Arithmetic Families of (φ, Γ)-Modules and Locally Analytic Representations of $GL_2(Q_p)$

Ildar Gaisin and Joaquín Rodrigues Jacinto

Received: September 12, 2017
Revised: July 26, 2018

Communicated by Otmar Venjakob

Abstract. Let A be an affinoid $\mathbf{Q}_p$-algebra, in the sense of Tate. We develop a theory of locally convex A-modules parallel to the treatment in the case of a field by Schneider and Teitelbaum. We prove that there is an integration map linking a category of locally analytic representations in A-modules and separately continuous *relative* distribution modules. There is a suitable theory of locally analytic cohomology for these objects and a version of Shapiro's Lemma. In the case of a field this has been substantially developed by Kohlhaase. As an application we propose a p-adic Langlands correspondence in *families*: For a *regular* trianguline (φ, Γ)-module of dimension 2 over the relative Robba ring $\mathscr{R}_A$ we construct a locally analytic $GL_2(Q_p)$-representation in A-modules.

2010 Mathematics Subject Classification: 11S37, 11S25, 20G05, 22E50

Keywords and Phrases: p-adic Langlands correspondence, (φ, Γ)-modules, affinoid algebras, locally analytic representations

Contents

1 Introduction

1.1 An extension of the p-adic Langlands correspondence

The aim of this article is to study the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ in arithmetic families. To put things into context, let us recall the general lines of this correspondence. In [11], [35] and [16], a bijection $V \mapsto \Pi(V)$ between absolutely irreducible 2-dimensional continuous L-representations[1] of the absolute Galois group $\mathcal{G}_{\mathbf{Q}_p}$ of $\mathbf{Q}_p$ and admissible unitary non-ordinary Banach L-representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ which are topologically absolutely irreducible is established.

The basic strategy of the construction of the functor $V \mapsto \Pi(V)$ is the following: by Fontaine's equivalence, the category of local Galois representations in L-vector spaces is equivalent to that of étale (φ, Γ)-modules over Fontaine's field $\mathscr{E}_L$ [2]. Any such (φ, Γ)-module D can be naturally seen as a P^+-equivariant sheaf [3] over $\mathbf{Z}_p$ whose global sections are D itself, where $P^+ = \begin{pmatrix} \mathbf{Z}_p - \{0\} & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix}$ is a sub-semi-group of the mirabolic subgroup $\begin{pmatrix} \mathbf{Q}_p^\times & \mathbf{Q}_p \\ 0 & 1 \end{pmatrix}$ of $\mathrm{GL}_2(\mathbf{Q}_p)$. If U is a compact open subset of $\mathbf{Z}_p$, we denote by $D \boxtimes U$ the sections over U of this sheaf so that, in particular, $D \boxtimes \mathbf{Z}_p = D$. In [11], a magical involution w_D acting on $D \boxtimes \mathbf{Z}_p^\times$ is defined, allowing one (noting that $\mathbf{P}^1(\mathbf{Q}_p)$ is built by glueing two

[1] During all this text, L will denote the coefficient field, which is a finite extension of $\mathbf{Q}_p$.

[2] The field $\mathscr{E}_L$ is defined as the Laurent series $\sum_{n \in \mathbf{Z}} a_n T^n$ such that $a_n \in L$ are bounded and $\lim_{n \to -\infty} a_n = 0$. $\mathscr{E}$ is equipped with a continuous action of the group $\Gamma = \mathbf{Z}_p^\times$ (we note σ_a, $a \in \mathbf{Z}_p^\times$, its elements) and an operator φ defined by the formulas $\sigma_a(T) = (1+T)^a - 1$ and $\varphi(T) = (1+T)^p - 1$. Recall that a (φ, Γ)-module is a free $\mathscr{E}$-module equipped with semi-linear continuous actions of Γ and φ.

[3] The matrix $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ codifies the action of φ, $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ the action of $\sigma_a \in \Gamma$ and $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ the multiplication by $(1+T)^b$.

copies of $\mathbf{Z}_p$ along $\mathbf{Z}_p^\times$) to extend D to a $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant sheaf over[4] $\mathbf{P}^1$, $U \mapsto D \boxtimes_\omega U$ with global sections $D \boxtimes_\omega \mathbf{P}^1$, where $\omega = (\det D)\chi^{-1}$ [5]. One then cuts out the desired Banach representation $\Pi(V)$ (and its dual) from the constituents of $D \boxtimes_\omega \mathbf{P}^1$. More precisely, we have a short exact sequence of topological $\mathrm{GL}_2(\mathbf{Q}_p)$-modules

$$0 \to \Pi(V)^* \otimes \omega \to D \boxtimes_\omega \mathbf{P}^1 \to \Pi(V) \to 0.$$

Let $\mathscr{R}_L$ denote the Robba ring [6] with coefficients in L. By a combination of results of Cherbonnier-Colmez [7] and Kedlaya [28], the categories of étale (φ, Γ)-modules over $\mathscr{E}_L$ and $\mathscr{R}_L$ are equivalent. Call $D \mapsto D_{\mathrm{rig}}$ this correspondence. We have analogous constructions as above for D_{rig} and, in particular, we have a $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant sheaf $U \mapsto D_{\mathrm{rig}} \boxtimes U$ over $\mathbf{P}^1$. If we note $\Pi(V)^{\mathrm{an}}$ the locally analytic vectors of $\Pi(V)$, we get an exact sequence

$$0 \mapsto (\Pi(V)^{\mathrm{an}})^* \otimes \omega \to D_{\mathrm{rig}} \boxtimes_\omega \mathbf{P}^1 \to \Pi(V)^{\mathrm{an}} \to 0.$$

However, the construction of $D_{\mathrm{rig}} \boxtimes_\omega \mathbf{P}^1$ is not a straightforward consequence of that of $D \boxtimes_\omega \mathbf{P}^1$. This is mainly because the formula defining the involution does not converge for a (φ, Γ)-module over $\mathscr{R}_L$ [7].
Inspired by the calculations of the p-adic local correspondence for trianguline [8] étale (φ, Γ)-modules, Colmez [14] has recently given a direct construction, for a (not necessarily étale) (φ, Γ)-module Δ (of rank 2) over $\mathscr{R}_L$, of a locally analytic L-representation $\Pi(\Delta)$ of $\mathrm{GL}_2(\mathbf{Q}_p)$. More precisely, we have the following theorem:

Theorem 1.1 ([14, Théorème 0.1]). *There exists a unique extension of* Δ *to a* $\mathrm{GL}_2(\mathbf{Q}_p)$*-equivariant sheaf of* $\mathbf{Q}_p$*-analytic type* [9] $\Delta \boxtimes_\omega \mathbf{P}^1$ *over* $\mathbf{P}^1$ *with central character* ω*. Moreover, there exists a unique admissible locally analytic* L*-representation* $\Pi(\Delta)$ *of* $\mathrm{GL}_2(\mathbf{Q}_p)$*, with central character* ω*, such that*

$$0 \to \Pi(\Delta)^* \otimes \omega \to \Delta \boxtimes_\omega \mathbf{P}^1 \to \Pi(\Delta) \to 0.$$

The purpose of the present work is to study this correspondence in the context of arithmetic families of (φ, Γ)-modules. Results in this direction on the ℓ-adic

[4]From now on, $\mathbf{P}^1$ will always mean $\mathbf{P}^1(\mathbf{Q}_p)$.

[5]The character $\det D$ is the character of $\mathbf{Q}_p^\times$ defined by the actions of φ and Γ on $\wedge^2 D$. If D is étale, it can also be seen as a Galois character via local class field theory. The character $\chi \colon x \mapsto x|x|$ denotes the cyclotomic character. We see both characters as characters of $\mathrm{GL}_2(\mathbf{Q}_p)$ by pre-composing with the determinant.

[6]It is defined as the ring of Laurent series $\sum_n a_n T^n$, $a_n \in L$, converging on some annulus $0 < v_p(T) \leq r$ for some $r > 0$.

[7]To construct the involution on D_{rig} in the étale case, one shows that w_D stabilises $D^\dagger \boxtimes \mathbf{Z}_p^\times$, where $D^\dagger$ is the (φ, Γ)-module over the overconvergent elements $\mathscr{E}_L^\dagger$ of $\mathscr{E}_L$ corresponding to D by the Cherbonnier-Colmez correspondence, and that it defines by continuity an involution on $D_{\mathrm{rig}} \boxtimes \mathbf{Z}_p^\times$.

[8]A rank 2 (φ, Γ)-module is trianguline if it is an extension of rank 1 (φ, Γ)-modules.

[9]A sheaf $U \mapsto M \boxtimes U$ is of $\mathbf{Q}_p$-analytic type if, for every open compact $U \subseteq \mathbf{P}^1$ and every compact $K \subseteq \mathrm{GL}_2(\mathbf{Q}_p)$ stabilizing U, the space $M \boxtimes U$ is of LF-type and a continuous $\mathcal{D}(K)$-module, where $\mathcal{D}(K)$ is the distribution algebra over K.

side (i.e. the *classical* local Langlands correspondence, cf. [25]) have been achieved by Emerton-Helm in [21]. The arguments in [14] strongly rely on the cohomology theory of locally analytic representations developed in [31], and specifically on Shapiro's lemma. Since the authors are not aware of any reference for these results in the relative setting, we develop, in an appendix (cf. §A), the necessary definitions and properties of locally analytic $\mathrm{GL}_2(\mathbf{Q}_p)$-representations in A-modules. Since this point might carry some interest on its own, we describe it in more detail in §1.3 below. We will exclusively work with affinoid spaces in the sense of Tate, rather than Berkovich or Huber. Let A be an affinoid $\mathbf{Q}_p$-algebra and let $\mathscr{R}_A$ be the relative Robba ring over A[10]. Our main result (Theorem 7.2) can be stated as follows:

Theorem 1.2. *Let A be an affinoid $\mathbf{Q}_p$-algebra and let Δ be a trianguline (φ, Γ)-module over $\mathscr{R}_A$ of rank 2 which is an extension of $\mathscr{R}_A(\delta_2)$ by $\mathscr{R}_A(\delta_1)$, where $\delta_1, \delta_2 : \mathbf{Q}_p^\times \to A^\times$ are locally analytic characters satisfying some regularity assumptions [11]. Then there exists a unique extension of Δ to a $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant sheaf of $\mathbf{Q}_p$-analytic type $\Delta \boxtimes_\omega \mathbf{P}^1$ over $\mathbf{P}^1$ with central character $\omega = \delta_1\delta_2\chi^{-1}$ and a locally analytic $\mathrm{GL}_2(\mathbf{Q}_p)$-representation [12] $\Pi(\Delta)$ in A-modules with central character ω, living in an exact sequence*

$$0 \to \Pi(\Delta)^* \otimes \omega \to \Delta \boxtimes_\omega \mathbf{P}^1 \to \Pi(\Delta) \to 0.$$

Moreover, if $\mathfrak{m} \subseteq A$ is a maximal ideal, $L = A/\mathfrak{m}$ and $\overline{\Delta} = \Delta \otimes_A L$, then $\Pi(\Delta) \otimes_A L = \Pi(\overline{\Delta})$, so that $\Pi(\Delta)$ interpolates at each closed point the construction of Theorem 1.1.

This result is expected to have applications to the study of eigenvarieties, however, in this paper we make no attempt to say anything in this direction, hoping to come back to it in the future.

1.2 The construction of the correspondence

The construction of the correspondence follows the general lines of [14], but several technical difficulties appear along the way. Let's briefly describe how to construct the correspondence $\Delta \mapsto \Pi(\Delta)$ and the additional problems that arise in the relative (affinoid) setting.

From the calculation of the locally analytic vectors of the unitary principal series [10, Théorème 0.7], one knows that, if D is an étale trianguline (φ, Γ)-module over $\mathscr{E}_L$ of dimension 2, then $\Pi(D)^{\mathrm{an}}$ is an extension of locally analytic principal series. The idea of [14] is to intelligently reverse this *dévissage* of $D_{\mathrm{rig}} \boxtimes_\omega \mathbf{P}^1$ in order to actually construct it from these pieces.

[10]The ring $\mathscr{R}_A$ can be defined in the same way as $\mathscr{R}_L$ by taking the coefficients a_n in A and replacing the p-adic valuation with the valuation defined by the norm of A.

[11]Precisely, we suppose that $\delta_1\delta_2^{-1}$ is pointwise never of the form χx^i or x^{-i} for some $i \geq 0$.

[12]See Definition A.26 for the definition of a locally analytic G-representation in A-modules.

For the rest of this introduction let $G = \mathrm{GL}_2(\mathbf{Q}_p)$ and $\overline{B}$ be its lower Borel subgroup and let δ_1, δ_2 and ω be as in Theorem 1.2. Using a relative version of the dictionary of p-adic functional analysis (cf. §2.1), we construct, for $? \in \{+, -, \emptyset\}$, G-equivariant sheaves $\mathscr{R}_A^?(\delta_1) \boxtimes_\omega \mathbf{P}^1$ (with central character ω) of $\mathbf{Q}_p$-analytic type living in an exact sequence

$$0 \to \mathscr{R}_A^+(\delta_1) \boxtimes_\omega \mathbf{P}^1 \to \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1 \to \mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1 \to 0.$$

Moreover, one can get identifications $B_A(\delta_2, \delta_1)^* \otimes \omega \cong \mathscr{R}_A^+(\delta_1) \boxtimes_\omega \mathbf{P}^1$ and $B_A(\delta_1, \delta_2) \cong \mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1$, where $B_A(\delta_1, \delta_2) = \mathrm{Ind}_{\overline{B}}^G(\delta_1 \chi^{-1} \otimes \delta_2)$ denotes the locally analytic principal series. These identifications allow us to consider the locally analytic principal series (and their duals) as (the global sections of) G-equivariant sheaves over $\mathbf{P}^1$.
We then construct the G-equivariant sheaf $\Delta \boxtimes_\omega \mathbf{P}^1$ over $\mathbf{P}^1$ as an extension of $\mathscr{R}_A(\delta_2) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$. This is done, as in [14], by showing that extensions of $\mathscr{R}_A(\delta_2)$ by $\mathscr{R}_A(\delta_1)$ correspond to extensions of $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$. One then shows that an extension of $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$ uniquely extends to an extension of $\mathscr{R}_A(\delta_2) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$. Once the sheaf $\Delta \boxtimes_\omega \mathbf{P}^1$ is constructed, one shows that the intermediate extension of $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1$ splits and thus one can separate the spaces that are Fréchets from those that are an inductive limit of Banach spaces so as to cut out the desired representation $\Pi(\Delta)$.
The fact that, for $? \in \{+, -, \emptyset\}$, the P^+-module $\mathscr{R}_A^?(\delta_1)$ can be seen as sections over $\mathbf{Z}_p$ of a G-equivariant sheaf over $\mathbf{P}^1$, and that the semi-group $\overline{P}^+ = \left(\begin{smallmatrix} \mathbf{Z}_p - \{0\} & 0 \\ p\mathbf{Z}_p & 1 \end{smallmatrix}\right)$ stabilizes $\mathbf{Z}_p$, show that $\mathscr{R}_A^?(\delta_1) = \mathscr{R}_A^?(\delta_1) \boxtimes_\omega \mathbf{Z}_p$ is automatically equipped with an extra action of the matrix $\left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right)$. We call

$$\mathscr{R}_A^?(\delta_1, \delta_2) := (\mathscr{R}_A^?(\delta_1) \boxtimes_\omega \mathbf{Z}_p) \otimes \delta_2^{-1}$$

the $\overline{P}^+$-module thus defined. The technical heart for proving Theorem 1.2 is a comparison result (Theorem 6.8) between the cohomology of the semi-groups $A^+ = \left(\begin{smallmatrix} \mathbf{Z}_p - \{0\} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\overline{P}^+$ with values in $\mathscr{R}_A(\delta_1 \delta_2^{-1})$ and $\mathscr{R}_A(\delta_1, \delta_2)$, respectively.

Theorem 1.3. *The restriction morphism from $\overline{P}^+$ to A^+ induces an isomorphism*

$$H^1(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2)) \to H^1(A^+, \mathscr{R}_A(\delta_1 \delta_2^{-1})).$$

The semi-group A^+ should be thought of as encoding the action of φ and Γ. The difficulty of course is to codify the action of the involution and this is the underlying idea for considering the semi-group $\overline{P}^+$. Indeed $\overline{P}^+$ should be thought of as getting closer to tracking the involution. Theorem 1.3 is (essentially) saying that a trianguline (φ, Γ)-module as in Theorem 1.2 admits an extension to a G-equivariant sheaf over $\mathbf{P}^1$.
Let us briefly describe the proof of Theorem 1.3. The main idea is to reduce this bijection to the case of a point (i.e to the case where $A = L$ is a

finite extension of $\mathbf{Q}_p$). The first step is to build a *Koszul* complex which calculates $\overline{P}^+$-cohomology (Proposition 4.8), which is analogue to but somewhat more involved than the Herr complex due to the non commutativity of $\overline{P}^+$. The construction of this complex allows us to show (Theorem 4.11) that $\mathscr{C}_{\tau,\varphi,\gamma}(\mathscr{R}_A(\delta_1, \delta_2))$ is a pseudo-coherent complex concentrated in degrees $[0, 3]$ and that, in particular, the cohomology groups $H^i(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2))$ are finite A-modules, which will be essential in the reduction of the relative case to the case of a point. It turns out that, for this reduction to work, we are forced to show that some higher relative cohomology groups are locally free. We do sow by explicitly calculating them in the case of a point in §5. This is carried out by the use of a complex calculating the Lie algebra cohomology of $\overline{P}^+$ and a result of Tamme, allowing one to calculate the former by taking invariants of the latter (Lemma 5.3).

The results described in the last paragraph allow for an analysis of a spectral sequence to take place and prove Theorem 1.3 in the case where A is reduced (Theorem 6.5). One then concludes via an induction argument on the index of nilpotency of the nilradical of A.

Via the complex $\mathscr{C}_{u^-,\varphi,a^+}(M)$ we also obtain an alternative proof of [14, Proposition 5.18] in the case of a cyclotomic (φ, Γ)-module, and correct a small mistake of *loc. cit.* in the pathological case (Proposition 5.2). Along the way we show a comparison isomorphism relating continuous cohomology and analytic cohomology for certain (φ, Γ)-modules (Proposition 3.3), in the spirit of Lazard's results.

1.3 Analytic families of locally analytic representations

Armed with Theorem 1.3, the reader may notice at this point however, that there is an absence of theory required to conclude (or even make sense of) Theorem 1.2. In an appendix, we develop the foundations of such a theory and prove some fundamental properties regarding the locally analytic cohomology theory of $\mathscr{D}(G, A)$-modules, which we describe now.

Recall that for a locally $\mathbf{Q}_p$-analytic group H, a theory of locally analytic representations of the group H in L-vector spaces has been developed by Schneider and Teitelbaum (cf. [39], [41], [40]). In order to construct the A-module $\Pi(\Delta)$ of Theorem 1.2, with a locally analytic action of G, we need to develop a reasonable framework to make sense of such an object. It turns out that, with some care, much of the existent theory can be extended without serious difficulties to the relative context.

Definition 1.4 (A1). A locally convex A-module is a topological A-module whose underlying topology is a locally convex $\mathbf{Q}_p$-vector space. We let LCS_A be the category of locally convex A-modules. Its morphisms are all continuous A-linear maps.

There is a notion of a strong dual in the category LCS_A, however outside of our applications, it is ill-behaved (in the sense that there are few reflexive objects which are not free A-modules). Let H be a locally $\mathbf{Q}_p$-analytic group.

Definition 1.5 (A2). We define the category $\mathrm{Rep}_A^{\mathrm{la}}(H)$ whose objects are barrelled, Hausdorff, locally convex A-modules M equipped with a topological A-linear action of H such that, for every $m \in M$, the orbit map $h \mapsto h \cdot m$ is a locally analytic function on H with values in M.

Denote $\mathrm{LA}(H, A)$ the space of locally analytic functions on H with values in A and $\mathscr{D}(H, A) = \mathrm{Hom}_{A,\mathrm{cont}}(\mathrm{LA}(H, A), A)$ (equipped with the strong topology) its strong A-dual, the space of A-valued distributions on H. Both $LA(H, A)$ and $\mathscr{D}(H, A)$ are locally convex A-modules. In order to algebrize the situation, one proceeds as in [39] and shows that a locally analytic representation of H is naturally a module over the relative distribution algebra. More precisely let $\mathrm{Rep}_A^{\mathrm{la},LB}(H) \subseteq \mathrm{Rep}_A^{\mathrm{la}}(H)$ denote the full subcategory consisting of spaces which are of A-LB-type (i.e inductive limit of Banach spaces whose transition morphism are A-linear) and complete. Then our main result in §A (Lemma A.36 and Corollary A.38) can be stated as follows:

Theorem 1.6. *Every locally analytic representation of H carries a separably continuous A-linear structure of $\mathscr{D}(H, A)$-module* [13]*. Moreover, the category $\mathrm{Rep}_A^{\mathrm{la},LB}(H)$ is equivalent to the category of complete, Hausdorff locally convex A-modules which are of A-LB-type equipped with a separately continuous $\mathscr{D}(H, A)$-action with morphisms all continuous $\mathscr{D}(H, A)$-linear maps.*

The idea to prove Theorem 1.6 is of course to reduce to the well known result of Schneider-Teitelbaum, cf. [39, Theorem 2.2]. To achieve this, the main intermediary result is the isomorphism (Proposition A.31)

$$\mathscr{D}(H, A) = \mathscr{D}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota} A.$$

Remark 1.7. Proposition A.31 would also follow immediately if $\mathrm{LA}(H, A)$ is complete (for H compact). To the best of our knowledge this seems to be an open question if the dimension of $H \geq 2$. If $H \cong \mathbf{Z}_p$, one can identity $\mathrm{LA}(\mathbf{Z}_p, A)$ with the negative powers of $\mathscr{R}_A$ and conclude the result, cf. Lemma A.16. In particular $\mathrm{LA}(\mathbf{Z}_p, A)$ is an example of an A-reflexive object, which is not free.

Finally, with the equivalence of Theorem 1.6 in mind, we switch our attention to cohomological questions concerning the category $\mathrm{Rep}_A^{\mathrm{la}}(H)$.

Definition 1.8. Let $\mathscr{G}_{H,A}$ denote the category of complete Hausdorff locally convex A-modules with the structure of a separately continuous A-linear $\mathscr{D}(H, A)$-module, taking as morphisms all continuous $\mathscr{D}(H, A)$-linear maps. More precisely we demand that the module structure morphism

$$\mathscr{D}(H, A) \times M \to M$$

is A-bilinear and separately continuous.

[13] More precisely, a separately continuous A-bilinear map $\mathscr{D}(H, A) \times M \to M$.

Following Kohlhaase ([31], [46]), one can develop a locally analytic cohomology theory for the category $\mathscr{G}_{H,A}$. One can define groups $H^i_{\rm an}(H, M)$ and $\mathrm{Ext}^i_{\mathscr{G}_{H,A}}(M, N)$ for $i \geq 0$ and objects M and N in $\mathscr{G}_{H,A}$. If H_2 is a closed locally $\mathbf{Q}_p$-analytic subgroup of H_1, we also have an induction functor[14] $\mathrm{ind}_{H_2}^{H_1}\colon \mathscr{G}_{H_2,A} \to \mathscr{G}_{H_1,A}$. Our main purpose in considering such a theory is to show the following relative version of Shapiro's lemma (Proposition A.56), which is crucially used in the construction of the correspondence $\Delta \mapsto \Pi(\Delta)$ of Theorem 1.2:

Proposition 1.9 (Relative Shapiro's Lemma). *Let H_1 be a locally $\mathbf{Q}_p$-analytic group and let H_2 be a closed locally $\mathbf{Q}_p$-analytic subgroup. If M and N are objects of $\mathscr{G}_{H_2,A}$ and $\mathscr{G}_{H_1,A}$, respectively, then there are A-linear bijections*

$$\mathrm{Ext}^q_{\mathscr{G}_{H_1,A}}(\mathrm{ind}_{H_2}^{H_1}(M), N) \to \mathrm{Ext}^q_{\mathscr{G}_{H_2,A}}(M, N)$$

for all $q \geq 0$.

Structure of the paper. In §2, we extend the dictionary of p-adic functional analysis to the relative setting. A key issue is to establish that the sheaf $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$ is G-equivariant over $\mathbf{P}^1$ and is $\mathbf{Q}_p$-analytic.
In §3, we show some general results on the cohomology of (φ, Γ)-modules, including a comparison between continuous and locally analytic cohomology, and we describe the (φ, Γ)-cohomology as in [6]. A key result for the subsequent chapter is the nullity of $H^2(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1})$ iff $\delta_1\delta_2^{-1}$ is (pointwise) never of the form χx^i or x^{-i} for some $i \geq 0$ (i.e. $\delta_1\delta_2^{-1}$ is regular).
In §4 and 5, the technical heart of the paper is carried out. We begin by proving the finiteness of $\overline{P}^+$-cohomology for $\mathscr{R}_A(\delta_1, \delta_2)$. Using the Lie-algebra complex we provide a different proof of [14, Proposition 5.18] (in the cyclotomic setting). We show that the dimension of the higher cohomology group $H^2(\overline{P}^+, \mathscr{R}_L(\delta_1, \delta_2))$ is constant (of dimension 1) when $\delta_1\delta_2^{-1}$ is regular.
In §6, Theorem 1.3 is then established.
In §7, the general machinery developed in [14, §6] is used to construct $\Pi(\Delta)$ from a regular trianguline (φ, Γ)-module of rank 2 Δ over $\mathscr{R}_A$.
In the appendix (§A) we establish a formal framework for the main result. We introduce the category of locally analytic G-representations in A-modules. We prove that there is a relationship between this category and a category of modules over the relative distribution algebra in the same spirit of [39]. There is a locally analytic cohomology theory extending that of [31] and we establish a relative version of Shapiro's Lemma. These results are used in §7.

Acknowledgments. The debt this paper owes to the work of Pierre Colmez will be obvious to the reader. The authors are grateful to him for suggesting this problem and would like to thank him for many discussions on various

[14]This is the dual of the *standard* Induction functor, typically denoted $\mathrm{Ind}_{H_2}^{H_1}$, cf. Lemma A.57.

aspects of this paper. The first author would like to thank Jean-François Dat for his continuous encouragement throughout. Most of this work has been done while the second author was completing his Ph.D. thesis under the supervision of Pierre Colmez, to whom he thanks heartily. He finally wants to express his gratitude to Sarah Zerbes. Next we want to thank Kiran Kedlaya for spending countless hours answering our questions on Robba rings and suggesting a crucial induction argument. We would also like to thank Jean-François Dat, Jan Kohlhaase and Peter Schneider for several helpful discussions on what the category of locally analytic G-representations in A-modules should be. Further thanks go to Gabriel Dospinescu and Arthur-César Le Bras for fruitful conversations on various topics. Finally we would like to thank the anonymous referee for her/his careful reading of the manuscript and for pointing out many corrections and suggestions in their attempt to improve the overall quality of the paper.

1.4 Notations

Let A be an affinoid $\mathbf{Q}_p$-algebra equipped with its Gauss-norm topology (making it a Banach space with norm $|\cdot|_A$ and $v_A = -\log_p |\cdot|_A$ a fixed valuation). We will denote

$$G = \mathrm{GL}_2(\mathbf{Q}_p), \quad A^+ = \begin{pmatrix} \mathbf{Z}_p\backslash\{0\} & 0 \\ 0 & 1 \end{pmatrix},$$

$$P^+ = \begin{pmatrix} \mathbf{Z}_p\backslash\{0\} & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix}, \ \overline{P}^+ = \begin{pmatrix} \mathbf{Z}_p\backslash\{0\} & 0 \\ p\mathbf{Z}_p & 1 \end{pmatrix}.$$

As usual we note $\Gamma = \mathbf{Z}_p^\times$, $\mathbf{P}^1 = \mathbf{P}^1(\mathbf{Q}_p)$ and we assume $p > 2$ throughout. For $n \geq 1$ we set $r_n := \frac{1}{(p-1)p^{n-1}}$ and denote the element $\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}$ by τ. For two continuous characters $\delta_1, \delta_2 \colon \mathbf{Q}_p^\times \to A^\times$ we will denote $\delta = \delta_1\delta_2^{-1}\chi^{-1}$ and $\omega = \delta_1\delta_2\chi^{-1}$ where $\chi(x) = x|x|$ corresponds to the cyclotomic character via local class field theory. We denote by $\kappa(\delta_1) := \delta_1'(1)$, the weight of δ_1.

2 Preliminaries

We start by recalling, in the relative case, some well-known constructions that will play a key role in the sequel.

2.1 Dictionary of relative functional analysis

Let us first set up some notation and definitions.

2.1.1 Relative Laurent series rings

The theory of relative Robba rings has been expounded by Kedlaya-Liu in [30]. For $0 < r < s \leq \infty$ (with r and s rational, except possibly $s = \infty$), the relative

Robba ring $\mathscr{R}_A$ is defined by setting,

$$\mathscr{R}_A^{[r,s]} = \mathscr{R}_{\mathbf{Q}_p}^{[r,s]} \widehat{\otimes}_{\mathbf{Q}_p} A; \quad \mathscr{R}_A^{]0,s]} = \varprojlim_{0<r<s} \mathscr{R}_A^{[r,s]}; \quad \mathscr{R}_A = \varinjlim_{s>0} \mathscr{R}_A^{]0,s]},$$

where $\mathscr{R}_{\mathbf{Q}_p}^{[r,s]}$ is the usual Banach ring of analytic functions on the rigid analytic annulus in the variable T with radii $r \leq v_p(T) \leq s$ with coefficients in $\mathbf{Q}_p$. The Banach ring $\mathscr{R}_A^{[r,s]}$ is equipped with valuation $v^{[r,s]}$ defined by:

$$v^{[r,s]} = \min\left(\inf_{k \in \mathbf{Z}} (v_A(a_k) + rk), \inf_{k \in \mathbf{Z}} (v_A(a_k) + sk) \right)$$

for $f = \sum_{k\in\mathbf{Z}} a_k T^k \in \mathscr{R}_A^{[r,s]}$. The Robba ring $\mathscr{R}_A$ is then equipped with the natural projective and inductive limit topology, under which it is a locally convex A-module (cf. Definition A.5 and Example A.13 for further discussion). For $s < r_1$ we have an A-linear ring endomorphism $\varphi : \mathscr{R}_A^{[r,s]} \to \mathscr{R}_A^{[r/p,s/p]}$, sending T to $(1+T)^p - 1$, inducing an action of the operator φ over $\mathscr{R}_A$ and we also have a continuous action of the group Γ, commuting with that of φ, whose action is given by the formula $\sigma_a(T) = (1+T)^a - 1$, $a \in \mathbf{Z}_p^\times$, over all rings defined above.

Lemma 2.1 (Lemme 1.3 (v) [6]). *For every interval $I \subseteq]0,\infty]$, the ring $\mathscr{R}_A^I$ is a flat A-module. In particular, $\mathscr{R}_A$ is flat over A.*

2.1.2 Locally analytic functions and distributions

The Robba ring $\mathscr{R}_A$ is well interpreted in terms of distributions and locally analytic functions. Define $\mathscr{R}_A^+ := \mathscr{R}_A^{]0,\infty]}$ which is stable under the action of φ and Γ (equipped with the subspace topology), and note $\mathscr{R}_A^- := \mathscr{R}_A / \mathscr{R}_A^+$ (with the induced action of φ and Γ equipped with the quotient topology).
We define the algebra of A-valued distributions as[15]

$$\mathscr{D}(\mathbf{Z}_p, A) := \mathscr{D}(\mathbf{Z}_p, \mathbf{Q}_p) \widehat{\otimes}_{\mathbf{Q}_p} A, \tag{1}$$

where the tensor product in (1) is independent of the choice of injective or projective tensor product (as $\mathscr{D}(\mathbf{Z}_p, \mathbf{Q}_p)$ is Fréchet and A is Banach). Let $\mathrm{LA}(\mathbf{Z}_p, A)$ be the space of locally analytic functions on $\mathbf{Z}_p$ with values in A. Both $\mathscr{D}(\mathbf{Z}_p, A)$ and $\mathrm{LA}(\mathbf{Z}_p, A)$ are equipped with actions of Γ and of an operator φ given by

$$\varphi(f)(x) = \begin{cases} f(x/p) & \text{if } x \in p\mathbf{Z}_p \\ 0 & \text{if not} \end{cases}, \quad \sigma_a(f)(x) = f(x/a);$$

[15] A priori this is different to Definition A.19 where the relative distribution algebra for a general locally $\mathbf{Q}_p$-analytic group is defined. Lemma A.31 says that these definitions are equivalent.

$$\int_{\mathbf{Z}_p} f(x)\cdot\varphi(\mu)=\int_{\mathbf{Z}_p} f(px)\cdot\mu,\quad \int_{\mathbf{Z}_p} f(x)\cdot\sigma_a(\mu)=\int_{\mathbf{Z}_p} f(ax)\cdot\mu;$$

where $a\in\mathbf{Z}_p^\times$, $f\in\mathrm{LA}(\mathbf{Z}_p,A)$ and $\mu\in\mathscr{D}(\mathbf{Z}_p,A)$.

If $\mu\in\mathscr{D}(\mathbf{Z}_p,A)$, its Amice transform is defined as

$$\mathscr{A}_\mu=\sum_{n\in\mathbf{N}}\int_{\mathbf{Z}_p}\binom{x}{n}T^n\cdot\mu(x)\in\mathscr{R}_A^+.$$

Finally, for $f\in\mathscr{R}_A$, we define its Colmez transform as (for all $x\in\mathbf{Z}_p$)

$$\phi_f(x)=\mathrm{res}_0((1+T)^{-x}f(T)\frac{dT}{1+T})=\mathrm{res}_0((1+T)^{-x}fdt),$$

where for $f=\sum_{n\in\mathbf{Z}}a_nT^n$, we put $\mathrm{res}_0(fdT)=a_{-1}$ (as usual we set $t:=\log(1+T)$). We then have the following result due to Chenevier, cf. [6, Proposition 2.8] (cf. also [27, Lemma 2.1.19]), generalizing those of Colmez, cf. [14, Théorème 2.3] (cf. also [38]):

Proposition 2.2.

- *The Amice transform* $\mu\mapsto\mathscr{A}_\mu$ *induces a topological isomorphism* $\mathscr{D}(\mathbf{Z}_p,A)\cong\mathscr{R}_A^+$.
- *The Colmez transform* $f\mapsto\phi_f(x)$ *induces a topological isomorphism* $\mathscr{R}_A^-\cong\mathrm{LA}(\mathbf{Z}_p,A)\otimes\chi^{-1}$.
- *If* $\mu\in\mathscr{D}(\mathbf{Z}_p,A)$ *and* $f\in\mathscr{R}_A$, *then* $\int_{\mathbf{Z}_p}\phi_f\cdot\mu=\mathrm{res}_0(\sigma_{-1}(\mathscr{A}_\mu)f\frac{dT}{1+T})$.
- *We have a* (φ,Γ)*-equivariant short exact sequence*

$$0\to\mathscr{D}(\mathbf{Z}_p,A)\to\mathscr{R}_A\to\mathrm{LA}(\mathbf{Z}_p,A)\otimes\chi^{-1}\to 0.$$

2.1.3 The operator ψ

The Robba ring $\mathscr{R}_A$ is equipped with a left inverse of φ constructed as follows: For $s<r_1$ the map $\oplus_{i=0}^{p-1}\mathscr{R}_A^{[r,s]}\to\mathscr{R}_A^{[r/p,s/p]}$ given by $(f_i)_{i=0,\dots,p-1}\mapsto\sum_{i=0}^{p-1}(1+T)^i\varphi(f_i)$ is a topological isomorphism and allows us to define $\psi:\mathscr{R}_A^{[r/p,s/p]}\to\mathscr{R}_A^{[r,s]}$ by $\psi(f)=f_0$, where $f=\sum_{i=0}^{p-1}(1+T)^i\varphi(f_i)$. In other words, since upon extending scalars the conjugates of T under the action of φ are $(1+T)\zeta+1$, $\zeta\in\mu_p$, we can write $\varphi\circ\psi(f)=p^{-1}\mathrm{Tr}_{\mathscr{R}_A^{[r/p,s/p]}/\varphi(\mathscr{R}_A^{[r,s]})}(f)=p^{-1}\sum_{\zeta\in\mu_p}f((1+T)\zeta-1)$. In particular we note that $\psi((1+T))=0$, $f_i=\psi((1+T)^{-i}f)$ and that $\psi(\varphi(f)g)=f\psi(g)$. We also note $\psi:\mathscr{R}_A\to\mathscr{R}_A$ the induced operator, which is continuous, surjective and is a left inverse of φ. Finally, ψ stabilizes $\mathscr{R}_A^+$ and hence defines an action on $\mathscr{R}_A^-$. Under the identifications of Proposition 2.2 we have

$$\int_{\mathbf{Z}_p}f(x)\cdot\psi(\mu)=\int_{p\mathbf{Z}_p}f(\frac{x}{p})\cdot\mu,\quad \psi(f)(x)=f(px),$$

for $f\in \mathrm{LA}(\mathbf{Z}_p,A)$ and $\mu\in\mathscr{D}(\mathbf{Z}_p,A)$.
If $\mu\in\mathscr{D}(\mathbf{Z}_p,A)$ and $f\in\mathscr{R}_A$ then we have (cf. [14, Proposition 2.2] for the case where $A=L$ is a finite extension of $\mathbf{Q}_p$, the other case follows by linearity)

$$\psi(\mathscr{A}_\mu)=\mathscr{A}_{\psi(\mu)} \text{ and } \phi_{\psi(f)}=\psi(\phi_f),$$

where $\psi(\mu)$ is given by $\int_{\mathbf{Z}_p}\phi\cdot\psi(\mu):=\int_{p\mathbf{Z}_p}\phi(x/p)\cdot\mu$, and for any $\phi\in\mathrm{LA}(\mathbf{Z}_p,A)$, we set $\psi(\phi)(x):=\phi(px)$. In particular, if $\mu\in\mathscr{D}(\mathbf{Z}_p,A)$ (resp. $\phi\in\mathrm{LA}(\mathbf{Z}_p,A)$), the condition $\psi(\mu)=0$ (resp. $\psi(\phi)=0$) translates into μ (resp. ϕ) being supported on $\mathbf{Z}_p^\times$. Since ψ is surjective, by the snake lemma we have a Γ-equivariant exact sequence

$$0\to\mathscr{D}(\mathbf{Z}_p^\times,A)\to\mathscr{R}_A^{\psi=0}\to\mathrm{LA}(\mathbf{Z}_p^\times,A)\otimes\chi^{-1}\to 0.$$

2.1.4 Multiplication by a function

If $\mu\in\mathscr{D}(\mathbf{Z}_p,A)$ and $\alpha\in\mathrm{LA}(\mathbf{Z}_p,A)$, we define the distribution $\alpha\mu$ by the formula

$$\int_{\mathbf{Z}_p}\phi\cdot\alpha\mu:=\int_{\mathbf{Z}_p}\alpha\phi\cdot\mu.$$

If $a\in\mathbf{Z}_p, n\in\mathbf{N}$ and if we take $\alpha=\mathbf{1}_{a+p^n\mathbf{Z}_p}$ the characteristic function of the compact open $a+p^n\mathbf{Z}_p\subseteq\mathbf{Z}_p$, then we note $\mathrm{Res}_{a+p^n\mathbf{Z}_p}$ the multiplication by α. Via the Amice transform we can write

$$\mathrm{Res}_{a+p^n\mathbf{Z}_p}\mathscr{A}_\mu:=\mathscr{A}_{\mathrm{Res}_{a+p^n\mathbf{Z}_p}(\mu)}=(1+T)^a\varphi^n\circ\psi^n(1+T)^{-a}\mathscr{A}_\mu,$$

where the last equality follows from [14, §2.1.1].

2.1.5 The differential operators ∂ and ∇

We define an A-linear differential operator $\partial:\mathscr{R}_A\to\mathscr{R}_A$ by the formula

$$\partial f:=(1+T)\frac{df(T)}{dT}.$$

This operator plays an important role in the subsequent constructions that we will consider. We have, for $f\in\mathscr{R}_A$ and $\mu\in\mathscr{D}(\mathbf{Z}_p,A)$, $\phi_{\partial f}(x)=x\phi_f(x)$ and $\partial\mathscr{A}_\mu=\mathscr{A}_{x\mu}$, and ∂ is bijective on $\mathscr{R}_A^{\psi=0}$ (cf. [14, Proposition 2.6 and Lemme 2.7] for the case when A is a finite extension of $\mathbf{Q}_p$, in our setup the same proof carries over). We have the following useful commutativity relations: $\partial\circ\sigma_a=a\sigma_a\circ\partial$, $a\in\mathbf{Z}_p^\times$, and $\partial\circ\varphi=p\varphi\circ\partial$.
Finally, let ∇ be the operator given by the action of (a generator of) the Lie algebra of Γ. Precisely, we have $\nabla=t\partial$, where $t=\log(1+t)$ is Fontaine's $2i\pi$. The differential operator ∇ commutes with Γ and φ.

