Big Cohen-Macaulay Modules, Morphisms of Perfect Complexes, and Intersection Theorems in Local Algebra

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Abstract. There is a well known link from the first topic in the title to the third one. In this paper we thread that link through the second topic. The central result is a criterion for the tensor nilpotence of morphisms of perfect complexes over commutative noetherian rings, in terms of a numerical invariant of the complexes known as their level. Applications to local rings include a strengthening of the Improved New Intersection Theorem, short direct proofs of several results equivalent to it, and lower bounds on the ranks of the modules in every finite free complex that admits a structure of differential graded module over the Koszul complex on some system of parameters.

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1. Introduction

A big Cohen-Macaulay module over a commutative noetherian local ring R is a (not necessarily finitely generated) R-module C such that some system of parameters of R forms a C-regular sequence. In [16] Hochster showed that the existence of such modules implies several fundamental homological properties of finitely generated R-modules. In [17], published in [18], he proved that big Cohen-Macaulay modules exist for algebras over fields, and conjectured their existence in the case of mixed characteristic. This was recently proved by Y. André in [2]; as a major consequence many "Homological Conjectures" in local algebra are now theorems.

A perfect R-complex is a bounded complex of finite projective R-modules. Its level with respect to R, introduced in [6] and defined in 2.3, measures the minimal number of mapping cones needed to assemble a quasi-isomorphic complex from bounded complexes of finite projective modules with differentials equal to zero.

The main result of this paper, which appears as Theorem 3.3, is the following

Tensor Nilpotence Theorem. *Let $f\colon G \to F$ be a morphism of perfect complexes over a commutative noetherian ring R.*

If f factors through a complex whose homology is I-torsion for some ideal I of R with $\operatorname{height} I \geq \operatorname{level}^R \operatorname{Hom}_R(G,F)$*, then the induced morphism*

$$\otimes_R^n f\colon\ \otimes_R^n G \to \otimes_R^n F$$

is homotopic to zero for some non-negative integer n.

Big Cohen-Macaulay modules play an essential, if discreet, role in the proof as a tool for constructing special morphisms in the derived category of R; see Proposition 3.7.

In applications to commutative algebra it is convenient to use another property of morphisms of perfect complexes: f is *fiberwise zero* if $\mathrm{H}(k(\mathfrak{p}) \otimes_R f) = 0$ holds for every $\mathfrak{p}$ in $\operatorname{Spec} R$. Hopkins [21] and Neeman [25] have shown that this is equivalent to tensor nilpotence; this is a key tool for the classification of the thick subcategories of perfect R-complexes.

It is easy to see that the level of a complex does not exceed its span; see 2.1. Due to these remarks, the Tensor Nilpotence Theorem is equivalent to the

Morphic Intersection Theorem. *If f is not fiberwise zero and factors through a complex with I-torsion homology for some ideal I of R, then there are inequalities:*

$$\operatorname{span} F + \operatorname{span} G - 1 \geq \operatorname{level}^R \operatorname{Hom}_R(G,F) \geq \operatorname{height} I + 1\,.$$

In Section 4 we use this result to prove directly, and sometimes to generalize and sharpen, several basic theorems in commutative algebra. These include the Improved New Intersection Theorem, the Monomial Theorem and several versions of the Canonical Element Theorem. All of them are equivalent, but we do not know if they imply the Morphic Intersection Theorem; a potentially

significant obstruction to that is that the difference $\operatorname{span} F - \operatorname{level}^R F$ can be arbitrarily big.
Another application, in Section 5, yields lower bounds for ranks of certain finite free complexes over local rings, related to a conjecture of Buchsbaum and Eisenbud, and Horrocks.
In [7] we prove a version of the Morphic Intersection Theorem for certain kinds of tensor triangulated categories. This has implications for morphisms of perfect complexes of sheaves and, more generally, of perfect differential sheaves over schemes.

2. Perfect complexes

Throughout this paper R will be a commutative noetherian ring.
This section is a recap on the various notions and construction, mainly concerning perfect complexes, used in this work. Pertinent references include [6, 26].

2.1. Complexes. In this work, an R-complex (a shorthand for 'a complex of R-modules') is a sequence of homomorphisms of R-modules

$$X := \quad \cdots \longrightarrow X_n \xrightarrow{\partial_n^X} X_{n-1} \xrightarrow{\partial_{n-1}^X} X_{n-2} \longrightarrow \cdots$$

such that $\partial^X \partial^X = 0$. We write $X^\natural$ for the graded R-module underlying X. The ith suspension of X is the R-complex $\Sigma^i X$ with $(\Sigma^i X)_n = X_{n-i}$ and $\partial_n^{\Sigma^i X} = (-1)^i \partial_{n-i}^X$ for each n. The *span* of X is the number

$$\operatorname{span} X := \sup\{i \mid X_i \neq 0\} - \inf\{i \mid X_i \neq 0\} + 1$$

Thus $\operatorname{span} X = -\infty$ if and only if $X = 0$, and $\operatorname{span} X = \infty$ if and only if $X_i \neq 0$ for infinitely many $i \geq 0$. The span of X is finite if and only if $\operatorname{span} X$ is an natural number. When $\operatorname{span} X$ is finite we say that X is *bounded*.
Complexes of R-modules are objects of two distinct categories.
In the category of complexes $\mathsf{C}(R)$ a morphism $f\colon Y \to X$ of R-complexes is a family $(f_i\colon Y_i \to X_i)_{i\in\mathbb{Z}}$ of R-linear maps satisfying $\partial_i^X f_i = f_{i-1}\partial_i^Y$. It is a *quasi-isomorphism* if $\mathrm{H}(f)$, the induced map in homology, is bijective. Complexes that can be linked by a string of quasi-isomorphisms are said to be *quasi-isomorphic*.
The derived category $\mathsf{D}(R)$ is obtained from $\mathsf{C}(R)$ by inverting all quasi-isomorphisms. For constructions of the localization functor $\mathsf{C}(R) \to \mathsf{D}(R)$ and of the derived functors $?\otimes^{\mathrm{L}}_R ?$ and $\operatorname{RHom}_R(?, ?)$, see e.g. [13, 31, 24]. When P is a complex of projectives with $P_i = 0$ for $i \ll 0$, the functors $P \otimes^{\mathrm{L}}_R ?$ and $\operatorname{RHom}_R(P, ?)$ are represented by $P \otimes_R ?$ and $\operatorname{Hom}_R(P, ?)$, respectively. In particular, the localization functor induces for each n a natural isomorphism of abelian groups

$$\mathrm{H}_{-n}(\operatorname{RHom}_R(P, X)) \cong \operatorname{Hom}_{\mathsf{D}(R)}(P, \Sigma^n X)\,. \tag{2.1.1}$$

2.2. Perfect complexes. In $\mathsf{C}(R)$, a *perfect* R-complex is a bounded complex of finitely generated projective R-modules. When P is perfect, the R-complex $P^* := \operatorname{Hom}_R(P, R)$ is perfect and the natural biduality map

$$P \longrightarrow P^{**} = \operatorname{Hom}_R(\operatorname{Hom}_R(P, R), R)$$

is an isomorphism. Moreover for any R-complex X the natural map

$$P^* \otimes_R X \longrightarrow \operatorname{Hom}_R(P, X)$$

is an isomorphism. In the sequel these properties are used without comment.

