows \mathrm{Tow}(R, \mathcal{PI}) : \lim$$

is a model approximation. □

9.3. Gorenstein homological algebra. This is again a vast and very active research subject, for which we refer for instance to Enochs-Jenda [14] and Holm [21].
Given a ring R an R-module E is said to be *Gorenstein injective* if there exists an exact complex of injective modules

$$I_\bullet : \cdots \longrightarrow I_2 \longrightarrow I_1 \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow \cdots$$

such that for any injective module J the complex $\mathrm{Hom}(J, I_\bullet)$ is acyclic and $E = \mathrm{Ker}(I_0 \to I_{-1})$. Denote the class of Gorenstein injective modules by $\mathcal{GI}$. We learn in [21, Theorem 2.6] that $\mathcal{GI}$ contains all injective modules, and that it is closed under arbitrary products and under direct summands.
The existence of enough Gorenstein injectives (a.k.a. Gorenstein injective pre-envelopes) for general modules is more problematic. Nevertheless Holm shows in [21, Theorem 2.15] that any R-module of finite Gorenstein injective dimension admits a Gorenstein injective pre-envelope and thus a Gorenstein injective

resolution in the sense of the present work (or "coproper right Gorenstein injective resolution" in Holm's terminology). Enochs and López-Ramos prove also in [13] that there are enough Gorenstein injectives in any Noetherian ring.

Proposition 9.4 ([13]). *The class $\mathcal{GI}$ of Gorenstein injective modules is an injective class for any Noetherian ring.*

If we wish to ensure that there are enough Gorenstein injectives, it is therefore enough to assume that all modules have finite Gorenstein injective dimension. It would be interesting to have conditions ensuring that for a given ring the relative version of axiom AB4*-n is satisfied, but for the moment we confine ourselves to the stronger condition that there is a bound on the Gorenstein injective dimension of all modules. By Enochs-Jenda [14] this characterizes Gorenstein rings. As above we readily deduce the following proposition. The (finite) dimension of the ring is the natural number n such that the category of R-modules satisfies AB4*-$\mathcal{GI}$-n.

Proposition 9.5. *Let R be a Gorenstein ring. Then the standard Quillen pair*

$$\mathrm{tow} : \mathrm{Ch}(R) \rightleftarrows \mathrm{Tow}(R, \mathcal{GI}) : \lim$$

is a model approximation. □

Appendix A. Relative homological algebra for left bounded complexes

In this section we work in a fixed abelian category $\mathcal{A}$, and we fix an injective class $\mathcal{I}$, as in Definition 1.3. In particular we assume that there are enough relative injectives. To show that one can equip $\mathrm{Ch}_{\leq 0}(\mathcal{A})$ with an $\mathcal{I}$-relative Quillen model structure we basically follow Quillen's arguments in [31]. We will use the terminology ($\mathcal{I}$-cofibrations, $\mathcal{I}$-weak equivalences, etc.) as introduced in Theorem 5.2. Before going into the homotopical subtleties, let us recall a couple of standard of constructions.

A.1. The cone construction. Let X be a chain complex in $\mathrm{Ch}_{\leq 0}(\mathcal{A})$. Define a complex CX as follows: $CX_0 = X_{-1}$ and $CX_n = X_n \oplus X_{n-1}$ for any $n < 0$. The differential $CX_0 \to CX_{-1}$ is (Id, d) and the lower ones $X_n \oplus X_{n-1} \to X_{n-1} \oplus X_{n-2}$ are given in matrix form by

$$\begin{bmatrix} d & (-1)^n Id \\ 0 & d \end{bmatrix}$$

There is a natural chain map $X \to CX$ given by the inclusion on the first factor, except in degree zero where we use the differential.

Lemma A.2. *The cone CX of any complex $X \in \mathrm{Ch}_{\leq 0}(\mathcal{A})$ is acyclic. The chain map $X \to CX$ is a split injection in strictly negative degrees, so in particular an $\mathcal{I}$-cofibration.* □

A.3. The mapping cylinder. Let $f : N \to M$ be a morphism of left bounded chain complexes. Denote by ∂ and d respectively the differentials

of the complexes N and M. We define a new complex $\mathrm{Cyl}(f)$ as follows : $\mathrm{Cyl}(f)_0 = N_0 \oplus M_0$ and $\mathrm{Cyl}(f)_i = N_i \oplus M_{i+1} \oplus M_i$ for $i < 0$.
The differentials are given as follows

$$\begin{array}{ccccc}
\mathrm{Cyl}(f)_i = & N_i & & M_{i+1} & & M_i \\
\downarrow & \downarrow \partial & \searrow^{(-1)^{i-1}f} & \downarrow d & \swarrow^{(-1)^i Id} & \downarrow d \\
\mathrm{Cyl}(f)_{i-1} = & N_{i-1} & & M_i & & M_{i-1}
\end{array}$$

We have a level-wise split injection $N \to \mathrm{Cyl}(f)$ given by (Id, f) whose cofiber is acyclic. The splitting is given by the projection on the first factor $\mathrm{Cyl}(f) \to N$ (a chain map). We have also a level-wise split epimorphism $\mathrm{Cyl}(f) \to M$, given by the projection on the last factor. This shows the following lemma.