2.1.6 $\mathbf{Q}_p$-Analytic sheaves and relative (φ, Γ)-modules

A notion which plays a greater role in the study of (φ, Γ)-modules is that of an analytic sheaf. We refer to [14, Définition 1.6] for the definition of an H-sheaf $\mathscr{M}$ over X with coefficients in A, where A is an affinoid $\mathbf{Q}_p$-algebra, H is a locally $\mathbf{Q}_p$-analytic semi-group and X an H-space (totally disconnected, compact space on which H acts by continuous endomorphisms), with the evident modifications replacing the coefficient field L by A. As in *loc. cit.*, we denote by $\mathscr{M} \boxtimes U$ the sections of the sheaf $\mathscr{M}$ over an open $U \subseteq X$.

2.1.7 Relative (φ, Γ)-modules

Recall the following definition.

Definition 2.3 ([27, Definition 2.2.12]). Let $r \in (0, r_1)$. A φ-module over $\mathscr{R}_A^{]0,r]}$ is a finite projective $\mathscr{R}_A^{]0,r]}$-module $M^{]0,r]}$ equipped with an isomorphism

$$M^{]0,r]} \otimes_{\mathscr{R}_A^{]0,r]}, \varphi} \mathscr{R}_A^{]0,r/p]} \cong M^{]0,r]} \otimes_{\mathscr{R}_A^{]0,r]}} \mathscr{R}_A^{]0,r/p]}$$

A (φ, Γ)-module over $\mathscr{R}_A^{]0,r]}$ is a φ-module $M^{]0,r]}$ over $\mathscr{R}_A^{]0,r]}$ equipped with a commuting semilinear continuous action of Γ. A (φ, Γ)-module over $\mathscr{R}_A$ is the base change to $\mathscr{R}_A$ of a (φ, Γ)-module over $\mathscr{R}_A^{]0,r]}$ for some r.

We denote by $\Phi\Gamma(\mathscr{R}_A)$ the category of (φ, Γ)-modules over $\mathscr{R}_A$. Morphisms are $\mathscr{R}_A$-linear morphisms commuting with the actions of φ and Γ.

Definition 2.4. For $(H, X) \in \{(P^+, \mathbf{Z}_p), (G, \mathbf{P}^1)\}$, we say that an H-sheaf $\mathscr{M}$ over X with coefficients in A is $\mathbf{Q}_p$-analytic if for all open compact $U \subset X$, $\mathscr{M} \boxtimes U$ is a locally convex A-module of A-LF-type (cf. Definition A.15) and such that the action of any open compact subgroup $K \subseteq H$ stabilizing U extends to a continuous action of $\mathscr{D}(K, A)$.

The point of Definition 2.4 is that, as pointed out in the introduction, a (φ, Γ)-module Δ over $\mathscr{R}_A$ naturally provides a $\mathbf{Q}_p$-analytic P^+-sheaf over $\mathbf{Z}_p$, which codifies its (φ, Γ)-structure, cf. [14, §1.3.3] for details, as well as [1, Lemme 4.1] or [27, Lemma 2.2.14(3)], for why the action of Γ is locally analytic. We say that Δ is analytic. Let us briefly recall the construction and basic properties of this sheaf for the commodity of the reader. The definition of the restriction $\mathrm{Res}_{a+p^n\mathbf{Z}_p}$ given in §2.1.4 in terms of the Amice transform makes sense for an arbitrary (φ, Γ)-module. The definition extends in the obvious way to any open compact subset $U \subseteq \mathbf{Z}_p$ and one checks that $U \mapsto D \boxtimes U := \mathrm{Res}_U D$ defines a P^+-equivariant sheaf on $\mathbf{Z}_p$. We have in particular $D \boxtimes \mathbf{Z}_p = D$ and $D \boxtimes \mathbf{Z}_p^\times = D^{\psi=0}$. The aim is to show that, for a trianguline (φ, Γ)-module over $\mathscr{R}_A$, we can extend the corresponding P^+-sheaf over $\mathbf{Z}_p$ to a G-sheaf over $\mathbf{P}^1$. Moreover the global sections of the latter will cut out the locally analytic G-representation in A-modules that we are attempting to attach to Δ.

2.1.8 Multiplication by a character on $\mathscr{R}_A$

Recall that, if $f \in \mathscr{R}_A \boxtimes \mathbf{Z}_p^\times = \mathscr{R}_A^{\psi=0}$, we can write

$$f = \sum_{i \in (\mathbf{Z}/p^N\mathbf{Z})^\times} (1+T)^i \varphi^N(f_i),$$

where $f_i = \psi^N(1+T)^{-i} f$. If $k \geq 0$, by the Leibniz rule we have

$$\begin{aligned} \partial^k f &= \partial^k \Big(\sum_{i \in (\mathbf{Z}/p^N\mathbf{Z})^\times} (1+T)^i \varphi^N(f_i) \Big) \\ &= \sum_{i \in (\mathbf{Z}/p^N\mathbf{Z})^\times} \sum_{j=0}^{k} \binom{k}{j} i^{k-j} (1+T)^i p^{Nj} \varphi^N(\partial^j f_i). \end{aligned}$$

This formula suggests the following relative version for the multiplication by a locally analytic function.

Proposition 2.5. *If $\delta : \mathbf{Z}_p^\times \to A^\times$ is a locally analytic character and $f \in \mathscr{R}_A^{\psi=0}$, the expression*

$$\sum_{i \in (\mathbf{Z}/p^N\mathbf{Z})^\times} \sum_{j=0}^{+\infty} \binom{\kappa(\delta)}{j} \delta(i) i^{-j} (1+T)^i p^{Nj} \varphi^N(\partial^j f_i),$$

where $\kappa(\delta) = \delta'(1)$ is the weight of δ [16]*, converges in $\mathscr{R}_A^{\psi=0}$ for N big enough (depending only on δ) to an element $m_\delta(f)$ that does not depend on N or on the choice of representatives of $(\mathbf{Z}/p^N\mathbf{Z})^\times$.*
Moreover, the map $m_\delta : \mathscr{R}_A \boxtimes \mathbf{Z}_p^\times \to \mathscr{R}_A \boxtimes \mathbf{Z}_p^\times$ thus defined, is continuous, with inverse $m_{\delta^{-1}}$, stabilizes $\mathscr{R}_A^+ \boxtimes \mathbf{Z}_p^\times$ and induces the multiplication by δ on $\mathscr{D}(\mathbf{Z}_p^\times, A)$ and $\mathrm{LA}(\mathbf{Z}_p^\times, A)$.

Proof. For $A = L$ a finite extension of $\mathbf{Q}_p$, this is [14, Proposition 2.9] (cf. also [23, §3.2], or [36, Proposition I.13] for a detailed proof in a more general setting). The same proof with minor modifications carries over for a general A. ☐

Remark 2.6. Recall that $\mathscr{R}_A = \varinjlim_{s>0} \varprojlim_{0<r<s} \mathscr{R}_A^{[r,s]}$ is an LF-space. Let $\mathfrak{X} := \mathrm{Hom}_{\mathrm{cont}}(\mathbf{Z}_p^\times, \mathbf{C}_p^\times)$ denote the p-adic weight space, which is a rigid analytic space. We say that a function $\xi : \mathfrak{X} \to \mathscr{R}_A$ is a rigid analytic function if $f \in \varinjlim_{s>0} \varprojlim_{0<r<s} \mathcal{O}(\mathfrak{X}) \widehat{\otimes} \mathscr{R}_A^{[r,s]}$, i.e., if f is a compatible system of rigid analytic functions on $\mathfrak{X}$ with values in the Banach space $\mathscr{R}_A^{[r,s]}$ for all $r < s$ for some small enough s. Observe that, since the characters $x \mapsto x^k$ are Zariski dense in $\mathfrak{X}$, such a function is determined by its values on them. One can show (cf. [36, Proposition I.13]) that, for f fixed, $\delta \mapsto m_\delta(f)$ defines a rigid analytic function on $\mathfrak{X}$ with values in $\mathscr{R}_A^{\psi=0}$.

[16] We have, for instance, that $\kappa(\delta) = 0$ if and only if δ is locally constant and $\kappa(\delta) \geq 1$ is an integer if and only if $\delta = \eta\chi^k$, where η is a locally constant character.

2.2 Duality

If $\Delta \in \Phi\Gamma(\mathscr{R}_A)$, we set $\check{\Delta} = \mathrm{Hom}_{\mathscr{R}_A}(\Delta, \mathscr{R}_A(\chi))$ and denote by

$$\langle\ ,\ \rangle \colon \check{\Delta} \times \Delta \to \mathscr{R}_A(\chi)$$

the natural pairing. We impose a (φ, Γ)-structure on $\check{\Delta}$ by setting

$$\langle g \cdot \check{z}, g \cdot z\rangle := g \cdot \langle \check{z}, z\rangle$$

for all $\check{z} \in \check{\Delta}$, $z \in \Delta$ and $g \in \{\sigma_a, \varphi\}$. Note that $\check{\Delta} \in \Phi\Gamma(\mathscr{R}_A)$.
The pairing $\langle\ ,\ \rangle$ defines a new pairing

$$\begin{aligned} \{\ ,\ \} \colon \check{\Delta} \times \Delta &\to A \\ (\check{z}, z) &\mapsto \mathrm{res}_0(\langle \sigma_{-1}(\check{z}), z\rangle), \end{aligned}$$

where $\mathrm{res}_0\left(\sum_{k\in\mathbf{Z}} a_k T^k dT\right) = a_{-1}$. Assuming that Δ is free over $\mathscr{R}_A$, the pairing $\{\ ,\ \}$ identifies $\check{\Delta}$ and Δ as topological A-duals of Δ and $\check{\Delta}$ respectively, cf. [12, Proposition III.2.3].

2.3 Principal series

Let $\delta_1, \delta_2 \colon \mathbf{Q}_p^\times \to A^\times$ be two locally analytic characters[17] and recall that $\delta = \delta_1\delta_2^{-1}\chi^{-1}$. We define $B_A(\delta_1, \delta_2)$ to be the space of locally analytic functions $\phi \colon \mathbf{Q}_p \to A$, such that $\delta(x)\phi\left(\frac{1}{x}\right)$ extends to an analytic function on a neighbourhood of 0. We equip $B_A(\delta_1, \delta_2)$ with an action of G defined by

$$\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix} \cdot \phi\right)(x) = \delta_2(ad - bc)\delta(a - cx)\phi\left(\frac{dx - b}{a - cx}\right). \tag{2}$$

Using that $\mathbf{Q}_p \subseteq \mathbf{P}^1 \cong G/B$, one can show that $B_A(\delta_1, \delta_2) = \mathrm{Ind}_{\overline{B}}^{G}(\delta_1\chi^{-1} \otimes \delta_2)$ (where $\overline{B}$ is the lower-half Borel subgroup of G). Here $\delta_1\chi^{-1} \otimes \delta_2$ denotes the character $\begin{pmatrix} a & 0 \\ c & d\end{pmatrix} \mapsto \delta_1\chi^{-1}(a)\delta_2(d)$ of $\overline{B}$. For the definition of $\mathrm{Ind}_{\overline{B}}^{G}(\delta_1\chi^{-1} \otimes \delta_2)$, cf. Remark A.27. The topology of $B_A(\delta_1, \delta_2)$ is by definition the topology coming from $\mathrm{LA}(G/\overline{B}, A)$, cf. Definition A.17. This makes $B_A(\delta_1, \delta_2)$ into a Hausdorff, complete, locally convex A-module, cf. Definition A.5 and Lemma A.16.
The strong topological dual (cf. Definition A.9) of $B_A(\delta_1, \delta_2)$ identifies with a space of distributions on $\mathbf{P}^1$ equipped with an action of G defined by

$$\int_{\mathbf{P}^1} \phi \cdot \begin{pmatrix} a & b \\ c & d\end{pmatrix}\mu = \delta_1^{-1}\chi(ad - bc)\int_{\mathbf{P}^1} \delta(cx + d)\phi\left(\frac{ax + b}{cx + d}\right) \cdot \mu(x). \tag{3}$$

[17] Recall that any continuous character $\delta : \mathbf{Q}_p^\times \to A^\times$ is automatically locally analytic.

2.4 The G-module $\mathscr{R}_A(\delta)\boxtimes_\omega \mathbf{P}^1$

Suppose $\Delta\in\Phi\Gamma(\mathscr{R}_A)$ is trianguline of rank 2. In this section, we follow [14, §4.3], to construct the G-modules $\mathscr{R}_A(\delta)\boxtimes_\omega \mathbf{P}^1$ which will be the constituents of the G-module $\Delta\boxtimes_\omega \mathbf{P}^1$. Once $\Delta\boxtimes_\omega \mathbf{P}^1$ is constructed, we will see that one of its constituents is the representation $\Pi(\Delta)$, which we are searching for. Since we are only interested in constructing representations of $\mathrm{GL}_2(\mathbf{Q}_p)$, the constructions from [14] (where representations of $\mathrm{GL}_2(F)$, for $F/\mathbf{Q}_p$ a finite extension, are constructed) simplify considerably.

We start by recalling a structure result for arithmetic families of (φ,Γ)-modules. Recall that $\Gamma\cong\mathbf{Z}_p^\times$ and let $\gamma\in\Gamma$ be a topological generator. Replacing T by $\gamma-1$, we can define the rings $\mathscr{R}_A^{[r,s]}(\Gamma)$, $\mathscr{R}_A^{]0,s]}(\Gamma)$, $\mathscr{R}_A(\Gamma)$ by following the usual construction.

Proposition 2.7 (Theorem 3.1.1,[27]). *Let A be an affinoid $\mathbf{Q}_p$-algebra and let $\Delta\in\Phi\Gamma(\mathscr{R}_A)$. There exists $r(\Delta)$ such that, for any $0<r<r(\Delta)$, $\gamma-1$ is invertible on $(\Delta^{]0,r]})^{\psi=0}$, and the $A[\Gamma,(\gamma-1)^{-1}]$-module structure on $(\Delta^{]0,r]})^{\psi=0}$ extends uniquely by continuity to a $\mathscr{R}_A^{]0,r]}(\Gamma)$-module structure for which $(\Delta^{]0,r]})^{\psi=0}$ is finite projective of rank $d=\mathrm{rank}_{\mathscr{R}_A}\Delta$. Moreover, if Δ is free over $\mathscr{R}_A$, then $(\Delta^{]0,r]})^{\psi=0}$ admits a set of d generators over $\mathscr{R}_A^{]0,r]}(\Gamma)$.*

Remark 2.8.

- Base changing we also get that $\Delta^{\psi=0}$ is a finite projective module over $\mathscr{R}(\Gamma)$ of rank d, admitting a set of m generators ($m=d$ if Δ is free).
- In the case when $\Delta=\mathscr{R}_A$, one can show that $\Delta^{\psi=0}$ is a free module of rank one over $\mathscr{R}_A(\Gamma)$ generated by $(1+T)$, cf. [6, Proposition 2.14 and Remarque 2.15].

Recall that, if $\Delta=\mathscr{R}_A$ we have a short exact sequence of Γ-modules

$$0\to(\mathscr{R}_A^+)^{\psi=0}\to\mathscr{R}_A^{\psi=0}\to(\mathscr{R}_A^-)^{\psi=0}\to 0,$$

that $(\mathscr{R}_A^+)^{\psi=0}=\mathscr{R}_A^+\boxtimes\mathbf{Z}_p^\times\cong\mathscr{D}(\mathbf{Z}_p^\times,A)$ via the Amice transform, and that we have a continuous involution w_* on it given by

$$\int_{\mathbf{Z}_p^\times}\phi(x)\cdot w_*\mu=\int_{\mathbf{Z}_p^\times}\phi(x^{-1})\cdot\mu.$$

The involution is Γ-anti-linear in the sense that we have $w_*\circ\sigma_a=\sigma_a^{-1}\circ w_*$ for all $a\in\mathbf{Z}_p^\times$. We denote by $\iota:\mathscr{R}_A(\Gamma)\to\mathscr{R}_A(\Gamma)$ the continuous involution defined by $\sigma_a\mapsto\sigma_a^{-1}$ on Γ.

Lemma 2.9. *There exists a unique continuous $\mathscr{R}_A(\Gamma)$-anti-linear involution w_* with respect to ι [18] on $\mathscr{R}_A\boxtimes\mathbf{Z}_p^\times$ extending that on $\mathscr{R}_A^+\boxtimes\mathbf{Z}_p^\times$. Moreover, w_* satisfies*

[18] i.e. satisfying $w_*\circ\lambda=\iota(\lambda)\circ w_*$ for all $\lambda\in\mathscr{R}_A(\Gamma)$.

- $w_* = \partial w_* \partial$.
- $\nabla \circ w_* = -w_* \circ \nabla$.
- $w_* \circ \mathrm{Res}_{a+p^n\mathbf{Z}_p} = \mathrm{Res}_{a^{-1}+p^n\mathbf{Z}_p} \circ w_*$, *for all* $a \in \mathbf{Z}_p^\times$, $n \geq 1$.

Proof. Take a generator e of the free $\mathscr{R}_A(\Gamma)$-module $\mathscr{R}_A \boxtimes \mathbf{Z}_p^\times$ of rank one such that $e \in \mathscr{R}_A^+ \boxtimes \mathbf{Z}_p^\times$ (e.g. $(1+T)$, cf. Remark 2.8). This forces

$$w_*(\lambda \cdot e) = \iota(\lambda) \cdot w_*(e)$$

for every $\lambda \in \mathscr{R}_A(\Gamma)$, where $w_*(e) \in \mathscr{R}_A^+ \boxtimes \mathbf{Z}_p^\times$ is well defined since $e \in \mathscr{R}_A^+ \boxtimes \mathbf{Z}_p^\times$. For the rest of the properties we can use [14, Lemme 2.14] which shows that they hold for w_* acting on $\mathscr{R}_A^+ \boxtimes \mathbf{Z}_p^\times$ (the same proof carries over for any A). We only show the first one, the other two being immediate. Let $z = \lambda \cdot e \in \mathscr{R}_A \boxtimes \mathbf{Z}_p^\times$ for some $\lambda \in \mathscr{R}_A(\Gamma)$. We have

$$\begin{aligned} \partial \circ w_* \circ \partial(\lambda \cdot e) &= \chi(\lambda)\, \partial \circ w_*(\lambda \cdot \partial e) = \chi(\lambda)\, \partial(\iota(\lambda) \cdot w_*(\partial e)) \\ &= \iota(\lambda) \cdot \partial \circ w_* \circ \partial(e) = \iota(\lambda) \cdot w_*(e) = w_*(z), \end{aligned}$$

where we have used: the identity $\partial \circ \lambda = \chi(\lambda)\lambda \circ \partial$ (which follows by continuity from the identity $\partial \circ \sigma_a = a\sigma_a \circ \partial$, cf. §2.1.5) for the first and third equalities, the $\mathscr{R}_A(\Gamma)$-anti-linearity of w_* for the second and the fifth, and the fact that $\partial \circ w_* \circ \partial = w_*$ on $\mathscr{R}_A^+$ for the fourth one. ☐

The following gives a relation between m_δ and w_*.

Lemma 2.10. *If* $\delta \colon \mathbf{Z}_p^\times \to A^\times$ *is a locally analytic character, then*

$$m_\delta \circ w_* = w_* \circ m_{\delta^{-1}}.$$

Proof. By Lemma 2.9, the identity is true for $\delta = x^k$ for all $k \in \mathbf{Z}$ (this is because $\partial^k = m_{x^k}$, cf. §2.1.5). Now the functions $\delta \mapsto m_\delta(z)$ and $\delta \mapsto w_* \circ m_{\delta^{-1}} \circ w_*(z)$, $z \in \mathscr{R}_A^{\psi=0}$, are rigid functions on $\mathfrak{X}$ (cf. Remark 2.6) and coincide on x^k for all $k \in \mathbf{Z}$. Thus they coincide for all δ by Zariski density. Observe that we could have argued by reducing to the case of a point, where the result is proved in [14, Lemme 2.22], as we do in Proposition 2.18 below, or by just imitating the proof of *loc. cit.*, which carries over in the relative case. ☐

If $\Delta \in \Phi\Gamma(\mathscr{R}_A)$, recall (cf. the discussion after Definition 2.4) that the restriction maps $\mathrm{Res}_{a+p^n\mathbf{Z}_p}$ allow us to see Δ as a $\left(\begin{smallmatrix} \mathbf{Z}_p - \{0\} & \mathbf{Z}_p \\ 0 & 1 \end{smallmatrix}\right)$-equivariant sheaf $U \mapsto \Delta \boxtimes U$ over $\mathbf{Z}_p$. If $\omega \colon \mathbf{Q}_p^\times \to A^\times$ (for applications ω will be $\delta_1\delta_2\chi^{-1}$ for any two continuous characters $\delta_1, \delta_2 \colon \mathbf{Q}_p^\times \to A^\times$) is a locally analytic character and ι is an involution on $\Delta \boxtimes \mathbf{Z}_p^\times$, we can define (cf. [14, §3.1.1] for details, from where we also borrow notations)

$$\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1 = \{(z_1, z_2) \in \Delta \times \Delta \, ; \, \mathrm{Res}_{\mathbf{Z}_p^\times}(z_1) = \iota(\mathrm{Res}_{\mathbf{Z}_p^\times}(z_2))\},$$

which is equipped with an action of a group $\tilde{G}$, of which G is a quotient, generated freely by:

- a group $\tilde{Z}$ isomorphic to the torus $\{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right), a \in \mathbf{Q}_p^\times\}$ acting on $\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1$ via multiplication by ω,
- a group $\tilde{A}^0 \cong \mathbf{Z}_p^\times$ acting by $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot (z_1, z_2) = (\sigma_a(z_1), \omega(a)\sigma_{a^{-1}}(z_2))$,
- a group $\tilde{U} \cong p\mathbf{Z}_p$ (encoding the multiplication by $(1+T)^b$, $b \in p\mathbf{Z}_p$),
- the element $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ (encoding the action of φ) and
- the element $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ acting via $w \cdot (z_1, z_2) = (z_2, z_1)$.

We will see that, when ι is suitably chosen, then the action of $\tilde{G}$ factorises through G.

Lemma 2.11. *The functor $M \mapsto M \boxtimes_{\omega,\iota} \mathbf{P}^1$ is an exact functor from P^+-modules living on $\mathbf{Z}_p$ to $\tilde{G}$-modules living on $\mathbf{P}^1$.*

Proof. Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of P^+-modules. We claim that we have an exact sequence $0 \to M' \boxtimes_{\omega,\iota} \mathbf{P}^1 \to M \boxtimes_{\omega,\iota} \mathbf{P}^1 \to M'' \boxtimes_{\omega,\iota} \mathbf{P}^1$. Let's show that the last arrow is surjective (for exactness in the middle and injectivity, the proof is easy). Let $(c,d) \in M'' \boxtimes_{\omega,\iota} \mathbf{P}^1$ and $(a,b) \in M \times M$ be any lifting. The element $\mathrm{Res}_{\mathbf{Z}_p^\times} a - \iota(\mathrm{Res}_{\mathbf{Z}_p^\times} b)$ maps to zero in M'' and so there exists an element $x \in (M')^{\psi=0}$ such that $\mathrm{Res}_{\mathbf{Z}_p^\times} a - \iota(\mathrm{Res}_{\mathbf{Z}_p^\times} b) = x$. The element $(a-x, b) \in M \boxtimes_{\omega,\iota} \mathbf{P}^1$ maps then to (c,d). □

For $\delta_1, \delta_2 : \mathbf{Q}_p^\times \to A^\times$ locally analytic characters, recall that we have set $\delta = \delta_1\delta_2^{-1}\chi^{-1}, \omega = \delta_1\delta_2\chi^{-1}$. We will soon be working with $\Delta \in \Phi\Gamma(\mathscr{R}_A)$ which is an extension of $\mathscr{R}_A(\delta_2)$ by $\mathscr{R}_A(\delta_1)$. Thus we need to *twist* appropriately the current involution w_*, cf. Lemma 2.9, on $\mathscr{R}_A \boxtimes \mathbf{Z}_p^\times$. We define an involution $\iota_{\delta_1,\delta_2}$[19] acting on the module $\mathscr{R}_A(\delta_1) \boxtimes \mathbf{Z}_p^\times$ by the formula[20]

$$\iota_{\delta_1,\delta_2}(f \otimes \delta_1) = (\delta_1(-1) w_* \circ m_{\delta^{-1}}(f)) \otimes \delta_1.$$

Definition 2.12. We define $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1 := \mathscr{R}_A(\delta_1) \boxtimes_{\omega,\iota_{\delta_1,\delta_2}} \mathbf{P}^1$ to be the $\tilde{G}$-module constructed in the above discussion using the involution $\iota_{\delta_1,\delta_2}$.

We show in what follows that this action of $\tilde{G}$ factorises through G. Recall that, for a finite extension L of $\mathbf{Q}_p$ and $\Delta \in \Phi\Gamma(\mathscr{R})$ of rank 1 or trianguline of rank 2, $\omega : \mathbf{Q}_p^\times \to L^\times$ locally analytic and ι an involution on $\Delta \boxtimes \mathbf{Z}_p^\times$, we have (cf. [14]) a G-module $\Delta \boxtimes_\omega \mathbf{P}^1$. the following lemma shows that, for $\Delta = \mathscr{R}_A(\delta_1)$, our construction specializes to that of Colmez.

Lemma 2.13. *Let $\mathfrak{m} \subseteq A$ be a maximal ideal of A, $L = A/\mathfrak{m}$, $\Delta \in \Phi\Gamma(\mathscr{R}_A)$ and denote $\overline{\omega}\colon \mathbf{Q}_p^\times \to L^\times$ the reduction modulo $\mathfrak{m}$ of ω. Assume that Δ is either of rank 1 or trianguline of rank 2. Then the $\tilde{G}$-module $(\Delta \boxtimes_\omega \mathbf{P}^1) \otimes_A L$ is canonically isomorphic to $(\Delta \otimes_A L) \boxtimes_{\overline{\omega}} \mathbf{P}^1$.*

[19] The fact that i_{δ_1,δ_2} is an involution follows from Lemma 2.10.
[20] By Proposition 2.5, this formula is well defined.

Proof. This is immediate. The uniqueness of both involutions w_* defined in Lemma 2.9 above and in [14, Proposition 2.19] shows that they both coincide (since they do on $\mathscr{R}^+ \boxtimes \mathbf{P}^1$). $\square$

The following result provides a link between the $-\boxtimes_\omega \mathbf{P}^1$ construction and principal series, cf. §2.3

Lemma 2.14. *We have*[21]

- $\mathscr{R}_A^+(\delta_1) \boxtimes_\omega \mathbf{P}^1 \cong B_A(\delta_2, \delta_1)^* \otimes \omega$
- $\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1 \cong B_A(\delta_1, \delta_2)$.

Moreover $\mathscr{R}_A^+(\delta_1) \boxtimes_\omega \mathbf{P}^1$ *and* $\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1$ *are* $\mathbf{Q}_p$*-analytic sheaves.*

Proof. This is essentially [14, Corollaire 4.11]. The same proof carries over with A in place of L (since one only checks that both actions of G coincide and the coefficient ring plays no role). The last part follows from Lemma A.57. $\square$

Definition 2.15. We note $\mathscr{R}_A(\delta_1, \delta_2)$ to be the $\overline{P}^+$-module [22]

$$\mathscr{R}_A(\delta_1, \delta_2) := (\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{Z}_p) \otimes \delta_2^{-1}.$$

We set $\mathscr{R}_A^+(\delta_1, \delta_2)$ the $\overline{P}^+$-submodule of $\mathscr{R}_A(\delta_1, \delta_2)$ corresponding to $\mathscr{R}_A^+$, and $\mathscr{R}_A^-(\delta_1, \delta_2)$ to be the quotient of $\mathscr{R}_A(\delta_1, \delta_2)$ by $\mathscr{R}_A^+(\delta_1, \delta_2)$.

Remark 2.16. As A^+-modules, $\mathscr{R}_A(\delta_1, \delta_2)$, $\mathscr{R}_A^+(\delta_1, \delta_2)$ and $\mathscr{R}_A^-(\delta_1, \delta_2)$ are respectively isomorphic to $\mathscr{R}_A(\delta_1 \delta_2^{-1})$, $\mathscr{R}_A^+(\delta_1 \delta_2^{-1})$ and $\mathscr{R}_A^-(\delta) \cong \mathrm{LA}(\mathbf{Z}_p, A) \otimes \delta$. The technical heart of this paper is to compare the $\overline{P}^+$-cohomology of $\mathscr{R}_A(\delta_1, \delta_2)$ and the A^+-cohomology of $\mathscr{R}_A(\delta_1 \delta_2^{-1})$, cf. §6.

Lemma 2.17. *Let N be the nilradical of A and let $j \geq 0$ be an integer such that $N^{j+1} = 0$. Then we have an exact sequence of $\tilde{G}$-modules*

$$0 \to (\mathscr{R}_{A/N} \boxtimes_\omega \mathbf{P}^1) \otimes_{A/N} N^j \to \mathscr{R}_A \boxtimes_\omega \mathbf{P}^1 \to \mathscr{R}_{A/N^j} \boxtimes_\omega \mathbf{P}^1 \to 0.$$

Proof. We have an exact sequence

$$0 \to N^j \to A \to A/N^j \to 0.$$

Since $\mathscr{R}_A$ is a flat A-module (cf. Lemma 2.1), we get an exact sequence of P^+-modules

$$0 \to \mathscr{R}_A \otimes_A N^j \to \mathscr{R}_A \to \mathscr{R}_A \otimes_A A/N^j \to 0.$$

[21] By $B_A(\delta_1, \delta_2)^*$ we mean $\mathrm{Hom}_{A,\mathrm{cont}}(B_A(\delta_1, \delta_2), A)$ equipped with the strong dual topology, cf. Defintition A.9, where $(-)^*$ is denoted by $(-)'_b$ there.

[22] Here δ_2^{-1} is seen as a character of $\overline{P}$, by setting $\delta_2^{-1}\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} = \delta_2^{-1}(a)$. We warn the reader that the module we call $\mathscr{R}_A(\delta_1, \delta_2)$ is not the one noted in the same way in [14, §4.3.2]. In our notation, $\mathscr{R}_A(\delta_1, \delta_2)$ corresponds to the module $\mathscr{R}_A(\delta_1, \delta_2, \eta)$ for $\eta = 1$ as defined in [14, §5.6].

Now, since tensor product commute with colimits, by [6, Lemme 1.3(iv)] we have $\mathscr{R}_A\otimes_A A/N^j\cong\mathscr{R}_A/N^j\mathscr{R}_A\cong\mathscr{R}_{A/N^j}$ and similarly $\mathscr{R}_A\otimes_A N^j\cong\mathscr{R}_A\otimes_A A/N\otimes_{A/N}N^j\cong\mathscr{R}_{A/N}\otimes_{A/N}N^j$. The result now follows from Lemma 2.11. ☐

The following is the main result of this section, which is a relative version of [14, Proposition 4.12].

Proposition 2.18. *The action of $\tilde{G}$ on $\mathscr{R}_A(\delta_1)\boxtimes_\omega\mathbf{P}^1$ factorises through G and we have an exact sequence of G-modules*

$$0\to B_A(\delta_2,\delta_1)^*\otimes\omega\to\mathscr{R}_A(\delta_1)\boxtimes_\omega\mathbf{P}^1\to B_A(\delta_1,\delta_2)\to 0.$$

Proof. We reduce the result to the case of a point, cf. [14, Proposition 4.12], using an inductive argument on the index $i\geq 0$ of nilpotency of A.
Suppose first that $i=0$, i.e that A is reduced. Take $z\in\mathscr{R}_A(\delta_1)\boxtimes_\omega\mathbf{P}^1$ and g in the kernel of $\tilde{G}\to G$. We need to show that $(g-1)z=(z_1,z_2)=0$. Let $\mathfrak{m}\subseteq A$ be any maximal ideal of A and note $L=A/\mathfrak{m}$. Since $(\mathscr{R}_A(\delta_1)\boxtimes_\omega\mathbf{P}^1)\otimes_A L=\mathscr{R}_L(\delta_1)\boxtimes_\omega\mathbf{P}^1$, then we know by [14, Proposition 4.12] that $z_i=0 \bmod \mathfrak{m}$. If we write $z_i=\sum_{n\in\mathbf{Z}}a_{n,i}T^n$, $i=1,2$, this means that $a_{n,i}=0 \bmod \mathfrak{m}$ and hence, since this holds for every maximal ideal $\mathfrak{m}$ and since A is reduced, we deduce that $a_{n,i}=0$ for every n and hence $z_i=0$ as desired.
Suppose now the result is true for every affinoid algebra of index of nilpotence $\leq j$ and let A be an affinoid $\mathbf{Q}_p$-algebra whose nilradical N satisfies $N^{j+1}=0$ and g be in the kernel of $\tilde{G}\to G$. We have the following short exact sequence (cf. Lemma 2.17)

$$0\to(\mathscr{R}_{A/N}\boxtimes_\omega\mathbf{P}^1)\otimes_{A/N}N^j\to\mathscr{R}_A\boxtimes_\omega\mathbf{P}^1\to\mathscr{R}_{A/N^j}\boxtimes_\omega\mathbf{P}^1\to 0.$$

By the base case of a reduced affinoid algebra and by the inductive hypothesis, the element $g-1$ induces a linear endomorphism of the short exact sequence above which vanishes on $(\mathscr{R}_{A/N}\boxtimes_\omega\mathbf{P}^1)\otimes_{A/N}N^j$ and $\mathscr{R}_{A/N^j}\boxtimes_\omega\mathbf{P}^1$ respectively. Therefore it vanishes on $\mathscr{R}_A\boxtimes\mathbf{P}^1$, which shows the desired result.

☐

Let us finish this chapter by observing that, by adapting the proof of [15, Proposition VI.7] (cf. also [11, §V.2], to where we refer for the details), for $\mathscr{R}_A(\delta_1)\boxtimes_\omega\mathbf{P}^1$, one obtains the lemma stated below. Let us observe that the arguments sketched below are almost exactly the same as [15, Proposition VI.7]: the only point that requires some different argument in the proof is the stability under the involution of the module $M_m^{\dagger,b}$ defined in the proof, which in our case is easy to verify since we are dealing with the case of rank 1.

Lemma 2.19. *$\mathscr{R}_A(\delta_1)\boxtimes_\omega\mathbf{P}^1$ is a $\mathbf{Q}_p$-analytic sheaf.*

Proof. Let $H\subseteq G$ be an open compact subgroup. We first make some standard reductions. Since $\mathscr{D}(H,A)=\mathscr{D}(H,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi}A$ (cf. Lemma A.31), by Lemma A.16, it suffices to show $\mathscr{R}_A(\delta_1)\boxtimes_\omega\mathbf{P}^1$ is equipped with a topological

$\mathscr{D}(H) := \mathscr{D}(H, \mathbf{Q}_p)$-action. However, since H and $K_m := I + p^m M_2(\mathbf{Z}_p)$ are commensurable, it is enough to show that $\mathscr{R}_A(\delta_1, \delta_2) \boxtimes_\omega \mathbf{P}^1$ is equipped with a topological $\mathscr{D}(K_m) := \mathscr{D}(K_m, \mathbf{Q}_p)$-action for some $m \geq 1$.

Denote $M = \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{Z}_p$. Since K_m is normal in $\mathrm{GL}_2(\mathbf{Z}_p)$ and $\mathbf{P}^1$ is the disjoint union of translates of $1+p^m\mathbf{Z}_p$ by $\mathrm{GL}_2(\mathbf{Z}_p)$ (cf. the paragraph following [11, Lemme V.2.11] for the explicit covering), one can reduce to showing that $M \boxtimes_\omega (1+p^m\mathbf{Z}_p) = (1+T)\varphi^m(M)$ (cf. the restriction formula in §2.1.4) admits a topological action of $\mathscr{D}(K_m)$ for some $m \geq 1$.

As is explained in [15, §V.2], the elements

$$a_m^+ = \begin{pmatrix} 1+p^m & 0 \\ 0 & 1 \end{pmatrix}, \quad a_m^- = \begin{pmatrix} 1 & 0 \\ 0 & 1+p^m \end{pmatrix}, \quad u_m^+ = \begin{pmatrix} 1 & p^m \\ 0 & 1 \end{pmatrix}, \quad u_m^- = \begin{pmatrix} 1 & 0 \\ p^m & 1 \end{pmatrix},$$

form a minimal system of topological generators of the uniform pro-p-group K_m. Moreover every element $\lambda \in \mathscr{D}(K_m)$ can be uniquely written in the form $\lambda = \sum_{\alpha=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in\mathbf{N}^4} c_\alpha b_m^\alpha$, where

$$b_m^\alpha = (a_m^+ - 1)^{\alpha_1} \cdot (a_m^- - 1)^{\alpha_2} \cdot (u_m^+ - 1)^{\alpha_3} \cdot (u_m^- - 1)^{\alpha_4} \in \mathbf{Z}_p[K_m]$$

and $c_\alpha \in \mathbf{Q}_p$ satisfying $\lim_{|\alpha|\to+\infty} v_p(c_\alpha) + \frac{|\alpha|}{p^h} = +\infty$ for all $h \in \mathbf{N}$ (cf. [11, §V.2.3]). We need to show that, if $\lambda = \sum_{\alpha\in\mathbf{N}^4} c_\alpha b_m^\alpha \in \mathscr{D}(K_m)$ and $f \in M \boxtimes_\omega (1+p^m\mathbf{Z}_p)$, then $\sum_{\alpha\in\mathbf{N}^4} c_\alpha(b_m^\alpha \cdot f)$ converges in $M \boxtimes_\omega (1+p^m\mathbf{Z}_p)$, i.e. that it converges for every valuation $v^{[r_a,r_b]}$, $a < b \in \mathbf{N}$ large enough, and this will be done by bounding the action of the b_m^α on some integral structures that we define below.

Fix $\mathcal{A} \subseteq A$ a model of A, i.e. a $\mathbf{Z}_p$-subalgebra of A topologically of finite type such that $\mathcal{A}[1/p] = A$. For $b \geq 1$, define $\mathscr{E}_{\mathcal{A}}^{\dagger,b}$ to be the p-adic completion of $\mathcal{A}[[T]][\frac{p}{T^{n_b}}]^\wedge$, where $n_b = p^{b-1}(p-1)$ and let $\mathscr{E}_{\mathcal{A}}^{(0,r_b]} = \mathscr{E}_{\mathcal{A}}^{\dagger,b}[\frac{1}{T}]$.

Let m be large enough such that $\delta_i(a) \in \mathcal{A}$ and $v_A(\delta_i(a) - 1) > 1$ for all $a \in 1+p^m\mathbf{Z}_p$, $i = 1, 2$. Let $D_0^{\dagger,b} = \mathscr{E}_{\mathcal{A}}^{\dagger,b}(\delta_1)$, $M_m^{\dagger,b} = (1+T)\varphi^m(D_0^{\dagger,b})$ and $D_0^{(0,r_b]} := \mathscr{E}_{\mathcal{A}}^{(0,r_b]}(\delta_1)$, and observe that we have $(1+T)\varphi^m(D_0^{(0,r_b]}) = \cup_{k\in\mathbf{Z}}(1+T)\varphi^m(T^k D_0^{\dagger,b}) = \cup_{k\in\mathbf{Z}}\varphi^m(T)^k M_m^{\dagger,b}$. Letting $\tau_m := a_m^+ - 1$, by [15, Proposition V.17], we have $\varphi^m(T)^k M_m^{\dagger,b} = \tau_m^k M_m^{\dagger,b}$.

We claim (cf. [11, Lemme V.2.11]) that, for any $g \in K_m$ and $n \in \mathbf{N}$, we have

$$(g-1)^n(\varphi^m(T)^k M_m^{\dagger,b}) \subseteq \varphi^m(T)^{k+n} M_m^{\dagger,b}. \tag{4}$$

This is done in three steps:

- One first deals with the case when $g = a_m^+$. This follows, for instance, by copying the proof of [11, Proposition V.1.14] (cf. also [6, Lemme 1.4(i); Proposition 1.7] or [27, Theorem 3.1.1] for similar estimations in the relative context). Moreover, the estimations made in [11, Proposition V.1.14] show that $M_m^{\dagger,b}$ is a free $\mathscr{E}_{\mathcal{A}}^{\dagger,b}(\Gamma_m)$-module of rank 1 generated by $(1+T) \otimes \delta_1$.