2.3. Levels. A *length l semiprojective filtration* of an R-complex P is a sequence of R-subcomplexes of finitely generated projective modules

$$0 = P(0) \subseteq P(1) \subseteq \cdots \subseteq P(l) = P$$

such that $P(i-1)^\natural$ is a direct summand of $P(i)^\natural$ and the differential of $P(i)/P(i-1)$ is equal to zero, for $i = 1, \dots, l$. For every R-complex F set

$$\operatorname{level}^R F = \inf \left\{ l \in \mathbb{N} \;\middle|\; \begin{array}{c} F \text{ is a retract in } \mathsf{D}(R) \text{ of some } R\text{-complex } P \\ \text{that has a semiprojective filtration of length } l \end{array} \right\}.$$

By [6, 2.4], this number is equal to the *level* of F with respect to R, as defined in [6, 2.1]. In particular, $\operatorname{level}^R F$ is finite if and only if F is quasi-isomorphic to some perfect complex. When F is quasi-isomorphic to a perfect complex P, one has

$$\operatorname{level}^R F \leq \operatorname{span} P\,. \tag{2.3.1}$$

Indeed, if $P := 0 \to P_b \to \cdots \to P_a \to 0$, then consider the filtration by subcomplexes $P(n) := P_{<n+a}$. The inequality can be strict; see 2.7 below.

When R is regular, any R-complex F with $\mathrm{H}(F)$ finitely generated satisfies

$$\operatorname{level}^R F \leq \dim R + 1\,. \tag{2.3.2}$$

For R-complexes X and Y one has

$$\begin{aligned} &\operatorname{level}^R(\Sigma^i X) = \operatorname{level}^R X \quad \text{for every integer } i, \text{ and} \\ &\operatorname{level}^R(X \oplus Y) = \max\{\operatorname{level}^R X, \operatorname{level}^R Y\}. \end{aligned} \tag{2.3.3}$$

These equalities follow easily from the definitions.

Lemma 2.4. *The following statements hold for every perfect R-complex P.*

(1) $\operatorname{level}^R(P^*) = \operatorname{level}^R P$.

(2) *For each perfect R-complex Q there are inequalities*

$$\operatorname{level}^R(P \otimes_R Q) \leq \operatorname{level}^R P + \operatorname{level}^R Q - 1\,.$$

$$\operatorname{level}^R \operatorname{Hom}_R(P, Q) \leq \operatorname{level}^R P + \operatorname{level}^R Q - 1\,.$$

Proof. (1) If P is a retract of P', then $\operatorname{level}^R P \leq \operatorname{level}^R P'$ and P^* is a retract of $(P')^*$. Thus, we can assume P itself has a finite semiprojective filtration $\{P(n)\}_{n=0}^{l}$. The inclusions $P(l-i) \subseteq P(l-i+1) \subseteq P$ define subcomplexes

$$P^*(i) := \operatorname{Ker}(P^* \longrightarrow P(l-i)^*) \subseteq \operatorname{Ker}(P^* \longrightarrow P(l-i+1)) =: P^*(i+1)$$

of finitely generated projective modules. They form a length l semiprojective filtration of P^*, as $P^*(i-1)^\natural$ is a direct summand of $P^*(i)^\natural$ and one has

$$\frac{P^*(n)}{P^*(n-1)} \cong \left(\frac{P(l-n+1)}{P(l-n)} \right)^*.$$

This gives $\operatorname{level}^R P^* \leq \operatorname{level}^R P$. The reverse inequality follows from $P \cong P^{**}$.
(2) Assume first that P has a semiprojective filtration $\{P(n)\}_{n=0}^{l}$ and Q has a semiprojective filtration $\{Q(n)\}_{n=0}^{m}$. For all h, i, we identity $P(h) \otimes_R Q(i)$ with a subcomplex of $P \otimes_R Q$.
For each integer $n \geq 0$ form the subcomplex

$$C(n) := \sum_{j \geqslant 0} P(j+1) \otimes_R Q(n-j)$$

of $P \otimes_R Q$. A direct computation yields an isomorphism of R-complexes

$$\frac{C(n)}{C(n-1)} \cong \sum_{j \geqslant 0} \frac{P(j+1)}{P(j)} \otimes_R \frac{Q(n-j)}{Q(n-j-1)}\,.$$

Thus $\{C(n)\}_{n=0}^{l+m-1}$ is a semiprojective filtration of $P \otimes_R Q$.
The second inequality in (2) follows from the first one, given (1) and the isomorphism $\operatorname{Hom}_R(P, Q) \cong P^* \otimes_R Q$. Next we verify the first inequality. There is nothing to prove unless the levels of P and Q are finite. Thus we may assume that P is a retract of a complex P' with a semiprojective filtration of length $l = \operatorname{level}^R P$ and Q is a retract of a complex Q' with a semiprojective filtration of length $m = \operatorname{level}^R Q$. Then $P \otimes_R Q$ is a retract of $P' \otimes_R Q'$, and—by what we have just seen—this complex has a semiprojective filtration of length $l+m-1$, as desired. $\square$

2.5. GHOST MAPS. A morphism $g\colon X \to Y$ in $\mathsf{D}(R)$ is a *ghost* if $\mathrm{H}(g) = 0$; see [11, §8]. Clearly, composing morphisms with ghosts yield ghosts.
The next result is a version of the "Ghost Lemma"; cf. [11, Theorem 8.3], [29, Lemma 4.11], and [5, Proposition 2.9].

LEMMA 2.6. *Let F be an R-complex and c an integer with $c \geq \operatorname{level}^R F$. Every composition $g\colon X \to Y$ of c ghosts induces morphisms*

$$\begin{gathered} F \otimes^{\mathrm{L}}_R g\colon F \otimes^{\mathrm{L}}_R X \longrightarrow F \otimes^{\mathrm{L}}_R Y \quad \textit{and} \\ \operatorname{RHom}_R(F, g)\colon \operatorname{RHom}_R(F, X) \longrightarrow \operatorname{RHom}_R(F, Y) \end{gathered}$$

Proof. For every R-complex W there is a canonical isomorphism

$$\operatorname{RHom}_R(F, R) \otimes^{\mathrm{L}}_R W \xrightarrow{\simeq} \operatorname{RHom}_R(F, W)\,,$$

so it suffices to prove the first assertion. For that, we may assume that F has a semiprojective filtration $\{F(n)\}_{n=0}^{l}$, where $l = \operatorname{level}^R F$. By hypothesis, $g = h \circ f$ where $f\colon X \to W$ is a $(c-1)$fold composition of ghosts and $h\colon W \to Y$ is a ghost. Tensoring these maps with the exact sequence of R-complexes

$$0 \longrightarrow F(1) \xrightarrow{\iota} F \xrightarrow{\pi} G \longrightarrow 0$$

