Cylindrical Wigner Measures

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Abstract. In this paper we study the semiclassical behavior of quantum states acting on the C*-algebra of canonical commutation relations, from a general perspective. The aim is to provide a unified and flexible approach to the semiclassical analysis of bosonic systems. We also give a detailed overview of possible applications of this approach to mathematical problems of both axiomatic relativistic quantum field theories and nonrelativistic many body systems. If the theory has infinitely many degrees of freedom, the set of Wigner measures, *i.e.* the classical counterpart of the set of quantum states, coincides with the set of all cylindrical measures acting on the algebraic dual of the space of test functions for the field, and this reveals a very rich semiclassical structure compared to the finite-dimensional case. We characterize the cylindrical Wigner measures and the *a priori* properties they inherit from the corresponding quantum states.

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1 Introduction

The study of semiclassical and effective behaviors in quantum mechanical systems with many particles plays an important role in mathematical, theoretical, and experimental physics. In particular, we may mention the recent widespread mathematical interest in the rigorous derivation of bosonic and fermionic effective theories from many-body non-relativistic Hamiltonians [it would be too long to provide here an extensive bibliography on the subject, the interested reader should refer, *e.g.*, to the reviews 16, 51, and references thereof contained]. For relativistic and semi-relativistic (bosonic) systems, where particles may be created or destroyed, there are fewer results [3, 4, 10, 31–33, 48, 60] since the situation is usually more involved (*e.g.*, due to renormalization issues). Most of the latter results, at least the most recent ones, make use of the semiclassical approach in the Fock representation developed by Ammari and Nier [6, 7, 8, 9], that is also suitable to study non-relativistic mean-field problems [see 11, 12, 64, 65, in addition to the ones just mentioned]. This approach overcomes the limitations of other techniques, related to the choice of (initial) quantum many-body state of the system: it is in fact possible to study the effective behavior of a general class of quantum states, in particular states that have no coherent semiclassical structure, or that are not "close" to one with such structure.

The aim of this paper is to take an even more general approach, in order to complete our mathematical understanding of bosonic semiclassical analysis, and to collect our knowledge in a unified description, valid for most situations of physical interest. To do so, we connect the algebraic formulation of quantum theories to semiclassical analysis. This has the advantage of taking into account at the same time all possible representations of the algebra of canonical commutation relations, thus allowing to study the semiclassical states of any given bosonic theory with fixed physical parameters such as mass and spin. As a consequence, we are able to apply our results, *e.g.*, to relativistic axiomatic field theories, for which inequivalent representations of the canonical commutation relations play a crucial role, and to thermodynamic states in quantum statistical mechanics. This and other applications are described in detail in § 2 [see also 28, 29, 35]. This paper can thus be seen as a continuation of the program started by Ammari and Nier, where we combine the representation-independent algebraic framework to the powerful tools of semiclassical analysis in a way that yields new physical applications, opening the way to interesting future directions. Mathematically, we also provide a bridge between noncommutative and commutative concepts, highlighting the way some properties of objects in a commutative theory may descend directly from the properties of the corresponding objects in the deformed (noncommutative) counterpart, when the deformation parameter converges to zero.

Throughout the paper, we may append the existential quantifier to mathematical objects as a superscript, if the range of quantification is unambiguous. That is, letting $m, n \in \mathbb{N}$, instead of $(\exists x_1 \in X_1)(\exists x_2 \in X_2)\cdots(\exists x_n \in X_n)R(x_1,\dots,x_n;y_1,\dots,y_m)$ we write $R({}^{\exists}x_1,\dots,{}^{\exists}x_n;y_1,\dots,y_m)$. In particular, the existential quantification appears as a superscript more than once only if appended to different symbols, and each time it means a distinct quantification; if an object is bounded by an existential quantifier and appears more than once in the same formula, the quantifier is appended to it only once, *e.g.*, ${}^{\exists}f \circ g = g \circ f = \mathrm{id}$. Even if this notation is somewhat unconventional, we think that here it helps to lighten the notation, thus improving readability. In § 1 and 2 an extensive use of footnotes is made, to quickly provide definitions of objects and properties that may be unfamiliar to the reader.

1.1 An overview of Wigner measures

Wigner semiclassical measures are a powerful tool in the effective or asymptotic description of quantum systems [see, *e.g.*, 27, 37, 45, 55, 66, 74, 83, and references thereof contained]. The standard Wigner measures are Radon measures on the cotangent bundle $\mathrm{T}^*\Sigma$ (phase space) of a *finite dimensional* real vector space Σ. They are the abelian counterpart of quantum states that are normal with respect to the irreducible representation of the corresponding Heisenberg group $\mathbb{H}(\mathrm{T}^*\Sigma)$. The semiclassical measures are naturally introduced in relation to pseudodifferential (Weyl) calculus. In fact, let $h \geq 0$ be the semiclassical parameter. Then for any bounded family of vectors $(u_h)_{0<h\leq h_0} \subset L^2(\Sigma)$, $h_0 \in \mathbb{R}^+$, there exists a subsequence $(u_{h_j})_{j\in\mathbb{N}}$, $h_j \to 0$, and a finite Radon measure μ on $\mathrm{T}^*\Sigma$ such that for any $a \in C_0^\infty(\mathrm{T}^*\Sigma)$

$$\lim_{j\to\infty} \langle u_{h_j}, \mathrm{Op}^{h_j}_{\frac{1}{2}}(a) u_{h_j}\rangle_2 = \int_{\mathrm{T}^*\Sigma} a(z)\mathrm{d}\mu(z) \ . \tag{1}$$

$\mathrm{Op}^h_{\frac{1}{2}}(a)$ denotes the h-dependent Weyl quantization of the symbol a. The physical interpretation of Eq. (1) is straightforward: as quantum states are non-commutative probabilities on which it is possible to evaluate quantum observables, classical states are classical probabilities on which it is possible to evaluate phase space functions (classical observables). In addition, the quantum expectation of a quantized classical observable converges in the classical limit to the classical expectation, with respect to the semiclassical measure, of the same observable.

To study the semiclassical behavior of many-body quantum systems, Heisenberg groups associated to *infinite dimensional* symplectic spaces play a crucial role, since they encode the group form of the canonical commutation relations. Pseudodifferential calculus for infinite dimensional symplectic phase spaces (X, ς) is notoriously difficult, due to the lack of a locally finite and translation-invariant measure. In addition, the Weyl C*-algebra associated to (X, ς) has uncountably many inequivalent irreducible representations. Nonetheless, as we will explain below, it is possible to use the finite-dimensional pseudodifferential calculus (quantizing the so-called cylindrical symbols) to obtain a semiclassical characterization of states in quantum field theory. In addition, a formula analogous to Eq. (1) holds for all smooth cylindrical symbols, and it can be adapted to suitable non-cylindrical and possibly non-smooth symbols as well [see 35, for additional details].

There have been attempts to construct directly a pseudodifferential calculus for symbols that are not cylindrical, if (X, ς) originates from a complex separable Hilbert space and the Weyl C*-algebra is represented on the Fock vacuum (Fock representation). Among these attempts, let us mention the Weyl calculus in abstract Wiener spaces [13–15]; the Wick quantization of polynomial symbols [6, 17], and the inductive approach adopted by Krée and Raczka [59].

In the Fock representation, the introduction of semiclassical Wigner measures as mean field or semiclassical counterparts of bosonic quantum many-body states is due to Ammari and Nier [6]. Analogous measures also appear in the formulation of the bosonic quantum de Finetti Theorem [61] (see [11, Proposition 3.2] for a link between the two points of view). We further develop the notion of Wigner measures introducing *cylindrical Wigner measures*. The additional adjective "cylindrical" is due to the fact that in general the classical counterpart of bosonic quantum states are not Radon measures on the space of classical fields, but rather cylindrical measures, *i.e.* finitely additive measures on the algebra of cylinders. The linear space of cylindrical measures includes the space of Radon measures, and the inclusion is often strict for infinite dimensional spaces. We also make precise in which sense quantum states converge to cylindrical measures in the semiclassical limit $h \to 0$, introducing two suitable topologies. More precisely, even if *a priori* quantum states and cylindrical measures may appear as different types of objects, they are both isomorphic to objects of the same type, and a notion of vicinity can therefore be introduced. We dedicate the rest of § 1 to the motivation of the basic ideas behind cylindrical Wigner measures, and the introduction of our main results. To improve readability, the results are stated in a simplified form, more complete statements and proofs can be found in § 3 to 6; § 2 is devoted to physical applications of the main results, and it is divided in two main subsections: in § 2.1 we describe applications to many-body quantum theories and non-relativistic quantum field theories, in § 2.2 we describe applications to relativistic quantum field theories.

1.2 Semiclassical quantum states

It is well known that, from an algebraic point of view, a quantum state is a positive and norm one functional on the algebra of observables. We relax the normalization assumption, so throughout this paper a *quantum state* is a positive functional on the C*-algebra of quantum observables. We also adopt the terminology *normalized quantum state* and *complex quantum state* to denote a normalized positive functional and a complex functional respectively.
The algebra of bosonic quantum observables $\mathfrak{B}_h$ should embed the h-dependent Weyl C*-algebra $\mathbb{W}_h(X, \varsigma)$ for some real symplectic space[1] $(X, \varsigma) \in \mathbf{Symp}_{\mathbb{R}}$, and should thus depend on the semiclassical parameter h. The Weyl C*-algebra $\mathbb{W}_h(X, \varsigma)$ is defined as the smallest C*-algebra containing the Weyl operators $\Big\{W_h(x), x \in X\Big\}$, *i.e.* the elements indexed by X satisfying the following three properties:

$$W_h(x) \neq 0 \qquad (\forall x \in X) \tag{i}$$

$$W_h(-x) = W_h(x)^* \qquad (\forall x \in X) \tag{ii}$$

$$W_h(x)W_h(y) = e^{-ih\varsigma(x,y)}W_h(x+y) \qquad (\forall x, y \in X) \tag{iii}$$

Let us emphasize that the dependence on the semiclassical parameter is given by the phase $e^{-ih\varsigma(x,y)}$ in Eq. (iii), and that it physically corresponds to the fact that the commutator between the canonical observables is of order h. Hence our analysis will apply to any such situation, and h may be in turn interpreted as a scale parameter, as an analogous of Planck's constant, or as a quantity proportional to

[1]Throughout the paper, we write categories in boldface. Let $\mathbf{C}$ be a category, by a slight abuse of notation we write $C \in \mathbf{C}$ for an object C of the category (in some cases, to avoid confusion, we explicitly write $C \in \mathrm{Obj}(\mathbf{C})$). Morphism are denoted by $\mathfrak{c} \in \mathrm{Morph}(\mathbf{C})$.

the inverse of the number of particles; for convenience, however, we always refer to the regime $h \to 0$ as the semiclassical regime. We are interested in characterizing the semiclassical behavior of families of quantum states[2] $(\omega_h)_{h\in(0,1)} \subset (\mathfrak{B}_h)'_+$, with $\mathbb{W}_h(X,\varsigma) \overset{\mathrm{w}_X}{\hookrightarrow} \mathfrak{B}_h$. In order to converge to a cylindrical Wigner measure, a family of quantum states should at least satisfy two properties: being uniformly bounded in norm with respect to h, and being a family of *regular* quantum states. A state $\omega_h \in (\mathfrak{B}_h)'_+$ is regular iff for any $x \in X$, the $\mathbb{R}$-action

$$\mathbb{R} \ni \lambda \mapsto \omega_h\big(W_h(\lambda x)\big) \in \mathbb{C}$$

is a *continuous map*. Therefore, we make the following definition.

Definition 1.1 (Semiclassical quantum states). *A (family of) quantum state(s)* $(\omega_h)_{h\in I\subseteq(0,1)} \subset (\mathfrak{B}_h)'_+$ *is* semiclassical *iff zero is adherent to* I*, and*

- $\sup_{h\in I} \omega_h\big(W_h(0)\big) < \infty$,
- ω_h *is* regular $(\forall h \in I)$.

Remark 1.2. All the definitions and results given in § 1.2 to 1.7 could be reproduced, *mutatis mutandis*, substituting Weyl C*-algebras with the corresponding *Resolvent algebras* [26], and the Fourier transform of a cylindrical measure with the Stieltjes transform. The resolvent algebra may be more convenient than the Weyl C*-algebra to study dynamical interacting theories; however since there is a (purity preserving) bijection from the regular states of one algebra to the regular states of the other [26], for semiclassical purposes it is sufficient to focus on one of the two.

From Definition 1.1, it follows that a very large class of quantum states admits a semiclassical description (*e.g.*, any family of *normalized* regular quantum states is semiclassical by Definition 1.1). In fact, the starting point of our analysis is to prove the following result. Let us denote by X^*_X the algebraic dual X^* of X endowed with the weak $\sigma(X^*, X)$ topology.

Theorem 1.3. *There exists a topology* $\mathfrak{P}$ *on the (disjoint) union of the set of all regular quantum states on* $\mathfrak{B}_h$, $h \in (0,1)$*, and of the set of all cylindrical measures on* X^*_X *such that any semiclassical quantum state* $(\omega_h)_{h\in I}$ *is relatively compact. In addition, every* $\mathfrak{P}$*-cluster point of* ω_h *as* $h \to 0$ *is a cylindrical measure on* X^*_X*, called a* cylindrical Wigner measure *of* ω_h*.*

Remark 1.4. A semiclassical quantum state ω_h has at least one cluster point, but it can have (infinitely) many cluster points, depending on which filter (or generalized sequence) converging to zero is taken as I. Although it is possible to construct semiclassical quantum states with different cluster points, such examples

[2] The range of the semiclassical parameter is chosen to be $(0,1)$, however 1 could be replaced with any strictly positive number. In addition, since states are elements of the continuous dual of the Banach space $\mathfrak{B}_h$, we can without loss of generality suppose that $\mathfrak{B}_h$ is *-isomorphic to $\mathbb{W}_h(X,\varsigma)$.

seem rather artificial and we could not think of a physically relevant example. One can therefore make the simplifying assumption that I is chosen to be a filter on $(0,1)$ converging to zero, and on which ω_h converges.

Remark 1.5. A family of normalized quantum states $(\omega_h)_{h\in I}$ is not semiclassical if the states are not regular on some filter $F \subset I$ (equivalently, on some generalized sequence) converging to 0. In this case it is not, in general, possible to accurately approximate such states with a cylindrical probability (for small h). This is not completely unexpected, since non-regular states are associated with inherently quantum physical phenomena [see, *e.g.*, 1].

1.3 Cylindrical measures

In order to understand Theorem 1.3 and the following results, it is important to review some properties of cylindrical measures, a concept the reader may not be completely familiar with. Let $A \in \mathbf{Set}$ be a set, and let $\mathcal{A} \subseteq \mathbb{R}^A$ be a subset of all the real-valued functions on A. Let us denote by $\hat{C}(A,\mathcal{A})$ the *initial σ-algebra* on A with respect to $\mathcal{A}$, *i.e.*, the smallest σ-algebra of A that makes all functions in $\mathcal{A}$ measurable. Equivalently, if $\mathcal{A}$ is a vector space then $\hat{C}(A,\mathcal{A})$ is the σ-algebra generated by the algebra $C(A,\mathcal{A})$ of cylinders, where a cylinder is defined as

$$C_{\alpha_1,\dots,\alpha_n;B_1,\dots,B_n} = \{a \in A, \forall 1 \leq j \leq n \ , \ \alpha_j(a) \in B_j\} \ ,$$

where $n \in \mathbb{N}_*$, $\alpha_j \in \mathcal{A}$ and $B_j \in \mathrm{Borel}(\mathbb{R})$ for any $1 \leq j \leq n$. Let us recall that if $\mathcal{A}$ is a vector space with a basis Δ of finite cardinality (finite-dimensional space), then $C(A,\mathcal{A}) = \hat{C}(A,\mathcal{A}) = \hat{C}(A,\Delta)$.

Definition 1.6 (Cylindrical measure). *Let $A \in \mathbf{Set}$, $\mathcal{A} \subseteq \mathbb{R}^A$ a vector space. A finitely additive measure M on $C(A,\mathcal{A})$ is a* cylindrical measure *iff it is finite and its restriction to every σ-algebra $C(A,\mathcal{F}) = \hat{C}(A,\mathcal{F})$, with $\mathcal{F} \subset \mathcal{A}$ a finite dimensional subspace, is countably additive. Let us denote by $\mathcal{M}_{\mathrm{cyl}}(A,\mathcal{A})$ the set of all cylindrical measures.*

It is clear that any (Radon) finite measure μ on the measurable space $\big(A,\hat{C}(A,\mathcal{A})\big)$ is a cylindrical measure, but the converse is not true in general (since there may be cylindrical measures that fail to be countably additive on $\hat{C}(A,\mathcal{A})$). If $A \in \mathbf{TVS}_{\mathbb{R}}$ is a real topological vector space and $\mathcal{A} = A'$ is its continuous dual, then the cylindrical measures can be equivalently defined as projective families of Borel measures $(\mu_\Phi, \pi_{\Phi,\Psi})_{\Phi\supset\Psi\in F(A)}$, where $F(A)$ is the set of $\sigma(A,\mathcal{A})$-closed subspaces of A with finite codimension, μ_Φ is a Borel measure on A/Φ, and $\pi_{\Phi,\Psi} : A/\Phi \leftarrow A/\Psi$ is the map obtained from the identity map passing to the quotients, that acts on measures in a projective way pushing them forward: $\mu_\Phi = \pi_{\Phi\Psi\,*}\,\mu_\Psi$. This is due to the fact that the polar of a finite dimensional subspace of A' is isomorphic to a weakly closed subspace of A of finite codimension, and all weakly closed finite codimensional subspaces are polars of some finite dimensional subspace of A'.
For the purpose of semiclassical analysis, cylindrical measures are identified by the vector space $\mathcal{A}$, rather than by the space A on which they act upon, and there is a good extent of arbitrariness in choosing the latter. In other words, there are

(infinitely) many classical effective theories corresponding to a given algebra of bosonic observables $\left(\mathbb{W}_h(X,\varsigma) \overset{w_X}{\hookrightarrow} \mathfrak{B}_h\right)_{h\in(0,1)}$. The arbitrariness is given by the fact that A can be any set such that X is linearly embedded in $\mathbb{R}^A$. In fact, if we denote by $e_X : X \xrightarrow{1\text{-}1} \mathbb{R}^A$ the linear embedding, then we can actually interpret the cluster points of Theorem 1.3 as cylindrical measures on A, with respect to the σ-algebra $\hat{C}\big(A, e_X(X)\big)$. However, all such spaces of cylindrical measures are *isomorphic to one another*, and in turn isomorphic to the space of cylindrical measures on the topological vector space[3] X^*_X. This yields the following result.

Proposition 1.7. *The semiclassical description of bosonic quantum systems, in terms of classical cylindrical probabilities and observables, always exists and it is unique up to isomorphisms.*

The isomorphisms among classical theories are given by Bochner's theorem[4] [see, *e.g.*, 19, 76, 80].

Theorem 1.8 (Bochner). *The Fourier transform is a bijection of the set of cylindrical measures on A, with respect to $\mathcal{A}$, onto the set of functions g from $\mathcal{A}$ to $\mathbb{C}$ such that:*

- $\displaystyle\sum_{j,k\in\mathfrak{F}} \bar{\zeta}_k \zeta_j g(\alpha_j - \alpha_k) \geq 0 \quad \Big(\forall \mathfrak{F} \textit{ finite set, } \forall(\zeta_j)_{j\in\mathfrak{F}} \subset \mathbb{C}\ ,\ \forall(\alpha_j)_{j\in\mathfrak{F}} \subset \mathcal{A}\Big)$
- $g\big|_{\mathcal{F}}$ *continuous* $\quad \Big(\forall \mathcal{F} \subset \mathcal{A}\ ,\ \mathcal{F} \textit{ finite dimensional}\Big)$

1.4 The topologies of semiclassical convergence

Now that we have introduced cylindrical Wigner measures, as cluster points of semiclassical quantum states, it is important to investigate the topology $\mathfrak{P}$ of convergence. As we will see, such topology has a natural physical interpretation, but it is at times not convenient to work with. We therefore introduce another topology for testing semiclassical convergence of quantum states, denoted by $\mathfrak{T}$, that is more convenient for the explicit characterization of Wigner measures.

The topology $\mathfrak{P}$ is a (Hausdorff) weak topology, where the convergence is tested with smooth cylindrical functions. In other words, a generalized sequence

$$\omega_{h_\beta} \xrightarrow[h_\beta\to 0]{\mathfrak{P}} M = (\mu_\Phi)_{\Phi\in F(X^*_X)}$$

[3] Let us remark that the continuous dual of X^*_X is isomorphic to X, thus the set of cylindrical measures on X^*_X is exactly $\mathcal{M}_{\mathrm{cyl}}(X^*, X)$ (where X is identified with $(X^*_X)'$).

[4] Applied to our context, Bochner's theorem is reformulated as the fact that the set of functions of positive type on the phase space X, continuous when restricted to any finite dimensional subspace of X, is isomorphic to the set of cylindrical measures on A with respect to $e_X(X)$ (and such set of functions clearly does not depend on A). This is why all the semiclassical descriptions are isomorphic, and why only X plays a role in their identification.

if and only if all the following expectations converge:

$$\begin{aligned}\lim_{h_\beta\to 0}\omega_{h_\beta}\Bigl(\mathrm{Op}^{h_\beta}_{\frac{1}{2}}(f)\Bigr):=\lim_{h_\beta\to 0}\omega_{h_\beta}\Bigl(\int_{\Phi^\circ}\hat{f}_\Phi(x)W_{h_\beta}(\pi x)\mathrm{d}x\Bigr)\\ =\int_{X^*_X/\Phi}f_\Phi(\xi)\mathrm{d}\mu_\Phi(\xi)=:\int_{X^*_X}f(z)\mathrm{d}M(z);\end{aligned} \tag{2}$$

where $\Phi\in F(X^*_X)$, $\Phi^\circ\subset X$ is its finite dimensional polar[5], $f:X^*_X\to\mathbb{C}$ is a smooth cylindrical function[6] with base function $f_\Phi\in C^\infty_0(X^*_X/\Phi)$, and $\mathrm{Op}^{h_\beta}_{\frac{1}{2}}(f)$ its Weyl quantization.

From the physical standpoint, the average of a cylindrical function with respect to a cylindrical Wigner measure is the number that approximates semiclassically the expectation on the quantum state of the quantization of the aforementioned function. This is a good starting point, in order to have a useful effective theory with some predictive power. It is, however, not easy to characterize the cylindrical Wigner measure explicitly using $\mathfrak{P}$-convergence. Ideally, one would like to test with the canonical quantum observables to characterize the measure. A convenient way to do that is using Weyl operators. The generating functional of a quantum state $\omega_h\in(\mathfrak{B}_h)'_+$ is defined as the numerical function characterizing the expectation of Weyl operators on the state:

$$X\ni x\mapsto \mathcal{G}_{\omega_h}(x)=\omega_h\bigl(W_h(x)\bigr)\in\mathbb{C}\;. \tag{3}$$

The map $\omega_h\mapsto\mathcal{G}_{\omega_h}$ is the *noncommutative Fourier transform*. The name is due to the fact that for *regular states* a noncommutative version of Bochner's theorem holds [77].

THEOREM 1.9 (Noncommutative Bochner). *The noncommutative Fourier transform is a bijection of the set of regular quantum states on* $\mathbb{W}_h(X,\varsigma)$ *onto the set of functions* $\mathcal{G}_h$ *from* X *to* $\mathbb{C}$ *such that:*

- $\displaystyle\sum_{j,k\in\mathfrak{F}}\bar{\zeta}_k\zeta_j\mathcal{G}_h(x_j-x_k)e^{ih\varsigma(x_j\,,\,x_k)}\geq 0$ $\Bigl(\forall\mathfrak{F}$ *finite set,* $\forall(\zeta_j)_{j\in\mathfrak{F}}\subset\mathbb{C}\;,$ $\forall(x_j)_{j\in\mathfrak{F}}\subset X\Bigr)$
- $\mathcal{G}_h\bigr|_Y$ *continuous* $\Bigl(\forall Y\subset X\;,\;Y$ *finite dimensional*$\Bigr)$

Noncommutative and commutative Fourier transforms provide therefore another way to treat quantum states and cylindrical Wigner measures on the same grounds, as complex-valued functions on X that are continuous on any finite-dimensional subspace. The (Hausdorff) topology $\mathfrak{T}$ is thus defined to be the preimage of the

[5] The polar of a closed subspace of X^*_X with finite codimension is a finite-dimensional subspace of X that is isomorphic to the dual of X^*_X/Φ [see, *e.g.*, 21].

[6] A function $f:X^*_X\to\mathbb{C}$ is a *smooth cylindrical function* iff there exists a $\Phi\in F(X^*_X)$ (the cylinder base) and a smooth function $f_\Phi\in C^\infty_0(X^*_X/\Phi)$ such that for any $z\in X^*_X$, $f(z)=f_\Phi(\pi_\Phi z)$, where $\pi_\Phi:X^*_X\to X^*_X/\Phi$ is the canonical projection. In other words, the function f is determined only by finitely many degrees of freedom.

topology of pointwise convergence on the complex-valued functions on X. In other words,

$$\omega_{h_\beta} \xrightarrow[h_\beta\to 0]{\mathfrak{T}} M = (\mu_\Phi)_{\Phi\in F(X_X^*)} \tag{4}$$

if and only if the corresponding generating functional $\mathcal{G}_{\omega_{h_\beta}}$ converges pointwise to $\hat{M}$ as $h_\beta \to 0$. This convergence provides a useful way to characterize the Wigner measure, since the generating functional can be often explicitly computed, and thus its limit as well, and the Fourier transform characterizes a cylindrical measure uniquely.

The topologies $\mathfrak{P}$ and $\mathfrak{T}$ are not comparable, so it may happen that a semiclassical quantum state converges to a cylindrical measure in $\mathfrak{T}$, but not in $\mathfrak{P}$, or that it converges to two different measures in the two topologies. However, the topology $\mathfrak{P}$ is the physically relevant one, since it guarantees the convergence of the expectation of quantum (cylindrical) observables. Therefore, we are only interested in semiclassical quantum states that converge either only in $\mathfrak{P}$, or in both $\mathfrak{P}$ and $\mathfrak{T}$ *to the same limit*. We therefore make use of the join topology $\mathfrak{P}\vee\mathfrak{T}$, *i.e.*, the coarsest topology that is finer than both $\mathfrak{P}$ and $\mathfrak{T}$. Not all semiclassical quantum states have limit in the $\mathfrak{P}\vee\mathfrak{T}$ topology, but statements can be formulated that are equivalent to such convergence; they are presented in the theorem below. Let us denote by $\psi_{\varepsilon,\Phi}\in C_0^\infty(X_X^*/\Phi)$ an approximate identity.

Theorem 1.10. *Let ω_h be a semiclassical quantum state on $\mathfrak{B}_h$ such that*

$$\omega_h \xrightarrow[h\to 0]{\mathfrak{P}} M = (\mu_\Phi)_{\Phi\in F(X_X^*)} \;.$$

Then the following statements are equivalent:

$$\omega_h \xrightarrow[h\to 0]{\mathfrak{P}\vee\mathfrak{T}} M \;; \tag{i}$$

$$\lim_{\varepsilon\to 0}\lim_{h\to 0}\omega_h\bigl(W_h(x)-\mathrm{Op}^h_{\frac12}(^\exists\psi_{\varepsilon,\Phi}\,e^{2ix(\cdot)})\bigr)=0 \quad \bigl(\forall\Phi\in F(X_X^*)\,,\forall x\in\Phi^\circ\bigr)\;; \tag{ii}$$

$$\lim_{\varepsilon\to 0}\lim_{h\to 0}\omega_h\bigl(1-\mathrm{Op}^h_{\frac12}(^\exists\psi_{\varepsilon,\Phi})\bigr)=0 \quad \bigl(\forall\Phi\in F(X_X^*)\bigr)\;; \tag{iii}$$

$$\lim_{h\to 0}\omega_h\bigl(W_h(0)\bigr)=M(X_X^*) \;. \tag{iv}$$

We call a $\mathfrak{P}$-convergent semiclassical quantum state that satisfies Eqs. (i) to (iv) a *semiclassical quantum state with no loss of mass.*

1.5 The set of all cylindrical Wigner measures

It is important to characterize the set of all possible semiclassical configurations of a given system. By Theorem 1.3 and Proposition 1.7, we know that all semiclassical configurations can be equivalently described as cylindrical measures acting on measurable or topological vector spaces. With cylindrical measures, however, it is only possible to integrate cylindrical functions. On the other hand, cylindrical measures can also be seen as Borel Radon measures, but usually on a "big" topological space. If we take the prototypical example of phase space in nonrelativistic quantum field theory, $\bigl(\mathbb{FF}L^2(\mathbb{R}^d), \mathrm{Im}\langle\,\cdot\,,\,\cdot\,\rangle_2\bigr)$[7], the cylindrical measures associated to it are Radon measures only when acting on the weak Hausdorff completion of $L^2(\mathbb{R}^d)$ (*i.e.*, on the completion of L^2 endowed with the $\sigma(L^2, L^2)$ topology). In addition, it is well known that there are true cylindrical measures on L^2, such as the Gaussian measure, that are concentrated outside of L^2 when considered as Radon measures [see, *e.g.*, the construction of abstract Wiener spaces, 52]. One may therefore hope that the set of cylindrical Wigner measures is contained strictly in the set of cylindrical measures, and that it coincides with the space of Radon measures on some topological vector space with nice properties. However, *this is not the case*: every cylindrical measure is the Wigner measure of at least one semiclassical quantum state.

Theorem 1.11. *The set of all cluster points, in the $\mathfrak{P}\vee\mathfrak{T}$ topology, of semiclassical quantum states on $(\mathfrak{B}_h)_{h\in(0,1)}$ coincides with the cone $\mathcal{M}_{\mathrm{cyl}}(X^*, X)$ of all cylindrical measures on X^*_X (or on any other set A for which $X \overset{e_X}{\hookrightarrow} \mathbb{R}^A$).*

This result is, in our opinion, quite interesting. It shows that cylindrical measures are indeed the states that emerge from a bosonic field theory in its classical approximation. This perhaps suggests that a classical field theory well-suited for quantization should either take into account such cylindrical structure, or that it should be set in a topological space where all cylindrical Wigner measures are Radon measures.

1.6 Maps, convolutions, products on cylindrical Wigner measures

Now that we know what is the classical counterpart of a regular (bosonic) state, we would like to characterize which structures and properties of quantum states are inherited by the corresponding classical states. First of all, let us consider the action of central group-homomorphisms between Heisenberg groups, acting on the corresponding Weyl C*-algebras as *-homomorphisms mapping Weyl operators into Weyl operators. A map of such type, from the Heisenberg group $\mathbb{H}(X,\varsigma)$ to $\mathbb{H}(Y,\tau)$, is induced by a *symplectic map* $s : X \to Y$, $\tau\bigl(s(x), s(x')\bigr) = \varsigma(x,x')$ $(\forall x, x' \in X)$. In turn, s induces the desired *-homomorphism between Weyl C*-algebras preserving Weyl operators:

$$\mathfrak{s}_h : \underset{W_h(x)}{\mathbb{W}_h(X,\varsigma)} \underset{\longmapsto}{\longrightarrow} \underset{W_h(s(x))}{\mathbb{W}_h(Y,\tau)}\ ; \tag{5}$$

that can be extended to a *-homomorphism from $\mathfrak{B}_h$ to $\mathfrak{C}_h$, whenever $\mathbb{W}_h(X,\varsigma) \overset{\mathrm{w}_X}{\hookrightarrow} \mathfrak{B}_h$ and $\mathbb{W}_h(Y,\tau) \overset{\mathrm{w}_Y}{\hookrightarrow} \mathfrak{C}_h$. As we discuss in § 1.7, these are the natural morphisms among quantum bosonic theories, induced by the (Segal) quantization functor. The map $\mathfrak{s}_h$ of quantum observables induces, by duality, a *positivity preserving continuous linear* map on quantum states, the transposed map:

$${}^{\mathrm{t}}\mathfrak{s}_h : (\mathfrak{C}_h)'_+ \longrightarrow (\mathfrak{B}_h)'_+\ .$$

[7] $\mathbb{FF}$ denotes the forgetful functor from complex to real vector spaces. In other words, $\mathbb{FF}L^2$ is the vector space that has the same elements as L^2, but only multiplication by real scalars is allowed (therefore its basis is "doubled"); $\mathbb{FF}L^2$ is a real separable Hilbert space with scalar product $\mathrm{Re}\langle\,\cdot\,,\,\cdot\,\rangle_2$, and a symplectic space with symplectic form $\mathrm{Im}\langle\,\cdot\,,\,\cdot\,\rangle_2$.