- We now claim that $\iota_{\delta_1,\delta_2}(\tau_m^k M_m^{\dagger,b}) \subseteq \tau_m^k M_m^{\dagger,b}$. Indeed, as we pointed out in the last point, $M_m^{\dagger,b}$ is a free $\mathscr{E}_A^{\dagger,b}(\Gamma_m)$-module of rank 1 generated by $(1+T)\otimes\delta_1$, and the claim follows from the anti-linearity of $\iota_{\delta_1,\delta_2}$ and the fact that w_* fixes $(1+T)$ (since $1+T$ corresponds to the Dirac measure Dir_1 via the Amice transform).

- From the identity $\varphi^m(T)^k M_m^{\dagger,b} = \tau_m^k M_m^{\dagger,b}$ and the second point, we deduce that the module $\varphi^m(T)^k M_m^{\dagger,b}$ is stable by $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and, since $a_m^- = wa_m^+w$ and $u_m^- = wu_m^+w$, one reduces to showing the inclusion (4) claim when $g \in \{a_m^+, u_m^+\}$. For $g = u_m^+$, this is trivial since $u_m^+ - 1$ acts via multiplication by $(1+T)^{p^m} - 1 = \varphi^m(T)$, and the case $g = a_m^+$ was already treated above.

Finally, one shows ([15, Corollaire VI.6]) that Equation (4) implies that, if $\lambda \in \mathscr{D}(K_m)$ and $f \in M \boxtimes_\omega (1+p^m\mathbf{Z}_p)$, then $\lambda \cdot f$ converges in $M \boxtimes_\omega (1+p^m\mathbf{Z}_p)$, thus finishing the proof.

□

3 Cohomology of (φ, Γ)-modules

In this section we recalculate some results of [6] using (φ, Γ)-cohomology. We calculate cohomology groups studied in [14, §5], in preparation to extend the results in loc.cit. to the affinoid setting. We also prove a comparison isomorphism between continuous and locally analytic cohomology, in the spirit of Lazard's results.

3.1 Definitions and preliminaries

Let H be a $\mathbf{Q}_p$-analytic semi-group (e.g. A^+, $\overline{P}^+$, G). Let M be a complete, Hausdorff locally convex A-module (cf. Definition A.5) with the structure of a separately continuous A-linear $\mathscr{D}(H, A)$-module (i.e. M is an object of the category $\mathscr{G}_{H,A}$, cf. Definition A.39). We denote by $H^i_{\mathrm{an}}(H, M)$ the i-th cohomology group of this module (cf. Definition A.48). As is explained in Remark A.51, for all the modules that will appear below, these cohomology groups can also be defined as the cohomology of the complex of pro-analytic cochains (for a detailed introduction to H^*_{an} when $A = L$ is a finite extension of $\mathbf{Q}_p$, we refer the reader to [31] or [23] and [2]). Finally $H^i(H, M)$ will denote continuous (semi-)group cohomology.
The following (semi-)groups will be of constant use in the sequel:

$$U^n = \begin{pmatrix} 1 & p^n\mathbf{Z}_p \\ 0 & 1 \end{pmatrix}, \quad A^0 = \begin{pmatrix} \mathbf{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix}, \quad \Phi^+ = \begin{pmatrix} p^{\mathbf{N}} & 0 \\ 0 & 1 \end{pmatrix}, \quad \overline{U}^n = \left(\begin{smallmatrix} 1 & 0 \\ p^n\mathbf{Z}_p & 1 \end{smallmatrix}\right),$$

where $n \in \mathbf{N}$. We identify $\varphi = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and we have that $A^+ = \Phi^+ \times A^0$. Recall that any $\Delta \in \Phi\Gamma(\mathscr{R}_A)$ is equipped with an action of the semigroup generated by U^n and A^+ by setting $\left(\begin{smallmatrix} p^k a & b \\ 0 & 1 \end{smallmatrix}\right) \cdot z = (1+T)^b \varphi^k(\sigma_a(z))$, $k \in \mathbf{N}$, $a \in \mathbf{Z}_p^\times$, $b \in \mathbf{Z}_p$.

Lemma 3.1. *If $\Delta \in \Phi\Gamma(\mathscr{R}_A)$, then $H^i_{\mathrm{an}}(U^n, \Delta) = 0$ $\forall n \in \mathbf{N}$, if $i = 0, 1$.*

Proof. For $i = 0$, we note that $H^0_{\mathrm{an}}(U^n, \Delta) = \Delta^{(1+T)^{p^n}=1} = 0$. Since the action of U^n is pro-analytic on Δ, the description of the $H^1_{\mathrm{an}}(U^n, \Delta)$ in terms of locally analytic cochains shows that $H^1_{\mathrm{an}}(U^n, \Delta) \subseteq H^1(U^n, \Delta)$. So it is enough to show that this last module vanishes. This follows since $H^1(U^n, \Delta) = \Delta/(1+T)^{p^n} - 1$ and $(1+T)^{p^n} - 1$ is invertible in $\mathscr{R}_A$ (as a rigid function on the unit disc, it only has a finite number of zeroes). ☐

Let Δ be a (φ, Γ)-module over $\mathscr{R}_A$. We construct a natural map

$$\Theta^\Delta : \mathrm{Ext}^1(\mathscr{R}_A, \Delta) \to H^1_{\mathrm{an}}(A^+, \Delta)$$

as follows. Let $\widetilde{\Delta}$ be an extension of $\mathscr{R}_A$ by Δ and let $e \in \widetilde{\Delta}$ be a lifting of $1 \in \mathscr{R}_A$. Then $g \mapsto (g-1)e$, $g \in A^+$, is an analytic 1-cocycle with values in Δ and induces an element of $H^1_{\mathrm{an}}(A^+, \Delta)$ independent of the choice of e.

Proposition 3.2 ([6, Lemme 2.2]). *For any (φ, Γ)-module Δ over $\mathscr{R}_A$, Θ^Δ is an isomorphism.*

Finally, we recall that locally analytic cohomology can be calculated as the invariants of the Lie algebra cohomology (cf. [23, Theorem 2.4] or [14, Lemme 5.6] for the A^+-cohomology of (φ, Γ)-modules, or [44] for a general locally analytic representation), which will be very useful for further calculations.

3.2 Continuous vs. analytic cohomology

In this section we show the following comparison between locally analytic cohomology and continuous group cohomology.

Proposition 3.3. *Let $\delta_1, \delta_2 \colon \mathbf{Q}_p^\times \to A^\times$ be locally analytic characters. If*

$$M \in \left\{ \mathscr{R}_A^+(\delta_1, \delta_2), \mathscr{R}_A^-(\delta_1, \delta_2), \mathscr{R}_A(\delta_1, \delta_2) \right\},$$

then the natural map

$$H^i_{\mathrm{an}}(\overline{P}^+, M) \to H^i(\overline{P}^+, M)$$

is an isomorphism for all $i \geq 0$ (here $H^i(\overline{P}^+, M)$ denotes continuous cohomology).

Proof. From the exact sequence of $\overline{P}^+$-modules

$$0 \to \mathscr{R}_A^+(\delta_1, \delta_2) \to \mathscr{R}_A(\delta_1, \delta_2) \to \mathscr{R}_A^-(\delta_1, \delta_2) \to 0,$$

it suffices to prove the result for $\mathscr{R}_A^+(\delta_1,\delta_2)$ and $\mathscr{R}_A^-(\delta_1,\delta_2)$. So suppose $M\in\mathscr{R}_A^+(\delta_1,\delta_2),\mathscr{R}_A^-(\delta_1,\delta_2)$. Note first that

$$H^j_{\mathrm{an}}(\overline{U}^1,M)\to H^j(\overline{U}^1,M) \tag{5}$$

is an isomorphism. Indeed by [31, Corollary 4.5] and Lemma A.34 the H^* (resp. H^j_{an}) is calculated by taking the $\mathrm{Hom}_{\mathscr{D}(\overline{U}^1)}(-,M)$[23] (resp. $\mathrm{Hom}_{\mathscr{D}(\overline{U}^1),\mathrm{cont}}(-,M)$) of the following resolution of 1

$$\ldots\to 0\to\mathscr{D}(\overline{U}^1)\xrightarrow{\tau-1}\mathscr{D}(\overline{U}^1)\xrightarrow{\varepsilon}1\to 0,$$

where $\tau-1$ denotes multiplication by $\tau-1$ and ε maps τ to 1 (cf. [31, (38)]), and computing the resulting cohomology. As it is explained in [31, §4] (cf. in particular the sentence immediately after the proof of [31, Corollary 4.3]), there is a natural bijection (called κ^q in *loc. cit.*) between the cohomology of both complexes, thus proving that the map of Equation (5) is a bijection.
Since Φ^+ is discrete we have an isomorphism

$$H^k_{\mathrm{an}}(\Phi^+,H^j_{\mathrm{an}}(\overline{U}^1,M))\to H^k(\Phi^+,H^j(\overline{U}^1,M)).$$

The same argument (for proving (5) is an isomorphism) gives an isomorphism

$$H^i_{\mathrm{an}}(A^0,H^k_{\mathrm{an}}(\Phi^+,H^j_{\mathrm{an}}(\overline{U}^1,M)))\to H^i(A^0,H^k(\Phi^+,H^j(\overline{U}^1,M))).$$

Finally for $?\in\{\mathrm{an},\emptyset\}$ we claim that we have spectral sequences

$$H^i_?(A^+,H^j_?(\overline{U}^1,M))\implies H^{i+j}_?(\overline{P}^+,M) \tag{6}$$

and for $j\geq 0$ fixed

$$H^i_?(A^0,H^k_?(\Phi^+,H^j_?(\overline{U}^1,M)))\implies H^{i+k}_?(A^+,H^j_?(\overline{U}^1,M)). \tag{7}$$

The main point to observe is that $H^j_{\mathrm{an}}(\overline{U}^1,M)$ (and therefore also $H^j(\overline{U}^1,M)$) admits the structure of a separately continuous $\mathscr{D}(A^+,A)$-module[24]. By Lemma A.34, it suffices to show that $H^j_{\mathrm{an}}(\overline{U}^1,M)$ admits the structure of a separately continuous $\mathscr{D}(A^+,\mathbf{Q}_p)$-module. This follows from the proof of [31, Theorem 6.8]. The two spectral sequences in (7) then follow from a similar discussion to that of [13, A.1] (one simply replaces L in loc.cit. by A). For $?=\emptyset$, the spectral sequence (6) follows from imitating the proof of [3, Chapter IX, Theorem 4.3], while for $?=\mathrm{an}$, one imitates the proof of [31, Theorem 6.8]. Spectral sequences (6) and (7) now give the result.

□

Remark 3.4. In the setting of Proposition 3.3, one cannot apply Lazard's classical result [32, Théorème 2.3.10] because M is not of finite type over A. Note also that a similar proof yields isomorphisms $H^i_{\mathrm{an}}(A^+,M)\xrightarrow{\sim}H^i(A^+,M)$ for all $i\geq 0$.

[23] Here $\mathrm{Hom}_{\mathscr{D}(\overline{U}^1)}(-,-)$ denotes morphisms in the category of abstract $\mathscr{D}(\overline{U}^1)$-modules.

[24] We do not claim that $H^j_{\mathrm{an}}(\overline{U}^1,M)$ is Hausdorff or complete.

3.3 The cohomology of A^+

We state here some results describing different relative cohomology groups and we collect some lemmas that will be useful later. This is an adaptation of the methods of [14] to the relative setting, using ideas from [6]. For the sake of brevity, and since many results are nowadays classical, we omit most of the proofs.

Proposition 3.5. *Let $\delta : \mathbf{Q}_p^\times \to A^\times$ be a locally analytic character.*

1. *Let $M = \mathscr{R}_A^- \otimes \delta$. Then*
   - *$H^0(A^+, M) = 0$;*
   - *If δ is pointwise never of the form x^i for any $i \geq 0$, then $H^1(A^+, M)$ is a free A-module of rank 1;*
   - *$H^2(A^+, M) = 0$ if and only if δ is pointwise never of the form x^i for any $i \geq 0$.*
2. *Let $M = \mathscr{R}_A^+ \otimes \delta$. If δ is pointwise never of the form x^{-i} for any $i \geq 0$, then $H^i(A^+, M) = 0$ for $i \in \{0, 1, 2\}$.*
3. *Let $M = \mathscr{R}_A \otimes \delta$. If δ is pointwise never of the form χx^i nor of the form x^{-i} for any $i \geq 0$, then*
   - *$H^0(A^+, \mathscr{R}_A \otimes \delta) = 0$;*
   - *$H^1(A^+, \mathscr{R}_A \otimes \delta)$ is a free A-module of rank 1;*
   - *$H^2(A^+, \mathscr{R}_A \otimes \delta) = 0$.*

Moreover, in all cases above, the A^+-cohomology is concentrated in degrees $[0, 2]$ and $H^i(A^+, M)$ is an A-module of finite type.

Proof. As we pointed out, these results can be proved by adapting the methods of [14, §5] to the relative setting, with the aid of the results of [6]. Nevertheless, all of the results stated here can be found in [6]. ☐

Remark 3.6.

- For $M \in \{\mathscr{R}_A^+, \mathscr{R}_A^-, \mathscr{R}_A\}$, the A-modules $H^i(A^+, M \otimes \delta)$ are finite (cf. [6]).
- $H^2(A^+, \mathscr{R}_A \otimes \delta) = 0$ if and only if δ is pointwise never of the form χx^i, $i \in \mathbf{N}$. Indeed, this is a necessary condition by Proposition 3.5(1) since $H^2(A^+, \mathscr{R}_A \otimes \delta)$ surjects to $H^2(A^+, \mathscr{R}_A^- \otimes \delta)$. For the converse first note that if δ is never of the form χx^i nor x^{-i} for any $i \in \mathbf{N}$, then $H^2(A^+, \mathscr{R}_A \otimes \delta) = 0$ by Proposition 3.5. On the other hand, if δ reduces to x^{-i} for some $i \geq 0$ at some point of $\mathrm{Sp}(A)$, we use the following argument to reduce to the case of a point (A a finite extension of $\mathbf{Q}_p$). The finiteness of the A-module $H^2(A^+, \mathscr{R}_A \otimes \delta)$, the vanishing of $H^3(A^+, \mathscr{R}_A \otimes \delta)$, the

fact that $\mathscr{R}_A$ is a flat A-module (cf. Lemma 2.1) and the Tor-spectral sequence

$$\mathrm{Tor}_{-p}(H^q(A^+, \mathscr{R}_A \otimes \delta), A/\mathfrak{m}) \Rightarrow H^{p+q}(A^+, \mathscr{R}_{A/\mathfrak{m}} \otimes \delta)$$

show that $H^2(A^+, \mathscr{R}_A \otimes \delta) \otimes A/\mathfrak{m} = H^2(A^+, \mathscr{R}_{A/\mathfrak{m}} \otimes \delta)$ for every maximal ideal $\mathfrak{m} \subseteq A$. Since $H^2(A^+, \mathscr{R}_{A/\mathfrak{m}} \otimes \delta) = 0$ (cf. [14, Théorème 5.16]), we can conclude that $H^2(A^+, \mathscr{R}_A \otimes \delta) = 0$ by Nakayama's lemma.

4 Relative cohomology

Over the next few sections we prove an isomorphism between the $\overline{P}^+$-cohomology and the A^+-cohomology with coefficients in $\mathscr{R}_A(\delta_1, \delta_2)$ assuming $\delta_1\delta_2^{-1}$ is regular. This is a generalization of a result of Colmez, who proves it for the case where A is a finite extension of $\mathbf{Q}_p$ and we indeed reduce the general result to that case using some arguments on derived categories inspired by [27].

4.1 Formalism of derived categories

Fix a noetherian ring A and let $\mathcal{D}^-(A)$ denote the derived category of A-modules bounded above. We begin by recalling the notion of a pseudo-coherent complex and some easy characterisations. For a detailed explanation we refer the reader to [42, Tag 064N] .

Definition 4.1.

1. An object $K^\bullet$ of $\mathcal{D}^-(A)$ is pseudo-coherent if it is quasi-isomorphic to a bounded above complex of finite free A-modules. We denote by $\mathcal{D}^-_{\mathrm{pc}}(A) \subseteq \mathcal{D}^-(A)$ the full subcategory of pseudo-coherent objects of $\mathcal{D}^-(A)$.

2. An A-module M is called pseudo-coherent if $M[0] \in \mathcal{D}^-_{\mathrm{pc}}(A)$.

We have the following simple Lemma detecting when a module is pseudo-coherent.

Lemma 4.2. *An A-module M is pseudo-coherent iff there exists an infinite resolution*

$$\cdots \to A^{\oplus n_1} \to A^{\oplus n_0} \to M \to 0$$

Since A is noetherian, Lemma 4.2 can be further strenghened to the following.

Lemma 4.3. *An A-module M is pseudo-coherent iff it is finite.*

The following Lemma allows us to use induction-type arguments when trying to prove results concerning pseudo-coherent complexes.

Lemma 4.4. *Let $K^\bullet \in \mathcal{D}^-(A)$. The following are equivalent*

1. $K^\bullet \in \mathcal{D}^-_{\mathrm{pc}}(A)$.

2. *For every integer m, there exists a bounded complex $E^\bullet$ (depending on m) of finite free A-modules and a morphism $\alpha : E^\bullet \to K^\bullet$ such that $H^i(\alpha)$ is an isomorphism for $i > m$ and $H^m(\alpha)$ is surjective.*

Remark 4.5. If a complex in $\mathcal{D}^-_{\mathrm{pc}}(A)$ has trivial cohomology in degrees strictly greater than b, then it is quasi-isomorphic to a complex $P^\bullet \in \mathcal{D}^-_{\mathrm{pc}}(A)$ with $P^i = 0\ \forall i \geq b+1$ and each P^j is finite free $\forall j \in \mathbf{Z}$. Moreover, $\mathcal{D}^-_{\mathrm{pc}}(A)$ is stable by extensions.

Since A is noetherian, we have the following simple criterion for detecting whether an object in $\mathcal{D}^-(A)$ is pseudo-coherent.

Proposition 4.6. *An object $K^\bullet \in \mathcal{D}^-(A)$ is pseudo-coherent iff $H^i(K^\bullet)$ is a finite A-module for all i.*

4.2 The Koszul complex

In this subsection we compute a complex that computes $\overline{P}^+$-cohomology, cf. Lemma 4.8. This is an analogue of the Koszul complex in a non-commutative setting. The reader can compare this with the complex constructed in [45, §1.5.1] which calculates Galois cohomology. In loc.cit. the construction is somewhat simpler because of the non-triviality of the center (of the group under consideration).

Let a be a topological generator of $\mathbf{Z}_p^\times$ and note

$$\gamma = \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \varphi = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \tau = \left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right),$$

which are topological generators of the (semi-)group $\overline{P}^+$ satisfying the relations

$$\varphi\gamma = \gamma\varphi,$$

$$\gamma\tau = \tau^{a^{-1}}\gamma,$$

$$\varphi\tau^p = \tau\varphi,$$

giving a finite presentation of it (proof: using those relations we can write any other relation as $\varphi^x\tau^y\gamma^z = 1$ which in turn implies $x = y = z = 0$).

Let M be a $\overline{P}^+$-module such that the action of $\overline{P}^+$ extends to an action of the Iwasawa algebra $\mathbf{Z}_p[[\overline{P}^+]]$ and define

$$\mathscr{C}_{\tau,\varphi,\gamma} : 0 \to M \xrightarrow{X} M \oplus M \oplus M \xrightarrow{Y} M \oplus M \oplus M \xrightarrow{Z} M \to 0 \tag{8}$$

where the arrows are defined as

$$\begin{aligned} X(x) &= ((1-\tau)x, (1-\varphi)x, (\gamma-1)x) \\ Y(x,y,z) &= ((1-\varphi\delta_p)x + (\tau-1)y, (\gamma\delta_a - 1)x + (\tau-1)z, (\gamma-1)y + (\varphi-1)z) \\ Z(x,y,z) &= (\gamma\delta_a - 1)x + (\varphi\delta_p - 1)y + (1-\tau)z \end{aligned}$$

and where we have noted,

$$\delta_p = \frac{1-\tau^p}{1-\tau} = 1+\tau+\ldots+\tau^{p-1},$$

and, for $a \in \mathbf{Z}_p^\times, b \in \mathbf{Z}_p$,

$$\frac{\tau^{ba}-1}{\tau^a-1} = \sum_{n\geq 1}\binom{ba}{n}(\tau^a-1)^{n-1} \in \mathbf{Z}_p[[\tau-1]],$$

$$\delta_a = \frac{\tau^a-1}{\tau-1},$$

which is a well defined element since, as $\tau^{p^n} \to 1$ as n tends to $+\infty$, $\tau-1$ is topologically nilpotent in the Iwasawa algebra $\mathbf{Z}_p[[\tau-1]] = \mathbf{Z}_p[[\overline{U}]] \subseteq \mathbf{Z}_p[[\overline{P}^+]]$. The construction of $\mathscr{C}_{\tau,\varphi,\gamma}$ is obtained from taking successive fibers of smaller complexes. Define

$$\mathscr{C}_\tau : 0 \to M \xrightarrow{D} M \to 0 \tag{9}$$

where

$$D(x) := (\tau-1)x$$

and

$$\mathscr{C}_{\tau,\varphi} : 0 \to M \xrightarrow{E} M \oplus M \xrightarrow{F} M \to 0 \tag{10}$$

where

$$\begin{aligned} E(x) &= ((\tau-1)x, (\varphi-1)x) \\ F(x,y) &= (\varphi\delta_p - 1)x + (1-\tau)y \end{aligned}$$

We now define morphisms between the complexes. We note by $[\varphi-1] : \mathscr{C}_\tau \to \mathscr{C}_\tau$ the morphism:

$$\begin{array}{ccccccc} \mathscr{C}_\tau : 0 & \longrightarrow & M & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi-1} & & \downarrow{\scriptstyle \varphi\delta_p-1} & & \\ \mathscr{C}_\tau : 0 & \longrightarrow & M & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

and $[\gamma-1] : \mathscr{C}_{\tau,\varphi} \to \mathscr{C}_{\tau,\varphi}$ the morphism:

$$\begin{array}{ccccccccc} \mathscr{C}_{\tau,\varphi} : 0 & \longrightarrow & M & \longrightarrow & M \oplus M & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma-1} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \gamma\delta_a-1} & & \\ \mathscr{C}_{\tau,\varphi} : 0 & \longrightarrow & M & \longrightarrow & M \oplus M & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where $s(x,y) = ((\gamma\delta_a - 1)x, (\gamma-1)y)$

Lemma 4.7. *There are distinguished triangles*

$$\mathscr{C}_{\tau,\varphi} \to \mathscr{C}_{\tau} \xrightarrow{[\varphi-1]} \mathscr{C}_{\tau}$$

and

$$\mathscr{C}_{\tau,\varphi,\gamma} \to \mathscr{C}_{\tau,\varphi} \xrightarrow{[\gamma-1]} \mathscr{C}_{\tau,\varphi}$$

in $\mathcal{D}^-(A)$.

Proof. This is evident from the definition of the cone of a morphism in $\mathcal{D}^-(A)$ and the relations $\varphi\delta_p \cdot \gamma\delta_a = \gamma\delta_a \cdot \varphi\delta_p$, $(\gamma\delta_a - 1)(\tau - 1) = (\tau - 1)(\gamma - 1)$ and $(\varphi\delta_p - 1)(\tau - 1) = (\tau - 1)(\varphi - 1)$. ☐

We now show that $C_{\tau,\varphi,\gamma}$ computes $\overline{P}^+$-cohomology. Recall that the Herr complex

$$\mathscr{C}_{\varphi,\gamma} : 0 \to M \xrightarrow{E'} M \oplus M \xrightarrow{F'} M \to 0,$$

where

$$E'(x) = ((1-\varphi)x, (\gamma-1)x)$$
$$F'(x,y) = (\gamma-1)x + (\varphi-1)y,$$

calculates the A^+-cohomology of M. There is an obvious restriction morphism $\mathscr{C}_{\tau,\varphi,\gamma} \to \mathscr{C}_{\varphi,\gamma}$ whose kernel (as a morphism in the abelian category of chain complexes) is

$$\mathscr{C}^{\text{twist}}_{\varphi,\gamma} : 0 \to M \xrightarrow{E''} M \oplus M \xrightarrow{F''} M \to 0$$

where

$$E''(x) = ((1-\varphi\delta_p)x, (\gamma\delta_a - 1)x)$$
$$F''(x,y) = ((\gamma\delta_a - 1)x + (\varphi\delta_p - 1)y$$

Lemma 4.8. *The complex* $\mathscr{C}_{\tau,\varphi,\gamma}$ *calculates the* $\overline{P}^+$*-cohomology groups. That is* $H^i(\mathscr{C}_{\tau,\varphi,\gamma}) = H^i(\overline{P}^+, M)$.

Proof. This is just a reinterpretation of the Hochschild-Serre spectral sequence. We have a distinguished triangle

$$\mathscr{C}_{\tau,\varphi,\gamma} \to \mathscr{C}_{\varphi,\gamma} \xrightarrow{1-\tau} \mathscr{C}^{\text{twist}}_{\varphi,\gamma} \tag{11}$$

in the derived category $\mathcal{D}^-(A)$, where the morphism

$$1-\tau \colon \mathscr{C}_{\varphi,\gamma} \to \mathscr{C}^{\text{twist}}_{\varphi,\gamma}$$

is componentwise just $1-\tau$. Let $\overline{U} = \overline{U}^1 = \begin{pmatrix} 1 & 0 \\ p\mathbf{Z}_p & 1 \end{pmatrix}$ so that $\overline{P}^+ = \overline{U} \rtimes A^+$. For a semi-group G we denote by R^G the derived functor of $(-)^G$. For M a $\overline{P}^+$-module, we claim that[25]

$$R^{\overline{U}}(M) = (0 \to M \xrightarrow{1-\tau} M^* \to 0)$$

where M^* is isomorphic to M as $\overline{U}$-modules, but is equipped with a *twisted* (φ, γ)-action (which we denote by $(\widetilde{\varphi}, \widetilde{\gamma})$):

$$\widetilde{\varphi} \cdot m := \varphi\delta_p \cdot m \text{ and } \widetilde{\gamma} \cdot m := \gamma\delta_a \cdot m.$$

First note that $1-\tau\colon M \to M^*$ is indeed a morphism of $\overline{P}^+$-modules (this follows from the relations $(\gamma\delta_a - 1)(\tau-1) = (\tau-1)(\gamma-1)$ and $(\varphi\delta_p - 1)(\tau-1) = (\tau-1)(\varphi-1)$). Now $H^1(\overline{U}, M)$ is equipped with a natural (φ, γ)-action (which we denote by (φ', γ')):

$$\varphi' \cdot c_\tau := \varphi \cdot c_{\tau^p} \text{ and } \gamma' \cdot c_\tau := \gamma \cdot c_{\tau^a},$$

where c_τ is the value of the 1-cocycle c with $[c] \in H^1(\overline{U}, M)$, at τ. To prove the claim it suffices to show that $\varphi \cdot c_{\tau^p} = \varphi\delta_p \cdot c_\tau$ and $\gamma \cdot c_{\tau^a} = \gamma\delta_a \cdot c_\tau$. However these follow from the fact that c is a 1-cocycle.
Thus, by the Hochschild-Serre spectral sequence, we have

$$R^{\overline{P}^+}(M) = R^{A^+}(0 \to M \xrightarrow{1-\tau} M^* \to 0).$$

Therefore applying R^{A^+} to the distinguished triangle

$$(0 \to M \xrightarrow{1-\tau} M^* \to 0) \to M \xrightarrow{1-\tau} M^*$$

gives the distinguished triangle

$$R^{\overline{P}^+}(M) \to R^{A^+}(M) \xrightarrow{1-\tau} R^{A^+}(M^*) \tag{12}$$

and it is easy to see that $R^{A^+}(M) = \mathscr{C}_{\varphi,\gamma}$ and $R^{A^+}(M^*) = \mathscr{C}^{\text{twist}}_{\varphi,\gamma}$. The result now follows from comparing the triangles (11) and (12). ☐

4.3 Finiteness of cohomology

We end this chapter by showing that the cohomology groups

$$H^i(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2))$$

are finite-type A-modules. The idea is to reduce the problem to finiteness of A^+-cohomology (cf. Proposition 3.5) and finiteness of *twisted* A^+-cohomology.

[25] Here $R^{\overline{U}}$ is viewed as a function from $\mathcal{D}^+(\overline{P}^+ - \text{Mod})$ to $\mathcal{D}^+(A^+ - \text{Mod})$.

Remark 4.9. We will freely use the identifications provided by Proposition 2.2 that we make explicit now for the commodity of the reader. The module $\mathscr{R}_A^+(\delta_1,\delta_2) = (\mathscr{R}_A^+(\delta_1) \boxtimes_\omega \mathbf{Z}_p) \otimes \delta_2^{-1}$ is isomorphic, as an A-module, to $\mathscr{R}_A^+$ and, by the Amice transform, to $\mathscr{D}(\mathbf{Z}_p, A)$. By Lemma 2.14, this isomorphism becomes $\overline{P}^+$-equivariant when we equip this last space with the action given by Equation (3) (twisted by the character $\omega\delta_2^{-1} = \delta_1\chi^{-1}$), namely

$$\int_{\mathbf{Z}_p} \phi \cdot \left(\begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \star \mu \right) = \delta_1\delta_2^{-1}(a) \int_{\mathbf{Z}_p} \delta(cx+1)\phi\left(\frac{ax}{cx+1}\right) \cdot \mu(x), \quad \left(\begin{smallmatrix} a & 0 \\ c & 1 \end{smallmatrix}\right) \in \overline{P}^+.$$

In the same way, we identify $\mathscr{R}_A^-(\delta_1,\delta_2)$ with $\mathrm{LA}(\mathbf{Z}_p, A)$ equipped with the action of $\overline{P}^+$ given by Equation (2) (twisted by the character δ_2^{-1}):

$$\left(\begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \star \phi \right)(x) = \delta(a-cx)\phi\left(\frac{x}{a-cx}\right), \quad \left(\begin{smallmatrix} a & 0 \\ c & 1 \end{smallmatrix}\right) \in \overline{P}^+.$$

Restricted to A^+, this is Remark 2.16.

The first thing to note is that the complexes $\mathscr{C}_{\tau,\varphi,\gamma}$ are well defined for $M \in \{\mathscr{R}_A^+(\delta), \mathscr{R}_A(\delta), \mathscr{R}_A^-(\delta)\}$, which is a consequence of the following lemma.

Lemma 4.10. *Let $M \in \{\mathscr{R}_A^-(\delta_1,\delta_2), \mathscr{R}_A(\delta_1,\delta_2), \mathscr{R}_A^+(\delta_1,\delta_2)\}$. The action of $\overline{P}^+$ extends by continuity to an action of the distribution algebra $\mathscr{D}(\overline{P}^+, A)$. In particular, M is equipped with an action of the Iwasawa algebra $\mathbf{Z}_p[[\overline{P}^+]]$.*

Proof. For the proof of this lemma, we use some facts of §2.4. For $M \in \{\mathscr{R}_A^+(\delta_1,\delta_2), \mathscr{R}_A^-(\delta_1,\delta_2)\}$, the result is a consequence of the isomorphisms $\mathscr{R}_A^+(\delta_1) \boxtimes_\omega \mathbf{P}^1 \cong B_A(\delta_2,\delta_1)^* \otimes \omega$ and $\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1 \cong B_A(\delta_1,\delta_2)$ of Lemma 2.14, the fact that the locally analytic principal series are equipped with an action of the distribution algebra $\mathscr{D}(G,A)$ and the fact that, since $\overline{P}^+$ stabilizes $\mathbf{Z}_p$, then $\mathscr{R}_A^{(\pm)}(\delta_1,\delta_2) = (\mathscr{R}_A^{(\pm)}(\delta_1) \boxtimes_\omega \mathbf{Z}_p) \otimes \delta_2^{-1}$ inherits an action of the distribution algebra $\mathscr{D}(\overline{P}^+, A)$, and in particular an action of the Iwasawa algebra $\mathbf{Z}_p[[\overline{P}^+]]$. For $M = \mathscr{R}_A(\delta_1,\delta_2)$, the result follows by the same arguments noting that, since $\mathscr{R}(\delta_1) \boxtimes_\omega \mathbf{P}^1$ is an extension of $\mathscr{R}_A^-(\delta) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A^+(\delta) \boxtimes_\omega \mathbf{P}^1$ in the category of separately continuous $\mathscr{D}(G,A)$-modules, it is also equipped with an action of $\mathscr{D}(G,A)$. □

The main theorem of this subsection is the following.

Theorem 4.11. *If $M = \mathscr{R}_A(\delta_1,\delta_2)$ then $\mathscr{C}_{\tau,\varphi,\gamma} \in \mathcal{D}^-_{\mathrm{pc}}(A)$. In particular, the A-modules $H^i(\overline{P}^+, \mathscr{R}(\delta_1,\delta_2))$ are finite.*

Proof. Recall that we have the distinguished triangle

$$\mathscr{C}_{\tau,\varphi,\gamma} \to \mathscr{C}_{\varphi,\gamma} \to \mathscr{C}^{\mathrm{twist}}_{\varphi,\gamma}$$

in the derived category $\mathcal{D}^-(A)$. By [27, Theorem 4.4.2], $\mathscr{C}_{\varphi,\gamma} \in \mathcal{D}^-_{\mathrm{pc}}(A)$. Thus, by Lemma 4.8, to prove the first assertion it is enough to show $\mathscr{C}^{\mathrm{twist}}_{\varphi,\gamma} \in \mathcal{D}^-_{\mathrm{pc}}(A)$, which now follows from Lemma 4.12. The last assertion follows from Lemma 4.8 and Proposition 4.6. $\square$

Lemma 4.12. *For $M = \mathscr{R}(\delta_1, \delta_2)$, the A-modules $H^i(\mathscr{C}^{\mathrm{twist}}_{\varphi,\gamma})$ are finite.*

Proof. To prove this lemma, we still proceed by a dévissage argument. We define a complex

$$\mathscr{C}_{\varphi\delta_p} : 0 \longrightarrow M \xrightarrow{1-\varphi\delta_p} M \longrightarrow 0$$

and we observe that we have a distinguished triangle

$$\mathscr{C}^{\mathrm{twist}}_{\varphi,\gamma} \to \mathscr{C}_{\varphi\delta_p} \xrightarrow{1-\gamma\delta_a} \mathscr{C}_{\varphi\delta_p}.$$

Moreover, by taking long exact sequences associated to the short exact sequence $0 \to \mathscr{R}^+_A(\delta_1, \delta_2) \to \mathscr{R}_A(\delta_1, \delta_2) \to \mathscr{R}^-_A(\delta_1, \delta_2) \to 0$, it is enough to show finiteness for $\mathscr{R}^+_A(\delta_1, \delta_2)$ and $\mathscr{R}^-_A(\delta_1, \delta_2)$. The lemma follows from 4.13. $\square$

Lemma 4.13. *For $M \in \{\mathscr{R}^+(\delta_1, \delta_2), \mathscr{R}^-(\delta_1, \delta_2)\}$, the A-modules $H^i(\mathscr{C}^{\mathrm{twist}}_{\varphi,\gamma})$ are finite.*

Proof. The case of $\mathscr{R}^+(\delta_1, \delta_2)$ follows directly from lemma 4.14 below, which shows that the cohomology of the complex $\mathscr{C}_{\varphi\delta_p}$ is already of finite type.
For $\mathscr{R}^-(\delta_1, \delta_2)$, the long exact sequence associated to the triangle $\mathscr{C}^{\mathrm{twist}}_{\varphi,\gamma} \to \mathscr{C}_{\varphi\delta_p} \xrightarrow{1-\gamma\delta_a} \mathscr{C}_{\varphi\delta_p}$ yields

$$\begin{aligned} 0 &\to H^0(\mathscr{C}^{\mathrm{twist}}_{\varphi,\gamma}) \to H^0(\mathscr{C}_{\varphi\delta_p}) \xrightarrow{1-\gamma\delta_a} H^0(\mathscr{C}_{\varphi\delta_p}) \\ &\to H^1(\mathscr{C}^{\mathrm{twist}}_{\varphi,\gamma}) \to H^1(\mathscr{C}_{\varphi\delta_p}) \xrightarrow{1-\gamma\delta_a} H^1(\mathscr{C}_{\varphi\delta_p}) \\ &\to H^2(\mathscr{C}^{\mathrm{twist}}_{\varphi,\gamma}) \to 0, \end{aligned}$$

and the result follows then from lemmas 4.15 and 4.16 $\square$

Lemma 4.14. *The operator $1 - \varphi\delta_p : \mathscr{R}^+_A(\delta_1, \delta_2) \to \mathscr{R}^+_A(\delta_1, \delta_2)$ has finite kernel and cokernel.*

Proof. Let N be big enough such that $|\delta(p)p^N| < 1$. We show that $1 - \varphi\delta_p : T^N\mathscr{R}^+_A(\delta_1, \delta_2) \to T^N\mathscr{R}^+_A(\delta_1, \delta_2)$ is bijective. For that, we construct an inverse of this operator by proving that $\sum_{k\geq 0}(\varphi\delta_p)^k$ converges. Observe that $(\varphi\delta_p)^k = \varphi^k\delta_{p^k}$ and that the operator $\delta_{p^k} = \frac{1-\tau^{p^k}}{1-\tau}$ is bounded independently of k. The result follows now by the arguments of the proof of [6, Lemme 2.9.(ii)]. $\square$

Lemma 4.15.

- *The operator* $1-\varphi\delta_p : \mathscr{R}_A^-(\delta_1,\delta_2) \to \mathscr{R}_A^-(\delta_1,\delta_2)$ *is injective.*
- *If* $N \geq 0$ *is big enough, then* $\mathscr{R}_A^-(\delta_1,\delta_2) = (1-\varphi\delta_p)\mathscr{R}_A^-(\delta_1,\delta_2) + \mathrm{LA}(\mathbf{Z}_p^\times, A) + \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, A)$.

Proof. For the first point, if $(1-\varphi\delta_p)f = 0$ then $\varphi\delta_p f = f$ and, applying this and the identity $(\varphi\delta_p)^n = \varphi^n\delta_{p^n}$ successively [26], we get $\varphi^n\delta_{p^n} f = f$ and so f is supported on $p^n\mathbf{Z}_p$ for all $n \geq 0$ and hence vanishes everywhere.
The second assertion is an easy combination of the arguments of the proofs of [6, Lemme 2.9(iii)] and [14, Lemme 5.9]. □

Lemma 4.16. *The operator* $1-\gamma\delta_a : \mathrm{LA}(\mathbf{Z}_p^\times, A) \to \mathrm{LA}(\mathbf{Z}_p^\times, A)$ *has finite kernel and cokernel (as* A*-modules)*

Proof. For the sake of brevity write $M = \mathrm{LA}(\mathbf{Z}_p^\times, A)$. We have a morphism of complexes (in the abelian category of chain complexes)

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{1-\gamma} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1-\tau} & & \downarrow{\scriptstyle 1-\tau} & & \\ 0 & \longrightarrow & M & \xrightarrow{1-\gamma\delta_a} & M & \longrightarrow & 0 \end{array}$$

Note that the cokernel of this morphism of complexes vanishes by Lemma 5.10 (the same proof carries over with L replaced by A). Since the A^0-cohomology of M is finite (cf. [14, Proposition 5.12(ii)], the same proof carries over for general A), it suffices to show that $M^{\tau=1}$ is a finite A-module. Take $f \in M^{\tau=1}$. Then by definition of the action of τ on $\mathscr{R}_A^-(\delta_1,\delta_2)$ we have

$$f(x) = \delta(1-px) f\left(\frac{x}{1-px}\right).$$

Repeating this procedure we see that the value of $f(x)$ determines the value of $f\left(\frac{x}{1-kpx}\right)$ for all $k \in \mathbf{Z}$. Now $1-p\mathbf{Z}$ is dense in $1-p\mathbf{Z}_p$ and so $(1-p\mathbf{Z})^{-1}$ is dense in $1-p\mathbf{Z}_p$. By continuity of f, this implies that the values $f(1)$, $f(2)$, $\ldots$, $f(p-1)$ determine f completely. This proves the result.