Lemma A.4. *The factorization $N \to \mathrm{Cyl}(f) \to M$ consists in a trivial $\mathcal{I}$-cofibration followed by a degreewise split epimorphism.* □

A.5. Path object. The path object of $X \in \mathrm{Ch}_{\leq 0}(\mathcal{A})$ is a chain complex $\mathrm{P}(X) \in \mathrm{Ch}_{\leq 0}(\mathcal{A})$ where $\mathrm{P}(X)_i := X_i \oplus X_i \oplus X_{i+1}$ and the differential $d_i : \mathrm{P}(X)_i \to \mathrm{P}(X)_{i-1}$ is given by the matrix:

$$d_i = \begin{bmatrix} d_i & 0 & 0 \\ 0 & d_i & 0 \\ (-1)^{i+1} & (-1)^i & d_{i+1} \end{bmatrix}$$

The projections $\mathrm{P}(X)_i = X_i \oplus X_i \oplus X_{i+1} \xrightarrow{\mathrm{pr}} X_i \oplus X_i$ define a morphism $\pi : \mathrm{P}(X) \to X \oplus X$ and the diagonals $(\mathrm{id}, \mathrm{id}, 0) : X_i \to X_i \oplus X_i \oplus X_{i+1} = \mathrm{P}(X)_i$ define a morphism $h : X \to \mathrm{P}(X)$. The factorization $X \xrightarrow{h} \mathrm{P}(X) \xrightarrow{\pi} X \oplus X$, of the diagonal $(\mathrm{id}, \mathrm{id}) : X \to X \oplus X$, will be also called the standard path object of X. The path object is a functorial construction and it commutes with arbitrary products and coproducts.
Two morphisms $f, g : X \to Y$ in $\mathrm{Ch}_{\leq 0}(\mathcal{A})$ are homotopic if there is a sequence of morphisms $s_i : X_i \to Y_{i+1}$ such that $f_i - g_i = d_{i+1}s_i + s_{i-1}d_i$ for any i. This is equivalent to the existence of a morphism $h : X \to \mathrm{P}(Y)$ for which the composition $X \xrightarrow{h} \mathrm{P}(Y) \xrightarrow{\pi} Y \oplus Y$ is given by $(f, g) : X \to Y \oplus Y$. If f and g are homotopic, then $H_i(f) = H_i(g)$.

A.6. Factorization axioms. We need a few preliminary results to construct the $\mathcal{I}$-relative factorizations. The property of $\mathcal{A}$ having enough $\mathcal{I}$-injectives can be extended to the following property of $\mathrm{Ch}(\mathcal{A})$. We do not claim any functoriality in this statement, as there are many choices involved in the construction.

Lemma A.7. *If $\mathcal{A}$ has enough $\mathcal{I}$-injectives, then for any chain complex X in $\mathrm{Ch}(\mathcal{A})$ there exists a map of chain complexes $X \to I$ such that $I_i \in \mathcal{I}$ and $X_i \to I_i$ is an $\mathcal{I}$-monomorphism for any i. Moreover we can choose I so that $I_i = 0$ whenever $X_i = 0$.*

Proof. For each i choose an $\mathcal{I}$-monomorphism $Z_i(X) \to J_i$ with $J_i \in \mathcal{I}$ and let $X_i \to Q_i$ be the base change of this $\mathcal{I}$-monomorphism along the inclusion

$Z_i(X) \hookrightarrow X_i$. Choose next an $\mathcal{I}$-monomorphism $Q_i \to I_i$ with $I_i \in \mathcal{I}$ and define $X_i \to I_i$ to be the composite $\mathcal{I}$-monomorphism $X_i \to Q_i \to I_i$. Finally, consider the base change of $Q_i \to I_i$ along $Q_i \to B_i(X)$. This is summarized in the following diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_i(X) & \longrightarrow & X_i & \longrightarrow & B_i(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & J_i & \longrightarrow & Q_i & \longrightarrow & B_i(X) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & I_i & \longrightarrow & R_i & \longrightarrow & 0.
\end{array}$$

