Rost Nilpotence and Free Theories

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Abstract. We introduce coherent cohomology theories $\underline{\mathrm{h}}$ and prove that if such a theory is moreover generically constant then the Rost nilpotence principle holds for projective homogeneous varieties in the category of $\underline{\mathrm{h}}$-motives. Examples of such theories are algebraic cobordism and its descendants the free theories.

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1 Introduction

Let X be a smooth projective quadric of dimension n over a field F of characteristic $\neq 2$. In [13] Rost showed that a correspondence $\alpha \in \mathrm{CH}_n(X \times_F X)$ which is rationally equivalent to zero over a field extension E of F is nilpotent. A (purely) algebraic consequence of this principle is that given a correspondence π of degree 0 on X, which becomes idempotent over an algebraic closure of F, then π corresponds to a direct summand of the (Chow-)motive of X. This implies in particular that there are no *phantom* direct summands, *i.e.* non trivial summands disappearing over an algebraic closure. Therefore getting a complete decomposition of the (Chow-)motive of a projective quadric is equivalent to finding a complete orthogonal set of rational, *i.e.* defined over the base field, idempotent correspondences. Using this Rost gave in *loc.cit.* a decomposition of the Chow motive of a norm quadric which plays a crucial role in the proof of the Milnor conjecture by Voevodsky [20].

It is conjectured that Rost nilpotence holds for all smooth projective varieties and so for all objects in the category of Chow motives with integral coefficients. This has been verified for projective homogeneous varieties by Chernousov, Merkurjev and the first author in [4] and with a different proof by Brosnan [2], for surfaces over fields of characteristic zero by the first author [7, 8], and for smooth projective varieties whose motive is generically split by Zainoulline and the second author [19]. Beside these examples and some trivial cases this conjecture is wide open.

The usefulness of the nilpotence principle is not limited to the proof of the Milnor conjecture. It implies uniqueness results, *i.e.* analogs of the Krull-Schmidt theorem, for the decomposition of motives, see for instance [3], and moreover some of the recent advances in the algebraic theory of quadratic forms rely on it. A refinement of these 'geometric' methods for more sophisticated cohomology theories than Chow theory should lead to further insights, but requires the verification of the Rost nilpotence principle for quadrics and – more general – for projective homogeneous varieties for such theories. We prove here this principle for free theories and projective homogeneous varieties, see Corollary 4.5. Moreover, we outline the general context where the nilpotence principle holds. It is the context of the in Definitions 2.4 and 2.6 introduced (generically) constant *coherent* theories. Such a theory describes an environment where a cohomology functor is defined not just for varieties of finite type over the ground field, but also for varieties over every finitely generated extension of it, and moreover, these functors fit nicely with each other for different extensions. A large supply of such theories are the so called *free theories*, *i.e.* theories $\underline{\mathrm{h}}_*$ which arise from algebraic cobordism by change of the group law: $\underline{\mathrm{h}}_*(X) = \underline{\mathrm{h}}_*(F) \otimes_{\mathbb{L}} \Omega_*(X)$, where $\mathbb{L}$ is the Lazard ring (cobordism of the point), Ω_* denotes algebraic cobordism of Levine-Morel, and $\underline{\mathrm{h}}_*(F)$ is the cohomology of the point, see Corollary 2.13. But the class of *coherent* theories is much bigger. In Example 4.6 we provide a construction which permits to produce such theories with very flexible properties.

On our way we demonstrate also the failure of various aspects of the nilpotence principle outside the *coherent* environment. This in particular shows that Rost nilpotence is not just a formal consequence of the axioms of oriented theories - see Remark 4.3 and Example 4.8.

The content of the paper is as follows. In the first section we introduce coherent extended oriented cohomology theories. We show then that algebraic cobordism and its near relatives, the free theories, are examples of such theories. In Section 3 we recall following Manin [12] the definition of $\underline{\mathrm{h}}$-motives, where $\underline{\mathrm{h}}$ is an oriented cohomology theory. The proof of the Rost nilpotence principle for projective homogenous varieties in the category of $\underline{\mathrm{h}}$-motives, where $\underline{\mathrm{h}}$ is a coherent extended oriented cohomology theory, is given in the last section. Here we give also counter examples to this principle for arbitrary oriented cohomology theories.

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2 Coherent theories

2.1 Notations and conventions

Throughout this article fields are assumed to be of characteristic zero.

For a field F we denote by $\mathbf{S}m_F$ the category of smooth quasi-projective schemes over $\mathrm{Spec}(F)$, and by $\mathfrak{R}ings^*$ the category of $\mathbb{Z}$-graded commutative rings.

Given a field k we denote by $\mathfrak{F}ields_k$ the category of finitely generated field extensions of k.

If X is a scheme over a field F and $L \supseteq F$ is a field extension we set $X_L := L \times_F X$. The function field of an integral F-scheme X is denoted by $F(X)$.

2.2 Oriented cohomology- and Borel-Moore homology theories

We follow the definitions and conventions in the book [10] on algebraic cobordism by Levine and Morel, *i.e.* an *oriented cohomology theory* over the field F is a functor

$$\underline{\mathrm{h}}^* : \mathbf{S}m_F{}^{op} \to Rings^*$$

with additional structure of push-forward maps $f_* : \underline{\mathrm{h}}^*(X) \longrightarrow \underline{\mathrm{h}}^{*+d}(Y)$ along projective morphisms $f : X \longrightarrow Y$ of (constant) relative codimension d subject to several axioms including a projective bundle theorem, see [10, Def 1.1.2].

As usual we denote the pull-back $\underline{\mathrm{h}}^*(f)$ by f^* for a morphism $f : X \longrightarrow Y$ in $\mathbf{S}m_F$.

If $X \in \mathbf{S}m_F$ with connected components $X_1, \ldots, X_l$ one sets

$$\underline{\mathrm{h}}_*(X) := \bigoplus_{i=1}^{l} \underline{\mathrm{h}}^{\dim X_i - *}(X_i)\,.$$

The assignment $X \mapsto \underline{\mathrm{h}}_*(X)$ defines then a Borel-Moore homology theory on $\mathbf{S}m_F$ in the sense of [10, Def. 5.1.3], and by [10, Prop. 5.2.1]

$$\underline{\mathrm{h}}_* \longleftrightarrow \underline{\mathrm{h}}^*$$

is a one-to-one correspondence between oriented cohomology- and oriented Borel-Moore homology theories on $\mathbf{S}m_F$.

Given an oriented Borel-Moore homology theory $\underline{h}_*$ on F we can extend it to schemes of finite type over F by setting:

$$\underline{h}_*(Y) := \operatorname{colim}_{V \to Y} \underline{h}_*(V),$$

where the limit runs over projective morphisms $V \to Y$ with $V \in \mathbf{S}m_F$ with push-forward maps as transition maps.

In the following we consider theories with localization sequence (these correspond to *oriented cohomology theories* of [17, Def. 2.1]). More precisely, we say an oriented Borel-Moore homology theory $\underline{h}_*$, or equivalently the respective oriented cohomology theory $\underline{h}^*$, over F has the *localization property* if given a smooth quasi-projective F-scheme X and a closed F-embedding $j : Z \to X$ with the open complement $i : U \to X$ then there is an exact sequence

$$\underline{h}_*(Z) \xrightarrow{j_*} \underline{h}_*(X) \xrightarrow{i^*} \underline{h}_*(U) \to 0. \qquad \text{(LOC)}$$

By [17, Sect. 2.2] the same property will hold for any quasi-projective scheme X over the base field F.

Examples of such theories are Chow groups, algebraic cobordism, and Grothendieck's K_0 (in the latter example we ignore by some abuse of notation the grading).

An immediate consequence of the localization property is the following useful lemma.

2.3 Lemma

Let $\underline{h}_*$ be an oriented Borel-Moore homology theory over the field F with localization property. Assume we have a projective morphism $\pi : X \longrightarrow Y$ between finite type F-schemes, whose restriction to $\pi^{-1}(U)$ is an isomorphism for some open set $U \subset Y$. Let $i : Z := Y \setminus U \hookrightarrow Y$ be the closed complement. Then

$$\pi_* + i_* : \underline{h}_*(X) \oplus \underline{h}_*(Z) \longrightarrow \underline{h}_*(Y)$$

is surjective.