□

5 The $\overline{P}^+$-cohomology

In this section we fix a finite extension L of $\mathbf{Q}_p$, two locally analytic characters $\delta_1, \delta_2 : \mathbf{Q}_p^\times \to L^\times$ and we consider the modules $\mathscr{R}_L^+(\delta_1,\delta_2), \mathscr{R}_L(\delta_1,\delta_2)$ and $\mathscr{R}_L^-(\delta_1,\delta_2)$ (for brevity we will omit the subscript L). We systematically

[26] We denote $\delta_{p^n} = \frac{1-\tau^{p^n}}{1-\tau}$.

calculate the $\overline{P}^+$-cohomology groups of these modules. This will be essential in comparing the relative $\overline{P}^+$ and A^+-cohomology by reducing the problem to the case over a point (cf. Theorem 6.8, which is the main technical result of this article), where the calculation of some higher cohomology groups turns out to be essential for our arguments. For the commodity of the reader, the main results of this section can be summarized as follows:

Proposition 5.1.

- *Let* $M_+ = \mathscr{R}^+(\delta_1, \delta_2)$.

  1. *If* $\delta_1\delta_2^{-1} \notin \{x^{-i}, i \in \mathbf{N}\}$, *then* $H^j(\overline{P}^+, M_+) = 0$ *for all* j.
  2. *If* $\delta_1\delta_2^{-1} = \mathbf{1}_{\mathbf{Q}_p^\times}$, *then* $\dim_L H^j(\overline{P}^+, M_+) = 1, 2, 1, 0$ *for* $j = 0, 1, 2, 3$.
  3. *If* $\delta_1\delta_2^{-1} = x^{-i}$, $i \geq 1$, *then* $\dim_L H^j(\overline{P}^+, M_+) = 1, 3, 3, 1$ *for* $j = 0, 1, 2, 3$.

- *Let* $M_- = \mathscr{R}^-(\delta_1, \delta_2)$.

  1. *If* $\delta_1\delta_2^{-1} \notin \{\chi x^i, i \in \mathbf{N}\}$, *then* $\dim_L H^j(\overline{P}^+, M_-) = 0, 1, 1, 0$ *for* $j = 0, 1, 2, 3$.
  2. *If* $\delta_1\delta_2^{-1} = \chi x^i$, $i \in \mathbf{N}$, *then* $\dim_L H^j(\overline{P}^+, M_-) = 0, 2, 3, 1$ *for* $j = 0, 1, 2, 3$.

- *Let* $M = \mathscr{R}(\delta_1, \delta_2)$.

  1. *If* $\delta_1\delta_2^{-1} \notin \{x^{-i}, i \in \mathbf{N}\} \cup \{\chi x^i, i \in \mathbf{N}\}$, *then* $\dim_L H^j(\overline{P}^+, M) = 0, 1, 1, 0$ *for* $j = 0, 1, 2, 3$.
  2. *If* $\delta_1\delta_2^{-1} = \mathbf{1}_{\mathbf{Q}_p^\times}$, *then* $\dim_L H^j(\overline{P}^+, M) = 1, 2, 1, 0$ *for* $j = 0, 1, 2, 3$.
  3. *If* $\delta_1\delta_2^{-1} = x^{-i}, i \geq 1$, *then* $\dim_L H^j(\overline{P}^+, M) = 1, 3, 2, 0$ *for* $j = 0, 1, 2, 3$.
  4. *If* $\delta_1\delta_2^{-1} = \chi x^i, i \in \mathbf{N}$, *then* $\dim_L H^j(\overline{P}^+, M) = 0, 2, 3, 1$ *for* $j = 0, 1, 2, 3$.

As a corollary from the results of this proposition, we will deduce the following.

Proposition 5.2. *The restriction morphism*

$$H^1(\overline{P}^+, \mathscr{R}(\delta_1, \delta_2)) \to H^1(A^+, \mathscr{R}(\delta_1, \delta_2))$$

is surjective. Moreover if $\delta_1\delta_2^{-1} \notin \{x^{-i}, i \geq 1\}$, *then it is an isomorphism.*

5.1 The Lie algebra complex

We note

$$a^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ a^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ u^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ u^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

the usual generators of the Lie algebra $\mathfrak{gl}_2$ of GL_2. We note that $[a^+, u^-] = -u^-$ and $p\varphi u^- = u^-\varphi$.
Denote by $H^i_{\mathrm{Lie}}(\overline{P}^+, M)$ the cohomology groups of the complex

$$\mathscr{C}_{u^-,\varphi,a^+} : 0 \to M \xrightarrow{X'} M \oplus M \oplus M \xrightarrow{Y'} M \oplus M \oplus M \xrightarrow{Z'} M \to 0 \quad (13)$$

where

$$X'(x) = ((\varphi - 1)x, a^+x, u^-x)$$
$$Y'(x, y, z) = (a^+x - (\varphi - 1)y, u^-y - (a^+ + 1)z, (p\varphi - 1)z - u^-x)$$
$$Z'(x, y, z) = u^-x + (p\varphi - 1)y + (a^+ + 1)z$$

Let $\tilde{P} := \begin{pmatrix} \mathbf{Z}_p^\times & 0 \\ p\mathbf{Z}_p & 1 \end{pmatrix}$. Note that $\tilde{P}$ is a subgroup of $\overline{P}^+$.

Lemma 5.3. *If $M = \mathscr{R}^-(\delta_1, \delta_2)$, the natural map*

$$H^i(\overline{P}^+, M) \to H^0(\tilde{P}, H^i_{\mathrm{Lie}}(\overline{P}^+, M))$$

is an isomorphism.

Proof. The same proof as Lemma 4.8 shows that there is a spectral sequence

$$H^i_{\mathrm{Lie}}(A^+, H^j_{\mathrm{Lie}}(\overline{U}, M)) \Rightarrow H^{i+j}_{\mathrm{Lie}}(\overline{P}^+, M)$$

where $H^i_{\mathrm{Lie}}(\overline{U}, M)$ is defined to be the cohomology of the complex

$$0 \to M \xrightarrow{u^-} M \to 0.$$

For the definition of $H^j_{\mathrm{Lie}}(A^+, -)$, cf. [14, §5.2]. The result now follows from [44, Corollary 21] by taking $\tilde{P}$-invariants on both sides. □

Remark 5.4. For future calculations, we need to render explicit the action of $\tilde{P}$ on the different Lie algebra cohomology groups. Recall that this group acts naturally on the module and by its adjoint action on the Lie algebra. Take for instance $(x, y, z) \in M^{\oplus 3}$ a 1-cocycle on the Lie algebra complex $\mathscr{C}_{u^-,\varphi,a^+}$ representing some cohomology class. An easy calculation shows that, if $\sigma_a \in A^0$, then, as cohomology classes

$$\sigma_a \cdot (x, y, z) = (\sigma_a x, \sigma_a y, a\sigma_a z).$$

If we want to calculate the action of τ, say, on 1-coycles, in the same way, we get

$$\tau(x,y,z) = (\tau x + \tau\varphi\frac{\tau^{1-p}-1}{\log(\tau)}z, \tau\, y - p\tau\, z, \tau z).$$

The formula for the first coordinate is obtained by using the fact that Lie algebra cohomology is calculated by 'differentiating locally analytic cocycles at the identity' (cf. [44]), and it can be taken as a formal formula (since there might be some convergence problems) but it will be enough for us (in general, one should replace τ by τ^n for some n big enough).

5.2 Deconstructing cohomology

In order to compute cohomology we build the complex $\mathscr{C}_{u^-,\varphi,a^+}$ from smaller complexes, in the same spirit as §4.2. Define

$$\mathscr{C}_{u^-}: 0 \to M \xrightarrow{D} M \to 0,$$

where

$$D(x) := u^- x$$

and

$$\mathscr{C}_{u^-,\varphi}: 0 \to M \xrightarrow{E} M \oplus M \xrightarrow{F} M \to 0,$$

where

$$E(x) = (u^- x, (\varphi-1)x)$$
$$F(x,y) = (p\varphi-1)x - u^- y.$$

We now define morphisms between the complexes. We note by $[\varphi-1]: \mathscr{C}_{u^-} \to \mathscr{C}_{u^-}$ the morphism:

$$\begin{array}{ccccccc}
\mathscr{C}_{u^-}: 0 & \longrightarrow & M & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi-1} & & \downarrow{\scriptstyle p\varphi-1} & & \\
\mathscr{C}_{u^-}: 0 & \longrightarrow & M & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

and $[a^+]: \mathscr{C}_{u^-,\varphi} \to \mathscr{C}_{u^-,\varphi}$ the morphism:

$$\begin{array}{ccccccccc}
\mathscr{C}_{u^-,\varphi}: 0 & \longrightarrow & M & \longrightarrow & M\oplus M & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a^+} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle a^+ +1} & & \\
\mathscr{C}_{u^-,\varphi}: 0 & \longrightarrow & M & \longrightarrow & M\oplus M & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

where $s(x,y) = ((a^+ + 1)x, a^+ y)$

Lemma 5.5. *We have the following distinguished triangles in $\mathcal{D}^-(L)$:*

$$\mathscr{C}_{u^-,\varphi} \to \mathscr{C}_{u^-} \xrightarrow{[\varphi-1]} \mathscr{C}_{u^-},$$

$$\mathscr{C}_{u^-,\varphi,a^+} \to \mathscr{C}_{u^-,\varphi} \xrightarrow{[a^+]} \mathscr{C}_{u^-,\varphi}.$$

Proof. This is evident from the definition of the cone of a morphism in $\mathcal{D}^-(L)$. □

The following lemma will be the cornerstone of our cohomology calculations.

Lemma 5.6.

1. $H^0(\mathscr{C}_{u^-,\varphi,a^+}) = H^0([a^+]\colon H^0(\mathscr{C}_{u^-,\varphi}))$.
2. *We have the following exact sequences in cohomology:*

$$0 \to H^1([a^+]\colon H^0(\mathscr{C}_{u^-,\varphi})) \to H^1(\mathscr{C}_{u^-,\varphi,a^+}) \to H^0([a^+]\colon H^1(\mathscr{C}_{u^-,\varphi})) \to 0, \quad (14)$$

$$0 \to H^1([\varphi-1]\colon H^0(\mathscr{C}_{u^-})) \to H^1(\mathscr{C}_{u^-,\varphi}) \to H^0([\varphi-1]\colon H^1(\mathscr{C}_{u^-})) \to 0. \quad (15)$$

3. *We have the following exact sequences in cohomology:*

$$0 \to H^1([a^+] : H^1(\mathscr{C}_{u^-,\varphi})) \to H^2(\mathscr{C}_{u^-,\varphi,a^+}) \to H^0([a^+] : H^2(\mathscr{C}_{u^-,\varphi})) \to 0, \quad (16)$$

$$H^2(\mathscr{C}_{u^-,\varphi}) \cong H^1([\varphi-1] : H^1(\mathscr{C}_{u^-})). \quad (17)$$

4. $H^3(\mathscr{C}_{u^-,\varphi,a^+}) = H^1([a^+]\colon H^2(\mathscr{C}_{u^-,\varphi}))$.

Proof. This follows from taking long exact sequences in cohomology from the triangles in Lemma 5.5. □

The module from which we are taking cohomology should be clear from context, and whenever it is not specified, it means that the result holds for any such module. Notations should be clear, for instance, we have $H^0([a^+]\colon H^0(\mathscr{C}_{u^-,\varphi})) = M^{u^-=0,\varphi=1,a^+=0}$, $H^1([a^+]\colon H^0(\mathscr{C}_{u^-,\varphi})) = \operatorname{coker}([a^+]\colon H^0(\mathscr{C}_{u^-,\varphi})) = \operatorname{coker}(a^+\colon M^{u^-=0,\varphi=1} \to M^{u^-=0,\varphi=1})$, et cetera desunt.

Remark 5.7. From the definition of the action of the group on the different terms of the Hochschild-Serre spectral sequence, or on the Lie algebra cohomology, as the composition of the natural action on the module with inner automorphisms on the group or on the Lie algebra, we can calculate the explicit action of $\tilde{P}$ on each constituent component of the exact sequences appearing in Lemma 5.6. For instance, the action of A^0 on the third term of Equation (15), on the terms of Equation (17) and on those of Lemma 5.6(4) is twisted by χ (this comes from the identity $\sigma_a^{-1}u^-\sigma = au^-$), while it acts as usual on the other terms (since A^0 commutes with itself and with φ). We can check that τ acts as usual on each separate term (but the sequences do not split as a sequence of $\overline{U}$-modules).

5.3 THE LIE ALGEBRA COHOMOLOGY OF $\mathscr{R}^-(\delta_1,\delta_2)$

The following conglomerate of technical lemmas on the action of the Lie algebra on $\mathscr{R}^-(\delta_1,\delta_2)$ will culminate in the main Proposition 5.24 of this section, calculating the $\overline{P}^+$-cohomology on this module, following the strategy suggested by Lemma 5.6. Bear in mind the identifications made in Remark 4.9 and recall (cf. Proposition 2.5) that, for a locally analytic character δ on $\mathbf{Q}_p^\times$, $\kappa(\delta)=\delta'(1)$ denotes the *weight* of δ.

LEMMA 5.8. *Call $M=\mathscr{R}^-(\delta_1,\delta_2)$ and let $f\in M$. Under the identification (as modules) $M=\mathrm{LA}(\mathbf{Z}_p,L)$, the infinitesimal actions of a^+, u^- and φ are given by* [27]

$$(a^+f)(x)=\kappa(\delta)f(x)-xf'(x),$$
$$(u^-f)(x)=\kappa(\delta)xf(x)-x^2f'(x),$$
$$(\varphi f)(x)=\delta(p)f\left(\frac{x}{p}\right).$$

Proof. The first and third formulas follow by a direct calculation. For the third one, use the formula (cf. [19, Théorème 1.1]) $u^-=-t^{-1}\nabla(\nabla+\kappa(\delta_2\delta_1^{-1}))$. □

We stress one last time that, by an inoffensive abuse of language, we will talk in the sequel about the action of the elements a^+ and u^- on $\mathrm{LA}(\mathbf{Z}_p,L)$ (resp. $\mathrm{LA}(\mathbf{Z}_p^\times,L)$), by which we mean their action on $\mathscr{R}^-(\delta_1,\delta_2)$ under the identification $\mathscr{R}^-(\delta_1,\delta_2)=\mathrm{LA}(\mathbf{Z}_p,L)$ (resp. $\mathscr{R}^-(\delta_1,\delta_2)\boxtimes\mathbf{Z}_p^\times=\mathrm{LA}(\mathbf{Z}_p^\times,L)$) as L-vector spaces.

5.3.1 CALCULATION OF $H^0(\mathscr{C}_{u^-,\varphi,a^+})$:

PROPOSITION 5.9. *Let $M=\mathscr{R}^-(\delta_1,\delta_2)$. We have $H^0(\mathscr{C}_{u^-,\varphi,a^+})=0$.*

Proof. This follows immediately from the injectivity of $1-\delta(p)\varphi$ on $\mathrm{LA}(\mathbf{Z}_p,L)$. □

5.3.2 CALCULATION OF $H^2(\mathscr{C}_{u^-,\varphi})$:

LEMMA 5.10. *The operator u^- restricted to $\mathrm{LA}(\mathbf{Z}_p^\times,L)$ is surjective on $\mathrm{LA}(\mathbf{Z}_p^\times,L)$.*

Proof. This is an easy exercise on power series that we leave to the reader. □

LEMMA 5.11. *If $M=\mathscr{R}^-(\delta_1,\delta_2)$ then:*

1. *If $\delta(p)\notin\{p^i|\ i\geq -1\}$, or if $\delta(p)=p^i$ for some $i\geq 0$ and $\kappa(\delta)\neq i$, then $H^2(\mathscr{C}_{u^-,\varphi})=0$.*

[27] Observe that, in the formula for φ below, $f\left(\frac{x}{p}\right)$ is taken to be zero whenever $z\in\mathbf{Z}_p^\times$, so the precise formula should be $(\varphi f)(x)=\mathbf{1}_{p\mathbf{Z}_p}(x)\delta(p)f\left(\frac{x}{p}\right)$.

2. *Otherwise* [28] $H^2(\mathscr{C}_{u^-,\varphi})$ *is of dimension* 1 *naturally generated by* $[x^{i+1}]$.

Proof. Suppose first $\delta(p) \notin \{p^i | \ i \geq -1\}$. Note that, by [14, Lemme 5.9],

$$M = \mathrm{LA}(\mathbf{Z}_p^\times) \oplus (p\varphi - 1)M.$$

By Lemma 5.6, $H^2(\mathscr{C}_{u^-,\varphi}) = \mathrm{coker}([\varphi - 1] : M/u^- M) = M/(u^-, (1-p\varphi))$ and the result follows from Lemma 5.10.
Suppose now $\delta(p) \in \{p^i | \ i \geq -1\}$. In this case, by [14, Lemme 5.9], we have

$$M = \big(\mathrm{LA}(\mathbf{Z}_p^\times) + (p\varphi - 1)M\big) \oplus L \cdot x^{i+1}, \tag{18}$$

where $\mathrm{LA}(\mathbf{Z}_p^\times) \cap (p\varphi - 1)M = L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^{i+1}$. If $i \geq 0$ and $\kappa(\delta) \neq i$ then $(\kappa(\delta) - i)^{-1} u^- x^i = x^{i+1}$. On the other hand, if $i \geq 0$ and $\kappa(\delta) = i$ or if $i = -1$, then x^{i+1} is not in the image of u^-. The result follows by Lemma 5.10 ☐

5.3.3 Calculation of $H^3(\mathscr{C}_{u^-,\varphi,a^+})$:

At this stage, we can already deduce the following.

Proposition 5.12. *Let* $M = \mathscr{R}^-(\delta_1, \delta_2)$.

- *If* $\delta(p) = p^i$, $i \geq -1$, *and* $\kappa(\delta) = i$, *then* $H^3(\mathscr{C}_{u^-,\varphi,a^+})$ *is of dimension* 1 *naturally generated by* $[x^{i+1}]$.
- *Otherwise* $H^3(\mathscr{C}_{u^-,\varphi,a^+}) = 0$.

Proof. This follows from Lemma 5.6(4) and Lemma 5.11, by observing that the action of $[a^+]$ on $H^2(\mathscr{C}_{u^-,\varphi}) = M/(u^-, p\varphi - 1)$ is given by $a^+ + 1$, and using the formula $(a^+ + 1)[x^{i+1}] = (\kappa(\delta) - i)[x^{i+1}]$. ☐

5.3.4 Calculation of $H^1(\mathscr{C}_{u^-,\varphi})$:

The following lemma describes the kernel of u^- acting on $\mathscr{R}^-(\delta_1, \delta_2)$ in the appropriate way so as to calculate (cf. Corollary 5.14) the left term of Equation (15) of Lemma 5.6. Define [29]

$$X_{\kappa(\delta)} := \Big\{ f \in \mathrm{LA}(\mathbf{Z}_\mathrm{p}^\times) | \ f(x) = \sum_{i \in (\mathbf{Z}/p^n\mathbf{Z})^\times} c_i \left(\frac{x}{i}\right)^{\kappa(\delta)} \mathbf{1}_{i + p^n \mathbf{Z}_p} \text{ for some } n > 0 \Big\}.$$

Lemma 5.13. *If* $M = \mathscr{R}^-(\delta_1, \delta_2) = \mathrm{LA}(\mathbf{Z}_p, L)$. *Then*

1. *If* $\delta(p) \notin \{p^i | \ i \geq 0\}$, *or if* $\delta(p) = p^i$ *for some* $i \geq 0$ *and* $\kappa(\delta) \neq i$, *then*

$$M^{u^- = 0} = X_{\kappa(\delta)} \oplus (1 - \varphi) M^{u^- = 0}.$$

[28] i.e if $\delta(p) = p^{-1}$, or if $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) = i$.

[29] Observe that, upon developing $(x/i)^{\kappa(\delta)} = \sum_{k \geq 0} \binom{\kappa(\delta)}{k} (x/i - 1)^k$ and observing that $v_p(\binom{\kappa(\delta)}{k}) \geq k(\min(\kappa(\delta), 0) - \frac{1}{p-1})$, we see that $(x/i)^{\kappa(\delta)}$ is a well defined analytic function on $i + p^n \mathbf{Z}_p$ for n big enough.

2. *Otherwise*

$$M^{u^-=0} = \left(X_{\kappa(\delta)} + (1-\varphi)M^{u^-=0}\right) \oplus L\cdot x^i,$$

Furthermore $X_{\kappa(\delta)} \cap (1-\varphi)M^{u^-=0}$ *is the line* $L\cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i$.

Proof. (1): First suppose that $\delta(p) \neq p^{-1}$. Take $f \in M^{u^-=0}$. Since $\delta(p) \notin \left\{p^i |\ i \geq -1\right\}$, by [14, Lemme 5.9], we can uniquely write $f = f_1 + (1-\varphi)f_2$ where f_1 is supported on $\mathbf{Z}_p^\times$ and $f_2 \in M$. Thus $0 = u^- f = u^- f_1 + u^-(1-\varphi)f_2 = u^- f_1 + (1-p\varphi)u^- f_2$. We deduce, again using [14, Lemme 5.9] (observe that $p\delta(p) \notin \{p^i, i \geq 0\}$), that $u^- f_1 = u^- f_2 = 0$. Solving the differential equation $u^- f_1 = 0$ gives precisely $f_1 \in X_{\kappa(\delta)}$.
Suppose now $\delta(p) = p^{-1}$. Repeating the same procedure as above, since $p\delta(p) = 1$, in this case [14, Lemme 5.9] gives $u^- f_1 = b\mathbf{1}_{\mathbf{Z}_p^\times}$ and $(1-p\varphi)u^- f_2 = -b\mathbf{1}_{\mathbf{Z}_p^\times}$ for some $b \in L$. This implies $u^- f_2 = -b\mathbf{1}_{\mathbf{Z}_p}$. This equation has no solution unless $b = 0$ (as follows by evaluating at zero), in which case we obtain again $u^- f_1 = u^- f_2 = 0$.
We leave the case when $\delta(p) = p^i$ and $\kappa(\delta) \neq i$, for $i \geq 0$, the proof of the fact that the sum is indeed a direct sum and the proof of the last assertion as an exercise, which involve the same kind of techniques.

□

Observing that $H^1([\varphi - 1]\colon H^0(\mathscr{C}_{u^-})) = M^{u^-=0}/(\varphi - 1)$, we get the following:

Lemma 5.14. *If* $M = \mathscr{R}^-(\delta_1, \delta_2)$ *then*

1. *If* $\delta(p) \notin \left\{p^i |\ i \geq 0\right\}$, *or if* $\delta(p) = p^i$ *for some* $i \geq 0$ *and* $\kappa(\delta) \neq i$, *then*

$$H^1([\varphi - 1]\colon H^0(\mathscr{C}_{u^-})) = X_{\kappa(\delta)}.$$

2. *Otherwise*

$$H^1([\varphi - 1]\colon H^0(\mathscr{C}_{u^-})) = \left(X_{\kappa(\delta)}/L\cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i\right) \oplus L\cdot [x^i].$$

Proof. This is an immediate consequence of Lemma 5.13. □

We now proceed to calculate the right side term of Equation (15) of Lemma 5.6. Note that the action of $[\varphi - 1]$ on $H^1(\mathscr{C}_{u^-}) = M/u^- M$ is given by $p\varphi - 1$.

Lemma 5.15. *If* $M = \mathscr{R}^-(\delta_1, \delta_2)$ *then*

1. *If* $\delta(p) \notin \left\{p^i |\ i \geq -1\right\}$, *or if* $\delta(p) = p^i$, $i \geq 0$, *and* $\kappa(\delta) \neq i$, *then* $H^0([\varphi - 1]\colon H^1(\mathscr{C}_{u^-})) = 0$.
2. *Otherwise* $H^0([\varphi - 1]\colon H^1(\mathscr{C}_{u^-})) = L\cdot [x^{i+1}]$.

Proof. We leave the proof as an exercise, using the same techniques as the proof of Lemma 5.13. Observe that, if $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) = i$, x^{i+1} is not in the image of u^- (expand around zero).

□

We can at this stage easily deduce the following corollary, which gives a complete description of $H^1(\mathscr{C}_{u^-,\varphi})$:

Proposition 5.16. *Let* $M = \mathscr{R}^-(\delta_1, \delta_2)$. *Then*

1. *If* $\delta(p) \notin \{p^i |\ i \geq -1\}$, *or if* $\delta(p) = p^i$, $i \geq 0$, *and* $\kappa(\delta) \neq i$, *then* $H^1(\mathscr{C}_{u^-,\varphi}) = X_{\kappa(\delta)}$.
2. *If* $\delta(p) = p^{-1}$, *then* $H^1(\mathscr{C}_{u^-,\varphi})$ *lives in an exact sequence*
$$0 \to X_{\kappa(\delta)} \to H^1(\mathscr{C}_{u^-,\varphi}) \to L \cdot [\mathbf{1}_{\mathbf{Z}_p}] \to 0.$$
3. *If* $\delta(p) = p^i$, $i \geq 0$, *and* $\kappa(\delta) = i$ *then* $H^1(\mathscr{C}_{u^-,\varphi})$ *lives in an exact sequence*
$$0 \to \left(X_{\kappa(\delta)}/L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i\right) \oplus L \cdot [x^i] \to H^1(\mathscr{C}_{u^-,\varphi}) \to L \cdot [x^{i+1}] \to 0.$$

Proof. This is an immediate consequence of Lemmas 5.6, 5.15 and Corollary 5.14. □

5.3.5 Calculation of $H^1(\mathscr{C}_{u^-,\varphi,\gamma})$.

We have already explicitly calculated the second exact sequence (15) of Lemma 5.6 in Proposition 5.16. The left hand side term of Equation (14) is easy to deal with:

Lemma 5.17. *If* $M = \mathscr{R}^-(\delta_1, \delta_2)$ *then* $H^0(\mathscr{C}_{u^-,\varphi}) = 0$.

Proof. This follows immediately from the injectivity of $\delta(p)\varphi - 1$ on $\mathrm{LA}(\mathbf{Z}_p, L)$, cf. [14, Lemme 5.9]. □

We calculate the kernel of $[a^+]$ on $H^0([\varphi - 1] \colon H^1(\mathscr{C}_{u^-})) = \ker(p\varphi - 1 \colon M/u^- M)$:

Lemma 5.18. *If* $M = \mathscr{R}^-(\delta_1, \delta_2)$ *then*

1. *If* $\delta(p) \notin \{p^i |\ i \geq -1\}$, *or if* $\delta(p) = p^i$ *for some* $i \geq -1$ *and* $\kappa(\delta) \neq i$, *then* $[a^+]$ *on* $H^0([\varphi - 1] \colon H^1(\mathscr{C}_{u^-}))$ *is injective.*
2. *Otherwise the kernel of* $[a^+]$ *on* $H^0([\varphi - 1] \colon H^1(\mathscr{C}_{u^-}))$ *is (naturally isomorphic to)* $L \cdot [x^{i+1}]$.

Proof. We use Lemma 5.15. If $\delta(p) \notin \{p^i |\ i \geq 0\}$ or if $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) \neq i$, then $H^0([\varphi - 1] \colon H^1(\mathscr{C}_{u^-})) = 0$ and the result is obvious. If $\delta(p) = p^i$, $i \geq 0$, and $\kappa(\delta) = i$ then
$$H^0([\varphi - 1] \colon H^1(\mathscr{C}_{u^-})) = L \cdot [x^{i+1}].$$
In this case $(a^+ + 1)[x^{i+1}] = 0$.

Suppose now $\delta(p) = p^{-1}$. In this case

$$H^0([\varphi - 1]\colon H^1(\mathscr{C}_{u^-})) = L \cdot [\mathbf{1}_{\mathbf{Z}_p}]$$

and, since $(a^+ + 1)\mathbf{1}_{\mathbf{Z}_p} = (\kappa(\delta) + 1)\mathbf{1}_{\mathbf{Z}_p}$, the result now follows depending on whether $\kappa(\delta) = -1$ or not. □

The last needed ingredient is the kernel of $[a^+]$ on $H^1([\varphi - 1]\colon H^0(\mathscr{C}_{u^-}))$.

Lemma 5.19. *If $M = \mathscr{R}^-(\delta_1, \delta_2)$ then*

1. *If $\delta(p) \notin \{p^i |\ i \geq 0\}$, or if $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) \neq i$, then*

$$\ker([a^+]\colon H^1([\varphi - 1]\colon H^0(\mathscr{C}_{u^-}))) = X_{\kappa(\delta)}.$$

2. *Otherwise*

$$\ker([a^+]\colon H^1([\varphi - 1]\colon H^0(\mathscr{C}_{u^-}))) = \left(X_{\kappa(\delta)}/L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i\right) \oplus L \cdot [x^i].$$

Proof. This is an immediate consequence of Lemma 5.14, noting that $a^+ X_{\kappa(\delta)} = 0$ and that $a^+ x^i = 0$ whenever $\kappa(\delta) = i$. □

Now we can calculate the first Lie algebra cohomology group with values in $\mathscr{R}^-(\delta_1, \delta_2)$.

Proposition 5.20. *Let $M = \mathscr{R}^-(\delta_1, \delta_2)$. Then*

1. *If $\delta(p) \notin \{p^i |\ i \geq -1\}$, or if $\delta(p) = p^i$ for some $i \geq -1$ and $\kappa(\delta) \neq i$, then*
$$H^1(\mathscr{C}_{u^-,\varphi,a^+}) = X_{\kappa(\delta)}.$$

2. *If $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) = i$, then $H^1(\mathscr{C}_{u^-,\varphi,a^+})$ lives in an exact sequence*

$$0 \to X_{\kappa(\delta)}/L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i \oplus L \cdot [x^i] \to H^1(\mathscr{C}_{u^-,\varphi,a^+}) \to L \cdot [x^{i+1}] \to 0$$

3. *If $\delta(p) = p^{-1}$ and $\kappa(\delta) = -1$ then $H^1(\mathscr{C}_{u^-,\varphi,a^+})$ lives in an exact sequence*

$$0 \to X_{\kappa(\delta)} \to H^1(\mathscr{C}_{u^-,\varphi,a^+}) \to L \cdot [\mathbf{1}_{\mathbf{Z}_p}] \to 0.$$

Proof. By Lemma 5.17 and the short exact sequence (14) of Lemma 5.6 we have
$$H^1(\mathscr{C}_{u^-,\varphi,a^+}) \cong \ker([a^+]\colon H^1(\mathscr{C}_{u^-,\varphi})).$$

We claim that the short exact sequence (15) splits as a sequence of a^+-modules. Indeed it suffices to produce an a^+-equivariant section of the morphism

$$H^1(\mathscr{C}_{u^-,\varphi}) \to H^0([\varphi - 1]\colon H^1(\mathscr{C}_{u^-})). \tag{19}$$

If $H^0([\varphi-1]\colon H^1(\mathscr{C}_{u^-}))=0$, this is trivial so assume $H^0([\varphi-1]\colon H^1(\mathscr{C}_{u^-}))\neq 0$. Then by Lemma 5.15, $H^0([\varphi-1]\colon H^1(\mathscr{C}_{u^-}))=L\cdot[x^{i+1}]$. Now $H^1(\mathscr{C}_{u^-,\varphi})=\frac{\{(f,g)\in M\oplus M \mid (p\varphi-1)f=u^-g\}}{\{(u^-f,(\varphi-1)f)\mid f\in M\}}$ and $H^0([\varphi-1]\colon H^1(\mathscr{C}_{u^-}))=\left(\frac{M}{u^-M}\right)^{p\varphi=1}$. In terms of these descriptions, the morphism (19) is explicitly given by

$$[(f,g)]\mapsto [f].$$

There is a (not necessarily unique) section given by

$$\left(\frac{M}{u^-M}\right)^{p\varphi=1}\to \frac{\{(f,g)\in M\oplus M \mid (p\varphi-1)f=u^-g\}}{\{(u^-f,(\varphi-1)f)\mid f\in M\}}$$
$$[x^{i+1}]\mapsto[(x^{i+1},g)],$$

where g is chosen such that $(p\varphi-1)x^{i+1}=u^-g$. Now $a^+\cdot[f]=[(a^++1)f]$ and $a^+\cdot[(f,g)]=[((a^++1)f,a^+g)]$. Thus the section is clearly a^+-equivariant and this proves the claim.

Since the short exact sequence (15) splits as a sequence of a^+-modules, we have

$$\begin{aligned}0&\to \ker(a^+\colon M^{u^-=0}/(\varphi-1))\to\ker([a^+]\colon H^1(\mathscr{C}_{u^-,\varphi}))\\&\to\ker(a^++1\colon (M/u^-M)^{p\varphi=1})\to 0.\end{aligned}$$

Now (1) (resp. (2), resp. (3)) follows from (1) (resp. (2), resp. (1)) of Lemma 5.19 and (1) (resp. (2), resp. (2)) of Lemma 5.18. □

5.3.6 Calculation of $H^2(\mathscr{C}_{u^-,\varphi,\gamma})$:

We start by calculating the left side term of equation (16) of Lemma 5.6:

Lemma 5.21. *If $M=\mathscr{R}^-(\delta_1,\delta_2)$ then*

1. *If $\delta(p)\notin\{p^i \mid i\geq -1\}$, or if $\delta(p)=p^i$ for some $i\geq -1$ and $\kappa(\delta)\neq i$, then*
$$\operatorname{coker}([a^+]:H^1(\mathscr{C}_{u^-,\varphi}))=X_{\kappa(\delta)}.$$
2. *If $\delta(p)=p^i$ for some $i\geq 0$ and $\kappa(\delta)=i$, we have a short exact sequence*
$$0\to X_{\kappa(\delta)}/L\cdot\mathbf{1}_{\mathbf{Z}_p^\times}x^i\oplus L\cdot[x^i]\to\operatorname{coker}([a^+]:H^1(\mathscr{C}_{u^-,\varphi}))\to L\cdot[x^{i+1}]\to 0.$$
3. *If $\delta(p)=p^{-1}$ and $\kappa(\delta)=-1$ then we have a short exact sequence*
$$0\to X_{\kappa(\delta)}\to\operatorname{coker}([a^+]:H^1(\mathscr{C}_{u^-,\varphi}))\to L\cdot[\mathbf{1}_{\mathbf{Z}_p}]\to 0.$$

Proof.

1. Suppose first that $\delta(p) \notin \{p^i | \, i \geq -1\}$, or that $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) \neq i$. By Proposition 5.16(1), we have $H^1(\mathscr{C}_{u^-,\varphi}) = X_{\kappa(\delta)}$. The result follows then by noting that the action of $[a^+]$ on this space is given by a^+, and that $a^+ X_{\kappa(\delta)} = 0$ (since $xa^+ f = u^- f$).

   We now deal with the case where $\delta(p) = p^{-1}$ and $\kappa(\delta) \neq -1$. In this case $H^1(\mathscr{C}_{u^-,\varphi})$ is described by (2) of Proposition 5.16 and the result follows from Lemma 5.22(1) below.

2. If $\delta(p) = p^i$ for some $i \geq 0$ and $\kappa(\delta) = i$, then $H^1(\mathscr{C}_{u^-,\varphi})$ is described by (3) of Proposition 5.16. Note that $[a^+]$ acts as a^+ on the left term of the exact sequence, and as $a^+ + 1$ on the right hand side term. The result follows then by Lemma 5.22(2) and by noting that $a^+ x^i = 0$.

3. This case follows in the same way, using Proposition 5.16(2) and Lemma 5.22(2).

□

The next lemma describes the cokernel of $[a^+]$ on $H^0([\varphi - 1] \colon H^1(\mathscr{C}_{u^-}))$.

LEMMA 5.22. *If $M = \mathscr{R}^-(\delta_1, \delta_2)$ and $\delta(p) = p^i$ for some $i \geq -1$ then*

1. *If $\kappa(\delta) \neq i$ then* $\operatorname{coker}([a^+] \colon H^0([\varphi - 1] \colon H^1(\mathscr{C}_{u^-}))) = 0$.
2. *If $\kappa(\delta) = i$ then* $\operatorname{coker}([a^+] \colon H^0([\varphi - 1] \colon H^1(\mathscr{C}_{u^-})))$ *is (naturally isomorphic to) $L \cdot [x^{i+1}]$.*

Proof. We use Lemma 5.15. Suppose first $i \geq 0$. If $\kappa(\delta) \neq i$ then $\ker(p\varphi - 1 : M/u^- M) = 0$ and the result is obvious. If $\kappa(\delta) = i$ then

$$\ker(p\varphi - 1 : M/u^- M) = L \cdot [\mathbf{1}_{\mathbf{Z}_p} x^{i+1}].$$

In this case $(a^+ + 1)x^{i+1} = 0$.
Suppose now $i = -1$. In this case

$$\ker(p\varphi - 1 : M/u^- M) = L \cdot [\mathbf{1}_{\mathbf{Z}_p}]$$

and so $(a^+ + 1)\mathbf{1}_{\mathbf{Z}_p} = (\kappa(\delta) + 1)\mathbf{1}_{\mathbf{Z}_p}$. The result now follows depending on whether $\kappa(\delta) = -1$ or not. □

We finally calculate the right hand side term $\ker([a^+] \colon H^2(\mathscr{C}_{u^-,\varphi}))$ of equation (16) of Proposition 5.6.

LEMMA 5.23. *If $M = \mathscr{R}^-(\delta_1, \delta_2)$. Then*

1. *If $\delta(p) \notin \{p^i | \, i \geq -1\}$, or if $\delta(p) = p^i$ for some $i \geq -1$ and $\kappa(\delta) \neq i$, then* $\ker([a^+] \colon H^2(\mathscr{C}_{u^-,\varphi})) = 0$.

2. *Otherwise* $\ker([a^+]\colon H^2(\mathscr{C}_{u^-,\varphi})) = L\cdot[x^{i+1}]$.

Proof.

1. If $\delta(p)\neq p^{-1}$ the result follows since, by Lemma 5.11, we know that $H^2(\mathscr{C}_{u^-,\varphi})=0$. If $\delta(p)=p^{-1}$ (and hence $\kappa(\delta)\neq -1$) then $H^2(\mathscr{C}_{u^-,\varphi}) = L\cdot \mathbf{1}_{\mathbf{Z}_p}$ and, since $(a^++1)\mathbf{1}_{\mathbf{Z}_p} = (\kappa(\delta)+1)\mathbf{1}_{\mathbf{Z}_p}$, it is injective.
2. Note first that $[a^+]$ acts on $H^2(\mathscr{C}_{u^-,\varphi})$ as a^++1. By (2) of Lemma 5.11, $H^2(\mathscr{C}_{u^-,\varphi}) = L\cdot x^{i+1}$ and the result follows since $(a^++1)x^{i+1}=0$.

□

We are now ready to compute $H^2(\mathscr{C}_{u^-,\varphi,a^+})$:

Proposition 5.24. *If $M=\mathscr{R}^-(\delta_1,\delta_2)$ then*

1. *If $\delta(p)\notin\{p^i|\ i\geq -1\}$, or if $\delta(p)=p^i$ for some $i\geq -1$ and $\kappa(\delta)\neq i$, then*
$$H^2(\mathscr{C}_{u^-,\varphi,a^+}) = X_{\kappa(\delta)}.$$
2. *If $\delta(p)=p^i$ for some $i\geq 0$ and $\kappa(\delta)=i$ then we have the following exact sequences*
$$0\to Y_i\to H^2(\mathscr{C}_{u^-,\varphi,a^+})\to L\cdot[x^{i+1}]\to 0,$$
where Y_i lives in an exact sequence
$$0\to X_{\kappa(\delta)}/L\cdot \mathbf{1}_{\mathbf{Z}_p^\times}x^i\oplus L\cdot[x^i]\to Y_i\to L\cdot[x^{i+1}]\to 0.$$
3. *If $\delta(p)=p^{-1}$ and $\kappa(\delta)=-1$ then we have the following exact sequences*
$$0\to Y_{-1}\to H^2(\mathscr{C}_{u^-,\varphi,a^+})\to L\cdot[\mathbf{1}_{\mathbf{Z}_p}]\to 0,$$
where Y_{-1} lives in an exact sequence
$$0\to X_{\kappa(\delta)}\to Y_{-1}\to L\cdot[\mathbf{1}_{\mathbf{Z}_p}]\to 0.$$

Proof. (1) (resp. (2), resp. (3)) follows from (1) (resp. (2), resp (3)) of Lemma 5.21 and (1) (resp. (2), resp. (2)) of Lemma 5.23. □

5.4 The $\overline{P}^+$-cohomology of $\mathscr{R}^-(\delta_1,\delta_2)$

We can now just calculate the $\tilde{P}$-invariants of the Lie algebra cohomology to calculate the $\overline{P}^+$-cohomology of $\mathscr{R}^-(\delta_1,\delta_2)$.