To define the boundary map $d_i : I_i \to I_{i-1}$, notice that since $B_i(X) \to R_i$ is an $\mathcal{I}$-monomorphism and J_{i-1} is in $\mathcal{I}$, the composite $B_i(X) \hookrightarrow Z_{i-1}(X) \to J_{i-1}$ admits a factorization through R_i and we get a map $R_i \to J_{i-1}$. Define d_i to be the composite $I_i \to R_i \to J_{i-1} \to Q_{i-1} \to I_{i-1}$. The composite $d_i d_{i+1}$ is zero since it factors through $J_i \to Q_i \to B_i(X)$. □

Lemma A.8. *For any object $M \in \mathrm{Ch}(\mathcal{A})_{\leq 0}$, the trivial map $M \to 0$ can be factored as a trivial cofibration followed by a fibration $M \overset{\sim}{\hookrightarrow} RM \twoheadrightarrow 0$.*

Proof. By Lemma A.7 we can find a degreewise $\mathcal{I}$-monomorphism $M \to I_0$ where the complex I_0 is made of $\mathcal{I}$-injectives. Let K_0 denote the cokernel of this map and choose again a degreewise $\mathcal{I}$-monomorphism $K \to I_1$. Repeating the process we construct a map from M to a double complex $I_{*,*}$ made of $\mathcal{I}$-injectives: $I_0 \to I_1 \to I_2 \to \ldots$. As a direct sum of $\mathcal{I}$-injectives is again an $\mathcal{I}$-injective the total complex $\mathrm{Tot}(I)_m = \oplus_{q-p=m} I_{p,q}$ is fibrant. The induced map $M \to \mathrm{Tot}(I) = RM$ is level-wise the sum of the maps $M_m \to I_{0,m}$ and zero maps and thus is an $\mathcal{I}$-monomorphism.
By construction, for any $W \in \mathcal{I}$, the functor $\mathcal{A}(-, W)$ transforms the sequence $K_{p,q} \to I_{p+1,q} \to K_{p+1,q} \to 0$ into an exact sequence. In particular, applying $\mathcal{A}(-, W)$ to the double complex $I_{p,q}$ yields a double complex which is acyclic in the p-direction. The spectral sequence of the complex $\mathcal{A}(\mathrm{Tot}(I_{*,*}), W) = \mathrm{Tot}(\mathcal{A}(I_{*,*}, W))$ collapses thus on one line, which shows that the induced map $M \to RM$ is an $\mathcal{I}$-equivalence. □

Lemma A.9. *For any object $M \in \mathrm{Ch}(\mathcal{A})_{\leq 0}$, the trivial map $M \to 0$ can be factored as a cofibration followed by a trivial fibration $M \hookrightarrow P \overset{\sim}{\twoheadrightarrow} 0$.*

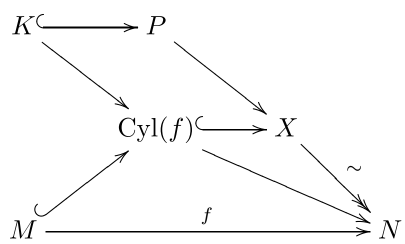
Proof. First factor $M \to 0$ as $M \hookrightarrow RM \twoheadrightarrow 0$ by Lemma A.8. Then perform the cone construction to get a chain map $RM \to C(RM)$ which is a cofibration to an acyclic complex by Lemma A.2. Finally, $P = C(RM)$ is degrewise a sum of $\mathcal{I}$-injectives, hence a fibrant object. □

We are now ready to prove the factorization axiom.

Proposition A.10. *Any map $M \to N$ can be factored as a cofibration followed by an acyclic fibration.*

Proof. First apply Lemma A.4 to get a factorization $M \overset{\sim}{\hookrightarrow} \mathrm{Cyl}(f) \to N$, where the map $\mathrm{Cyl}(f) \to N$ is a split epimorphism in each degree. Let K denote the kernel of $\mathrm{Cyl}(f) \to N$ and factor the trivial map $K \to 0$ as $K \to P \overset{\sim}{\twoheadrightarrow} 0$ by Lemma A.9.