where $G := F/F(1)$, yields a commutative diagram of graded R-modules

$$\begin{tikzcd}
\mathrm{H}(F(1)\otimes_R X) \arrow[r] \arrow[d, "\mathrm{H}(F(1)\otimes f)"'] & \mathrm{H}(F\otimes_R X) \arrow[r] \arrow[d, "\mathrm{H}(F\otimes f)"] & \mathrm{H}(G\otimes_R X) \arrow[d, "\mathrm{H}(G\otimes f)"] \\
\mathrm{H}(F(1)\otimes_R W) \arrow[r, "\mathrm{H}(\iota\otimes W)"] \arrow[d, "\mathrm{H}(F(1)\otimes h)"'] & \mathrm{H}(F\otimes_R W) \arrow[r, "\mathrm{H}(\pi\otimes W)"] \arrow[d, "\mathrm{H}(F\otimes h)"] & \mathrm{H}(G\otimes_R W) \arrow[d] \\
\mathrm{H}(F(1)\otimes_R Y) \arrow[r, "\mathrm{H}(\iota\otimes Y)"'] & \mathrm{H}(F\otimes_R Y) \arrow[r] & \mathrm{H}(G\otimes_R Y)
\end{tikzcd}$$

where the rows are exact. Since $\operatorname{level}^R G \leq l-1 \leq c-1$, the induction hypothesis implies $G\otimes f$ is a ghost; that is to say, $\mathrm{H}(G\otimes f)=0$. The commutativity of the diagram above and the exactness of the middle row implies that

$$\operatorname{Im}\mathrm{H}(F\otimes f) \subseteq \operatorname{Im}\mathrm{H}(\iota\otimes W)$$

This entails the inclusion below.

$$\begin{aligned}
\operatorname{Im}\mathrm{H}(F\otimes g) &= \mathrm{H}(F\otimes h)(\operatorname{Im}\mathrm{H}(F\otimes f)) \\
&\subseteq \mathrm{H}(F\otimes h)(\operatorname{Im}\mathrm{H}(\iota\otimes W)) \\
&\subseteq \operatorname{Im}\mathrm{H}(\iota\otimes Y)\,\mathrm{H}(F(1)\otimes h)) \\
&= 0
\end{aligned}$$

The second inclusion comes from the commutativity of the diagram. The last equality holds because $F(1)$ is graded-projective and hence $\mathrm{H}(h)=0$ implies $\mathrm{H}(F(1)\otimes h)=0$. $\square$

2.7. Koszul complexes. Let $\boldsymbol{x} := x_1,\dots,x_n$ be elements in R.
We write $K(\boldsymbol{x})$ for the Koszul complex on $\boldsymbol{x}$. Thus $K(\boldsymbol{x})^\natural$ is the exterior algebra on a free R-module $K(\boldsymbol{x})_1$ with basis $\{\widetilde{x}_1,\dots,\widetilde{x}_n\}$, and ∂^K is the unique map that satisfies the Leibniz rule and has $\partial(\widetilde{x}_i)=x_i$ for $i=1,\dots,n$. Thus $K(\boldsymbol{x})$ is a DG (differential graded) algebra, and so its homology $\mathrm{H}(K(\boldsymbol{x}))$ is a graded algebra with $\mathrm{H}_0(K(\boldsymbol{x}))=R/(\boldsymbol{x})$. This implies $(\boldsymbol{x})\,\mathrm{H}(K(\boldsymbol{x}))=0$.
Evidently $K(\boldsymbol{x})$ is a perfect R-complex; it is indecomposable when R is local; see [1, 4.7]. As $K(\boldsymbol{x})_i$ is non-zero precisely for $0,\dots,n$, from (2.3.1) one gets

$$\operatorname{level}^R K(\boldsymbol{x}) \leq \operatorname{span} K(\boldsymbol{x}) = n+1\,.$$

Equality holds if R is local and $\boldsymbol{x}$ is a system of parameters (see Theorem 4.2 below), but $\operatorname{span} K(\boldsymbol{x}) - \operatorname{level}^R K(\boldsymbol{x})$ can be arbitrarily large; see [1, Section 3].
For any Koszul complex K on n elements, there are isomorphisms

$$K^* \cong \Sigma^{-n} K \quad \text{and} \quad K\otimes_R K \cong \bigoplus_{i=0}^{n} \Sigma^i K^{\binom{n}{i}}\,.$$

See [8, Propositions 1.6.10 and 1.6.21]. It thus follows from (2.3.3) that

$$\operatorname{level}^R \operatorname{Hom}_R(K,K) = \operatorname{level}^R (K\otimes_R K) = \operatorname{level}^R K\,. \tag{2.7.1}$$

In particular, the inequalities in Lemma 2.4(2) can be strict.

3. Tensor nilpotent morphisms

In this section we prove the Tensor Nilpotence Theorem announced in the introduction. We start by reviewing the properties of interest.

3.1. Tensor nilpotence. Let $f\colon Y \to X$ be a morphism in $\mathsf{D}(R)$.
The morphism f is said to be *tensor nilpotent* if for some $n \in \mathbb{N}$ the morphism

$$\underbrace{f \otimes^{\mathsf{L}}_R \cdots \otimes^{\mathsf{L}}_R f}_{n}\colon Y \otimes^{\mathsf{L}}_R \cdots \otimes^{\mathsf{L}}_R Y \longrightarrow X \otimes^{\mathsf{L}}_R \cdots \otimes^{\mathsf{L}}_R X$$

is equal to zero in $\mathsf{D}(R)$; when the R-complexes X, Y are perfect this means that the morphism $\otimes^n f\colon\ \otimes^n_R Y \to \otimes^n_R X$ is homotopic to zero.
When X is perfect and $f\colon X \to \Sigma^l X$ is a morphism such that $\otimes^n f$ homotopic to zero, the n-fold composition

$$X \longrightarrow \Sigma^l X \xrightarrow{\Sigma^l f} \Sigma^{2n} X \xrightarrow{\Sigma^{2l} f} \cdots \xrightarrow{\Sigma^{nl}} \Sigma^{nl} X$$

is also homotopic to zero. The converse does not hold, even when R is a field for in that case tensor nilpotent morphisms are zero.

3.2. Fiberwise zero morphisms. A morphism $f\colon Y \to X$ that satisfies

$$k(\mathfrak{p}) \otimes^{\mathsf{L}}_R f = 0 \quad \text{in} \quad \mathsf{D}(k(\mathfrak{p})) \quad \text{for every } \mathfrak{p} \in \operatorname{Spec} R$$

is said to be *fiberwise zero.* This is equivalent to requiring $k \otimes^{\mathsf{L}}_R f = 0$ in $\mathsf{D}(k)$ for every homomorphism $R \to k$ with k a field. In $\mathsf{D}(k)$ a morphism is zero if (and only if) it is a ghost, so the latter condition is equivalent to $\mathrm{H}(k \otimes^{\mathsf{L}}_R f) = 0$. A morphism in $\mathsf{D}(k)$ is tensor nilpotent exactly when it is zero. Thus if f is tensor nilpotent, it is fiberwise zero. There is a partial converse: *If a morphism $f\colon G \to F$ of perfect R-complexes is fiberwise zero, then it is tensor nilpotent.* This was proved by Hopkins [21, Theorem 10] and Neeman [25, Theorem 1.1].
The next result is the Tensor Nilpotence Theorem from the Introduction. Recall that an R-module is said to be I-torsion if each one of its elements is annihilated by some power of I.