5.4.1 Calculation of $H^0(\overline{P}^+,\mathscr{R}^-(\delta_1,\delta_2))$:

Lemma 5.25. *Let $M=\mathscr{R}^-(\delta_1,\delta_2)$. Then $H^0(\overline{P}^+,M)=0$.*

Proof. Obvious from Proposition 5.9. □

5.4.2 Calculation of $H^1(\overline{P}^+, \mathscr{R}^-(\delta_1,\delta_2))$:

Lemma 5.26. *If $M = \mathscr{R}^-(\delta_1,\delta_2)$ then*

1. *If $\delta(x) \neq x^i$ for any $i \geq 0$ then $H^1(\overline{P}^+, M)$ is of dimension 1 and generated by $\mathbf{1}_{\mathbf{Z}_p^\times}\delta\otimes\delta$.*

2. *If $\delta(x) = x^i$ for some $i \geq 0$ then $H^1(\overline{P}^+, M)$ is of dimension 2.*

Proof. Start observing that the action of $\tilde{P}$ on the Lie algebra cohomology is explicitly given in Remark 5.4 (cf. also Remark 5.7): the action of τ on each of the extremities of the exact sequences of Lemma 5.6 is the usual one, while the action of A^0 is given by the usual one, except for the term $\ker([\varphi-1]\colon H^1(\mathscr{C}_{u^-}))$, on which its action is given by the usual action twisted by χ.
- Suppose we are under the hypothesis of Proposition 5.20(1). Then

$$H^1(\mathscr{C}_{u^-,\varphi,a^+}) = X_{\kappa(\delta)}.$$

Note $\gamma = \sigma_a \in A^0$ a topological generator. Let $f \in H^0(A^0, X_{\kappa(\delta)})$ and write

$$f(x) = \sum_{i\in(\mathbf{Z}/p^n\mathbf{Z})^\times} c_i \left(\frac{x}{i}\right)^{\kappa(\delta)} \mathbf{1}_{i+p^n\mathbf{Z}_p}, \quad n \geq 0.$$

Since $\gamma f = f$, we have

$$\begin{aligned}\sum_{i\in(\mathbf{Z}/p^n\mathbf{Z})^\times} c_i \left(\frac{x}{i}\right)^{\kappa(\delta)} \mathbf{1}_{i+p^n\mathbf{Z}_p} &= \delta(a) \sum_{i\in(\mathbf{Z}/p^n\mathbf{Z})^\times} c_i \left(\frac{x}{ia}\right)^{\kappa(\delta)} \mathbf{1}_{ia+p^n\mathbf{Z}_p} \\ &= \delta(a) \sum_{i\in(\mathbf{Z}/p^n\mathbf{Z})^\times} c_{ia^{-1}} \left(\frac{x}{i}\right)^{\kappa(\delta)} \mathbf{1}_{i+p^n\mathbf{Z}_p}.\end{aligned}$$

Thus $\delta(a)c_{ia^{-1}} = c_i$ which implies $c_a = c_1\delta(a)$, for any $a \in \mathbf{Z}_p^\times$. This implies

$$f(x) = c_1\delta(x)\mathbf{1}_{\mathbf{Z}_p^\times}.$$

Since $\delta\mathbf{1}_{\mathbf{Z}_p^\times}$ is fixed by τ, the result follows from Lemma 5.3.
- We now place ourselves under the hypothesis of Proposition 5.20(2). We have

$$0 \to X_{\kappa(\delta)}/L\cdot\mathbf{1}_{\mathbf{Z}_p^\times}x^i \oplus L\cdot[x^i] \to H^1(\mathscr{C}_{u^-,\varphi,a^+}) \to L\cdot[x^{i+1}] \to 0. \tag{20}$$

To calculate the A^0-invariants of $X := X_{\kappa(\delta)}/L\cdot\mathbf{1}_{\mathbf{Z}_p^\times}x^i$, we just consider the short exact sequence of Γ-modules

$$0 \to L\cdot\mathbf{1}_{\mathbf{Z}_p^\times}x^i \to X_{\kappa(\delta)} \to X \to 0$$

and take the associated long exact sequence. One easily sees that

- if $\delta(x) \neq x^i$, then $H^0(A^0, L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i) = H^1(A^0, L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} x^i) = 0$ so that $H^0(A^0, X) = H^0(A^0, X_{\kappa(\delta)}) = L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} \delta$.

- If $\delta(x) = x^i$, then A^0 fixes $\mathbf{1}_{\mathbf{Z}_p^\times} x^i$, so we get a long exact sequence

$$0 \to L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} \delta \to L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} \delta \to H^0(A^0, X) \to L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} \delta$$
$$\xrightarrow{\alpha} H^1(A^0, X_{\kappa(\delta)}) \to H^1(A^0, X) \to 0.$$

  Now $H^1_{\mathrm{Lie}}(A^0, X_{\kappa(\delta)}) = X_{\kappa(\delta)}$ and so $H^1(A^0, X_{\kappa(\delta)}) = H^0(A^0, X_{\kappa(\delta)}) = L \cdot 1_{\mathbf{Z}_p^\times} \delta$. It follows that α is an isomorphism and so $H^0(A^0, X) = 0$. Note also that this implies $H^1(A^0, X) = 0$.

Thus, if $\delta(x) \neq x^i$, $H^1(\mathscr{C}_{u^-,\varphi,a^+})^{\tilde{P}} = L \cdot \mathbf{1}_{\mathbf{Z}_p^\times} \delta$. If $\delta(x) = x^i$, since the action of A^0 on each of the terms of Equation (20) is locally constant, taking invariants is exact and we obtain

$$0 \to L \cdot [x^i] \to H^1(\mathscr{C}_{u^-,\varphi,a^+})^{A^0} \to L \cdot [x^{i+1}] \to 0.$$

taking $\overline{U}$-invariants of this exact sequence (note that $\tau([x^{i+1}]) = [\frac{x^{i+1}}{1-px}] = [x^{i+1}] + p[x^{i+2}] + p^2[x^{i+3}] + \ldots = [x^{i+1}] \mod u^-$ if $\delta(x) = x^i$), we obtain the desired result.
- Finally, suppose that hypothesis of Proposition 5.20(3) hold. We have

$$0 \to X_{\kappa(\delta)} \to H^1(\mathscr{C}_{u^-,\varphi,a^+}) \to L \cdot \mathbf{1}_{\mathbf{Z}_p} \to 0,$$

and the result follows similarly. □

5.4.3 Calculation of $H^2(\overline{P}^+, \mathscr{R}^-(\delta_1, \delta_2))$:

Lemma 5.27. *If $M = \mathscr{R}^-(\delta_1, \delta_2)$ then*

1. *If $\delta \neq x^i$ for any $i \geq 0$, then $H^2(\overline{P}^+, M)$ is of dimension 1 and generated by $\mathbf{1}_{\mathbf{Z}_p^\times} \delta \otimes \delta$.*

2. *If $i \geq 0$ and $\delta(x) = x^i$ then $H^2(\overline{P}^+, M)$ is of dimension 3.*

Proof. The result is treated in a similar way as Proposition 5.26, considering the different cases of Proposition 5.24. □

5.4.4 Calculation of $H^3(\overline{P}^+, \mathscr{R}^-(\delta_1, \delta_2))$:

Lemma 5.28. *Let $M = \mathscr{R}^-(\delta_1, \delta_2)$.*

- *If $\delta = x^i$, $i \geq 0$, then $H^3(\overline{P}^+, \mathscr{R}^-(\delta_1, \delta_2))$ is of dimension 1 naturally generated by $[x^{i+1}]$.*

- *Otherwise* $H^3(\overline{P}^+, \mathscr{R}^-(\delta_1, \delta_2)) = 0$.

Proof. This follows by taking $\tilde{P}$-invariants to the results of Proposition 5.12, by observing that the action of τ is the natural one, and that of A^0 is twisted by χ. ☐

5.5 The $\overline{P}^+$-cohomology of $\mathscr{R}^+(\delta_1, \delta_2)$: a first reduction

In this section we calculate all $\overline{P}^+$-cohomology groups of $\mathscr{R}^+(\delta_1, \delta_2)$ as described in Proposition 5.1. We first start with a lemma that allows us to reduce the calculation of $H^i(\overline{P}^+, \mathscr{R}^+(\delta_1, \delta_2))$ to that of $H^i(\overline{P}^+, \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*(\delta_1, \delta_2))$ for $N \geq 0$ big enough, where $\mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*(\delta_1, \delta_2)$ denotes the submodule of $\mathscr{R}^+(\delta_1, \delta_2)$ corresponding to $\mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*$ under the identification (as L-vector spaces) $\mathscr{R}^+(\delta_1, \delta_2) = \mathscr{R}^+$. We also recall that, under the Amice transform, we have an identification $\mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*(\delta_1, \delta_2) = \oplus_{i=0}^N L \cdot t^i$. This module is stable under the action of $\overline{P}^+$ by [14, Lemme 5.20] [30], and the action of $\overline{P}^+$ is explicitly given by

$$\sigma_a(t^j) = \delta_1\delta_2^{-1}(a)a^j t^j, \quad \varphi(t^j) = \delta_1\delta_2^{-1}(p)p^j t^j, \quad \tau(t^j) = \sum_{h=0}^{j} \binom{\kappa - h}{j - h} p^{j-h}\, t^h,$$

where we have set $\kappa = -\kappa(\delta_1\delta_2^{-1}) - 1$. Observe that, if $\kappa \in \{0, 1, \dots, N-1\}$ and $j = \kappa + 1$, then $\tau t^j = t^j$.

Lemma 5.29. *We have, for every i and for N big enough,*

$$H^i(\overline{P}^+, \mathscr{R}^+(\delta_1, \delta_2)) = H^i(\overline{P}^+, \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*(\delta_1, \delta_2)).$$

Proof. This follows from the Hochschild-Serre spectral sequence and the same arguments of [6, Lemme 2.9(ii)]. Observe that it suffices to take N such that $|\delta(p)| < p^N$. ☐

5.6 The $\overline{P}^+$-cohomology of $\mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*(\delta_1, \delta_2)$

We now proceed to calculate the $\overline{P}^+$-cohomology of $\mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*(\delta_1, \delta_2)$, which turns out to be a much simpler task since we have an explicit description of the action of τ on this module. We will be using the Hochschild-Serre spectral sequence. We follow closely [14, Lemme 5.21].

From now on until the end of this section, call $M_{+,N} = \mathrm{Pol}_{\leq N}(\mathbf{Z}_p, L)^*(\delta_1, \delta_2)$, which we identify with $\oplus_{i=0}^N L \cdot t^i$ equipped with the corresponding action of $\overline{P}^+$.

[30] The first $\overline{P}^+$-cohomology group of this module is calculated in [14, Lemme 5.21] but, as we mentioned earlier, there are some small mistakes loc.cit., whence the incompatibility with our results.

Lemma 5.30. *We have*

1. $H^0(\overline{U}, M_{+,N}) = L \cdot t^0 \oplus L \cdot t^{\kappa+1}$ *and* $H^1(\overline{U}, M_{+,N}) = L \cdot [t^N] \oplus L \cdot [t^\kappa]$ *if* $\kappa \in \{0, \dots, N-1\}$.
2. $H^0(\overline{U}, M_{+,N}) = L \cdot t^0$ *and* $H^1(\overline{U}, M_{+,N}) = L \cdot [t^N]$ *otherwise.*

Proof. This is an easy calculation from the explicit description of τ on the module $M_{+,N}$. □

Remark 5.31. Observe that, as an A^+-module, we have that $H^0(\overline{U}, M_{+,N}) \cong L(\delta_1\delta_2^{-1}) \oplus L(\delta_1\delta_2^{-1}x^{\kappa+1})$ or $L(\delta_1\delta_2^{-1})$ accordingly to whether $\kappa \in \{0, \dots, N-1\}$ or not. On the other hand, recall that the action of A^+ on a 1-cocycle $c : \overline{U} \to M_{+,N}$ representing a cohomology class in $H^1(\overline{U}, M_{+,N})$ is given by $(\gamma c)(\tau) = \gamma c_{\gamma^{-1}\tau\gamma}$, where $\gamma \in A^+$. Since for any $a \in \mathbf{Z}_p^\times$ we have $\sigma_a^{-1}\tau\sigma_a = \tau^a$, we deduce

$$c_{\sigma_a^{-1}\tau\sigma_a} = c_{\tau^a} = \delta_a c_\tau,$$

where

$$\begin{aligned}\delta_a = \frac{\tau^a - 1}{\tau - 1} &= \sum_{n \geq 1} \binom{a}{n} (\tau - 1)^{n-1} \\ &= a + \binom{a}{2}(\tau - 1) + \binom{a}{3}(\tau - 1)^2 + ... \equiv a \bmod (\tau - 1)M_{+,N}.\end{aligned}$$

We deduce that, as an A^+-module, we have $H^1(\overline{U}, M_{+,N}) = L(\delta_1\delta_2^{-1}x^{N+1}) \oplus L(\delta_1\delta_2^{-1}x^{\kappa+1})$ or $L(\delta_1\delta_2^{-1}x^{N+1})$ accordingly to whether $\kappa \in \{0, \dots, N-1\}$ or not.

We can now deduce the results of Proposition 5.1 concerning the module $\mathscr{R}^+(\delta_1, \delta_2)$. Call $M_+ = \mathscr{R}^+(\delta_1, \delta_2)$.

Proposition 5.32. *We have*

$$\dim_L H^i(\overline{P}^+, M_+) = \begin{cases} 1, 3, 3, 1 \, (i = 0, 1, 2, 3) & \textit{if } \delta_1\delta_2^{-1} = x^{-i}, i \geq 1, \\ 1, 2, 1, 0 \, (i = 0, 1, 2, 3) & \textit{if } \delta_1\delta_2^{-1} \textit{ is trivial,} \\ 0, 0, 0, 0 \, (i = 0, 1, 2, 3) & \textit{otherwise.} \end{cases}$$

Proof. By Lemma 5.29, it suffices to deal with the cohomology of $M_{+,N}$ for N large enough (in the case $\delta_1\delta_2^{-1} = x^{-i}$ for $i \geq 0$, we can take $N = i + 1$). The result now follows from the Hochschild-Serre spectral sequence, by applying Lemma 5.30, Remark 5.31 and [14, Lemme 5.11] for calculating all the cohomology groups $H^j(A^+, H^i(\overline{U}, M_{+,N}))$ appearing in the spectral sequence. Precisely, considering the different cases, we see that:

$$\dim_L H^i(A^+, H^0(\overline{U}, M_{+,N}))) = \begin{cases} 1, 2, 1 \, (i = 0, 1, 2) & \text{if } \delta_1\delta_2^{-1} = x^{-i}, i \geq 0, \\ 0, 0, 0 \, (i = 0, 1, 2) & \text{otherwise;} \end{cases}$$

$$\dim_L H^i(A^+, H^1(\overline{U}, M_{+,N}))) = \begin{cases} 1,2,1\,(i=0,1,2) & \text{if } \delta_1\delta_2^{-1} = x^{-i}, i \geq 1, \\ 0,0,0\,(i=0,1,2) & \text{otherwise;} \end{cases}$$

If $\delta_1\delta_2^{-1} \neq x^{-i}, i \geq 1$ then the result follows by dimension counting, since the Hochschild-Serre spectral sequence degenerates at the second page (since $H^0(A^+, H^1(\overline{U}, M_{+,N})) = 0$).

In the case $\delta_1\delta_2^{-1} = x^{-i}$ for $i \geq 1$, we need to understand the transgression map

$$tr\colon H^0(A^+, H^1(\overline{U}, M_{+,N}))) \to H^2(A^+, H^0(\overline{U}, M_{+,N}))).$$

This is a morphism of one-dimensional L-vector spaces, so it is either the zero morphism, or it is an isomorphism. We claim that it is the zero morphism.

We first do the following general observation: let G be a semi-group, H a normal subgroup of G and M a G-module. Suppose that there exists a submodule M' of M which is stable by the action of G and such that the natural map $H^0(G/H, H^1(H, M')) \to H^0(G/H, H^1(H, M))$ induced by the inclusion $M' \subseteq M$ is an isomorphism. Then we claim that the transgression map tr_M : $H^0(G/H, H^1(H, M)) \to H^2(G/H, H^0(H, M))$ factors through

$$H^0(G/H, H^1(H, M)) \cong H^0(G/H, H^1(H, M')) \xrightarrow{tr_{M'}} H^2(G/H, H^0(H, M')) \\ \to H^2(G/H, H^0(H, M)).$$

Indeed, if c is a 2-cocycle with values in M whose cohomology class $[c]$ comes from the class of a 2-cocycle c' with values in M', then one can use indifferently c or c' in the recipe of [34, (1.6.6) Proposition] to calculate the transgression map of $[c]$. Using c', this recipe will actually provide an element in $H^2(G/H, H^0(H, M'))$, whose natural image in $H^2(G/H, H^0(H, M))$ is the transgression map of $[c]$.

Now we use the above observation with $M = M_{+,N}$ and $M' = M_{+,\kappa}$, with $\kappa = i - 1$, which is a submodule of $M_{+,N}$ stable by $\overline{P}^+$. Observe that, by Lemma 5.30, we have $H^1(\overline{U}, M_{+,N}) = L \cdot [t^N] \oplus L \cdot [t^\kappa]$ and thus $H^0(A^+, H^1(\overline{U}, M_{+,N})) = L \cdot [t^\kappa]$, while for $M_{+,\kappa}$, applying Lemma 5.30 with $N = \kappa = i - 1$ so that $\kappa \notin \{0, \ldots, N-1\}$, we get $H^1(\overline{U}, M_{+,\kappa}) = L \cdot [t^N] = L \cdot [t^\kappa]$, and hence $H^0(A^+, H^1(\overline{U}, M_{+,\kappa})) = L \cdot [t^\kappa]$. So we have $H^0(A^+, H^1(\overline{U}, M_{+,N})) = H^0(A^+, H^1(\overline{U}, M_{+,\kappa}))$ and hence a commutative diagram

$$\begin{array}{ccc} H^0(A^+, H^1(\overline{U}, M_{+,N})) & \xrightarrow{tr} & H^2(A^+, H^0(\overline{U}, M_{+,N})) \\ \downarrow\cong & & \uparrow \\ H^0(A^+, H^1(\overline{U}, M_{+,\kappa})) & \xrightarrow{tr} & H^2(A^+, H^0(\overline{U}, M_{+,\kappa})), \end{array}$$

where the left vertical map, which is the natural map, is an isomorphism. On the other hand, by Lemma 5.30 applied with $N = \kappa$, we get that $H^0(\overline{U}, M'_+) = L \cdot [t^0]$ and hence $H^2(A^+, H^0(\overline{U}, M'_+)) = 0$, which shows that the lower right term is zero and finishes the proof. □

5.7 The $\overline{P}^+$-cohomology of $\mathscr{R}(\delta_1, \delta_2)$

Call, for $* \in \{+, -, \emptyset\}$, M_* the module $\mathscr{R}^*(\delta_1, \delta_2)$. We use the short exact sequence

$$0 \to M_+ \to M \to M_- \to 0 \tag{21}$$

in order to calculate the $\overline{P}^+$-cohomology of M. We restate one of the propositions announced at the beginning of this section.

Proposition 5.33. *Let $M = \mathscr{R}(\delta_1, \delta_2)$. Then*

1. *If $\delta_1\delta_2^{-1} \notin \{x^{-i}, i \in \mathbf{N}\} \cup \{\chi x^i, i \in \mathbf{N}\}$, then $\dim_L H^j(\overline{P}^+, M) = 0, 1, 1, 0$ for $j = 0, 1, 2, 3$.*
2. *If $\delta_1\delta_2^{-1} = \mathbf{1}_{\mathbf{Q}_p^\times}$, then $\dim_L H^j(\overline{P}^+, M) = 1, 2, 1, 0$ for $j = 0, 1, 2, 3$.*
3. *If $\delta_1\delta_2^{-1} = x^{-i}, i \geq 1$, then $\dim_L H^j(\overline{P}^+, M) = 1, 3, 2, 0$ for $j = 0, 1, 2, 3$.*
4. *If $\delta_1\delta_2^{-1} = \chi x^i, i \in \mathbf{N}$, then $\dim_L H^j(\overline{P}^+, M) = 0, 2, 3, 1$ for $j = 0, 1, 2, 3$.*

Proof. For a $\overline{P}^+$-module N we note, for simplicity, $H^i(N) = H^i(\overline{P}^+, N)$. Consider the long exact sequence on cohomology associated to the sequence of Equation 21

$$\begin{aligned} 0 &\to H^0(M_+) \to H^0(M) \to H^0(M_-) \to H^1(M_+) \to H^1(M) \to H^1(M_-) \\ &\to H^2(M_+) \to H^2(M) \to H^2(M_-) \to H^3(M_+) \to H^3(M) \to H^3(M_-) \to 0. \end{aligned}$$

Recall that we have already calculated (cf. Lemmas 5.25, 5.26, 5.27, 5.28 and Proposition 5.32) all of the $H^j(M_\pm)$.
If $\delta_1\delta_2^{-1} \notin \{x^{-i}, i \in \mathbf{N}\}$, then $H^j(M_+) = 0$ for all j, which implies that $H^j(M) = H^j(M_-)$.
If $\delta_1\delta_2^{-1} = x^{-i}$, $i \in \mathbf{N}$, then $H^0(M_-) = H^3(M_-) = 0$ and the map $H^1(M) \to H^1(M_-)$ is the zero map (cf. [14, Corollaire 5.23(ii)], which is independent of [14, Lemme 5.21]). We hence have $H^1(M) = H^1(M_+)$, and an exact sequence

$$0 \to H^1(M_-) \to H^2(M_+) \to H^2(M) \to H^2(M_-) \to H^3(M_+) \to H^3(M) \to 0.$$

By the same arguments as the ones in the proof of [14, Théorème 5.16], we can show that in this case the map $H^3(M_+) \to H^3(M)$ is the zero map, hence we deduce $H^3(M) = 0$. Finally one can easily deduce $\dim_L H^2(M)$, depending on whether $i = 0$ or not. □

We finish this chapter with a proof of Theorem 5.2.

Proof of Theorem 5.2. Suppose first that $\delta_1\delta_2^{-1} \notin \{x^{-i}, i \geq 1\}$. Injectivity of $H^1(\overline{P}^+, \mathscr{R}(\delta_1, \delta_2)) \to H^1(A^+, \mathscr{R}(\delta_1, \delta_2))$ follows from [14, Lemme 5.22]. The result now follows since the dimensions of both sides (as L-vector spaces) are

the same (compare the dimensions from Proposition 5.1 and those from [14, Théorème 5.16]).
Now suppose $\delta_1\delta_2^{-1} = x^{-i}, i \geq 1$. In this case the morphisms (coming from taking cohomology of the short exact sequence $0 \to \mathscr{R}^+(\delta_1, \delta_2) \to \mathscr{R}(\delta_1, \delta_2) \to \mathscr{R}^-(\delta_1, \delta_2) \to 0$)

$$H^1(\overline{P}^+, \mathscr{R}^+(\delta_1, \delta_2)) \to H^1(\overline{P}^+, \mathscr{R}(\delta_1, \delta_2))$$

and

$$H^1(A^+, \mathscr{R}^+(\delta_1, \delta_2)) \to H^1(A^+, \mathscr{R}(\delta_1, \delta_2))$$

are isomorphisms (cf. proof of Proposition 5.33 and [14, Remarque 5.17], respectively). Thus it suffices to show that the restriction morphism

$$H^1(\overline{P}^+, \mathscr{R}^+(\delta_1, \delta_2)) \to H^1(A^+, \mathscr{R}^+(\delta_1, \delta_2)) \tag{22}$$

is surjective. By the proof of Proposition 5.32, we have an injection

$$H^1(A^+, H^0(\overline{U}, \mathscr{R}^+(\delta_1, \delta_2))) \to H^1(\overline{P}^+, \mathscr{R}^+(\delta_1, \delta_2)).$$

The composition with (22) gives an isomorphism

$$H^1(A^+, H^0(\overline{U}, \mathscr{R}^+(\delta_1, \delta_2))) \to H^1(A^+, \mathscr{R}^+(\delta_1, \delta_2))$$

since it induces the natural inclusion and dimensions coincide (cf. proof of Proposition 5.32 and [14, Proposition 5.13(ii)]). Therefore (22) is surjective. □

6 A relative cohomology isomorphism

In this chapter we prove our main technical result, namely the comparison between the relative $\overline{P}^+$ and A^+-cohomology of $\mathscr{R}_A(\delta_1, \delta_2)$. We recall one last time (cf. Remark 2.16) that, as A^+-modules, $\mathscr{R}_A(\delta_1, \delta_2)$ (resp. $\mathscr{R}_A^+(\delta_1, \delta_2)$, resp. $\mathscr{R}_A^-(\delta_1, \delta_2)$) is isomorphic to $\mathscr{R}_A(\delta_1\delta_2^{-1})$ (resp. $\mathscr{R}_A^+(\delta_1\delta_2^{-1})$, resp. $\mathscr{R}_A^-(\delta) \cong \mathrm{LA}(\mathbf{Z}_p, A) \otimes \delta$). We will freely use these identifications.

Definition 6.1. We say that a locally analytic character $\delta : \mathbf{Q}_p^\times \to A^\times$ is *regular* if it is pointwise (meaning the reduction for each maximal ideal $\mathfrak{m} \subset A$) never of the form χx^i or x^{-i} for some $i \geq 0$.

Remark 6.2. By [6, Corollaire 2.11], if δ is regular then $H^2(A^+, \mathscr{R}_A(\delta)) = 0$. Moreover in the setting of a point (A is a finite extension of $\mathbf{Q}_p$), $\delta_1\delta_2^{-1} : \mathbf{Q}_p^\times \to L^\times$ is regular implies the pair (δ_1, δ_2) is generic in the sense of [14].

6.1 The reduced case

The following is a relative version of [14, Proposition 5.18].

Proposition 6.3. *Suppose A is reduced. Let δ_1, $\delta_2 : \mathbf{Q}_p^\times \to A^\times$ such that $\delta_1\delta_2^{-1}$ is regular. Then the restriction morphism from $\overline{P}^+$ to A^+, induces a surjection:*

$$H^1(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2)) \to H^1(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1})).$$

Proof. We work at the derived level. For the sake of brevity let $C^\bullet_{\overline{P}^+}$ denote the Koszul complex $\mathscr{C}_{\tau,\varphi,\gamma}$ of §4.2, and similarly let $C^\bullet_{A^+}$ denote the complex $\mathscr{C}_{\varphi,\gamma}$. We have a canonical morphism:

$$C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2)) \to C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1}))$$

in $\mathcal{D}^-_{\mathrm{pc}}(A)$. Let

$$C^\bullet := \mathrm{Cone}(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2)) \to C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1})))$$

and note that $C^\bullet \in \mathcal{D}^-_{\mathrm{pc}}(A)$ by Remark 4.5. The distinguished triangle

$$C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2)) \to C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1})) \to C^\bullet$$

induces a long exact sequence in cohomology

$$\cdots \to H^1(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2))) \to H^1(C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1}))) \to H^1(C^\bullet) \to \cdots$$

Observe that, since $H^i(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2)))$ and $H^i(C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1})))$ are finitely generated modules (cf. Proposition 3.5 and Theorem 4.11), so are the modules $H^i(C^\bullet)$. Moreover since $\mathscr{R}_A(\delta_1, \delta_2)$ is a flat A-module and $\mathscr{R}_A(\delta_1, \delta_2) \otimes_A A/\mathfrak{m} \cong \mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2)$ for any maximal ideal $\mathfrak{m} \subset A$, it follows that

$$C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2)) \otimes^{\mathbf{L}} A/\mathfrak{m} \cong C^\bullet_{\overline{P}^+}(\mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2)).$$

Similarly we have

$$C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1})) \otimes^{\mathbf{L}} A/\mathfrak{m} \cong C^\bullet_{A^+}(\mathscr{R}_{A/\mathfrak{m}}(\delta_1\delta_2^{-1})).$$

Hence the morphism $A \to A/\mathfrak{m}$ induces a morphism of distinguished triangles which by the functoriality of the truncation operators gives a morphism of long exact sequences

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^1(C^\bullet_{\overline{P}^+}(X_A)) & \longrightarrow & H^1(C^\bullet_{A^+}(Y_A)) & \xrightarrow{\gamma} & H^1(C^\bullet) & \xrightarrow{\gamma_1} & \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & H^1(C^\bullet_{\overline{P}^+}(X_{A/\mathfrak{m}})) & \xrightarrow{\alpha} & H^1(C^\bullet_{A^+}(Y_{A/\mathfrak{m}})) & \xrightarrow{\beta} & H^1(C^\bullet \otimes^{\mathbf{L}} A/\mathfrak{m}) & \xrightarrow{\beta_1} & \cdots
\end{array}$$

where, for $R = A$ or $R = A/\mathfrak{m}$ a finite extension of $\mathbf{Q}_p$, we have noted $X_R = \mathscr{R}_R(\delta_1, \delta_2)$ and $Y_R = \mathscr{R}_R(\delta_1\delta_2^{-1})$.
By [14, Proposition 5.18] (see also Proposition 5.2), α is an isomorphism and so β is the zero morphism. We claim that γ is the zero morphism as well. To do this, we take advantage of the spectral sequence:

$$\mathrm{Tor}_{-p}(H^q(C^\bullet), A/\mathfrak{m}) \implies H^{p+q}(C^\bullet \otimes^{\mathbf{L}} A/\mathfrak{m}),$$

whose 2nd page takes the form

$$\begin{array}{ccc}
0 \searrow & 0 & 0 \\
\mathrm{Tor}_2(H^2(C^\bullet), A/\mathfrak{m}) \searrow & \mathrm{Tor}_1(H^2(C^\bullet), A/\mathfrak{m}) & H^2(C^\bullet) \otimes_A A/\mathfrak{m} \\
\mathrm{Tor}_2(H^1(C^\bullet), A/\mathfrak{m}) \searrow & \mathrm{Tor}_1(H^1(C^\bullet), A/\mathfrak{m}) & H^1(C^\bullet) \otimes_A A/\mathfrak{m} \\
\mathrm{Tor}_2(H^0(C^\bullet), A/\mathfrak{m}) & \mathrm{Tor}_1(H^0(C^\bullet), A/\mathfrak{m}) & H^0(C^\bullet) \otimes_A A/\mathfrak{m}
\end{array}$$

The long exact sequence in cohomology

$$\cdots \to H^3(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2))) \to H^3(C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1}))) = 0 \to H^3(C^\bullet) \to 0 \to \cdots$$

implies that $H^3(C^\bullet) = 0$ hence explaining the top row. Moreover since $\delta_1\delta_2^{-1}$ is regular, $H^2(C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1}))) = 0$ by [6, Corollaire 2.11] (see also Remark 3.6). By Proposition 5.1, $H^3(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2))) = 0$ and thus from the long exact sequence

$$\cdots \to 0 \to H^2(C^\bullet) \to H^3(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2))) = 0 \to 0 \to \cdots$$

we deduce that $H^2(C^\bullet) = 0$. Hence the spectral sequence degenerates at the 2nd page in degree 1 cohomology and so $H^1(C^\bullet \otimes^{\mathbf{L}} A/\mathfrak{m}) = H^1(C^\bullet) \otimes_A A/\mathfrak{m}$. Similarly the spectral sequence

$$\mathrm{Tor}_{-p}(H^q(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1\delta_2^{-1}))), A/\mathfrak{m}) \implies H^{p+q}(C^\bullet_{\overline{P}^+}(\mathscr{R}_{A/\mathfrak{m}}(\delta_1\delta_2^{-1})))$$

implies $H^2(C^\bullet_{\overline{P}^+}(\mathscr{R}_{A/\mathfrak{m}}(\delta_1\delta_2^{-1}))) = H^2(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1\delta_2^{-1}))) \otimes_A A/\mathfrak{m}$. Thus in the diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow H^1(C^\bullet_{A^+}(Y_A)) & \xrightarrow{\gamma} & H^1(C^\bullet) & \xrightarrow{\gamma_1} & H^2(C^\bullet_{\overline{P}^+}(X_A)) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow H^1(C^\bullet_{A^+}(Y_{A/\mathfrak{m}})) & \xrightarrow{\beta} & H^1(C^\bullet \otimes^{\mathbf{L}} A/\mathfrak{m}) & \xrightarrow{\beta_1} & H^2(C^\bullet_{\overline{P}^+}(X_{A/\mathfrak{m}})) & \longrightarrow & 0
\end{array}$$

we have $\beta_1 = \gamma_1 \otimes A/\mathfrak{m}$, where we have noted again $X_R = \mathscr{R}_R(\delta_1, \delta_2)$ and $Y_R = \mathscr{R}_R(\delta_1\delta_2^{-1})$, for $R \in \{A, A/\mathfrak{m}\}$. By Proposition 5.2, β_1 is an isomorphism of dimension 1 vector spaces over $A/\mathfrak{m}$ for every $\mathfrak{m} \subset A$. By [27, Lemma 2.1.8(1)] the modules $H^1(C^\bullet)$ and $H^2(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2)))$ are flat and, since they are finitely generated, they are locally free. Thus γ_1 is a surjective morphism of locally free A-modules which are locally of dimension 1. Hence it is an isomorphism and so γ is the zero morphism, as desired. This completes the proof. □

Proposition 6.4. *Suppose A is reduced. Let $\delta_1,\ \delta_2 : \mathbf{Q}_p^\times \to A^\times$ such that $\delta_1\delta_2^{-1}$ is regular. Then the restriction morphism from $\overline{P}^+$ to A^+, induces an injection:*

$$H^1(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2)) \to H^1(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1})).$$

Proof. Keeping the notation used in the proof of Proposition 6.3, since $H^1(C^\bullet)$ is locally free, $\mathrm{Tor}_1(H^1(C^\bullet), A/\mathfrak{m}) = 0$. But the spectral sequence

$$\mathrm{Tor}_{-p}(H^q(C^\bullet), A/\mathfrak{m}) \implies H^{p+q}(C^\bullet \otimes^{\mathbf{L}} A/\mathfrak{m})$$

abuts to 0 in degree 0 as α is an isomorphism. This implies that $H^0(C^\bullet) \otimes A/\mathfrak{m} = 0$ for every maximal ideal $m \subset A$. By Nakayama's Lemma it follows that $H^0(C^\bullet)$ is 0 and we deduce the result. □

Theorem 6.5. *Suppose A is reduced. Let $\delta_1,\ \delta_2 : \mathbf{Q}_p^\times \to A^\times$ such that $\delta_1\delta_2^{-1}$ is regular. Then the restriction morphism from $\overline{P}^+$ to A^+, induces an isomorphism:*

$$H^1(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2)) \to H^1(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1})).$$

Proof. This is a consequence of Propositions 6.3 and 6.4. □

6.2 The non-reduced case

We are now ready to handle the case when A is non-reduced. We begin by proving a slightly enhanced version of Theorem 6.5.

Proposition 6.6. *Suppose A is reduced and M a finite A-module (equipped with trivial $\overline{P}^+$-action). Let $\delta_1,\ \delta_2 : \mathbf{Q}_p^\times \to A^\times$ such that $\delta_1\delta_2^{-1}$ is regular. Then the restriction morphism from $\overline{P}^+$ to A^+, induces an isomorphism:*

$$H^1(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2) \otimes_A M) \to H^1(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1}) \otimes_A M).$$

Proof. To prove surjectivity, we follow the proof of Proposition 6.3. Note that by the spectral sequence:

$$\mathrm{Tor}_{-p}(H^q(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2))), M) \implies H^{p+q}(C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2) \otimes_A M))$$

we get that the $H^i(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2) \otimes_A M)$ are finitely generated A-modules. Similarly the groups $H^i(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1}) \otimes_A M)$ are finitely generated A-modules. Thus one obtains that the cone of the morphism

$$C^\bullet_{\overline{P}^+}(\mathscr{R}_A(\delta_1, \delta_2) \otimes_A M) \to C^\bullet_{A^+}(\mathscr{R}_A(\delta_1\delta_2^{-1}) \otimes_A M)$$

also has cohomology which are finitely generated A-modules. Therefore as in the proof of Proposition 6.3, one obtains that the H^1 of this cone is a locally free A-module.

The only thing that remains to be checked is that

$$H^1(\overline{P}^+, \mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2) \otimes_A M) \to H^1(A^+, \mathscr{R}_{A/\mathfrak{m}}(\delta_1\delta_2^{-1}) \otimes_A M)$$

is an isomorphism. Denote by $M' := M \otimes_A A/\mathfrak{m}$, so that the above is equivalent to showing

$$H^1(\overline{P}^+, \mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2) \otimes_{A/\mathfrak{m}} M') \to H^1(A^+, \mathscr{R}_{A/\mathfrak{m}}(\delta_1\delta_2^{-1}) \otimes_{A/\mathfrak{m}} M'), \quad (23)$$

is an isomorphism. Now M' is an $A/\mathfrak{m}$-vector space of finite dimension and, since the first cohomology is an additive functor, we have

$$H^1(\overline{P}^+, \mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2) \otimes_{A/\mathfrak{m}} M') \cong H^1(\overline{P}^+, \mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2)) \otimes_{A/\mathfrak{m}} M'$$

and similarly

$$H^1(A^+, \mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2) \otimes_{A/\mathfrak{m}} M') \cong H^1(A^+, \mathscr{R}_{A/\mathfrak{m}}(\delta_1, \delta_2)) \otimes_{A/\mathfrak{m}} M'.$$

Thus the morphism 23 is an isomorphism by [14, Proposition 5.18]. The rest of the proof of Proposition 6.3 goes through with $C^\bullet$ replaced by $C^\bullet \otimes^{\mathbf{L}} M$. For injectivity the proof of Proposition 6.4 remains unchanged except with $C^\bullet$ replaced by $C^\bullet \otimes^{\mathbf{L}} M$. This completes the proof. □

We now need a lemma that guarantees that the connection morphisms are 0 in a certain long exact sequence.