On the other hand, at the classical level, the map $s : X \to Y$ induces by duality a *continuous linear* map between Y^*_Y and X^*_X:

$$^{\mathrm{t}}s : Y^*_Y \longrightarrow X^*_X \ ,$$

that acts on cylindrical measures pushing them forward[8]: given a cylindrical measure M_Y on Y^*_Y, then $M_X := {}^{\mathrm{t}}s_{\,*}\, M_Y$ is a cylindrical measure on X^*_X. It is therefore natural to ask whether ${}^{\mathrm{t}}\mathfrak{s}_h$ converges in some sense to ${}^{\mathrm{t}}s$. The answer is positive when the quantum map acts on semiclassical quantum states converging in the $\mathfrak{P} \vee \mathfrak{T}$ topology:

$$\varpi_{h_\beta} \underset{h_\beta \to 0}{\overset{\mathfrak{P}\vee\mathfrak{T}}{\longrightarrow}} M \quad \Longrightarrow \quad {}^{\mathrm{t}}\mathfrak{s}_{h_\beta}\varpi_{h_\beta} \underset{h_\beta \to 0}{\overset{\mathfrak{P}\vee\mathfrak{T}}{\longrightarrow}} {}^{\mathrm{t}}s_{\,*}\, M \quad \bigl(\forall \varpi_{h_\beta} \in (\mathfrak{C}_{h_\beta})'_+ \text{ semiclassical quantum state}\bigr). \tag{6}$$

As it will be outlined in § 2.2, such result is important to study the semiclassical behavior of symmetry transformations of relativistic bosonic fields.
The product of (cylindrical) measures is related to statistical independence of classical systems. A system of classical bosonic fields (or particles) composed of two subsystems can be modeled by a topological vector space $W = V_1 \times V_2$ that is product of two spaces $V_1, V_2 \in \mathbf{TVS}_{\mathbb{R}}$, each one describing a subsystem. The state of the system is given in statistical mechanics by a (cylindrical) measure M on W. If such measure can be written as the (tensor) product[9] $M = M_1 \otimes M_2$ of a measure on V_1 and one on V_2, then the two subsystems are independent. This is due to the fact that in such case an event in one of the two subsystems would not affect the outcome of events in the other. At the quantum level, a bosonic composite system is described by the tensor product of two C*-algebras $\mathfrak{B}_h \otimes_\alpha \mathfrak{C}_h$, where the index α stands for a suitable choice of cross norm for the product algebra, and $\mathbb{W}_h(X,\varsigma) \overset{\mathrm{w}_X}{\hookrightarrow} \mathfrak{B}_h$, $\mathbb{W}_h(Y,\tau) \overset{\mathrm{w}_Y}{\hookrightarrow} \mathfrak{C}_h$ (from which it follows that $\mathbb{W}_h(X \oplus Y, \varsigma \oplus \tau) \overset{\exists \mathrm{w}_{XY}}{\hookrightarrow} \mathfrak{B}_h \otimes_\alpha \mathfrak{C}_h$). If a quantum state ξ_h on $\mathfrak{B}_h \otimes_\alpha \mathfrak{C}_h$ is of the form $\xi_h = \omega_h \otimes \varpi_h$, with $\omega_h \in (\mathfrak{B}_h)'_+$ and $\varpi_h \in (\mathfrak{C}_h)'_+$, then the two subsystems are statistically independent. The lack of statistical independence in quantum systems is often called quantum entanglement. The following proposition is a well-known physical fact about entanglement, that we can rigorously prove in our framework.

Proposition 1.12. *Entanglement can at most be destroyed by the classical limit.*

[8] The push-forward of a cylindrical measure $M = (\mu_\Phi)_{\Phi \in F(V)}$ by a weakly continuous linear function $u : V \to W$ on a topological vector space V is defined as follows. For any $\Xi \in F(W)$, let $u^{-1}(\Xi) \in F(V)$; and let $u_\Xi : V/u^{-1}(\Xi) \to W/\Xi$ be the linear application obtained from u passing to the quotients. Then $u_{\,*}\, M = (u_{\Xi\,*}\, \mu_{u^{-1}(\Xi)})_{\Xi \in F(W)}$.

[9] Let V_1 and V_2 be topological vector spaces. The (tensor) product $M_1 \otimes M_2$ of two cylindrical measures M_1 on V_1 and M_2 on V_2 is the cylindrical measure on $V_1 \times V_2$ defined as follows. $M_1 \otimes M_2 = \bigl(\mu_{V_1/\Phi_1} \otimes \mu_{V_2/\Phi_2}\bigr)_{\Phi_1 \in F(V_1), \Phi_2 \in F(V_2)}$, where $\mu_{V_1/\Phi_1} \otimes \mu_{V_2/\Phi_2}$ is the usual product measure on the finite dimensional space $(V_1/\Phi_1) \times (V_2/\Phi_2) \cong V_1 \times V_2/\Phi_1 \times \Phi_2$. The family $\bigl(\mu_{V_1/\Phi_1} \otimes \mu_{V_2/\Phi_2}\bigr)_{\Phi_1 \in F(V_1), \Phi_2 \in F(V_2)}$ is projective, and the set of subspaces of the form $\Phi_1 \times \Phi_2$ is cofinal with the set of all weakly closed subspaces of finite codimension. Therefore $M_1 \otimes M_2$ defines a cylindrical measure on $V_1 \times V_2$.

The proof, whose details are given in § 2.2.3, consists of two parts. The first is to construct an entangled semiclassical quantum state whose limit is a product cylindrical measure, and thus statistically independent. This can be easily done, *e.g.*, using an entangled "perturbation" of order h of a non-entangled state. The second part consists in proving that all possible cylindrical Wigner measures of a semiclassical quantum state $\omega_h \otimes \varpi_h$ are of the form $M_X \otimes M_Y$, with M_X a cylindrical measure on X^*_X and M_Y a cylindrical measure on Y^*_Y.
Another useful notion in classical probability is that of convolution of (cylindrical) measures[10]. Signed or complex cylindrical measures are an algebra, if we use the convolution as product. The Fourier transform on cylindrical measures behaves as the usual Fourier transform with respect to convolution (and is thus an algebra homomorphism from the algebra of signed/complex cylindrical measures to the algebra of complex-valued functions):

$$\big(\widehat{M_1 \circledast M_2}\big)(\cdot) = \hat{M}_1(\cdot)\hat{M}_2(\cdot) \; .$$

This suggests an analogous definition of quantum (noncommutative) convolution, that would make the complex quantum states an algebra (and the noncommutative Fourier transform an algebra homomorphism). Let $\omega_h, \varpi_h \in \mathbb{W}_h(X,\varsigma)'$, and define the action $\omega_h \star \varpi_h$ on Weyl operators by

$$(\omega_h \star \varpi_h)\big(W_h(x)\big) = \omega_h\big(W_h(x)\big)\varpi_h\big(W_h(x)\big) \; .$$

This action is extended by linearity to finite combinations

$$(\omega_h \star \varpi_h)\Big(\sum_{j\in\mathfrak{F}} \zeta_j W_h(x_j)\Big) = \sum_{j\in\mathfrak{F}} \zeta_j (\omega_h \star \varpi_h)\big(W_h(x_j)\big)$$

and hence to a (complex) state on $\mathbb{W}_h(X,\varsigma)$. Semiclassically, we have the following result:

$$\omega_{h_\beta} \xrightarrow[h_\beta\to 0]{\mathfrak{P}\vee\mathfrak{T}} M_1 \; , \; \varpi_{h_\beta} \xrightarrow[h_\beta\to 0]{\mathfrak{P}\vee\mathfrak{T}} M_2 \quad \Longrightarrow \quad \omega_{h_\beta} \star \varpi_{h_\beta} \xrightarrow[h_\beta\to 0]{\mathfrak{P}\vee\mathfrak{T}} M_1 \circledast M_2 \; .$$

1.7 Categorical interpretation

The framework of cylindrical Wigner measures admits an elegant reformulation in the language of category theory. Such reformulation is well-adapted to algebraic formulations of relativistic quantum field theory, such as the Locally Covariant Quantum Field Theory, and thus we present it here. Some results of semiclassical analysis in LCQFT are given in § 2.
Let $\mathbf{Symp}_{\mathbb{R}}$ be the category of real symplectic spaces, *i.e.*, the category with objects the real symplectic vector spaces, and morphisms the linear symplectic maps; and let $\mathbf{C^*alg}$ be the category of C*-algebras, *i.e.*, the category with C*-algebras as objects and *-homomorphisms as morphisms.

[10]The convolution $M_1 \circledast M_2$ of two cylindrical measures M_1 and M_2 on $V \in \mathbf{TVS}_{\mathbb{R}}$, is again a cylindrical measure on V, defined as follows. It is the pushforward of the product $M_1 \otimes M_2$ by the addition map $\tilde{+} : V\times V \to V$ defined by $(v,w) \mapsto v+w$. In other words, $M_1 \circledast M_2 = \tilde{+}_* M_1 \otimes M_2$.

Definition 1.13 (Segal bosonic quantization). *Let $h \in (0,1)$. The functor*

$$\mathbb{W}_h : \mathbf{Symp}_{\mathbb{R}} \to \mathbf{C^*alg}$$

that associates:

- *to each symplectic space (X, ς) the Weyl C*-algebra $\mathbb{W}_h(X, \varsigma)$,*
- *to each linear symplectomorphism $s : (X, \varsigma) \to (Y, \tau)$ the associated Weyl-operator-preserving *-homomorphism $\mathbb{W}_h(s) := \mathfrak{s}_h$, defined by Eq. (5) ,*

is called Segal bosonic quantization.

The Segal quantization associates to each classical phase space the corresponding algebra of observables generated by the canonical commutation relations, and to linear maps between symplectic spaces the corresponding *-homomorphism of observables, that maps Weyl operators into Weyl operators[11]. Now, let **BanCone** be the category with objects the pointed and generating cones in real Banach spaces, and with morphisms the linear continuous cone preserving maps between underlying Banach spaces. Hence the contravariant duality functor $\mathbb{D}_+ : \mathbf{C^*alg} \to \mathbf{BanCone}$ associates to each C*-algebra the corresponding cone of quantum states (positive linear functionals), and to *-homomorphisms their transposed maps. It follows that

$$\mathbb{S}_h = \mathbb{D}_+ \circ \mathbb{W}_h : \mathbf{Symp}_{\mathbb{R}} \to \mathbf{BanCone}$$

is the (contravariant) *functor of bosonic quantum states*, associating to each classical phase space the corresponding quantum states in the algebra of canonical commutation relations.
The classical limit of the functor of bosonic quantum states is the *functor of cylindrical Wigner measures*, defined as follows. Let **CylM** be the category that has as objects the sets of cylindrical measures, and as morphisms the compatible maps[12].

Definition 1.14 (Functor of cylindrical Wigner measures). *A contravariant functor*

$$\mathbb{S}_0 : \mathbf{Symp}_{\mathbb{R}} \to \mathbf{CylM}$$

is a functor of cylindrical Wigner measures *iff* [13]*:*

$$\mathbb{S}_0(X, \varsigma) = \mathcal{M}_{\mathrm{cyl}}\big({}^{\exists}A, {}^{\exists}e_X(X)\big)\ ,\ X \overset{e_X}{\hookrightarrow} \mathbb{R}^A \qquad \Big(\forall (X, \varsigma) \in \mathbf{Symp}_{\mathbb{R}}\Big) \tag{i}$$

$$e_Y\big(s(x)\big) = e_X(x) \circ \mathbb{S}_0(s) \qquad \Big(\forall x \in X\ ,\ \forall s \in \mathrm{Morph}(\mathbf{Symp}_{\mathbb{R}})\ ,\ s : (X, \varsigma) \to (Y, \tau)\Big) \tag{ii}$$

[11] The Segal functor should not be confused with the second quantization functor [70], that could be considered as a special case of the former.

[12] A map $f : A \to B$, $A, B \in \mathbf{Set}$ is compatible with the sets of cylindrical measures $\mathcal{M}_{\mathrm{cyl}}(A, \mathcal{A})$ and $\mathcal{M}_{\mathrm{cyl}}(B, \mathcal{B})$, $\mathcal{A} \subseteq \mathbb{R}^A, \mathcal{B} \subseteq \mathbb{R}^B$, iff for any $\beta \in \mathcal{B}$, $\beta \circ f \in \mathcal{A}$. In that case, it is possible to define the pushforward $f_* M \in \mathcal{M}_{\mathrm{cyl}}(B, \mathcal{B})$ for any $M \in \mathcal{M}_{\mathrm{cyl}}(A, \mathcal{A})$, and therefore f induces a homomorphism between sets of cylindrical measures (again denoted by f).

[13] With a slight abuse of notation, we use the quantifier $\forall$ for categories that are not small. The corresponding statements should be intended in the appropriate sense.

Definition 1.14 takes into account the fact that it is possible to equivalently describe Wigner measures as cylindrical measures in different spaces, as discussed in § 1.3. However, the semiclassical description always exists and it is essentially unique, as stated in Proposition 1.7. In the categorical setting, such statement takes the following form.

PROPOSITION 1.15. *The functor of cylindrical Wigner measures always exists. If $\mathbb{S}_0$ and $\mathbb{T}_0$ are two functors of cylindrical Wigner measures, then there exists a natural isomorphism $\nu : \mathbb{S}_0 \xrightarrow{\text{iso}} \mathbb{T}_0$.*

Proof. The existence is guaranteed by the fact that it is possible to choose, for all $(X,\varsigma) \in \mathbf{Symp}_{\mathbb{R}}$, $A = X^*_X$, identifying X with $(X^*_X)'$. This choice satisfies (i). The transformed symplectic morphisms are also easily defined as $\mathbb{S}_0(s) = {}^{\mathrm{t}}s$, satisfying (ii). Uniqueness is proved as follows. Let $\mathbb{S}_0$ and $\mathbb{T}_0$ be two functors of cylindrical Wigner measures. Let us show that there is a natural isomorphism $\nu : \mathbb{S}_0 \to \mathbb{T}_0$. Let $(X,\varsigma) \in \mathbf{Symp}_{\mathbb{R}}$. Define the component $\nu_X : \mathbb{S}_0(X,\varsigma) \to \mathbb{T}_0(X,\varsigma)$ in the following way. By definition of classical functor, $\mathbb{S}_0(X,\varsigma) = \mathcal{M}_{\mathrm{cyl}}\bigl(A, e_X(X)\bigr)$ and $\mathbb{T}_0(X,\varsigma) = \mathcal{M}_{\mathrm{cyl}}\bigl(B, f_X(X)\bigr)$ with $X \overset{e_X}{\hookrightarrow} \mathbb{R}^A$ and $X \overset{f_X}{\hookrightarrow} \mathbb{R}^B$. By Bochner's theorem, Theorem 1.8, the Fourier transform induces an isomorphism

$$\nu_X : \mathcal{M}_{\mathrm{cyl}}\bigl(A, e_X(X)\bigr) \to \mathcal{M}_{\mathrm{cyl}}\bigl(B, f_X(X)\bigr) \; .$$

The commutativity of the diagram

$$\begin{array}{ccc} \mathbb{S}_0(X,\varsigma) & \xleftarrow{\mathbb{S}_0(s)} & \mathbb{S}_0(Y,\tau) \\ {\scriptstyle \nu_X}\big\downarrow & & \big\downarrow{\scriptstyle \nu_Y} \\ \mathbb{T}_0(X,\varsigma) & \xleftarrow{\mathbb{T}_0(s)} & \mathbb{T}_0(Y,\tau) \end{array}$$

follows since by (ii) of Definition 1.14, $e_Y\bigl(s(x)\bigr) = e_X(x) \circ \mathbb{S}_0(s)$ and $f_Y\bigl(s(x)\bigr) = f_X(x) \circ \mathbb{T}_0(s)$. ⊣

All the results of § 1.2 to 1.6 can conveniently be reinterpreted as a "semiclassical convergence" of $\mathbb{S}_h$ to $\mathbb{S}_0$, as $h \to 0$.

THEOREM-DEFINITION 1.16.

$$\mathbb{S}_h \underset{h\to 0}{\rightarrowtail} \mathbb{S}_0 \; .$$

Proof. The semiclassical convergence of functors $\underset{h\to 0}{\rightarrowtail}$ holds in the following sense in any *small* subcategory $\mathbf{S} \subset \mathbf{Symp}_{\mathbb{R}}$:

- *Convergence within objects* (Theorem 1.3). There exists a topology for the element-wise semiclassical convergence of objects:

$$\mathfrak{P}_{\mathbf{S}} := \prod_{(X,\varsigma)\in\mathbf{S}} \mathfrak{P}_X \; ,$$

where $\mathfrak{P}_X$ is the topology of semiclassical convergence defined in § 1.4. In other words, any semiclassical family of maps $\Big(\mathbf{S} \ni (X,\varsigma) \mapsto \omega_{h,X} \in \mathbb{S}_h(X,\varsigma)\Big)_{h\in I\subseteq(0,1)}$, with zero adherent to I and $\omega_{h,X}$ semiclassical, is relatively compact in the $\mathfrak{P}_{\mathbf{S}}$ topology. Therefore the element-wise convergence of semiclassical objects is given by

$$\Big((X,\varsigma) \mapsto \omega_{h_\beta,X}\Big) \xrightarrow[h_\beta\to 0]{\mathfrak{P}_{\mathbf{S}}} \Big((X,\varsigma) \mapsto M_X\Big) ,$$

with $M_X \in \mathbb{S}_0(X,\varsigma) \cong \mathcal{M}_{\mathrm{cyl}}(X^*,X)$ for any $(X,\varsigma) \in \mathbf{S}$.

- *Convergence of morphisms* (Eq. (6)). There exists a topology $\mathfrak{T}_{\mathbf{S}}$ on objects for the pointwise semiclassical convergence of morphisms:

$$\mathfrak{T}_{\mathbf{S}} := \prod_{(X,\varsigma)\in\mathbf{S}} \mathfrak{T}_X ,$$

where $\mathfrak{T}_X$ is the other topology of semiclassical convergence defined in § 1.4. In fact, let $\Big(\mathrm{Morph}(\mathbf{S}) \ni \big(s:(X,\varsigma)\to(Y,\tau)\big) \longmapsto \mathbb{S}_h(s) = {}^{\mathrm{t}}\mathfrak{s}_h \in \mathcal{L}\big(\mathbb{S}_h(Y,\tau),\mathbb{S}_h(X,\varsigma)\big)\Big)_{h\in(0,1)}$, be a family of maps of morphisms. Then on any semiclassical family of $(\mathfrak{P}_{\mathbf{S}} \vee \mathfrak{T}_{\mathbf{S}})$-convergent maps

$$\Big((X,\varsigma) \mapsto \omega_{h_\beta,X}\Big) \xrightarrow[h_\beta\to 0]{\mathfrak{P}_{\mathbf{S}}\vee\mathfrak{T}_{\mathbf{S}}} \Big((X,\varsigma) \mapsto M_X\Big) ,$$

we have the pointwise convergence $\Big(s \mapsto \mathbb{S}_{h_\beta}(s)\Big) \xrightarrow[h_\beta\to 0]{} \Big(s \mapsto \mathbb{S}_0(s)\Big)$:

$$\Big((Y,\tau) \mapsto \mathbb{S}_{h_\beta}(s)\omega_{h_\beta,Y}\Big) \xrightarrow[h_\beta\to 0]{\mathfrak{P}_{\mathbf{S}}\vee\mathfrak{T}_{\mathbf{S}}} \Big((Y,\tau) \mapsto \mathbb{S}_0(s)_* M_Y\Big) .$$

- $\mathrm{Ran}\big(\mathbb{S}_h\big|_{\mathbf{S}}\big) \underset{h\to 0}{\rightarrowtail} \mathrm{Ran}\big(\mathbb{S}_0\big|_{\mathbf{S}}\big)$ (Theorem 1.11). We define the range of a functor $\mathbb{F}:\mathbf{A}\to\mathbf{B}$, $\mathbf{A}$ small, as the set[14]

$$\mathrm{Ran}(\mathbb{F}) = \bigcup\Big\{\mathbb{F}(a)\,,\, a\in\mathbf{A}\Big\} .$$

The notation $\mathrm{Ran}\big(\mathbb{S}_h\big|_{\mathbf{S}}\big) \underset{h\to 0}{\rightarrowtail} \mathrm{Ran}\big(\mathbb{S}_0\big|_{\mathbf{S}}\big)$ should be interpreted as the fact that every element $M_{\exists X} \in \mathrm{Ran}\big(\mathbb{S}_0\big|_{\mathbf{S}}\big)$ is the limit point (in the $\mathfrak{P}_X \vee \mathfrak{T}_X$ topology) of at least one semiclassical state ${}^{\exists}\omega_{h,M_X} \in \mathrm{Ran}\big(\mathbb{S}_h\big|_{\mathbf{S}}\big)$.

⊣

[14] We use here the following standard set-theoretic notation [see, *e.g.*, 57]: given a set S, we denote by $\bigcup S$ the set

$$\bigcup S = \Big\{t\,,\, t\in{}^{\exists}s\in S\Big\} .$$

Remark 1.17. By means of the $\mathfrak{P}$ topology, we could also formulate a result of "weak semiclassical convergence" of the Segal quantization functor $\mathbb{W}_h$ to a functor of classical observables, at least when restricted to cylindrical observables (and tested on semiclassical quantum states).

2 Physical applications

In this section we outline some applications of the results introduced in § 1, to physical systems described by bosonic field theories. Some applications are new, and some are developed in detail elsewhere; we provide more details for the new ones, and mainly refer to the literature for the others. The section is divided in two main parts, § 2.1 dealing with the semiclassical and mean-field analysis of nonrelativistic (or semirelativistic) systems, and § 2.2 dealing with semiclassical relativistic systems.

2.1 Nonrelativistic and semirelativistic Quantum Field Theories

Nonrelativistic and semirelativistic quantum field theories are the ones either describing nonrelativistic particles (invariant with respect to Galilei symmetry transformations), or describing the interaction of nonrelativistic particles with relativistic force-carrier bosonic fields (invariant with respect to Lorentz symmetry transformations). In the first case, the quantum field theoretic description comes into play when the limit of a large number of particles is considered (mean field description), and if particles can be pumped in or absorbed by the environment. In the second case, the relativistic force-carriers can be created and destroyed by the interaction, and are thus naturally described by quantum fields.

2.1.1 Fock representation

Since most of the nonrelativistic and semirelativistic quantum field theories are studied in the Fock representation, let us briefly recall some well known results about it here, mostly to fix the notation. Let $\mathfrak{H}$ be a separable complex Hilbert space. As briefly recalled in § 1.5, a complex Hilbert space induces a real Hilbert and symplectic space *via* the forgetful functor $\mathbb{FF}$ from complex to real vector spaces. Therefore, $\mathbb{FF}\mathfrak{H}$ is a real vector space with the same elements[15] as $\mathfrak{H}$, complete with respect to the inner product $\mathrm{Re}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{H}}$ and with symplectic form $\mathrm{Im}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{H}}$. Let us denote the associated Weyl C*-algebra by

$$\mathbb{W}_h(\mathfrak{H}) := \mathbb{W}_h(\mathbb{FF}\mathfrak{H}, \mathrm{Im}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{H}}) \; .$$

The *Fock vacuum* ϖ_h on $\mathbb{W}_h(\mathfrak{H})$ is the regular quantum state with generating functional

$$\mathcal{G}_{\varpi_h}(\mathbb{FF}\eta) = e^{-\frac{h}{2}\|\eta\|^2_{\mathfrak{H}}} \qquad (\forall \eta \in \mathfrak{H}) \; .$$

[15] If $\eta \in \mathfrak{H}$, let us denote by $\mathbb{FF}\eta$ the corresponding element on $\mathbb{FF}\mathfrak{H}$.

DEFINITION 2.1 (Fock representation). *The Fock representation of* $\mathbb{W}_h(\mathfrak{H})$ *is the GNS representation* $(\mathscr{H}_{\varpi_h}, \pi_{\varpi_h}, \Omega_h)$ *generated by the Fock vacuum. The Hilbert space* $\mathscr{H}_{\varpi_h} = \Gamma_{\mathrm{s}}(\mathfrak{H})$ *is the* symmetric Fock space*, and the cyclic vector* $\Omega_h \in \Gamma_{\mathrm{s}}(\mathfrak{H})$ *is the* (bosonic) Fock vacuum vector.

The Fock space $\Gamma_{\mathrm{s}}(\mathfrak{H})$ is defined as

$$\Gamma_{\mathrm{s}}(\mathfrak{H}) := \mathbb{C} \oplus \bigoplus_{n\in\mathbb{N}_*} \bigvee_{j=1}^{n} \mathfrak{H} \;,$$

where $\bigvee$ stands for the symmetric tensor product. The vacuum vector Ω_h is the vector

$$\Omega_h = (1, 0, \dots, 0, \dots) \;.$$

The dependence on the semiclassical parameter h is given by the canonical variables of the Fock representation, the so-called *creation and annihilation operators*. For any $\eta, \xi \in \mathfrak{H}$, let $a_h^*(\eta)$ and $a_h(\xi)$ be the creation and annihilation (closed) operators on $\Gamma_{\mathrm{s}}(\mathfrak{H})$ satisfying the commutation relations

$$[a_h^*(\eta), a_h(\xi)] = -h\langle \xi, \eta\rangle_{\mathfrak{H}} \;.$$

They are explicitly defined by the action on Fock space vectors $\psi_h = (\psi_{0,h}, \psi_{1,h}, \dots, \psi_{n,h}, \dots)$ with finitely many non-zero components, and each component of the form

$$\psi_{n,h} = \bigvee_{j=1}^{n} \eta_{j,h}^{(n)} \;. \tag{7}$$

The set of such vectors is a core for every $a_h^*(\eta)$ and $a_h(\eta)$.

$$\begin{aligned} \bigl(a_h^*(\eta)\psi_h\bigr)_n &= \sqrt{hn}\,\eta \vee \psi_{n-1,h} = \sqrt{hn}\,\eta \vee \bigvee_{j=1}^{n-1} \eta_{j,h}^{(n-1)} \;; \\ \bigl(a_h(\eta)\psi_h\bigr)_n &= \sqrt{h(n+1)}\,\langle \eta, \psi_{n+1,h}\rangle_{\mathfrak{H}} = \sqrt{\frac{h}{n+1}} \sum_{k=1}^{n+1} \langle \eta, \eta_{k,h}^{(n+1)}\rangle \bigvee_{j\neq k} \eta_{j,h}^{(n+1)} \;. \end{aligned}$$

Intuitively, the spaces $\bigvee_{j=1}^{n} \mathfrak{H}$ represent the subspaces with n particles (and $\mathbb{C}$ the subspace with no particles). With this interpretation in mind, it is easy to see that by definition the creation operator $a_h^*(\eta)$ creates a particle in the configuration $\eta \in \mathfrak{H}$, and $a_h(\eta)$ annihilates a particle (in the same configuration). The Weyl operators $W_h\bigl(\mathbb{F}\eta\bigr)$, $\eta \in \mathfrak{H}$, are Fock-represented by the following unitary operators on $\Gamma_{\mathrm{s}}(\mathfrak{H})$:

$$\pi_{\varpi_h}\Bigl(W_h\bigl(\mathbb{F}\eta\bigr)\Bigr) = e^{i\,\left(a_h^*(\eta) + a_h(\eta)\right)} \;.$$

The Fock representation is the most used, since it describes free quantum field theories, both from the nonrelativistic and relativistic standpoint. Let us also remark that if $\mathfrak{H} \subset \mathcal{S}'(G)$, for some locally compact abelian group G (*e.g.*, $G = \mathbb{R}^d$), then the Fock representation is also a representation of the Weyl C*-algebra of test functions $\mathbb{W}_h\bigl(\mathbb{F}\mathbb{F}\mathcal{S}(G),\operatorname{Im}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{H}}\bigr)$, where $\mathcal{S}(G)\subset\mathfrak{H}$ is the nuclear space of rapid decrease functions on G. The non-Fock representations of $\mathbb{W}_h\bigl(\mathbb{F}\mathbb{F}\mathcal{S}(G),\operatorname{Im}\langle\,\cdot\,,\,\cdot\,\rangle_{\mathfrak{H}}\bigr)$ play an important role in interacting relativistic field theories, as it is discussed in § 2.2.

2.1.2 Thermodynamic limit of trapped ideal Bose gases

A system of N non-relativistic d-dimensional bosons in a harmonic trap is usually described by the following Hamiltonian on $\bigvee_{j=1}^{N} L^2(\mathbb{R}^d)$:

$$H_{N,V_N}=\sum_{j=1}^{N}(-\Delta_j+\omega_N^2x_j^2)+\frac{1}{N}\sum_{j<k}V_N(x_j-x_k)\;; \tag{8}$$

where $\omega_N\in\mathbb{R}$ is proportional to the frequency of the trap, and V_N is a symmetric two-body interaction potential, with suitable regularity properties yielding the self-adjointness of H_{N,V_N} on a suitable domain. The two body interaction V_N may be, *e.g.*, independent of N (mean field regime), or of the form $V_N(\cdot)=N^2V(N\,\cdot)$ for $d=3$ (Gross-Pitaevskii regime). The Hamiltonian H_{N,V_N} agrees with the restriction to the N-particle sector of a particle-preserving Hamiltonian on the Fock representation[16] $\Gamma_{\mathrm{s}}\bigl(L^2(\mathbb{R}^d)\bigr)$, provided that $h=N^{-1}$:

$$H_{h,V_h}=\mathrm{d}\Gamma_h\bigl(k_{\omega_{h^{-1}}}\bigr)+\mathrm{d}\Gamma_h^{(2)}\bigl(\tfrac{1}{2}V_{h^{-1}}\bigr)\;, \tag{9}$$

where $k_\lambda(\nabla_x,x)=-\Delta_x+\lambda^2x^2$ is a differential self-adjoint operator on $L^2(\mathbb{R}^d)$. In this context, the semiclassical parameter h is therefore interpreted as a quantity proportional to the inverse of the expected number of particles in the system.
Let us consider now a trapped *ideal* nonrelativistic Bose gas in d dimensions. In the Fock representation, the system is therefore described by the Hamiltonian

$$H_{h,0}=\mathrm{d}\Gamma_h(k_{\omega_{h^{-1}}})\;,$$

with $k_{\omega_{h^{-1}}}$ the one-particle harmonic oscillator of frequency $\omega_{h^{-1}}\in\mathbb{R}^+$. The corresponding grand-canonical Gibbs state γ_h at inverse temperature $\beta_h>0$ and

[16] The one-particle and two-particle second quantizations $\mathrm{d}\Gamma_h$ and $\mathrm{d}\Gamma_h^{(2)}$ are maps from the one or two-particle self-adjoint operators on $\mathfrak{H}$ and $\mathfrak{H}\vee\mathfrak{H}$ respectively to the self-adjoint operators on $\Gamma_{\mathrm{s}}(\mathfrak{H})$. They are defined by the action on core vectors $\psi_h=(\psi_{0,h},\psi_{1,h},\dots,\psi_{n,h},\dots)$ with finitely many non-zero components, and each component of the form (7) (with each $\eta_{j,h}$ or $\eta_{j,h}\vee\eta_{j',h}$ in the domain of the one or two-particle self-adjoint operator):

$$\bigl(\mathrm{d}\Gamma_h(K^{(1)})\psi\bigr)_n=h\sum_{k=1}^{n}(K^{(1)}\eta_k)\vee\bigvee_{j\neq k}\eta_j \qquad \Bigl(\forall n\in\mathbb{N}^*\,,\;\bigl(\mathrm{d}\Gamma_h(K^{(1)})\psi\bigr)_0=0\Bigr)$$

$$\bigl(\mathrm{d}\Gamma_h^{(2)}(K^{(2)})\psi\bigr)_n=h^2\sum_{k\neq l=1}^{n}(K^{(2)}\eta_k\vee\eta_l)\vee\bigvee_{j\neq k,l}\eta_j \qquad \Bigl(\forall n\geq 2\,,\;\bigl(\mathrm{d}\Gamma_h^{(2)}(K^{(2)})\psi\bigr)_0=\bigl(\mathrm{d}\Gamma_h^{(2)}(K^{(2)})\psi\bigr)_1=0\Bigr)$$

chemical potential[17] $\mu_h \in \mathbb{R}$ has the following form:

$$\gamma_h(\cdot) = \frac{\mathrm{Tr}_{\Gamma_s}\Big(\,\cdot\, \; e^{-\beta_h H_{h,0} + \mu_h \mathrm{d}\Gamma_h(1)}\Big)}{\mathrm{Tr}_{\Gamma_s}\Big(e^{-\beta_h H_{h,0} + \mu_h \mathrm{d}\Gamma_h(1)}\Big)} \; .$$

It corresponds to the generating functional [see 23, Proposition 5.2.28]

$$\mathcal{G}_{\gamma_h}(\eta) = \exp\biggl(-\tfrac{h}{2}\Bigl\langle \eta \, , \, \Bigl(e^{-\beta_h(k_{\omega_{h^{-1}}} - \mu_h)}\Bigl(1 - e^{-\beta_h(k_{\omega_{h^{-1}}} - \mu_h)}\Bigr)^{-1}\Bigr) \eta \Bigr\rangle_2\biggr) \; . \quad (10)$$

We are interested in the behavior of the grand-canonical Gibbs state in the thermodynamic limit. The thermodynamic limit is usually defined as the limit $N, V \to \infty$ (N expected number of particles, V volume), with $\frac{N}{V} = \rho$ constant. In this case, $N = h^{-1}$, and therefore $N \to \infty$ is equivalent to $h \to 0$. The effective volume occupied by the trapped system depends on the frequency of the trap, and more precisely it is proportional to $\omega_{h^{-1}}^{-d}$. Therefore, to achieve the thermodynamic limit, one should relax the trap in the following way: $\omega_{h^{-1}} = \omega h^{\frac{1}{d}}$, $\omega \in \mathbb{R}^+$, as $h \to 0$ [see, *e.g.*, 71, or any other book on Bose-Einstein condensation]. The thermodynamic limit can therefore be reinterpreted as a semiclassical limit $h \to 0$. Now, let $h \in I \subseteq (0,1)$, with zero adherent to I. Since any state normal with respect to the Fock representation is regular[18], and

$$\sup_{h \in I} \mathcal{G}_{\gamma_h}(0) = 1 \; ,$$

it follows that the grand canonical Gibbs state γ_h is a *semiclassical quantum state*, and thus it has at least one cylindrical Wigner measure associated to it. We can without loss of generality suppose that

$$\gamma_h \xrightarrow[h \to 0]{\mathfrak{P}} G \;\; \text{(otherwise change } I\text{)},$$

with[19] $G \in \mathcal{M}_{\mathrm{cyl}}\big(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)\big)$.