One of the points we would like to stress in our article is that the Rost Nilpotence result can't be proven using only varieties of finite type over the ground field k (as various counter-examples provided below demonstrate). One has to work over various (finitely generated) extensions of k as well. And the cohomology theory should be "adjusted" for such a context. It should have a strong property of *coherence*, which assures that not only $\underline{h}$ is functorial with respect to field extensions, but also that, for any finitely generated field extensions $E/F/k$, the restriction of $\underline{h}$ to E-varieties can be reconstructed out of its values on F-varieties.

2.4 Definition

An *extended oriented cohomology theory* over the field k is a family of oriented cohomology theories with localization property

$$\underline{\mathrm{h}}^*_F \;:\; \mathbf{S}m_F \longrightarrow \mathfrak{R}ings^* \,, \quad F \in \mathfrak{F}ields_k \,,$$

with the following additional data:

Given $F \in \mathfrak{F}ields_k$ and a smooth morphism $\rho : Y \longrightarrow X$ in $\mathbf{S}m_F$ with X integral then there is a homomorphism of rings

$$\underline{\mathrm{h}}^*_F(Y) \xrightarrow{\theta_{Y/X}} \underline{\mathrm{h}}^*_{F(X)}(F(X) \times_X Y) \,.$$

(The map $\theta_{Y/X}$ might be interpreted as pull-back along the upper row in the cartesian square

$$\begin{array}{ccc} F(X) \times_X Y & \longrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle \rho} \\ \operatorname{Spec} F(X) & \xrightarrow[\iota]{} & X \end{array} \,,$$

where $\iota : \operatorname{Spec} F(X) \longrightarrow X$ is the generic point of X.)

These data are subject to the following four axioms:

(EC0) $\theta_{X/\operatorname{Spec} F}$ is the identity map for all $F \in \mathfrak{F}ields_k$ and all $X \in \mathbf{S}m_F$.

Given a commutative diagram

$$\begin{array}{ccc} Y_1 & \xrightarrow{\;f\;} & Y_2 \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

in $\mathbf{S}m_F$ with X integral for some finitely generated field extension F of k we denote by $\tilde{f} : F(X) \times_X Y_1 \longrightarrow F(X) \times_X Y_2$ the induced morphism on the generic fibers. Then:

(EC1) For any morpism f we have

$$\theta_{Y_1/X} \circ f^* \;=\; \tilde{f}^* \circ \theta_{Y_2/X} \,;$$

(EC2) If f is projective of constant relative dimension

$$\tilde{f}_* \circ \theta_{Y_1/X} \;=\; \theta_{Y_2/X} \circ f_* \,.$$

(EC3) Given a diagram of cartesian squares

$$\begin{array}{ccccc}
\widetilde{Z} & \longrightarrow & F(X)\times_X Z & \longrightarrow & Z \\
\downarrow & & {\scriptstyle\tilde{\sigma}}\downarrow & & \downarrow{\scriptstyle\sigma} \\
\operatorname{Spec} F(Y) & \longrightarrow & F(X)\times_X Y & \longrightarrow & Y \\
 & & \downarrow & & \downarrow{\scriptstyle\rho} \\
 & & \operatorname{Spec} F(X) & \longrightarrow & X
\end{array}$$

with X, Y integral and ρ, σ morphisms in $\mathbf{S}m_F$ for some $F \in \mathfrak{F}ields_k$ (note that $F(Y)$ and the function field of $F(X) \times_X Y$ coincide). Then in this situation the following holds:

$$\theta_{(F(X)\times_X Z)/(F(X)\times_X Y)} \circ \theta_{Z/X} = \theta_{Z/Y} \; : \; \underline{\mathrm{h}}^*_F(Z) \; \longrightarrow \; \underline{\mathrm{h}}^*_{F(Y)}(\widetilde{Z})\,.$$

2.5 The group $\underline{\mathrm{h}}^*_F(Y/X)$

Let $\underline{\mathrm{h}}^*$ be an extended oriented cohomology theory over the field k and F a finitely generated field extension of k.

For a smooth morphism $Y \to X$ in $\mathbf{S}m_F$ with X integral we set

$$\underline{\mathrm{h}}^*_F(Y/X) \; := \; \operatorname{colim}_{U\to X} \underline{\mathrm{h}}^*_F(U \times_X Y)\,,$$

where the colimit runs over all open subschemes U of X.

Let $U \subseteq X$ be an open subscheme. Then we have a homomorphism

$$\theta_{U\times_X Y/X} \; : \; \underline{\mathrm{h}}^*_F(U \times_X Y) \; \longrightarrow \; \underline{\mathrm{h}}^*_{F(X)}(\widetilde{Y})\,,$$

where we used that $\widetilde{Y} := F(X) \times_X Y = F(X) \times_X (U \times_X Y)$. By axiom (EC1) we have $\theta_{Y/X} = \theta_{U\times_X Y/X} \circ q^*$, where $q : U \times_X Y \longrightarrow Y$ is the projection, and more generally

$$\theta_{U\times_X Y/X} \; = \; \theta_{V\times_X Y/X} \circ (i \times \mathrm{id}_Y)^*$$

if $i : V \hookrightarrow U$ is another open subscheme of X contained in U. Hence we have a canonical map

$$\varphi_{Y/X} \; : \; \underline{\mathrm{h}}^*_F(Y/X) \; \longrightarrow \; \underline{\mathrm{h}}^*_{F(X)}(\widetilde{Y})\,.$$

2.6 Definition

An extended oriented cohomology theory over k is called *coherent* if the map $\varphi_{Y/X}$ is an isomorphism for all smooth F-morphisms $Y \longrightarrow X$ and all $F \in \mathfrak{F}ields_k$.

2.7 Examples

(i) Let Ω^* be algebraic cobordism as defined by Levine and Morel [10], $\rho : Y \longrightarrow X$ a morphism in $\mathbf{S}m_F$ with X integral for some $F \in \mathfrak{Fields}_k$. Set $E := F(X)$ and $\widetilde{Y} := E \times_X Y$. Given a cobordism cycle

$$[g : W \longrightarrow Y\,,\ L_1, \dots, L_r]$$

over Y, see [10, Def. 2.16] for the definition, then the pull-back

$$\big[\ \operatorname{Spec} E \times_X W \longrightarrow \widetilde{Y}\,,\ p_W^*(L_1), \dots, p_W^*(L_r)\,\big]\,,$$

where $p_W : \operatorname{Spec} E \times_X W \longrightarrow W$ is the projection, is a cobordism cycle over $\widetilde{Y}$. It is straightforward to check that this induces a homomorphism $\theta_{Y/X} : \Omega^*(Y) \longrightarrow \Omega^*(\widetilde{Y})$. We will show below, see Corollary 2.10, that this map obeys the axioms (EC1)-(EC3), and so algebraic cobordism is an extended oriented cohomology theory over all fields of characteristic 0. In Theorem 2.11 we prove then that algebraic cobordism is also coherent.

(ii) From the fact that algebraic cobordism is an extended oriented cohomology theory it follows also that Chow groups with rational or algebraic equivalence and arbitrary coefficients, and also Grothendieck's K_0 are extended oriented cohomology theories. Note that this can be seen directly as well and is also true over fields of positive characteristic. All of these theories, except for CH^*_{alg}, are coherent by Corollary 2.13.