Lemma 6.7. *Let A be an affinoid $\mathbf{Q}_p$-algebra and $I \subset A$ an ideal of A. Let δ_1, $\delta_2 : \mathbf{Q}_p^\times \to A^\times$ such that $\delta_1\delta_2^{-1}$ is regular. The short exact sequence*

$$0 \to I \to A \to A/I \to 0$$

induces an injective morphism

$$H^2(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2) \otimes_A I) \to H^2(\overline{P}^+, \mathscr{R}_A(\delta_1, \delta_2)).$$

Proof. Note first that $H^i(\overline{P}^+, \mathscr{R}_L^+(\delta_1, \delta_2)) = 0$ for every finite extension L of $\mathbf{Q}_p$ and for all i. Thus, as in Remark 3.6, the Tor-spectral sequence and Nakayama's Lemma imply that $H^i(\overline{P}^+, \mathscr{R}_A^+(\delta_1, \delta_2) \otimes_A I) = H^i(\overline{P}^+, \mathscr{R}_A^+(\delta_1, \delta_2)) = 0$. Then, using the 4-lemma, one reduces to showing that the morphism

$$H^2(\overline{P}^+, \mathscr{R}_A^-(\delta_1, \delta_2) \otimes_A I) \to H^2(\overline{P}^+, \mathscr{R}_A^-(\delta_1, \delta_2))$$

is injective.
Recall the Koszul complex from §4.2:

$$\mathscr{C}_{\tau,\varphi,\gamma} : 0 \to M \xrightarrow{X} M \oplus M \oplus M \xrightarrow{Y} M \oplus M \oplus M \xrightarrow{Z} M \to 0 \qquad (24)$$

where the arrows are defined as

$$\begin{aligned} X(x) &= ((1-\tau)x, (1-\varphi)x, (\gamma-1)x) \\ Y(x,y,z) &= ((1-\varphi\delta_p)x + (\tau-1)y, (\gamma\delta_a - 1)x + (\tau-1)z, (\gamma-1)y + (\varphi-1)z) \\ Z(x,y,z) &= (\gamma\delta_a - 1)x + (\varphi\delta_p - 1)y + (1-\tau)z. \end{aligned}$$

Noting $M_A = \mathscr{R}_A^-(\delta_1, \delta_2)$, we need to show that if $(x,y,z) \in M_A^{\oplus 3}$ such that $Y(x,y,z) \in M_I^{\oplus 3}$, where $M_I := M_A \otimes_A I$, then $(x,y,z) \in M_I$.
Now $H^2(A^+, \mathscr{R}_L^-(\delta_1,\delta_2)) = 0$ for every finite extension L of $\mathbf{Q}_p$ and hence once more the Tor-spectral sequence and Nakayama's Lemma imply that $H^2(A^+, M_I) = H^2(A^+, M_A) \otimes_A I = 0$. It follows that

$$H^2(A^+, M_I) \to H^2(A^+, M_A)$$

is trivially injective and so we can assume $(x,y) \in M_I^{\oplus 2}$. Thus the problem is the following: if $f \in M_A$ is such that

$$(1-\gamma\delta_a)f \in M_I \text{ and } (1-\varphi\delta_p)f \in M_I$$

then one must show that $f \in M_I$.
Bearing in mind Remark 2.16 and the identifications of A-moudules $M_A = \mathrm{LA}(\mathbf{Z}_p, A)$, $M_I = \mathrm{LA}(\mathbf{Z}_p, I)$, if $\phi \in \mathrm{LA}(\mathbf{Z}_p, A)$ is such that $(1-\varphi\delta_p)\phi \in \mathrm{LA}(\mathbf{Z}_p, I)$, then call $\overline{\phi} \in \mathrm{LA}(\mathbf{Z}_p, A/I)$ the reduction of ϕ modulo I. We have then $(1-\varphi\delta_p)(\overline{\phi}) = 0$, but $1-\varphi\delta_p$ is injective on $\mathrm{LA}(\mathbf{Z}_p, A/I)$ (cf. Lemma 2.16), so $\overline{\phi} = 0$, which translates into $\phi \in \mathrm{LA}(\mathbf{Z}_p, I)$. This finishes the proof. □

Theorem 6.8. *Let δ_1, $\delta_2 : \mathbf{Q}_p^\times \to A^\times$ such that $\delta_1\delta_2^{-1}$ is regular. Then the restriction morphism from $\overline{P}^+$ to A^+, induces an isomorphism:*

$$H^1(\overline{P}^+, \mathscr{R}_A(\delta_1,\delta_2)) \to H^1(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1})).$$

Proof. Since A is in particular noetherian its nilradical N is nilpotent. Thus it is natural to proceed via induction on the index of nilpotence $i \geq 0$. The base case $i = 0$ (meaning A is reduced) is Theorem 6.5. Suppose by induction the result is true for index i and suppose now $N^{i+1} = 0$. For the sake of brevity denote $X_{N^i} := \mathscr{R}_A(\delta_1,\delta_2) \otimes_A N^i$, $X_A := \mathscr{R}_A(\delta_1,\delta_2)$ and $X_{A/N^i} := \mathscr{R}_A(\delta_1,\delta_2) \otimes_A A/N^i$. By flatness of $\mathscr{R}_A(\delta_1,\delta_2)$ (cf. Lemma 2.1 and observe that, as A-modules, $\mathscr{R}_A(\delta_1,\delta_2)$ is isomorphic to $\mathscr{R}_A$) we have a short exact sequence

$$0 \to X_{N^i} \to X_A \to X_{A/N^i} \to 0$$

that gives a commutative diagram

$$\begin{array}{ccccccccc} H^0(\overline{P}^+, X_{A/N^i}) & \xrightarrow{\beta'} & H^1(\overline{P}^+, X_{N^i}) & \longrightarrow & H^1(\overline{P}^+, X_A) & \longrightarrow & H^1(\overline{P}^+, X_{A/N^i}) & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\ H^0(A^+, X_{A/N^i}) & \xrightarrow{\beta} & H^1(A^+, X_{N^i}) & \longrightarrow & H^1(A^+, X_A) & \longrightarrow & H^1(A^+, X_{A/N^i}) & \longrightarrow & 0. \end{array}$$

Indeed, the two rows come from long exact sequences in cohomology, commutativity comes from functoriality of the restriction morphism $H^i(\overline{P}^+, -) \to H^i(A^+, -)$, and exactness at $H^1(\overline{P}^+, X_{A/N^i})$ in the top line follows from Lemma 6.7. Identifying X_{A/N^i} with $\mathscr{R}_{A/N^i}(\delta_1, \delta_2)$ and X_{N^i} with $\mathscr{R}_{A/N}(\delta_1, \delta_2) \otimes_{A/N} N^i$ (cf. the proof of Lemma 2.17); we see that α_1 is an isomorphism by Proposition 6.6 and α_3 is an isomorphism by the inductive step. On the other hand, by [6, Proposition 2.10], the morphism

$$H^0(A^+, X_A) \to H^0(A^+, X_{A/N^i})$$

is surjective. Thus β is the zero morphism. By the commutativity of the first square, this implies that β' is the zero morphism. Thus by the 5-Lemma, α_2 is an isomorphism and this proves the result. □

7 Construction of the correspondence

In this section we construct, following [14, Chapter 6], the correspondence $\Delta \mapsto \Pi(\Delta)$ for a regular (φ, Γ)-module Δ over the relative Robba ring $\mathscr{R}_A$, interpolating the analogous construction of loc. cit. at the level of points. The construction is inspired from the calculation of the locally analytic vectors in the unitary principal series case (corresponding to the case when Δ is trianguline and étale), cf. [8], [10], [13] and involves a detailed study of the Jordan-Hölder components of the G-module $\Delta \boxtimes_\omega \mathbf{P}^1$. We will see that the sought-after representation $\Pi(\Delta)$ is cut out from these constituents.

7.1 The main result

We begin with a definition.

Definition 7.1. Let Δ be a trianguline (φ, Γ)-module over $\mathscr{R}_A$, which is an extension of $\mathscr{R}_A(\delta_2)$ by $\mathscr{R}_A(\delta_1)$. We say that Δ is regular if $\delta_1\delta_2^{-1} \colon \mathbf{Q}_p^\times \to A^\times$ is regular in the sense of Definition 6.1.

The following theorem is a relative version of [14, Theorem 6.11] in the case when the pair (δ_1, δ_2) is regular'.

Theorem 7.2. *Suppose Δ is a regular (φ, Γ)-module over $\mathscr{R}_A$ such that*

$$0 \to \mathscr{R}_A(\delta_1) \to \Delta \to \mathscr{R}_A(\delta_2) \to 0.$$

Then there exists a locally analytic A-representation $\Pi(\Delta)$[31] *of* $\mathrm{GL}_2(\mathbf{Q}_p)$*, with central character* ω*, such that we have an exact sequence*

$$0 \to \Pi(\Delta)^* \otimes \omega \to \Delta \boxtimes_\omega \mathbf{P}^1 \to \Pi(\Delta) \to 0.$$

Moreover $\Pi(\Delta)$ *is an extension of* $B_A(\delta_2, \delta_1)$ *by* $B_A(\delta_1, \delta_2)$*. Furthermore if* Δ *is a non-trivial extension of* $\mathscr{R}_A(\delta_2)$ *by* $\mathscr{R}_A(\delta_1)$ *then* $\Pi(\Delta)$ *is a non-trivial extension of* $B_A(\delta_2, \delta_1)$ *by* $B_A(\delta_1, \delta_2)$*.*

Remark 7.3. Contrary to [14, Theorem 6.11], unless A is a finite extension of $\mathbf{Q}_p$, then $\Pi(\Delta)$ will not be of compact type (this is because A is not of compact type as a locally convex $\mathbf{Q}_p$-vector space). Thus $\Pi(\Delta)$ is almost never an admissible G-representation in the sense of [40]. It is however of A-LB-type, cf. Definition A.15.

Before we begin to prove Theorem 7.2 we need to define and construct the G-module $\Delta \boxtimes_\omega \mathbf{P}^1$.

7.2 Notations

We let $\mathrm{Ext}^1(\mathscr{R}_A(\delta_2), \mathscr{R}_A(\delta_1))$ denote the group of extensions of $\mathscr{R}_A(\delta_2)$ by $\mathscr{R}_A(\delta_1)$ in the category of (φ, Γ)-modules over $\mathscr{R}_A$. Note that, since every (φ, Γ)-module over $\mathscr{R}_A$ is analytic (cf. the paragraph following Definition 2.4), this last group coincides with the extension group $\mathrm{Ext}^1_{\mathrm{an}}(\mathscr{R}_A(\delta_2), \mathscr{R}_A(\delta_1))$ in the category of analytic (φ, Γ)-modules over $\mathscr{R}_A$ [32].
Let H be a finite dimensional locally $\mathbf{Q}_p$-analytic group. We refer the reader to the appendix for the necessary definitions and properties of the theory of locally analytic H-representations in A-modules. We just recall that $\mathscr{G}_{H,A}$ (cf. Definition A.39) denotes the category of complete Hausdorff locally convex A-modules equipped with a separately continuous A-linear $\mathscr{D}(H, A)$-module structure and we let $\mathrm{Ext}^1_H(M, N)$ [33] denote the group of extensions of M by N in the category $\mathscr{G}_{H,A}$.

Example 7.4. From Lemma 2.14, Proposition 2.18 and Lemma 2.19, it follows that, if $? \in \{+, -, \emptyset\}$, then the spaces $\mathscr{R}^?_A(\delta_i) \boxtimes_\omega \mathbf{P}^1$ are objects of the category $\mathscr{G}_{G,A}$.

If H_2 is a closed locally $\mathbf{Q}_p$-analytic subgroup of a locally $\mathbf{Q}_p$-analytic group H_1, we have an induction functor $\mathrm{ind}_{H_1}^{H_2} \colon \mathscr{G}_{H_2,A} \to \mathscr{G}_{H_1,A}$, cf. Lemma A.54. We cite the following fact from the appendix that will be of much use to us, cf. Proposition A.56.

[31] It is probably unique but we are unable to show this. This comes down to knowing that the Ext^1 of certain principal series is a free A-module of rank 1.

[32] This fact can also be seen by using the bijections $\mathrm{Ext}^1(\mathscr{R}_A(\delta_2), \mathscr{R}_A(\delta_1)) = H^1(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1})) = H^1_{\mathrm{an}}(A^+, \mathscr{R}_A(\delta_1\delta_2^{-1})) = \mathrm{Ext}^1_{\mathrm{an}}(\mathscr{R}_A(\delta_2), \mathscr{R}_A(\delta_1))$ where the equalities follow from [6, Lemme 2.2], Proposition 3.3 and Proposition 3.2, respectively.

[33] Note that this is called $\mathrm{Ext}^1_{\mathscr{G}_{H,A}}(M, N)$ in the appendix. We warn the reader that $\mathscr{G}_{H,A}$ is not an abelian category and so one needs to define precisely what the group of extensions means, cf. Definition A.47.

Proposition 7.5 (Relative Shapiro's Lemma). *Let H_1 be a locally $\mathbf{Q}_p$-analytic group and let H_2 be a closed locally $\mathbf{Q}_p$-analytic subgroup. If M and N are objects of $\mathscr{G}_{H_2,A}$ and $\mathscr{G}_{H_1,A}$, respectively, then there are A-linear bijections*

$$\mathrm{Ext}^q_{H_1}(\mathrm{ind}_{H_2}^{H_1}(M), N) \to \mathrm{Ext}^q_{H_2}(M, N)$$

for all $q \geq 0$.

7.3 Extensions of $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$

Denote by $\overline{P} = \begin{pmatrix} \mathbf{Q}_p^\times & 0 \\ \mathbf{Q}_p & 1 \end{pmatrix}$ the lower-half mirabolic subgroup of the lower-half Borel $\overline{B} = \begin{pmatrix} \mathbf{Q}_p^\times & 0 \\ \mathbf{Q}_p & \mathbf{Q}_p^\times \end{pmatrix}$ and $\overline{U}^1 = \begin{pmatrix} 1 & 0 \\ p\mathbf{Z}_p & 1 \end{pmatrix}$. We are now ready to state the first result toward a proof of Theorem 7.2, which is essentially a formal consequence of Theorem 6.8.

Theorem 7.6. *Let $\delta_1,\ \delta_2 \colon \mathbf{Q}_p^\times \to A^\times$ such that $\delta_1\delta_2^{-1}$ is regular. Then there is a natural isomorphism*

$$\mathrm{Ext}^1_G(\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1, \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1) \cong \mathrm{Ext}^1(\mathscr{R}_A(\delta_2), \mathscr{R}_A(\delta_1)).$$

Proof. Denote by $\mathscr{R}_A(\delta_1, \delta_2) \boxtimes \mathbf{P}^1$ the $\overline{P}$-module $(\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1) \otimes \delta_2^{-1}$, so that $\mathscr{R}_A(\delta_1, \delta_2)$ is identified with the $\overline{P}^+$-submodule $(\mathscr{R}(\delta_1) \boxtimes_\omega \mathbf{Z}_p) \otimes \delta_2^{-1}$ of $\mathscr{R}_A(\delta_1, \delta_2) \boxtimes \mathbf{P}^1$. The proof is done in several steps and follows the proof of [14, Théorème 6.1] in the case where A is a finite extension of $\mathbf{Q}_p$.

(Step 1) We first descend from G to $\overline{P}$ using Shapiro's Lemma. Since $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1 \cong \mathrm{Ind}_{\overline{B}}^G(\delta_1\chi^{-1} \otimes \delta_2)^* \otimes \omega$, cf. Lemma 2.14, using Lemma A.57 we get that

$$\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1 \cong \mathrm{Ind}_{\overline{B}}^G(\delta_2^{-1} \otimes \delta_1^{-1}\chi)^* \cong \mathrm{ind}_{\overline{B}}^G(\delta_2 \otimes \chi^{-1}\delta_1).$$

So by Proposition 7.5 (for $q = 1$) we get

$$\mathrm{Ext}^1_G(\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1, \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1) \cong \mathrm{Ext}^1_{\overline{B}}(\delta_2 \otimes \chi^{-1}\delta_1, \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1).$$

Since we are only interested in locally analytic representations with a central character ω, we don't loose any information by passing from $\overline{B}$ to $\overline{P}$ (since both $\delta_2 \otimes \chi^{-1}\delta_1$ and $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$ have the same central character, namely ω) and thus we have

$$\mathrm{Ext}^1_{\overline{B}}(\delta_2 \otimes \chi^{-1}\delta_1, \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1) \cong \mathrm{Ext}^1_{\overline{P}}(\delta_2 \otimes \chi^{-1}\delta_1, \mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1).$$

Then, since (as $\overline{P}$-modules) $\mathscr{R}_A(\delta_1, \delta_2) \boxtimes \mathbf{P}^1 \cong (\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1) \otimes (\delta_2^{-1} \otimes \chi\delta_1^{-1})$, the RHS in the above equality is also equal to

$$H^1_{\mathrm{an}}(\overline{P}, \mathscr{R}_A(\delta_1, \delta_2) \boxtimes \mathbf{P}^1).$$

(Step 2) We now descend from $\overline{P}$ to $\overline{P}^+$. That is the restriction of $\mathscr{R}_A(\delta_1,\delta_2)\boxtimes\mathbf{P}^1$ to a $\overline{P}^+$-module induces an isomorphism

$$H^1_{\mathrm{an}}(\overline{P},\mathscr{R}_A(\delta_1,\delta_2)\boxtimes\mathbf{P}^1)=H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A(\delta_1,\delta_2)\boxtimes\mathbf{P}^1).$$

This is shown in the exact same way as in [14, Lemme 6.4].

(Step 3) Finally we descend from $\mathscr{R}_A(\delta_1,\delta_2)\boxtimes\mathbf{P}^1$ to $\mathscr{R}_A(\delta_1,\delta_2)$. More precisely we show that the inclusion $\mathscr{R}_A(\delta_1,\delta_2)\subset\mathscr{R}_A(\delta_1,\delta_2)\boxtimes\mathbf{P}^1$ (as $\overline{P}^+$-modules) induces an isomorphism

$$H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A(\delta_1,\delta_2))\cong H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A(\delta_1,\delta_2)\boxtimes\mathbf{P}^1).$$

Indeed by the long exact sequence in cohomology associated to the short exact sequence

$$0\to\mathscr{R}_A(\delta_1,\delta_2)\to\mathscr{R}_A(\delta_1,\delta_2)\boxtimes\mathbf{P}^1\to Q\to 0,\qquad(25)$$

where we define $Q:=\mathscr{R}_A(\delta_1,\delta_2)\boxtimes\mathbf{P}^1)/\mathscr{R}_A(\delta_1,\delta_2)$ as a $\overline{P}^+$-module, it suffices to show that $H^0_{\mathrm{an}}(\overline{P}^+,Q)=H^1_{\mathrm{an}}(\overline{P}^+,Q)=0$.

First observe that $Q=\mathscr{R}_A(\delta_1,\delta_2)\boxtimes(\mathbf{P}^1-\mathbf{Z}_p)$ as $\overline{U}^1$-modules and that $H^1_{\mathrm{an}}(\overline{U}^1,Q)=0$. Indeed, since $\overline{U}^1=\begin{pmatrix}0&1\\p&0\end{pmatrix}U^0\begin{pmatrix}0&p^{-1}\\1&0\end{pmatrix}$, by the equivariance of the sheaf, and since $\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)(\mathbf{P}^1-\mathbf{Z}_p)=\mathbf{Z}_p$, the matrix $\left(\begin{smallmatrix}0&1\\p&0\end{smallmatrix}\right)$ induces an isomorphism between the pairs $(\overline{U}^1,Q)$ and $(U^0,\mathscr{R}_A)$. It is then enough to show that $H^1_{\mathrm{an}}(U^0,\mathscr{R}_A)=0$, which follows from Lemma 3.1. For the same reason $H^0_{\mathrm{an}}(\overline{U}^1,Q)=0$ and so $H^0_{\mathrm{an}}(\overline{P}^+,Q)=0$.

Finally, let $g\mapsto c_g$ be a locally analytic 1-cocycle over $\overline{P}^+$ with values in Q. By adding a coboundary we can assume that $c_g=0$ for every $g\in\overline{U}^1$. For $a\in\mathbf{Z}_p\backslash\{0\}$, let $\alpha(a)=\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)$. By the relation $\alpha(a)\left(\begin{smallmatrix}1&0\\ap&1\end{smallmatrix}\right)=\left(\begin{smallmatrix}1&0\\p&1\end{smallmatrix}\right)\alpha(a)$ we get $c_{\alpha(a)}=\left(\begin{smallmatrix}1&0\\p&1\end{smallmatrix}\right)c_{\alpha(a)}$ and thus $c_{\alpha(a)}=0$ for every $a\in\mathbf{Z}_p\backslash\{0\}$ since $\left(\begin{smallmatrix}1&0\\p&1\end{smallmatrix}\right)-1$ is injective on Q (since $w\left(\left(\begin{smallmatrix}1&0\\p&1\end{smallmatrix}\right)-1\right)w=\left(\begin{smallmatrix}1&p\\0&1\end{smallmatrix}\right)-1$ is injective on $\mathscr{R}(\delta_1,\delta_2)\boxtimes_\omega p\mathbf{Z}_p$). Thus $c_g=0$ for all $g\in\overline{P}^+$.

Steps 1-3 show that

$$\mathrm{Ext}^1_G(\mathscr{R}^+_A(\delta_2)\boxtimes_\omega\mathbf{P}^1,\mathscr{R}_A(\delta_1)\boxtimes_\omega\mathbf{P}^1)\cong H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A(\delta_1,\delta_2)).$$

The result now follows from Theorem 6.8 and Propositions 3.2, 3.3 and Remark 3.4.

□

7.4 THE G-MODULE $\Delta \boxtimes_\omega \mathbf{P}^1$

In this section, following [14, §6.3], we show that, for a trianguline (φ,Γ)-module $\Delta \in \mathrm{Ext}^1(\mathscr{R}_A(\delta_2),\mathscr{R}_A(\delta_1))$, there exists a unique extension $\Delta \boxtimes_\omega \mathbf{P}^1$ of $\mathscr{R}_A(\delta_2) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$ extending the construction given by Theorem 7.6. As in the introduction, we observe that working in the context of cyclotomic (φ,Γ) simplifies considerably several proofs and constructions. We begin by a lemma permitting to extend the involution.

PROPOSITION 7.7. *Let $\Delta, \Delta_1 \in \Phi\Gamma(\mathscr{R}_A)$ be in an exact sequence $0 \to \Delta_1 \to \Delta \xrightarrow{\alpha} \mathscr{R}_A(\delta) \to 0$, for $\delta\colon \mathbf{Q}_p^\times \to A^\times$ locally analytic, and let $\Delta_+ = \alpha^{-1}(\mathscr{R}_A^+(\delta)) \subseteq \Delta$. Let $j_+\colon \mathscr{R}_A^+(\Gamma) \to \mathscr{R}_A^+(\Gamma)$ and $j\colon \mathscr{R}_A(\Gamma) \to \mathscr{R}_A(\Gamma)$ be the involutions defined by $\sigma_a \mapsto \delta(a)\sigma_a^{-1}$. Then any $\mathscr{R}_A^+(\Gamma)$-anti-linear involution $\iota\colon \Delta_+ \boxtimes \mathbf{Z}_p^\times \to \Delta_+ \boxtimes \mathbf{Z}_p^\times$ with respect to j_+[34] stabilizing $\Delta_1 \boxtimes \mathbf{Z}_p^\times$ extends uniquely to an $\mathscr{R}_A(\Gamma)$-anti-linear involution with respect to j on $\Delta \boxtimes \mathbf{Z}_p^\times$.*

Proof. As $\mathscr{R}_A^+(\Gamma)$-modules we have

$$\Delta_+ \boxtimes \mathbf{Z}_p^\times \cong (\Delta_1 \boxtimes \mathbf{Z}_p^\times) \oplus \mathscr{R}_A^+(\Gamma) \cdot e_2,$$

for some $e_2 \in \Delta_+ \boxtimes \mathbf{Z}_p^\times$, with $\sigma_a(e_2) = \delta(a)e_2$, $a \in \mathbf{Z}_p^\times$; and similarly, as $\mathscr{R}_A(\Gamma)$-modules we have

$$\Delta \boxtimes \mathbf{Z}_p^\times \cong (\Delta_1 \boxtimes \mathbf{Z}_p^\times) \oplus \mathscr{R}_A(\Gamma) \cdot e_2,$$

so that, in particular, the module $\Delta_+ \boxtimes \mathbf{Z}_p^\times$ contains a basis of $\Delta \boxtimes \mathbf{Z}_p^\times$ as a $\mathscr{R}_A(\Gamma)$-module. Since the involution ι' we are looking for (on $\Delta \boxtimes \mathbf{Z}_p^\times$) extends ι and is $\mathscr{R}_A(\Gamma)$-anti-linear, we are forced to set, for any $z = z_1 + \lambda e_2$, $z_1 \in \Delta_1$, $\lambda \in \mathscr{R}_A(\Gamma)$,

$$\iota'(z) = \iota(z_1) + j(\lambda)\iota(e_2).$$

Since every element of Δ can be uniquely written in this way, we deduce the result.

□

Denote by $G^+ = \begin{pmatrix} \mathbf{Z}_p & \mathbf{Z}_p \\ p\mathbf{Z}_p & \mathbf{Z}_p^\times \end{pmatrix} \cap G$, $\overline{B}^+ = \begin{pmatrix} \mathbf{Z}_p\backslash\{0\} & 0 \\ p\mathbf{Z}_p & \mathbf{Z}_p^\times \end{pmatrix}$ and note that $P^+ \subset G^+$, $\overline{B}^+ \subset G^+$ and that G^+ stabilizes $\mathbf{Z}_p$ so that, if M is a G-equivariant sheaf over $\mathbf{P}^1$, then $M \boxtimes \mathbf{Z}_p$ inherits an action of G^+. The next result explicitly describes the isomorphism of Theorem 7.6 and gives the construction of the G-module $\Delta \boxtimes_\omega \mathbf{P}^1$ for a regular (φ,Γ)-module Δ over $\mathscr{R}_A$.

PROPOSITION 7.8. *Let M be a non-trivial extension of $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{P}^1$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$. Then:*

1. *M constains a unique G^+-submodule Δ_+ which is an extension of $\mathscr{R}_A^+(\delta_2) \boxtimes_\omega \mathbf{Z}_p$ by $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{Z}_p$.*

[34] i.e. satisfying $\iota \circ \lambda = j_+(\lambda) \circ \iota$

2. *There exists a unique* $\Delta \in \Phi\Gamma(\mathscr{R}_A)$ *which is an extension of* $\mathscr{R}_A(\delta_2)$ *by* $\mathscr{R}_A(\delta_1)$ *such that* Δ_+ *is identified with the inverse image of* $\mathscr{R}_A^+(\delta_2)$ *in* Δ.

3. $\Delta_+ \boxtimes \mathbf{Z}_p^\times$ *is stable under* w *and, if we denote by* ι *the involution of* $\Delta_+ \boxtimes \mathbf{Z}_p^\times$ *induced by* w*, then* $M = \Delta_+ \boxtimes_{\omega,\iota} \mathbf{P}^1$.

4. *The involution* ι *extends uniquely to a* $\mathscr{R}_A(\Gamma)$*-anti-linear involution (with respect to* j *defined above) on* $\Delta \boxtimes \mathbf{Z}_p^\times$ *and* $\Delta \boxtimes_{w,\iota} \mathbf{P}^1$ *is a* G*-module which is an extension of* $\mathscr{R}_A(\delta_2) \boxtimes_\omega \mathbf{P}^1$ *by* $\mathscr{R}_A(\delta_1) \boxtimes_\omega \mathbf{P}^1$.

Proof. The first three points follow as in the proof of [14, Proposition 6.7], which is the non-relative version of this result, and observing that all the notions of locally analytic representations used there make sense in the relative setting.
We prove (4). The existence and uniqueness of ι follows from Proposition 7.7. For the last part it suffices to show that the action of $\tilde{G}$ on $\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1$ factorizes via G. First note that if A is a finite extension of $\mathbf{Q}_p$ the result follows from [14, Proposition 6.7(iv)].
We now proceed by induction on the index $i \geq 0$ of nilpotence of A. Suppose first that A is reduced. Take $(z_1, z_2) \in \Delta \boxtimes_{\omega,\iota} \mathbf{P}^1$ and g in the kernel of $\tilde{G} \to G$. It suffices to show that $y = (g-1)z = 0$. Call $y = (y_1, y_2)$. Let $\mathfrak{m} \subset A$ be a maximal ideal. By Lemma 2.13 and by the result for the case of a point, $y_i = 0$ mod $\mathfrak{m}$. If we write $y_i = \sum_{n \in \mathbf{Z}} a_{n,i} T^n \oplus \sum_{n \in \mathbf{Z}} a_{n,i} T^n$ for $i = 1, 2$ we see that $a_{n,i} = 0$ mod $\mathfrak{m}$ and hence $y_i = 0$ so that $y = 0$, as desired.
Suppose now the result is true for every affinoid algebra of index of nilpotence $\leq j$ and let A be an affinoid algebra whose nilradical N satisfies $N^{j+1} = 0$ and g be in the kernel of $\tilde{G} \to G$. We have the following short exact sequence (note that Δ is a flat A-module because it is an extension of flat A-modules and so is $\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1$ who is topologically isomorphic to two copies of Δ)

$$0 \to (\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1) \otimes_A N^j \to \Delta \boxtimes_{\omega,\iota} \mathbf{P}^1 \to (\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1) \otimes_A A/N^j \to 0.$$

We can identify $(\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1) \otimes_A A/N^j$ with $(\Delta \otimes_A A/N^j) \boxtimes_{\omega,\iota} \mathbf{P}^1$ and $(\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1) \otimes_A N^j$ with $(\Delta \otimes_A A/N) \boxtimes_{\omega,\iota} \mathbf{P}^1 \otimes_{A/N} N^j$. The result now follows by the inductive hypothesis and the base case. ☐

From now on we denote by $\Delta \boxtimes_\omega \mathbf{P}^1$ the module $\Delta \boxtimes_{\omega,\iota} \mathbf{P}^1$ constructed in Proposition 7.8.

7.5 The representation $\Pi(\Delta)$

We are now almost ready to construct the representation $\Pi(\Delta)$ and prove Theorem 7.2. We will need some preparation results. We start by showing that $H^1(\overline{P}^+, \mathscr{R}_A^-(\delta_1, \delta_2))$ is a free A-module in the *quasi*-regular case[35].

[35] Here quasi-regular means that pointwise $\delta_1 \delta_2^{-1}$ is never of the form χx^i for some $i \geq 0$. Clearly regular implies quasi-regular.

Proposition 7.9. *Let $\delta_1,\delta_2\colon \mathbf{Q}_p^\times \to A^\times$ be locally analytic characters such that $\delta_1\delta_2^{-1}$ is quasi-regular. Then $H^1(\overline{P}^+,\mathscr{R}_A^-(\delta_1,\delta_2))$ is a free A-module of rank 1.*

Proof. This will follow from Proposition 3.5(1) and Lemma 7.11 below. □

Lemma 7.10. *$\mathscr{R}_A^-$ is a flat A-module.*

Proof. For $0<r\leq s\leq\infty$ rings $\mathscr{R}_A^{[r,s]}$ are Banach A-algebras of countable type. Thus by [29, Lemma 1.3.8] we have

$$\mathscr{R}_A^{[r,s]}/\mathscr{R}_A^{[r,\infty]}=\mathscr{R}_{\mathbf{Q}_p}^{[r,s]}/\mathscr{R}_{\mathbf{Q}_p}^{[r,\infty]}\widehat{\otimes}_{\mathbf{Q}_p,\iota}A=(\widehat{\oplus}_{i\in I}\mathbf{Q}_p\cdot e_i)\widehat{\otimes}_{\mathbf{Q}_p,\iota}A.$$

Since the first factor is a locally convex direct limit of finite dimensional vector spaces and since A is Banach, by [20, Lemma 1.1.30] tensor product and direct sum commute, showing that we can identify $\mathscr{R}_A^{[r,s]}/\mathscr{R}_A^{[r,\infty]}$ with the completed direct sum $\widehat{\oplus}_{i\in I}Ae_i$ where $(e_i)_{i\in I}$ form a potentially orthonormal basis. We note in the following $(\mathscr{R}_A^{[r,s]})^-$ the module $\mathscr{R}_A^{[r,s]}/\mathscr{R}_A^{[r,\infty]}$. Under this identification, if $I\subseteq A$ is a finitely generated ideal of A, then

$$I\otimes_A(\mathscr{R}_A^{[r,s]})^-\cong\widehat{\oplus}_{i\in I}Ie_i.$$

This implies that the morphism $I\otimes_A(\mathscr{R}_A^{[r,s]})^-\to(\mathscr{R}_A^{[r,s]})^-$ is injective. Thus $(\mathscr{R}_A^{[r,s]})^-$ is a flat A-module. Observe that, in fact, the quotient $\mathscr{R}_A^{[r,s]}/\mathscr{R}_A^{[r,\infty]}$ does not depend on r and coincides with $\mathscr{R}_A^{]0,s]}/\mathscr{R}_A^{]0,\infty]}$ so that this last module is flat. Finally, since filtered colimits are exact, this implies that

$$\varinjlim_{s>0}\mathscr{R}_A^{]0,s]}/\mathscr{R}_A^{]0,\infty]}=\mathscr{R}_A/\mathscr{R}_A^+=\mathscr{R}_A^-$$

is a flat A-module.

□

Lemma 7.11. *Let $\delta_1,\delta_2\colon\mathbf{Q}_p^\times\to A^\times$ such that $\delta_1\delta_2^{-1}$ is quasi-regular. The restriction morphism*

$$H^1(\overline{P}^+,\mathscr{R}_A^-(\delta_1,\delta_2))\to H^1(A^+,\mathscr{R}_A^-(\delta))$$

is an isomorphism.

Proof. This is precisely the same proof as Theorem 6.8 with $\mathscr{R}_A^-$ replacing $\mathscr{R}_A$. The key points are the following

- The morphism
$$C^\bullet_{\overline{P}^+}(\mathscr{R}_A^-(\delta_1,\delta_2))\to C^\bullet_{A^+}(\mathscr{R}_A^-(\delta))$$
is in $\mathcal{D}^-_{\mathrm{pc}}(A)$ by Proposition 3.5(1) and Lemma 4.13.

- $\mathscr{R}_A^-$ is flat A-module, cf. Lemma 7.10. This means that for any maximal ideal $\mathfrak{m} \subset A$ we have

$$C^\bullet_{\overline{P}^+}(\mathscr{R}_A^-(\delta_1, \delta_2)) \otimes^{\mathbf{L}} A/\mathfrak{m} \cong C^\bullet_{\overline{P}^+}(\mathscr{R}_{A/\mathfrak{m}}^-(\delta_1, \delta_2))$$

and

$$C^\bullet_{A^+}(\mathscr{R}_A^-(\delta)) \otimes^{\mathbf{L}} A/\mathfrak{m} \cong C^\bullet_{A^+}(\mathscr{R}_{A/\mathfrak{m}}^-(\delta)).$$

- Since $\delta_1\delta_2^{-1}$ is quasi-regular, $H^2(C^\bullet_{A^+}(\mathscr{R}_A^-(\delta))) = 0$ by Proposition 3.5(1).
- The result is true when A is a finite extension of $\mathbf{Q}_p$, cf. [14, Lemme 5.24].

□

This completes the proof of Proposition 7.9. Finally we need a lemma which identifies $\check{\Delta} \boxtimes_{\omega^{-1}} \mathbf{P}^1$ as the topological dual of $\Delta \boxtimes_\omega \mathbf{P}^1$ (equipped with the strong topology).

Lemma 7.12. *If Δ is an extension of $\mathscr{R}_A(\delta_2)$ by $\mathscr{R}_A(\delta_1)$ and if the G-module $\Delta \boxtimes_\omega \mathbf{P}^1$ exists, then its dual is $\check{\Delta} \boxtimes_{\omega^{-1}} \mathbf{P}^1$.*

Proof. This is the same proof as [14, Proposition 3.2]. □

Proof of Theorem 7.2. By Proposition 2.18, Theorem 7.6 and Proposition 7.8 we have that[36]

$$\Delta \boxtimes_\omega \mathbf{P}^1 = [B_A(\delta_2, \delta_1)^* \otimes \omega - B_A(\delta_1, \delta_2) - B_A(\delta_1, \delta_2)^* \otimes \omega - B_A(\delta_2, \delta_1)].$$

We begin by showing that the middle extension $[B_A(\delta_1, \delta_2) - B_A(\delta_1, \delta_2)^* \otimes \omega]$ is split in the category $\mathscr{G}_{G,A}$. We compute

$$\begin{aligned}\mathrm{Ext}^1_G(B(\delta_1, \delta_2)^* \otimes \omega, \mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1) &\overset{(i)}{\cong} \mathrm{Ext}^1_{\overline{B}}(\delta_2 \otimes \chi^{-1}\delta_1, \mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1) \\ &\overset{(ii)}{\cong} \mathrm{Ext}^1_{\overline{P}}(\delta_2 \otimes \chi^{-1}\delta_1, \mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1)\end{aligned}$$

where (i) follows from Proposition 7.5 and (ii) follows from the fact that both $(\delta_2 \otimes \chi^{-1}\delta_1)$ and $\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1$ have the same central character. Then as $\overline{P}$-modules

$$(\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1) \otimes (\delta_2^{-1} \otimes \chi\delta_1^{-1}) \cong (\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1) \otimes \delta_2^{-1}.$$

Let us denote by $\mathscr{R}_A^-(\delta_1, \delta_2) \boxtimes \mathbf{P}^1$ the $\overline{P}$-module $(\mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1) \otimes \delta_2^{-1}$, so that we get

$$\mathrm{Ext}^1_{\overline{P}}(\delta_2 \otimes \chi^{-1}\delta_1, \mathscr{R}_A^-(\delta_1) \boxtimes_\omega \mathbf{P}^1) = H^1_{\mathrm{an}}(\overline{P}, \mathscr{R}_A^-(\delta_1, \delta_2) \boxtimes \mathbf{P}^1).$$

[36]The notation $M = [M_1 - M_2 - \ldots - M_n]$ means that M admits an increasing filtration $0 \subseteq F_1 \subseteq \ldots \subseteq F_n = M$ by subobjects such that $M_i = F_i/F_{i-1}$ for $i = 1, \ldots, n$.

Finally, as in [14, Lemme 6.4], we have

$$H^1_{\mathrm{an}}(\overline{P},\mathscr{R}_A^-(\delta_1,\delta_2)\boxtimes \mathbf{P}^1) = H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A^-(\delta_1,\delta_2)\boxtimes \mathbf{P}^1).$$

Putting this calculations together, we conclude that

$$\mathrm{Ext}^1_G(B(\delta_1,\delta_2)^*\otimes\omega,\mathscr{R}_A^-(\delta_1)\boxtimes_\omega \mathbf{P}^1) = H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A^-(\delta_1,\delta_2)\boxtimes \mathbf{P}^1).$$

Consider the commutative diagram:

$$\begin{array}{ccc} H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A(\delta_1,\delta_2)) & \xrightarrow{\sim} & H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A(\delta_1,\delta_2)\boxtimes \mathbf{P}^1) \\ \downarrow & & \downarrow \\ H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A^-(\delta_1,\delta_2)) & \longrightarrow & H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A^-(\delta_1,\delta_2)\boxtimes \mathbf{P}^1), \end{array}$$

where the top horizontal arrow is an isomorphism by Theorems 6.8 and 7.6. By reinterpreting the extensions in terms of cohomology classes, we see that showing that the middle extension $[B_A(\delta_1,\delta_2)-B_A(\delta_1,\delta_2)^*\otimes\omega]$ splits is equivalent to showing that the right vertical arrow is the zero morphism. Now, by Proposition 7.9, $H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A^-(\delta_1,\delta_2))$ is a free A-module of rank 1 and the same proof as the first point in [14, Remarque 5.26] now shows that the bottom horizontal arrow

$$H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A^-(\delta_1,\delta_2)) \to H^1_{\mathrm{an}}(\overline{P}^+,\mathscr{R}_A^-(\delta_1,\delta_2)\boxtimes \mathbf{P}^1)$$

is the zero morphism and so is the right vertical arrow, proving the claim.
It follows that $\Delta\boxtimes_\omega \mathbf{P}^1$ is an extension of Π_1 by $\Pi_2^*\otimes\omega$, where Π_1 and Π_2 are extensions of $B_A(\delta_2,\delta_1)$ by $B_A(\delta_1,\delta_2)$. By Lemma 7.12, it follows that $\Pi_1=\Pi_2$. We now define $\Pi(\Delta):=\Pi_2$.
Furthermore if Δ is a non-trivial extension of $\mathscr{R}_A(\delta_2)$ by $\mathscr{R}_A(\delta_1)$, then so is the extension of $B_A(\delta_1,\delta_2)^*\otimes\omega$ by $\mathscr{R}_A(\delta_1)\boxtimes_\omega \mathbf{P}^1$ (recalling that

$$\mathrm{Ext}^1_G(\mathscr{R}_A^+(\delta_2)\boxtimes_\omega \mathbf{P}^1,\mathscr{R}_A(\delta_1)\boxtimes_\omega \mathbf{P}^1)\cong \mathrm{Ext}^1(\mathscr{R}_A(\delta_2),\mathscr{R}_A(\delta_1)),$$

cf. Theorem 7.6). This implies that $\Pi(\Delta)$ is a non-trivial extension of $B_A(\delta_2,\delta_2)$ by $B_A(\delta_1,\delta_2)$ and finishes the proof.

□

A Locally analytic G-representations in A-modules

Let A be an affinoid $\mathbf{Q}_p$-algebra (in the sense of Tate). Unless otherwise stated H will be a locally $\mathbf{Q}_p$-analytic group (for applications H will be a closed locally $\mathbf{Q}_p$-analytic subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$). We attempt to generalise some definitions from [37] in the case where the base coefficient is an affinoid $\mathbf{Q}_p$-algebra. There are some results in this direction in [26, §3] although our approach is different.

In particular, our aim is to give a reasonable definition of the category of locally analytic H-representations in A-modules, $\mathrm{Rep}_A^{\mathrm{la}}(H)$, analogous to the definition in [39, §3] (in the case where A is a finite extension of $\mathbf{Q}_p$), and to study the (locally analytic) cohomology of such a representation.

Remark A.1. Note that the paper deals with semi-groups such as $A^+ = \Phi^+ \times A^0$ and $\overline{P}^+ = \overline{U}^1 \rtimes A^+$, where the only component which is a semi-group is Φ^+, which is discrete. Hence these semi-groups have a covering by pairwise disjoint open subsets which carry a structure of a locally $\mathbf{Q}_p$-analytic group. This is enough to extend the main theorems of this appendix (in particular Lemma A.34, Corollary A.38) to this situation.

A.1 Preliminaries and definitions

In what follows if V and W are two locally convex $\mathbf{Q}_p$-vector spaces and in the situation that the bijection

$$V \otimes_{\mathbf{Q}_p,\iota} W \to V \otimes_{\mathbf{Q}_p,\pi} W \tag{26}$$

is a topological isomorphism, we write simply $V \otimes_{\mathbf{Q}_p} W$ and $V \widehat{\otimes}_{\mathbf{Q}_p} W$ to denote the topological tensor product and its completion, respectively. In many contexts, the bijection (26) is a topological isomorphism. For example if V and W are Fréchet spaces (resp. of compact type), cf. [37, Proposition 17.6] (resp. [20, Proposition 1.1.31]). Note that H is strictly paracompact (this means that every open covering of H admits a locally finite refinement of pairwise disjoint open subsets) and so it admits a covering of pairwise disjoint open compact subsets.
We need to define a notion of a locally convex A-module. First we recall the definition of a *topological* A-module.