We would like to reinterpret the well-known analysis of condensation for ideal Bose gases using the thermodynamic (classical) state G. Let us consider the orthonormal basis $\Bigl\{\varphi^{(m)}_{\omega_{h^{-1}}}\Bigr\}_{m \in \mathbb{N}^d}$ of $L^2(\mathbb{R}^d)$ given by the eigenvectors of the harmonic oscillator $k_{\omega_{h^{-1}}}$, satisfying

$$k_{\omega_{h^{-1}}} \varphi^{(m)}_{\omega_{h^{-1}}} = \omega_{h^{-1}}(m_1 + \cdots + m_d + 1)\varphi^{(m)}_{\omega_{h^{-1}}} \; ,$$

[17] The chemical potential $\mu_h \in \mathbb{R}$ should be suitably chosen, *i.e.* it should be s.t. $\beta_h\bigl(k_{\omega_{h^{-1}}} - \mu_h\bigr) > 0$ uniformly with respect to $h \in (0,1)$.

[18] A state ω on a C*-algebra is normal with respect to a given representation $\mathscr{H}$ iff $\omega(\cdot) = \mathrm{Tr}_{\mathscr{H}}(\,\cdot\, ^{\exists}\rho)$, with $\rho \in \mathfrak{S}^1_+(\mathscr{H})$ (positive and trace class).

[19] To be precise, $M_{\gamma_h} \in \mathcal{M}_{\mathrm{cyl}}\bigl(\mathbb{RF}L^2(\mathbb{R}^d), \mathbb{RF}L^2(\mathbb{R}^d)\bigr)$ (with the usual implicit identification $\mathbb{RF}L^2(\mathbb{R}^d) \cong \mathbb{RF}L^2(\mathbb{R}^d)'$), however $\mathcal{M}_{\mathrm{cyl}}\bigl(L^2(\mathbb{R}^d), L^2(\mathbb{R}^d)\bigr)$ is the equivalent description on the original complex vector spaces.

where we have used the notation $m = (m_1, \dots, m_d)$. The quantum observable $a_h^*(\varphi^{(m)}_{\omega_{h^{-1}}})a_h(\varphi^{(m)}_{\omega_{h^{-1}}})$, $m \in \mathbb{N}^d$, counts the relative number of particles in the configuration $\varphi^{(m)}_{\omega_{h^{-1}}}$. From Eq. (10), it is not difficult to see that

$$\begin{aligned}\gamma_h\Bigl(a_h^*(\varphi^{(m)}_{\omega_{h^{-1}}})a_h(\varphi^{(m)}_{\omega_{h^{-1}}})\Bigr) &= h\frac{e^{-\beta_h\bigl(\omega_{h^{-1}}(m_1+\dots+m_d+1)-\mu_h\bigr)}}{1-e^{-\beta_h\bigl(\omega_{h^{-1}}(m_1+\dots+m_d+1)-\mu_h\bigr)}}\\ &= \frac{h}{e^{\beta_h\bigl(\omega h^{\frac{1}{d}}(m_1+\dots+m_d+1)-\mu_h\bigr)}-1}\,.\end{aligned} \tag{11}$$

Let us recall that h is interpreted as the inverse of the expected number of particles, *i.e.* it satisfies the consistency condition

$$h^{-1} = \gamma_h\Bigl(\sum_{m\in\mathbb{N}^d} a_1^*(\varphi^{(m)}_{\omega_{h^{-1}}})a_1(\varphi^{(m)}_{\omega_{h^{-1}}})\Bigr)\,.$$

Therefore, the quantity in Eq. (11) is a number between zero and one uniformly in h (as expected).

Now, if $m = (0, \dots, 0)$, Eq. (11) counts the relative number of particles in the ground state. If, in the thermodynamic limit $h \to 0$, this number is bigger than zero, then a macroscopic fraction of particles is in the ground state, and thus the system exhibits condensation. We characterize such macroscopic fraction of particles in the condensate indirectly[20]: we would like to find suitable cylindrical symbols s_m, $m \neq (0, \dots, 0)$, such that

$$\lim_{h\to 0}\gamma_h\Bigl(\sum_{m\neq(0,\dots,0)} a_h^*(\varphi^{(m)}_{\omega_{h^{-1}}})a_h(\varphi^{(m)}_{\omega_{h^{-1}}})\Bigr) = \sum_{m\neq(0,\dots,0)}\int_{L^2(\mathbb{R}^d)} s_m(z)\mathrm{d}G(z)\,. \tag{12}$$

If $\varphi^{(m)}_{\omega_{h^{-1}}}$ was independent of h, the symbol would be

$$s_m(z) = \Bigl|\langle\varphi^{(m)}_{\omega_{h^{-1}}}\,,\,z\rangle_2\Bigr|^2\,.$$

In the thermodynamic limit, however, we are relaxing the harmonic trap, hence $\varphi^{(m)}_{\omega_{h^{-1}}}$ depends on h, through $\omega_{h^{-1}} \underset{h\to 0}{\longrightarrow} 0$. On the other hand, each $\varphi^{(m)}_{\omega_{h^{-1}}}$ belongs to the unit ball of $L^2(\mathbb{R}^d)$ uniformly with respect to $h \in I$, and thus by Banach-Alaoglu's theorem it has a weak limit $\varphi^{(m)}_0 \in L^2(\mathbb{R}^d)$ (up to a redefinition of I), and therefore

$$s_m(z) = \Bigl|\langle\varphi^{(m)}_0\,,\,z\rangle_2\Bigr|^2\,.$$

The quantity

$$f_0(G) = 1 - \sum_{m\neq(0,\dots,0)}\int_{L^2(\mathbb{R}^d)} s_m(z)\mathrm{d}G(z)$$

[20] The indirect approach is customary in physics. In addition, it is related to the fact that, in the thermodynamic limit, the measure G loses mass, and such mass corresponds to the condensed fraction [see 12, for additional details].

$0 \leq f_0(G) \leq 1$, is the *fraction of condensed particles in the (classical) thermodynamic state* G. Such fraction is easily calculated using the following physical assumptions: $\beta_h = \beta h^{\frac{d-1}{d}}$, and $\mu_h = \omega h^{\frac{1}{d}}$, $\beta \in \mathbb{R}^+$. With these assumptions the thermodynamic state G only depends on the macroscopic inverse temperature β, and thus we denote it by G_β. It is well-known that there is a $\beta_* \in \mathbb{R}^+$ (the inverse of the critical temperature) for which there is a phase transition: $f_0(G_\beta) = 0$ for any $\beta \leq \beta_*$ (no condensation), and $f_0(G_\beta) > 0$ (condensation) for any $\beta > \beta_*$, with $f_0(G_\beta) \underset{\beta\to\infty}{\longrightarrow} 1$ (complete condensation at temperature zero).

2.1.3 High-temperature limit and Gibbs measures

The effective behavior of grand-canonical quantum Gibbs states as classical measures has also been recently studied, in order to provide a microscopic derivation of Gibbs measures [40, 41, 62, 63]. The scaling used in these works is, however, very different from the one used in § 2.1.2 above: they analyze a *high-temperature limit*, in which there is no condensation (and thus the classical state could be a Gibbs measure, that has no atoms, *i.e.* no macroscopic occupation of the ground state). Nonetheless, their results could be reinterpreted in our general framework as the convergence (in the $\mathfrak{P} \vee \mathfrak{T}$ topology) of grand canonical Gibbs states to their unique cylindrical Wigner measures, the corresponding classical Gibbs measures. Without entering into the details, one interesting fact is that in dimension $d = 2, 3$, the limit measures are, already in the non-interacting case, truly cylindrical in $L^2(\Lambda_d)$ (where Λ_d is either $\mathbb{R}^d$ or the d-dimensional torus), and concentrated as Radon measures in $H^{r(d)}(\Lambda_d) \smallsetminus L^2(\Lambda^d)$, for a suitable $r(d) < 0$. Of course by Theorem 1.11 it is known that there are semiclassical quantum states that converge to true cylindrical measures, but these grand-canonical Gibbs states provide a physically relevant example (other such examples will be given in § 2.1.4 below).

2.1.4 Rough effective potentials in the quasi-classical limit of semi-relativistic theories

Using semiclassical analysis for semirelativistic quantum field theories it is possible to study how controllable effective external potentials acting on quantum particles can be created by making the latter interact with semiclassical radiation fields [28, 29], as it is common practice in experimental physics. In order to produce interesting potentials, such as harmonic traps and uniform magnetic potentials, it is necessary to use semiclassical quantum states whose Wigner measures are *truly cylindrical*. This is due to the fact that the effective potentials generated by semiclassical quantum states whose cylindrical Wigner measures are Radon measures cannot be singular functions (they are continuous and vanishing at infinity).

Let us briefly overview the main ideas. The interaction of nonrelativistic particles with radiation is modeled by composite systems described by self-adjoint Hamiltonian operators acting on spaces of the form $\mathscr{H} \otimes \Gamma_s(\mathfrak{H})$, with $\mathscr{H}, \mathfrak{H}$ separable Hilbert. The *quasi-classical approximation* is the regime in which only the field behaves semiclassically, while the quantum nature of the particles is still relevant. Mathematically, it amounts to say that the semiclassical parameter h appears only in the canonical commutation relations of the field, *i.e.*,

$$\left[1 \otimes a_h^*(\eta), 1 \otimes a_h(\xi)\right] = -h\langle \eta, \xi\rangle_{\mathfrak{H}} \qquad (\forall \eta, \xi \in \mathfrak{H}) .$$

Fock-normal semiclassical states are therefore represented by density matrices $\rho_h \in \mathfrak{S}_+^1(\mathscr{H} \otimes \Gamma_s(\mathfrak{H}))$, with trace uniformly bounded with respect to $h \in I \subseteq (0,1)$. The additional degrees of freedom given by $\mathscr{H}$ are reflected in the fact that the Wigner measures associated to $\mathrm{Tr}\big(\cdot \rho_h\big)$ are in general not scalar-valued, but vector-valued (with values in the Banach cone of positive states on the C*-algebra $\mathcal{L}(\mathscr{H})$). This generalization is taken into account below in § 3 to 7, where we prove (generalizations of) the results discussed in § 1. However, if the semiclassical quantum state is of the form

$$\big|^{\exists}\Psi \otimes {}^{\exists}\psi_h\big\rangle\big\langle \Psi \otimes \psi_h \big| \; , \; \Psi \in \mathscr{H} \; , \; \psi_h \in \Gamma_s(\mathfrak{H}) \; ,$$

then the corresponding cylindrical measure factorizes to the product $\mathrm{Tr}_{\mathscr{H}}\big(\; \cdot \; |\Psi\rangle\langle\Psi|\big) \times M$, where M is the scalar cylindrical Wigner measure of $\mathrm{Tr}_{\Gamma_s}\big(\cdot |\psi_h\rangle\langle\psi_h|\big)$. Now, let $\psi_h \in \Gamma_s(\mathfrak{H})$ be a semiclassical quantum vector (*i.e.*, $\mathrm{Tr}_{\Gamma_s}\big(\cdot |\psi_h\rangle\langle\psi_h|\big)$ is a semiclassical quantum state) converging to $M \in \mathcal{M}_{\mathrm{cyl}}(\mathfrak{H},\mathfrak{H})$ in the $\mathfrak{P} \vee \mathfrak{T}$ topology. As discussed above we are interested in studying the quasi-classical behavior of a composite system consisting of particles and a bosonic field; in particular we would like to characterize the effects of the interaction on the subsystem of particles. A simple model for the interaction is the following. Let $\mathscr{H} = L^2(\mathbb{R}^{Nd})$ be the Hilbert space of N nonrelativistic particles (without spin), and let

$$H_h = \big(-\Delta + V(x)\big) \otimes 1 + 1 \otimes \mathrm{d}\Gamma_h(k) + a_h^*\big(\lambda(x)\big) + a_h\big(\lambda(x)\big) \; ,$$

where V, when considered as a multiplication operator, is a small Kato perturbation of $-\Delta$, $k \geq 0$ is self-adjoint on $\mathfrak{H}$, and $\lambda \in L^\infty(\mathbb{R}^{Nd},\mathfrak{H})$. These types of models were first introduced in the community of mathematical physics by Nelson [69]. If we take the partial trace of H_h with respect to $|\psi_h\rangle\langle\psi_h|$, we obtain an operator $\langle H_h\rangle_{\psi_h}$ acting only on $\mathscr{H} = L^2(\mathbb{R}^{Nd})$. In addition, we subtract from $\langle H_h\rangle_{\psi_h}$ the multiple of the identity $c_h = \langle \mathrm{d}\Gamma_h(k)\rangle_{\psi_h}$, since it only amounts to a spectral shift. Then in the quasi-classical limit $h \to 0$ we are able to prove in [28] that, as long as the semiclassical vector ψ_h is sufficiently regular[21], the following convergence holds in the norm resolvent sense:

$$\langle H_h\rangle_{\psi_h} - c_h \xrightarrow[h\to 0]{\text{norm-res}} H(\mu) = -\Delta + V(x) + 2\mathrm{Re}\int_{\mathfrak{H}} \langle \lambda(x), z\rangle_{\mathfrak{H}} \mathrm{d}\mu(z) \; .$$

Therefore, the interaction with the field generates, in the quasi-classical limit, an effective potential $2\mathrm{Re}\int_{\mathfrak{H}}\langle\lambda(x),z\rangle_{\mathfrak{H}}\mathrm{d}\mu(z)$ acting on the particles. Such potential is controllable by tuning the quantum field configuration ψ_h. One drawback in requiring that the Wigner measure of ψ_h is a Radon measure on $\mathfrak{H}$ is that it is not possible to generate "physically interesting" potentials, *e.g.*, unbounded ones that

[21]In particular, we have that $M = \mu \in \mathcal{M}_{\mathrm{rad}}(\mathfrak{H})$, with $\int_{\mathfrak{H}} \|z\|_{\mathfrak{H}} \mathrm{d}\mu(z) < \infty$.

could trap the particles (such as $\omega^2 x^2$, that played an important role in § 2.1.2 to study condensation). In fact, for every $\mu \in \mathcal{M}_{\mathrm{rad}}(\mathfrak{H})$ the effective potential is either continuous and vanishing at infinity (if $\|\cdot\|_{\mathfrak{H}}$ is μ-integrable), or undefined (if $\int_{\mathfrak{H}} \|z\|_{\mathfrak{H}} \mathrm{d}\mu(z)$ diverges). In order to obtain interesting potentials it is therefore necessary to consider semiclassical vectors whose cylindrical Wigner measures *are not* Radon measures on $\mathfrak{H}$. In fact, it is, *e.g.*, possible to prove that given any potential $W \in L^2_{\mathrm{loc}}(\mathbb{R}^{Nd}, \mathbb{R}_+)$, then there exists at least one semiclassical vector $\psi_{h,W} \in \Gamma_{\mathrm{s}}\bigl(L^2(\mathbb{R}^d)\bigr)$, with explicit form and converging to a "true" cylindrical measure, and (infinitely many) coupling functions $\lambda_W \in L^\infty\bigl(\mathbb{R}^{Nd}, L^2(\mathbb{R}^d)\bigr)$ such that

$$\langle H_{h,\lambda_W}\rangle_{\psi_{h,W}} - c_h \underset{h\to 0}{\overset{\text{strong-res}}{\longrightarrow}} H_W = -\Delta + V(x) + W(x) \; .$$

Therefore this procedure describes a simple way of producing any given external potential, acting on quantum particles, exploiting their interaction with a semiclassical bosonic field (*e.g.*, a phonon field, a radiation field, ...); and it relies on semiclassical states whose semiclassical measures are truly cylindrical to produce strong potentials.

2.2 Relativistic Quantum Field Theories

If the applications described in § 2.1 motivate the importance of cylindrical measures as Wigner measures, the ones described in this section motivate the abstract, representation independent, algebraic approach to semiclassical states taken in § 1. In fact, there are features of relativistic quantum field theories that make such an approach necessary. One is that the phase space is often taken to be a space of test functions, such as the nuclear space $\mathcal{S}(G)$ or $\mathcal{D}(G) = C_0^\infty(G)$ with the inductive limit topology (where G in both cases is some locally compact abelian group). Such choice actually plays an important role in defining the right representation of the canonical commutation relations, at least in the few rigorously definable interacting theories that we know of [see, *e.g.*, 49, and references thereof contained]. It is therefore too restrictive, in relativistic quantum field theories, to assume the phase spaces to be (pre)Hilbert (with the natural induced topology), and to focus solely on Fock-normal states. In addition, since the phase space is considered, from the physical standpoint, as a space of test functions, it becomes clear why the natural space of classical fields (the one on which the cylindrical Wigner measures should act upon) is a space of "distributions", *i.e.* a space in duality with the phase space[22]. Another feature that emphasizes the importance of a representation-independent analysis of the semiclassical states is the so-called *Haag's theorem* [see, *e.g.*, 53], asserting that if we consider two relativistic invariant states[23] of the same Weyl C*-algebra (*i.e.* two states that are invariant with respect to the action of the Poincar group on the C*-algebra), then they are either

[22] In the aforementioned examples, it would naturally be respectively $\mathcal{S}'(G)$ with the $\sigma\bigl(\mathcal{S}'(G), \mathcal{S}(G)\bigr)$ topology, and $\mathcal{D}'(G)$ with the analogous weak topology.

[23] A typical example would be to consider the ground state of a non-interacting theory (the Fock vacuum for the scalar field), and the ground state of an interacting theory (such as $(\varphi^4)_2$).

equal or disjoint[24]. Since the ground states of a free and an interacting theory cannot be the same, this means that a relativistically covariant interacting theory should be in a representation that is inequivalent to the free (Fock) representation. Since the results of § 1 are all *representation-independent*, they are well suited for application to relativistic quantum field theories.

2.2.1 Semiclassical Axiomatic Quantum Field Theory

There are various axiomatic formulations of relativistic quantum field theories. The most used sets of axioms are the "Hilbert space" axioms known as Grding-Wightman [78, 82], and the "algebraic" axioms known as Haag-Kastler [53, 54]. Due to the algebraic formulation of semiclassical analysis given in § 1, it is natural for us to choose algebraic axioms as a starting point. In order to describe also quantum field theories in non-Minkowski spacetimes, we use the so-called *locally covariant* algebraic axioms [see, *e.g.*, 24, 25, 39].
The starting point of the locally covariant approach to algebraic bosonic quantum field theory is a *local bosonic quantization functor*. Let $\mathbf{A}$ be a *small* category, to be interpreted as the category that has as objects local spacetimes, *e.g.*, local (bounded) regions of a given spacetime, and as morphisms suitably regular mappings between local spacetimes. The local quantization functor is then the composition of a functor from $\mathbf{A}$ to (local) phase spaces (more accurately, to spaces of local test functions) and the Segal bosonic quantization functor.

Definition 2.2 (Local bosonic quantization functor). *A functor* $\mathbb{LW}_h : \mathbf{A} \to \mathbf{C^*alg}$ *is a local bosonic quantization functor iff*

$$\mathbb{LW}_h = \mathbb{W}_h \circ {}^{\exists}\mathbb{L} \qquad \left(\mathbb{L} : \mathbf{A} \to \mathbf{Symp_{\mathbb{R}}}\right) .$$

The local bosonic quantization functor associates to each local spacetime a C*-algebra of canonical commutation relations, the local bosonic observables[25]. By another composition with the duality functor $\mathbb{D}_+ : \mathbf{C^*alg} \to \mathbf{BanCone}$, defined in § 1.7, we obtain the *functor of local bosonic quantum states*

$$\mathbb{LS}_h := \mathbb{D}_+ \circ \mathbb{LW}_h = \mathbb{D}_+ \circ \mathbb{W}_h \circ \mathbb{L} .$$

Clearly, the results formulated in § 1.7 for $\mathbb{S}_h$ hold, *mutatis mutandis*, for $\mathbb{LS}_h$. To this extent, let us define a *functor of local cylindrical Wigner measures* to be any functor of the form

$$\mathbb{LS}_0 := \mathbb{S}_0 \circ \mathbb{L} ,$$

where $\mathbb{S}_0$ is any functor of cylindrical Wigner measures, see Definition 1.14. By Proposition 1.15, it follows that for any $\mathbb{L}$ a functor of local cylindrical Wigner

[24] Two states ω_1, ω_2 on a C*-algebra are disjoint iff ω_1 is not normal with respect to the GNS representation of ω_2, and vice-versa. It follows that the GNS representations given by ω_1 and ω_2 are *inequivalent*, *i.e.* they are not related by a unitary isomorphism.

[25] The algebras of local bosonic observables are in general taken to be larger, containing the algebras of canonical commutation relations as subalgebras. However, since we are interested in semiclassical regular quantum states, we can without loss of generality restrict the observables to the ones in the Weyl C*-algebra.

measures exists and it is unique up to natural isomorphisms. In addition, Theorem 1.16 yields

$$\mathbb{LS}_h \underset{h\to 0}{\rightarrowtail} \mathbb{LS}_0 \;. \tag{13}$$

It is now time to introduce the four axioms of locally covariant algebraic quantum field theory: *isotony*, *covariance*, *Einstein causality*, and *time-slice*. The *spectrum condition*, that together with the existence of a preferred *ground state* (or vacuum) is very important in Minkowski spacetime, is locally implemented in suitable non-Minkowski spacetimes using *Hadamard states*. We discuss the semiclassical behavior of ground states in § 2.2.5.

In a relativistic theory, the quantum fields should be locally covariant. In the Grding-Wightman formulation (in Minkowski global spacetime), this corresponds to requesting the existence of Poincar invariant operator-valued field distributions [see *e.g.* 73, IX.8 Property 6]. In the locally covariant algebraic formulation, the covariance axiom has a very simple form.

$$\mathbb{L} \text{ is a covariant functor.} \tag{Cov}$$

An immediate consequence of (Cov) is that $\mathbb{LW}_h$ is a covariant functor, and that $\mathbb{LS}_h$ is a contravariant functor (since $\mathbb{D}_+$ is contravariant).

The isotony axiom formalizes the fact that observables of a local spacetime region are also observables of any region that includes the former. In mathematical terms, the functor of local quantum observables preserves injective morphisms $\hookrightarrow$ (embeddings):

$$\mathfrak{a} : A \hookrightarrow B \;\Rightarrow\; \mathbb{L}(\mathfrak{a}) : \mathbb{L}(A) \hookrightarrow \mathbb{L}(B) \qquad \big(\forall \mathfrak{a} \in \mathrm{Morph}(\mathbf{A})\big) \;. \tag{Iso}$$

Lemma 2.3. *If the* (Iso) *axiom holds, then* $\mathbb{LW}_h(\mathfrak{a}) : \mathbb{LW}_h(A) \hookrightarrow \mathbb{LW}_h(B)$ *for any embedding* $\mathfrak{a} : A \hookrightarrow B$ *of* $\mathbf{A}$.

Proof. $\mathbb{LW}_h(\mathfrak{a})$ is the Weyl-operator-preserving *-homomorphism of Weyl C*-algebras induced by an injective homomorphism of Heisenberg groups. More precisely, it is induced by the group homomorphism between $\mathbb{H}\big(\mathbb{L}(A)\big)$ and $\mathbb{H}\big(\mathbb{L}(B)\big)$, in turn induced by the injective symplectomorphism $\mathbb{L}(\mathfrak{a})$. Therefore, $\mathbb{LW}_h(\mathfrak{a})$ is injective as well. ⊣

In relativistic theories, quantum fields should also respect causality. The best-known formulation of the causality axioms is that local observables belonging to space-like separated regions of spacetime should always commute. As showed by Roos [75], given two commuting subalgebras of observables statistical independence is equivalent to a non-vanishing condition on the product of elements from each different subalgebra, that is always satisfied by tensor products (provided the embedding algebra also embeds the tensor product of the two subalgebras). In fact, the sensible notion of Einstein causality in the locally covariant formulation turns out to be the fact that the local functor $\mathbb{L}$ *preserves tensor structures*. A category is *monoidal* (or tensor) if there exists a monoidal structure on it[26]. A functor that preserves monoidal structures is called *homomorphic*, or *monoidal*[27]. On local spacetimes there is a natural monoidal structure, therefore we suppose that $\mathbf{A}^{\otimes}$ is monoidal. On symplectic spaces, the monoidal structure is given by direct products: $(X,\varsigma)\otimes(Y,\tau)=(X\oplus Y,\varsigma\oplus\tau)$, for any $(X,\varsigma),(Y,\tau)\in\mathbf{Symp}_{\mathbb{R}}$ (with the space $\{0\}$ as identity).

$$\mathbb{L}^{\otimes}:\mathbf{A}^{\otimes}\to\mathbf{Symp}_{\mathbb{R}}^{\otimes}\ \text{ is homomorphic.} \tag{Ein}$$

On C*-algebras, there is a directed set of possible monoidal structures, each one corresponding to a different choice of C*-cross-norm [see 79, for an introduction to tensor products on C*-algebras]. For every monoidal structure on $\mathbf{C^*alg}$ the identity is $\mathbb{C}\in\mathbf{C^*alg}$. Clearly, we denote by $\mathbf{C^*alg}^{\otimes_\alpha}$ the monoidal category of C*-algebras with a given choice $\otimes_\alpha$ of C*-cross-norm ($\alpha_{\min}\leq\alpha\leq\alpha_{\max}$, where $\alpha_{\min}$ and $\alpha_{\max}$ are the injective and projective C*-cross-norms respectively). This also induces a tensor structure on the space of states, and the functor $\mathbb{D}_+$ is homomorphic. It is also not difficult to see that the functor $\mathbb{W}_h$ is homomorphic for any $h\in(0,1)$, and for any choice of monoidal structure $\mathbf{C^*alg}^{\otimes_\alpha}$. Therefore the following lemma is true.

LEMMA 2.4. *If* (Ein) *holds, then both* $\mathbb{LW}_h^{\otimes_\alpha}$ *and* $\mathbb{LS}_h^{\otimes_\alpha}$ *are homomorphic for any* $\alpha_{\min}\leq\alpha\leq\alpha_{\max}$.

Let us remark that the formulation above is *equivalent* to the requirement that space-like separated local observables commute if the C*-cross-norm is the inductive one $\alpha_{\min}$ [25].

The last axiom to be introduced is time-slice. The idea behind the time slice axiom is that a quantum field theory should be a quantum evolution theory, *i.e.* it should be determined completely by fields "at one fixed time", and by the action on them of an evolution operator. In curved spacetimes, the concept of space at a fixed time is given by *Cauchy surfaces*, that is surfaces that are intersected by every non-extensible, causal spacetime path exactly once. Since we do not use here the

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[26] More precisely, a category $\mathbf{C}$ is *monoidal* (usually denoted by $\mathbf{C}^{\otimes}$) if there exists a monoidal structure:

- $\exists\otimes:\mathbf{C}\times\mathbf{C}\to\mathbf{C}$ (tensor or monoidal product);
- $\exists I\in\mathbf{C}$ (identity);
- $\exists\,\nu^{\mathrm{ass}},\nu^{\mathrm{l}},\nu^{\mathrm{r}}$ natural isomorphisms with components:
  - $\nu^{\mathrm{ass}}_{A,B,C}:(A\otimes B)\otimes C\xrightarrow{\text{iso}}A\otimes(B\otimes C)$,
  - $\nu^{\mathrm{l}}_A:I\otimes A\xrightarrow{\text{iso}}A$,
  - $\nu^{\mathrm{r}}_A:A\otimes I\xrightarrow{\text{iso}}A$.

[27] A functor $\mathbb{F}^{\otimes}:\mathbf{C}^{\otimes}\to\mathbf{D}^{\otimes}$ is homomorphic iff:

- $\mathbb{F}^{\otimes}(A\otimes B)\cong\mathbb{F}^{\otimes}(A)\otimes\mathbb{F}^{\otimes}(B)$;
- $\mathbb{F}^{\otimes}\big(\mathfrak{c}_1\otimes\mathfrak{c}_2:A\otimes B\to C\otimes D\big)\cong\mathbb{F}^{\otimes}(\mathfrak{c}_1)\otimes\mathbb{F}^{\otimes}(\mathfrak{c}_2)$;
- $\mathbb{F}^{\otimes}(I_{\mathbf{C}})\cong I_{\mathbf{D}}$.

properties of Cauchy surfaces, we do not discuss the conditions a spacetime has to satisfy to guarantee their existence. Given the small category of local spacetimes $\mathbf{A}$, supposed to be based on sets, we define a Cauchy surface of $\mathbf{A}$ to be any set Σ with the following property

$$\Sigma = \bigcap_{\Sigma\subset A\in\mathbf{A}} A \neq \emptyset \; . \tag{14}$$

Let us also suppose that the set of objects of $\mathbf{A}$ that include Σ is directed, when ordered by inclusion. Then if we denote by $\mathfrak{i}_{AB} : A \hookleftarrow B$, $A \supseteq B$, the canonical inclusion morphisms, it follows that

$$\bigl(A \supset \Sigma, \mathfrak{i}_{AB}\bigr)_{A\supseteq B\in\mathbf{A}}$$

is a projective family that satisfies

$$\varprojlim_{\Sigma\subset A\in\mathbf{A}} A = \bigcap_{\Sigma\subset A\in\mathbf{A}} A = \Sigma \; .$$

By (Cov) and (Iso), it also follows that $\mathbb{LW}_h(\mathfrak{i}_{AB}) : \mathbb{LW}_h(A) \hookleftarrow \mathbb{LW}_h(B)$ are embeddings, whenever $A \supseteq B$. It appears then natural to define the algebra of observables in the Cauchy surface Σ as the C*-projective limit of the algebras of observables on the local spacetimes containing Σ [see, *e.g.*, 38][28]:

$$\mathfrak{O}(\Sigma) := \varprojlim_{\Sigma\subset A\in\mathbf{A}} \mathbb{LW}_h(A) \; . \tag{15}$$

One should however be careful since such limit may be the empty set. By construction, the algebra of observables $\mathfrak{O}(\Sigma)$ has natural projections $P_{A,\Sigma} : \mathfrak{O}(\Sigma) \to \mathbb{LW}_h(A)$. The time slice axiom then consists of two parts: the first is that there exists a Weyl C*-algebra $\mathbb{W}_h(X_\Sigma, \varsigma_\Sigma)$ of time-sliced (or time-zero) bosonic fields in the Cauchy surface (this *a fortiori* also guarantees that the algebra of observables in the Cauchy surface is not empty); and the second is that all projections P_A are *-isomorphisms.

$$\mathbb{W}_h({}^{\exists}X_\Sigma, {}^{\exists}\varsigma_\Sigma) \overset{\exists w_\Sigma}{\hookrightarrow} \mathfrak{O}(\Sigma) \;\text{ and } P_{A,\Sigma} \text{ is a *-isomorphism} \quad \bigl(\forall\Sigma \in \mathrm{Cau}_{\mathbf{A}} \; , \; \forall\mathbf{A} \ni A \supset \Sigma\bigr) \; . \tag{TS}$$

In § 2.2.2 to 2.2.4 we discuss the semiclassical implications of the four axioms above. For the reader's convenience, such implications are summarized schematically in Table 1.

2.2.2 Covariance and isotony

The covariance axiom (Cov) has the semiclassical consequence that the classical limit fields behave properly with respect to spacetime transformations. More precisely, consider the convergence $\mathbb{LS}_h \underset{h\to 0}{\rightarrowtail} \mathbb{LS}_0$, applied to a spacetime morphism

[28] More generally, the algebra of observables in the Cauchy surface should contain $\varprojlim \mathbb{LW}_h(a)$ as a subalgebra.

Quantum property	Semiclassical consequence	Ref.
Local structure	Loc. functor of cl. fields; convergence of loc. states and linear maps (q. to cl.); surjectivity of convergence	Eq. (13)
(Cov)	Covariance of loc. cl. fields; convergence of spacetime transformations represented on states (q. to cl.)	§ 2.2.2
(Iso)	Projective structure of loc. cl. states; definition of global cl. states as projective limits	§ 2.2.2
(Ein)	Tensoriality of cl. spacelike-separated fields; at most destruction of spacelike entanglement by cl. limit	Pr. 2.6
(TS)	Definition of cl. time-sliced fields; (case-by-case) characterization of cl. flow, and Egorov theorems	§ 2.2.4

Table 1: Semiclassical consequences of (locally covariant) QFT axioms for bosonic theories. List of relevant abbreviations: (Iso) – isotony axiom; (Cov) – covariance axiom; (Ein) – Einstein causality; (TS) – time-slice axiom; loc. – local; cl. – classical; q. – quantum; stat. – statistical.

$\mathfrak{a} : A \to B$, $A, B \in \mathbf{A}$. The latter could be interpreted, *e.g.*, as a spacetime symmetry transformation, that induces a covariant transformation $\mathbb{L}(\mathfrak{a})$ of local test functions, and in turn by quantization a transformation of quantum observables $\mathbb{LW}_h(\mathfrak{a})$. By duality, it also induces a contravariant transformation of quantum states $\mathbb{LS}_h(\mathfrak{a})$. At the classical level, it induces a transformation $\mathbb{LS}_0(\mathfrak{a})$ of cylindrical Wigner measures. The transformed semiclassical quantum states converge, as they should, to cylindrical Wigner measures pushed forward by the corresponding classical fields' transformation:

$$\omega_{h_\beta,B} \xrightarrow[h_\beta\to 0]{\mathfrak{P}_{\mathbb{L}(B)}\vee\mathfrak{T}_{\mathbb{L}(B)}} M_B \implies \mathbb{LS}_h(\mathfrak{a})\omega_{h_\beta,B} \xrightarrow[h_\beta\to 0]{\mathfrak{P}_{\mathbb{L}(A)}\vee\mathfrak{T}_{\mathbb{L}(A)}} \mathbb{LS}_0(\mathfrak{a})_* M_B \; .$$

The classical transformation is contravariant since it pushes forward measures acting on fields, dual to test functions (and $\mathbb{L}$ is covariant on test functions). Taking the concrete example of a Lorentz transformation between Minkowski local spacetimes, this could be interpreted as the fact that the unitary representation of Lorentz transformations on local quantum bosonic fields converges "in the Schrdinger picture" to the representation of Lorentz transformations on the corresponding local classical fields.