Perform now the cobase change of $K \to \mathrm{Cyl}(f)$ along the cofibration $K \hookrightarrow P$, a situation we sum up in the following diagram:



This yields a cofibration $\mathrm{Cyl}(f) \hookrightarrow X$. The map $X \to N$ is induced by $\mathrm{Cyl}(f) \to N$ and the zero map $P \to N$; it is a split epimorphism since $\mathrm{Cyl}(f) \to N$ is so. Moreover its kernel is the fibrant complex P by construction. This complex is $\mathcal{I}$-trivial so that $X \to N$ is a trivial $\mathcal{I}$-fibration. □

Proposition A.11. *Any map $M \to N$ can be factored as an acyclic cofibration followed by a fibration.*

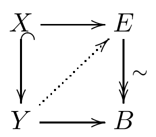
Proof. As above, first apply Lemma A.4 to factor $f : M \overset{\sim}{\hookrightarrow} \mathrm{Cyl}(f) \to N$, where we point out that the first map is an acyclic cofibration. Consider now the kernel K of $\mathrm{Cyl}(f) \to N$, and factor the map $K \to 0$ as in Lemma A.8 $K \overset{\sim}{\hookrightarrow} RK \twoheadrightarrow 0$. Perform next the cobase change of $K \to RK$ along the map $K \to \mathrm{Cyl}(f)$. Since cofibrations and weak equivalences are preserved under cobase change we get an acyclic cofibration $\mathrm{Cyl}(f) \overset{\sim}{\hookrightarrow} X$. We conclude just as in Proposition A.10 that the induced map $X \to N$ is an $\mathcal{I}$-fibration. □

A.12. Lifting axioms. We prove here the left lifting property for cofibrations with respect to trivial fibrations, and then the right lifting property for fibrations with respect to trivial cofibrations.

Lemma A.13. *Let $p : E \to B$ be an acyclic fibration and denote by K its kernel. Then $E = K \oplus B$, p is the second projection and K splits as a direct sum of complexes of the form $0 \longrightarrow W \overset{}{=\!=\!=} W \longrightarrow 0$ with $W \in \mathcal{I}$.*

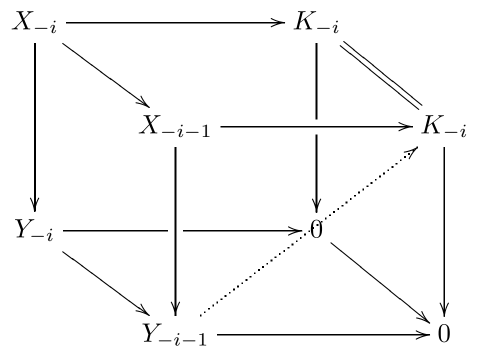
Proof. As K is $\mathcal{I}$-trivial and made of $\mathcal{I}$-injectives it splits as a direct sum of complexes of the form $0 \longrightarrow W =\!=\!= W \longrightarrow 0$ with $W \in \mathcal{I}$. Such complexes are both projective and injective in the category of chain complexes, therefore E splits as $\mathrm{Ker} p \oplus \mathrm{Im}\, p = K \oplus B$. □

Proposition A.14. *Let $p : E \overset{\sim}{\twoheadrightarrow} B$ be an acyclic fibration and $X \hookrightarrow Y$ a cofibration. In any commutative square*



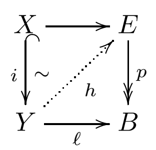
there is a dotted arrow making both triangles commutative.

Proof. As $E \to B$ is an acyclic fibration, the problem reduces by Lemma A.13 to find a lift $Y \to K$, where K is the kernel of $E \to B$, and hence of the form $0 \longrightarrow K_{-i} = K_{-i} \longrightarrow 0$ with $K_{-i} \in \mathcal{I}$ for any $i \leq 0$. In the following cube



The lift $h : Y_{-i-1} \to K_{-i}$ exists because the morphism $X_{-i-1} \to Y_{-i-1}$ is an $\mathcal{I}$-monomorphism and $K_{-i} \in \mathcal{I}$. Define $Y_{-i} \to K_{-i}$ as the composite $Y_{-i} \to Y_{-i-1} \overset{h}{\to} K_{-i}$. One easily checks that this gives a chain map $Y \to K$ with the desired properties. □

Proposition A.15. *Let $p \colon E \twoheadrightarrow B$ be a fibration and $i \colon X \overset{\sim}{\hookrightarrow} Y$ a trivial cofibration. In any commutative square*



there is a dotted arrow making both triangles commutative.