Theorem 3.3. *Let R be a commutative noetherian ring and $f\colon G \to F$ a morphism of perfect R-complexes. If for some ideal I the following conditions hold*

(1) *f factors through some complex with I-torsion homology, and*
(2) $\operatorname{level}^R \operatorname{Hom}_R(G, F) \le \operatorname{height} I\,$*,*

then f is fiberwise zero. In particular, f is tensor nilpotent.

The proof of the theorem is given after Proposition 3.7.

Remark 3.4. Lemma 2.4 shows that the inequality (2) is implied by

$$\operatorname{level}^R F + \operatorname{level}^R G \le \operatorname{height} I + 1\,;$$

the converse does not hold; see (2.7.1).
On the other hand, condition (2) cannot be weakened: Let $(R, \mathfrak{m}, k)$ be a local ring and G the Koszul complex on some system of parameters of R and let

$$f\colon G \longrightarrow (G/G_{\leqslant d-1}) \cong \Sigma^d R$$

be the canonical surjection with $d = \dim R$. The complex G is $\mathfrak{m}$-torsion and $\operatorname{level}^R G = d+1$; see 2.7. Evidently $\mathrm{H}(k \otimes_R f) \neq 0$, so f is not fiberwise zero.

In the proof of Theorem 3.3 we exploit the functorial nature of I-torsion.

3.5. Torsion complexes. The derived I-torsion functor assigns to every X in $\mathsf{D}(R)$ an R-complex $\mathrm{R}\Gamma_I X$; when X is a module it computes its local cohomology: $\mathrm{H}^n_I(X) = \mathrm{H}_{-n}(\mathrm{R}\Gamma_I X)$ holds for each integer n. There is a natural morphism $t\colon \mathrm{R}\Gamma_I X \longrightarrow X$ in $\mathsf{D}(R)$ that has the following universal property: Every morphism $Y \to X$ such that $\mathrm{H}(Y)$ is I-torsion factors uniquely through t; see Lipman [24, Section 1]. It is easy to verify that the following conditions are equivalent.

(1) $\mathrm{H}(X)$ is I-torsion.
(2) $\mathrm{H}(X)_{\mathfrak{p}} = 0$ for each prime ideal $\mathfrak{p} \not\supseteq I$.
(3) The natural morphism $t\colon \mathrm{R}\Gamma_I X \to X$ is a quasi-isomorphism.

When they hold, we say that X is *I-torsion*. Note a couple of properties:

(3.5.1) If X is I-torsion, then $X \otimes^{\mathrm{L}}_R Y$ is I-torsion for any R-complex Y.

(3.5.2) There is a natural isomorphism $\mathrm{R}\Gamma_I(X \otimes^{\mathrm{L}}_R Y) \cong (\mathrm{R}\Gamma_I X) \otimes^{\mathrm{L}}_R Y$.

Indeed, $\mathrm{H}(X_{\mathfrak{p}}) \cong \mathrm{H}(X)_{\mathfrak{p}} = 0$ holds for $\mathfrak{p} \not\supseteq I$, giving $X_{\mathfrak{p}} = 0$ in $\mathsf{D}(R)$. Thus

$$(X \otimes^{\mathrm{L}}_R Y)_{\mathfrak{p}} \cong X_{\mathfrak{p}} \otimes^{\mathrm{L}}_R Y \cong 0$$

holds in $\mathsf{D}(R)$. It yields $\mathrm{H}(X \otimes^{\mathrm{L}}_R Y)_{\mathfrak{p}} \cong \mathrm{H}((X \otimes^{\mathrm{L}}_R Y)_{\mathfrak{p}}) = 0$, as desired.
A proof of the isomorphism in (3.5.2) can be found in [24, 3.3.1].

3.6. Big Cohen-Macaulay modules. Let $(R, \mathfrak{m}, k)$ be a local ring.
A (not necessarily finitely generated) R-module C is *big Cohen-Macaulay* if every system of parameters for R is a C-regular sequence, in the sense of [8, Definition 1.1.1]. In the literature the name is sometimes used for R-modules C that satisfy the property for *some* system of parameters for R; however, the $\mathfrak{m}$-adic completion of C is big Cohen-Macaulay in the sense above; see [8, Corollary 8.5.3].
The existence of big Cohen-Macaulay was proved by Hochster [16, 17] in case when R contains a field as a subring, and by André [2] when it does not; for the latter case, see also Heitmann and Ma [15].
In this paper, big Cohen-Macaulay modules are visible only in the next result, and in the proofs of Theorems 3.3 and 3.10.

Proposition 3.7. *Let I be an ideal in a local ring R and set $c :=$ height I. When C is a big Cohen-Macaulay R-module the following assertions hold.*

(1) *The canonical morphism $t\colon \mathrm{R}\Gamma_I C \to C$ from the I-torsion complex $\mathrm{R}\Gamma_I C$ (see 3.5) is a composition of c ghosts.*
(2) *If a morphism $g\colon G \to C$ of R-complexes with $\operatorname{level}^R G \leq c$ factors through some I-torsion complex, then $g = 0$.*

Proof. (1) There exist elements $\boldsymbol{x} = \{x_1, \dots, x_c\}$ in I such that $\operatorname{height}(\boldsymbol{x}) = c$; see [8, Theorem A.2]. It is well known that such a sequence $\boldsymbol{x}$ can be extended to system of parameters for R; see [8, Proposition A.4]. As any I-torsion complex is $(\boldsymbol{x})$-torsion, and the canonical morphism $\mathrm{R}\Gamma_I C \to C$ factors through the morphism $\mathrm{R}\Gamma_{(\boldsymbol{x})} C \to C$, we can assume $I = (\boldsymbol{x})$. The morphism t factors as

$$\mathrm{R}\Gamma_{(x_1,\dots,x_c)}(C) \longrightarrow \mathrm{R}\Gamma_{(x_1,\dots,x_{c-1})}(C) \longrightarrow \cdots \longrightarrow \mathrm{R}\Gamma_{(x_1)}(C) \longrightarrow C\,.$$

Since the sequence $\boldsymbol{x}$ is part of a system of parameters for R, it is C-regular and hence $\mathrm{H}_i(\mathrm{R}\Gamma_{(x_1,\dots,x_j)}(C)) = 0$ for $i \neq -j$; see [8, (3.5.6) and (1.6.16)]. Thus every one of the arrows above is a ghost, and hence t is a composition of c ghosts, as desired.

(2) Suppose g factors as $G \to X \to C$ with X an I-torsion R-complex. As noted in 3.5, the morphism $X \to C$ factors through t, so g factors as

$$G \xrightarrow{g'} X \xrightarrow{g''} \mathrm{R}\Gamma_I C \xrightarrow{t} C\,.$$

In view of the hypothesis $\operatorname{level}^R G \leq c$ and part (1), Lemma 2.6 shows that

$$\mathrm{RHom}_R(G,t)\colon\ \mathrm{RHom}_R(G, \mathrm{R}\Gamma_I C) \longrightarrow \mathrm{RHom}_R(G,C)$$

is a ghost. Using brackets to denote cohomology classes, we get

$$[g] = [tg''g'] = \mathrm{H}_0(\mathrm{RHom}_R(G,t))([g''g']) = 0\,.$$

Due to the isomorphism (2.1.1), this means that g is zero in $\mathsf{D}(R)$. □

Lemma 3.8. *Let R be a commutative noetherian ring and $f\colon G \to F$ a morphism of R-complexes, where G is finite free with $G_i = 0$ for $i \ll 0$ and F is perfect. Let $f'\colon F^* \otimes_R G \to R$ denote the composed morphism in the next display, where e is the evaluation map:*

$$F^* \otimes_R G \xrightarrow{F^* \otimes_R f} F^* \otimes_R F \xrightarrow{e} R\,.$$

If f factors through some I-torsion complex, then so does f'.
The morphism f' is fiberwise zero if and only if so is f.