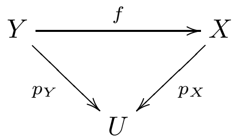
2.8 Remark

Not all extended oriented cohomology theories are *coherent*. An example is provided by CH^*_{alg}. Indeed, consider the smooth projection $C \times C \to C$, where C is an elliptic curve over k. Then in $\mathrm{CH}^*_{alg}(C \times C)$ the class of the diagonal $[\Delta]$ is not equal to the linear combination of $[C \times p]$ and $[p \times C]$, for a k-rational point p, since it is not so even modulo homological equivalence. Hence, from the localization sequence (and the fact that classes of all rational points on C are equivalent modulo algebraic equivalence) we obtain that the restriction of $[\Delta]$ to $\left(\mathrm{CH}^*_{alg}\right)_k (C \times C/C)$ is different from that of $[C \times p]$. On the other hand, the restriction of these two classes to $\left(\mathrm{CH}^*_{alg}\right)_{k(C)} (C_{k(C)})$ are just classes of two rational points which are equal. Hence, the map $\varphi_{C\times C/C}$ is not injective.

But below we will show that all the theories obtained from Ω^* by change of coefficients (the so-called *free theories*) are coherent.

2.9 Lemma

Let $\underline{\mathrm{h}}^* := \Omega^*$ *be algebraic cobordism and let* F *be a field. Assume we have a commutative diagram*



in $\mathbf{S}m_F$ *with* U *integral and* p_X, p_Y *smooth. Denote by* $\widetilde{X}$ *and* $\widetilde{Y}$ *the generic fiber of* p_X *and* p_Y*, respectively, and let* $\widetilde{f} : \widetilde{Y} \longrightarrow \widetilde{X}$ *be the morphism induced by* f*. Then we have:*

(i) $\widetilde{f}^*(\theta_{X/U}(\alpha)) = \theta_{Y/U}(f^*(\alpha))$ *for all* $\alpha \in \Omega^*(X)$*;*

(ii) if f *is projective, then* $\widetilde{f}_*(\theta_{Y/U}(\gamma)) = \theta_{X/U}(f_*(\gamma))$ *for all* $\gamma \in \Omega^*(Y)$*; and*

(iii) $\theta_{X/U}(\alpha \cdot \beta) = \theta_{X/U}(\alpha) \cdot \theta_{X/U}(\beta)$ *for all* $\alpha, \beta \in \Omega^*(X)$*;*

Proof. We recall first that given a closed regular embedding $j : V \hookrightarrow W$ in $\mathbf{S}m_F$ for some field F then $\Omega^*(W)$ is generated by cobordism cycles $[h : Z \longrightarrow W]$ with h transversal to j, see Levine and Morel [10, Prop. 3.3.1]. The image of such a cycle under j^* is then just a pull-back of cycles. In fact, since V, W are smooth over F the fiber product $Z \times_W V$ is regular and so is smooth, as $\operatorname{char} F = 0$. Hence we can apply [10, Cor. 6.5.5] and get

$$j^*\big([h : Z \longrightarrow W]\big) \;=\; [h' : Z \times_W V \longrightarrow V]\,,$$

where h' is the projection. This implies (i) if f is a regular embedding.

For the general case of (i) observe that f factors as $Y \xrightarrow{f\times \mathrm{id}} X \times_F Y \to X$ and so, since X and Y are smooth, into a composition of a regular embedding and a smooth morphism. This reduces the proof to the case of a smooth equidimensional morphism of constant relative dimension, where it follows from the definition of the pull-back in this case, see [10, Sect. 2.1].

(ii) is an immediate consequence of the definition of the push-forward of cobordism cycles along projective maps.

Finally we prove (iii). Denote by $\Delta_{X/U} : X \longrightarrow X \times_U X$ and $\Delta_{X/F} : X \longrightarrow X \times_F X$ the respective diagonal maps. Consider $\Delta_{X/U}$ as a U-morphism and denote $\widetilde{\Delta}_{X/U}$ the induced map between the generic fibers. Now $F(U) \times_U (X \times_U X)$ is naturally isomorphic to the fiber product of $\widetilde{X} = F(U) \times_U X$ with itself over $F(U)$, and under this identification $\widetilde{\Delta}_{X/U}$ corresponds to the diagonal map $\Delta_{\widetilde{X}/F(U)} : \widetilde{X} \longrightarrow \widetilde{X} \times_{F(U)} \widetilde{X}$. Hence by (ii) we have

$$\theta_{X/U} \circ \Delta^*_{X/U} = \Delta^*_{\widetilde{X}/F(U)} \circ \theta_{X \times_U X/U}\,. \tag{1}$$

By definition of the product in algebraic cobordism we have

$$\alpha \cdot \beta \;=\; \Delta^*_{X/F}(\alpha \times \beta)$$

and

$$\theta_{X/U}(\alpha) \cdot \theta_{X/U}(\beta) \;=\; \Delta^*_{\widetilde{X}/F(U)}\big(\theta_{X/U}(\alpha) \times \theta_{X/U}(\beta)\big)\,.$$

Hence by (1) it remains to show

$$\theta_{X\times_U X/U}\big(\delta^*_{X/U}(\alpha \times \beta)\big) \;=\; \theta_{X/U}(\alpha) \times \theta_{X/U}(\beta) \tag{2}$$

for all $\alpha, \beta \in \Omega^*(X)$, where $\delta_{X/U}$ is the regular immersion $X \times_U X \longrightarrow X \times_F X$.

If $\rho : U' \to U$ is an open embedding and $\chi : \;\; X' := U' \times_U X \longrightarrow X$ the projection then we have

$$\delta^*_{X'/U'} \circ (\chi \times \chi)^* \;=\; (\chi \times_\rho \chi)^* \circ \delta^*_{X/U} \quad \text{and} \quad \theta_{X\times_U X/U} = \theta_{X'\times_{U'} X'/U'} \circ (\chi \times_\rho \chi)^*.$$

Thus to prove (2) we can replace U by an open subscheme of U.

After these reductions and remarks we can finish the proof of (iii). We can assume that α is represented by $[v : V \longrightarrow X]$ and β by $[w : W \longrightarrow X]$ for some $V, W \in \mathbf{S}m_F$. The generic fiber $F(U) \times_U W$ is (as localization of a regular scheme) also regular and therefore smooth over $F(U)$ since (by our general assumption) $\operatorname{char} F = 0$. Therefore replacing U by an open subscheme we can assume that the composition of morphisms $W \longrightarrow X \xrightarrow{p_X} U$ is smooth. Then $V \times_F W \longrightarrow X \times_F X$ is transversal to the regular embedding $\delta_{X/U} : X \times_U X \longrightarrow X \times_F X$ and $V \times_U W$ is smooth over F. Hence by [10, Cor. 6.5.5] the pull-back $\delta^*_{X/U}(\alpha \times \beta)$ is represented by the cartesian product $[V \times_U W \to X \times_U X]$, and so we have $\theta_{X\times_U X/U}\big(\delta^*_{X/U}(\alpha \times \beta)\big) = \theta_{X/U}(\alpha) \times \theta_{X/U}(\beta)$. From this the claim follows. □

2.10 Corollary

Algebraic cobordism (of Levine and Morel) is an extended oriented cohomology theory.

Proof. The axiom (EC0) is obvious, and by Lemma 2.9 above $\theta_{Y/X}$ is a homomorphism of rings satisfying also (EC1) and (EC2).

For (EC3): It is enough to check this on 'standard' cobordism cycles $[w : W \longrightarrow X]$, where it is straightforward and amounts to the identification of two different ways to express the generic fiber of a map to a fibration. □

For the proof of the following theorem we use the generators and relations of algebraic cobordism.

2.11 Theorem

Algebraic cobordism is a coherent theory.