The isotony axiom (Iso) is used to provide a notion of *global* observables that is in accordance with the local structure. The quantum global structure induces a corresponding classical global structure as well, together with the associated semiclassical properties. Let us define a partial ordering on the set of local regions of spacetime $\mathrm{Obj}(\mathbf{A})$ (the set of objects of the small category $\mathbf{A}$): $\preceq$ is defined as follows for any $A, B \in \mathrm{Obj}(\mathbf{A})$

$$A \preceq B \iff {}^{\exists}\mathfrak{i}_{AB} : A \hookrightarrow B \; .$$

Usually the set of spacetimes is *directed*[29] by $\preceq$, and we suppose this to be the case. From (Iso) it then follows that

$$\Bigl(\mathbb{LW}_h(A, \mathfrak{i}_{AB})\Bigr)_{A\preceq B\in \mathrm{Obj}(\mathbf{A})}$$

is an *inductive* family of C*-algebras (or, using the terminology introduced by Haag [see, *e.g*, 53], a local net of observables). In addition, both

$$\Bigl(\mathbb{LS}_h(A, \mathfrak{i}_{AB})\Bigr)_{A\preceq B\in \mathrm{Obj}(\mathbf{A})} \quad \text{and} \quad \Bigl(\mathbb{LS}_0(A, \mathfrak{i}_{AB})\Bigr)_{A\preceq B\in \mathrm{Obj}(\mathbf{A})}$$

[29] A partially ordered set $(S, \preceq)$ is directed iff for any $s, t \in S$, there exists $u \in S$ such that $s \preceq u$ and $t \preceq u$.

are *projective* families, of Banach cones and vector spaces respectively. Since the partial order is essentially the one given by inclusion (embedding) of one local spacetime into another, it is natural to define the C*-algebra of quantum global observables as the inductive limit of the net of local C*-algebras:

$$\varinjlim_{A\in \mathrm{Obj}(\mathbf{A})} \mathbb{LW}_h(A) \; .$$

Analogously, is it possible to characterize the global classical states as a projective limit of the local classical states? The answer is *affirmative*, and it is a consequence of the fact that, since $(\mathrm{Obj}(\mathbf{A}),\preceq)$ is directed,

$$\varprojlim_{A\in \mathrm{Obj}(\mathbf{A})} \Bigl(\mathbb{C}^{\mathbb{L}(A)}\Bigr) \cong \Bigl(\mathbb{C}\Bigr)^{\varinjlim_{A\in \mathrm{Obj}(\mathbf{A})} \mathbb{L}(A)} \; .$$

Let us denote $\underline{X} := \varinjlim \mathbb{L}(A)$ the *global phase space*. From the above property, it follows that the projective limit of the spaces of local cylindrical measures is isomorphic to the space of cylindrical Wigner measures associated to the global phase space $\underline{X}$:

$$\varprojlim_{A\in \mathrm{Obj}(\mathbf{A})} \mathcal{M}_{\mathrm{cyl}}\bigl(\mathbb{L}(A)^*,\mathbb{L}(A)\bigr) \cong \mathcal{M}_{\mathrm{cyl}}\bigl(\underline{X}^*,\underline{X}\bigr) \; .$$

On the other hand, by an analogous reasoning, it is also possible to identify the projective limit of the set of local regular quantum states as states on the global algebra of observables $\varinjlim \mathbb{LW}_h(A)$. Since in addition we have the convergence of functors $\mathbb{S}_h \underset{h\to 0}{\rightarrowtail} \mathbb{S}_0$ (see Theorem 1.16), it is clear that global states defined by projective families of convergent semiclassical states converge semiclassically to the corresponding measure on $\mathcal{M}_{\mathrm{cyl}}\bigl(\underline{X}^*,\underline{X}\bigr)$.

2.2.3 Einstein causality

As we have discussed in the previous section, Einstein causality can be abstractly formulated as the fact that the local functor preserves tensor structures (Ein). As a consequence, also the local bosonic quantization functor and the functor of local quantum states preserve tensor structures (Lemma 2.4). At the classical level, there is a tensor structure for cylindrical measures **CylM**, given by the product measures: for any $\mathcal{A}\subseteq \mathbb{R}^A$, $\mathcal{B}\subseteq \mathbb{R}^B$, $A,B\in \mathbf{Set}$, then

$$\mathcal{M}_{\mathrm{cyl}}(A,\mathcal{A})\otimes \mathcal{M}_{\mathrm{cyl}}(B,\mathcal{B}) := \mathcal{M}_{\mathrm{cyl}}(A\times B,\mathcal{A}\oplus\mathcal{B}) \; . \tag{16}$$

In addition, let

$$\mathfrak{a}: A_1\to A_2 \quad , \quad \mathfrak{b}: B_1\to B_2 \quad (A_1,A_2,B_1,B_2\in \mathbf{Set})$$

be compatible with $\mathcal{M}_{\mathrm{cyl}}(A_1,\mathcal{A}_1),\mathcal{M}_{\mathrm{cyl}}(A_2,\mathcal{A}_2)$ and $\mathcal{M}_{\mathrm{cyl}}(B_1,\mathcal{B}_1),\mathcal{M}_{\mathrm{cyl}}(B_2,\mathcal{B}_2)$ respectively. Then

$$\mathfrak{a}\otimes\mathfrak{b} := \mathfrak{a}\times\mathfrak{b} \quad , \quad \mathfrak{a}\times\mathfrak{b}: A_1\times B_1\to A_2\times B_2$$

is compatible with $\mathcal{M}_{\text{cyl}}(A \times B, \mathcal{A} \oplus \mathcal{B})$. The identity is $i_{\mathbf{CylM}} = \mathcal{M}_{\text{cyl}}(\{\emptyset\}, \{\mathbf{0}\})$, where $\mathbf{0}(\varnothing) = 0$ is the zero function. Using this structure, $\mathbf{CylM}^{\otimes}$ is a monoidal category. The classical counterpart of Lemma 2.4 is then the following lemma.

LEMMA 2.5. *If* (Ein) *holds, then* $\mathbb{LS}_0^{\otimes}$ *is homomorphic.*

The convergence of functors (13) can then be rewritten as a convergence of homomorphic functors $\mathbb{LS}_h^{\otimes_\alpha} \underset{h\to 0}{\rightarrowtail} \mathbb{LS}_0^{\otimes}$, for any $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$. One important consequence of this convergence is the fact that spacelike entanglement of local regions can only be destroyed in the classical limit. In other words, given a quantum state describing the fields localized in two spacelike separated regions of spacetime, the corresponding entanglement could only disappear in the classical limit $h \to 0$. The most convenient way to prove that is to show that on one hand, given any semiclassical quantum state with no entanglement, all its corresponding cylindrical Wigner measures are statistically independent as well; on the other hand, that there exist entangled quantum states whose classical limit is not entangled. Let us remark that the proof of what we have just discussed (that is given after the proposition below), could be immediately adapted to prove the more general Proposition 1.12.

PROPOSITION 2.6. *If* (Ein) *holds, then*

$$\mathbb{LS}_h^{\otimes_\alpha} \underset{h\to 0}{\rightarrowtail} \mathbb{LS}_0^{\otimes} \; ,$$

for any $\alpha_{\min} \leq \alpha \leq \alpha_{\max}$*, with both* $\mathbb{LS}_h^{\otimes_\alpha}$ *and* $\mathbb{LS}_0^{\otimes}$ *homomorphic. In addition, entanglement between spacelike separated regions can only be destroyed by the classical limit* $h \to 0$*.*

Proof. The convergence of functors has already been discussed, let us prove the fact that entanglement can only be destroyed in the classical limit. Let us consider two (spacelike separated) local regions of spacetime $A, B \in \mathbf{A}^{\otimes}$. Then

$$\mathbb{LW}_h(A \otimes B) = \mathbb{W}_h\bigl(\mathbb{L}(A) \oplus \mathbb{L}(B)\bigr) \; .$$

Now, let $\omega_{h,A}$ be a semiclassical quantum state on $\mathbb{W}_h\bigl(\mathbb{L}(A)\bigr)$, $\omega_{h,B}$ a semiclassical quantum state on $\mathbb{W}_h\bigl(\mathbb{L}(B)\bigr)$, and $\omega_{h,A\otimes B}$ an *entangled* semiclassical quantum state on $\mathbb{W}_h\bigl(\mathbb{L}(A) \oplus \mathbb{L}(B)\bigr)$, *i.e.* the latter cannot be written as the tensor product of one state acting on $\mathbb{W}_h\bigl(\mathbb{L}(A)\bigr)$ and another acting on $\mathbb{W}_h\bigl(\mathbb{L}(B)\bigr)$. Then

$$\omega_{h,A} \otimes \omega_{h,B} \in \mathbb{W}_h\bigl(\mathbb{L}(A) \oplus \mathbb{L}(B)\bigr)'$$

is a *non-entangled* semiclassical quantum state. Let us consider one of its cluster points in either the $\mathfrak{P}_{\mathbb{L}(A)\oplus\mathbb{L}(B)}$ or the $\mathfrak{P}_{\mathbb{L}(A)\oplus\mathbb{L}(B)} \vee \mathfrak{T}_{\mathbb{L}(A)\oplus\mathbb{L}(B)}$ topology:

$$M_{AB} \in \mathcal{M}_{\text{cyl}}\Bigl(\mathbb{L}(A)^* \times \mathbb{L}(B)^*, \mathbb{L}(A) \oplus \mathbb{L}(B)\Bigr) \; .$$

The cluster point can indeed be written as a cylindrical measure on $\mathbb{L}(A)^*_{\mathbb{L}(A)} \times \mathbb{L}(B)^*_{\mathbb{L}(B)}$, *i.e.* on the set $\mathbb{L}(A)^* \times \mathbb{L}(B)^*$ with the weak $\sigma\bigl(\mathbb{L}(A)^*, \mathbb{L}(A)\bigr) \times$

$\sigma\big(\mathbb{L}(B)^*, \mathbb{L}(B)\big)$ topology, since

$$\sigma\big(\mathbb{L}(A)^* \times \mathbb{L}(B)^*, \mathbb{L}(A) \oplus \mathbb{L}(B)\big) = \sigma\big(\mathbb{L}(A)^*, \mathbb{L}(A)\big) \times \sigma\big(\mathbb{L}(B)^*, \mathbb{L}(B)\big) \; .$$

Now, from the fact that $\omega_{h,A} \otimes \omega_{h,B}$ is a tensor product it follows that

$$M_{AB} = M_A \otimes M_B \quad , \quad M_A \in \mathcal{M}_{\mathrm{cyl}}\big(\mathbb{L}(A)^*, \mathbb{L}(A)\big) \; , \; M_B \in \mathcal{M}_{\mathrm{cyl}}\big(\mathbb{L}(B)^*, \mathbb{L}(B)\big) \; .$$

This is due to a general property of tensor products, however let us prove the result explicitly for the $\mathfrak{T}_{\mathbb{L}(A)\oplus\mathbb{L}(B)}$ convergence. Let $\mathcal{G}_{\omega_{h_\beta,A}\otimes\omega_{h_\beta,B}}(\cdot)$ be the family of generating functionals converging pointwise to $\hat{M}_{AB}(\cdot)$. Since the quantum state is a tensor product, it follows that for any $a_1, a_2 \in \mathbb{L}(A)$ and $b_1, b_2 \in \mathbb{L}(B)$,

$$\begin{aligned}\mathcal{G}_{\omega_{h_\beta,A}\otimes\omega_{h_\beta,B}}(a_1,b_1)\mathcal{G}_{\omega_{h_\beta,A}\otimes\omega_{h_\beta,B}}(a_2,b_2) &= \\ \mathcal{G}_{\omega_{h_\beta,A}\otimes\omega_{h_\beta,B}}(a_1,b_2)\mathcal{G}_{\omega_{h_\beta,A}\otimes\omega_{h_\beta,B}}(a_2,b_1)&\end{aligned}$$

Then, in the limit $h_\beta \to 0$,

$$\hat{M}_{AB}(a_1,b_1)\hat{M}_{AB}(a_2,b_2) = \hat{M}_{AB}(a_1,b_2)\hat{M}_{AB}(a_2,b_1) \; .$$

Hence there exist two functions of positive type (continuous on finite dimensional subspaces) $\hat{M}_A : \mathbb{L}(A) \to \mathbb{C}$ and $\hat{M}_B : \mathbb{L}(B) \to \mathbb{C}$ such that

$$\hat{M}_{AB} = \hat{M}_A \otimes \hat{M}_B \; .$$

Hence $\hat{M}_{AB}$ is the Fourier transform of the tensor product of two cylindrical measures.

It remains to show that there exist entangled quantum states whose classical limit is not entangled. The easiest way to do it is "perturbing" a convergent non-entangled semiclassical state. Let $\omega_{h_\beta,A}\otimes\omega_{h_\beta,B}$ be the non-entangled semiclassical state converging in the $\mathfrak{P}_{\mathbb{L}(A)\oplus\mathbb{L}(B)} \vee \mathfrak{T}_{\mathbb{L}(A)\oplus\mathbb{L}(B)}$ topology to $M_A \otimes M_B$. Then let us consider the semiclassical quantum state

$$\varpi_{h_\beta} := \omega_{h_\beta,A} \otimes \omega_{h_\beta,B} + h_\beta \omega_{h_\beta,A\otimes B} \; ;$$

where $\omega_{h_\beta,A\otimes B}$ is the entangled state defined above at the beginning of this proof. Since $\omega_{h_\beta,A\otimes B}$ is entangled, also ϖ_{h_β} is entangled. In addition, it converges in the $\mathfrak{P}_{\mathbb{L}(A)\oplus\mathbb{L}(B)} \vee \mathfrak{T}_{\mathbb{L}(A)\oplus\mathbb{L}(B)}$ topology to $M_A \otimes M_B$ (since the non-entangled perturbation becomes small as $h_\beta \to 0$). $\dashv$

2.2.4 Time slice and Egorov type theorems

The time slice axiom (TS) defines a non-empty algebra of bosonic observables on any Cauchy surface, and states that the knowledge of such algebra is sufficient to describe all local observables of regions containing the aforementioned Cauchy surface. A Cauchy surface is a "slice of time". Therefore, if the time slice axiom holds true, one should define the (Heisenberg-picture) evolution of the system as a map between the observables of two different Cauchy surfaces. Let $\Sigma, \Theta \in \mathrm{Cau}_{\mathbf{A}}$

such that $^{\exists}A \supset \Sigma \cup \Theta$. Then (TS) implies that a Θ-time generator $W_h(x_\Theta) \in \mathbb{W}_h(X_\Theta, \varsigma_\Theta)$ is evolved to some Σ-observable $O_h(x_\Theta) = U_h^{(\Sigma\Theta)}\bigl(W_h(x_\Theta)\bigr) \in \mathfrak{O}(\Sigma)$ by the *-isomorphism $U_h^{(\Sigma\Theta)}$ defined by

$$U_h^{(\Sigma\Theta)} := P_{A,\Sigma}^{-1} \circ P_{A,\Theta} .$$

For interacting theories *we cannot expect* the latter to be a Σ-time-zero field, and it could even not belong to the Weyl C*-algebra [for interesting examples, in non-relativistic particle quantum mechanics, see 36]. By duality, it also follows that

$$^{\mathrm{t}}U_h^{(\Sigma\Theta)} : \mathfrak{O}(\Sigma)' \to \mathfrak{O}(\Theta)'$$

transforms states of $\mathbb{W}_h(X_\Sigma, \varsigma_\Sigma)$ in complex states of $\mathbb{W}_h(X_\Theta, \varsigma_\Theta)$. Physically, we may think of $^{\mathrm{t}}U_h^{(\Sigma\Theta)}$ as the *evolution in the Schrdinger picture* (not necessarily positivity preserving), mapping a quantum state at a given time (described by the Cauchy surface Σ), to a state at another time (described by the Cauchy surface Θ).

Consider now the semiclassical complex quantum state[30]

$$\omega_h^{(\Sigma)} \in \mathfrak{O}(\Sigma)' \overset{^{\mathrm{t}}\mathrm{w}_\Sigma}{\to} \mathbb{W}_h(X_\Sigma, \varsigma_\Sigma)' ,$$

and suppose that

$$^{\mathrm{t}}U_h^{(\Sigma\Theta)}\omega_h^{(\Sigma)} \in \mathfrak{O}(\Theta)' \overset{^{\mathrm{t}}\mathrm{w}_\Theta}{\to} \mathbb{W}_h(X_\Theta, \varsigma_\Theta)'$$

is semiclassical as well. In addition, suppose that both $\omega_h^{(\Sigma)}$ and $^{\mathrm{t}}U_h^{(\Sigma\Theta)}\omega_h^{(\Sigma)}$ converge[31], as $h \to 0$, in the $\mathfrak{P}$ or $\mathfrak{P} \vee \mathfrak{T}$ topologies, to $M_\Sigma \in \mathcal{M}_{\mathrm{cyl}}\bigl(^{\exists}S_\Sigma, ^{\exists}e_{X_\Sigma}(X_\Sigma)\bigr)_{\mathbb{C}}$ and $M_\Theta \in \mathcal{M}_{\mathrm{cyl}}\bigl(^{\exists}S_\Theta, ^{\exists}f_{X_\Theta}(X_\Theta)\bigr)_{\mathbb{C}}$ respectively. Hence it would be desirable to have a *compatible* map

$$\Phi_{\Theta\Sigma} : S_\Theta \leftarrow S_\Sigma$$

such that

$$M_\Theta = \Phi_{\Theta\Sigma\,*} M_\Sigma .$$

The map $\Phi_{\Theta\Sigma}$ could then interpreted as the *classical flow*, evolving the fixed-time classical fields from one Cauchy surface to the other. In particular, let us call an *Egorov theorem* the following sentence:

$$\bigl(^{\exists}\Phi_{\Theta\Sigma} : S_\Theta \leftarrow S_\Sigma\bigr)_{(\Sigma,\Theta)\in \mathrm{Cau}^2_{\mathbf{A}}} \text{ , with } \Phi_{\Theta\Sigma} \text{ compatible and } \Phi_{\Sigma\Sigma} = \mathrm{id} \text{ , such that}$$
$$\Bigl(\omega_h^{(\Sigma)} \xrightarrow[h\to 0]{(\mathfrak{P}\vee\mathfrak{T})_\Sigma} M_\Sigma \iff \forall (\Sigma,\Theta) \in \mathrm{Cau}^2_{\mathbf{A}} \text{ , } {}^{\mathrm{t}}U_h^{(\Sigma\Theta)}\omega_h^{(\Sigma)} \xrightarrow[h\to 0]{(\mathfrak{P}\vee\mathfrak{T})_\Theta} \Phi_{\Theta\Sigma\,*} M_\Sigma\Bigr) .$$

The terminology is borrowed from standard semiclassical analysis, although the Egorov theorem for finite dimensional systems holds in a stronger form. In some

[30] A complex quantum state can always be decomposed in four positive states. A complex state is semiclassical if each of the four positive states of its decomposition are semiclassical.

[31] The convergence of a semiclassical complex state to a complex cylindrical measure has to be intended as follows: all the four positive states of its decomposition converge separately to positive cylindrical measures (whose combination is a complex cylindrical measure).

semi-relativistic and non-relativistic quantum field theories, Egorov theorems have been proved rigorously [see e.g. 3–6, 9, 11]. They are proved to hold for a suitable set of Fock-normal families of states whose cylindrical Wigner measures are concentrated as Radon measures (on a space $\mathcal{Z}$ with some structure, usually a Hilbert space) [see 6, 35, for *a priori* conditions that guarantee concentration]. The classical flow is the same for all families of states, and it is explicitly identified to be the Hamiltonian flow $\Phi_{t-t_0,\mathcal{Z}}$ corresponding to a suitable classical nonlinear PDE, globally well-posed on $\mathcal{Z}$. Interestingly, in [4] the Egorov theorem holds for a *renormalized* quantum system (the Nelson model) and the corresponding naf classical flow (Schrdinger-Klein-Gordon system with Yukawa coupling), so at least in that case the renormalization does not affect the classical limit. On specific quantum states, in particular on coherent states, it is possible to prove Egorov-type results more easily exploiting the special semiclassical structure of such configurations [see *e.g.* 31, 32, 46–48, 56]. Let us conclude remarking that it does not seem possible to infer Egorov theorems from the (TS) axiom in a systematic fashion: they have to be proved on a case-by-case basis.

2.2.5 Invariant, KMS, and Ground States

The notions of Invariant, KMS, and Ground States play an important role in statistical and relativistic quantum mechanics (in Minkwoski spacetime). In order to introduce them we should define the concept of quantum symmetries and quantum dynamical systems. We will not do it in full generality, but we will consider only symmetries of quantum fields. Let $(X,\varsigma)\in\mathbf{Symp}_{\mathbb{R}}$ be a phase space, and let $(G,\cdot)$ be a (Lie) group that is represented by[32] $\bigl(s(g)\bigr)_{g\in G}$ on (X,ς). This induces a representation of the group on quantum fields (by action on generators, see § 1.6), that is extended uniquely to a group of *-automorphisms $\bigl(\mathfrak{s}_h(g)\bigr)_{g\in G}=\Bigl(\mathbb{W}_h\bigl(s(g)\bigr)\Bigr)_{g\in G}$ on $\mathbb{W}_h(X,\varsigma)$. Clearly, we have that

$$\mathfrak{s}_h(g_1)\mathfrak{s}_h(g_2)^{-1}W_h(x)=W_h\bigl(s(g_1\cdot g_2^{-1})x\bigr)\quad\bigl(\forall x\in X,\forall g_1,g_2\in G\bigr)\ .$$

Definition 2.7 (Symmetry group of quantum fields). *Let $(X,\varsigma)\in\mathbf{Symp}_{\mathbb{R}}$; and let $\bigl(\mathfrak{s}_h(g)\bigr)_{g\in G}$ be a representation of the group G on $\mathbb{W}_h(X,\varsigma)$. Then $\bigl(\mathfrak{s}_h(g)\bigr)_{g\in G}$ is a* group of symmetry transformations of quantum fields *iff $\bigl(^\exists s(g)\bigr)_{g\in G}$ representation of G on (X,ς) such that*

$$\mathfrak{s}_h(g)=\mathbb{W}_h\bigl(s(g)\bigr)\qquad\qquad(\forall g\in G)\ .$$

Symmetries of quantum fields play an important role in both nonrelativistic and relativistic quantum field theories. In particular, it is one of the Wightman axioms that the proper, orthochronous Poincar group is a symmetry group of quantum fields. It is another axiom that the theory is set in the GNS representation of an *invariant state* of the aforementioned Poincar group. The definition of an invariant state is rather intuitive, and it is the following.

[32] As it is customary, we are interested in automorphic representations. Therefore, in the category of symplectic spaces, the representation is linear and symplectic.

Definition 2.8 (Invariant State). *Let* $(X,\varsigma)\in\mathbf{Symp}_{\mathbb{R}}$ *; and let* $\big(\mathfrak{s}_h(g)\big)_{g\in G}$ *be a representation of the group* G *on* $\mathbb{W}_h(X,\varsigma)$*. Then a quantum state* $\omega_h\in\mathbb{S}_h(X,\varsigma)$ *is* G-invariant *iff*

$$^{\mathrm{t}}\mathfrak{s}_h(g)\omega_h := \omega_h\circ\mathfrak{s}_h(g)=\omega_h \qquad (\forall g\in G)\ .$$

Analogously, it is possible to give a definition of invariant classical states, *i.e.* invariant cylindrical measures associated to a phase space.

Definition 2.9 (Invariant measure). *Let* G *be a group, represented by* $\big(s(g)\big)_{g\in G}$ *on* $(X,\varsigma)\in\mathbf{Symp}_{\mathbb{R}}$*. Then a measure* $M\in\mathcal{M}_{\mathrm{cyl}}(X^*,X)$ *is* G-invariant *iff*

$$^{\mathrm{t}}s(g)_*M=M \qquad (\forall g\in G)\ .$$

Lemma 2.10. *A measure* $M\in\mathcal{M}_{\mathrm{cyl}}(X^*,X)$ *is* G*-invariant iff*

$$\hat{M}\big(s(g)x\big)=\hat{M}(x) \qquad (\forall x\in X,\forall g\in G)\ . \tag{17}$$

Using Theorem 1.8 and Eq. (17) it is then possible to define invariant measures for any space of cylindrical measures $\mathcal{M}_{\mathrm{cyl}}\big({}^{\exists}A,{}^{\exists}e_X(X)\big)$ *associated to* X*.*

Proof. The lemma is an easy consequence of Bochner's theorem, Theorem 1.8. ⊣

From Definitions 2.8 to 2.9, it is clear that we can apply Theorem 1.16 to nicely characterize the cylindrical Wigner measures of semiclassical invariant states with no loss of mass.

Proposition 2.11. *Let* $\big(\mathfrak{s}_h(g)\big)_{g\in G}$ *be a group of symmetry transformations of quantum fields on* $\mathbb{W}_h(X,\varsigma)$*, and* $\big(s(g)\big)_{g\in G}$ *the corresponding representation of* G *on* (X,ς)*. Then* the cylindrical Wigner measure *of any* G*-invariant semiclassical quantum state with no loss of mass* is G-invariant.

Proof. By Eq. (6), it follows that

$$\omega_{h_\beta}\xrightarrow[h_\beta\to 0]{\mathfrak{P}\vee\mathfrak{T}}M \quad\Longrightarrow\quad (\forall g\in G)\ {}^{\mathrm{t}}\mathfrak{s}_{h_\beta}(g)\omega_{h_\beta}\xrightarrow[h_\beta\to 0]{\mathfrak{P}\vee\mathfrak{T}}{}^{\mathrm{t}}s(g)_*M\ .$$

Hence from ${}^{\mathrm{t}}\mathfrak{s}_{h_\beta}(g)\omega_{h_\beta}=\omega_{h_\beta}$ it follows that ${}^{\mathrm{t}}s(g)_*M=M$. ⊣

In addition, by Theorem 1.11 we know that any G-invariant measure is the limit of at least one semiclassical quantum state, but such quantum state *may not be* G*-invariant*, and there may be invariant classical measures that are not the limit of any invariant semiclassical state (in fact, there are systems in which no invariant quantum state exists). For extremal (*i.e.* pure) invariant states, the so-called Haag's theorem [see, *e.g.*, 23, Corollaries 5.3.41-42] holds, and it plays a very important role in relativistic theories. In fact, one consequence of Haag's theorem is that free and interacting theories must correspond to inequivalent representations of the canonical commutation relations.

Theorem 2.12 (Haag). *Let $\omega_h, \varpi_h \in \mathbb{W}_h(X,\varsigma)'_+$ be two G-Abelian pure normalized states*[33]*. Then they are either equal or disjoint.*

At the classical level, however, it may happen that two different (therefore disjoint) G-Abelian pure normalized regular (therefore semiclassical) states converge to the same cylindrical Wigner measure. It would be interesting to find explicit examples in which it is the case, or examples in which they converge to two mutually singular Wigner measures.
A special set of invariant states that has been extensively studied in statistical mechanics is the subset of so-called *KMS states*.

Definition 2.13 (KMS state). *Let $\big(\mathfrak{s}_h(t)\big)_{t\in\mathbb{R}}$ be a representation of the abelian group $\mathbb{R}$ on $\mathfrak{B}_h \overset{\exists \mathrm{w}_X}{\hookleftarrow} \mathbb{W}_h(\exists X, \exists\varsigma)$. Then a quantum state $\omega_h \in \mathfrak{B}'_h$ is a $\mathfrak{s}_h$-KMS state at value $\beta\in\mathbb{R}$ iff*

$$\omega_h(b_h a_h) = \omega_h\big(a_h \mathfrak{s}_h(ih\beta) b_h\big) \qquad (\forall a_h, b_h \in \mathfrak{B}_h) \ . \tag{18}$$

Remark 2.14. It is sufficient to test the KMS condition on a norm dense and $\mathfrak{s}_h$-invariant subalgebra.

A KMS state is a state that is almost a trace (it is a trace only if $\beta = 0$), the deviation being measured by $\mathfrak{s}_h$. In this context, the abelian group $\mathbb{R}$ is usually interpreted as the group of time translations, and thus $\mathfrak{s}_h$ is the quantum dynamical map (and β is the inverse temperature). Semiclassically, one would like to prove that the cylindrical Wigner measures of semiclassical quantum states satisfy an equation of the following type:

$$\int \big\{a(z), b(z)\big\} \mathrm{d}M(z) = \beta \int b(z) \big\{a(z), \mathfrak{h}(z)\big\} \mathrm{d}M(z) \ , \tag{19}$$

for any a, b in a suitable set of classical observables, where $\mathfrak{h}$ is the classical Hamiltonian observable, and $\{\cdot,\cdot\}$ is a Poisson bracket. This "static" semiclassical KMS condition has been studied for systems with finitely many degrees of freedom [42, 43], but its origin from the quantum KMS condition was justified only formally. It is possible to derive Eq. (19) from Eq. (18) in our framework; however, unless X is finite dimensional, additional properties should hold, and they have to be proved case-by-case. For any $x, y \in X$, $\varsigma(x,\cdot) \in X^*$ and $\varsigma(x,\cdot) \neq \varsigma(y,\cdot)$ if $x \neq y$ (by the non-degeneracy of ς), and thus $X \overset{\mathrm{s}}{\hookrightarrow} X^*_X$. In addition, $\varsigma(\cdot,\cdot)$ extends to a symplectic form $\tilde{\varsigma}(\cdot,\cdot)$ on $\mathrm{s}(X)$ by $\tilde{\varsigma}(\cdot,\cdot) = \varsigma(\mathrm{s}^{-1}\,\cdot, \mathrm{s}^{-1}\,\cdot\,)$. The map s is a bijection iff X is finite dimensional. Since X^*_X is locally convex, there is a notion of smooth maps and therefore a Poisson bracket $\{\cdot,\cdot\}$ can be defined on $\big(\mathrm{s}(X), \tilde{\varsigma}\big)$. These two types of results can therefore be expected to be provable in suitable systems:

- If the generator[34] of $\big(\mathfrak{s}_h(t)\big)_{t\in\mathbb{R}}$ is $\delta_h = \frac{i}{h}[\mathfrak{h}_h,\,\cdot\,]$, with $\mathfrak{h}_h = \mathrm{Op}^h_{\frac{1}{2}}(\mathfrak{h})$, $\mathfrak{h} : X^*_X \to \mathbb{R}$ a cylindrical function with base Φ, and ω_h is a semiclassical quantum state

[33] The G-abelian states are a subset of the set of G-invariant states that satisfy an additional property of commutativity, let us omit the precise definition here [see, *e.g.*, 22, Definition 4.3.6].

[34] The generator δ_h of a (strongly continuous) group $\big(\mathfrak{s}_h(t)\big)_{t\in\mathbb{R}}$ (with suitable continuity properties) is a map from a dense domain $D(\delta_h) \subset \mathfrak{B}_h$ to $\mathfrak{B}_h$ such that for any $a_h \in \mathfrak{B}_h$, $\mathfrak{s}_h(t)a_h$ is differentiable with respect to t, and $\delta_h a_h = \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{s}_h(t)a_h\big|_{t=0}$.

such that $\omega_h(\mathfrak{h}_h) \leq C$ uniformly with respect to h; then

$$\omega_h \xrightarrow[h\to 0]{\mathfrak{P}} M$$
$$\Downarrow$$
$$\begin{aligned}\omega_h\Big(b_h a_h\Big) = \omega_h\Big(a_h \mathfrak{s}_h(ih\beta) b_h\Big) &\xrightarrow[h\to 0]{\mathfrak{P}} \int_{X_X^*/\Phi} \big\{a_\Phi(z), b_\Phi(z)\big\} \mathrm{d}\mu_\Phi(z) \\ &= \beta \int_{X_X^*/\Phi} b_\Phi(z)\big\{a_\Phi(z), \mathfrak{h}_\Phi(z)\big\}\mathrm{d}\mu_\Phi(z)\end{aligned}$$

for any $a_h = \mathrm{Op}^h_{\frac{1}{2}}(a)$ and $b_h = \mathrm{Op}^h_{\frac{1}{2}}(b)$, with $a, b : X_X^* \to \mathbb{R}$ smooth cylindrical functions with the same base Φ as $\mathfrak{h}$.

- If the generator of $\big(\mathfrak{s}_h(t)\big)_{t\in\mathbb{R}}$ is $\delta_h = \frac{i}{h}[\mathfrak{h}_h, \cdot\,]$, with $\mathfrak{h}_h$ a suitable quantization of some symbol $\mathfrak{h} : \mathrm{s}(X) \to \mathbb{R}$ (possibly only densely defined with domain $D(\mathfrak{h}) \subseteq \mathrm{s}(X)$ carrying a suitable topology), and ω_h is a semiclassical quantum state[35] with no loss of mass such that $\big\langle \Omega_{\omega_h}, \pi_{\omega_h}(\mathfrak{h}_h)\Omega_{\omega_h}\big\rangle_{\mathscr{H}_{\omega_h}} \leq C$ uniformly with respect to h (more regularity of the state could be necessary); then

$$\omega_h \xrightarrow[h\to 0]{\mathfrak{P}\vee\mathfrak{T}} \mu \in \mathcal{M}_{\mathrm{rad}}\big(D(\mathfrak{h})\big)$$
$$\Downarrow$$
$$\begin{aligned}\omega_h\Big(b_h a_h\Big) = \omega_h\Big(a_h \mathfrak{s}_h(ih\beta) b_h\Big) &\xrightarrow[h\to 0]{\mathfrak{P}\vee\mathfrak{T}} \int_{D(\mathfrak{h})} \big\{a(z), b(z)\big\} \mathrm{d}\mu(z) \\ &= \beta \int_{D(\mathfrak{h})} b(z)\big\{a(z), \mathfrak{h}(z)\big\}\mathrm{d}\mu(z)\end{aligned}$$

for any a_h, b_h that are suitable quantizations of smooth symbols $a, b : \mathrm{s}(X) \to \mathbb{R}$.