Definition A.2. A topological A-module is an A-module endowed with a topology such that module addition $+\colon M \times M \to M$ and scalar multiplication $\cdot\colon A \times M \to M$ are continuous functions (where the domains of these functions are endowed with product topologies).

Definition A.3. Let M be an A-module. A seminorm q on M is a function $q\colon M \to \mathbf{R}$ such that

- $q(am) = |a|q(m)$ for all $a \in A$ and $m \in M$, where $|\cdot|$ is some non-zero multiplicative seminorm on A.
- $q(m+n) \le \max\{q(m), q(n)\}$ for any $m, n \in M$.

Let $(q_i)_{i\in I}$ be a family of seminorms on an A-module M. We define a topology on M to be the coarsest topology on M such that

1. All $q_i\colon M \to \mathbf{R}$, for $i \in I$, are continuous.
2. All translation maps $m + -\colon M \to M$, for $m \in M$, are continuous.

Remark A.4. One would at a first glance be tempted to define a locally convex A-module as a topological A-module whose underlying topology is given by a family of seminorms in the above sense. The problem with this definition is twofold. The affinoid $\mathbf{Q}_p$-algebra A equipped with the topology defined by the Gauss norm say, will not necessarily be a locally convex A-module (unless A is reduced). This is essentially due to the fact that the Gauss norm is not necessarily multiplicative on A. On the other hand the topology on A defined by the seminorms coming from the Berkovich spectrum $\mathcal{M}(A)$ coincides with the topology induced by the spectral seminorm $(f \in A \mapsto \max_{x \in \mathcal{M}(A)} |f(x)|)$. Under this topology A will indeed be a locally convex A-module but not necessarily Hausdorff.

Due to Remark A.4 we define a locally convex A-module in the following way.

Definition A.5. A locally convex A-module is a topological A-module whose underlying topology is a locally convex $\mathbf{Q}_p$-vector space. We let LCS_A be the category of locally convex A-modules. Its morphisms are all continuous A-linear maps.

Remark A.6. Let us show that this definition is coherent in the case when $A = L$ is a finite extension of $\mathbf{Q}_p$. That is, that a locally convex $\mathbf{Q}_p$-vector space equipped with a continuous multiplication by L is also an L-locally convex vector space.
We employ the notion of [37, §4]. Let L be a finite extension of $\mathbf{Q}_p$. It is clear that any locally convex L-vector space in the sense of loc.cit. satisfies the conditions of Definition A.5. On the other hand, let M be a locally convex $\mathbf{Q}_p$-vector space (whose topology is defined by a family of lattices $\mathscr{B}$) equipped with a continuous multiplication by L. We show that we can equip M with a system of lattices $\mathscr{B}'$ satisfying conditions (lc1) and (lc2) of [37, §4] with K there replaced by L defining the same topology as $\mathscr{B}$. Let x_i be a $\mathbf{Z}_p$-basis of $\mathcal{O}_L$. For $U \in \mathscr{B}$, set $U' = \sum_i x_i U$ and denote $\mathscr{B}' = \{U' : U \in \mathscr{B}\}$. It is easy to see that this family of $\mathcal{O}_L$-lattices satisfies conditions (lc1) and (lc2). We show that the topology defined by $\mathscr{B}'$ coincides with that defined by $\mathscr{B}$.
- Let $U \in \mathscr{B}$. Since multiplication by x_i is an homeomorphism (multiplication by x_i^{-1} being a continuous inverse) $x_i U$ for all i is open (in the topology defined by $\mathscr{B}$). Thus so is $\sum x_i U = U'$, which shows that the topology defined by $\mathscr{B}$ is finer than that defined by $\mathscr{B}'$.
- On the otherhand, let $U \in \mathscr{B}$. We show now that there exists $V' \in \mathscr{B}'$ such that $V' \subseteq U$. Let V be such that $V \subseteq x_i^{-1} U$ for all i (again we use the fact that multiplication by x_i^{-1} is open and (lc2)). Then $\sum x_i V \subseteq \sum_i x_i x_i^{-1} U \subseteq U$. This completes the proof of the claim.

Lemma A.7. *A equipped with its norm topology is a barrelled, complete Hausdorff locally convex A-module.*

Proof. Consider the induced Gauss norm on A, $|\cdot|$. By [4, §3.1, Proposition 5(i)], $|\cdot|$ is a $\mathbf{Q}_p$-algebra norm. Thus A equipped with its norm topology is a locally convex $\mathbf{Q}_p$-vector space. Moreover since $|\cdot|$ is sub-multiplicative ($|ab| \leq |a| \cdot |b|$), multiplication by A is continuous. Finally since the topology is defined by a norm, it is Hausdorff and A is a $\mathbf{Q}_p$-Banach algebra for this norm. Note that all Banach spaces are barrelled, cf. [37, page 40]. □

We now prove that there is a well defined Hausdorff completion for a locally convex A-module.

Lemma A.8. *For any locally convex A-module M there exists, up to unique topological isomorphism, a unique complete Hausdorff topological A-module $\widehat{M}$ together with a continuous A-linear map*

$$c_M : M \to \widehat{M}$$

such that the following universal property holds: For any continuous A-linear map $f : M \to N$ into a complete Hausdorff locally convex A-module N there is a unique continuous A-linear map $\widehat{f} : \widehat{M} \to N$ such that

$$f = \widehat{f} \circ c_M.$$

Proof. Uniqueness follows from the universal property. For the existence, replacing M by $M/\overline{\{0\}}$ if necessary, we may assume that M is Hausdorff (note that $\overline{\{0\}}$ is a locally convex A-module and thus so is $M/\overline{\{0\}}$). We consider M as a locally convex $\mathbf{Q}_p$-vector space and let $\widehat{M}$ be as in [37, Proposition 7.5]. We show that $\widehat{M}$ is a topological A-module. It is easy to see that $\widehat{A \times M} = A \times \widehat{M}$ (as locally convex $\mathbf{Q}_p$-vector spaces). By the universal property in loc.cit. the A-module structure $A \times M \to M$ extends to a continuous morphism $\alpha : A \times \widehat{M} \to \widehat{M}$. Since $M \to \widehat{M}$ is an injection and M is dense in $\widehat{M}$, it follows that α exhibits $\widehat{M}$ as a topological A-module. Finally since c_M is an injection it is A-linear. □

From now on when A is considered as a locally convex A-module, we will assume it is equipped with its Gauss-norm topology, $(A, |\cdot|_A)$.
We need the notion of the *strong* dual of a locally convex A-module. This will be much less well behaved compared to the classical situation when A is a finite extension of $\mathbf{Q}_p$. For example if the dimension of H is greater than 1 then we do not even know if $\mathrm{LA}(H, A)$ is A-reflexive, cf. Conjecture A.20 and Remark A.32.

Definition A.9. Let M be a locally convex A-module. As an abstract A-module, we define

$$M'_b := \mathrm{Hom}_{A,\mathrm{cont}}(M, A)$$

equipped with the following topology. We equip $\mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(M, A)$ with the strong $\mathbf{Q}_p$-locally convex topology and give

$$M'_b \subseteq \mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(M, A)$$

the induced subspace topology. We call M'_b, equipped with this topology, the strong dual of M. We say M is A-reflexive if the canonical morphism $M \to (M'_b)'_b$ is a topological isomorphism.

REMARK A.10. Let M be a locally convex A-module. The strong dual M'_b can equivalently be defined with the topology obtained by taking the sets $\{f\colon M \to A | f(B) \subseteq U\}$ for $B \subseteq M$ a bounded set and $U \subseteq A$ open, as a system of neighbourhoods of 0.

REMARK A.11. Let $R \in \{\mathbf{Q}_p, A\}$. For M and N locally convex R-modules, we will sometimes write $\mathcal{L}_{R,b}(M, N) := \mathrm{Hom}_{R,\mathrm{cont}}(M, N)$ equipped with the strong topology. If M is a locally convex $\mathbf{Q}_p$-vector space, then we will denote the *classical* strong dual of M, cf. [37, Chapter 1, §9], by $M'_{\mathbf{Q}_p,b}$.

We now prove that M'_b as given by Definition A.9 is indeed a locally convex A-module.

LEMMA A.12. *If M is a locally convex A-module, then so is its strong dual M'_b.*

Proof. By [37, §5], M'_b is a locally convex $\mathbf{Q}_p$-vector space. So it suffices to show that multiplication by A is continuous. This comes down to chasing definitions. Let $|\cdot|$ be the Gauss norm on A. For any bounded set B of M (viewing M as a locally convex $\mathbf{Q}_p$-vector space), we have the seminorm (on $\mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(M, A)$)

$$p_B(f) := \sup_{v \in B} |f(v)|.$$

The locally convex topology on $\mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(M, A)$ is then defined by the family of seminorms $\{p_B\}_{B \in \mathscr{B}}$ where $\mathscr{B}$ is the set of all bounded subsets of M. For any finitely many seminorms $p_{B_1}, p_{B_2}, \ldots, p_{B_r}$ in the given family and any real number $\epsilon > 0$ the open sets

$$\{f \in \mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(M, A) |\ p_{B_1}, p_{B_2}, \ldots, p_{B_r}(f) \leq \epsilon\}$$

form a basis around 0 of $\mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(M, A)$. For any $a \in A$ we have $p_B(af) \leq |a| p_B(f)$ and now it is easy to see that multiplication by A is continuous on $\mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(M, A)$. Thus it is also continuous on $\mathrm{Hom}_{A,\mathrm{cont}}(M, A)$. ☐

EXAMPLE A.13. Recall the definition of the relative Robba ring $\mathscr{R}_A$, its subquotients $\mathscr{R}_A^+$ and $\mathscr{R}_A^-$ and their topologies, as described in §2.1.1, §2.1.2. For more details on the topologies on $\mathscr{R}_A$ and $\mathscr{R}_A^+$, cf. [6, §1.1]. By [27, Lemma 2.1.19] (cf. also [17, 5.5 Proposition]), we have topological isomorphisms [37]

$$\mathcal{L}_{A,b}(\mathscr{R}_A^-, A) = \mathscr{R}_A^+, \quad \mathcal{L}_{A,b}(\mathscr{R}_A, A) = \mathscr{R}_A.$$

[37]Note that, identifying $\mathscr{R}_A^- = \varinjlim_s \mathscr{R}_A^{]0,s]} / \mathscr{R}_A^{]0,+\infty]}$, one can see (cf. Lemma A.25 and Lemma A.30 below) that $\mathcal{L}_{A,b}(\mathscr{R}_A^-, A) = \varprojlim_s \mathcal{L}_{A,b}(\mathscr{R}_A^{]0,s]} / \mathscr{R}^{]0,+\infty]}, A)$ is Fréchet (observe that the space $\mathscr{R}_A^{]0,s]} / \mathscr{R}_A^{]0,+\infty]}$ with its topology defined by $v^{]0,s]}$, where $v^{]0,s]}$ is the valuation induced by $v^{[r,s]}$ for any $r < s$, is a Banach space).

Note that, if we denote by $\mathscr{R}_A^\sim$ the A-submodule of $\mathscr{R}_A$ given by Laurent series whose non-negative powers of T vanish, equipped with it's induced topology, then we have $(\mathscr{R}_A^+)^\perp = \mathscr{R}_A^+$ and $(\mathscr{R}_A^\sim)^\perp = \mathscr{R}_A^\sim$ (the orthogonal is taken with respect to the natural separately continuous pairing $\mathscr{R}_A \times \mathscr{R}_A \to A$, $(f,g) \mapsto \mathrm{res}_0(f(T)g(T)dT)$). This shows that $\mathscr{R}_A^+$ and $\mathscr{R}_A^\sim$ are closed A-submodules of $\mathscr{R}_A$ and that we have a topological decomposition $\mathscr{R}_A = \mathscr{R}_A^+ \oplus \mathscr{R}_A^\sim$ (cf. [37, Proposition 8.8(ii)]). We can conclude, by using [37, Lemma 5.3(ii)], that we have a topological isomorphism of locally convex A-modules $\mathscr{R}_A^- \cong \mathscr{R}_A^\sim$ and $\mathscr{R}_A^+ \cong \mathscr{R}_A/\mathscr{R}_A^\sim$. On the other hand, the same argument of [27, Lemma 2.1.19] shows that

$$\mathcal{L}_{A,b}(\mathscr{R}_A^+, A) = \mathscr{R}_A^-.$$

Indeed, as in loc. cit., the inverse to the natural map $\mathscr{R}_A \to \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{R}_A, A)$ induced by the pairing res_0 is given by associating, to any $\mu \in \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{R}_A, A)$, the power series $\sum_{n\in\mathbf{Z}} \mu(T^{-1-n})T^n$. This series lies in $\mathscr{R}_A^\sim$ if and only if $\mu(T^{-1-n}) = 0$ for all $n \geq 0$, or equivalently $\mu(\mathscr{R}_A^\sim) = 0$. This shows in particular that the spaces $\mathscr{R}_A^+$ and $\mathscr{R}_A^-$ are A-reflexive.

We need to recall some classical definitions adapted to our context:

Definition A.14. A Cauchy net in a locally convex A-module M is a net $(m_i)_{i\in I}$ in M (a family of vectors m_i in M where the index set I is directed) such that for every $\epsilon > 0$ and every continuous seminorm q_α, $\exists \kappa$ such that for all $\lambda, \mu > \kappa$, $q_\alpha(m_\lambda - m_\mu) < \epsilon$. M is complete if and only if every Cauchy net converges.

Definition A.15. Let $R \in \{\mathbf{Q}_p, A\}$ and let M be a locally convex R-module. We say M is a Fréchet space if it is metrizable and complete. We say M is R-LB-type if it can be written as a countable filtered union of R-Banach spaces. We say M is R-LF-type if it can be written as a countable filtered union of locally convex R-modules which are Fréchet spaces.

A.2 Relative non-archimedean functional analysis

Here are our first (and main) examples of locally convex A-modules:

Lemma A.16. *$\mathscr{R}_A^+$, $\mathscr{R}_A^+ \boxtimes \mathbf{P}^1$, $\mathscr{R}_A^-$, $\mathscr{R}_A^- \boxtimes \mathbf{P}^1$, $\mathscr{R}_A$ and $\mathscr{R}_A \boxtimes \mathbf{P}^1$ are complete Hausdorff locally convex A-modules. Moreover $\mathscr{R}_A^+$ and $\mathscr{R}_A^+ \boxtimes \mathbf{P}^1$ are Fréchet spaces, $\mathscr{R}_A^-$ and $\mathscr{R}_A^- \boxtimes \mathbf{P}^1$ are of A-LB-type and $\mathscr{R}_A$ and $\mathscr{R}_A \boxtimes \mathbf{P}^1$ are of A-LF-type. For $? \in \{+,-,\emptyset\}$ we have that*

$$\mathscr{R}_A^? = \mathscr{R}_{\mathbf{Q}_p}^? \widehat{\otimes}_{\mathbf{Q}_p} A,$$

$$\mathscr{R}_A^? \boxtimes \mathbf{P}^1 = (\mathscr{R}_{\mathbf{Q}_p}^? \boxtimes \mathbf{P}^1) \widehat{\otimes}_{\mathbf{Q}_p} A.$$

Proof. We first prove the statement for $\mathscr{R}_A^?$, $? \in \{+,-,\emptyset\}$. By [6, lemme 1.3(i)] we have an isomorphism as Fréchet spaces (in the category of locally convex $\mathbf{Q}_p$-vector spaces) $\mathscr{R}_A^+ = \mathscr{R}_{\mathbf{Q}_p}^+ \widehat{\otimes}_{\mathbf{Q}_p} A$. Thus $\mathscr{R}_A^+$ is a locally convex $\mathbf{Q}_p$-vector

space. Multiplication by A is clearly continuous on $\mathscr{R}^+_{\mathbf{Q}_p} \otimes_{\mathbf{Q}_p} A$ (the latter is also a locally convex $\mathbf{Q}_p$-vector space) and so by Lemma A.8, the completion $\mathscr{R}^+_A$ is a locally convex A-module.
By example A.13, $\mathscr{R}^-_A$ is A-reflexive and so, by Remark A.32 together with Proposition 2.2, we have an isomorphism $\mathscr{R}^-_A = \mathscr{R}^-_{\mathbf{Q}_p} \widehat{\otimes}_{\mathbf{Q}_p} A$. Moreover $\mathscr{R}^-_A = \varinjlim_s \mathscr{R}^{]0,s]}_A / \mathscr{R}^{]0,+\infty]}_A$ is of A-LB-type.
Finally by definition $\mathscr{R}_A$ is A-LF-type and since $\mathscr{R}_A = \mathscr{R}^+_A \oplus \mathscr{R}^-_A$ (as topological A-modules), it is also Hausdorff and complete. We compute

$$\begin{aligned}\mathscr{R}_A &= \mathscr{R}^+_A \oplus \mathscr{R}^-_A \\ &= (\mathscr{R}^+_{\mathbf{Q}_p} \widehat{\otimes}_{\mathbf{Q}_p} A) \oplus (\mathscr{R}^-_{\mathbf{Q}_p} \widehat{\otimes}_{\mathbf{Q}_p} A) \\ &= (\mathscr{R}^+_{\mathbf{Q}_p} \oplus \mathscr{R}^-_{\mathbf{Q}_p}) \widehat{\otimes}_{\mathbf{Q}_p} A \\ &= \mathscr{R}_{\mathbf{Q}_p} \widehat{\otimes}_{\mathbf{Q}_p} A.\end{aligned}$$

We finally observe that, if $M \in \{\mathscr{R}^+_A, \mathscr{R}^-_A, \mathscr{R}_A\}$, then $M \boxtimes \mathbf{P}^1$ (with its topology induced by the inclusion $M \boxtimes \mathbf{P}^1 \subseteq M \times M$) is topologically isomorphic to $M \times M$, the isomorphism given by $z \mapsto (\mathrm{Res}_{\mathbf{Z}_p}(z), \psi(\mathrm{Res}_{\mathbf{Z}_p}(wz)))$ with inverse $(z_1, z_2) \mapsto (z_1, \varphi(z_2) + w(\mathrm{Res}_{\mathbf{Z}_p^\times}(z_1)))$. Thus the last assertions follow. □

Let M be a Hausdorff locally convex A-module. We define the locally convex A-module $\mathrm{LA}(H, M)$ of M-valued *locally analytic* functions on H, adapting the classical definition.

Definition A.17. An M-index $\mathcal{I}$ on H is a family of triples

$$\{(H_i, \phi_i, M_i)\}_{i \in I}$$

where the H_i are pairwise disjoint open subsets of H which cover H, each $\phi_i \colon H_i \to \mathbf{Q}_p^d$ is a chart for the manifold H whose image is an affinoid ball[38] and $M_i \hookrightarrow M$ is a continuous linear injection from an A-Banach space M_i into M. Let $\mathcal{F}_{\phi_i}(M_i)$ be the set of all functions $f \colon H_i \to M_i$ such that $f \circ \phi_i^{-1}$ is an M_i-valued holomorphic function on the affinoid ball $\phi_i(H_i)$. Note that $\mathcal{F}_{\phi_i}(M_i)$ is an A-Banach space. We set

$$\mathcal{F}_{\mathcal{I}}(M) := \prod_{i \in I} \mathcal{F}_{\phi_i}(M_i),$$

where $\mathcal{F}_{\mathcal{I}}(M)$ is equipped with the direct product topology (in particular it is a locally convex A-module). We then define[39]

$$\mathrm{LA}(H, M) := \varinjlim_{\mathcal{I}} \mathcal{F}_{\mathcal{I}}(M)$$

equipped with the $\mathbf{Q}_p$-locally convex inductive limit topology.

[38] To be more precise one takes $\mathbb{H}_i$ an affinoid rigid analytic space over $\mathbf{Q}_p$ isomorphic to a closed ball, so that ϕ_i induces an isomorphism $\phi'_i \colon H_i \xrightarrow{\sim} \mathbb{H}_i(\mathbf{Q}_p)$.
[39] This colimit is taking place in the category of $\mathbf{Q}_p$-locally convex vector spaces.

Remark A.18. In Definition A.17, in order to see that $\mathrm{LA}(H, M)$ is a locally convex A-module, one needs to check that multiplication by A on $\mathrm{LA}(H, M)$ is continuous. Indeed since, $\cdot : A \times M \to M$ is continuous, then so is $A \times \mathcal{F}_{\phi_i}(M_i) \to \mathcal{F}_{\phi_i}(M_i)$. This implies that multiplication $B_{\mathcal{I}} : A \times \mathcal{F}_{\mathcal{I}}(M) \to \mathcal{F}_{\mathcal{I}}(M)$ is continuous. Denote by $B : A \times \mathrm{LA}(H, M) \to \mathrm{LA}(H, M)$, the multiplication by A on $\mathrm{LA}(H, M)$. The continuity of B follows from the continuity of the $B_{\mathcal{I}}$ (cf. the 3rd paragraph of the proof of Lemma A.25 where a similar problem is proved).

We now define the space of relative distributions.

Definition A.19. We define the space of distributions on H with values in A as the strong dual of $\mathrm{LA}(H, A)'_b$ (cf. Definition A.9)

$$\mathscr{D}(H, A) := \mathrm{LA}(H, A)'_b.$$

To kick-start our study of locally convex A-modules and their relationship to $\mathrm{Rep}^{\mathrm{la}}_A(H)$ (cf. Definition A.26) we need to know that $\mathrm{LA}(H, A)$ is well-behaved. Féaux states explicitly in his thesis (cf. [22]) that he does not know if $\mathrm{LA}(H, A)$ is complete. The completeness of $\mathrm{LA}(H, A)$ has since become somewhat of a folklore conjecture:

Conjecture A.20. If H is a compact locally $\mathbf{Q}_p$-analytic group then $\mathrm{LA}(H, A)$ is complete.

Remark A.21. If Conjecture A.20 is true then one can show that $\mathrm{LA}(H, A) \cong \mathrm{LA}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi} A \cong \mathrm{LA}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota} A$, cf. Remark A.32. If H is of dimension 1 then Conjecture A.20 is true, cf. lemma A.16.

Although we are unable to prove Conjecture A.20 we show that $\mathrm{LA}(H, A)$ is sufficiently well-behaved for applications.

Definition A.22. Let $R \in \{\mathbf{Q}_p, A\}$. We call a Hausdorff space of R-LB-type $V = \varinjlim_n V_n$, *R-regular* if, for every bounded subset B of V, there exists an n such that V_n contains B and B is bounded in V_n.

Remark A.23. Let $R \in \{\mathbf{Q}_p, A\}$. By [20, Proposition 1.1.10 and 1.1.11], a Hausdorff semi-complete R-LB-type is R-regular.

The following lemma will be of use.

Lemma A.24. *Let $(V, ||\cdot||)$ be a $\mathbf{Q}_p$-Banach space and let $V'_0 \subset V'_{\mathbf{Q}_p,b}$ be the unit ball. Given a constant $C \in |\mathbf{Q}_p\backslash\{0\}|_p = p^{\mathbf{Z}}$ and a vector $v \in V$, if $|l(v)|_p < C$ for all $l \in V'_0$ then $||v|| \leq C$.*

Proof. This is a direct consequence of the Hahn-Banach theorem. Suppose that $||v|| > C$. Applying [37, Proposition 9.2] with $U := V$, $q := ||\cdot||$, $U_o := \mathbf{Q}_p \cdot v$ and $l_0 : U_0 \to \mathbf{Q}_p$ the linear form defined by $l_0(av) = ac$ for any $a \in \mathbf{Q}_p$, and $c \in \mathbf{Q}_p$ such that $C = |c|_p$, we deduce that there exists a continuous linear form l on V extending l_0. Then $|l(v)|_p = |l_0(v)|_p = |c|_p = C$. ∎

Before we state our next result we need some notation. Via Mahler expansions (cf. [32, III. Théorème 1.2.4]), the set of continuous functions from $\mathbf{Z}_p^d$ to $\mathbf{Q}_p$ can be viewed as the space of all series

$$\sum_{\alpha \in \mathbf{N}^d} c_\alpha \binom{x}{\alpha}$$

with $c_\alpha \in \mathbf{Q}_p$ such that $|c_\alpha| \to 0$ as $|\alpha| \to 0$. Here as usual

$$\binom{x}{\alpha} := \binom{x_1}{\alpha_1} \cdots \binom{x_d}{\alpha_d}$$

and

$$|\alpha| := \sum_{i=1}^{d} \alpha_i$$

for $x = (x_1, \ldots, x_d) \in \mathbf{Z}_p^d$ and multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{N}^d$.

For $f \in \mathrm{LA}(\mathbf{Z}_p^d, A)$, write $f = \sum_{n \in \mathbf{N}^d} a_n(f) \binom{x}{\alpha}$, $a_n(f) \in A$, the Mahler expansion of the continuous function f. Recall that, for $h \in \mathbf{N}$, $\mathrm{LA}_h(\mathbf{Z}_p^d, \mathbf{Q}_p)$ denotes the space of functions which are analytic on every ball of poly-radius $(h, \ldots, h)$. By Amice's theorem (cf. [32, III. Théorème 1.3.8] or [9, Théorème I.4.7]), we have

$$\mathrm{LA}_h(\mathbf{Z}_p^d, \mathbf{Q}_p) = \widehat{\bigoplus_{n \in \mathbf{N}^d}} \mathbf{Q}_p \cdot k_{n,h} \binom{x}{n}$$

where $k_{n,h} := \lfloor p^{-h} n_1 \rfloor ! \ldots \lfloor p^{-h} n_d \rfloor !$. One also obtains

$$\mathrm{LA}_h(\mathbf{Z}_p^d, A) = \widehat{\bigoplus_{n \in \mathbf{N}^d}} A \cdot k_{n,h} \binom{x}{n}.$$

By definition, we also have

$$\mathrm{LA}(\mathbf{Z}_p^d, A) = \varinjlim_{h \in \mathbf{N}} \mathrm{LA}_h(\mathbf{Z}_p^d, A).$$

We denote the norms on $\mathrm{LA}_h(\mathbf{Z}_p^d, \mathbf{Q}_p)$ and $\mathrm{LA}_h(\mathbf{Z}_p^d, A)$ by $|\cdot|_{h,\mathbf{Q}_p}$ and $|\cdot|_{h,A}$, respectively and $v_{h,\mathbf{Q}_p}$ and $v_{h,A}$ their respective valuations.

We are grateful for G. Dospinescu for supplying us with the idea for the following lemma.

Lemma A.25. *If H is a compact locally $\mathbf{Q}_p$-analytic group then $\mathrm{LA}(H, A)$ is A-regular.*

Proof. By choosing a covering of H by a finite number of open subsets isomorphic (as locally $\mathbf{Q}_p$-analytic manifolds) to $\mathbf{Z}_p^d$ for some $d \in \mathbf{N}$, we suppose that $H = \mathbf{Z}_p^d$, cf. [18, Corollary 8.34].

Consider the $\mathbf{Q}_p$-bilinear map

$$B\colon A'_{\mathbf{Q}_p,b} \times \mathrm{LA}(H,A) \to \mathrm{LA}(H,\mathbf{Q}_p)$$
$$(l,f) \mapsto l \circ f.$$

Note that the Mahler coefficients of $l \circ f$ are then just given by $a_n(l \circ f) = l(a_n(f))$.

We show that the map B above is continuous. For this, it is enough to note that B is the composition of the continuous morphisms

$$\begin{aligned} B\colon A'_{\mathbf{Q}_p,b} \times \mathrm{LA}(H,A) &\xrightarrow{\mathrm{id}\times\alpha} A'_{\mathbf{Q}_p,b} \times (A \widehat{\otimes}_{\mathbf{Q}_p,\pi} \mathrm{LA}(H,\mathbf{Q}_p)) \\ &\to A'_{\mathbf{Q}_p,b} \widehat{\otimes}_{\mathbf{Q}_p,\pi} (A \widehat{\otimes}_{\mathbf{Q}_p,\pi} \mathrm{LA}(H,\mathbf{Q}_p)) \\ &= (A'_{\mathbf{Q}_p,b} \widehat{\otimes}_{\mathbf{Q}_p,\pi} A) \widehat{\otimes}_{\mathbf{Q}_p,\pi} \mathrm{LA}(H,\mathbf{Q}_p) \\ &\xrightarrow{\beta} \mathrm{LA}(H,\mathbf{Q}_p), \end{aligned}$$

where α is the morphism of [20, Proposition 2.2.10] (cf. the discussion immediately after loc.cit.) which is a continuous bijection. The second arrow is the canonical continuous bilinear map (cf. [37, §17.B]). The last morphism β is induced from the natural continuous pairing of A and $A'_{\mathbf{Q}_p,b}$.

Let $T \subset \mathrm{LA}(H,A)$ be a bounded subset and consider

$$S := A'_0 \times T \subset A'_{\mathbf{Q}_p,b} \times \mathrm{LA}(H,A)$$

where $A'_0 \subset A'_{\mathbf{Q}_p,b}$ is the unit ball. Then S is bounded and so is its image in $A'_{\mathbf{Q}_p,b} \widehat{\otimes}_{\mathbf{Q}_p,\pi} \mathrm{LA}(H,A)$ by definition of the projective tensor product topology (cf. [37, §17.B]). Therefore $B(S)$ is bounded, since B is continuous (and hence factorizes through $A'_{\mathbf{Q}_p,b} \widehat{\otimes}_{\mathbf{Q}_p,\pi} \mathrm{LA}(H,A)$).

By [20, Proposition 1.1.11], $\mathrm{LA}(H,\mathbf{Q}_p)$ is $\mathbf{Q}_p$-regular and so for some $h \geq 1$, $B(S)$ is contained in $\mathrm{LA}_h(H,\mathbf{Q}_p)$ and there exists a constant $C \in \mathbf{Z}$ such that

$$v_{h,\mathbf{Q}_p}(l \circ f) = \inf_{n \in \mathbf{N}^d} v_p(l(a_n(f))) - v_p(k_{n,h}) > C$$

for all $l \in A'_0$. Lemma A.24 now implies that $v_A(a_n(f)) - v_p(k_{n,h}) \geq C$ for all $n \in \mathbf{N}^d$. Now

$$\begin{aligned} v_A(a_n(f)) - v_p(k_{n,h+1}) &= v_A(a_n(f)) - v_p(k_{n,h}) + v_p(k_{n,h}) - v_p(k_{n,h+1}) \\ &\geq C + v_p(k_{n,h}) - v_p(k_{n,h+1}) \xrightarrow{|n|\to+\infty} +\infty. \end{aligned}$$

This implies that $f \in \mathrm{LA}_{h+1}(\mathbf{Z}_p^d, A)$ for all $f \in T$. We now compute

$$\begin{aligned} v_{h+1,A}(f) &= \inf_{n\in\mathbf{N}^d} v_A(a_n(f)) - v_p(k_{n,h+1}) \\ &= \inf_{n\in\mathbf{N}^d} v_A(a_n(f)) - v_p(k_{n,f}) + v_p(k_{n,h}) - v_p(k_{n,h+1}) \\ &\geq C \end{aligned}$$

since $v_p(k_{n,h}) - v_p(k_{n,h+1}) \geq 0$. This shows that T is contained and bounded in $\mathrm{LA}_{h+1}(\mathbf{Z}_p^d, A)$. Thus $\mathrm{LA}(H,A)$ is A-regular. □

A.3 Relative locally analytic representations

In this section we define the category $\mathrm{Rep}^{\mathrm{la}}_A(H)$ and we study the structure of a locally analytic representation over the relative distribution algebras, generalizing some fundamental work of Schneider and Teitelbaum to our relative setting.
The following definition is similar to [5, Définition 3.2].

Definition A.26. An object M in $\mathrm{Rep}^{\mathrm{la}}_A(H)$ is a barrelled, Hausdorff, locally convex A-module equipped with a topological[40] A-linear action of H, such that, for each $m \in M$, the orbit map $h \mapsto h \cdot m$ is an element in $\mathrm{LA}(H, M)$. Morphisms are given by continuous A-linear H-maps.

Remark A.27 (locally analytic induced representation). Let G be a locally $\mathbf{Q}_p$-analytic group, H a closed locally $\mathbf{Q}_p$-analytic subgroup and suppose that G/H is compact. Let M be an object of $\mathrm{Rep}^{\mathrm{la}}_A(H)$, which is Banach. Then

$$\mathrm{Ind}^G_H(M) := \left\{ f \in \mathrm{LA}(G, M) \mid \forall h \in H, g \in G: \ f(gh) = h^{-1} \cdot f(g) \right\}$$

identifies (as topological A-modules) with $\mathrm{LA}(G/H, M)$, cf. [22, Satz 4.3.1]. Moreover $\mathrm{Ind}^G_H(M)$ (equipped with the natural action of G: $(g \cdot f)(x) := f(g^{-1}x)$) is an object of $\mathrm{Rep}^{\mathrm{la}}_A(G)$, cf. Satz 4.1.5 in loc.cit.
To track the action of $\mathscr{D}(G, A)$ on $\mathrm{LA}(G/H, M)$ induced by the above isomorphism, we need to make explicit this isomorphism. Any choice of a section $G/H \to G$ gives an isomorphism of locally $\mathbf{Q}_p$ analytic manifolds $G \cong G/H \times H$. This gives an isomorphism $\mathrm{LA}(G, M) \cong \mathrm{LA}(G/H \times H, M)$ and the space $\mathrm{Ind}^G_H(M) \subseteq \mathrm{LA}(G, M)$ is identified with the submodule $\{f : f(\overline{g}, h) = h^{-1} \cdot f(\overline{g}, 1),\ \overline{g} \in G/H, h \in H\}$ of $\mathrm{LA}(G/H \times H, M)$. On the other hand, the composition[41]

$$\mathrm{LA}(G/H, M) \to \mathrm{LA}(H, \mathrm{End}(M)) \times \mathrm{LA}(G/H, M) \to \mathrm{LA}(G/H \times H, M)$$

$$\tilde{f} \mapsto (\rho^{-1}, \tilde{f}) \mapsto [(\overline{g}, h) \mapsto \rho(h)^{-1} \cdot \tilde{f}(\overline{g})],$$

where we have noted $\rho^{-1} : H \to \mathrm{GL}(M)$ the representation of H on M, induces an isomorphism between $\mathrm{LA}(G/H, M)$ and the image of $\mathrm{Ind}^G_H(M)$ in $\mathrm{LA}(G, M)$.

It will turn out that every complete object of $\mathrm{Rep}^{\mathrm{la}}_A(H)$ carries a structure of a $\mathscr{D}(H, A)$ -module. The following lemma is essentially [46, Proposition 1.3].

Lemma A.28. *Let M be a locally convex $\mathbf{Q}_p$-module and let N be a locally convex A-module. Then $\tilde{f}(a \otimes x) = af(x)$ defines an A-linear bijection*

$$\mathcal{L}_{\mathbf{Q}_p, b}(M, N) \xrightarrow{\sim} \mathcal{L}_{A, b}(M \otimes_{\mathbf{Q}_p, \pi} A, N).$$

[40] We say that the H-action on M is *topological* if H induces continuous endomorphisms of M.
[41] we refer the reader to the proof of [22, Satz 4.3.1].

Proof. For $f \in \mathcal{L}_{\mathbf{Q}_p,b}(M,N)$, the map $\tilde{f}$ is given by the composition of the continuous map $M \otimes_{\mathbf{Q}_p,\pi} A \to N \otimes_{\mathbf{Q}_p,\pi} A$ induced by $x \otimes a \mapsto f(x) \otimes a$ and the (continuous) map $N \otimes_{\mathbf{Q}_p,\pi} A \to N$ induced by the A-module structure on N, so it is well defined. The inverse of $f \mapsto \tilde{f}$ is given by $g \mapsto g_0$, where $g_0(x) = g(x \otimes 1)$. This shows that the map of the statement induces an A-linear bijection.

□

Corollary A.29. *In the setting of Lemma A.28, if in addition M and N are Banach spaces, then*

$$\mathcal{L}_{\mathbf{Q}_p,b}(M,N) \xrightarrow{\sim} \mathcal{L}_{A,b}(M \widehat{\otimes}_{\mathbf{Q}_p,\pi} A, N)$$

is a topological isomorphism.

Proof. By Lemma A.28 and the universal property of Hausdorff completion, cf. [37, Corollary 7.7], the map is a bijection. By [37, Remark 6.7, Proposition 7.16], we see that $\mathcal{L}_{\mathbf{Q}_p,b}(M,N)$ is a Banach space. Similarly it is easy to check that $\mathcal{L}_{A,b}(M \widehat{\otimes}_{\mathbf{Q}_p,\pi} A, N)$ is a Banach space. Thus by the open mapping theorem, cf. Proposition 8.6 in loc.cit. it suffices to show that the inverse map

$$\begin{aligned} \mathcal{L}_{A,b}(M \otimes_{\mathbf{Q}_p,\pi} A, N) &\to \mathcal{L}_{\mathbf{Q}_p,b}(M,N) \\ g &\mapsto g_0 \end{aligned}$$

is continuous. Let $\alpha \colon M \to M \otimes_{\mathbf{Q}_p,\pi} A$ defined by $m \mapsto m \otimes 1$, which, by [37, Corollary 17.5(iii)], induces an homeomorphism of M onto its image. To prove continuity of the above map, note that if $B \subseteq M$ is bounded and $U \subseteq N$ is open, then the inverse image of $\{f \colon M \to N \mid f(B) \subseteq U\}$ by the map $g \mapsto g_0$ is $\{g \colon M \otimes_{\mathbf{Q}_p,\pi} A \to N \mid g(\alpha(B)) \subseteq U\}$, which is open since $\alpha(B)$ is bounded in $M \otimes_{\mathbf{Q}_p,\pi} A$. This completes the proof. □

Lemma A.30. *Let $R \in \{\mathbf{Q}_p, A\}$. If $V = \varinjlim_n V_n$ is R-regular (cf. Definition A.22) then for any locally convex R-module, the natural map $\mathcal{L}_{R,b}(V,W) \to \varprojlim_n \mathcal{L}_{R,b}(V_n,W)$ is a topological isomorphism which is R-linear.*

Proof. This is the same proof as [20, Proposition 1.1.22]. The crucial point in loc.cit. is that a Hausdorff semi-complete of R-LB-type is R-regular. □

Lemma A.31. *Let H be a compact locally $\mathbf{Q}_p$-analytic group. We have an isomorphism of locally convex A-modules*

$$\mathscr{D}(H,A) = \mathscr{D}(H,\mathbf{Q}_p) \widehat{\otimes}_{\mathbf{Q}_p,\pi} A.$$

Proof. By [37, Proposition 20.9] we have

$$\mathscr{D}(H,\mathbf{Q}_p) \widehat{\otimes}_{\mathbf{Q}_p,\pi} A = \mathcal{L}_{\mathbf{Q}_p,b}(\mathscr{D}(H,\mathbf{Q}_p)'_{\mathbf{Q}_p,b}, A).$$

We conclude by observing that (we use the notation from the proof of Lemma A.25)

$$\begin{aligned}
\mathcal{L}_{\mathbf{Q}_p,b}(D(H,\mathbf{Q}_p)'_{\mathbf{Q}_p,b},A) &\overset{(i)}{=} \mathcal{L}_{\mathbf{Q}_p,b}(\mathrm{LA}(H,\mathbf{Q}_p),A) \\
&\overset{(ii)}{=} \varprojlim_n \mathcal{L}_{\mathbf{Q}_p,b}(\mathrm{LA}_n(H,\mathbf{Q}_p),A) \\
&\overset{(iii)}{=} \varprojlim_n \mathcal{L}_{A,b}(\mathrm{LA}_n(H,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi}A,A) \\
&\overset{(iv)}{=} \varprojlim_n \mathcal{L}_{A,b}(\mathrm{LA}_n(H,A),A) \\
&\overset{(v)}{=} \mathcal{L}_{A,b}(\mathrm{LA}(H,A),A) \\
&\overset{(vi)}{=} \mathscr{D}(H,A),
\end{aligned}$$

where (i) follows by reflexivity of $\mathrm{LA}(H,\mathbf{Q}_p)$ (cf. [39, Lemma 2.1 and Theorem 1.1]), (ii) follows from Lemma A.30, (iii) follows from Corollary A.29, (iv) is an immediate consequence of the definition of $\mathrm{LA}_n(H,A)$, (v) is a consequence of Lemmas A.25 and A.30 and (vi) is by definition. The last assertion follows since $\mathscr{D}(H,\mathbf{Q}_p)$ is Fréchet and A is Banach so their completed projective tensor product is Fréchet. This completes the proof. ☐

Remark A.32. For H a compact locally $\mathbf{Q}_p$-analytic group, the natural morphism

$$\alpha\colon \mathrm{LA}(H,A) \to \mathrm{LA}(H,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi}A$$

is (cf. the discussion immediately after [20, Proposition 2.2.10]) a continuous bijection. We do not know whether it is actually a topological isomorphism, cf. Conjecture A.20. By [33, Theorem 2], this is equivalent to $\mathrm{LA}(H,A)$ being complete (note that $\mathrm{LA}_h(H,A)$ has a Schauder basis, by Amice's theorem, so it has the approximation property). We claim that α is a topological isomorphism if $\mathrm{LA}(H,A)$ is a reflexive A-module. Indeed, suppose $\mathrm{LA}(H,A)$ is a reflexive A-module. It suffices to show that $\mathcal{L}_{A,b}(\mathscr{D}(H,A),A)$ is complete. By [37, Proposition 7.16], we see that $\mathcal{L}_{\mathbf{Q}_p,b}(\mathscr{D}(H,A),A)$ is complete. But $\mathcal{L}_{A,b}(\mathscr{D}(H,A),A)$ is a closed subspace of $\mathcal{L}_{\mathbf{Q}_p,b}(\mathscr{D}(H,A),A)$ (imitate the proof of [37, Lemma 6.10]) and so it is complete as well.