Proof. Let F be a field of characteristic zero, and $Y \stackrel{\pi}{\longrightarrow} X$ be a smooth morphism in $\mathbf{S}m_F$ with X integral, $E = F(X)$ and $\widetilde{Y}$ the generic fiber. Let $\Omega^*_F(Y/X) = \operatorname{colim}_{U \to X} \Omega^*(Y \times_X U)$ over all open subschemes of X and

$$\varphi_{Y/X} : \Omega^*_F(Y/X) \longrightarrow \Omega^*_E(\widetilde{Y})$$

be the canonical map. We need to show that $\varphi_{Y/X}$ is an isomorphism.
Combining the definitions of Ω^* from [10] and [11] we have an exact sequence:

$$\mathcal{R}_*(Q) \to \mathcal{Z}_*(Q) \to \Omega_*(Q) \to 0.$$

Here $\mathcal{Z}_*(Q)$ is a free abelian group generated by isomorphism classes of *cobordism cycles* $[v : V \to Q, L_1, \ldots, L_r]$, where v is projective, V is smooth, and $L_1, \ldots, L_r$ are (non-ordered) line bundles on V, see [10, Def 2.1.6].
The relations $\mathcal{R}_*(Q)$ form a free abelian group which splits as a direct sum

$$\mathcal{R}_*(Q) \,=\, \mathcal{R}^{Sect}_* \oplus \mathcal{R}^{Dim}_* \oplus \mathcal{R}^{MPR}_* \,.$$

The three groups on the right hand side and their maps to $\mathcal{Z}_*(Q)$ are given by:

(Rel1) $\mathcal{R}^{Dim}_*(Q)$ is the free abelian group generated by isomorphism classes of cobordism cycles $[v : V \to Q, L_1, \ldots, L_r]$ with $r > \dim V$ with the obvious map to $\mathcal{Z}_*(Q)$.

(Rel2) $\mathcal{R}^{Sect}_*(Q)$ is the free abelian group generated by the isomorphism classes of pairs $[Z \stackrel{i}{\subset} V \stackrel{v}{\to} Q, L_1, \ldots, L_r]$ of a cobordism cycle plus a smooth divisor on V. Such a pair is mapped to the difference

$$[V \stackrel{v}{\to} Q, L_1, \ldots, L_r, O(Z)] - [Z \stackrel{v \circ i}{\to} Q, i^*L_1, \ldots, i^*L_r]$$

in $\mathcal{Z}_*(Q)$.

(Rel3) Finally, $\mathcal{R}^{MPR}_*$ is the free abelian group generated by the isomorphism classes of *multiple point relations* $[w : W \to Q \times \mathbb{P}^1, M_1, \ldots, M_s]$, where w is projective, W is smooth, $W_0 = w^{-1}(Q \times 0)$ and $W_1 = w^{-1}(Q \times 1)$ are divisors with strict normal crossings on W, see [10, Def 3.1.4]), and $M_1, \ldots, M_s$ are line bundles on W.

Such an element is mapped to $w_*([W_0 \to W, M_1, \ldots, M_s] - [W_1 \to W, M_1, \ldots, M_s])$, where $[W_l \to W]$ is the *combinatorial divisor class*, see [10, Def 3.15], and where we lift the coefficients of the universal formal group law to cobordism cycles from $\mathcal{Z}_*(\operatorname{Spec}(k))$.

Recall here, that if $D = \sum_{i=1}^{m} n_i \cdot D_i$ is a divisor with strict normal crossing on W, with $L_i = O(D_i)$ and $d_J : D_J = \cap_{j \in J} D_j \to W$ for $J \subset \{1, \ldots, m\}$, the faces of our divisor, we can write down the formal sum

$$[n_1] \cdot_F u_1 +_F \ldots +_F [n_m] \cdot_F u_m \quad \text{as} \quad \sum_{J \subset \{1,\ldots,m\}} u^J \cdot F_J^{n_1,\ldots,n_m}(u_1, \ldots, u_m)\,,$$

where

$$F_J^{n_1,\ldots,n_m}(u_1, \ldots, u_m) \in \mathbb{L}[[u_1, \ldots, u_m]]$$

are some power series with $\mathbb{L}$-coefficients, which can be chosen canonically. Then the combinatorial divisor class $[D \to W]$ is just the sum

$$\sum_{J \subset \{1,\ldots,m\}} [d_J : D_J \to W, F_J^{n_1,\ldots,n_m}(L_1^J, \ldots, L_m^J)],$$

where $L_i^J = d_J^* L_i$. Here we ignore all the terms of degree $> \dim W$ (these are covered by $\mathcal{R}_*^{Dim}$), and so, it is a finite sum. Note, that $\mathcal{R}_*^{MPR}$ contains the *geometric cobordism relations*, *cf.* [10, Def 2.3.2], as well as *double point relations*, see [11], and so, the (FGL) relations of [10].

Finally, the map $\mathcal{Z}_*(Q) \to \Omega_*(Q)$ is given by

$$[v : V \to Q, L_1, \ldots, L_r] \mapsto v_*(c_1(L_1) \cdot \ldots \cdot c_1(L_r)).$$

It follows from [10, Thm 2.3.13] that in $\mathcal{R}_*^{Dim}$ we can take any number $N \geq \dim V$ instead of $\dim V$ (modifying accordingly the interpretation of $\mathcal{R}_*^{MPR}$).

We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\tilde{\mathcal{R}}_*(\widetilde{Y}) & \longrightarrow & \mathcal{Z}_*(\widetilde{Y}) & \longrightarrow & \Omega_*^E(\widetilde{Y}) & \longrightarrow & 0 \\
\uparrow \rho_{Y/X} & & \uparrow \psi_{Y/X} & & \uparrow \varphi_{Y/X} & & \\
\operatorname*{colim}_{U \to X} \mathcal{R}_*(Y \times_X U) & \longrightarrow & \operatorname*{colim}_{U \to X} \mathcal{Z}_*(Y \times_X U) & \longrightarrow & \Omega_*^F(Y/X) & \longrightarrow & 0,
\end{array}$$

where in the definition of $\tilde{\mathcal{R}}_*^{Dim}(\widetilde{Y})$ we swap $\dim \widetilde{V}$ by $\dim \widetilde{V} + \dim X$.

We show now that $\rho_{Y/X}$ is onto and $\psi_{Y/X}$ is an isomorphism, from which it follows that $\varphi_{Y/X}$ is an isomorphism too, hence finishing the proof.

We start showing $\psi_{Y/X}$ is onto. In fact, if we have a projective map $\widetilde{v} : \widetilde{V} \to \widetilde{Y}$ in $\mathbf{Sm}_E$, we can always extend it to some projective map $v : V \to Y$. Moreover, using the resolution of singularities, we can make V smooth. Also, any line bundle can be extended from $\widetilde{V}$ to V. This shows that $\psi_{Y/X}$ is surjective.

If two maps $v : V \to Y \times_X U$ and $v' : V' \to Y \times_X U'$ are isomorphic over the generic fiber $\widetilde{Y}$, then this isomorphism is actually defined over some open neighborhood $U'' \subset U \cap U'$ of the $\mathrm{Spec}(E)$. And if two line bundles L and

L' on V are isomorphic when restricted to $\widetilde{V}$, then these differ in the Picard group by a class of a line bundle coming from U (since $\widetilde{V}$ is obtained from V by removing preimages of some divisors from U). So, these become isomorphic over the preimage of some neighborhood of Spec(E). Hence, the map $\psi_{Y/X}$ is injective, and so an isomorphism.

By the same arguments, the modified relations $\tilde{\mathcal{R}}_*^{Dim}(\widetilde{Y})$ are covered by $\mathcal{R}_*^{Dim}(Y)$.

If we have some projective map $\widetilde{v} : \widetilde{V} \to \widetilde{Y}$ with the smooth divisor $\widetilde{Z} \subset \widetilde{V}$, as above, using the results of Hironaka [9], we can extend $\widetilde{v}$ to a projective map $v : V \to Y$ from a smooth V, and can extend $\widetilde{Z}$ to a smooth divisor Z on V. Also, we can extend any line bundle from $\widetilde{V}$ to V. Hence, the map $\rho_{Y/X}$ is surjective on $\mathcal{R}_*^{Sect}$.