Proof. We define a lifting $h \colon Y \to E$ step by step. Let K be the kernel of the chain map p. As p is a fibration, in each degree $E_n = B_n \oplus K_n$ and p_n is the first projection. Denote by f_n the composite $X_n \to B_n \oplus K_n \to K_n$. To define the lift h we only need to extend the map $f_n\colon X_n \to K_n$ along $X_n \to Y_n$:

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & K_n \\ \downarrow & \overset{\exists k_n}{\nearrow} & \\ Y_n & & \end{array}$$

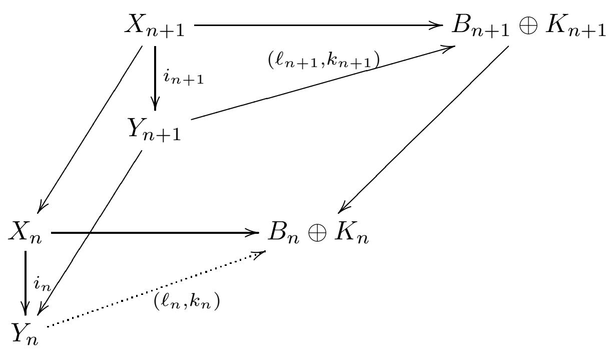
in such a way that the map $h = (\ell, k)$ is a chain map. For this we proceed by induction on n. When $n = 0$, observe that, since K_0 is $\mathcal{I}$-injective and i is a cofibration, we have a quasi-isomorphism of cochain complexes

$$\begin{array}{ccccccc} 0 & \longleftarrow & \operatorname{Hom}(X_0, K_0) & \longleftarrow & \operatorname{Hom}(X_{-1}, K_0) & \longleftarrow & \cdots \\ & & \uparrow{\scriptstyle i_0^*} & & \uparrow{\scriptstyle i_{-1}^*} & & \\ 0 & \longleftarrow & \operatorname{Hom}(Y_0, K_0) & \longleftarrow & \operatorname{Hom}(Y_{-1}, K_0) & \longleftarrow & \cdots \end{array}$$

In particular there exists $\xi \in \operatorname{Hom}(X_{-1}, K_0)$ and $\phi \in \operatorname{Hom}(Y_0, K_0)$ such that $f_0 + \xi\partial_X = i_{-1}^*\phi$. Since $X_{-1} \to Y_{-1}$ is an $\mathcal{I}$-monomorphism, there exists $\zeta\colon Y_{-1} \to K_0$ such that ξ factors through Y_{-1} as $X_{-1} \to Y_{-1} \overset{\zeta}{\to} K_0$. Define $k_0\colon Y_0 \to K_0$ to be $\phi - \zeta\partial_Y$. The desired lift $h\colon Y_0 \to B_0 \oplus K_0$ is then $l_0 \oplus k_0$. For $n \leq -1$, we assume that k_{n+1} has been constructed. The differential of the complex E written according to the degree-wise splitting $E = B \oplus K$ has the form:

$$\begin{pmatrix} \partial_B^{n+1} & 0 \\ \Delta_{n+1} & \partial_K^{n+1} \end{pmatrix} : B_{n+1} \oplus K_{n+1} \longrightarrow B_n \oplus K_n$$

We also have a commutative diagram of solid arrows:



Finally the trivial cofibration i induces as above a quasi-isomorphism of cochain complexes:

$$\begin{array}{ccccc} \operatorname{Hom}(X_{n+1}, K_n) & \longleftarrow & \operatorname{Hom}(X_n, K_n) & \longleftarrow & \operatorname{Hom}(X_{n-1}, K_n) \\ \uparrow{\scriptstyle i_{n+1}^*} & & \uparrow{\scriptstyle i_n^*} & & \uparrow{\scriptstyle i_{n-1}^*} \\ \operatorname{Hom}(Y_{n+1}, K_n) & \longleftarrow & \operatorname{Hom}(Y_n, K_n) & \longleftarrow & \operatorname{Hom}(Y_{n-1}, K_n) \end{array}$$

We are looking for a map k_n that is firstly a chain map, and secondly extends f_n. This translates into the following equations:

$$\Delta_{n+1} \circ \ell_{n+1} + \partial_K^{n+1} k_{n+1} = k_n \circ \partial_Y^{n+1} \tag{4}$$

$$k_n \circ i_n = f_n \tag{5}$$

Observe that Equation (5) expresses an equality in $\mathrm{Hom}(X_n, K_n)$ while Equation (4) is an equality in $\mathrm{Hom}(Y_{n+1}, K_n)$. Precompose the latter by $i_{n+1} : X_{n+1} \to Y_{n+1}$, to get:

$$\Delta_{n+1} \circ \ell_{n+1} \circ i_{n+1} + \partial_K^{n+1} k_{n+1} \circ i_{n+1} = k_n \circ \partial_Y^{n+1} \circ i_{n+1} \tag{6}$$

By commutativity of the back face of the commutative diagram above, the left hand side of this equation is equal to

$$f_n \circ \partial_X^{n+1}$$

which is a trivial cocycle in $\mathrm{Hom}(X_{n+1}, K_n)$. Since i^* is a quasi-isomorphism, this implies that the left hand side of Equation (4) is a trivial cocycle in $\mathrm{Hom}(Y_{n+1}, K_n)$. In particular there is a map $\phi_n : Y_n \to K_n$ such that

$$\Delta_{n+1} \circ \ell_{n+1} + \partial_K^{n+1} k_{n+1} = \phi_n \circ \partial_Y^{n+1}$$