Proof. For the first assertion, note that if f factors through an I-torsion complex X, then $F^* \otimes_R f$ factors through $F^* \otimes_R X$, and the latter is I-torsion.

For the second assertion, let k be a field and $R \to k$ be a homomorphism of rings. Let $\overline{(-)}$ and $(-)^\vee$ stand for $k \otimes_R (-)$ and $\operatorname{Hom}_k(-,k)$, respectively. The goal is to prove that $\overline{f} = 0$ is equivalent to $\overline{f'} = 0$.

Since F is perfect, there are canonical isomorphisms

$$F^* \otimes k \xrightarrow{\cong} \operatorname{Hom}_R(F,k) \cong \operatorname{Hom}_k(k \otimes_R F, k) = (\overline{F})^\vee\,.$$

Given this, it follows that $\overline{f'}$ can be realized as the composition of morphisms

$$(\overline{F})^\vee \otimes_k \overline{G} \xrightarrow{(\overline{F})^\vee \otimes_k \overline{f}} (\overline{F})^\vee \otimes_k \overline{F} \xrightarrow{\overline{e}} k\,.$$

If $\overline{F}$ is zero, then $\overline{f} = 0$ and $\overline{f'} = 0$ hold. When $\overline{F}$ is nonzero, it is easy to verify that $\overline{f} \neq 0$ is equivalent to $\overline{f'} \neq 0$, as desired. □

Proof of Theorem 3.3. Given morphisms of R-complexes $G \to X \to F$ such that F and G are perfect and X is I-torsion for an ideal I with

$$\operatorname{level}^R \operatorname{Hom}_R(G,F) \le \operatorname{height} I\,,$$

we need to prove that f is fiberwise zero. This implies the tensor nilpotence of f, as recalled in 3.1.

By Lemma 3.8, the morphism $f'\colon F^* \otimes_R G \to R$ factors through an I-torsion complex, and if f' is fiberwise zero, so is f. The isomorphisms of R-complexes

$$(F^* \otimes_R G)^* \cong G^* \otimes_R F \cong \operatorname{Hom}_R(G,F)$$

and Lemma 2.4 yield $\operatorname{level}^R (F^* \otimes_R G) = \operatorname{level}^R \operatorname{Hom}_R(G,F)$. Thus, replacing f by f', it suffices to prove that if $f\colon G \to R$ is a morphism that factors through an I-torsion complex and satisfies $\operatorname{level}^R G \le \operatorname{height} I$, then f is fiberwise zero.

Fix $\mathfrak{p}$ in $\operatorname{Spec} R$. When $\mathfrak{p} \not\supseteq I$ we have $X_\mathfrak{p} = 0$, by 3.5(2). For $\mathfrak{p} \supseteq I$ we have

$$\operatorname{level}^{R_\mathfrak{p}} G_\mathfrak{p} \le \operatorname{level}^R G \le \operatorname{height} I \le \operatorname{height} I_\mathfrak{p}\,,$$

where the first inequality follows directly from the definitions; see [6, Proposition 3.7]. It is easy to verify that $X_\mathfrak{p}$ is $I_\mathfrak{p}$-torsion. Thus, localizing at $\mathfrak{p}$, we may further assume $(R,\mathfrak{m},k)$ is a local ring, and we have to prove that $\mathrm{H}(k \otimes_R f) = 0$ holds.

Let C be a big Cohen-Macaulay R-module. It satisfies $\mathfrak{m}C \ne C$, so the canonical map $\pi\colon R \to k$ factors as $R \xrightarrow{\gamma} C \xrightarrow{\varepsilon} k$. The composition $G \xrightarrow{f} R \xrightarrow{\gamma} C$ is zero in $\mathsf{D}(R)$, by Proposition 3.7. We get $\pi f = \varepsilon\gamma f = 0$, whence $\mathrm{H}(k \otimes_R \pi)\,\mathrm{H}(k \otimes_R f) = 0$. Since $\mathrm{H}(k \otimes_R \pi)$ is bijective, this implies $\mathrm{H}(k \otimes_R f) = 0$, as desired. □

The following consequence of Theorem 3.3 is often helpful.

Corollary 3.9. *Let $(R,\mathfrak{m},k)$ be a local ring, F a perfect R-complex, and G an R-complex of finitely generated free modules.*

If a morphism of R-complexes $f\colon G \to F$ satisfies the conditions

(1) *f factors through some $\mathfrak{m}$-torsion complex, and*
(2) *$\sup F^\natural - \inf G^\natural \le \dim R - 1$,*

then $\mathrm{H}(k \otimes_R f) = 0$.

Proof. An $\mathfrak{m}$-torsion R-complex X satisfies $k(\mathfrak{p}) \otimes^{\mathsf{L}}_R X = 0$ for every prime $\mathfrak{p}$ in $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$. Thus a morphism, g, of R-complexes that factors through X is fiberwise zero if and only if $k \otimes^{\mathsf{L}}_R g = 0$. This remark is used in what follows.

Condition (2) implies $G_n = 0$ for $n \ll 0$. Let f' denote the composition

$$F^* \otimes_R G \xrightarrow{F^* \otimes_R f} F^* \otimes_R F \xrightarrow{e} R\,,$$

where e is the evaluation map. Since $\inf\,(F^* \otimes_R G)^\natural = -\sup F^\natural + \inf G^\natural$, it suffices to prove the corollary for morphisms $f\colon G \to R$; see Lemma 3.8.

As f factors through some $\mathfrak{m}$-torsion complex, so does the composite morphism

$$G_{\leqslant 0} \hookrightarrow G \xrightarrow{f} R$$

It is easy to check that if the induced map $\mathrm{H}(k\otimes_R G_{\leqslant 0}) \to \mathrm{H}(k\otimes_R R) = k$ is zero, then so is $\mathrm{H}(k\otimes_R f)$. Thus we may assume $G_n = 0$ for $n \notin [-d+1, 0]$, where $d = \dim R$. This implies $\operatorname{level}^R G \leq d$, so Theorem 3.3 yields the desideratum. □

The following result is a variant of Theorem 3.3—the hypothesis is weaker as is the conclusion. It is not used in this work. The example in Remark 3.4 shows that the result cannot be strengthened to conclude that f is fiberwise zero.

THEOREM 3.10. *Let R be a local ring and let $f\colon G \to F$ be a morphism of R-complexes.*
If there exists an ideal I of R such that

(1) *f factors through an I-torsion complex, and*
(2) *$\operatorname{level}^R F \leq \operatorname{height} I$,*

then $\mathrm{H}(C \otimes^{\mathrm{L}}_R f) = 0$ for every big Cohen-Macaulay module C.