If we have some multiple point relation $\widetilde{w} : \widetilde{W} \to \widetilde{Y} \times \mathbb{P}^1$, then we can extend $\widetilde{w}$ to a projective map $w : W \to Y \times \mathbb{P}^1$. Using the results of Hironaka [9] we can resolve the singularities of W without changing the generic fiber of $\widetilde{w}$ (since the singularities are outside this fiber) and then further resolve the pre-images of $Y \times 0$ and $Y \times 1$ to make divisors with strict normal crossings of these. Thus, the relations $\mathcal{R}_*^{MPR}(\widetilde{Y})$ are covered by $\rho_{Y/X}$. □

2.12 Free theories

Any formal group law $\mathcal{F}$ over a ring R (possibly graded) is induced from the universal one $(\mathbb{L}_*, \mathcal{F}_U)$ via the unique homomorphism of rings $\mathbb{L}_* \longrightarrow R$ and one can assign to it a theory

$$X \longmapsto \underline{\mathrm{h}}_{\mathcal{F}}(X) := R \otimes_{\mathbb{L}} \Omega_*(X).$$

Such a theory is called *free* (see [10, Rem 2.4.14]) and it inherits from algebraic cobordism all basic properties of an oriented cohomology theory. Examples of free theories are cobordism itself with arbitrary coefficients, Chow groups with rational equivalence and arbitrary coefficients, K_0, and Morava K-theories. The latter are defined and investigated for instance in Sechin [14, 15].

Since all our structural maps in Ω_* are $\mathbb{L}$-linear, it follows that a free theory $\underline{\mathrm{h}}_{\mathcal{F}}$ is an *extended oriented cohomology theory.* And since colimits commute with tensor products we obtain:

2.13 Corollary

Free theories are coherent.

2.14 Base Change

Let $\underline{\mathrm{h}}^*$ be an extended oriented cohomology theory over the field k.

Given $L \supseteq F$ in $\mathfrak{Fields}_k$ there exists a smooth integral F-scheme U, such that $L = F(U)$. Let $\iota : \operatorname{Spec} L = \operatorname{Spec} F(U) \hookrightarrow U$ be the embedding of the generic point.

Let now $X \in \mathbf{S}m_F$. We have a smooth morphism $U \times_F X \to U$, and we will denote the corresponding map $\theta_{U \times X/U}$ by

$$(\iota \times \mathrm{id}_X)^* : \underline{\mathrm{h}}_i(U \times_F X) \longrightarrow \underline{\mathrm{h}}_{i-\dim U}(X_{F(U)}) = \underline{\mathrm{h}}_{i-\dim U}(X_L) .$$

Composing it with the map π^*, where $\pi : U \times_F X \to X$ is the projection we obtain the *base change homomorphism*:

$$\underline{\mathrm{h}}_*(X) \longrightarrow \underline{\mathrm{h}}_*(X_L) , \; \alpha \longmapsto \alpha_L ,$$

which we denote by $\mathrm{res}_{L/F}$. This homomorphism does not depend on the choice of the model U. In fact, if U' is another model then U' is birational to U, and so by (EC1) we are reduced to show that if $j : U' \hookrightarrow U$ is an open subscheme then $\theta_{U \times X/U} = \theta_{U' \times X/U'} \circ (j \times \mathrm{id}_X)^*$. This can be seen as follows: Applying (EC3) to the sequence of morphisms

$$U' \times_F X \xrightarrow{p} U' \xrightarrow{j} U ,$$

where p is the projection, and taking (EC0) into account we obtain $\theta_{U' \times X/U'} = \theta_{U' \times X/U}$. From this equality we get $\theta_{U \times X/U} = \theta_{U' \times X/U'} \circ (j \times \mathrm{id}_X)^*$ by (EC1). From the fact that $\theta_{X \times U/U}$ and π^* are ring homomorphisms and from (EC1) and (EC2) we get:

2.15 Lemma

Let $\underline{\mathrm{h}}_*$ be a coherent extended oriented cohomology theory over a field k, $F \in \mathfrak{Fields}_k$, X and Y smooth F-schemes, and $L = F(U) \supseteq F$ a field extension, where U is a smooth and integral F-scheme. Denote by ι the inclusion of the generic point of U into U. Let further $f, g : Y \longrightarrow X$ be a morphism, respectively, projective morphism in $\mathbf{S}m_F$.

Then:

(i) $(\iota \times \mathrm{id}_X)^*(\alpha \cdot \beta) = (\iota \times \mathrm{id}_X)^*(\alpha) \cdot (\iota \times \mathrm{id}_X)^*(\beta)$ *for all* $\alpha, \beta \in \underline{\mathrm{h}}_*(U \times X)$.

(ii) $f_L^* \circ (\iota \times \mathrm{id}_X)^* = (\iota \times \mathrm{id}_Y)^* \circ (\mathrm{id}_U \times f)^*$.

(iii) $g_{L*} \circ (\iota \times \mathrm{id}_Y)^* = (\iota \times \mathrm{id}_X)^* \circ (\mathrm{id}_U \times g)_*$.

(iv) $(\alpha \cdot \beta)_L = \alpha_L \cdot \beta_L$ *for all* $\alpha, \beta \in \underline{\mathrm{h}}_*(X)$.

3 $\underline{\mathrm{h}}$-motives

3.1 $\underline{\mathrm{h}}$-correspondences of degree 0.

The following definitions and assertions are an adaption of Manin's [12] for Chow groups, see also Fulton [6, Chap. 16] or Elman, Karpenko and Merkurjev [5, Chap. XII]. We recall them here for the sake of completeness.

Let $\underline{\mathrm{h}}_*$ be an oriented Borel-Moore homology theory over a field k and X, Y smooth projective k-schemes. Denote $X_1, \ldots, X_l$ the connected components of X. An $\underline{\mathrm{h}}$-*correspondence* between X and Y is an element of $\underline{\mathrm{h}}_*(X \times_k Y)$, and an $\underline{\mathrm{h}}$-*correspondence of degree* 0 between X and Y is an element

$$\alpha \in \bigoplus_{i=1}^{l} \underline{\mathrm{h}}_{\dim X_i}(X_i \times_k Y) \subseteq \underline{\mathrm{h}}_*(X \times_k Y) .$$

We indicate the latter situation by $\alpha : X \rightsquigarrow Y$.

The *composition* of two correspondences $\alpha : X \rightsquigarrow Y$ and $\beta : Y \rightsquigarrow Z$ is defined for connected smooth and projective schemes X and Y as follows:

$$\beta \circ \alpha := p_{XZ\,*}\big(p_{XY}^*(\alpha) \cdot p_{YZ}^*(\beta)\big) : X \rightsquigarrow Z ,$$

where $\cdot$ denotes the product in $\underline{\mathrm{h}}_*(X \times_k Y \times_k Z)$ and p_{XY}, p_{YZ}, and p_{XZ} denote the indicated projections from $X \times_k Y \times_k Z$ to $X \times_k Y$, $Y \times_k Z$, and $X \times_k Z$, respectively. This composition extends then linearly to non connected schemes. It is associative with the image of the $1 \in \underline{\mathrm{h}}_*(X)$ under the push-forward along the diagonal embedding $X \longrightarrow X \times_k X$ acting as identity.

We denote the set of $\underline{\mathrm{h}}$-correspondences $X \rightsquigarrow Y$ of degree 0 by $\mathrm{Hom}_k(X, Y)_{\underline{\mathrm{h}}}$, respectively by $\mathrm{End}_k(X)_{\underline{\mathrm{h}}}$ if $X = Y$.

Given an $\underline{\mathrm{h}}$-correspondence $\alpha : X \rightsquigarrow X$ of degree 0, where X is a smooth and projective k-scheme, it acts on $\underline{\mathrm{h}}_i(X)$ by

$$\alpha_*(\gamma) := \alpha \circ \gamma .$$

3.2 $\underline{\mathrm{h}}$-motives

These are constructed completely analogous as the category of (effective) Chow motives as outlined in Manin [12].

One observers first, see *loc.cit.* , that the category whose objects are the smooth projective k-schemes and whose groups of morphisms are given by $\mathrm{Hom}_k(X, Y)_{\underline{\mathrm{h}}}$ is an additive category, called the category of $\underline{\mathrm{h}}$-correspondences of degree 0. The idempotent completion of this category, see *e.g.* [5, Sect. 64], is the *category of* $\underline{\mathrm{h}}$*-motives over* k, which we will denote by $\mathfrak{Mot}_{\underline{\mathrm{h}}}(k)$.