A special subset of the set of KMS states is the set of ground states. There are general algebraic definitions of ground states as KMS states with special properties, let us focus however on a more physical definition. Let $(\mathfrak{B}_h, \mathfrak{s}_h)$, $\mathfrak{B}_h \overset{\exists_{\mathrm{W}X}}{\hookleftarrow} \mathbb{W}_h(^{\exists}X, {}^{\exists}\varsigma)$, be an algebra of bosonic observables with evolution group $\mathfrak{s}_h$ whose generator is δ_h.

DEFINITION 2.15 (Ground state). *A state* $\omega_h \in \mathfrak{B}'_h$ *is a* ground state *iff* $\pi_{\omega_h}(\delta_h) = \frac{i}{h}[^{\exists}\mathfrak{h}_h, \cdot\,]$, *with the Hamiltonian* $\mathfrak{h}_h$ *self-adjoint and bounded from below on* $\mathscr{H}_{\omega_h}$, *and such that*

$$\mathfrak{h}_h \Omega_{\omega_h} = \lambda_0 \Omega_{\omega_h} \ , \ \lambda_0 = \min\big\{\lambda \in \mathrm{spec}(\mathfrak{h}_h)\big\} \ .$$

Taking into account the semiclassical properties of Invariant and KMS states, it is natural to expect that semiclassical quantum ground states would converge to

[35] Let us denote by $(\mathscr{H}_{\omega_h}, \pi_{\omega_h}, \Omega_{\omega_h})$ the GNS representation given by ω_h.

classical ground states. Suppose that ω_h is a semiclassical quantum ground state, and that $\mathfrak{h}_h$ is the quantization of some bounded from below and densely defined symbol $\mathfrak{h} : \mathrm{s}(X) \to \mathbb{R}$ (the classical energy, with domain $D(\mathfrak{h}) \subseteq \mathrm{s}(X)$). Then it is interesting to prove whether the following statement is true:

$$\omega_h \xrightarrow[h\to 0]{\mathfrak{P}\vee\mathfrak{T}} \mu \in \mathcal{M}_{\mathrm{rad}}\bigl(D(\mathfrak{h})\bigr) \implies \int_{D(\mathfrak{h})} \mathfrak{h}(z)\mathrm{d}\mu(z) = \inf_{z\in D(\mathfrak{h})} \mathfrak{h}(z) \; . \tag{20}$$

Suppose that (20) holds, and that $\mathfrak{h}$ has minimizers, *i.e.* suppose that $\varnothing \neq {}^{\exists}\mathrm{Min}(\mathfrak{h}) \subset D(\mathfrak{h})$ such that

$$\mathfrak{h}(z_0) = \inf_{z\in D(\mathfrak{h})} \mathfrak{h}(z) \qquad \bigl(\forall z_0 \in \mathrm{Min}(\mathfrak{h})\bigr) \; .$$

It is also interesting to see whether in this case it follows that

$$\mu = \delta_{{}^{\exists}z_0} \; , \; z_0 \in \mathrm{Min}(\mathfrak{h}) \; , \tag{21}$$

i.e. whether the cylindrical Wigner measure has to be concentrated on a classical minimizer. While statements that yield (20) has been proved at least in one suitable case [see, *e.g.*, 3], to the author's knowledge there are no results of type (21) in the literature.

2.2.6 Other perspectives

There are many other interesting problems that could be studied within this framework. Let us mention very briefly some of them. One is to study the convergence of quantum to classical ergodicity, and mixing. The idea is that under suitable conditions the cylindrical Wigner measures of ergodic (mixing) semiclassical quantum states should be ergodic (mixing) as well. Another is to study the classical limit of Haag-Ruelle scattering theory for quantum fields. Scattering theory in the classical limit has only been studied for coherent states in some specific cases [32, 46, 47], and it would be interesting to make a more systematic study for relativistic theories in the Haag-Ruelle framework using cylindrical Wigner measures and semiclassical techniques. Finally, to complement for curved spacetimes what we discussed in § 2.2.2 to 2.2.4, it would be interesting to study the semiclassical behavior of Hadamard states, that play the role of ground states in curved spacetimes [see 30, 44, 58, 72, 81, and references thereof contained for more information about Hadamard states]. It would also be interesting to develop a similar framework for the semiclassical analysis of fermionic quantum field theories, such as Dirac quantum fields. However, for fermions the semiclassical description is rather different, and it is better captured by a multiscale analysis [see 12, for additional details].

3 Weyl C*-algebras

In this section we introduce the Weyl C*-algebra of canonical commutation relations corresponding to an infinite dimensional Heisenberg group. We then describe some of the properties of composite quantum systems, consisting of a semiclassical and a fixed part.

3.1 Infinite-dimensional Heisenberg groups and the algebra of CCR

Let $(X, \varsigma) \in \mathbf{Symp}_{\mathbb{R}}$ be of arbitrary (infinite) dimension. Let us recall that the symplectic form $\varsigma : X \times X \to \mathbb{R}$ is non-degenerate, bilinear and antisymmetric. The Heisenberg group $\mathbb{H}(X, \varsigma)$ associated to (X, ς) is the space $X \times \mathbb{R}$ endowed with the group structure

$$\bigl(x, t\bigr) \cdot \bigl(y, s\bigr) = \bigl(x + y, t + s - \varsigma(x, y)\bigr) \ .$$

The set of elements $Z = \bigl\{(0, t), t \in \mathbb{R}\bigr\}$ is the center of the group. Among the representations of the Heisenberg group, one plays a prominent role in quantum theories and semiclassical analysis: the *Weyl C*-algebra*. From a physical standpoint, this algebra encodes in a natural way the canonical commutation relations of (bosonic) quantum systems. In semiclassical analysis, it is the starting point to define quantizations and pseudodifferential calculus. The Weyl C*-algebra is uniquely defined, up to *-isomorphisms, as the smallest C*-algebra containing the set

$$\bigl\{W(x), x \in X\bigr\} \ ,$$

together with the following three properties for its elements

- $W(x) \neq 0 \qquad (\forall x \in X)$
- $W(-x) = W(x)^* \qquad (\forall x \in X)$
- $W(x)W(y) = e^{-i\varsigma(x,y)} W(x + y) \qquad (\forall x, y \in X)$

As in § 1, we adopt the notation $\mathbb{W}_1(X, \varsigma) = \mathrm{C}^*\bigl(\bigl\{W(x), x \in X\bigr\}\bigr)$. From the definition, it follows that $W(0) = 1$ (identity element), and that each $W(x)$ is unitary. Therefore $\mathbb{W}_1(X, \varsigma)$ is a non abelian unital C*-algebra generated by unitaries. The map

$$\begin{aligned} X \times \mathbb{R} &\longrightarrow \mathbb{W}_1(X, \varsigma) \\ (x, t) &\longmapsto e^{it}\, W(x) \end{aligned} \ ,$$

together with the identification of the group product with the C*-algebra product, provides the unitary representation of the Heisenberg group in the Weyl C*-algebra. Therefore from now on we will focus on the Weyl C*-algebra, keeping in mind the underlying Heisenberg group structure.

The non abelian nature of the Heisenberg group, or equivalently of the Weyl algebra, is given by the symplectic factors $-i\varsigma(x, y)$ in the product. Borrowing an idea from deformation theory, it is natural to "measure" the noncommutativity of the Weyl algebra introducing a real parameter $h \geq 0$ such that when $h > 0$ the algebra is non-abelian, and when $h = 0$ it becomes abelian. This justifies the following definition.

Definition 3.1 (Weyl deformation). *Let* $(X, \varsigma) \in \mathbf{Symp}_{\mathbb{R}}$. *Then the* Weyl deformation $\bigl(\mathbb{W}_h(X, \varsigma)\bigr)_{h \geq 0}$ *is a family of C*-algebras. For any* $h \geq 0$, *the algebra* $\mathbb{W}_h(X, \varsigma)$ *is generated by the set*

$$\bigl\{W_h(x), x \in X\bigr\} \ ,$$

together with the three properties of its elements

- $W_h(x) \neq 0$ $(\forall x \in X)$
- $W_h(-x) = W_h(x)^*$ $(\forall x \in X)$
- $W_h(x)W_h(y) = e^{-ih\varsigma(x,y)}W_h(x+y)$ $(\forall x, y \in X)$

For any $h > 0$, the algebra $\mathbb{W}_h(X,\varsigma)$ is *-isomorphic to $\mathbb{W}_1(X,\varsigma)$, since $W_h(x) = W(h^{1/2}x)$. When $h = 0$ however, the algebra $\mathbb{W}_0(X,\varsigma)$ is an abelian unital C*-algebra of almost periodic functions [see § 7; and 18, for an introduction to almost periodic functions]. In other words, a Weyl deformation contains infinitely many identical copies of the Weyl C*-algebra, and a single abelian algebra of almost periodic functions.

3.2 Tensor product of C*-algebras and partial evaluation

For physical reasons [see, *e.g.*, 28, 29], we couple the Weyl algebra with another C*-algebra that represent some additional degrees of freedom that do not behave semiclassically (and therefore do not depend on h). Instead of $\bigl(\mathbb{W}_h(X,\varsigma)\bigr)_{h\geq 0}$, we consider the deformation with the additional degrees of freedom given by a C*-algebra of physical (either quantum or classical) observables $\mathfrak{A} \in \mathbf{C^*alg}$.

$$(\mathfrak{W}_h)_{h\geq 0} = \bigl(\mathbb{W}_h(X,\varsigma) \otimes_{\gamma_h} \mathfrak{A}\bigr)_{h\geq 0} \; ; \tag{22}$$

where the index γ_h stands for a suitable choice of cross norm for the tensor product C*-algebra [see, *e.g.*, 79]. There are some differences depending on what norm is chosen, as it will be highlighted in the following. In applications, it is sometimes important to consider the enveloping von Neumann algebra $\mathbb{W}_h(X,\varsigma)''$ in place of $\mathbb{W}_h(X,\varsigma)$, or other algebras that embed $\mathbb{W}_h(X,\varsigma)$. The majority of our results extend to any deformation $(\mathfrak{B}_h \otimes_{\gamma_h} \mathfrak{A})_{h\geq 0}$ such that $\mathbb{W}_h(^\exists X, {}^\exists\varsigma) \overset{\exists_{\mathrm{w}_h}}{\hookrightarrow} \mathfrak{B}_h$. It will be pointed out explicitly in the text whether a result extends or not to the aforementioned case. Finally, if one is interested only in the deformation $\bigl(\mathbb{W}_h(X,\varsigma)\bigr)_{h\geq 0}$, it suffices to take $\mathfrak{A}$ to be the trivial C*-algebra generated by a single element. On $\mathfrak{W}_h$, there are two natural maps that play an important role, and we call them partial evaluations. We recall that for any topological linear space $V \in \mathbf{TVS}$, we denote by V' the continuous dual of V (while V^* stands for the algebraic dual).

Definition 3.2 (Partial evaluation). *For any $h \geq 0$, define the* partial evaluation map

$$\mathbb{E}^{(\cdot)}_{h,1} : \mathfrak{W}_h' \to \mathcal{B}\bigl(\mathbb{W}_h(X,\varsigma), \mathfrak{A}'\bigr) \; ,$$

by its action

$$\mathbb{E}^{\omega_h}_{h,1}(w_h)(a) = \omega_h(w_h \otimes a) \quad \bigl(\forall \omega_h \in \mathfrak{W}_h' \, , \, \forall w_h \in \mathbb{W}_h(X,\varsigma) \, , \, \forall a \in \mathfrak{A}\bigr) \; .$$

The partial trace *of the complex quantum state $\omega_h \in \mathfrak{W}_h'$ is the partial evaluation of the identity element $\mathbb{E}^{\omega_h}_{h,1}(1) \in \mathfrak{A}'$. The partial evaluation $E^{(\cdot)}_{h,2} : \mathfrak{W}_h' \to \mathcal{B}\bigl(\mathfrak{A}, \mathbb{W}_h(X,\varsigma)'\bigr)$ is defined in a symmetric fashion.*

In the definition above, $\mathcal{B}(X,Y)$ stands for the space of continuous linear maps from X to Y. We chose to emphasize (perhaps with an heavy notation) the dependence on the semiclassical parameter h, for it will play an important role. The partial evaluation does what its name suggests: given a (complex) state on the tensor algebra, it evaluates any observable of the first algebra and gives as output a (complex) state acting on the second algebra alone. The partial evaluation map has some important properties that are summarized in the following proposition [see, *e.g.*, 79, for a proof].

Proposition 3.3. *For any $h \geq 0$, the evaluation map $\mathbb{E}_{h,1}$ is an isometry of $\mathfrak{W}'_h$ into $\mathcal{B}\bigl(\mathbb{W}_h(X,\varsigma),\mathfrak{A}'\bigr)$. In addition, an element $\omega_h \in \mathfrak{W}'_h$ is a state of total mass m_h* – i.e. *$\omega_h \in (\mathfrak{W}_h)'_+$ and $\lVert\omega_h\rVert_{\mathfrak{W}'_h} = m_h$ – if the resulting evaluation $\mathbb{E}^{\omega_h}_{h,1} : \mathbb{W}_h(X,\varsigma) \to \mathfrak{A}'$ is completely positive and the partial trace $\mathbb{E}^{\omega_h}_{h,1}(1) \in \mathfrak{A}'_+$ satisfies*

$$\Bigl\lVert \mathbb{E}^{\omega_h}_{h,1}(1) \Bigr\rVert_{\mathfrak{A}'} = m_h \; .$$

If γ_h is the maximal norm $\gamma_{\max}$, then $\mathbb{E}_{h,1}$ is onto *$\mathcal{B}\bigl(\mathbb{W}_h(X,\varsigma),\mathfrak{A}'\bigr)$ and the converse of the second statement holds.*

3.3 The generating map and regular states.

Given a state on the Weyl C*-algebra, it is possible to define its generating functional [see 77] or noncommutative Fourier transform; in our framework it is not a functional, but a map from X to $\mathfrak{A}'$. Throughout this section, we take $h > 0$ if not specified otherwise.

Definition 3.4 (Generating map). *Let $\omega_h \in (\mathfrak{W}_h)'_+$ be a state, we define the* generating map *$\mathcal{G}_{\omega_h} : X \to \mathfrak{A}'$ by*

$$\mathcal{G}_{\omega_h}(x) = \mathbb{E}^{\omega_h}_{h,1}\bigl(W_h(x)\bigr) \qquad\qquad (\forall x \in X) \; .$$

The generating map is used to define a very important class of states (and hence its name), the so-called regular states. Regular states are those that allow for a natural semiclassical description in term of Wigner measures.

Definition 3.5 (Regular states). *Let $\omega_h \in (\mathfrak{W}_h)'_+$ be a state, $\mathcal{G}_{\omega_h}$ its generating map. Then ω_h is* regular *iff for any $x \in X$, the $\mathbb{R}$-action*

$$\mathcal{G}_{\omega_h}(\,\cdot\, x) : \mathbb{R} \to \mathfrak{A}'$$

is continuous when $\mathfrak{A}'$ is endowed with the $\sigma(\mathfrak{A}',\mathfrak{A})$ topology (ultraweakly continuous).

There are many equivalent definitions of regular states. We also make use of the following one, that can be proved, *e.g.*, using the properties of the map $\mathbb{E}_{h,2}$ and the equivalent result for trivial $\mathfrak{A}$ [see 23, § 5.2.3]. Let (R,ς) be a finite dimensional real symplectic vector space. We say that a state ϱ_h on $\mathbb{W}_h(R,\varsigma) \otimes_{\gamma_h} \mathfrak{A}$ is *normal* iff for any $a \in \mathfrak{A}_+$, $\mathbb{E}^{\varrho_h}_{h,2}(a)$ is a (positive) trace class operator in the unique irreducible representation of $\mathbb{W}_h(R,\varsigma)$ (the uniqueness up to unitary equivalence of such representation is guaranteed by Stone-von Neumann's theorem).

Proposition 3.6. *A state ω_h is regular iff for any finite dimensional $R \subset X$ its restriction ϱ_h to $\mathbb{W}_h(R,\varsigma) \otimes_{\gamma_h} \mathfrak{A}$ is a normal state. In particular, it follows that the generating map of a regular state is ultraweakly continuous when restricted to any finite dimensional subspace of X.*

The following result is an extension to our setting of the main result of the aforementioned paper of Segal [77]. The idea is that regular states are uniquely determined by the generating map, and the latter is "almost" completely positive (up to a complex phase factor) and ultraweakly continuous on finite dimensional subspaces.

Proposition 3.7. *For any $h > 0$, a map $\mathcal{G}_h : X \to \mathfrak{A}'$ is the generating map of a regular state $\omega_h \in (\mathfrak{W}_h)'_+$ of partial trace $\alpha_h \in \mathfrak{A}'_+$ only if all the restrictions of $\mathcal{G}_h$ to finite dimensional subspaces of X are ultraweakly continuous, $\mathcal{G}_h(0) = \alpha_h$ and*

$$\sum_{j,k \in J} \mathcal{G}_h(x_j - x_k) e^{ih\varsigma(x_j,x_k)}(a_k^* a_j) \geq 0 \,;$$

where the $x_j \in X$ are arbitrary as well as the $a_j \in \mathfrak{A}$, and J is any finite index set. If in addition γ_h is the maximal norm, the converse holds and the map $\mathcal{G}_h$ uniquely determines ω_h.

Remark 3.8. If in $\mathfrak{W}_h$, with maximal norm, we replace $\mathbb{W}_h(X,\varsigma)$ by its enveloping von Neumann algebra or any algebra that embeds the Weyl C*-algebra as a subalgebra, $\mathcal{G}_h$ does not determine ω_h uniquely.

Proof. Let us start with the easy "only if" part for generic cross norms. Ultraweak continuity follows from Proposition 3.6, the other two properties follow from Proposition 3.3: in fact $W_h(0) = 1$;

$$\sum_{j,k \in F} \mathbb{E}^{\omega_h}_{h,1}\big(W_h(x_k)^* W_h(x_j)\big)(a_k^* a_j) \geq 0$$

by complete positivity of $\mathbb{E}^{\omega_h}_{h,1}$; and $W_h(-x)W_h(y) = e^{ih\varsigma(x,y)} W_h(x-y)$ by definition of the Weyl algebra. To prove the "if" part and uniqueness when γ_h is the maximal cross norm, we act with the generating map on an arbitrary $a \in \mathfrak{A}_+$. Since $\mathfrak{A} = \mathfrak{A}_+ - \mathfrak{A}_+ + i(\mathfrak{A}_+ - \mathfrak{A}_+)$, this suffices to characterize the map $\mathcal{G}_h : X \to \mathfrak{A}'$ by linearity. Let us denote by $\mathcal{G}_h^a(\cdot) = \mathcal{G}_h(\cdot)(a) : X \to \mathbb{C}$. By Theorem 1 of [77], to $\mathcal{G}_h^a$ there corresponds a unique regular state $\varrho_h^a \in \big(\mathbb{W}_h(X,\varsigma)\big)'_+$ such that $\mathcal{G}_h^a(\cdot) = \varrho_h^a\big(W_h(\cdot)\big)$. By the last property of $\mathcal{G}_h$ this defines a unique completely positive map $\varrho_h^{(\cdot)} : \mathfrak{A} \to \mathbb{W}_h(X,\varsigma)'$. Therefore the analogous of Proposition 3.3 for $\mathbb{E}_{h,2}$ yields that $\omega_h = \mathbb{E}_{h,2}^{-1}(\varrho_h^{(\cdot)})$ is a positive regular measure of total mass $\mathcal{G}_h(0)$, uniquely determined by $\mathcal{G}_h$. ⊣

4 Semiclassical compactness for families of regular states

In this section we study the semiclassical behavior $h \to 0$ of families of regular states. In particular, we prove compactness, in a suitable topology, of families of generating maps.

4.1 Finite dimensional Weyl algebra

By Proposition 3.6, if X is finite dimensional then any state that is normal with respect to the Schrdinger representation is regular. In other words, the semiclassical analysis of regular states on $\mathbb{W}_h(X,\varsigma)$, with $X \cong \mathbb{R}^d \times (\mathbb{R}^d)'$, reduces to the semiclassical analysis of families in $\mathfrak{S}^1\bigl(L^2(\mathbb{R}^d)\bigr)_+$, the cone of positive trace class operators. By linearity, it is sufficient to study families of vectors $(u_h)_{h>0} \subset L^2(\mathbb{R}^d)$ as $h \to 0$ (with norms uniformly bounded with respect to h). The commutative objects corresponding to the family $(u_h)_{h>0}$ are called Wigner measures, and are finite Borel measures on $\mathbb{R}^d \times (\mathbb{R}^d)'$ (equivalently on X). In addition, to any family $(u_h)_{h>0}$ with uniformly bounded norms there corresponds at least one Wigner measure. We formulate this well-known result [66, Thorme III.1] in a slightly different way, that is better suited for our algebraic approach.
Let T be a topological space, and S a set; we denote by T^S_{s} the space of functions from S to T with the topology of simple (pointwise) convergence, that coincides with the product topology. We use the following notation for semiclassical generalized sequences (nets): $(\diamond_{h_\beta})_{\beta\in B}$, $h_\beta \to 0$, is a net of objects indexed by the directed set B, such that $\diamond_{h_\beta} = \diamond_{h_{\beta'}}$ whenever $h_\beta = h_{\beta'}$, $(h_\beta)_{\beta\in B}$ is relatively compact in the topology of $\mathbb{R}$, and

$$\lim_{\beta\in B} h_\beta = 0 \; .$$

A point x is a cluster point for $(x_\beta)_{\beta\in B}$ if there exists a subnet $(x_b)_{b\in\underline{B}}$ such that $\lim_{b\in\underline{B}} x_b = x$. A point x is a sequential cluster point for $(x_\beta)_{\beta\in B}$ if it is a cluster point and the corresponding subnet is a subsequence (*i.e.* if $\underline{B} = \mathbb{N}$).
A uniformly bounded net of regular states $(\varrho_{h_\beta})_{\beta\in B}$ on $\mathbb{W}_{h_\beta}(X,\varsigma) \otimes_{\gamma_{h_\beta}} \mathfrak{A}$, with $X \cong X'$ finite dimensional, defines a net of linear maps $(H_{\varrho_{h_\beta}})_{\beta\in B}$ from $\mathfrak{A}$ to the dual $C_0^\infty(X')'$ of smooth compactly supported functions by the Bochner integral

$$H_{\varrho_h}(a)(\varphi) = \varrho_h\Bigl(\int_X \hat{\varphi}(x) W_h(\pi x) \mathrm{d}L_X(x) \otimes a\Bigr) \quad \bigl(\forall a \in \mathfrak{A}\,,\, \forall \varphi \in C_0^\infty(X')\bigr) \; , \quad (23)$$

where L_X is the Lebesgue measure of X. In addition, each $H_{\varrho_{h_\beta}}$ is also a continuous linear map from $\mathfrak{A}$ to $C_0(X')'$, the dual of the compactly supported continuous functions.

Proposition 4.1. *Let $(\varrho_{h_\beta})_{\beta\in B}$, $h_\beta \to 0$, be a net of regular states that acts on $\bigl(\mathbb{W}_{h_\beta}(X,\varsigma) \otimes_{\gamma_{h_\beta}} \mathfrak{A}\bigr)_{\beta\in B}$ with X* of finite dimension. *If*

$$\sup_{\beta\in B} \varrho_{h_\beta}\bigl(W_{h_\beta}(0)\bigr) < \infty \; ,$$

then for any $a \in \mathfrak{A}$, $\Bigl\{H_{\varrho_{h_\beta}}(a), \beta \in B\Bigr\}$ is relatively compact in the $\sigma\bigl(C_0(X')', C_0(X)\bigr)$ topology, and the cluster points of any subnet $\bigl(H_{\varrho_{h_b}}(a_+)\bigr)_{b\in\underline{B}}$, $h_b \to 0$, with $a_+ \in \mathfrak{A}_+$ are positive *linear functionals. Thus by Riesz-Markov's theorem they can be identified with finite Radon measures on X'.*

Let us fix now $a \in \mathfrak{A}_+$, and a subnet $\big(H_{\varrho_{h_b}}(a)\big)_{b\in\underline{B}}$, $h_b \to 0$, converging to $\mu_a \in \mathcal{M}_{\mathrm{rad}}(X')$. In addition, denote by $(\psi_\varepsilon)_{\varepsilon\in(0,1)} \subset C_0^\infty(X')$ an approximate identity on X'. The following four statements are equivalent:

$$\big(\mathcal{G}_{\varrho_{h_b}}(\,\cdot\,)(a)\big)_{b\in\underline{B}} \xrightarrow[h_b\to 0]{\mathfrak{C}_{\mathrm{s}}^X} \hat{\mu}_a(\,\cdot\,) \qquad ; \qquad \text{(i)}$$

$$\lim_{\varepsilon\to 0}\lim_{b\in\underline{B}} \varrho_{h_b}\big(W_{h_b}(x)\otimes a - \mathrm{Op}_{\frac{1}{2}}^{h_b}(^{\exists}\psi_\varepsilon\, e^{2ix(\cdot)})\otimes a\big) = 0 \qquad \big(\forall x\in X\big)\,; \qquad \text{(ii)}$$

$$\lim_{\varepsilon\to 0}\lim_{b\in\underline{B}} \varrho_{h_b}\big(a - \mathrm{Op}_{\frac{1}{2}}^{h_b}(^{\exists}\psi_\varepsilon)\otimes a\big) = 0 \qquad ; \qquad \text{(iii)}$$

$$\lim_{b\in\underline{B}} \varrho_{h_b}\big(W_{h_b}(0)\otimes a\big) = \mu_a(X') \qquad . \qquad \text{(iv)}$$

The cluster points μ_a of Proposition 4.1, considered as Radon measures, are the so-called Wigner or semiclassical measures.

4.2 Compactness and convergence in infinite-dimensional Weyl algebras

Let $(X,\varsigma) \in \mathbf{Symp}_{\mathbb{R}}$ be of arbitrary dimension, and let

$$(X_\lambda)_{\lambda\in F} \subset 2^X$$

be a collection of finite dimensional symplectic subspaces, indexed by a set F. In addition, let ω_h be a regular state on $\mathfrak{W}_h$. In analogy with (23), for any $\lambda \in F$, define the map $H_{\omega_h}^{(\lambda)}$ from $\mathfrak{A}$ to $C_0^\infty(X'_\lambda)'$ by

$$H_{\omega_h}^{(\lambda)}(a)(\varphi_\lambda) = \omega_h\Big(\int_{X_\lambda} \hat{\varphi}_\lambda(x) W_h(\pi x) \mathrm{d}L_{X_\lambda}(x)\otimes a\Big) \quad \big(\forall a\in\mathfrak{A}\,, \forall \varphi_\lambda \in C_0^\infty(X'_\lambda)\big)\,. \tag{24}$$

Let us consider now the set

$$H_B = \prod_{\lambda\in F}\prod_{a\in\mathfrak{A}_+}\Big\{H_{\omega_{h_\beta}}^{(\lambda)}(a), \beta\in B\Big\}\,, \tag{25}$$

and denote by $\mathfrak{P}_F$ the product topology on H_B, with each $\Big\{H_{\omega_{h_\beta}}^{(\lambda)}(a), \beta \in B\Big\}$ endowed with the weak $\sigma\big(C_0(X'_\lambda)', C_0(X_\lambda)\big)$ topology.

Lemma 4.2. *Let $(\omega_{h_\beta})_{\beta\in B}$, $h_\beta\to 0$, be a net of regular states on $(\mathfrak{W}_{h_\beta})_{\beta\in B}$. If*

$$\sup_{\beta\in B} \omega_{h_\beta}\big(W_{h_\beta}(0)\big) < \infty\,,$$

then H_B is $\mathfrak{P}_F$-relatively compact, and the cluster points of any subnet

$$\Big(\mathfrak{h}_{h_b}\Big)_{b\in\underline{B}} \subset H_B\,,$$

$h_b\to 0$, can be identified with a unique family of vector valued measures

$$(\mu_\lambda)_{\lambda\in F}\,,\ \forall\lambda\in F\,,\ \mu_\lambda \in \mathcal{M}_{\mathrm{rad}}(X'_\lambda;\mathfrak{A}'_+)\,.$$

Proof. Compactness in the product topology follows immediately from Proposition 4.1. In addition, the cluster points $\mathfrak{h}$ of $(\mathfrak{h}_{h_b})_{b\in\underline{B}}$, $h_b \to 0$, have the form

$$a \mapsto \mathfrak{h}(a)\ ,\ \mathfrak{h}(a) = (\mathfrak{h}_\lambda(a))_{\lambda\in F}\ ,$$

with each $\mathfrak{h}_\lambda(a)$ a positive element of $C_0(X'_\lambda)'$. Therefore by Riesz-Markov's theorem, for any $a \in \mathfrak{A}_+$ there exists a unique $\mu_\lambda(a) \in \mathcal{M}_{\mathrm{rad}}(X'_\lambda; \mathbb{R}^+)$ whose action on $C_0(X'_\lambda)$ agrees with $\mathfrak{h}_\lambda(a)$. In other words, there exists an injective map $\mathcal{R}$ such that

$$\bigl(a \mapsto \mathfrak{h}_\lambda(a)\bigr)_{\lambda\in F} \overset{\mathcal{R}}{\longmapsto} \bigl(a \mapsto \mu_\lambda(a)\bigr)_{\lambda\in F}\ . \tag{26}$$

In addition, by linearity it follows that for any $\lambda \in F$,

$$\bigl(a \mapsto \mu_\lambda(a)\bigr) \in \mathrm{Hom}_{\mathrm{mon}}\bigl(\mathfrak{A}_+, \mathcal{M}_{\mathrm{rad}}(X'_\lambda; \mathbb{R}^+)\bigr)\ .$$

Therefore by Theorem A.3 there exists a bijection $\mathcal{P}$ such that, with $\mathcal{Q} = \mathcal{P} \circ \mathcal{R}$,

$$\bigl(a \mapsto \mathfrak{h}_\lambda(a)\bigr)_{\lambda\in F} \overset{\mathcal{Q}}{\longmapsto} \bigl(\mu_\lambda\bigr)_{\lambda\in F}\ ;\ \forall\lambda \in F\ ,\ \mu_\lambda \in \mathcal{M}_{\mathrm{rad}}(X'_\lambda; \mathfrak{A}'_+)\ . \tag{27}$$

⊣

Let us now turn attention to the converging nets whose cluster points have no loss of mass. For these nets, there is an easier characterization of cluster points by means of the generating maps, introduced in § 3.3. The set of generating maps is always relatively compact with respect to the topology of simple convergence. In fact, let $(\mathfrak{W}_h)_{h\geq 0}$ be the tensor Weyl deformation introduced in § 3.2, with X of arbitrary (infinite) dimension. We are interested in *semiclassical states* (see Definition 1.1), *i.e.* generalized bounded sequences of regular states

$$(\omega_{h_\beta})_{\beta\in B}\quad ,\quad \lim_{\beta\in B} h_\beta = 0\quad ,\quad \sup_{\beta\in B}\ \omega_{h_\beta}(1) = m < \infty\ .$$

Let $\mathfrak{A}'_{\mathfrak{A}}$ be the continuous dual of $\mathfrak{A}$ endowed with the ultraweak $\sigma(\mathfrak{A}', \mathfrak{A})$ topology. Let us denote by $G_B \subset (\mathfrak{A}'_{\mathfrak{A}})^X$ and $G_B(x) \subset \mathfrak{A}'_{\mathfrak{A}}$ the following sets:

$$G_B = \Bigl\{\mathcal{G}_{\omega_{h_\beta}}, \beta \in B\Bigr\}\ ;\ G_B(x) = \Bigl\{\mathcal{G}_{\omega_{h_\beta}}(x), \beta \in B\Bigr\}\ ,\ x \in X\ .$$

The first result is that the family of images $G_B(x)$ is pointwise compact for any $x \in X$, and therefore G_B is relatively compact as a subset of the space of functions from X to $\mathfrak{A}'_{\mathfrak{A}}$.