Since by construction $f_n \circ \partial_X^{n+1} = \phi_n \circ \partial_Y^{n+1} \circ i_{n+1} = \phi_n \circ i_n \circ \partial_X^{n+1}$, there is a map $\zeta_{n-1} \in \mathrm{Hom}(X_{n-1}, K_n)$ such that $f_n = \phi_n \circ i_n + \zeta_{n-1} \circ \partial_X^n$. By surjectivity of i_{n-1}^* we may lift this map to $\xi_{n-1} \in \mathrm{Hom}(Y_{n-1}, K_n)$ such that $\xi_{n-1} \circ i_{n-1} = \zeta_{n-1}$, and the map $k_n = \phi_n + \xi_{n-1} \circ \partial_Y^n$ is the one we are looking for. □

Theorem A.16. *Assume that $\mathcal{A}$ has enough $\mathcal{I}$-injectives. Then the choice of $\mathcal{I}$-weak equivalences, $\mathcal{I}$-cofibrations, and $\mathcal{I}$-fibrations gives* $\mathrm{Ch}_{\leq 0}(\mathcal{A})$ *the structure of a model category.*

Proof. The category $Ch_{\leq 0}(\mathcal{A})$ is clearly closed under both limits and colimits, which proves (MC1). Since quasi-isomorphisms satisfy the "2 out of 3" property so do $\mathcal{I}$-weak equivalences, this is (MC2).

Let us prove (MC3). Retracts of epimorphisms and of quasi-isomorphisms are epimorphisms and quasi-isomorphisms respectively, so $\mathcal{I}$-cofibrations and $\mathcal{I}$-weak equivalences are preserved under retracts. As for $\mathcal{I}$-fibrations, notice that the retract of a map with a section also has a section. Moreover since $\mathcal{I} = \overline{\mathcal{I}}$ is stable under retracts we conclude that the retract of an $\mathcal{I}$-fibration is again an $\mathcal{I}$-fibration.

Finally, the factorization axiom (MC4) and the lifting axiom (MC5) have been established in the preceding propositions. □

Appendix B. Elementary algebra for topologists

This appendix contains a few elementary and well-known facts about localization, injective envelopes, and local cohomology; none of these is new but we need it explicitly to describe relative resolutions in the case we restrict the notion of injectives. For a prime ideal $\mathfrak{p}$, we denote by $M_{\mathfrak{p}}$ the localization of an R-module M at $\mathfrak{p}$. The first lemma will allow us to reduce certain problems to the case of a local ring, namely$R_{\mathfrak{p}}$.

Lemma B.1. *An R-module M is zero if and only if $M_{\mathfrak{p}}$ is zero for all prime ideals $\mathfrak{p}$.*

Proof. Let us assume that M is non-zero, but $M_{\mathfrak{p}} = 0$ for any prime ideal $\mathfrak{p}$. We choose a non-zero element $x \in M$ and consider its annihilator. This ideal is contained in a maximal ideal $\mathfrak{m}$ and since $M_{\mathfrak{m}} = 0$, there must exist an element $r \in R \setminus \mathfrak{m}$ such that $rx = 0$, a contradiction. □

A theorem of Matlis, [26], describes the injective modules as direct sums of injective hulls $E(R/\mathfrak{p})$ of quotients of the ring by prime ideals. The following two lemmas give some properties of these indecomposable injective modules.

Lemma B.2. *If $\mathfrak{q} \subset \mathfrak{p}$, the module $E(R/\mathfrak{q})$ is $\mathfrak{p}$-local, and $E(R/\mathfrak{q})_{\mathfrak{p}} = 0$ otherwise.*

Proof. Assume $\mathfrak{q} \subset \mathfrak{p}$ and fix $r \notin \mathfrak{p}$. The multiplication by r on $E(R/\mathfrak{q})$ is an isomorphism, so $E(R/\mathfrak{q})$ is $\mathfrak{p}$-local. Assume now that $\mathfrak{q} \not\subset \mathfrak{p}$. Then $\mathfrak{q}^m \not\subset \mathfrak{p}$ for any $m \geq 1$. If x is any element of $E(R/\mathfrak{q})$, its annihilator is $\mathfrak{q}^m$ for some positive integer m since $E(R/\mathfrak{q})$ is $\mathfrak{q}$-torsion. There exists thus an element $s \in \mathfrak{q}^m$ which does not belong to $\mathfrak{p}$ and such that $sx = 0$. Hence $x_{\mathfrak{p}} = 0$. This shows that $E(R/\mathfrak{q})_{\mathfrak{p}} = 0$. □