If $\underline{\mathrm{h}}_* = \mathrm{CH}_*$ this is the well known category of effective Chow motives $\mathfrak{Chow}(k)$.

In this category the projective line decomposes into two summands, one isomorphic to the motive of the point $\operatorname{Spec} k$, which is denoted by $\underline{\mathbb{Z}_{\mathrm{h}}}$, and the other summand is called the *Tate-* or *Lefschetz motive* and denoted by $\underline{\mathbb{Z}_{\mathrm{h}}}\{1\}$.

The cartesian product induces a 'tensor' product on the motives. In particular we have the (positive) *Tate twists*

$$\underline{\mathbb{Z}_{\mathrm{h}}}\{r\} := \underline{\mathbb{Z}_{\mathrm{h}}}\{1\}^{\otimes r} \qquad \text{and} \qquad X\{r\} := \underline{\mathbb{Z}_{\mathrm{h}}}\{r\} \otimes X$$

for a smooth and projective k-scheme X and $r \geq 0$ an integer.

3.3 The Rost nilpotence principle

Let now $\underline{\mathrm{h}}_*$ be an extended oriented cohomology theory. By some abuse of notation we suppress the lower supscript in $\underline{\mathrm{h}}_{*\,E}$ and write $\underline{\mathrm{h}}_*$ instead.

We say that a smooth projective variety X over the field k satisfies the *Rost nilpotence principle* for the theory $\underline{\mathrm{h}}_*$ if for all field extensions $F \supseteq k$ the kernel of the base change map

$$\mathrm{res}_{F/k} \,:\, \mathrm{End}_k(X)_{\underline{\mathrm{h}}} \longrightarrow \mathrm{End}_F(F \times_k X)_{\underline{\mathrm{h}}}\,,\; \alpha \longmapsto \alpha_F$$

consists of nilpotent correspondences, *i.e.* correspondences α, such that $\alpha^{\circ N} = 0$ for some natural number N.

In the article [18] of Yagita and the second named author it is shown that a decomposition in the category of Chow motives implies a decomposition in the category of Ω-motives and so also in the category of $\underline{\mathrm{h}}$-motives for all oriented cohomology theories $\underline{\mathrm{h}}_*$ by the universality of algebraic cobordism, see [10, Chap. 7]. This is a consequence of the fact that the kernel of the canonical homomorphism $\Omega_*(X) \longrightarrow \mathrm{CH}_*(X)$ consists of nilpotent elements.

This result can be applied to projective homogeneous varieties over semisimple linear algebraic groups. Recall here that given such an algebraic group G then a G-variety X is called *projective homogeneous* if G acts transitively on X and the stabilizer of every point of X is a parabolic subgroup of G.

Now by the main result of [4], see also Brosnan [2], an isotropic projective homogeneous variety for a semisimple linear algebraic group decomposes in the category of Chow motives into Tate twists of other projective homogeneous varieties. This implies by the remarks above the following decomposition theorem for $\underline{\mathrm{h}}_*$-motives of such varieties.

3.4 Theorem

Let $\underline{\mathrm{h}}_*$ be any oriented cohomology theory (in the sense of [10, Def. 1.1.2]), k a field, and X an isotropic projective homogeneous k-variety for a semisimple linear algebraic group G over the field k. Then there exist projective homogeneous

k-varieties $Y_1, \ldots, Y_l$, $l \geq 2$, and integers $n_i \geq 0$, $1 \leq i \leq l$, such that

$$X \simeq \bigoplus_{i=1}^{l} Y_i\{n_i\}$$

in $\mathfrak{Mot}_{\underline{\mathrm{h}}}(k)$.

3.5 Remark

The arguments of Yagita and the second author in [18] also show that if Rost nilpotence holds for a smooth projective variety X in the category of Chow motives then X satisfies this principle in the category of Ω-motives as well.

4 Proof of the Rost nilpotence principle for generically constant coherent theories and projective homogeneous varieties

Most proofs of the Rost nilpotence principle for a variety in the category of Chow motives use a lemma proven by Rost [13, Prop. 1], which we call the *Rost lemma*. (An exception is the argument of the second author for projective quadrics in [16, Lem. 3.10].) We prove here the analog of this lemma for a coherent theory $\underline{\mathrm{h}}_*$ following Brosnan's [1] arguments for $\underline{\mathrm{h}}_* = \mathrm{CH}_*$. However there are some subtle technical difficulites, which are addressed by the following lemma. We overcome them using resolutions of singularities, *i.e.* by taking advantage of our assumption that the base field k has characteristic 0.

4.1 Lemma

Let $\underline{\mathrm{h}}_*$ be a coherent theory over the field k and X, W be smooth schemes over k with X projective. Assume W is integral with generic point w. Denote by ι the inclusion of the generic point $\operatorname{Spec} k(w) \hookrightarrow W$. Let $\alpha \in \operatorname{End}_k(X)_{\underline{\mathrm{h}}}$ and $\gamma \in \underline{\mathrm{h}}_i(W \times X)$ for some $i \in \mathbb{Z}$. If

$$\alpha_{k(w)\,*}\big((\iota \times \mathrm{id}_X)^*(\gamma)\big) \;=\; 0$$

then there exists a closed subscheme $j : Z \hookrightarrow W$, which is $\neq W$, such that $\alpha_*(\gamma)$ is in the image of

$$(j \times \mathrm{id}_X)_* \,:\, \underline{\mathrm{h}}_i(Z \times X) \longrightarrow \underline{\mathrm{h}}_i(W \times X)\,.$$

Proof. We give a comprehensive proof indicating all properties of coherent theories which we use.

We show first

$$\alpha_{k(w)\,*}\big((\iota \times \mathrm{id}_X)^*(\gamma)\big) \;=\; (\iota \times \mathrm{id}_X)^*\big(\alpha_*(\gamma)\,\big)\,. \tag{3}$$

To prove this equation we use the following two commutative diagrams, where we have set $\times = \times_k$:

$$\begin{array}{ccccc} & & X \times X & & \\ & \overset{q}{\nearrow} & & \overset{p}{\nwarrow} & \\ k(w) \times X \xleftarrow{\;q_{13}\;} k(w) \times X \times X & & \xrightarrow{\;\iota \times \mathrm{id}_{X \times X}\;} & & W \times X \times X \\ \Big\downarrow{\scriptstyle q_{12}} & & & & \Big\downarrow{\scriptstyle p_{12}} \\ k(w) \times X & & \xrightarrow{\;\iota \times \mathrm{id}_X\;} & & W \times X \end{array} \tag{4}$$

and

$$\begin{array}{ccc} k(w) \times X \times X & \xrightarrow{\;\iota \times \mathrm{id}_{X \times X}\;} & W \times X \times X \\ \Big\downarrow{\scriptstyle q_{13}} & & \Big\downarrow{\scriptstyle p_{13}} \\ k(w) \times X & \xrightarrow{\;\iota \times \mathrm{id}_X\;} & W \times X \;\;, \end{array} \tag{5}$$

where p_{ij} and q_{ij} denote the projections to the respective ij-components.

We compute now:

$\alpha_{k(w)\,*}\big((\iota \times \mathrm{id}_X)^*(\gamma)\big)$

$= q_{13\,*}\big(\alpha_{k(w)} \cdot q_{12}^*((\iota \times \mathrm{id}_X)^*(\gamma))\big)$ by def.

$= q_{13\,*}\big((\iota \times \mathrm{id}_{X\times X})^*(p^*(\alpha)) \cdot (\iota \times \mathrm{id}_{X\times X})^*(p_{12}^*(\gamma))\big)$ by def., Diag. (4), and Lem. 2.15 (ii)

$= q_{13\,*}\Big((\iota \times \mathrm{id}_{X\times X})^*\big(p^*(\alpha) \cdot p_{12}^*(\gamma)\big)\Big)$ by Lem. 2.15 (i)

$= (\iota \times \mathrm{id}_X)^*\Big(p_{13\,*}\big(p^*(\alpha) \cdot p_{12}^*(\gamma)\big)\Big)$ by Diag. (5) and Lem. 2.15 (iii)

$= (\iota \times \mathrm{id}_X)^*\big(\alpha_*(\gamma)\big)$ by definition.