Proof. Set $c := \operatorname{height} I$ and let $t\colon \mathrm{R}\Gamma_I C \to C$ be the canonical morphism. It follows from (3.5.1) that $C \otimes^{\mathrm{L}}_R f$ also factors through an I-torsion R-complex. The quasi-isomorphism (3.5.2) and the universal property of derived I-torsion, see 3.5, imply that $C \otimes^{\mathrm{L}}_R f$ factors as a composition of the morphisms:

$$C \otimes^{\mathrm{L}}_R G \longrightarrow (\mathrm{R}\Gamma_I C) \otimes^{\mathrm{L}}_R F \xrightarrow{t\otimes^{\mathrm{L}}_R F} C \otimes^{\mathrm{L}}_R F$$

By Proposition 3.7(1) the morphism t is a composition of c ghosts. Thus condition (2) and Lemma 2.6 imply $t\otimes^{\mathrm{L}}_R F$ is a ghost, and hence so is $C\otimes^{\mathrm{L}}_R f$. □

4. APPLICATIONS TO LOCAL ALGEBRA

In this section we record applications the Tensor Nilpotence Theorem to local algebra. To that end it is expedient to reformulate it as the Morphic Intersection Theorem from the Introduction, restated below.

THEOREM 4.1. *Let R be a commutative noetherian ring and $f\colon G \to F$ a morphism of perfect R-complexes.*
If f is not fiberwise zero and factors through a complex with I-torsion homology for some ideal I of R, then there are inequalities:

$$\operatorname{span} F + \operatorname{span} G - 1 \geq \operatorname{level}^R \operatorname{Hom}_R(G, F) \geq \operatorname{height} I + 1\,.$$

Proof. The inequality on the left comes from Lemma 2.4 and (2.3.1). The one on the right is the contrapositive of Theorem 3.3. □

Here is one consequence.

THEOREM 4.2. *Let R be a local ring and F a complex of finite free R-modules:*

$$F := \quad 0 \to F_d \to F_{d-1} \to \cdots \to F_0 \to 0$$

For each ideal I such that $I \cdot \mathrm{H}_i(F) = 0$ for $i \geq 1$ and $I \cdot z = 0$ for some element z in $\mathrm{H}_0(F) \setminus \mathfrak{m}\,\mathrm{H}_0(F)$, where $\mathfrak{m}$ is the maximal ideal of R, one has

$$d + 1 \geq \operatorname{span} F \geq \operatorname{level}^R F \geq \operatorname{height} I + 1\,.$$

Proof. The first two inequalities are clear from the definitions. As to third one, pick $\widetilde{z} \in F_0$ representing z in $\mathrm{H}_0(F)$ and consider the morphism of complexes $f\colon R \to F$ given by $r \mapsto r\widetilde{z}$. Since z is not in $\mathfrak{m}\,\mathrm{H}_0(F)$, one has

$$\mathrm{H}_0(k \otimes_R f) = k \otimes_R \mathrm{H}_0(f) \neq 0$$

for $k = R/\mathfrak{m}$. In particular, $k \otimes_R f$ is nonzero. On the other hand, f factors through the inclusion $X \subseteq F$, where X is the subcomplex defined by

$$X_i = \begin{cases} F_i & i \geq 1 \\ R\widetilde{z} + \partial(F_1) & i = 0 \end{cases}$$

By construction, we have $\mathrm{H}_i(X) = \mathrm{H}_i(F)$ for $i \geq 1$ and $I\,\mathrm{H}_0(X) = 0$, so $\mathrm{H}(X)$ is I-torsion. The desired inequality follows from Theorem 4.1 applied to f. $\square$

The preceding result is a stronger form of the Improved New Intersection Theorem[3] of Evans and Griffith [12]; see also [19, §2]. First, the latter is in terms of spans of perfect complexes whereas the one above is in terms of levels with respect to R; second, the hypothesis on the homology of F is weaker. Theorem 4.2 also subsumes prior extensions of the New Intersection Theorem to statements involving levels, namely [6, Theorem 5.1], where it was assumed that $I \cdot \mathrm{H}_0(F) = 0$ holds, and [1, Theorem 3.1] which requires $\mathrm{H}_i(F)$ to have finite length for $i \geq 1$.

In the influential paper [18], Hochster identified certain *canonical elements* in the local cohomology of local rings, conjectured that they are never zero, and proved that statement in the equal characteristic case. He also gave several reformulations that do not involve local cohomology. The relations between these statements and the histories of their proofs have been the subject of a number of detailed discussions; see [20, 28] for the most recent ones.

Some of those statements concern properties of morphisms from the Koszul complex on some system of parameters to resolutions of various R-modules. This makes them particularly amenable to approaches from the Morphic Intersection Theorem. In the rest of this section we uncover direct paths to various forms of the Canonical Element Theorem and related results.

We first prove a version of [18, 2.3]. The conclusion there is that f_d is not zero, but the remarks in [18, 2.2(6)] show that it is equivalent to the result below.

Theorem 4.3. *Let $(R, \mathfrak{m}, k)$ be a local ring, $\boldsymbol{x}$ a system of parameters, K the Koszul complex on $\boldsymbol{x}$, and F a complex of finitely generated free R-modules. If $f\colon K \to F$ is a morphism of R-complexes with $\mathrm{H}_0(k \otimes_R f) \neq 0$, then one has*

$$\mathrm{H}_d(S \otimes_R f) \neq 0 \quad \textit{for} \quad S = R/(\boldsymbol{x}) \quad \textit{and} \quad d = \dim R\,.$$

Proof. It is easy to verify the result when $d = 0$ so we assume $d \geq 1$.

Recall from 2.7 that $K^\natural$ is the exterior algebra on K_1, which has a basis $\widetilde{x}_1, \ldots, \widetilde{x}_d$, and that $\partial(K)$ lies in $(\boldsymbol{x})K$. In particular, K_d is a free R-module

[3] Prior to the appearance of [2], this statement, the Canonical Element Theorem, and the Monomial Theorem had been proved in equal characteristics and conjectured in general.

with basis $x = \widetilde{x}_1 \wedge \cdots \wedge \widetilde{x}_d$ and $\mathrm{H}_d(S \otimes_R K) = S(1 \otimes x)$, so we need to prove

$$f(K_d) \not\subseteq (\boldsymbol{x})F_d + \partial(F_{d+1})\,.$$

Arguing by contradiction, suppose that the contrary holds. This means

$$f(x) = x_1 y_1 + \cdots + x_d y_d + \partial^F(y)$$

with $y_1, \dots, y_d \in F_d$ and $y \in F_{d+1}$. Let $h\colon K \to F$ denote the degree one R-linear map defined by setting $h_d(\widetilde{x}_1 \wedge \cdots \wedge \widetilde{x}_d) = y$,

$$h_{d-1}(\widetilde{x}_1 \wedge \cdots \wedge \widetilde{x}_{j-1} \wedge \widetilde{x}_{j+1} \wedge \cdots \wedge \widetilde{x}_d) = (-1)^{j-1} y_j \quad \text{for} \quad j = 1, \dots, d\,,$$

and $h_i = 0$ for $i \neq d-1, d$. We use h to produce a morphism of complexes

$$g := f - \partial^F h - h\partial^K \colon K \to F \quad \text{that has} \quad g_d = 0\,.$$

The last condition implies that g factors as a composition of morphisms

$$K \xrightarrow{g'} F_{<d} \hookrightarrow F\,.$$

Since f and g are homotopic, and $g_i = g'_i$ for $i = 0, 1$ (as $d \geq 1$), we have

$$\mathrm{H}_0(k \otimes_R f) = \mathrm{H}_0(k \otimes_R g) = \mathrm{H}_0(k \otimes_R g')\,.$$

As K is $\mathfrak{m}$-torsion (see 2.7), Corollary 3.9 applies to g' and gives $\mathrm{H}(k \otimes_R g') = 0$. Thus we get $\mathrm{H}_0(k \otimes_R f) = 0$; this is the desired contradiction. □

A first specialization is the Canonical Element Theorem.