Lemma 4.3. *Let $(\omega_{h_\beta})_{\beta\in B}$ be a semiclassical state in the Weyl deformation $(\mathfrak{W}_{h_\beta})_{\beta\in B}$. Then $G_B(x)$ is relatively compact for any $x \in X$. It then follows that G_B is relatively compact as a subset of $(\mathfrak{A}'_{\mathfrak{A}})^X_{\mathrm{s}}$.*

Proof. It follows from Definition 3.4 of the generating map – since the Weyl operators are unitary – that for any $x \in X$, $\beta \in B$ and $a \in \mathfrak{A}$

$$\bigl|G_{\omega_{h_\beta}}(x)(a)\bigr| \leq \|\omega_{h_\beta}\|_{\mathfrak{W}'_{h_\beta}} \|a\|_{\mathfrak{A}} \leq m\|a\|_{\mathfrak{A}}\ .$$

Therefore $G_B(x)$ is contained in the ball of radius m of $\mathfrak{A}'$, and therefore it is relatively compact in the ultraweak topology by Banach-Alaoglu's theorem. ⊣

Let us now consider a converging net $(\mathcal{G}_{\omega_{h_b}})_{b\in\underline{B}}$, $h_b \to 0$, in the relatively compact set G_B, and denote its restriction to any finite dimensional subspace $R \subset X$ by $(\mathcal{G}_{\omega_{h_b}}|_R)_{b\in\underline{B}}$. Now, let $a \in \mathfrak{A}$; then

$$a = a^+_{\mathrm{R}} - a^-_{\mathrm{R}} + i(a^+_{\mathrm{I}} - a^-_{\mathrm{I}})\ , \quad a^+_{\mathrm{R}}, a^-_{\mathrm{R}}, a^+_{\mathrm{I}}, a^-_{\mathrm{I}} \in \mathfrak{A}_+\ .$$

A function $f \in (\mathfrak{A}'_{\mathfrak{A}})^R$ is continuous iff $f(\cdot)(a) : R \to \mathbb{C}$ is continuous for any $a \in \mathfrak{A}$. By linearity it follows, using the decomposition above, that $f \in (\mathfrak{A}'_{\mathfrak{A}})^R$ is continuous iff $f(\cdot)(a_+)$ is continuous for any $a_+ \in \mathfrak{A}_+$. For any $b \in \underline{B}$, define

$$\mathcal{G}^{a_+}_{\omega_{h_b}}\big|_R : R \to \mathbb{C} \quad , \quad \mathcal{G}^{a_+}_{\omega_{h_b}}\big|_R(r) = \mathcal{G}_{\omega_{h_b}}(r)(a_+)\ (\forall r \in R)\ .$$

Lemma 4.4. *For any $a_+ \in \mathfrak{A}_+$ and $(R_0, \varsigma) \subset (X, \varsigma)$ a symplectic subspace, $\mathcal{G}^{a_+}_{\omega_{h_b}}\big|_{R_0}$ is the generating functional of the normal state $\mathbb{E}^{\omega_{h_b}}_{h_b,2}(a_+)|_{\mathbb{W}_{h_b}(R_0,\varsigma)}$.*

Proof. By Proposition 3.6, since ω_{h_b} is regular then the restricted states

$$\mathbb{E}^{\omega_{h_b}}_{h_b,2}(a_+)|_{\mathbb{W}_{h_b}(R_0,\varsigma)}$$

are normal with respect to the Schrdinger representation. It is easy to check that, for any $r \in R_0$,

$$\mathcal{G}_{\omega_{h_b}}(r)(a_+) = \mathcal{G}_{\mathbb{E}^{\omega_{h_b}}_{h_b,2}(a_+)}(r)\ .$$

⊣

As in Proposition 4.1, let us always denote by

$$(\psi^{(\lambda)}_\varepsilon)_{\varepsilon\in(0,1)} \subset C^\infty_0(X'_\lambda) =: \mathcal{D}(X'_\lambda) \tag{28}$$

a smooth function, approximating the identity in the sense of distributions.

Proposition 4.5. *Let $(X_\lambda)_{\lambda\in F} \subset 2^X$ be a family of finite dimensional symplectic subspaces such that $\bigcup_{\lambda\in F} X_\lambda = X$, and $(\omega_{h_\beta})_{\beta\in B}$, a semiclassical quantum state on $(\mathfrak{W}_{h_\beta})_{\beta\in B}$. In addition, suppose that $(\mathfrak{h}_{h_b})_{b\in\underline{B}} \subset H_B$, $h_b \to 0$, is $\mathfrak{P}_F$-convergent, with limit $(\mu_\lambda)_{\lambda\in F}$ (see Lemma 4.2). Then the following four statements are equivalent:*

$$\mathcal{G}_{\omega_{h_b}} \xrightarrow[h_b\to 0]{\mathbb{C}^X_{\mathrm{s}}} g \quad , \quad g\big|_{X_\lambda} = \hat{\mu}_\lambda \in C(X_\lambda, \mathfrak{A}'_{\mathfrak{A}}) \qquad ; \quad \text{(i)}$$

$$\lim_{\varepsilon\to 0}\lim_{b\in\underline{B}} \omega_{h_b}\big(W_{h_b}(x)\otimes a - \mathrm{Op}^{h_b}_{\frac{1}{2}}(^\exists\psi^{(\lambda)}_\varepsilon\ e^{2ix(\cdot)})\otimes a\big) = 0 \qquad \big(\forall\lambda\in F\,,\ \forall x \in X_\lambda\,,\ \forall a \in \mathfrak{A}_+\big) \quad ; \quad \text{(ii)}$$

$$\lim_{\varepsilon\to 0}\lim_{b\in\underline{B}} \omega_{h_b}\big(a - \mathrm{Op}^{h_b}_{\frac{1}{2}}(^\exists\psi^{(\lambda)}_\varepsilon)\otimes a\big) = 0 \qquad \big(\forall\lambda\in F\,,\ \forall a \in \mathfrak{A}_+\big) \quad ; \quad \text{(iii)}$$

$$\lim_{b\in\underline{B}} \omega_{h_b}\big(W_{h_b}(0)\big) = \mu_\lambda(0) \qquad . \quad \text{(iv)}$$

A semiclassical quantum state that satisfies Eqs. (i) to (iv) is called a state with no loss of mass.

Proof. For any $\lambda \in F$, $g\big|_{X_\lambda}$ is continuous iff $g\big|_{X_\lambda}(a_+) \in C(X_\lambda, \mathbb{C})$ for any $a_+ \in \mathfrak{A}_+$. Let us consider the net $(\mathcal{G}^{a_+}_{\omega_{h_b}}|_{X_\lambda})_{b\in\underline{B}}$. By Lemma 4.4 it is the family of generating functionals of the net of normal states $\big(\mathbb{E}^{\omega_{h_b}}_{h_b,2}(a_+)|_{\mathbb{W}_{h_b}(X_\lambda,\varsigma)}\big)_{b\in\underline{B}}$ on $\big(\mathbb{W}_{h_b}(X_\lambda,\varsigma)\big)_{b\in\underline{B}}$. Therefore by Proposition 4.1 it follows that the statements (i) to (iv) are equivalent. ⊣

Hence we have proved that given a semiclassical net of quantum states, and a family $(X_\lambda)_{\lambda\in F}$ of finite dimensional subspaces of X, there is always at least one family of Radon vector measures, on each X'_λ, associated to it. In addition, if $\bigcup_{\lambda\in F} X_\lambda = X$, and each measure μ_λ does not lose mass, then the corresponding net of generating maps converges, and the limit is ultraweakly continuous when restricted to the finite dimensional subsets $(X_\lambda)_{\lambda\in F}$. Let us formulate this generalization of Proposition 4.1 to spaces of arbitrary dimension as a theorem. This theorem is a generalization of Theorems 1.3 and 1.10, as it will become clearer after the introduction of the topologies of semiclassical convergence in § 5.

THEOREM 4.6. *Let $(X,\varsigma) \in \mathbf{Symp}_{\mathbb{R}}$, $\mathfrak{A} \in \mathbf{C^*alg}$, and $(\mathfrak{W}_h)_{h\geq 0}$ the corresponding Weyl deformation (22). For any semiclassical quantum state $(\omega_{h_\beta})_{\beta\in B}$, and for any collection $(X_\lambda)_{\lambda\in F}$ of finite dimensional symplectic subspaces of X, there exists a nonempty set of cluster points for H_B in the $\mathfrak{P}_F$ product topology, and any cluster point is identified with a family $(\mu_\lambda)_{\lambda\in F}$ of finite $\mathfrak{A}'_+$-valued Radon measures on each X'_λ.*
In addition, there exists a nonempty set of cluster points of the family of generating maps $(\mathcal{G}_{\omega_{h_\beta}})_{\beta\in B}$. Each cluster point g satisfies:

$$\sum_{j,k\in J} g(x_j - x_k)(a_k^* a_j) \geq 0 \; ;$$

where the $x_j \in X$ are arbitrary as well as the $a_j \in \mathfrak{A}$, and J is any finite index set. For a convergent net $\mathfrak{h}_{h_b} \to_{\mathfrak{P}_F} (\mu_\lambda)_{\lambda\in F}$, if $\bigcup_{\lambda\in F} X_\lambda = X$ then the following four statements are equivalent:

$$\mathcal{G}_{\omega_{h_b}} \xrightarrow[h_b\to 0]{\mathbb{C}^X_{\mathrm{s}}} g \quad , \quad g\big|_{X_\lambda} = \hat{\mu}_\lambda \in C(X_\lambda, \mathfrak{A}'_{\mathfrak{A}}) \qquad ; \quad \text{(i)}$$

$$\lim_{\varepsilon\to 0}\lim_{b\in\underline{B}} \omega_{h_b}\big(W_{h_b}(x)\otimes a - \mathrm{Op}^{h_b}_{\frac{1}{2}}({}^{\exists}\psi^{(\lambda)}_\varepsilon\; e^{2ix(\cdot)})\otimes a\big) = 0 \qquad \big(\forall \lambda \in F\,,\ \forall x \in X_\lambda\,, \forall a \in \mathfrak{A}_+\big) \quad ; \quad \text{(ii)}$$

$$\lim_{\varepsilon\to 0}\lim_{b\in\underline{B}} \omega_{h_b}\big(a - \mathrm{Op}^{h_b}_{\frac{1}{2}}({}^{\exists}\psi^{(\lambda)}_\varepsilon)\otimes a\big) = 0 \qquad \big(\forall \lambda \in F\,, \forall a \in \mathfrak{A}_+\big) \quad ; \quad \text{(iii)}$$

$$\lim_{b\in\underline{B}} \omega_{h_b}\big(W_{h_b}(0)\big) = \mu_\lambda(0) \quad . \quad \text{(iv)}$$

COROLLARY 4.7. *If $F = \mathbb{N}$, and the resulting sequence $(X_n,\varsigma)_{n\in\mathbb{N}}$ of symplectic subspaces of (X,ς) satisfies $\bigcup_{n\in\mathbb{N}} X_n = X$, then from any net $(\omega_{h_\beta})_{\beta\in B}$ it is possible to extract a* subsequence *$(\omega_{h_j})_{j\in\mathbb{N}}$ such that*

$$\lim_{j\to\infty} \mathfrak{h}_{h_j} = \mathfrak{h}\,,\ \lim_{j\to\infty} \mathcal{G}_{\omega_{h_j}} = g\,.$$

Proof. The proof is the standard diagonal trick. Let us prove it for the generating map $\mathcal{G}$. On X_0, it is possible to extract the subsequence $h^0_{\beta_k}$ such that $\mathcal{G}_{\omega_{h^0_{\beta_k}}}\big|_{X_0} \to g\big|_{X_0} \in C(X_0, \mathfrak{A}'_{\mathfrak{A}})$. On X_1 it is possible to extract a subsequence $h^1_{\beta_k}$ of $h^0_{\beta_k}$ such that $\mathcal{G}_{\omega_{h^1_{\beta_k}}}\big|_{X_1} \to g\big|_{X_1} \in C(X_1, \mathfrak{A}'_{\mathfrak{A}})$, and so on. Therefore the diagonal sequence $h_j = h^j_{\beta_j}$ yields the pointwise convergence to $g \in (\mathfrak{A}'_{\mathfrak{A}})^X$, such that $g|_{X_n} \in C(X_n, \mathfrak{A}'_{\mathfrak{A}})$ for any $n \in \mathbb{N}$. ⊣

5 Cylindrical Wigner measures and topologies of semiclassical convergence

Theorem 4.6 above proves that any semiclassical quantum state has at least a classical counterpart. However, it would be suitable to identify such classical object more explicitly. If X is finite dimensional, the classical characterization given by Proposition 4.1 is satisfactory, and we would like to obtain an analogous one for spaces X of arbitrary dimension. If X is infinite dimensional however, the structure is richer, and the situation is more involved. First of all, there are many inequivalent topologies that are admissible on X, and it is therefore possible to put X in duality with many spaces V in a way such that $X \cong V'$. Analogously, it is possible to interpret X as a subset $\mathcal{A}$ of the set of functions on some set A in many ways. In this section, we discuss mostly how duality in linear spaces can be used to identify the classical counterparts of semiclassical states with *cylindrical measures*. Let us recall that a cylindrical measure on a topological vector space $V \in \mathbf{TVS}$ can be seen equivalently as a finitely additive inner regular measure on the algebra of cylinders (induced by the duality of V and V'), or as a projective family of Borel measures in the space of finite dimensional quotients of V. A concise definition of vector valued cylindrical measures in topological vector spaces is given in Appendix A.2.2. In the same section, vector-valued Bochner's theorem is also discussed (Theorem A.17). Thanks to Theorem A.17, it is possible to identify in a unique fashion cylindrical measures with their Fourier transform (or characteristic maps), *i.e.* with completely positive maps on V' that are continuous when restricted to any finite dimensional subspace of V'. In the light of Theorem 4.6, it is clear why they can be considered as the natural classical counterpart of semiclassical states. Since, as it is discussed in § 1.3 (Proposition 1.7), the semiclassical description in terms of cylindrical measures always exists and it is unique up to isomorphisms, we focus on measures acting on topological vector spaces. The corresponding results for measures defined on more general sets can be easily deduced using the isomorphisms yielded by Bochner's theorem.

5.1 Dual topologies and the definition of semiclassical measures

Let $W, Y \in \mathbf{VS}_{\mathbb{R}}$ be two real vector spaces; W and Y are in B-duality if there exists a bilinear form $B : W \times Y \to \mathbb{R}$. Given W and Y in B-duality, the duality separates points in W (respectively Y) iff $W \ni w \mapsto B(w, \cdot) \in Y^*$ (respectively $Y \ni y \mapsto B(\cdot, y) \in W^*$) is injective. Let us recall that Z^* denotes the algebraic dual of the linear space Z. The duality between linear spaces induces many important and useful topologies. Before discussing the definition of cylindrical Wigner measures, let us briefly outline some classical results about duality in linear spaces. Given a linear space W, the algebraic dual W^* is in duality with W, by the canonical bilinear form $B(w,\xi) = \xi(w)$ on $W \times W^*$. In addition, the duality separates points in both W and W^*. Using the injective maps $d_B : y \mapsto B(\cdot, y)$ and $s_B : w \mapsto B(w, \cdot)$, it is often convenient to identify Y or W (in separating duality with W or Y) with a subspace of W^* or Y^*.

In our setting, given a dual pair W and Y it is useful to characterize the topologies on W that are *compatible with the duality*, *i.e.* the topologies on W such that d_B is a bijection of Y onto the continuous dual $W' \subset W^*$. It turns out that there are coarsest and finest locally convex topologies on W compatible with the duality. Such topologies are the weak $\sigma(W,Y)$ and the Mackey $\tau(W,Y)$ topology respectively. Mackey's theorem proves that any locally convex topology on W that is compatible with the duality is finer than the weak and coarser than the Mackey topology. The strong topology $\beta(W,Y)$ is in general finer than the Mackey topology and therefore not compatible with the duality. However if W has a given (initial) topology, $\beta(W,W') = \tau(W,W')$ whenever W is barrelled (*e.g.*, if W is a Frchet space). If the topological spaces W and Y are in compatible duality, the bijection $Y \cong W'$ is implicitly understood.

Let us now consider the setting of Theorem 4.6. As it will become clear in the following, if $X \cong V'$ for some topological real linear space V, then we can identify the classical counterpart of regular quantum states with cylindrical measures on V. This can be done, however, in many different ways. First of all, there may be (infinitely) many spaces V for which there exists a $B : V \times X \to \mathbb{R}$ that puts V and X in B-duality, separating points in X (the latter requirement is necessary to have compatible topologies on V). For example X^*, or any of its subspaces that separates points in X. Once we have fixed such V, all the locally convex topologies finer than $\sigma(V,X)$ and coarser than $\tau(V,X)$ are compatible with the identification $X \cong V'$. From the physical perspective, this indicates that there may exist many equivalent effective descriptions of a quantum field theory as a classical theory, due to the rich structure of infinite dimensional linear spaces. Since the starting point is the quantum Heisenberg group associated to (X,ς), it could seem natural to choose $V = X^*$ with the coarsest locally convex topology $\sigma(X^*,X)$ as the classical space for the definition of Wigner measures (and *a fortiori* this choice proves that there always exists a space of classical fields). As we will see, there are situations in which such a choice implies that all the Wigner cylindrical measures are in fact true (Borel) Radon measures on V.

Let us now introduce the appropriate topologies to prove convergence of regular states to Wigner measures. Let V be a topological space in compatible duality with X. Let us denote by $\mathcal{R}_h \subset (\mathfrak{W}_h)'_+$ the set of regular states on $\mathfrak{W}_h$, and by

$$\mathcal{M}_{\mathrm{cyl}}(V;\mathfrak{A}'_+) := \mathcal{M}_{\mathrm{cyl}}(V,V';\mathfrak{A}'_+)$$

the cone-valued cylindrical measures on V – see Appendices A.1 and A.4 for a precise definition of cone-valued measures and for the admissibility of $\mathfrak{A}'_+$ as a cone. Let us denote the union of the aforementioned sets by

$$S(V,X,\mathfrak{A}) = \mathcal{M}_{\mathrm{cyl}}(V;\mathfrak{A}'_+) \cup \bigsqcup_{h>0} \mathcal{R}_h \;. \tag{29}$$

Let us denote by $F(V)$ the set of $\sigma(V,V')$-closed subspaces of V with finite codimension. For any $\Phi \in F(V)$, $V' \supset \Phi^\circ \cong (V/\Phi)'$ is its finite dimensional polar. Finally, given a locally compact real vector space L let us denote by $C_0(L)'_{C_0(L)}$ the continuous dual of the space of its compactly supported continuous functions endowed with the $\sigma\bigl(C_0(L)',C_0(L)\bigr)$ topology. It is possible to define a map $\mathscr{H}$ from $S(V,X,\mathfrak{A})$ to $\prod_{\Phi\in F(V)}\bigl(C_0(V/\Phi)'_{C_0(V/\Phi)}\bigr)^{\mathfrak{A}_+}$ in the following way:

$$\mathscr{H}(s) = \begin{cases} \Bigl(a \mapsto \int_{V/\Phi} \cdot \,\mathrm{d}\mu_{\Phi,a}\Bigr)_{\Phi\in F(V)} & s = M \in \mathcal{M}_{\mathrm{cyl}}(V;\mathfrak{A}'_+) \\ \Bigl(a \mapsto H^{(\Phi^\circ)}_{\omega_h}(a)(\cdot)\Bigr)_{\Phi\in F(V)} & s = (\omega_h,h) \in \mathcal{R}_h \times \mathbb{R}^+_* \end{cases} ,$$

where $H^{(\Phi^\circ)}_{\omega_h}(a)$ is defined by (24), and $\mathbb{R}^+_* = \mathbb{R}^+ \smallsetminus \{0\}$. The product topology $\mathfrak{P}_{F(V)}$ on

$$\prod_{\Phi\in F(V)} \bigl(C_0(V/\Phi)'_{C_0(V/\Phi)}\bigr)^{\mathfrak{A}_+}_{\mathrm{s}} ,$$

where each $\bigl(C_0(V/\Phi)'_{C_0(V/\Phi)}\bigr)^{\mathfrak{A}_+}_{\mathrm{s}}$ is endowed with the topology of simple convergence, induces a topology on $S(V,X,\mathfrak{A})$: the preimage topology $\mathfrak{P} = \mathscr{H}^{-1}\mathfrak{P}_{F(V)}$. On the other hand, by Bochner's theorem A.17 and Proposition 3.7, there is a map $\mathscr{F} : S(V,X,\mathfrak{A}) \to (\mathfrak{A}'_{\mathfrak{A}})^X$ defined as follows:

$$\mathscr{F}(s) = \begin{cases} \hat{M}(\cdot) & s = M \in \mathcal{M}_{\mathrm{cyl}}(V;\mathfrak{A}'_+) \\ \mathcal{G}_{\omega_h}(\cdot) & s = (\omega_h,h) \in \mathcal{R}_h \times \mathbb{R}^+_* \end{cases} .$$

The topology $\mathfrak{T}_{\mathrm{s}}$ of simple convergence on $(\mathfrak{A}'_{\mathfrak{A}})^X$ induces therefore the topology $\mathfrak{T} = \mathscr{F}^{-1}\mathfrak{T}_{\mathrm{s}}$ on $S(V,X,\mathfrak{A})$. Finally, let us denote by $\mathfrak{P}\vee\mathfrak{T}$ the join topology, *i.e.* the coarsest topology on $S(V,X,\mathfrak{A})$ that is finer than both $\mathfrak{P}$ and $\mathfrak{T}$. Hence Theorem 4.6 can be reformulated as a convergence result, in suitable topologies, for semiclassical quantum states, as $h \to 0$. In order to do that, we have only to take into account the projective structure induced by $F(V)$.

Let $F_\varsigma(X)$ be the set of all finite dimensional symplectic subspaces of X. First of all, let us remark that given any finite dimensional vector subspace R of X, there is always an $X_R \in F_\varsigma(X)$ such that R is a (closed) subspace of X_R. Therefore if $g \in (\mathfrak{A}'_{\mathfrak{A}})^X$ is continuous when restricted to every symplectic subspace in $F_\varsigma(X)$, then it is continuous when restricted to every finite dimensional subspace. In addition, let V be a topological real vector space in compatible duality with X; then the set $F^\circ_\varsigma(X) = \bigl\{\Xi^\circ, \Xi \in F_\varsigma(X)\bigr\}$, where the polar is taken with respect to the duality between X and V, is the subset of $F(V)$ consisting of the weakly closed subspaces with *even* codimension. If we partially order the set $F(V)$ by $\supset$, then $F^\circ_\varsigma(X)$ is a *cofinal* subset of $F(V)$. In § 4.2, given a semiclassical quantum state $(\omega_{h_\beta})_{\beta\in B}$ on $(\mathfrak{W}_{h_\beta})_{\beta\in B}$, we defined the family of relatively compact and nonempty sets

$$\big(H_B(\Phi)\big)_{\Phi\in F_\varsigma^\circ(X)} = \Big(\prod_{a\in\mathfrak{A}_+}\big\{H^{(\Phi)}_{\omega_{h_\beta}}(a), \beta\in B\big\}\Big)_{\Phi\in F_\varsigma^\circ(X)} ,$$

by Eqs. (24) and (25), with $X_\Phi = \Phi^\circ$; and since the definition of $H^{(\Phi)}_{\omega_{h_\beta}}(a)$ can be extended to any $\Phi\in F(V)$, we can also define the family

$$\big(H_B(\Phi)\big)_{\Phi\in F(V)} = \Big(\prod_{a\in\mathfrak{A}_+}\big\{H^{(\Phi)}_{\omega_{h_\beta}}(a), \beta\in B\big\}\Big)_{\Phi\in F(V)} .$$

In addition, if $R\subset X$ is a finite dimensional vector subspace of X, we define the subalgebra $\mathbb{W}_h(R)\subset\mathbb{W}_h(X,\varsigma)$ by

$$\mathbb{W}_h(R) = \mathrm{C}^*\big(\{W_h(x), x\in R\}\big) .$$

By this definition, it follows that for any $a\in\mathfrak{A}_+$ and $\Phi\in F(V)$,

$$H^{(\Phi)}_{\omega_{h_\beta}}(a) = H_{\omega_{h_\beta}\big|_{\mathbb{W}_{h_\beta}(\Phi^\circ)\otimes\gamma_{h_\beta}\mathfrak{A}}}(a) ,$$

where the latter is defined by Eq. (23). For any $\Phi\supset\Psi\in F(V)$, let us now introduce the (continuous) maps $\pi_{\Phi\Psi} : H_B(\Psi)\to H_B(\Phi)$:

$$\pi_{\Phi\Psi}\Big(a\mapsto H^{(\Psi)}_{\omega_{h_\beta}}(a)\Big) = a\mapsto H^{(\Phi)}_{\omega_{h_\beta}}(a) ,$$

where the definition is justified by the fact that Φ° is injected canonically in Ψ° by the map ${}^{\mathrm{t}}p_\Psi\circ{}^{\mathrm{t}}p_{\Phi\Psi}\circ{}^{\mathrm{t}}p_\Phi^{-1}$, where $p_\Phi : V\to V/\Phi$ is the canonical map and $p_{\Phi\Psi} : V/\Psi\to V/\Phi$ is the map obtained from the identity passing to the quotients. It is easy to see that $p_{\Phi\Omega} = p_{\Phi\Psi}\circ p_{\Psi\Omega}$ for any $\Phi\supset\Psi\supset\Omega\in F(V)$. Therefore the families

$$\big(H_B(\Phi),\pi_{\Phi\Psi}\big)_{\Phi\supset\Psi\in F_\varsigma^\circ(X)} , \; \big(H_B(\Phi),\pi_{\Phi\Psi}\big)_{\Phi\supset\Psi\in F(V)}$$

are projective systems with cofinal index set and thus they share the same projective limit. In addition, the limit is nonempty and relatively compact since $H_B(\Phi)$ is nonempty and relatively compact for any $\Phi\in F_\varsigma^\circ(X)$. Therefore for $F = F_\varsigma(X)$, the set H_B of Lemma 4.2 can be restricted to the projective limit

$$H_B = \varprojlim H_B(\Phi) .$$

It then follows by Theorem 4.6 that the $\mathfrak{P}$-cluster points of semiclassical quantum states are *projective families* of Radon measures on the finite dimensional quotients of V, and therefore cylindrical measures. A convergent net $\omega_{h_b}\underset{h_b\to 0}{\overset{\mathfrak{P}}{\longrightarrow}} M$, that satisfies one of the following four equivalent conditions has no loss of mass:

$$\omega_{h_b} \xrightarrow[h_b\to 0]{\mathfrak{P}\vee\mathfrak{T}} M \qquad ; \quad \text{(i)}$$

$$\lim_{\varepsilon\to 0}\lim_{b\in\underline{B}} \omega_{h_b}\big(W_{h_b}(x)\otimes a - \mathrm{Op}^{h_b}_{\frac{1}{2}}({}^{\exists}\psi_{\varepsilon,\Phi}\, e^{2ix(\cdot)})\otimes a\big) = 0 \quad \big(\forall \Phi\in F^{\circ}_{\varsigma}(X)\,, \forall x\in\Phi^{\circ}\,, \forall a\in\mathfrak{A}_+\big) \qquad ; \quad \text{(ii)}$$

$$\lim_{\varepsilon\to 0}\lim_{b\in\underline{B}} \omega_{h_b}\big(a - \mathrm{Op}^{h_b}_{\frac{1}{2}}({}^{\exists}\psi_{\varepsilon,\Phi})\otimes a\big) = 0 \qquad \big(\forall\Phi\in F^{\circ}_{\varsigma}(X)\,, \forall a\in\mathfrak{A}_+\big) \qquad ; \quad \text{(iii)}$$

$$\lim_{b\in\underline{B}} \omega_{h_b}\big(W_{h_b}(0)\big) = M(0) \qquad . \quad \text{(iv)}$$

From a practical point of view, it is easier to study the semiclassical behavior of quantum states with no loss of mass, since it is very convenient to characterize the limit measure by means of the noncommutative and commutative Fourier transforms.

THEOREM 5.1. *Let* $(X,\varsigma)\in\mathbf{Symp}_{\mathbb{R}}$, $\mathfrak{A}\in\mathbf{C^*alg}$, *and* $(\mathfrak{W}_h)_{h\geq 0}$ *the corresponding Weyl deformation* (22). *In addition, let* V *be any topological space in compatible duality with* X. *Consider a semiclassical state* $(\omega_{h_\beta})_{\beta\in B}$ *on* $(\mathfrak{W}_{h_\beta})_{\beta\in B}$. *Then there exists a subnet* $(\omega_{h_b})_{b\in\underline{B}}$ *such that*

$$\omega_{h_b} \xrightarrow[h_b\to 0]{\mathfrak{P}} M\,, \tag{30}$$

where $M\in\mathcal{M}_{\mathrm{cyl}}(V;\mathfrak{A}'_+)$ *is a* cylindrical Wigner measure. *In other words, if we denote by* $\mathscr{W}\big(\omega_{h_\beta},\beta\in B\big)$ *the set of cluster points of* $(\omega_{h_\beta})_{\beta\in B}\subset S(V,X,\mathfrak{A})_{\mathfrak{P}}$, *then* $s\in\mathscr{W}\big(\omega_{h_\beta},\beta\in B\big)$ *implies* $s\in\mathcal{M}_{\mathrm{cyl}}(V;\mathfrak{A}'_+)$. *If in addition* $(\omega_{h_b})_{b\in\underline{B}}$ *has no loss of mass then*

$$\omega_{h_b} \xrightarrow[h_b\to 0]{\mathfrak{P}\vee\mathfrak{T}} M\,. \tag{31}$$

The cylindrical measures are not countably additive Borel measures, even if each countably additive Borel measure defines a cylindrical measure. A cylindrical measure on the topological space V is a projective system of Borel measures on finite dimensional vector spaces, indexed by the weakly closed subspaces of V of finite codimension. Let us denote by $F(V)$ the set of $\sigma(V,V')$-closed subspaces with finite codimension. The set $F(V)$ is identified uniquely by the finite dimensional subspaces of V'. In fact, $\Phi\in F(V)$ iff there exists a finite dimensional subspace $F\subset V'$ such that $F^{\circ}=\Phi$, where F° is the polar (orthogonal) of F (with respect to the canonical duality between V and V'). The *if* part is proved as follows: the bipolar $F^{\circ\circ}$ is isomorphic to V/Φ, and it is also the closure of F with respect to the $\sigma(V',V)$ topology; since the duality between V and V' separates points in V', it follows that the finite dimensional subspace F is closed with respect to $\sigma(V',V)$, and therefore $F=F^{\circ\circ}$, thus proving $\Phi\in F(V)$. The *only if* part is proved as follows: let $\Phi\in F(V)$, and choose $F=\Phi^{\circ}$; then $F^{\circ}=\Phi^{\circ\circ}$ is the closure of Φ with respect to the $\sigma(V,V')$ topology, and therefore $\Phi^{\circ\circ}=\Phi$ since $\Phi\in F(V)$. This further motivates the fact that it is the phase space X (of test functions) that

determines the cylindrical measures, notwithstanding the possibility of choosing the measurable space of classical fields V in many different ways.
To sum up, the semiclassical quantum states always have classical counterparts, the cylindrical Wigner measures, equivalently acting on some suitably defined topological space of fields. On the other hand, families of non-regular states cannot be identified, in general, with cylindrical measures, because their generating map could fail to be ultraweakly continuous on every finite dimensional subspace. Therefore non-regular states are not suitable for a classical effective description. From a physical standpoint this is not unreasonable, since non-regular states appear in relation to typically quantum physical behaviors, such as infrared divergence [see, *e.g.*, 1, 2].

5.2 Phase space and Wigner measures

The symplectic space (X, ς) has a natural interpretation in physics, especially when finite dimensional, as the phase space of a given theory. The symplectic structure is extremely useful to study dynamical properties (Hamiltonian flows). The usual Wigner measures for finite-dimensional Heisenberg groups can be interpreted straightforwardly as (probability) measures on the phase space, since (X, ς) is in compatible duality with itself whenever $\dim X < \infty$. For infinite-dimensional Heisenberg groups, it is possible only in few favorable cases, that may be called *admissible phase spaces*. Explicitly, Wigner measures are cylindrical measures on the phase space if X is a topological space in compatible duality with itself. Interestingly enough, this allows to consider the geometric phase spaces constructed from reflexive spaces (seen as topological manifolds). Given a *reflexive* locally convex space Σ, its cotangent bundle $\mathrm{T}^*\Sigma \cong \Sigma \oplus \Sigma'$ is a symplectic space with the canonical symplectic form. Since Σ is reflexive, $\Sigma \oplus \Sigma'$ is in compatible duality with itself, and therefore $\mathrm{T}^*\Sigma$ is an admissible phase space.
As already discussed in § 2, in quantum field theories the phase space (X, ς) is usually interpreted as a space of *test functions*, and the space of classical fields is in duality with it (it is therefore a space of distributions). The Wigner measures act on the latter, and this is physically reasonable, since the measures should describe the configuration of classical fields and not of test functions. The symplectic structure of test functions is carried out on the space of classical fields, if the latter is taken to be X^*_X: for any $x, y \in X$, $\varsigma(x, \cdot) \in X^*$ and $\varsigma(x, \cdot) \neq \varsigma(y, \cdot)$ if $x \neq y$ (by the non-degeneracy of ς), and thus $X \overset{\mathrm{s}}{\hookrightarrow} X^*_X$, and there is a symplectic form $\tilde{\varsigma}(\cdot, \cdot)$ on $\mathrm{s}(X)$ defined by $\tilde{\varsigma}(\cdot, \cdot) = \varsigma(\mathrm{s}^{-1}\,\cdot, \mathrm{s}^{-1}\,\cdot)$. In order to define an Hamiltonian dynamics on the space of classical fields it is therefore necessary to either restrict to measures concentrated on $\mathrm{s}(X)$ (but as discussed in § 5.4, this point of view may be too restrictive), or to extend $\tilde{\varsigma}$ to the whole X^*_X.