Equation (3) implies the lemma: since $\underline{\mathrm{h}}_*$ is coherent, $\underline{\mathrm{h}}_*(k(w) \times X)$ is isomorphic to the direct limit of all $\underline{\mathrm{h}}_*(U \times X)$, where U runs through the open subschemes of W. Hence the assumption $\alpha_{k(w)\,*}\big((\iota\times\mathrm{id}_X)^*(\gamma)\big) = 0$ and (3) imply the existence of an open $U \subseteq W$, such that $\alpha_*(\gamma)|_{U\times X} = 0$. Let $j : Z \hookrightarrow W$

be the closed complement of U. We have $Z \neq W$, and by the localization sequence

$$\underline{\mathrm{h}}_i(Z \times X) \xrightarrow{(j\times \mathrm{id}_X)_*} \underline{\mathrm{h}}_i(W \times X) \longrightarrow \underline{\mathrm{h}}_i(U \times X) \longrightarrow 0$$

there exists $\delta \in \underline{\mathrm{h}}_i(Z \times X)$, such that $\alpha_*(\gamma) = (j \times \mathrm{id}_X)_*(\delta)$. We are done. □

4.2 The Rost lemma for coherent theories

Let $\underline{\mathrm{h}}_*$ be a coherent theory over the field k, X, Y smooth and projective k-scheme, and $\alpha \in \mathrm{End}_k(X)_{\underline{\mathrm{h}}}$. If

$$\alpha_{k(y)\,*}\big(\underline{\mathrm{h}}_*(X_{k(y)})\big) \;=\; 0$$

for all $y \in Y$ then

$$(\alpha^{\circ(1+\dim Y)})_*\big(\underline{\mathrm{h}}_*(Y \times_k X)\big) \;=\; 0.$$

Proof. Let $n = \dim Y$.

We define a filtration on $\underline{\mathrm{h}}_*(Y \times_k X)$ as follows. Let $\mathfrak{F}_l \subseteq \underline{\mathrm{h}}_*(Y \times_k X)$ for all $0 \leq l \leq n$ be the subgroup generated by all images

$$(j_V \times \mathrm{id}_X)_* \,:\, \underline{\mathrm{h}}_*(V \times_k X) \longrightarrow \underline{\mathrm{h}}_*(Y \times_k X)\,,$$

where $j_V : V \longrightarrow Y$ is a closed k-subscheme of Y of dimension $\leq l$, equivalently we can take only the images with V a closed integral k-subscheme.

We have

$$\underline{\mathrm{h}}_*(Y \times_k X)) \;=\; \mathfrak{F}_n \;\supseteq\; \mathfrak{F}_{n-1} \ldots \supseteq \mathfrak{F}_1 \;\supseteq\; \mathfrak{F}_0 \;\supseteq\; \mathfrak{F}_{-1} \;:=\; \{0\}\,,$$

and so it is enough to show that $\alpha_*(\mathfrak{F}_l) \subseteq \mathfrak{F}_{l-1}$ for all $-1 \leq l \leq n := \dim Y$. We verify this by induction on l. The induction beginning $l = -1$ is clear, so let $l \geq 0$.

Let for this $a \in \mathfrak{F}_l \subseteq \underline{\mathrm{h}}_*(Y \times_k X)$ be equal to $(j_V \times \mathrm{id}_X)_*(b)$, where

$$(j_V \times \mathrm{id}_X)_* \,:\, \underline{\mathrm{h}}_*(V \times X) \longrightarrow \underline{\mathrm{h}}_*(Y \times_k X)\,,$$

and $j_V : V \hookrightarrow Y$ is a closed embedding of a subscheme of dimension $\leq l$. We have to show $\alpha_*(a) \in \mathfrak{F}_{l-1}$, and can assume for this that V is a closed integral subscheme.

Let $\pi : W \longrightarrow V$ be a projective birational k-morphism with W smooth projective and integral, which exists since k has characteristic 0 by Hironaka [9]. By Lemma 2.3 there exists then $c \in \underline{\mathrm{h}}_*(W \times_k X)$, a closed subscheme $j : V' \hookrightarrow V$, and $b' \in \underline{\mathrm{h}}_*(V' \times_k X)$, such that

$$b \;=\; (\pi \times \mathrm{id}_X)_*(c) + (j \times \mathrm{id}_X)_*(b')\,.$$

Hence

$$a \;=\; (\varepsilon\times \mathrm{id}_X)_*(c) + (j_{V'}\times \mathrm{id}_X)_*(b'),$$

where $\varepsilon = j_V\circ\pi$ and $j_{V'} = j_V\circ j$. Note now that $\dim V' < \dim V \le l$ since V is irreducible, and so we have that $\big(j_{V'}\times \mathrm{id}_X\big)_*(b')\in \mathfrak{F}_{l-1}$, which by induction implies

$$\alpha_*\big(\,(j_{V'}\times \mathrm{id}_X)_*(b')\big) \;\subseteq\; \mathfrak{F}_{l-2}\subseteq \mathfrak{F}_{l-1}\,.$$

Hence it is enough to show that $\alpha_*\big((\varepsilon\times \mathrm{id}_X)_*(c)\big)$ is in $\mathfrak{F}_{l-1}$.

It follows from the standard compatibilities of pull backs and push forwards that the diagram

$$\begin{array}{ccc} \underline{\mathrm{h}}_*(W\times_k X) & \xrightarrow{(\varepsilon\times \mathrm{id}_X)_*} & \underline{\mathrm{h}}_*(Y\times_k X) \\ \downarrow{\scriptstyle\alpha_*} & & \downarrow{\scriptstyle\alpha_*} \\ \underline{\mathrm{h}}_*(W\times_k X) & \xrightarrow[(\varepsilon\times \mathrm{id}_X)_*]{} & \underline{\mathrm{h}}_*(Y\times_k X) \end{array}$$

is commutative. Hence $\alpha_*\big((\varepsilon\times \mathrm{id}_X)_*(c)\big) = (\varepsilon\times \mathrm{id}_X)_*(\alpha_*(c))$.
Let v,w be the generic points of V and W, respectively. We have $k(v)=k(w)$ and so by our assumption

$$\alpha_{k(w)\,*}\big(\,(\iota\times \mathrm{id}_X)^*(c)\big) \;=\; 0\,,$$

where $\iota:\operatorname{Spec} k(w)\hookrightarrow W$ is the embedding of the generic point. Therefore by Lemma 4.1 there exists a closed embedding $q: Z\hookrightarrow W$ with $Z\neq W$ and so $\dim Z<\dim W\le l$, such that $\alpha_*(c)$ is in the image of

$$(q\times \mathrm{id}_X)_* \;:\; \underline{\mathrm{h}}_*(Z\times_k X)\;\longrightarrow\; \underline{\mathrm{h}}_*(W\times_k X)\,.$$

Then $\alpha_*\big((\varepsilon\times \mathrm{id}_X)_*(c)\big)$ is in the image of $(p\times \mathrm{id}_X)_*$, where $p=\varepsilon\circ q: Z\to Y$. And hence it is in the image of $(j_{\overline{Z}}\times \mathrm{id}_X)_*$, where $\overline{Z}$ is the reduced image of p and $j_{\overline{Z}}:\overline{Z}\to Y$ is the respective closed embedding. Since $\dim\overline{Z}<l$, this element belongs to $\mathfrak{F}_{l-1}$. □

4.3 Remark

For non-coherent theories, the Rost Lemma is in general wrong. Here is an example. Let $\underline{\mathrm{h}}_*$ be CH^*_{alg}, and $X=Y$ be an elliptic curve C. Let $\alpha = [\Delta]-[C\times p]-[p\times C]$, where Δ is the diagonal and p is some k-rational point on C. Then, for any L/k, $\mathrm{CH}^*_{alg}(X_L)=\mathbb{Z}\cdot 1\oplus\mathbb{Z}\cdot[p]$, and so $\alpha_{L*}(\underline{\mathrm{h}}_*(X_L))=0$. At the same time, α is a *projector* $\alpha^{\circ 2}=\alpha$. Hence, for any natural N, $\alpha_*^{\circ N}(\underline{\mathrm{h}}_*(Y\times X))\neq 0$, since $\alpha\neq 0$.