Lemma B.3. *The R-module of homomorphisms $\mathrm{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{q}))$ is non-zero if and only if $\mathfrak{p} \subset \mathfrak{q}$.*

Proof. Since $E(R/\mathfrak{q})$ is $\mathfrak{q}$-local by the previous lemma, any homomorphism factors through the $\mathfrak{q}$-localization of $E(R/\mathfrak{p})$, which is zero unless $\mathfrak{p} \subset \mathfrak{q}$. This proves one implication. In this case the quotient morphism $R/\mathfrak{p} \to \mathfrak{q}$ extends to the injective envelopes showing the other implication. □

Let us now introduce local cohomology, a good reference for which is [22].

Definition B.4. Given an ideal $\mathfrak{p}$ in R, the $\mathfrak{p}$-*torsion* of an R-module M is the submodule $\Gamma_{\mathfrak{p}}(M)$ of elements with annihilator $\mathfrak{p}^m$ for some positive integer m. The *local cohomology* modules $H^*_{\mathfrak{p}}(-)$ with support in $\mathfrak{p}$ are the right derived functors of $\Gamma_{\mathfrak{p}}$.

Explicitly, to compute the local cohomology of a module M, we construct an injective resolution $I_\bullet$ of M and compute $H^j_{\mathfrak{p}}(M) = H^j(\Gamma_{\mathfrak{p}}(I_\bullet))$. Our last lemma helps us to understand how this $\mathfrak{p}$-torsion injective complex look like.

Lemma B.5. *The $\mathfrak{p}$-torsion module $\Gamma_{\mathfrak{p}}(E(R/\mathfrak{q})) = E(R/\mathfrak{q})$ if $\mathfrak{p} \subset \mathfrak{q}$ and is zero otherwise.*

Proof. Again this follows from the fact that $E(R/\mathfrak{q})$ is $\mathfrak{q}$-torsion. □

Remark B.6. Let R be a local ring with maximal ideal $\mathfrak{m}$ and let us consider the generization closed subset of *Spec*(R) given by $S = \{\mathfrak{q} \,|\, \mathfrak{q} \neq \mathfrak{m}\}$. It yields the injective class $\mathcal{I}$ generated by all injective envelopes $E(R/\mathfrak{q})$ with $\mathfrak{q} \neq \mathfrak{m}$

see [6, Proposition 3.1]. Given a module M and an injective resolution $I_\bullet$, we have a triangle in the derived category $\Gamma_{\mathfrak{m}}(I_\bullet) \to I_\bullet \to W_\bullet$, where $W_\bullet$ is an $\mathcal{I}$-relative injective resolution of M. In particular $H_k(W_\bullet) \cong H^{k+1}_{\mathfrak{m}}(M)$ for $k \geq 2$.

Proposition B.7. *Let R be a Noetherian ring and $\mathfrak{p}$ be the radical of $(x_1, \ldots, x_n)$. Then $H^k_{\mathfrak{p}}(M) = 0$ for any $k > n$ and any module M.*

Proof. Since the torsion functor does not see the difference between an ideal and its radical, we can assume that $\mathfrak{p} = (x_1, \ldots, x_n)$. Then the local cohomology can be computed by means of the Čech complex $\otimes_i \check{C}(x_i, R) \otimes M$, [22, Theorem 7.13]. Here $\check{C}(x, R)$ is the complex $0 \to R \to R_x \to 0$ concentrated in degrees 0 and 1. The Čech complex is thus concentrated in degrees $\leq n$. □

Remark B.8. If R is a Noetherian local ring of dimension d, then the maximal ideal can always be expressed as the radical of an ideal generated by n elements, see [22, Theorem 1.17].