5.3 Cylindrical Wigner measures as Radon measures

Cylindrical measures are not completely satisfactory from a practical standpoint. Integration of functions with respect to cylindrical measures is possible only if the functions are cylindrical as well. It is possible to interpret cylindrical measures on V as Radon measures only on a space that is "bigger" than V. This is done exploiting Prokhorov's tightness of the projective family of measures [76]. For any $\Phi \in F(V)$, let us consider the Čech compactification $\overline{V/\Phi}$ of V/Φ. From the canonical injection $j_\Phi : V/\Phi \to \overline{V/\Phi}$ it is possible to construct a canonical injection, let us call it j, of V into the product space

$$\overline{V} = \prod_{\Phi \in F(V)} \overline{V/\Phi} \,.$$

We endow $\overline{V}$ with the product topology. The push-forward $\overline{M} = j_* M$ of any $M \in \mathcal{M}_{\mathrm{cyl}}(V; \mathfrak{A}'_+)$ is defined as the family $\overline{M} = (j_{\Phi\,*}\,\mu_\Phi)_{\Phi \in F(V)}$. It is a tight projective family of measures on $\overline{V}$, and therefore it originates from a finite Borel Radon measure $\mu_{\overline{M}}$. It follows that the set of cylindrical measures on V can be identified with a subset of finite Radon measures on $\overline{V}$. If M does not originate from a Radon measure on V then $\mu_{\overline{M}}(j(V)) \neq m$ (where $m = \mu(\overline{V})$ is the total mass of the measure), and there are cylindrical measures for which $\mu_{\overline{M}}(j(V)) = 0$. As it will be proved in the next section, every cylindrical measure, on any space V in compatible duality with X, is the Wigner measure of at least one generalized sequence of regular quantum states. Therefore cylindrical measures that are not Radon measures play an important role in the semiclassical description. In other words, the spaces $\overline{V}$, rather than V or X, are the most suitable candidates to accommodate a complete classical description of field theories (as classical probability theories). The non-uniqueness of $\overline{V}$ may seem on one hand disappointing, on the other hand it allows for additional freedom in the choice of classical effective description. We have already highlighted that there seems to be one "preferred" choice of V, *i.e.* choosing it to be the algebraic dual X^* of X, with the $\sigma(X^*, X)$ topology. A classical result [21, II.54, Proposition 9] lets us identify X^*_X with the Hausdorff completion of $\big(V, \sigma(V, X)\big)$ for any V and X in a duality that separates points of both V and X. Finally, for any V in a compatible duality with X that separates points in V, there is a natural bijection [76] (equivalently yielded by Bochner's theorem)

$$\mathcal{M}_{\mathrm{cyl}}(V; \mathfrak{A}'_+) \cong \mathcal{M}_{\mathrm{cyl}}(X^*_X; \mathfrak{A}'_+) \,. \tag{32}$$

The bijection (32) becomes of particular importance whenever X is a second countable topological space. In fact, in such case any cylindrical measure originates from a Radon measure:

$$\mathcal{M}_{\mathrm{cyl}}(X^*_X; \mathfrak{A}'_+) \cong \mathcal{M}_{\mathrm{rad}}(X^*_X; \mathfrak{A}'_+) \,,$$

where the latter is the set of all finite Borel Radon measures. The Wigner measures are then exactly the finite Borel Radon measures on X^*_X. Let us remark that also for a second countable X Wigner measures can be seen as Radon measures on $\overline{X^*_X}$, however they are precisely the set of measures that are concentrated as Borel Radon measures on X^*_X.

5.4 Characterization of the set of all Wigner measures

As anticipated in Theorem 1.11 and § 5.3, any cylindrical measure is the Wigner measure of some semiclassical quantum state. To prove this very interesting result, let us define the set $\mathscr{W}$ of all possible Wigner measures associated to the Weyl deformation *with maximal cross norm.* Let us denote the aforementioned Weyl deformation by

$$(\mathfrak{W}_h^{\mathrm{m}})_{h>0} = \bigl(\mathbb{W}_h(X,\varsigma) \otimes_{\gamma_{\max}} \mathfrak{A}\bigr)_{h>0} .$$

Then, using the notation introduced in Theorem 5.1, the set of all Wigner measures is defined as

$$\mathscr{W} = \bigcup_{\underline{h}\in\mathbb{R}_*^+} \bigcup_{(\omega_h)_{h\in(0,\underline{h})}\in 2^{S(V,X,\mathfrak{A})}} \mathscr{W}\bigl(\omega_h, h\in(0,\underline{h})\bigr) .$$

The aim is to prove, for any V in compatible duality with X, $\mathscr{W} = \mathcal{M}_{\mathrm{cyl}}(V;\mathfrak{A}'_+)$. By Theorem 5.1, it suffices to prove that $\mathscr{W} \supset \mathcal{M}_{\mathrm{cyl}}(V;\mathfrak{A}'_+)$, *i.e.* that for any cylindrical measure M there exists a semiclassical quantum state $(\omega_h^{(M)})_{h\in(0,\underline{h})}$ such that $\omega_h^{(M)} \xrightarrow[h\to 0]{\mathfrak{P}\vee\mathfrak{T}} M$ on $S(V,X,\mathfrak{A})$.

We make use of the following result of standard semiclassical analysis. Its proof relies on the fact that squeezed coherent states on $L^2(\mathbb{R}^d)$ converge to measures concentrated on a point of $\mathbb{R}^{2d}$; and that linear combinations of point measures are dense in the space of finite measures $\mathcal{M}_{\mathrm{rad}}(\mathbb{R}^{2d})_{\mathbb{C}}$, endowed with the weak topology [see 34, for additional details]. The extension to general finite dimensional symplectic spaces (R,ς) and to tensor products $\mathbb{W}_h(R,\varsigma)\otimes_{\gamma_{\max}}\mathfrak{A}$ does not present difficulties.

Lemma 5.2. *Let* $(R_0,\varsigma) \in \mathbf{Symp}_{\mathbb{R}}$. *For any* $\mu \in \mathcal{M}_{\mathrm{rad}}(R'_0;\mathfrak{A}'_+)$, *there exists a semiclassical state* $(\tilde{\varrho}_h)_{h\in(0,1)}$ *such that for any* $h\in(0,1)$, $\tilde{\varrho}_h \in \bigl(\mathbb{W}_h(R_0,\varsigma)\otimes_{\gamma_{\max}} \mathfrak{A}\bigr)'_+$ *is normal,*

$$\|\varrho_h\|_{R_0} := \tilde{\varrho}_h\bigl(W_h(0)\otimes 1\bigr) = \|\hat{M}(0)\|_{\mathfrak{A}'} ,$$

and on $S(R'_0,R_0,\mathfrak{A})$

$$\tilde{\varrho}_h \xrightarrow[h\to 0]{\mathfrak{P}\vee\mathfrak{T}} \mu .$$

It is then sufficient to combine Lemma 5.2 with a projective argument, obtaining the following theorem for any topological vector space V in compatible duality with the real symplectic space (X,ς).

Theorem 5.3. $\mathscr{W} = \mathcal{M}_{\mathrm{cyl}}(V;\mathfrak{A}'_+) : {}^{\exists}\omega_h^{(M)} \xrightarrow[h\to 0]{\mathfrak{P}\vee\mathfrak{T}} M \ \bigl(\forall M\in\mathcal{M}_{\mathrm{cyl}}(V;\mathfrak{A}'_+)\bigr)$.

Proof. Let us start with the simpler case where $X = \bigcup_{m\in\mathbb{N}} X_m$, for a countable directed sequence of finite dimensional symplectic subspaces $(X_m)_{m\in\mathbb{N}}$, ordered by inclusion. The family $(X_m)_{m\in\mathbb{N}}$, together with the identities $i_{mn} : X_m \to X_n$, $m\leq n$ is an inductive family of finite dimensional vector spaces. Therefore $\varinjlim X_m = X$, and

$$(\mathfrak{A}'_{\mathfrak{A}})^X = \varprojlim\, (\mathfrak{A}'_{\mathfrak{A}})^{X_m} . \tag{33}$$

Now given a cylindrical measure $M = (\mu_\Phi)_{\Phi\in F(V)}$, the subfamily $(\mu_{X_m^\circ})_{m\in\mathbb{N}}$ is sufficient to characterize it completely (the polar is intended with respect to the

duality between V and X). In fact, $\hat{M} : X \to \mathfrak{A}'_{\mathfrak{A}}$ can be characterized uniquely by the projective family of restrictions $(\hat{M}|_{X_m})_{m\in\mathbb{N}}$.
Consider X_0. We define the following set of generating maps, corresponding to *regular* states on the maximal cross product algebra $\mathbb{W}_h(X_0, \varsigma) \otimes_{\gamma_{\max}} \mathfrak{A}$:

$$G(\mu_{X_0^\circ}) = \left\{ \mathcal{G}_{\varrho_h} \ , \ h \in (0,1) \ , \ \varrho_h \xrightarrow[h\to 0]{\mathfrak{P}\vee\mathfrak{T}} \mu_{X_0^\circ} \right\} .$$

From $G(\mu_{X_0^\circ})$, construct (recursively) the compatible projective family of generating maps of regular states $\big(G_m, \imath_{mn}\big)_{m\leq n\in\mathbb{N}}$, defined as follows. $G_0 = G(\mu_{X_0^\circ})$, and for any $m > 0$

$$G_m = \left\{ \mathcal{G}_{\varrho_h} \ , h \in (0,1) \ , \mathcal{G}_{\varrho_h}\big|_{X_{m-1}} \in \mathcal{G}_{m-1} \right\} .$$

The *surjective* maps $\imath_{mn} : G_n \to G_m$, $m \leq n$, are defined by $\imath_{mn}(\mathcal{G}_{\varrho_h}) = \mathcal{G}_{\varrho_h} \circ i_{mn}$, where i_{mn} are the identities defined above for the inductive family $(X_m)_{m\in\mathbb{N}}$. It is easy to prove that for any $m \in \mathbb{N}$, there exists $(\varrho_h^{(m)})_{h\in(0,1)}$, $\big(\mathcal{G}_{\varrho_h^{(m)}}\big)_{h\in(0,1)} \subset G_m$, such that $\varrho_h^{(m)} \xrightarrow[h\to 0]{\mathfrak{P}\vee\mathfrak{T}} \mu_{X_m^\circ}$. For $m = 0$ it is true by definition, and for any $m > 0$ it follows from the fact that for any $h \in (0,1)$, $\mathcal{G}_{\tilde{\varrho}_h}\big|_{X_{m-1}} \in G_{m-1}$, where $(\tilde{\varrho}_h)_{h\in(0,1)}$ is the convergent net of Lemma 5.2. Therefore, *a fortiori*, all G_m are non-empty. By Eq. (33) and [20, III.58 Proposition 5], the projective limit $G = \varprojlim G_m$ is a non-empty set of nets of generating maps $(\mathcal{G}_{\omega_h})_{h\in(0,1)}$ of regular states on $(\mathfrak{W}_h^{\mathrm{m}})_{h\in(0,1)}$ (with maximal cross norm on the tensor products). In addition, by the reasoning above, there exists $(\mathcal{G}_{\omega_h^{(M)}})_{h\in(0,1)} \subset G$ such that $\omega_h^{(M)} \xrightarrow[h\to 0]{\mathfrak{P}\vee\mathfrak{T}} M$.
In the general case, it is possible to follow the same guidelines. Given a finite dimensional symplectic subspace $R_0 \subset X$, it is always possible to find a family $(X_\lambda)_{\lambda\in J(V)}$ of finite dimensional symplectic subspaces – where $J(V)$ is some directed set with partial order $\leq$ and least element 0 – such that $X_0 = R_0$, $X_\eta \subset X_\lambda$ for any $\eta \leq \lambda \in J(V)$, and $\bigcup_{\lambda\in J(V)} X_\lambda = X$. Therefore $(X_\lambda)_{\lambda\in J(V)}$ is an inductive family, and the following identity holds:

$$(\mathfrak{A}'_{\mathfrak{A}})_{\mathrm{s}}^{X} = \varprojlim\, (\mathfrak{A}'_{\mathfrak{A}})_{\mathrm{s}}^{X_\lambda} \ . \tag{34}$$

In addition, any cylindrical measure $M = (\mu_\Phi)_{\Phi\in F(V)}$ can be equivalently characterized by the family of restricted Fourier transforms $(\hat{M}|_{X_\lambda})_{\lambda\in J(V)}$. Similarly to the countable case, we define the set of generating maps of regular states

$$G_0 = \left\{ \mathcal{G}_{\varrho_h} \ , \ h \in (0,1) \ , \ \varrho_h \xrightarrow[h\to 0]{\mathfrak{P}\vee\mathfrak{T}} \mu_{X_0'} \ , \|\varrho_h\|_{X_0} = \|\hat{M}(0)\|_{\mathfrak{A}'} \right\} ,$$

where $\|\cdot_h\|_{X_0}$ is the norm of the Banach space $\big(\mathbb{W}_h(X_0, \varsigma) \otimes_{\gamma_{\max}} \mathfrak{A}\big)'$. The set $G_0 \subset (\mathfrak{A}'_{\mathfrak{A}})_{\mathrm{s}}^{X_0}$ is relatively compact and non-empty, by Banach-Alaoglu's theorem and Lemma 5.2. Recursively, we define the sets $G_\lambda \subset (\mathfrak{A}'_{\mathfrak{A}})_{\mathrm{s}}^{X_\lambda}$, $\lambda \in J(V)$, by

$$G_\lambda = \left\{ \mathcal{G}_{\varrho_h} \ , h \in (0,1) \ , \forall \eta \leq \lambda \ , \ \mathcal{G}_{\varrho_h}\big|_{X_\eta} \in G_\eta \right\} .$$

The sets G_λ are all relatively compact and non-empty as well. In addition, the projections $\imath_{\lambda\eta} : G_\eta \to G_\lambda$, $\lambda \leq \eta$ defined by $\imath_{\lambda\eta}(\mathcal{G}_{\varrho_h}) = \mathcal{G}_{\varrho_h} \circ i_{X_\lambda X_\eta}$ are continuous. Therefore the projective system $(G_\lambda, \imath_{\lambda\eta})_{\lambda\leq\eta\in J(V)}$ has a non-empty limit [20], and $G = \varprojlim G_\lambda$ is a set of generating maps $(\mathcal{G}_{\omega_h})_{h\in(0,1)}$ of regular states on $(\mathfrak{W}_h^{\mathrm{m}})_{h\in(0,1)}$. In addition, as in the countable case, there exists $(\mathcal{G}_{\omega_h^{(M)}})_{h\in(0,1)} \subset G$ such that $\omega_h^{(M)} \xrightarrow[h\to 0]{\mathfrak{P}\vee\mathfrak{T}} M$. ⊣

6 Push-forward and convolution of Wigner measures

In this section we study transformations on the cylindrical Wigner measures induced by transformations of the regular quantum states, and prove the results outlined in § 1.6. Throughout the section, we set $\mathfrak{A} = \mathbb{C}$ for simplicity.

6.1 Group homomorphisms

Endomorphisms of $\mathbb{W}_h(X,\varsigma)$ – and perhaps more interestingly endomorphisms of $\pi\bigl(\mathbb{W}_h(X,\varsigma)\bigr)''$ (its bicommutant in some irreducible representation) – play a crucial role in defining quantum dynamical systems, as briefly discussed in § 2.2.4 and 2.2.5. Therefore, it is interesting to study the semiclassical action induced by *-homomorphisms in Weyl algebras. A systematic study of *-homomorphisms would require, at least to some extent, the development of pseudodifferential calculus for infinite dimensional phase spaces, and it is beyond the scope of this work. We restrict our attention to *-homomorphisms induced by a class of central Heisenberg group homomorphisms (field morphisms). In this way it is possible to characterize the induced semiclassical action on cylindrical Wigner measures in a natural way. Let $(X,\varsigma),(Y,\tau) \in \mathbf{Symp}_{\mathbb{R}}$. For any $\tilde{f} \in X^{Y\times\mathbb{R}}$ and $\tilde{F} \in \mathbb{R}^{Y\times\mathbb{R}}$,

$$(\tilde{f},\tilde{F}) \in \mathrm{Hom}_{\mathrm{gr}}\bigl(\mathbb{H}(Y,\tau),\mathbb{H}(X,\varsigma)\bigr)$$

iff for any $\varphi,\psi \in Y \times \mathbb{R}$:

$$\begin{aligned}\tilde{f}(\varphi\cdot\psi) &= \tilde{f}(\varphi) + \tilde{f}(\psi)\\ \tilde{F}(\varphi\cdot\psi) &= \tilde{F}(\varphi) + \tilde{F}(\psi) - \varsigma(\tilde{f}(\varphi),\tilde{f}(\psi))\ .\end{aligned}$$

In addition, $(\tilde{f},\tilde{F})$ is central if for any $t \in \mathbb{R}$, $\tilde{f}(0,t) = 0$. Among the central homomorphisms, we restrict our attention to the ones of the following form: there exists $f \in X^Y$ such that $\tilde{f}(y,t) = f(y)$. Let us denote such homomorphisms by $\tilde{\mathfrak{F}}$. On the Weyl C*-algebra representation $\mathbb{W}_h(Y,\tau)$ of the Heisenberg group, the induced action of $\tilde{\mathfrak{F}}$ is then

$$\tilde{\mathfrak{F}}\bigl(e^{it}W_h(y)\bigr) = e^{i\tilde{F}(y,t)}W_h(f(y))\ , \quad \forall y \in Y\ ,\ \forall t \in \mathbb{R}\ .$$

As it will become clear in the following, it is useful to consider h-dependent homomorphisms of the form $\tilde{\mathfrak{F}}_h = (f,\tilde{F}_h)$. In order to obtain a linear transformation on the Weyl C*-algebra, we "forget" the t-dependence of $\tilde{\mathfrak{F}}_h$ setting $F_h(x) = \tilde{F}_h(x,0)$, and consider the associated homomorphism $\mathfrak{F}_h = (f, F_h)$. Since $\mathfrak{F}_h$ does not act on scalars anymore, it can be extended to a (isometric) *-homomorphism

$$\mathfrak{F}_h \in {}^*\mathrm{Hom}\big(\mathbb{W}_h(Y,\tau), \mathbb{W}_h(X,\varsigma)\big) ,$$

with the following conditions on f and F_h:

$$\begin{aligned} &f(y+y') = f(y) + f(y') && (\forall y, y' \in Y) \\ &F_h(y+y') = F_h(y) + F_h(y') - ih\big(\varsigma(f(y), f(y') - \tau(y,y')\big) && (\forall y, y' \in Y) \end{aligned}$$

By duality, the adjoint map ${}^{\mathrm{t}}\mathfrak{F}_h \in \mathcal{B}\big(\mathbb{W}_h(X,\varsigma)', \mathbb{W}_h(Y,\tau)'\big)$. The action of ${}^{\mathrm{t}}\mathfrak{F}_h$ on a regular state $\omega_h \in \mathbb{W}_h(X,\varsigma)'_+$ is defined by the generating map

$$\mathcal{G}_{{}^{\mathrm{t}}\mathfrak{F}_h(\omega_h)}(\cdot) = e^{iF_h(\cdot)}\mathcal{G}_{\omega_h}(f(\cdot)) .$$

In general $\mathcal{G}_{{}^{\mathrm{t}}\mathfrak{F}_h(\omega_h)}$ fails to be of almost positive type, and thus ${}^{\mathrm{t}}\mathfrak{F}_h\big(\mathbb{W}_h(X,\varsigma)'_+\big) \not\subseteq \mathbb{W}_h(Y,\tau)'_+$.

Nonetheless, consider a semiclassical quantum state $(\omega_{h_\beta})_{\beta\in B}$ such that

$$\omega_{h_\beta} \xrightarrow[h_\beta\to 0]{\mathfrak{P}\vee\mathfrak{T}} M \in \mathcal{M}_{\mathrm{cyl}}(V) .$$

If $\lim_{\beta\in B} F_{h_\beta} = F_0$ (pointwise), then

$$g(y) := \lim_{\beta\in B} \mathcal{G}_{{}^{\mathrm{t}}\mathfrak{F}_{h_\beta}(\omega_{h_\beta})}(y) = e^{iF_0(y)}\hat{M}\big(f(y)\big) \quad (\forall y \in Y) . \tag{35}$$

This can also be extended to other maps that are not homomorphisms. Let $\xi \in X$ be fixed; then the map

$$\mathfrak{F}_h^{(\xi)}\big(W_h(x)\big) = W_h(\xi)W_h(x) = e^{-ih\varsigma(\xi,x)}W_h(x+\xi)$$

extends by linearity to a map $\mathfrak{F}_h^{(\xi)} : \mathbb{W}_h(X,\varsigma) \to \mathbb{W}_h(X,\varsigma)$. For $\mathfrak{F}_h^{(\xi)}$, Eq. (35) becomes

$$g^{(\xi)}(x) = \hat{M}(x+\xi) , \quad \forall x \in X . \tag{36}$$

Eq. (36) defines a (complex) signed cylindrical measure on V, that we may denote (with a slight abuse of notation) by $e^{i\xi(z)}\mathrm{d}M(z)$ or $e^{i\xi(\cdot)}M$. In other words, we have the following characterization of the Wigner measures associated to $\big({}^{\mathrm{t}}\mathfrak{F}_{h_\beta}^{(\xi)}\omega_{h_\beta}\big)_{\beta\in B}$:

$$\mathscr{W}_{\mathfrak{P}\vee\mathfrak{T}}\Big(\omega_{h_\beta}(W_{h_\beta}(\xi)\,\cdot\,), \beta\in B\Big) = \Big\{e^{i\xi(\cdot)}M\ ,\ M \in \mathscr{W}_{\mathfrak{P}\vee\mathfrak{T}}(\omega_{h_\beta}, \beta \in B)\Big\} . \tag{37}$$

We are interested in giving a similar characterization for other mappings. The part $f^{(\xi)}$ of $\mathfrak{F}_h^{(\xi)}$ is non-linear, and for more general non-linear transformations it may be difficult to obtain an explicit formula of type (37). We restrict to linear transformations f. Let us remark that if $F_h = 0$, then f should be a symplectic transformation for $\mathfrak{F}_h$ to be a *-homomorphism; in that case Proposition 6.1 is equivalent to Eq. (6).

Proposition 6.1. *Let* $(X,\varsigma),(Y,\tau)\in \mathbf{Symp}_{\mathbb{R}}$, $V,W\in \mathbf{TVS}_{\mathbb{R}}$ *in compatible duality separating points with* X *and* Y *respectively. For any weakly continuous*[36] *linear map* $\mathrm{u}:Y\to X$, *and for any* $F_h\in\mathbb{R}^Y$ *such that for any* $y\in Y$, $\lim_{h\to 0}F_h(y)=0$, *define* $\mathfrak{F}_h^{(\mathrm{u})}=(\mathrm{u},F_h)$. *Then for any semiclassical quantum state* $(\omega_{h_\beta})_{\beta\in B}$ *on* $\bigl(\mathbb{W}_{h_\beta}(X,\varsigma)\bigr)_{\beta\in B}$ *with no loss of mass:*

$$\mathscr{W}_{\mathfrak{P}\vee\mathfrak{T}}\Bigl({}^{\mathrm{t}}\mathfrak{F}_{h_\beta}^{(\mathrm{u})}(\omega_{h_\beta})\,,\,\beta\in B\Bigr)=\Bigl\{{}^{\mathrm{t}}\mathrm{u}_*M\,,\,M\in\mathscr{W}_{\mathfrak{P}\vee\mathfrak{T}}(\omega_{h_\beta},\beta\in B)\Bigr\}\,, \tag{38}$$

where ${}^{\mathrm{t}}\mathrm{u}:V\to W$ *is the transposed map of* u.

Proof. The definition of push-forward (image) of a cylindrical measure is briefly recalled in § 1.6 and Appendix A.2.2 [see 19, 76, 80, for additional details]; let us recall that with the assumptions above the transposed map ${}^{\mathrm{t}}\mathrm{u}$ is continuous with respect to the $\sigma(V,X)$ and $\sigma(W,Y)$ topologies [see, *e.g.*, 21, EVT II.50]. Eq. (35) for $\mathfrak{F}_h^{(\mathrm{u})}$ becomes

$$g(y)=\hat{M}\bigl(u(y)\bigr)=\widehat{{}^{\mathrm{t}}\mathrm{u}_*M}(y)\qquad(\forall y\in Y)\,.$$

⊣

Remark 6.2. Choosing $V=X^*_X$ and $W=Y^*_Y$, Proposition 6.1 holds *for any linear map* $\mathrm{u}:Y\to X$, since all linear maps are weakly continuous with respect to $\sigma(Y,Y^*)$ and $\sigma(X,X^*)$.

6.2 Quantum convolution and convolution of measures

From Theorem 5.1 it also follows that the Wigner measures associated to the quantum convolution of regular quantum states are convoluted measures. The convolution $\circledast$ of two cylindrical measures $M,N\in\mathcal{M}_{\mathrm{cyl}}(V)$ is defined as the pushforward of the product cylindrical measure $M\otimes N\in\mathcal{M}_{\mathrm{cyl}}(V\times V)$ by means of the addition map $\alpha:V\times V\to V$ defined by $(v,w)\mapsto v+w$. In other words,

$$M\circledast N=\alpha_*(M\otimes N)\,.$$

It is a well-known result that

$$\widehat{M\circledast N}=\hat{M}\cdot\hat{N}\,,$$

where the dot stands for the multiplication of complex-valued functions.
On the other hand, we define the *quantum convolution* as the quantum counterpart of the convolution of cylindrical measures. Let $\omega_h,\varpi_h\in\mathbb{W}_h(X,\varsigma)'_+$ be regular quantum states, their quantum convolution $\omega_h\star\varpi_h\in\mathbb{W}_h(X,\varsigma)'$ is the state defined as follows. On generators, define the action

$$(\omega_h\star\varpi_h)\bigl(W_h(\xi)\bigr)=\omega_h\bigl(W_h(\xi)\bigr)\varpi_h\bigl(W_h(\xi)\bigr)\,.$$

[36] Weak continuity here means continuity with respect to the $\sigma(Y,W)$ and $\sigma(X,V)$ topologies.

On finite linear combinations one has

$$(\omega_h \star \varpi_h)\Big(\sum_{j=1}^{n} \lambda_j W_h(\xi_j)\Big) = \sum_{j=1}^{n} \lambda_j \,(\omega_h \star \varrho_h)\big(W_h(\xi_j)\big) \ .$$

Then $\omega_h \star \varpi_h$ extends to a state on $\mathbb{W}_h(X, \varsigma)$.

Proposition 6.3. *Let $(X, \varsigma) \in \mathbf{Symp}_{\mathbb{R}}$, $V \in \mathbf{TVS}_{\mathbb{R}}$ in compatible duality with X. Consider two semiclassical quantum states with no loss of mass $(\omega_{h_\beta})_{\beta\in B}$ and $(\varpi_{h_\beta})_{\beta\in B}$ on $\big(\mathbb{W}_{h_\beta}(X, \varsigma)\big)_{\beta\in B}$. If*

$$\omega_{h_\beta} \xrightarrow[h_\beta\to 0]{\mathfrak{P}\vee\mathfrak{T}} M \ , \ \varpi_{h_\beta} \xrightarrow[h_\beta\to 0]{\mathfrak{P}\vee\mathfrak{T}} N \ ;$$

then

$$\omega_{h_\beta} \star \varpi_{h_\beta} \xrightarrow[h_\beta\to 0]{\mathfrak{P}\vee\mathfrak{T}} M \circledast N \ .$$

Proof. By definition of quantum convolution, the generating functional of $\omega_h \star \varpi_h$ satisfies:

$$\mathcal{G}_{\omega_h\star\varpi_h}(\cdot) = \mathcal{G}_{\omega_h}(\cdot)\mathcal{G}_{\varpi_h}(\cdot) \ .$$

Therefore pointwise

$$\lim_{\beta\in B} \mathcal{G}_{\omega_{h_\beta}\star\varpi_{h_\beta}}(\cdot) = \hat{M}(\cdot)\hat{N}(\cdot) = \widehat{M \circledast N}\,(\cdot) \ .$$

⊣

7 States on the C*-algebra of almost periodic functions

In § 5 we discussed how it is possible to identify the limit points of semiclassical quantum states, as $h \to 0$, with cylindrical measures on some topological vector space. In this section we provide an alternative identification, that is perhaps not so interesting for semiclassical analysis, but fits with the ideas of deformation theory.

In the definition of Weyl deformation $(\mathfrak{W}_h)_{h\geq 0}$, the C*-algebra at $h = 0$ is the tensor product of $\mathfrak{A}$ with an abelian C*-algebra of almost periodic functions. Let G be a topological abelian group, let us denote by $\mathbb{AP}(G)$ the algebra of (continuous) almost periodic functions. It has the following characterization:

$$\mathbb{AP}(G) = \mathrm{C}^*\big\{\hat{G}\big\} \ ,$$

where the completion is intended with respect to the supremum norm. $\hat{G}$ is here the character group of G, *i.e.* it is the set of continuous group homomorphisms from G to the multiplicative group $\mathbb{C}_1^\times = \big(\{z \in \mathbb{C}, |z| = 1\}, \times\big)$. Consider now a topological vector space $V \in \mathbf{TVS}_{\mathbb{R}}$ as an abelian topological group with respect to the addition operation (more precisely, apply the suitable forgetful functor $\mathbb{FF}$). There is a natural subalgebra $\mathbb{LAP}(\mathbb{FF}V) \subset \mathbb{AP}(\mathbb{FF}V)$, defined as

$$\mathbb{LAP}(V) := \mathbb{LAP}(\mathbb{FF}V) = \mathrm{C}^*\big\{e^{2i\xi(\cdot)} \ , \ \xi \in V'\big\} \ .$$

Therefore the $h = 0$ algebra of the Weyl deformation defined in (22) can be identified as follows: $\mathfrak{W}_0 \cong \mathbb{LAP}(V) \otimes_{\gamma_0} \mathfrak{A}$ for any topological vector space V in compatible duality with X. The idea is to show that the cylindrical measures $\mathcal{M}_{\mathrm{cyl}}(V; \mathfrak{A}'_+)$ can be identified with algebraic states of $\mathfrak{W}_0$ whenever $\gamma_0 = \gamma_{\max}$, *i.e.* they can be identified as (positive) elements of $(\mathbb{LAP}(V) \otimes_{\gamma_{\max}} \mathfrak{A})'$.

LEMMA 7.1. *Let V be a locally convex space in compatible duality with X. In addition, let $\mathfrak{A} \in$ **C*alg**; and $M \in \mathcal{M}_{\mathrm{cyl}}(V; \mathfrak{A}'_+)$. Then there exists a completely positive map $\mathscr{F}_M \in \mathcal{B}\big(\mathbb{LAP}(V), \mathfrak{A}'\big)$.*

Proof. Let us define the map $\mathscr{F}_M$ by its action on the generators: let $x \in \Phi^\circ \subset X$,

$$\mathscr{F}_M(e^{2ix(\cdot)}) = \int_{V/\Phi} e^{2ix(v)} \mathrm{d}\mu_\Phi(v) = \hat{M}(x) \ .$$

It then follows that $\mathscr{F}_M$ is linear. Let

$$f_n(\cdot) = \sum_{j=1}^{n} z_j e^{2ix_j(\cdot)}$$

be a complex linear combination of generators, then

$$\mathscr{F}_M(f_n) = \sum_{j=1}^{n} z_j \hat{M}(x_j) = \int_{V/\Phi_n} \sum_{j=1}^{n} z_j e^{2ix_j(v)} \mathrm{d}\mu_{\Phi_n}(v) \ ,$$

where $\Phi_n \in F(V)$ is such that $\{x_1, \dots, x_n\} \subset \Phi_n^\circ$. Using the corresponding result for standard measures $\mu_{\Phi_n,\kappa}$ (see Appendix A for the notation), it is not difficult to prove that

$$\Big\| \mathscr{F}_M(f_n) \Big\|_{\mathfrak{A}'} \leq \| f_n \|_\infty \| \hat{M}(0) \|_{\mathfrak{A}'} \ ,$$

where $\| \cdot \|_\infty$ denotes the supremum norm. Now let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to the supremum norm, that converges to $f \in \mathbb{LAP}(V)$. Then it is possible to define $\mathscr{F}_M(f)$ as the strong limit of $\mathscr{F}_M(f_n)$ in $\mathfrak{A}'$, and

$$\Big\| \mathscr{F}_M(f) \Big\|_{\mathfrak{A}'} \leq \| f \|_\infty \| \hat{M}(0) \|_{\mathfrak{A}'} \ .$$

Finally, $\mathscr{F}_M$ is completely positive since it is completely positive on linear combinations of generators by Bochner's theorem,Theorem A.17. ⊣

COROLLARY 7.2. *If $\gamma_0 = \gamma_{\max}$, then by means of the map $\mathscr{F}_M$ it is possible to associate a state $\Omega_M \in \big(\mathbb{LAP}(V) \otimes_{\gamma_{\max}} \mathfrak{A}\big)'_+$ to any cylindrical measure $M \in \mathcal{M}_{\mathrm{cyl}}(V; \mathfrak{A}'_+)$:*

$$\Omega_M = (\mathbb{E}_{0,1})^{-1} \mathscr{F}_M \ .$$

Proof. By Lemma 7.1, we can associate to M the map $\mathscr{F}_M$, and the latter is a completely positive element of $\mathcal{B}\big(\mathbb{LAP}(V), \mathfrak{A}'\big)$. Therefore by Proposition 3.3, $\Omega_M = (\mathbb{E}_{0,1})^{-1} \mathscr{F}_M$ is a state. ⊣

Finally, the application of Tietze's extension theorem yields an extension $\tilde{\Omega}_M \in \big(\mathbb{AP}(\mathbb{FF}V) \otimes_{\gamma_{\max}} \mathfrak{A}\big)'_+$ of Ω_M to the algebra of almost periodic functions.