Corollary 4.4. *Let I be an ideal in R containing a system of parameters $\boldsymbol{x}$ and K the Koszul complex on $\boldsymbol{x}$. If F is a free resolution of R/I, any morphism $f\colon K \to F$ of R-complexes lifting the surjection $R/(\boldsymbol{x}) \to R/I$ has $f(K_d) \neq 0$ for $d = \dim R$.* □

As usual, when A is a matrix, $I_d(A)$ denotes the ideal of its minors of size d.

Corollary 4.5. *Let $(R, \mathfrak{m}, k)$ be a local ring, $\boldsymbol{x}$ a system of parameters for R, and $\boldsymbol{y}$ a finite subset of $\mathfrak{m}$ with $(\boldsymbol{y}) \supseteq (\boldsymbol{x})$.*
If A is a matrix such that $A\boldsymbol{y} = \boldsymbol{x}$, then $I_d(A) \not\subseteq (\boldsymbol{x})$ for $d = \dim R$.

Proof. Let K and F be the Koszul complexes on $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively. The matrix A defines a unique morphism of DG R-algebras $f\colon K \to F$. Evidently, $\mathrm{H}_0(k \otimes_R f)$ is the identity map on k, and hence is not zero. Since f_d can be represented by a column matrix whose entries are the various $d \times d$ minors of A, the desired statement is a direct consequence of Theorem 4.3. □

A special case of the preceding result yields the Monomial Theorem.

Corollary 4.6. *If $y_1, \dots, y_d$ is a system of parameters in a local ring, one has*

$$(y_1 \cdots y_d)^n \notin (y_1^{n+1}, \dots, y_d^{n+1}) \quad \textit{for every integer} \quad n \geq 1\,.$$

Proof. Apply Corollary 4.5 to the inclusion $(y_1^{n+1}, \dots, y_d^{n+1}) \subseteq (y_1, \dots, y_d)$ and

$$A := \begin{bmatrix} y_1^n & 0 & \cdots & 0 \\ 0 & y_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_d^n \end{bmatrix}. \qquad \square$$

For the next application of Theorem 4.3 we recall that for any pair (S,T) of commutative R-algebras the graded module $\operatorname{Tor}^R(S,T)$ carries a natural structure of graded-commutative R-algebra, given by the $\pitchfork$-product of Cartan and Eilenberg [10, Section XI.4].

Lemma 4.7. *Let R be a commutative ring, I an ideal of R, and set $S := R/I$. Let $G \to S$ be an R-free resolution, K be the Koszul complex on some finite generating set of I, and $g\colon K \to G$ a morphisms of R-complexes lifting the identity of S.*
For every surjective homomorphism $\psi\colon S \to T$ of of commutative rings there is a natural in T commutative diagram of strictly graded-commutative S-algebras

$$\begin{tikzcd} S \otimes_R K \arrow[r, equal] & \bigwedge_S \mathrm{H}_1(S \otimes_R K) \arrow[r, two heads, "\alpha"] & \bigwedge_S \operatorname{Tor}_1^R(S,S) \arrow[r, "\mu^S"] \arrow[d, two heads, "\bigwedge_\psi \operatorname{Tor}_1^R(S,\psi)"'] & \operatorname{Tor}^R(S,S) \arrow[d, "\operatorname{Tor}^R(S,\psi)"] \\ & & \bigwedge_T \operatorname{Tor}_1^R(S,T) \arrow[r, "\mu^T"] & \operatorname{Tor}^R(S,T) \end{tikzcd}$$

where $\alpha_1 = \mathrm{H}_1(S \otimes_R g)$, and the maps α and $\mu^?$ are defined, respectively, by the functoriality and the universal property of exterior algebras.

Proof. The equality follows from $\partial^K(K) \subseteq IK$ and $K^\natural = \bigwedge_R K_1$. The resolution G can be chosen to have $G_{\leqslant 1} = K_{\leqslant 1}$; this makes α_1 surjective, and the surjectivity of α follows. The map $\operatorname{Tor}_1^R(S,\psi)$ is surjective because it can be identified with the natural map $I/I^2 \to I/IJ$, where $J = \operatorname{Ker}(R \to T)$; the surjectivity of $\bigwedge_\psi \operatorname{Tor}_1^R(S,\psi)$ follows. The square commutes by the naturality of $\pitchfork$-products. $\square$

The result below is another form of the Canonical Element Theorem. Roberts [27] proposed the statement and proved that it is equivalent to the Canonical Element Theorem; a different proof appears in Huneke and Koh [22].

Theorem 4.8. *Let $(R, \mathfrak{m}, k)$ be a local ring, I an ideal, and $S := R/I$. Let $S \to T$ be a surjective homomorphism of rings and let*

$$\mu^T\colon \bigwedge_T \operatorname{Tor}_1^R(S,T) \longrightarrow \operatorname{Tor}^R(S,T)$$

be the morphism of graded T-algebras defined in Lemma 4.7.
If I is a parameter ideal, then $\mu^T \otimes_T k$ is injective; in particular, μ^k is injective.

Proof. By Lemma 4.7 the canonical surjection $\pi\colon T \to k$ induces a commutative diagram of graded-commutative algebras

$$\begin{array}{ccc} \bigwedge_T \operatorname{Tor}_1^R(S,T) & \xrightarrow{\mu^T} & \operatorname{Tor}^R(S,T) \\ {\scriptstyle \bigwedge_\pi \operatorname{Tor}_1^R(S,\pi)}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Tor}^R(S,\pi)} \\ \bigwedge_k \operatorname{Tor}_1^R(S,k) & \xrightarrow{\mu^k} & \operatorname{Tor}^R(S,k) \end{array}$$

It is easy to verify that π induces a bijective map

$$\operatorname{Tor}_1^R(S,\pi)\otimes_T k\colon\ \operatorname{Tor}_1^R(S,T)\otimes_T k \to \operatorname{Tor}_1^R(S,k)\,,$$

so $(\wedge_\pi \operatorname{Tor}_1^R(S,\pi))\otimes_T k$ is an isomorphism. Thus it suffices to show that the map μ^k is injective.