Using the decomposition from Theorem 3.4 the Rost lemma for $\underline{\mathrm{h}}_*$ implies the Rost nilpotence principle for projective homogeneous varieties and generically constant coherent theories. The proof is word by word the same as [4, Proof of Thm. 8.2] (only replacing CH_* by $\underline{\mathrm{h}}_*$). However to indicate where we need that the product in a coherent theory is compatible with base change we give the details.

4.4 Corollary

Let X be a projective homogeneous variety for a semisimple linear algebraic group. Then X satisfies the Rost nilpotence principle for every generically constant coherent theory $\underline{\mathrm{h}}_*$.

Proof. If X is split then it is a cellular variety and so by [18, Cor. 2.9] a direct sum of twists of Tate motives. Since $\underline{\mathrm{h}}_*$ is *generically constant*, $\underline{\mathrm{h}}_*(k) \longrightarrow \underline{\mathrm{h}}_*(L)$ is an isomorphism for all field extensions $L \supseteq k$, and so Rost nilpotence holds trivially in this case.

If X is not split then it still has a decomposition as in Theorem 3.4:

$$X \simeq \bigoplus_{i=1}^{l} Y_i\{n_i\}$$

for projective homogeneous varieties Y_i and integers $n_i \geq 0$ (it certainly has such a decomposition with $l = 1$ and $Y_1 = X$, $n_1 = 0$). We prove now by down going induction on the number of components l that there exists an integer $m \geq 0$ which only depends on the dimension of the projective homogenous variety and on the number l, such that $\alpha^{\circ m} = 0$ if α is in the kernel of

$$\mathrm{End}_k(X)_{\underline{\mathrm{h}}} \longrightarrow \mathrm{End}_L(X_L)_{\underline{\mathrm{h}}}$$

for some field extension $L \supseteq k$. As the number of possible components is bounded by the number of Tate motives appearing in a decomposition over the algebraic closure of k, we have the base of induction.

Let $\alpha \in \mathrm{End}_k(X)_{\underline{\mathrm{h}}}$, such that $\alpha_L = 0$ for some field extension L of k. If y is a point of Y_i then Y_i is isotropic over $k(y)$ and so by Theorem 3.4 the motive $Y_{i\,k(y)}\{n_i\}$ is a direct sum of at least two Tate twisted motives of projective homogenous varieties over $k(y)$. Hence over $k(y)$ the motive of X decomposes in more than l summands and so by induction we have $\alpha_{k(y)}^{\circ m} = 0$ for an integer $m \geq 1$ which depends only on the number of components of the decomposition of X over $k(y)$ and the dimension of X. As the number of possible components is bounded we can choose m, such that $\alpha_{k(y)}^{\circ m} = 0$ for all $y \in Y_i$ and all $1 \leq i \leq l$.

By Lemma 2.15 (iv) we have

$$0 = \alpha_{k(y)}^{\circ m} = (\alpha^{\circ m})_{k(y)} ,$$

and so $(\alpha^{\circ m})_{k(y)\,*}\big(\underline{\mathrm{h}}_*(X_{k(y)})\big) = 0$ for all $y \in Y_i$ and all $1 \leq i \leq l$. By the Rost lemma this implies

$$(\alpha^{\circ m \cdot (1+\dim Y_i)})_*\big(\mathrm{Hom}_k(Y_i\{n_i\}, X)_{\underline{\mathrm{h}}} \big) = 0$$

for all $1 \leq i \leq l$. Let t be the maximum of the dimensions of the Y_i. Then since the motive of X is equal the direct sum of the motives $Y_i\{n_i\}$'s in the category of $\underline{\mathrm{h}}$-motives we get $\alpha^{\circ m \cdot (1+t)} = 0$. We are done. □

From [10, Cor. 4.4.3] we know that *free* theories are generically constant and from Corollary 2.13 these are also coherent. Hence, we obtain:

4.5 Corollary

Let X be a projective homogeneous variety for a semisimple linear algebraic group over a field. Then X satisfies the Rost nilpotence principle for every free theory $\underline{\mathrm{h}}_*$.

Note that not all *generically constant coherent* theories are *free* (= of *rational type* [17]). Here is a large supply of examples.

4.6 Example

Let A^* be a generically constant coherent theory over k, and $\{(Q_\lambda, a_\lambda)\}_{\lambda\in\Lambda}$ a collection of smooth projective varieties Q_λ defined over k (!) with some classes $a_\lambda \in A^*(Q_\lambda)$. In other words, we have a collection of A^*-correspondences $\rho_\lambda : Q_\lambda \rightsquigarrow \mathrm{Spec}(k)$. Suppose that the map $\rho_{\lambda\,*} : A_*(Q_{\lambda,F}) \to A_*(F)$ sending v to $\pi_\lambda^*(v \cdot a_\lambda)$ is zero, for any field extension F/k, where π_λ is the projection $Q_{\lambda,F} \longrightarrow \mathrm{Spec}\, F$. For any F/k, and any smooth variety X/F we have natural projections:

$$Q_{\lambda,F} \xleftarrow{\pi_1} Q_{\lambda,F} \times_F X \xrightarrow{\pi_2} X$$

Now, in $A^*(X)$ we can mod-out the images of the maps from $\rho_{\lambda\,*} : A^*(Q_{\lambda,F} \times_F X) \to A^*(X)$ sending u to $(\pi_2)_*(u \cdot \pi_1^*(a_\lambda))$. It is not difficult to check that the resulting theory $\widetilde{A}^*$ is coherent and generically constant. As long as not all a_λ's are zero, it will not be free, since it has the same coefficient ring as A^*, but the natural projection $A^* \to \widetilde{A}^*$ has a non-trivial kernel, and so, there is a non-trivial kernel of the projection $\widetilde{A}^*(k) \otimes_{\mathbb{L}} \Omega^* \twoheadrightarrow \widetilde{A}^*$.

Here is the simplest such example. Take $A^* = \mathrm{CH}^*$ and a single elliptic curve C over k with $a = [p_1] - [p_0]$, where p_0, p_1 are two distinct k-rational points on C. Then the resulting theory $\widetilde{A}^*$ will be a constant coherent theory in-between CH^* and CH^*_{alg}.

4.7 Failure of the Rost nilpotence principle for non-coherent theories

For non-coherent theories it does not make much sense to speak about Rost nilpotence principle. If the theory is non-constant, counterexamples are obvious: just take $A_k^* = \mathrm{CH}^*$, and $A_F^* = \mathrm{CH}^*/2$, for any non-trivial extension F/k. But even for constant ones we have:

4.8 Example

Let C be some elliptic curve over k, and p a k-rational point on it. Take $B_k^* = \mathrm{CH}^*/2$. Consider the B^*-correspondence $\rho : C \times C \rightsquigarrow \mathrm{Spec}(k)$ given by the class $\alpha = [\Delta] - [C \times p] - [p \times C]$. Notice, that it has the property, that, for all F/k, the map $\rho_* : \mathrm{CH}^*((C \times C)_F)/2 \to \mathrm{CH}^*(F)/2$ is zero.

For a non-trivial extension F/k, and smooth X/F let us take $B^*_F(X)$ to be the quotient of $\mathrm{CH}^*(X)/2$ modulo the image of $\rho_* : \mathrm{CH}^*(C \times C \times X)/2 \to \mathrm{CH}^*(X)/2$. Then B^* will be an extended constant oriented cohomology theory, and we have a B^*-correspondence $\alpha : C \rightsquigarrow C$ which vanishes over all non-trivial field extensions by construction. At the same time, α itself is a projector. This projector is clearly non-trivial, since the motive of an elliptic curve (even with $\mathbb{Z}/2$-coefficients) is not a direct sum of two Tate-motives. Hence, the Rost nilpotence principle fails for the theory B^*.

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