References

1. I. T. Adamson, *Cohomology theory for non-normal subgroups and non-normal fields*, Proc. Glasgow Math. Assoc. 2 (1954), 66–76.
2. D. Barnes, J. P. C. Greenlees, M. Kedziorek, and B. Shipley, *Rational $SO(2)$-equivariant spectra*, Preprint available at :https://arxiv.org/abs/1511.03291, 2015.
3. J. E. Bergner, *Homotopy limits of model categories and more general homotopy theories*, Bull. Lond. Math. Soc. 44 (2012), no. 2, 311–322. MR 2914609
4. A. K. Bousfield, *Cosimplicial resolutions and homotopy spectral sequences in model categories*, Geom. Topol. 7 (2003), 1001–1053 (electronic).
5. K. S. Brown, *Abstract homotopy theory and generalized sheaf cohomology*, Trans. Amer. Math. Soc. 186 (1973), 419–458. MR 0341469 (49 #6220)
6. W. Chachólski, W. Pitsch, and J. Scherer, *Injective classes of modules*, J. Algebra Appl. 12 (2013), no. 4, 1250188, 13. MR 3037263
7. W. Chachólski and J. Scherer, *Homotopy theory of diagrams*, Mem. Amer. Math. Soc. 155 (2002), no. 736, x+90.
8. C. Chevalley and S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. 63 (1948), 85–124.
9. J. D. Christensen and M. Hovey, *Quillen model structures for relative homological algebra*, Math. Proc. Cambridge Philos. Soc. 133 (2002), no. 2, 261–293.
10. W. G. Dwyer, P. S. Hirschhorn, D. M. Kan, and J. H. Smith, *Homotopy limit functors on model categories and homotopical categories*, Mathematical Surveys and Monographs, vol. 113, American Mathematical Society, Providence, RI, 2004. MR 2102294
11. W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126.
12. S. Eilenberg and J. C. Moore, *Foundations of relative homological algebra*, Mem. Amer. Math. Soc. No. 55 (1965), 39.
13. E. Enochs and J. A. López-Ramos, *Kaplansky classes*, Rend. Sem. Mat. Univ. Padova 107 (2002), 67–79. MR 1926201
14. E. E. Enochs and O. M. G. Jenda, *Relative homological algebra*, de Gruyter Expositions in Mathematics, vol. 30, Walter de Gruyter & Co., Berlin, 2000. MR 1753146
15. E. Dror Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Mathematics, vol. 1622, Springer-Verlag, Berlin, 1996.

16. J. P. C. Greenlees and B. Shipley, *Homotopy theory of modules over diagrams of rings*, Proc. Amer. Math. Soc. Ser. B 1 (2014), 89–104. MR 3254575
17. A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. 9 (1957), no. 2, 119–221.
18. L. Gruson and C. U. Jensen, *Modules algébriquement compacts et foncteurs* $\varprojlim^{(i)}$, C. R. Acad. Sci. Paris Sér. A-B 276 (1973), A1651–A1653. MR 0320112
19. Y. Harpaz and M. Prasma, *The Grothendieck construction for model categories*, Adv. Math. 281 (2015), 1306–1363. MR 3366868
20. G. Hochschild, *Relative homological algebra*, Trans. Amer. Math. Soc. 82 (1956), 246–269.
21. H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra 189 (2004), no. 1-3, 167–193. MR 2038564
22. S. B. Iyengar, G. J. Leuschke, A. Leykin, C. Miller, E. Miller, A. K. Singh, and U. Walther, *Twenty-four hours of local cohomology*, Graduate Studies in Mathematics, vol. 87, American Mathematical Society, Providence, RI, 2007.
23. M. Kashiwara and P. Schapira, *Categories and sheaves*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 332, Springer-Verlag, Berlin, 2006. MR 2182076 (2006k:18001)
24. R. Kiełpiński and D. Simson, *On pure homological dimension*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 23 (1975), 1–6. MR 0407089
25. S. Mac Lane, *Homology*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1975 edition. MR 1344215 (96d:18001)
26. E. Matlis, *Injective modules over Noetherian rings*, Pacific J. Math. 8 (1958), 511–528.
27. M. Nagata, *Local rings*, Interscience Tracts in Pure and Applied Mathematics, No. 13, Interscience Publishers a division of John Wiley & Sons New York-London, 1962.
28. A. Neeman, *The chromatic tower for* $D(R)$, Topology 31 (1992), no. 3, 519–532, With an appendix by Marcel Bökstedt.
29. ———, *Non-left-complete derived categories*, Math. Res. Lett. 18 (2011), no. 5, 827–832. MR 2875857
30. M. Prest, *Purity, spectra and localisation*, Encyclopedia of Mathematics and its Applications, vol. 121, Cambridge University Press, Cambridge, 2009. MR 2530988
31. D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967.
32. J.-E. Roos, *Derived functors of inverse limits revisited*, J. London Math. Soc. (2) 73 (2006), no. 1, 65–83. MR 2197371 (2006i:18002)
33. C. Serpé, *Resolution of unbounded complexes in grothendieck categories*, J. Pure Appl. Algebra 177 (200), no. 1, 103–112.
34. N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. 65 (1988), no. 2, 121–154.
35. S. Virili, *On the exactness of products in the localization of* $AB4^*$ *Grothendieck categories*, to appear in J. of Algebra (2017).

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