The aim is now to obtain a version of Lemma A.31 without the assumption that H is compact.

Lemma A.33. *Let $\{H_i\}_{i\in I}$ be pairwise disjoint compact open subsets which cover H. Then there is an A-linear topological isomorphism*

$$\mathscr{D}(H,A) = \bigoplus_i \mathscr{D}(H_i,A).$$

Moreover $\mathscr{D}(H,A)$ is complete and Hausdorff.

Proof. We have a topological isomorphism

$$\mathrm{LA}(H, A) = \prod_i \mathrm{LA}(H_i, A).$$

The claim now follows from the fact that there is a topological isomorphism

$$(\prod_i \mathrm{LA}(H_i, A))'_b = \bigoplus_i \mathrm{LA}(H_i, A)'_b$$

To see this last fact, one repeats the same proof for [37, Proposition 9.11]. Finally $\mathscr{D}(H, A)$ is complete and Hausdorff follows from [37, Corollary 5.4 and Lemma 7.8]. □

Lemma A.34. *Let H be a locally $\mathbf{Q}_p$-analytic group. We have an isomorphism of locally convex A-modules*

$$\mathscr{D}(H, A) = \mathscr{D}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota} A.$$

Proof. This is an immediate consequence of Lemmas A.31 and A.33. Let $\{H_i\}_{i\in I}$ be pairwise disjoint compact open subsets which cover H. We have a commutative diagram

$$\begin{array}{ccc} \bigoplus_i \left[\mathscr{D}(H_i, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p,\iota} A\right] & \xrightarrow{\sim} & \mathscr{D}(H, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p,\iota} A \\ \downarrow & & \downarrow \\ \bigoplus_i \left[\mathscr{D}(H_i, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota} A\right] & \longrightarrow & \mathscr{D}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota} A. \end{array}$$

Now [20, Lemma 1.1.30] implies that the top horizontal arrow is a topological isomorphism. By definition the right vertical arrow is a topological embedding (since $\mathscr{D}(H, \mathbf{Q}_p)$ is Hausdorff, so is $\mathscr{D}(H, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p} A$, cf. [37, Corollary 17.5(i)]) that identifies its target with the completion of its source. We will show that the same is true for the left vertical arrow which will imply that the bottom horizontal arrow is a topological isomorphism, as required. Since the composite of the top horizontal arrow and the right vertical arrow is a topological embedding, the same is true for the left vertical arrow. It clearly has dense image and the target ($= \mathscr{D}(H, A)$) is complete. This completes the proof. □

Remark A.35. In the setting of Lemma A.34, [24, I.1.3 Proposition 6][42] shows that

$$\mathscr{D}(H, A) = \mathscr{D}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi} A$$

is a topological isomorphism.

[42] The proposition essentially states that direct sums commute with (completion) of projective tensor product.

The following is a relative version of the integration map constructed in [39, Theorem 2.2].

Lemma A.36. *Let H be a locally $\mathbf{Q}_p$-analytic group and let M be a complete Hausdorff locally convex A-module. There is a unique A-linear map*

$$I : \mathrm{LA}(H, M) \to \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{D}(H, A), M),$$

satisfying $I(\phi)(\delta_h) = \phi(h)$ for all $\phi \in \mathrm{LA}(H, M)$ and all $h \in H$. Here $\delta_h \in \mathscr{D}(H, A)$ is such that $\delta_h(f) := f(h)$ for all $f \in \mathrm{LA}(H, A)$. Moreover, if M is of A-LB-type (cf. Definition A.15) then this map is a bijection.

Proof. By [39, Theorem 2.2] (cf. also the comment immediately after its proof), one has a unique map

$$I_{\mathbf{Q}_p} : \mathrm{LA}(H, M) \to \mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(\mathscr{D}(H, \mathbf{Q}_p), M),$$

satisfying $I_{\mathbf{Q}_p}(\phi)(\delta_h) = \phi(h)$, $h \in H$, and which is bijective if M is of $\mathbf{Q}_p$-LB-type. Note that this map is clearly A-linear.
By Lemma A.33, one reduces to show the result for H compact. So assume H is compact. By Lemma A.28, since M is Hausdorff and complete, there is an A-linear bijection

$$r\colon \mathrm{Hom}_{\mathbf{Q}_p,\mathrm{cont}}(\mathscr{D}(H, \mathbf{Q}_p), M) \xrightarrow{\sim} \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{D}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi} A, M).$$

Moreover, Lemma A.31 gives an isomorphism

$$s\colon \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{D}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\pi} A, M) \xrightarrow{\sim} \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{D}(H, A), M).$$

The composition of all these maps $(s \circ r \circ I_{\mathbf{Q}_p})$ gives the desired map

$$I : \mathrm{LA}(H, M) \to \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{D}(H, A), M).$$

The result follows. ∎

Before we state the main result of this section we need to equip $\mathscr{D}(H, A)$ with a ring structure. We show that the convolution product on $(\mathscr{D}(H, \mathbf{Q}_p), *)$, cf. [39, §2], extends to $\mathscr{D}(H, A)$. Indeed by Lemma A.34 we have an isomorphism of locally convex A-modules

$$\mathscr{D}(H, A) = \mathscr{D}(H, \mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota} A.$$

We define an A-bilinear, separately continuous map

$$\begin{aligned} *_A\colon (\mathscr{D}(H, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p,\iota} A) \times (\mathscr{D}(H, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p,\iota} A) &\to (\mathscr{D}(H, \mathbf{Q}_p) \otimes_{\mathbf{Q}_p,\iota} A) \\ (\delta_1 \otimes a_1, \delta_2 \otimes a_2) &\mapsto (\delta_1 * \delta_2, a_1 a_2) \end{aligned}$$

Since the Dirac distributions δ_h for $h \in H$ are dense in $\mathscr{D}(H, \mathbf{Q}_p)$, cf. [39, Lemma 3.1], $*_A$ is well defined. Note that $*_A$ is separately continuous since $*$

is separately continuous, cf. [39, Proposition 2.3]. It is clear that $*_A$ extends uniquely to an A-bilinear, separately continuous map (which we denote by $*$, abusing notation)

$$*: \mathscr{D}(H, A) \times \mathscr{D}(H, A) \to \mathscr{D}(H, A).$$

The following lemma summarizes the above discussion.

Lemma A.37. *$(\mathscr{D}(H, A), *)$ is an associative A-algebra with δ_1 ($1 \in H$ is the unit element) as the unit element. Furthermore the convolution $*$ is separately continuous and A-bilinear.*

Let $\mathrm{Rep}_A^{\mathrm{la},LB}(H) \subseteq \mathrm{Rep}_A^{\mathrm{la}}(H)$ be the full subcategory consisting of spaces which are A-LB-type and complete. As a result we obtain the following corollary.

Corollary A.38. *The category of $\mathrm{Rep}_A^{\mathrm{la},LB}(H)$ is equivalent to the category of complete, Hausdorff locally convex A-modules which are of A-LB-type equipped with a separately continuous $\mathscr{D}(H, A)$-action (more precisely the module structure morphism $\mathscr{D}(H, A) \times M \to M$ is A-bilinear and separately continuous) with morphisms all continuous $\mathscr{D}(H, A)$-linear maps.*

Proof. This is an immediate consequence of Lemma A.36. □

A.4 Locally analytic cohomology and Shapiro's lemma

In this section we prove Shapiro's Lemma for a relative version of the cohomology theory developed by Kohlhaase in [31]. We should warn the reader that Lazard's definition of locally analytic cohomology of a locally $\mathbf{Q}_p$-analytic group via analytic cochains, cf. [32, Chapitre V, §2.3] (or [43] for a modern treatment), does not always coincide with the cohomology groups defined in [31].

Let us first explain the setup. Let H be a locally $\mathbf{Q}_p$-analytic group (for applications H will be a closed locally $\mathbf{Q}_p$-analytic subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$). We will follow closely the treatment in [31], albeit in a relative setting. In particular we are able to reduce many of the arguments to the case considered in loc.cit. The key is lemma A.34.

Definition A.39. Let $\mathscr{G}_{H,A}$ denote the category of complete Hausdorff locally convex A-modules with the structure of a separately continuous A-linear $\mathscr{D}(H, A)$-module, taking as morphisms all continuous $\mathscr{D}(H, A)$-linear maps. More precisely we demand that the module structure morphism

$$\mathscr{D}(H, A) \times M \to M$$

is A-bilinear and separately continuous.

Remark A.40. Alternatively, one sees that $\mathscr{G}_{H,A}$ can be also defined as the category of complete Hausdorff locally convex $\mathbf{Q}_p$-modules with the structure of a separately continuous $\mathscr{D}(H, A)$-module, taking as morphisms all continuous $\mathscr{D}(H, A)$-linear maps.

As a consequence of Lemma A.37 and the fact that $\mathscr{D}(H, A)$ is complete and Hausdorff (cf. Lemma A.33) the convolution product $(\mathscr{D}(H, A), *)$ induces a unique continuous A-linear map[43]

$$\mathscr{D}(H, A)\widehat{\otimes}_{A,\iota}\mathscr{D}(H, A) \to \mathscr{D}(H, A). \tag{27}$$

We now endow $\mathscr{G}_{H,A}$ (and LCS_A) with the structure of an exact category. A sequence in $\mathscr{G}_{H,A}$ (or LCS_A)

$$\cdots \to M_{i-1} \xrightarrow{\alpha_{i-1}} M_i \xrightarrow{\alpha_i} M_{i+1} \xrightarrow{\alpha_{i+1}} \cdots$$

is called *s-exact* if $M_i = K_i \oplus L_i$ (as topological A-modules) where $K_i := \ker(\alpha_i)$ and α_i induces an isomorphism (as topological A-modules) between L_i and K_{i+1}.

Remark A.41. A sequence in $\mathscr{G}_{H,A}$

$$0 \to M \to N \to P \to 0$$

is s-exact iff it is split in the category of topological A-modules.

Definition A.42. An object P of $\mathscr{G}_{H,A}$ is called *s-projective* if the functor $\mathrm{Hom}_{\mathscr{G}_{H,A}}(P, \cdot)$ transforms all short s-exact sequences

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

in $\mathscr{G}_{H,A}$ into exact sequences of A-modules.

Lemma A.43. *If M is any complete Hausdorff locally convex A-module, then*

$$\mathscr{D}(H, A)\widehat{\otimes}_{A,\iota}M$$

is an object of $\mathscr{G}_{H,A}$.

Proof. Indeed $\mathscr{D}(H, A)\widehat{\otimes}_{A,\iota}M$, being Hausdorff and complete by definition, it suffices to remark that by tensoring the identity map on M with (27) we obtain a continuous A-linear map

$$\begin{aligned}\mathscr{D}(H, A)\widehat{\otimes}_{A,\iota}(\mathscr{D}(H, A)\widehat{\otimes}_{A,\iota}M) &\cong (\mathscr{D}(H, A)\widehat{\otimes}_{A,\iota}\mathscr{D}(H, A))\widehat{\otimes}_{A,\iota}M \\ &\to \mathscr{D}(H, A)\widehat{\otimes}_{A,\iota}M.\end{aligned}$$

□

[43] The tensor product $\mathscr{D}(H, A)\widehat{\otimes}_{A,\iota}\mathscr{D}(H, A)$ deserves some explanation. First one forms the abstract tensor product $\mathscr{D}(H, A) \otimes_A \mathscr{D}(H, A)$ and equips it with the *injective* tensor product topology. This means that the topology is universal for separately continuous A-bilinear maps $\beta\colon V \times W \to U$ where V, W and U are locally convex A-modules. Then one takes the Hausdorff completion.

We'll call an object of the form $\mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}M$ (for M any complete Hausdorff locally convex A-module) in $\mathscr{G}_{H,A}$ *s-free*. Notice that s-free does not imply it is free as an A-module. As one expects, s-projective modules can be viewed as direct summands of an s-free module.

Lemma A.44. *An object P of $\mathscr{G}_{H,A}$ is s-projective if and only if it is a direct summand (in $\mathscr{G}_{H,A}$) of an s-free module.*

Proof. First note that for any complete Hausdorff locally convex A-module M and any object N of $\mathscr{G}_{H,A}$ there is a natural A-linear bijection

$$\mathrm{Hom}_{\mathscr{G}_{H,A}}(\mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}M,N)\to \mathrm{Hom}_{A,\mathrm{cont}}(M,N).$$

This is the same proof as the first paragraph of the proof of Lemma A.28 (with A replaced by $\mathscr{D}(H,A)$). The result now follows from [46, Proposition 1.4]. □

We will be interested in considering the cohomology of objects in $\mathscr{G}_{H,A}$ and so we need the notion of a resolution.

Definition A.45. If M is an object of $\mathscr{G}_{H,A}$ then by an *s-projective s-resolution* of M we mean an s-exact sequence

$$\cdots\to X_1\xrightarrow{d_1}X_0\xrightarrow{d_0}M\to 0$$

in $\mathscr{G}_{H,A}$ in which all objects X_i are s-projective.

For an object M of $\mathscr{G}_{H,A}$ let $B_{-1}(H,M):=M$ and for $q\geq 0$ let

$$B_q(H,M):=\mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}B_{q-1}(H,M)$$

with its structure of an s-free module. For $q\geq 0$ define

$$d_q(\delta_0\otimes\cdots\otimes\delta_q\otimes m):=\sum_{i=0}^{q-1}(-1)^i\delta_0\otimes\dots\delta_i\delta_{i+1}\otimes\cdots\otimes\delta_q\otimes m+(-1)^q\delta_0\otimes\cdots\otimes\delta_{q-1}\otimes\delta_q m.$$

Lemma A.46. *For any object M of $\mathscr{G}_{H,A}$ the sequence $(B_q(H,M),d_q)_{q\geq 0}$ is an s-projective s-resolution of M in $\mathscr{G}_{H,A}$.*

Proof. This is an immediate consequence of [31, Proposition 2.4]. The critical point is that by Lemma A.34

$$B_q(H,M)=\mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}B_{q-1}(H,M)=\mathscr{D}(H,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota}B_{q-1}(H,M)$$

and so the $B_q(H,M)$ defined above coincide with the ones defined in [31]. One proves that the splitting is as A-modules and not just as $\mathbf{Q}_p$-vector spaces by exhibiting a contracting homotopy consisting of continuous A-linear maps, cf. [46, §2]. □

Definition A.47. If M and N are objects of $\mathscr{G}_{H,A}$ we define $\mathrm{Ext}^q_{\mathscr{G}_{H,A}}(M,N)$ to be the qth cohomology group of the complex $\mathrm{Hom}_{\mathscr{G}_{H,A}}(B_\bullet(H,M),N)$ for any $q\geq 0$.

Even though we do not need it in the appendix, cohomology groups are thoroughly used in the main body of the text.

Definition A.48. For $M \in \mathscr{G}_{H,A}$, we define the locally analytic cohomology groups $H^q_{\mathrm{an}}(H,M)$ to be $\mathrm{Ext}^q_{\mathscr{G}_{H,A}}(1,M)$, where 1 denotes the object of $\mathscr{G}_{H,A}$ determined by the trivial character $1\colon \mathscr{D}(H,A) \to A$ (which exists by [31, (14)] and Lemma A.34).

Remark A.49. Definition A.47 is independent of the s-projective s-resolution one takes for M. To see this it suffices to show that if M is s-projective and N any object of $\mathscr{G}_{H,A}$ then $\mathrm{Ext}^q_{\mathscr{G}_{H,A}}(M,N)$ is trivial for all $q > 0$. The proof is the same as [46, Proposition 2.2(b)].

Remark A.50. For any two objects M and N of $\mathscr{G}_{H,A}$, as usual $\mathrm{Ext}^1_{\mathscr{G}_{H,A}}(M,N)$ is the set of equivalence classes of s-exact sequences

$$0 \to N \to E \to M \to 0$$

with objects E of $\mathscr{G}_{H,A}$. The point is that for P an object of $\mathscr{G}_{H,A}$, there are natural maps $\delta^*\colon \mathrm{Ext}^q_{\mathscr{G}_{H,A}}(P,M) \to \mathrm{Ext}^{q+1}_{\mathscr{G}_{H,A}}(P,N)$ such that

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathscr{G}_{H,A}}(P,N) \to \mathrm{Hom}_{\mathscr{G}_{H,A}}(P,E) \to \mathrm{Hom}_{\mathscr{G}_{H,A}}(P,M) \\ \xrightarrow{\delta^*} \mathrm{Ext}^1_{\mathscr{G}_{H,A}}(P,N) \to \cdots \end{aligned}$$

is exact. To construct a bijection between $\mathrm{Ext}^1_{\mathscr{G}_{H,A}}(M,N)$ and the set of equivalence classes of s-exact sequences $S\colon 0 \to N \to E \to M \to 0$, one takes $P = M$ and sends S to $\delta^*(\mathrm{id}_M)$. One then checks that this gives a bijection.

Remark A.51. The study of locally analytic cohomology started with Lazard [32] for a compact $\mathbf{Q}_p$-analytic group H and module M of the form $M = \mathbf{Z}_p^r \times \mathbf{Q}_p^s$, $r, s \geq 0$, equipped with a locally analytic action of H. In this setting, locally analytic cohomology is defined by considering the cohomology of the complex of locally analytic cochains. We now compare, following [31, Remark 2.17], Lazard's cohomology groups and the cohomology groups defined here (cf. Definition A.48).

We first recall some facts. Let $V = \varinjlim_j \varprojlim_i V_{i,j}$ be a locally convex A-module of LF-type equipped with a continuous A-linear action of a $\mathbf{Q}_p$-analytic semigroup H extending to a continuous action of $\mathscr{D}(H,A)$. For any $q \geq 0$, we say that a function $H^q \to V$ is pro-analytic (cf. [2, §2.1]) if it locally takes values in some $\varprojlim_i V_{i,j}$ and if each projection $H^q \to V_{i,j}$ is (locally) analytic. We call $\mathscr{C}^{\mathrm{pro-an}}(H^q, V)$ the space of pro-analytic functions from H^q to V. Observe that, if V is of LB-type, then a function is pro-analytic if and only if it is locally analytic, but the definitions of pro-analytic and locally analytic differ when V is Fréchet. One can define (cf. [14, §V], [23, §IV], [2, §2.1]) a well behaved cohomology theory by considering the complex

$$0 \to V \to \mathscr{C}^{\mathrm{pro-an}}(H,V) \to \mathscr{C}^{\mathrm{pro-an}}(H^2,V) \to \ldots,$$

of pro-analytic cochains, where the differentials are given as usual.
Assume first that $V \in \mathrm{Rep}_A^{\mathrm{la},LB}(H)$. Then we have

$$\begin{aligned} \mathrm{Hom}_{\mathscr{G}_{H,A}}(B_q(H,\mathbf{1}),V) &= \mathrm{Hom}_{\mathscr{G}_{H,A}}(\widehat{\otimes}^{q+1}_{A,\iota}\mathscr{D}(H,A),V) \\ &= \mathrm{Hom}_{A,\mathrm{cont}}(\widehat{\otimes}^{q}_{A,\iota}\mathscr{D}(H,A),V) \\ &= \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{D}(H^q,A),V) \\ &\xleftarrow{\sim} \mathrm{LA}(H^q,V), \end{aligned}$$

where we have used, respectively, the definition of $B_q(H,\mathbf{1})$, the A-linear bijection from the proof of Lemma A.44, the isomorphism $\widehat{\otimes}^q_{A,\iota}\mathscr{D}(H,A) \cong \big(\widehat{\otimes}^q_{\mathbf{Q}_p,\iota}\mathscr{D}(H,\mathbf{Q}_p)\big)\widehat{\otimes}_{\mathbf{Q}_p,\iota}A \cong \mathscr{D}(H^q,A)$ (cf. [31, Remark 2.17] for the last isomorphism) and the A-linear bijection of Lemma A.36. We deduce that locally analytic cohomology is in bijection with the cohomology of the complex of locally analytic cochains.
Assume now that $V = \varprojlim V_i$ is a Fréchet A-module, with V_i Banach, equipped with a separately continuous action of $\mathscr{D}(H,A)$ (e.g. the dual of an element of $\mathrm{Rep}_A^{\mathrm{la},LB}(H)$, as $\mathscr{D}(H,A)$ itself). We clearly have $\mathscr{C}^{\mathrm{pro-an}}(H^q,V) = \varprojlim_i \mathrm{LA}(H^q,V_i)$. Then

$$\begin{aligned} \mathrm{Hom}_{\mathscr{G}_{H,A}}(B_q(H,\mathbf{1}),V) &= \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{D}(H^q,A),V) \\ &= \varprojlim_i \mathrm{Hom}_{A,\mathrm{cont}}(\mathscr{D}(H^q,A),V_i) \\ &= \varprojlim_i \mathrm{LA}(H^q,V_i) \\ &= \mathscr{C}^{\mathrm{pro-an}}(H^q,V). \end{aligned}$$

This shows that locally analytic cohomology can be calcualted via the complex of pro-analytic cochains.
Finally, assume that $V = \varinjlim_j \varprojlim_i V_{i,j}$ is a locally convex A-module of A-LF-type which is an extension (in the category of locally convex A-modules) of an A-LB-space by an A-Fréchet, equipped with a separately continuous $\mathscr{D}(H,A)$-action. Then by considering long exact sequences of cohomology, we check again that locally analytic cohomology of V is in bijection with the cohomology calculated with the complex of pro-analytic cochains.

As in the setting of [31], one can identify the categories of separately continuous left and right $\mathscr{D}(H,A)$-modules. If M and N are objects of $\mathscr{G}_{H,A}$, M a right module, we define $M\widetilde{\otimes}_{\mathscr{D}(H,A),\iota}N$ to be the quotient of $M\widehat{\otimes}_{A,\iota}N$ by the image of the natural map

$$\begin{aligned} M\widehat{\otimes}_{A,\iota}\mathscr{D}(H,A)\widehat{\otimes}_{A,\iota}N &\to M\widehat{\otimes}_{A,\iota}N \\ m\otimes\delta\otimes n &\mapsto m\delta\otimes n - m\otimes\delta n, \end{aligned}$$

where $m \in M$, $n \in N$ and $\delta \in \mathscr{D}(H,A)$. The induced topology is the quotient topology.

Lemma A.52. *For any complete Hausdorff locally convex A-module M and any object N of $\mathscr{G}_{H,A}$ there is a natural A-linear topological isomorphism*

$$(M\widehat{\otimes}_{A,\iota}\mathscr{D}(H,A))\widetilde{\otimes}_{\mathscr{D}(H,A),\iota}N \cong M\widehat{\otimes}_{A,\iota}N.$$

If the object P of $\mathscr{G}_{H,A}$ is s-projective then the functor $P\widetilde{\otimes}_{\mathscr{D}(H,A),\iota}(\cdot)$ takes s-exact sequences in $\mathscr{G}_{H,A}$ to exact sequences of A-modules. If P is s-free this functor takes s-exact sequences in $\mathscr{G}_{H,A}$ to s-exact sequences in LCS_A.

Proof. The first part is [46, Proposition 1.5]. The second part follows from that fact that $(-)\widehat{\otimes}_{\mathbf{Q}_p,\iota}M$ preserves the s-exactness of sequences of locally convex A-modules and Lemma A.44. □

Let H_1 be a locally $\mathbf{Q}_p$-analytic group and let H_2 be a closed locally $\mathbf{Q}_p$-analytic subgroup. For an object M of $\mathscr{G}_{H_2,A}$ we set

$$\mathrm{ind}_{H_2}^{H_1}(M) := \mathscr{D}(H_1,A)\widetilde{\otimes}_{\mathscr{D}(H_2,A),\iota}M. \tag{28}$$

For (28) to be a functor[44], we need the following lemma.

Lemma A.53. *The (right) $\mathscr{D}(H_2,A)$-module $\mathscr{D}(H_1,A)$ is s-free. In particular there is an A-linear topological isomorphism*

$$\mathscr{D}(H_1,A) \cong \mathscr{D}(H_1/H_2,A)\widehat{\otimes}_{A,\iota}\mathscr{D}(H_2,A).$$

Proof. The proof of [31, Lemma 5.2] gives that

$$\mathscr{D}(H_1,\mathbf{Q}_p) \cong \mathscr{D}(H_1/H_2,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota}\mathscr{D}(H_2,\mathbf{Q}_p) \tag{29}$$

We now compute

$$\begin{aligned}
\mathscr{D}(H_1,A) &\overset{(i)}{=} \mathscr{D}(H_1,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota}A\\
&\overset{(ii)}{=} \mathscr{D}(H_1/H_2,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota}\mathscr{D}(H_2,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota}A\\
&\overset{(iii)}{=} \mathscr{D}(H_1/H_2,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota}\mathscr{D}(H_2,A)\\
&= \mathscr{D}(H_1/H_2,\mathbf{Q}_p)\widehat{\otimes}_{\mathbf{Q}_p,\iota}A\widehat{\otimes}_{A,\iota}\mathscr{D}(H_2,A)\\
&\overset{(iv)}{=} \mathscr{D}(H_1/H_2,A)\widehat{\otimes}_{A,\iota}\mathscr{D}(H_2,A)
\end{aligned}$$

where (i), (iii) and (iv) follows from Lemma A.34, and (ii) follows from (29). This completes the proof. □

We are now ready to prove the following lemma. The proof is similar to the proof of [31, Proposition 5.1].

[44] It is a priori not clear that $\mathrm{ind}_{H_2}^{H_1}(M)$ is an object of $\mathscr{G}_{H_1,A}$.

Lemma A.54. *The functor*

$$\begin{aligned}\operatorname{ind}_{H_2}^{H_1}\colon \mathscr{G}_{H_2,A} &\to \mathscr{G}_{H_1,A}\\ M &\mapsto \operatorname{ind}_{H_2}^{H_1}(M)\end{aligned}$$

takes s-exact sequences to s-exact sequences and s-projective objects to s-projective objects.

Proof. Lemmas A.52 and A.53 imply that there is a natural A-linear topological isomorphism

$$\operatorname{ind}_{H_2}^{H_1}(M) = \mathscr{D}(H_1/H_2, A)\widehat{\otimes}_{A,\iota} M. \tag{30}$$

Thus $\operatorname{ind}_{H_2}^{H_1}(M)$ is Hausdorff and complete. Its structure of a separately continuous $\mathscr{D}(H_1, A)$-module is the one induced from the s-free module $\mathscr{D}(H_1, A)\widehat{\otimes}_{A,\iota} M$. The final assertion follows from Lemmas A.52 and A.53, and the fact that $\operatorname{ind}_{H_2}^{H_1}(M)$ respects direct sums. □

Lemma A.55 (Relative Frobenius reciprocity). *If M and N are objects of $\mathscr{G}_{H_2,A}$ and $\mathscr{G}_{H_1,A}$, respectively, then there is an A-linear bijection*

$$\operatorname{Hom}_{\mathscr{G}_{H_1,A}}(\operatorname{ind}_{H_2}^{H_1}(M), N) \to \operatorname{Hom}_{\mathscr{G}_{H_2,A}}(M, N)$$

Proof. From the proof of Lemma A.44 we have an A-linear bijection

$$\begin{aligned}\alpha\colon \operatorname{Hom}_{\mathscr{G}_{H_1,A}}(\mathscr{D}(H_1, A)\widehat{\otimes}_{A,\iota} M, N) &\to \operatorname{Hom}_{A,\mathrm{cont}}(M, N)\\ g &\mapsto \alpha(g)\end{aligned}$$

where $\alpha(g)(m) := g(1 \otimes m)$. It follows directly from the definitions that a continuous $\mathscr{D}(H_1, A)$-linear map g from the left factors through $\operatorname{ind}_{H_2}^{H_1}(M)$ $(= \mathscr{D}(H_1/H_2, A)\widehat{\otimes}_{A,\iota} M)$ if and only if $\alpha(g)$ is $\mathscr{D}(H_2, A)$-linear. □

Proposition A.56 (Relative Shapiro's Lemma). *Let H_1 be a locally $\mathbf{Q}_p$-analytic group and let H_2 be a closed locally $\mathbf{Q}_p$-analytic subgroup. If M and N are objects of $\mathscr{G}_{H_2,A}$ and $\mathscr{G}_{H_1,A}$, respectively, then there are A-linear bijections*

$$\operatorname{Ext}^q_{\mathscr{G}_{H_1,A}}(\operatorname{ind}_{H_2}^{H_1}(M), N) \to \operatorname{Ext}^q_{\mathscr{G}_{H_2,A}}(M, N)$$

for all $q \geq 0$.

Proof. Choose an s-projective s-resolution $X_\bullet \to M$ in $\mathscr{G}_{H_2,A}$ (e.g. $(B_q(H, M), d_q)_{q\geq 0}$). By Lemma A.54, the complex $\operatorname{ind}_{H_2}^{H_1}(X_\bullet) \to \operatorname{ind}_{H_2}^{H_1}(M)$ is an s-projective s-resolution of $\operatorname{ind}_{H_2}^{H_1}(M)$. By Lemma A.55 there is an A-linear isomorphism of complexes

$$\operatorname{Hom}_{\mathscr{G}_{H_1,A}}(\operatorname{ind}_{H_2}^{H_1}(X_\bullet), N) \to \operatorname{Hom}_{\mathscr{G}_{X_{H_2},A}}(X_\bullet, N).$$

The result now follows. □

The next result relates locally analytic induction $\mathrm{Ind}_{H_2}^{H_1}$, cf. Remark A.27 and the functor $\mathrm{ind}_{H_2}^{H_1}$.

Lemma A.57. *Let $\delta\colon H_2 \to A^\times$ be a locally analytic character and suppose H_1/H_2 is compact and of dimension 1. Then $\mathrm{Ind}_{H_2}^{H_1}\delta$ and $\left(\mathrm{Ind}_{H_2}^{H_1}\delta\right)'_b$ are naturally objects of $\mathscr{G}_{H_1,A}$ (where $\left(\mathrm{Ind}_{H_2}^{H_1}\delta\right)'_b$ is equipped with H_1-action: $(h_1 \cdot F)(f) := F(h_1^{-1}\cdot f)$ for $F \in \left(\mathrm{Ind}_{H_2}^{H_1}\delta\right)'_b$, $f \in \mathrm{Ind}_{H_2}^{H_1}\delta$ and $h_1 \in H_1$) , and we have an isomorphism*

$$\left(\mathrm{Ind}_{H_2}^{H_1}\delta\right)'_b \cong \mathrm{ind}_{H_2}^{H_1}\delta^{-1}$$

in the category $\mathscr{G}_{H_1,A}$.

Proof. Indeed by Remark A.27, $\mathrm{Ind}_{H_2}^{H_1}\delta \cong \mathrm{LA}(H_1/H_2, A)$ as locally convex A-modules. By Remark A.21[45] the latter is a complete locally convex A-module of A-LB-type. Thus $\mathrm{Ind}_{H_2}^{H_1}\delta$ is an object of $\mathrm{Rep}_A^{\mathrm{la},LB}(H_1)$. By Corollary A.38, it follows that $\mathrm{Ind}_{H_2}^{H_1}\delta$ is an object of $\mathscr{G}_{H_1,A}$ as claimed.
Now as locally convex A-modules

$$\begin{aligned}\left(\mathrm{Ind}_{H_2}^{H_1}\delta\right)'_b &\cong \mathrm{LA}(H_1/H_2, A)'_b\\ &\overset{(i)}{\cong} \mathscr{D}(H_1/H_2, A)\\ &\overset{(ii)}{\cong} \mathrm{ind}_{H_2}^{H_1}\delta^{-1},\end{aligned}$$

where (i) is by definition and (ii) follows from (30). One verifies that the above isomorphism is H_1-equivariant, which shows that $\left(\mathrm{Ind}_{H_2}^{H_1}\delta\right)'_b$ is naturally an object of $\mathscr{G}_{H_1,A}$ and the above isomorphism holds in the category $\mathscr{G}_{H_1,A}$. This finishes the proof.

□

References

[1] Berger, L. Représentations p-adiques et équations différentielles. *Invent. Math. 148*, 2 (2002), 219–284.

[2] Berger, L., and Fourquaux, L. Iwasawa theory and F-analytic Lubin-Tate (φ,Γ)-modules. *Doc. Math. 22* (2017), 999–1030.

[3] Borel, A., and Wallach, N. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, second ed., vol. 67 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2000.

[45]It is important that H_1/H_2 is of dimension 1 here.

[4] Bosch, S. *Lectures on formal and rigid geometry.* Springer-Verlag, 2014.

[5] Breuil, C., Hellmann, C. B. E., and Schraen, B. Une interprétation modulaire de la variété trianguline. *Math. Annalen 367* (2017), 1587-1645.

[6] Chenevier, G. Sur la densité des représentations cristallines du groupe de galois absolu de $\mathbb{Q}_p$. *Math. Ann. 355*, 4 (2013), 1469–1525.

[7] Cherbonnier, F., and Colmez, P. Représentations p-adiques surconvergentes. *Invent. Math. 133*, 3 (1998), 581–611.

[8] Colmez, P. Représentations triangulines de dimension 2. *Astérisque 319* (2008), 213–258.

[9] Colmez, P. Fonctions d'une variable p-adique. *Astérisque*, 330 (2010), 13–59.

[10] Colmez, P. La série principale unitaire de $\mathrm{GL}_2(\mathbf{Q}_p)$. *Astérisque*, 330 (2010), 213–262.

[11] Colmez, P. Representations de $\mathrm{GL}_2(\mathbf{Q}_p)$ et (φ, Γ)-modules. *Astérisque 330* (2010), 281–509.

[12] Colmez, P. (φ, Γ)-modules et représentations du mirabolique de $\mathrm{GL}_2(\mathbf{Q}_p)$. *Astérisque*, 330 (2010), 61–153.

[13] Colmez, P. La série principale unitaire de $\mathrm{GL}_2(\mathbf{Q}_p)$: vecteurs localement analytiques. *Automorphic Forms and Galois Representations*, 1 (2014), 286–358.

[14] Colmez, P. Représentations localement analytique de $\mathrm{GL}_2(\mathbf{Q}_p)$ et (φ, Γ)-modules. *Representation Theory*, 20 (2016), 187–248.

[15] Colmez, P., and Dospinescu, G. Complétés universels de représentations de $\mathrm{GL}_2(\mathbf{Q}_p)$. *Algebra Number Theory 8*, 6 (2014), 1447–1519.

[16] Colmez, P., Dospinescu, G., and Paškūnas, V. The p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$. *Camb. J. Math. 2*, 1 (2014), 1–47.

[17] Crew, R. Finiteness theorems for the cohomology of an overconvergent isocrystal on a curve. *Ann. Sci. École Norm. Sup. (4) 31*, 6 (1998), 717–763.

[18] Dixon, J. D., du Sautoy, M. P. F., Mann, A., and Segal, D. *Analytic pro-p groups*, second ed., vol. 61 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1999.

[19] Dospinescu, G. Actions infinitésimales dans la correspondance de Langlands locale p-adique pour $\mathrm{GL}_2(\mathbf{Q}_p)$. *Math. Ann. 354* (2012), 627–657.

[20] Emerton, M. Locally analytic vectors in representations of locally p-adic analytic groups. *Memoirs of the AMS 248* (2017), no. 1175.

[21] Emerton, M., and Helm, D. The local Langlands correspondence for GL_n in families. *Ann. Sci. Éc. Norm. Supér. (4) 47*, 4 (2014), 655–722.

[22] Féaux de Lacroix, C. T. Einige Resultate über die topologischen Darstellungen p-adischer Liegruppen auf unendlich dimensionalen Vektorräumen über einem p-adischen Körper. In *Schriftenreihe des Mathematischen Instituts der Universität Münster. 3. Serie, Heft 23*, vol. 23 of *Schriftenreihe Math. Inst. Univ. Münster 3. Ser.* Univ. Münster, Münster, 1999, pp. x+111.

[23] Fourquaux, L., and Xie, B. Triangulable $\mathcal{O}_F$-analytic (φ_q, Γ)-modules of rank 2. *Algebra & Number Theory 7* (2013), 2545–2592.

[24] Grothendieck, A. Produits tensoriels topologiques et espaces nucléaires. In *Séminaire Bourbaki, Vol. 2*. Soc. Math. France, Paris, 1995, pp. Exp. No. 69, 193–200.

[25] Harris, M., and Taylor, R. *The geometry and cohomology of some simple Shimura varieties*, vol. 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich.

[26] Johansson, C., and Newton, J. Extended eigenvarieties for overconvergent cohomology. 2016.

[27] Kedlaya, K., Pottharst, J., and Xiao, L. Cohomology of arithmetic families of (φ, Γ)-modules. *Journal of the American Mathematical Society 27* (2014), 1043–1115.

[28] Kedlaya, K. S. A p-adic local monodromy theorem. *Ann. of Math. (2) 160*, 1 (2004), 93–184.

[29] Kedlaya, K. S. *p-adic differential equations*, vol. 125 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[30] Kedlaya, K. S., and Liu, R. Relative p-adic Hodge theory: foundations. *Astérisque*, 371 (2015), 239.

[31] Kohlhasse, J. The cohomology of locally analytic representations. *J. Reine Angew. Math. 651* (2011), 187–240.

[32] Lazard, M. Groupes analytiques p-adiques. *Inst. Hautes Études Sci. Publ. Math.*, 26 (1965), 389–603.

[33] Mangino, E. M. Complete projective tensor product of (LB)-spaces. *Arch. Math. (Basel) 64*, 1 (1995), 33–41.

[34] Neukirch, J., Schmidt, A., and Wingberg, K. *Cohomology of number fields*, second ed., vol. 323 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2008.

[35] Paškūnas, V. The image of Colmez's Montreal functor. *Publ. Math. Inst. Hautes Études Sci. 118* (2013), 1–191.

[36] Rodrigues Jacinto, J. (φ, Γ)-modules de de Rham et fonctions L p-adiques. *Algebra & Number Theory 12*, 4 (2018), 885–934.

[37] Schneider, P. *Nonarchimedean Functional Analysis*. Springer Monographs in Mathematics. Springer, 2002.

[38] Schneider, P., and Teitelbaum, J. p-adic Fourier theory. *Doc. Math. 6* (2001), 447–481.

[39] Schneider, P., and Teitelbaum, J. Locally analytic distributions and p-adic representation theory, with applications to GL_2. *J. Amer. Math. Soc. 15*, 2 (2002), 443–468.

[40] Schneider, P., and Teitelbaum, J. Algebras of p-adic distributions and admissible representations. *Invent. Math. 153*, 1 (2003), 145–196.

[41] Schneider, P., Teitelbaum, J., and Prasad, D. $U(\mathfrak{g})$-finite locally analytic representations. *Represent. Theory 5* (2001), 111–128. With an appendix by Dipendra Prasad.

[42] Stacks Project Authors, T. Stacks project. http://stacks.math.columbia.edu, 2016.

[43] Symonds, P., and Weigel, T. Cohomology of p-adic analytic groups. In *New horizons in pro-p groups*, vol. 184 of *Progr. Math.* Birkhäuser Boston, Boston, MA, 2000, pp. 349–410.

[44] Tamme, G. On an analytic version of Lazard's isomorphism. *Algebra Number Theory 9*, 4 (2015), 937–956.

[45] Tavares Ribeiro, F. An explicit formula for the Hilbert symbol of a formal group. *Ann. Inst. Fourier (Grenoble) 61*, 1 (2011), 261–318.

[46] Taylor, J. L. Homology and cohomology for topological algebras. *Advances in Math. 9* (1972), 137–182.

Ildar Gaisin
University of Tokyo
7 Chome-3-1 Hongo,
Bunkyō, Tokyo 113-8654
ildar@ms.u-tokyo.ac.jp

Joaquín Rodrigues Jacinto
University College London
25 Gordon street
WC1H 0AY London
j.jacinto@ucl.ac.uk