Let K be the Koszul complex on a minimal generating set of I. Let $G \xrightarrow{\simeq} S$ and $F \xrightarrow{\simeq} k$ be R-free resolutions of S and k, respectively. Lift the identity map of S and the canonical surjection $\psi\colon S \to k$ to morphisms $g\colon K \to G$ and $h\colon G \to F$, respectively. We have $\mu^S\alpha = \mathrm{H}(S\otimes_R g)$ and $\operatorname{Tor}^R(S,\psi) = \mathrm{H}(S\otimes_R h)$. This implies the second equality in the string

$$\mu_d^k \operatorname{Tor}_d^R(S,\pi)\alpha_d = \operatorname{Tor}_d^R(S,\psi)\mu_d^S\alpha_d = \mathrm{H}_d(S\otimes_R hg) \neq 0\,.$$

The first equality comes from Lemma 4.7, with $T = k$, and the non-equality from Theorem 4.3, with $f = hg$. In particular, we get $\mu_d^k \neq 0$. We have an isomorphism $\operatorname{Tor}_1^R(S,k) \cong \bigwedge_k k^d$ of graded k-algebras, so μ^k is injective by the next remark. □

Remark 4.9. If Q is a field, d is a non-negative integer, and $\lambda\colon \bigwedge_Q Q^d \to B$ is a homomorphism of graded Q-algebras with $\lambda_d \neq 0$, then λ is injective.

Indeed, the graded subspace $\bigwedge_Q^d Q^d$ of exterior algebra $\bigwedge_Q Q^d$ is contained in every non-zero ideal and has rank one, so $\lambda_d \neq 0$ implies $\operatorname{Ker}(\lambda) = 0$.

5. Ranks in finite free complexes

This section is concerned with DG modules over Koszul complexes on sequences of parameters. Under the additional assumptions that R is a domain and F is a resolution of some R-module, the theorem below was proved in [3, 6.4.1], and earlier for cyclic modules in [9, 1.4]; background is reviewed after the proof. The Canonical Element Theorem, in the form of Theorem 4.3 above, is used in the proof.

Theorem 5.1. *Let $(R,\mathfrak{m},k)$ be a local ring, set $d = \dim R$, and let F be a complex of finite free R-modules with $\mathrm{H}_0(F) \neq 0$ and $F_i = 0$ for $i < 0$.*

If F admits a structure of DG module over the Koszul complex on some system of parameters of R, then there is an inequality

$$\operatorname{rank}_R(F_n) \geq \binom{d}{n} \quad \textit{for each} \quad n\in\mathbb{Z}\,. \tag{5.1.1}$$

Proof. The desired inequality holds when $d = 0$, for $\mathrm{H}_0(F) \neq 0$ implies $F_0 \neq 0$. We can thus assume $d \geq 1$. Let $\boldsymbol{x}$ be the said system of parameters of R and K the Koszul complex on $\boldsymbol{x}$. Since F is a DG K-module, each $\mathrm{H}_i(F)$ is an $R/(\boldsymbol{x})$-module, and hence of finite length.

First we reduce to the case when R is a domain. To that end, let $\mathfrak{p}$ be a prime ideal of R such that $\dim(R/\mathfrak{p}) = d$. Evidently, the image of $\boldsymbol{x}$ in R is a system of parameters for $R/\mathfrak{p}$. By base change, $(R/\mathfrak{p}) \otimes_R F$ is a DG module over $(R/\mathfrak{p}) \otimes_R K$, the Koszul complex on $\boldsymbol{x}$ with coefficients in $R/\mathfrak{p}$, with

$$\mathrm{H}_0((R/\mathfrak{p}) \otimes_R F) \cong R/\mathfrak{p} \otimes \mathrm{H}_0(F) \neq 0\,.$$

Moreover, the rank of $F^\natural$ as an R-module equals the rank of $(R/\mathfrak{p}) \otimes_R F^\natural$ as an $R/\mathfrak{p}$-module. Thus, after base change to $R/\mathfrak{p}$ we can assume R is a domain.

Choose a cycle $z \in F_0$ that maps to a minimal generator of the R-module $\mathrm{H}_0(F)$. Since F is a DG K-module, this yields a morphism of DG K-modules

$$f\colon K \to F \quad \text{with} \quad f(a) = az\,.$$

This is, in particular, a morphism of complexes. Since $k \otimes_R \mathrm{H}_0(F) \neq 0$, by the choice of z, Theorem 4.3 applies, and yields $f(K_d) \neq 0$. As R is a domain, this implies $f(Q \otimes_R K_d)$ is non-zero, where Q is the field of fractions of R.

Set $\Lambda := (Q \otimes_R K)^\natural$ and consider the homomorphism of graded Λ-modules

$$\lambda := Q \otimes_R f^\natural \colon \Lambda \to Q \otimes_R F^\natural\,.$$

As Λ is isomorphic to $\bigwedge_Q Q^d$, Remark 4.9 gives the inequality in the display

$$\operatorname{rank}_R(F_n) = \operatorname{rank}_Q(Q \otimes_R F_n) \geq \operatorname{rank}_Q(\Lambda_n) = \binom{d}{n}\,.$$

Both equalities are clear. □

The inequalities (5.1.1) are related to a major topic of research in commutative algebra. We discuss it for a local ring $(R, \mathfrak{m}, k)$ and a bounded R-complex F of finite free modules with $F_{<0} = 0$, homology of finite length, and $\mathrm{H}_0(F) \neq 0$.

5.2. Ranks of syzygies. The celebrated and still open Rank Conjecture of Buchsbaum and Eisenbud [9, Proposition 1.4], and Horrocks [14, Problem 24] predicts that (5.1.1) holds if F is a *resolution* of some module of finite length. The Rank Conjecture is known to hold for $d \leq 4$. Its validity would imply that $\sum_n \operatorname{rank}_R F_n \geq 2^d$ holds in all dimensions. For $d = 5$ and equicharacteristic R, this inequality was proved in [4, Proposition 1] by using Evans and Griffth's Syzygy Theorem [12]; in view of [2], it holds for all R.

In a breakthrough, M. Walker [30] used methods from K-theory to prove that $\sum_n \operatorname{rank}_R F_n \geq 2^d$ holds when R contains $\frac{1}{2}$, and is complete intersection (in particular, regular) or contains a finite field.

5.3. DG module structures on resolutions. Let F be a *minimal resolution* of an R-module M of nonzero finite length and $\boldsymbol{x}$ a parameter set for R with $\boldsymbol{x}M = 0$.

When F admits a DG module structure over $K(\boldsymbol{x})$ the Rank Conjecture holds, by Theorem 5.1. It was conjectured in [9, 1.2′] that such a structure exists for all F and $\boldsymbol{x}$. An obstruction to its existence was found in [3, 1.2], and examples when that obstruction is not zero were produced in [3, 2.4.2].

On the other hand, in [3, 1.8] it is proved that that obstruction vanishes when $\boldsymbol{x}$ lies in $\mathfrak{m}\operatorname{ann}_R(M)$, and the question was raised if F supports a DG $K(\boldsymbol{x})$-module structure for some *special choice* of $\boldsymbol{x}$; in particular, for high powers of systems of parameters contained in $\operatorname{ann}_R(M)$. The answer is not known.

For complexes that are not resolutions the situation is different.

5.4. DG module structures on complexes. Theorem 5.1 provides a series of obstructions for the existence of DG module structures on F. In particular, it implies that if $\operatorname{rank}_R F < 2^d$ holds with $d = \dim R$, then F supports no DG module structure over $K(\boldsymbol{x})$ for *any* system of parameters $\boldsymbol{x}$. Complexes satisfying the restriction on ranks were recently constructed in [23, 4.1]. These complexes are not resolutions of R-modules as they have nonzero homology in degrees 0 and 1.

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