Proposition 7.3. *Let $(X,\varsigma) \in \mathbf{Symp}_{\mathbb{R}}$, and V a locally convex space in compatible duality with X. In addition, let $\mathfrak{A} \in \mathbf{C^*alg}$, and $(\mathfrak{W}_h)_{h\geq 0}$ the associated Weyl deformation (22). Consider a semiclassical quantum state $(\omega_{h_\beta})_{\beta\in B}$, then for any $M \in \mathscr{W}(\omega_{h_\beta}, \beta \in B) \subset \mathcal{M}_{\mathrm{cyl}}(V;\mathfrak{A}'_+)$, $\Omega_M = (\mathbb{E}_{0,1})^{-1}\mathscr{F}_M$ is a state on $\mathbb{LAP}(V)\otimes_{\gamma_{\max}}\mathfrak{A}$. In addition, the state Ω_M can be extended continuously to a state $\tilde{\Omega}_M \in \big(\mathbb{AP}(\mathbb{FF}V) \otimes_{\gamma_{\max}} \mathfrak{A}\big)'_+$.*

Remark 7.4. In general, the state Ω_M may have a mass defect, *i.e.* $\Omega_M(1) \neq \lim_{h_b} \omega_{h_b}(1)$. However, if the converging sequence $(\omega_{h_b})_{b\in \underline{B}}$ has no loss of mass, then $\Omega_M(1) = \lim_{h_b} \omega_{h_b}(1)$.

Remark 7.5. One could define the "no-quantization" functor $\mathbb{W}_0 : \mathbf{Symp}_{\mathbb{R}} \to \mathbf{C^*alg}$ by $\mathbb{W}_0(X,\varsigma) = \mathbb{AP}(\mathbb{FF}X^*_X)$.

A Elements of cone-valued measure theory

In this appendix, we outline some results of vector integration as a reference. For our purpose, vector measures with values in cones behave essentially as standard measures, and in particular Bochner's theorem is valid for cylindrical cone-valued measures.

A.1 Definition of cone-valued measures

- Given a measurable space E, we denote by Σ its σ-algebra.
- We will always denote by X a real vector space, and by C a *pointed and generating* convex cone in X containing 0. This means that

$$C \cap -C = \{0\}\ ;\ C - C = X\ .$$

- As before, we denote by X^* the algebraic dual of X, and for any $X' \subset X^*$ we denote by C' the dual cone of C defined by $C' = \big\{\kappa \in X', \kappa(C) \subseteq \mathbb{R}^+\big\}$. If X is locally convex, X' its continuous dual and C is closed, then the Hahn-Banach separation theorem yields

$$C = C'' = \big\{x \in X, x(C') \subseteq \mathbb{R}^+\big\}\ . \tag{c1}$$

  We will consider only triples (X, X', C) satisfying (c1).

- We denote by $\mathbb{R}^+_\infty = [0,\infty]$ the extended real semi-line considered as an additive semigroup with the additional rule

$$(\forall x \in \mathbb{R}^+_\infty)\ \infty + x = x + \infty = \infty\ .$$

  We also denote by $\mathbb{R} \cup \{-\infty,+\infty\} = [-\infty,+\infty]$ the (compact) complete lattice of extended reals, and by $\mathbb{C} \cup \{\infty\}$ the extended complex numbers (one-point compactification of $\mathbb{C}$).

- We denote by $C_\infty = \mathrm{Hom}_{\mathrm{mon}}(C', \mathbb{R}^+_\infty)$ the subset of $(\mathbb{R}^+_\infty)^{C'}$ consisting of monoid homomorphisms. C_∞ is a monoid with respect to pointwise addition.

- We denote by $\mathfrak{i}_C$ the natural monoid morphism
$$\mathfrak{i}_C : C \to C_\infty \; , \; (\forall c \in C)(\forall \kappa \in C')\mathfrak{i}_C(c)(\kappa) = \kappa(c) \; .$$
$\mathfrak{i}_C(c_1) = \mathfrak{i}_C(c_2)$ yields $(\forall \kappa \in C')\kappa(c_1 - c_2) = \kappa(c_2 - c_1) = 0$. Therefore (c1) implies $c_1 - c_2 \in C \cap -C$ and by the pointedness of C we have $c_1 - c_2 = 0$. Thus $\mathfrak{i}_C$ is injective and $C \cong \mathfrak{i}_C(C)$ is a submonoid of C_∞.

- The next condition is important to define cone-valued measures:
$$\mathfrak{i}_C(C) = \mathrm{Hom}_{\mathrm{mon}}(C', \mathbb{R}^+) \; . \tag{c2}$$
We discuss later some explicit example of triples that satisfy (c1) and (c2).

Cone valued measures are vector measures generalizing the concept of positive measures. They can be seen as suitable collections of the latter, and therefore they share many interesting properties with “usual” positive measures.

DEFINITION A.1 (C-valued measures). *Let (X, X', C) be a triple that satisfies (c1)-(c2), and E a measurable space. Then $\mu \in (C_\infty)^\Sigma$ is a C-valued measure on E iff it is countably additive and $\mu(\emptyset) = 0$.*

REMARK A.2. In the definition above, 0 is the trivial monoid morphism that maps every $\kappa \in C'$ to $0 \in \mathbb{R}^+_\infty$. In addition, countable additivity is intended in the following sense. Let $\{K_j\}_{j\in\mathbb{N}} \subset C_\infty$ be a subset of C_∞; then the countable combination $\sum_{j\in\mathbb{N}} K_j \in C_\infty$ is defined by pointwise convergence – in the topology of extended reals – of partial sums, i.e. by convergence of the sequences
$$\mathbb{R}^+_\infty \supset (w^\kappa_n)_{n\in\mathbb{N}} = \Bigl(\sum_{j=0}^{n} K_j(\kappa)\Bigr)_{n\in\mathbb{N}} \; , \; \kappa \in C' \; .$$
Therefore a function $\mu \in (C_\infty)^\Sigma$ is countably additive iff for any collection $\{b_j\}_{j\in\mathbb{N}} \subset \Sigma$ of mutually disjoint measurable sets,
$$\mu\Bigl(\bigcup_{j\in\mathbb{N}} b_j\Bigr) = \sum_{j\in\mathbb{N}} \mu(b_j) \; .$$
If $C = \mathbb{R}^+$, Definition A.1 corresponds to that of positive Borel measures. As it was stated before, a key feature of C-valued measures is that they are in fact families of positive measures, indexed by the dual cone C'. The precise statement is the following, whose proof follows almost directly from Definition A.1 above.

THEOREM A.3 (Neeb [68]). *There is a bijection between C-valued measures μ on E and families of positive measures $(\mu_\kappa)_{\kappa\in C'}$ on E such that for any $b \in \Sigma$, $\bigl(\kappa \mapsto \mu_\kappa(b)\bigr) \in \mathrm{Hom}_{\mathrm{mon}}(C', \mathbb{R}^+_\infty)$,* i.e. *the map $\kappa \mapsto \mu_\kappa(b)$ belongs to C_∞.*

In light of Theorem A.3, we define a C-valued measure μ *finite* if μ_κ is a finite positive measure for any $\kappa \in C'$.

We turn now to integration of (scalar) functions with respect to cone-valued measures. As usual, it is convenient to start with the integration of non-negative functions. Theorem A.3 is very convenient in this context, since we can simply define cone-valued integration by means of usual integration. Let $f : E \to \mathbb{R}^+_\infty$ be a non-negative measurable function with values on the extended reals. Let $b \in \Sigma$; then we define for any $\kappa \in C'$,

$$\mathbb{R}^+_\infty \ni I_\kappa = \int_b f(x) \mathrm{d}\mu_\kappa(x) \ .$$

The map $\kappa \mapsto I_\kappa$ is a monoid morphism, and therefore an element of C_∞, that we denote by $\mu_b(f)$. This leads to the following natural definition.

Definition A.4 (μ-integrable functions). *Let (X, X', C) be a triple that satisfies (c1)-(c2); and μ a C-valued measure on a measurable space E. The* measure *of a* non-negative *measurable function $f \in (\mathbb{R}^+_\infty)^E$ is defined by*

$$C_\infty \ni \mu_b(f) = \Big(\kappa \mapsto \int_b f(x) \mathrm{d}\mu_\kappa(x)\Big) \ .$$

A non-negative measurable function $f \in (\mathbb{R}^+_\infty)^E$ is μ-integrable *on the measurable set $b \in \Sigma$ iff $\mu_b(f) \in \mathfrak{i}_C(C) = \mathrm{Hom}_{\mathrm{mon}}(C', \mathbb{R}^+)$. In this case, we denote the* integral *by*

$$C \ni \int_b f(x) \mathrm{d}\mu(x) = \mathfrak{i}_C^{-1}\big(\mu_b(f)\big) \ .$$

A complex function *$f \in \mathbb{C}^E$ is μ-integrable on the measurable set $b \in \Sigma$ iff $|f|$ is μ-integrable, and*

$$\begin{aligned} X_\mathbb{C} \ni \int_b f(x) \mathrm{d}\mu(x) &= \int_b (\Re f)_+(x) \mathrm{d}\mu(x) - \int_b (\Re f)_-(x) \mathrm{d}\mu(x) \\ &\quad + i\Big(\int_b (\Im f)_+(x) \mathrm{d}\mu(x) - \int_b (\Im f)_-(x) \mathrm{d}\mu(x)\Big); \end{aligned}$$

where $X_\mathbb{C}$ is the complexification of X, and $f = (\Re f)_+ - (\Re f)_- + i\Big((\Im f)_+ - (\Im f)_-\Big)$ with

$$\big\{(\Re f)_+, (\Re f)_-, (\Im f)_+, (\Im f)_-\big\} \subset (\mathbb{R}^+)^E \ .$$

Remark A.5. If μ is finite, then any $f \in \bigcap_{\kappa \in C'} L^\infty(E, \mathrm{d}\mu_\kappa)$ is integrable. In particular, any continuous and bounded function is Borel integrable on a topological space E.

In the next proposition we state the important linearity property of the integral $\int_b f(x) \mathrm{d}\mu(x)$. The proof is trivial.

Proposition A.6. *The mapping* $f \mapsto \int_b f(x)\mathrm{d}\mu(x)$ *is a linear, cone-homomorphism. In other words, for any complex-valued* μ*-integrable functions* f_1, f_2 *and* $z \in \mathbb{C}$*:*

$$\int_b \big(f_1(x) + zf_2(x)\big)\mathrm{d}\mu(x) = \int_b f_1(x)\mathrm{d}\mu(x) + z\int_b f_2(x)\mathrm{d}\mu(x) \ .$$

In addition, the cone of non-negative μ*-integrable functions is mapped into the cone* $(C + i\{0\}) \subset X_{\mathbb{C}}$*.*

A.2 Bochner's theorem

We are now ready to prove a result that is crucial in our framework: Bochner's theorem for finite C-valued measures. To prove the theorem we follow closely [50, 68].

A.2.1 Locally compact abelian groups

In this subsection – if not specified otherwise – we take as measure space G a locally compact abelian group with character group $\hat{G}$; and (X, X', C) a triple satisfying (c1)-(c2) and some or all of the following additional conditions. Let K be a pointed and generating cone in a real vector space A. Then an involution * on $A_{\mathbb{C}}$ agrees with K if: $a^*a \in K + i\{0\}$ for any $a \in A_{\mathbb{C}}$, and for any $k \in K$ there exists an $a_k \in A_{\mathbb{C}}$ such that $k = a_k^* a_k$. Then we define the following conditions:

$$X \text{ and } X' \text{ locally convex ;} \tag{c3}$$

$$C' \text{ pointed and generating in } X' \text{ ;} \tag{c4}$$

$$(X')_{\mathbb{C}} \text{ is an involutive algebra with involution agreeing with } C' \text{ ;} \tag{c5}$$

$$X_{\mathbb{C}} \text{ is the continuous dual of } (X')_{\mathbb{C}} \text{ .} \tag{c6}$$

Given a locally convex real vector space T, there are (infinitely) many ways to endow $T_{\mathbb{C}}$ with a topology in a "natural" way (*i.e.* satisfying some suitable properties). Therefore one may ask if it is always possible to endow $X_{\mathbb{C}}$ and $(X')_{\mathbb{C}}$ with suitable topologies such that (c6) is satisfied. If X' is a Banach space and X its continuous dual, the answer is that for any so-called natural complexification of X' there is a so-called reasonable complexification of X such that (c6) is satisfied.

Definition A.7 (Completely positive functions). *Let* G *be an abelian group,* (X, X', C) *a triple satisfying* (c1) *and* (c3)-(c6)*. A function* $f \in (X_{\mathbb{C}})^G$ *is* completely positive *iff for any* $n \in \mathbb{N}_*$*, for any* $\{g_i\}_{i=1}^n \subset G$ *and* $\{\tilde{\kappa}_i\}_{i=1}^n \in (X')_{\mathbb{C}}$*:*

$$\sum_{i,j=1}^n \tilde{\kappa}_j^* \tilde{\kappa}_i \Big(f\big(g_i g_j^{-1}\big)\Big) \geq 0 \ .$$

The definition above is the analogous of positive-definiteness for the cone C. In order to study completely positive functions, it is convenient to introduce a slight generalization. Let $f \in (X_{\mathbb{C}})^G$, where G is an abelian group. Then there exist an associated kernel $F_f(\cdot,\cdot) : G \times G \to X_{\mathbb{C}}$ defined by $F_f(g_1, g_2) = f(g_1 g_2^{-1})$. Hence it is natural to have the following definition.

Definition A.8 (Completely positive kernels). *Let A be a set, (X, X', C) a triple satisfying (c1) and (c3)-(c6). A kernel $F : A \times A \to X_{\mathbb{C}}$ is completely positive iff for any $n \in \mathbb{N}_*$, for any $\{a_i\}_{i=1}^n \subset A$ and $\{\tilde{\kappa}_i\}_{i=1}^n \in (X')_{\mathbb{C}}$:*

$$\sum_{i,j=1}^{n} \tilde{\kappa}_j^* \tilde{\kappa}_i \big(F(a_i, a_j) \big) \geq 0 \ .$$

The equivalence of the two definitions for groups is given by the following trivial result.

Lemma A.9. *Let G be an abelian group, (X, X', C) a triple satisfying (c1) and (c3)-(c6). A function $f \in (X_{\mathbb{C}})^G$ is completely positive iff the associated kernel $F_f \in (X_{\mathbb{C}})^{G \times G}$ is completely positive.*

In order to prove Bochner's theorem, we prove a couple of preliminary results related to complete positivity.

Lemma A.10. *Let (X, X', C) be a triple satisfying (c1)-(c6), and μ a finite C-valued Borel measure on a topological space E. If we denote by $L^2(E, \mathrm{d}\mu) \subset \mathbb{C}^E$ the space of μ-a.e. square integrable functions,* i.e.

$$L^2(E, \mathrm{d}\mu) = \bigcap_{\kappa \in C'} L^2(E, \mathrm{d}\mu_k) \ ;$$

then the integral map $I_\mu : L^2(E, \mathrm{d}\mu) \times L^2(E, \mathrm{d}\mu) \to X_{\mathbb{C}}$, defined by

$$I_\mu(f, g) = \int_E f(x) \bar{g}(x) \mathrm{d}\mu(x) \ ,$$

is well-defined and a completely positive kernel.

Proof. The fact that the kernel I_μ is well-defined is easy to prove using the corresponding property of I_{μ_κ}, $\kappa \in C'$. To prove complete positivity, we proceed as follows. Let $n \in \mathbb{N}_*$, $\{f_i\}_{i=1}^n \subset L^2(E, \mu)$ and $\{\tilde{\kappa}_i\}_{i=1}^n \in (X')_{\mathbb{C}}$. Using the decomposition $X'_{\mathbb{C}} = C' - C' + i(C' - C')$, we see that the map

$$X'_{\mathbb{C}} \ni \tilde{\kappa} \mapsto \mu_{\tilde{\kappa}} = \mu_{\kappa_{\mathrm{R}}^+} - \mu_{\kappa_{\mathrm{R}}^-} + i(\mu_{\kappa_{\mathrm{I}}^+} - \mu_{\kappa_{\mathrm{I}}^-})$$

defines a linear morphism from $X'_{\mathbb{C}}$ to the standard signed measures. Now let $\mu_{\tilde{\kappa}}$ be a signed measure, f an everywhere $\mu_{\tilde{\kappa}}$-integrable function. Then there is a signed measure $\mu_{\tilde{\kappa}(f)}$ defined by $d\mu_{\tilde{\kappa}(f)}(x) = f(x) d\mu_{\tilde{\kappa}}(x)$. If we define in addition

$$\mu_{\tilde{\kappa}(f)^*} = \mu_{\tilde{\kappa}^*(\bar{f})} \ , \ \mu_{\tilde{\kappa}_1(f_1) + \tilde{\kappa}_2(f_2)} = \mu_{\tilde{\kappa}_1(f_1)} + \mu_{\tilde{\kappa}_2(f_2)} \ , \ \mu_{\tilde{\kappa}_1(f_1)\tilde{\kappa}_2(f_2)} = \mu_{\tilde{\kappa}_1 \tilde{\kappa}_2(f_1 f_2)} \ ;$$

then it is easy to see, using property (c5), that for any Borel set $b \in \mathscr{B}(E)$, $\tilde{\kappa} \in X'_{\mathbb{C}}$ and everywhere $\mu_{\tilde{\kappa}}$-integrable f:

$$\int_b \mathrm{d}\mu_{\tilde{\kappa}(f)^* \tilde{\kappa}(f)} \geq 0 \ .$$

Then

$$\sum_{i,j=1}^{n} \tilde{\kappa}_j^* \tilde{\kappa}_i \big(I_\mu(f_i, f_j)\big) = \sum_{i,j=1}^{n} \int_E f_i(x) \bar{f}_j(x) \mathrm{d}\mu_{\tilde{\kappa}_j^* \kappa_i} = \sum_{i,j=1}^{n} \int_E \mathrm{d}\mu_{\tilde{\kappa}_j(f_j)^* \tilde{\kappa}_i(f_i)}$$
$$= \int_E \mathrm{d}\mu_{\left(\sum_{i=1}^n \tilde{\kappa}_i(f_i)\right)^* \left(\sum_{i=1}^n \tilde{\kappa}_i(f_i)\right)} \geq 0 \,.$$

⊣

Corollary A.11. *Let (X, X', C) be a triple satisfying (c1)-(c6), and μ a finite C-valued Borel measure on a topological space E. If we denote by $L^\infty(E, \mathrm{d}\mu) \subset \mathbb{C}^E$ the space of μ-a.e. bounded functions,* i.e.

$$L^\infty(E, \mathrm{d}\mu) = \bigcap_{\kappa \in C'} L^\infty(E, \mathrm{d}\mu_k) \; ;$$

then the integral $I_\mu : L^\infty(E, \mathrm{d}\mu) \to X_{\mathbb{C}}$ is a completely positive function – considering $L^\infty(E, \mathrm{d}\mu)$ as an abelian multiplicative group.

The last ingredient needed to formulate Bochner's theorem is the Fourier transform. The Fourier transform extends quite naturally to cone-valued measures.

Definition A.12 (Fourier transform of a C-valued measure). *Let G be a locally compact abelian group, (X, X', C) a triple satisfying (c1)-(c2), and $\mathcal{M}(\hat{G}, C)$ the set of finite C-valued Borel measures on the character group $\hat{G}$. The* Fourier transform *is a map $\hat{} : \mathcal{M}(\hat{G}, C) \to (X_{\mathbb{C}})^G$, defined by*

$$\hat{\mu}(g) = \int_{\hat{G}} \gamma(g) \mathrm{d}\mu(\gamma) \qquad\qquad (\forall g \in G) \,.$$

Using the definitions above, Bochner's theorem is written in a rather familiar form.

Theorem A.13 (Bochner). *Let G be a locally compact abelian group, (X, X', C) a triple satisfying (c1)-(c6). The Fourier transform is a bijection between finite C-valued measures on $\hat{G}$ and completely positive ultraweakly continuous functions from G to $X_{\mathbb{C}}$.*

Proof. Let μ be a finite C-valued measure on $\hat{G}$. Finiteness of the measure implies the integrability of $\gamma(g)$, since $(\forall \gamma \in \hat{G})(\forall g \in G)|\gamma(g)| = 1$. In addition, γ is a representation of the abelian group G on the functions $L^\infty(G, \mu)$. Hence it follows by Corollary A.11 that $\hat{\mu}(\cdot)$ is completely positive. To prove ultraweak continuity, let $\kappa \in C' + i\{0\}$. By Definition A.4

$$\kappa\big(\hat{\mu}(\cdot)\big) = \int_{\hat{G}} \gamma(\cdot) \mathrm{d}\mu_\kappa(\gamma)$$

is the Fourier transform of the finite measure μ_κ, hence continuous. Now by (c4), $(X')_{\mathbb{C}} = C' - C' + i(C' - C')$ and therefore for any $\tilde{\kappa} \in (X')_{\mathbb{C}}$, $\tilde{\kappa}\big(\hat{\mu}(\cdot)\big) \in \mathbb{C}^G$ is continuous. By (c6), this yields the ultraweak continuity of $\hat{\mu}(\cdot)$.

Now let us consider a completely positive ultraweakly continuous function f from G to $X_{\mathbb{C}}$. Then for any $\kappa \in C' + i\{0\}$, $\kappa\bigl(f(\cdot)\bigr)$ is a positive definite continuous $\mathbb{C}$-valued function. Continuity trivially follows from ultraweak continuity (since $\kappa \in (X')_{\mathbb{C}}$). To prove positive-definiteness, we exploit complete positivity. By Definition A.7, for any $n \in \mathbb{N}_*$, $\{g_i\}_{i=1}^n \subset G$ and $\{\tilde{\kappa}_i\}_{i=1}^n \subset (X')_{\mathbb{C}}$,

$$\sum_{i,j=1}^n \tilde{\kappa}_j^* \tilde{\kappa}_i \Bigl(f\bigl(g_i g_j^{-1}\bigr)\Bigr) \geq 0 \, .$$

Then by property (c5), there exists $\tilde{\kappa}_\kappa \in (X')_{\mathbb{C}}$ such that $\kappa = \tilde{\kappa}_\kappa^* \tilde{\kappa}_\kappa$. So we can choose $\tilde{\kappa}_i = z_i \tilde{\kappa}_\kappa$ for any $i \in \{1, \dots, n\}$, where $z_i \in \mathbb{C}$. Therefore by linearity we obtain

$$\sum_{i,j=1}^n \bar{z}_j z_i \kappa \Bigl(f\bigl(g_i g_j^{-1}\bigr)\Bigr) \geq 0 \, ;$$

and hence positive-definiteness of $\kappa\bigl(f(\cdot)\bigr)$.
The classical Bochner's theorem for locally compact abelian groups [see, *e.g.*, 67] implies the existence of a unique positive, finite measure μ_κ such that $\kappa\bigl(f(\cdot)\bigr) = \hat{\mu}_\kappa(\cdot)$. Therefore we have a unique family of positive and finite measures $(\mu_\kappa)_{\kappa \in C'}$. In order for it to define a unique finite C-valued measure, it is necessary that $\kappa \mapsto \mu_\kappa$ is additive. Let $\kappa_1, \kappa_2 \in C'$. Then $\kappa_1 + \kappa_2 \in C'$, and there is a unique measure $\mu_{\kappa_1+\kappa_2}$ such that $\hat{\mu}_{\kappa_1+\kappa_2}(\cdot) = (\kappa_1 + \kappa_2)\bigl(f(\cdot)\bigr) = \kappa_1\bigl(f(\cdot)\bigr) + \kappa_2\bigl(f(\cdot)\bigr) = \hat{\mu}_{\kappa_1}(\cdot) + \hat{\mu}_{\kappa_2}(\cdot)$. However since the Fourier transform is a linear bijection, it follows that $\mu_{\kappa_1+\kappa_2} = \mu_{\kappa_1} + \mu_{\kappa_2}$. Hence by Theorem A.3 we have defined a unique C-valued measure μ. In addition, by Definition A.4 for any $\kappa \in C'$

$$\kappa\bigl(f(\cdot)\bigr) = \int_{\hat{G}} \gamma(\cdot) \mathrm{d}\mu_\kappa(\gamma) = \kappa\biggl(\int_{\hat{G}} \gamma(\cdot) \mathrm{d}\mu(\gamma)\biggr) \, .$$

Now by (c4), it follows that for any $\tilde{\kappa} \in (X')_{\mathbb{C}}$

$$\tilde{\kappa}\bigl(f(\cdot)\bigr) = \tilde{\kappa}\biggl(\int_{\hat{G}} \gamma(\cdot) \mathrm{d}\mu(\gamma)\biggr) \, ,$$

and therefore by (c6) it follows that

$$f(\cdot) = \int_{\hat{G}} \gamma(\cdot) \mathrm{d}\mu(\gamma) \, .$$

⊣

A.2.2 Topological vector spaces

Bochner's Theorem, Theorem A.13 can be applied to finite dimensional real vector spaces (seen as abelian groups under addition). In that context, the Fourier transform takes the following form. Let V be a finite dimensional vector space, V'

its continuous dual. Given a C-valued measure on V, then its Fourier transform is a function from V' to $X_{\mathbb{C}}$ defined by

$$\hat{\mu}(\omega) = \int_V e^{2i\omega(v)} \mathrm{d}\mu(v) \ .$$

Using cylinders, we obtain a variant of Bochner's theorem for cylindrical measures on topological real vector spaces with arbitrary dimension.

DEFINITION A.14 (C-valued cylindrical measure). *Let V be a topological real vector space, $F(V)$ the set of its $\sigma(V, V')$-closed subspaces with finite codimension (ordered by inclusion); and (X, X', C) a triple satisfying (c1)-(c2). A family of measures $M = (\mu_\Phi)_{\Phi\in F(V)}$ is a cylindrical measure iff it is a projective system of C-valued measures on the family $Q(V)$ of finite dimensional quotients of V.*
In other words, the family $(\mu_\Phi)_{\Phi\in F(V)}$ satisfies:

- $\bigl(\forall\Phi \in F(V)\bigr)$ μ_Φ *is a C-valued measure on V/Φ;*
- *Define for any $b \in \mathscr{B}(V/\Phi)$; $p^{-1}_{\Phi\Psi}(b) = \bigl\{\xi \in V/\Psi, p_{\Phi\Psi}(\xi) \in b\bigr\}$, and $p_{\Phi\Psi}(\mu_\Psi)(b) = \mu_\Psi\bigl(p^{-1}_{\Phi\Psi}(b)\bigr)$. Then*

$$\bigl(\forall\, \Phi \supset \Psi \in F(V)\bigr)\ \mu_\Phi = p_{\Phi\Psi}(\mu_\Psi) = p_{\Phi\Psi\,*}\,\mu_\Psi \ .$$

REMARK A.15. The compatibility condition of Definition A.14 implies that for any $\Phi, \Psi \in F(V)$,

$$\mu_\Phi(V/\Phi) = \mu_\Psi(V/\Psi) = m \in C_\infty \ .$$

We call m the *total mass* of the cylindrical measure M. A cylindrical measure $M = (\mu_\Phi)_{\Phi\in F(V)}$ is *finite* if for any $\Phi \in F(V)$, the measure μ_Φ is finite.

We recall that every C-valued measure μ on V induces a cylindrical measure $M_\mu = (\mu_\Phi)_{\Phi\in F(V)}$. In fact, let $p_\Phi : V \to V/\Phi$ be the canonical projection, then it is sufficient to choose $\mu_\Phi = p_{\Phi\,*}\,\mu$, the push-forward (image) of μ by means of the canonical projection. On the other hand, for any *finite dimensional* V, every cylindrical measure $M = (\mu_\Phi)_{\Phi\in F(V)}$ induces a measure $\mu^{(M)} = \mu_{\{0\}}$ (this fails to be true if V is infinite dimensional). Now we are almost ready to define the Fourier transform of cylindrical measures. In order to do that, let us recall that the push-forward can be extended to cylindrical measures. Consider a cylindrical measure $M = (\mu_\Phi)_{\Phi\in F(V)}$, and a linear continuous map $u : V \to W$. For any $\Xi \in F(W)$, $u^{-1}(\Xi) \in F(V)$, and $u_\Xi : V/u^{-1}(\Xi) \to W/\Xi$ is the linear application obtained from u passing to the quotients. Then

$$u_*\,M = (u_{\Xi\,*}\,\mu_{u^{-1}(\Xi)})_{\Xi\in F(W)} \ .$$

Finally, we denote by $\Phi^\circ \subset V'$ the polar of Φ, since Φ is a vector space, $\Phi^\circ = \bigl\{\xi \in V', (\forall v \in V)\xi(v) = 0\bigr\}$. It is possible to identify $(V/\Phi)'$ and Φ° by means of the adjoint map ${}^{\mathrm{t}}p_\Phi$.

Definition A.16 (Fourier transform of cylindrical measures). *Let V be a topological vector space, (X, X', C) a triple satisfying* (c1)-(c2)*, and let $\mathcal{M}_{\mathrm{cyl}}(V, C)$ be the set of finite C-valued cylindrical measures on V. The* Fourier transform *is a map* $\hat{}: \mathcal{M}_{\mathrm{cyl}}(V, C) \to (X_{\mathbb{C}})^{V'}$*, defined by*

$$\hat{M}(\xi) = \int_{\mathbb{R}} e^{2it} \mathrm{d}(\xi_* M)(t) \ .$$

The Fourier transform can be equivalently defined as

$$(\forall \xi \in \Phi^\circ)\ \hat{\mu}(\xi) = \int_{V/\Phi} e^{2i\xi(v)} \mathrm{d}\mu_\Phi(v) \ .$$

We remark that

$$V' = \bigcup_{\Phi \in F(V)} \Phi^\circ \ , \tag{39}$$

and the consistency condition of Definition A.14 ensure the above definition is consistent. With the aid of Theorem A.13 and a projective argument, it is possible to prove the following result. A more direct proof, for scalar measures, can be found in [80].

Theorem A.17 (Bochner for cylindrical measures). *Let V be a topological vector space, (X, X', C) a triple satisfying* (c1)-(c6)*. The Fourier transform is a bijection between finite C-valued cylindrical measures on V and completely positive functions from V' to $X_{\mathbb{C}}$ that are ultraweakly continuous when restricted to any finite dimensional subspace of V'.*

A.3 Signed and complex vector measures

As in the scalar case, it is possible to introduce signed and complex vector measures.

- Given a convex pointed cone C of a real vector space X, we define the relation $\leq_C$ by
$$x \leq_C y \text{ iff } y - x \in C \ .$$
If for any $\{x, y\} \subset X$, there exist the supremum $x \vee_C y$ with respect to the partial order $\leq_C$, then $(X, \leq_C)$ is a Riesz space. In this case, we say that C is a *lattice cone* of X. Every pointed and generating cone is a lattice cone, if there are two elements in C with either infimum or supremum.

- The extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$ is not an additive monoid, since $+\infty - \infty$ is not defined. However both $(-\infty, +\infty]$ and $[-\infty, +\infty)$ are additive monoids.

- We define $X_\infty = \mathrm{Hom}_{\mathrm{mon}}(C', \mathbb{R} \cup \{-\infty, +\infty\})$ as the subset of functions $f \in (\mathbb{R} \cup \{-\infty, +\infty\})^{C'}$ satisfying the following properties:
  - If $\pm\infty \in \mathrm{Ran}\, f$, then $\mp\infty \notin \mathrm{Ran}\, f$;

– $f : C' \to \operatorname{Ran} f$ is a monoid homomorphism.

This definition is justified by the fact that since $+\infty - \infty$ is not defined, signed measures may only take either $+\infty$ or $-\infty$ as a value (in order to be additive). This has also to be the case for signed vector measures, and therefore they will have X_∞ as target space, see Definition A.18 below.

- $(X_{\mathbb{C}})_\infty = \operatorname{Hom}_{\mathrm{mon}}(C', \mathbb{C} \cup \{\infty\})$.

Let us consider the extension to vector measures of the concept of signed measures. This is easily done by means of X_∞ defined above.

DEFINITION A.18 (Signed vector measures). *Let (X, X', C) be a triple that satisfies (c1)-(c2), E a measurable space. A function $\mu \in (X_\infty)^\Sigma$ is a signed vector measure on E iff it is countably additive and $\mu(\emptyset) = 0$.*

The following useful lemma follows directly from the definition of signed measures.

LEMMA A.19. *Every C-valued measure is also a signed measure. Any real linear combination of two C-valued measures is a signed measure, provided at least one of the two measures is finite.*

The important Theorem A.3 can be easily adapted to hold for signed measures as well.

PROPOSITION A.20. *There is a bijection between signed vector measures μ on E and families of signed measures $(\mu_\kappa)_{\kappa \in C'}$ on E such that for any $b \in \Sigma$, $\big(\kappa \mapsto \mu_\kappa(b)\big) \in X_\infty$.*

A signed measure μ is *finite* iff for any $\kappa \in C'$, μ_κ is finite. The idea behind signed vector measures is that, as in the case of standard measures, they are the sum of two cone-valued measures. Therefore it is reasonable to define them as a collection indexed only by the dual cone C', in order to prevent possible "sign incongruences" on μ_κ due to the action of a $\kappa \notin C'$. As a matter of fact, with this definition we can indeed prove the existence of a unique Jordan decomposition for signed vector measures. The precise statement is contained in the following result.

PROPOSITION A.21. *Let (X, X', C) be a triple satisfying (c1)-(c2); and μ a signed vector measure on a measurable space E. Then there exist three C-valued measures $\mu^+, \mu^-, |\mu|$ such that:*

- *$\mu = \mu^+ - \mu^-$, and the decomposition is unique;*
- *$|\mu| = \mu^+ + \mu^-$;*
- *At least one between μ^+ and μ^- is finite;*
- *μ is finite iff $|\mu|$ is finite.*

If in addition $(X, \leq_C)$ is a Riesz space, then $\mu^+ = \mu \vee_C 0$, $\mu^- = \mu \wedge_C 0$ and $|\mu| = |\mu|_C$. The operations $+$, $-$, $\vee_C$, $\wedge_C$ and $|\cdot|_C$ on measures are defined pointwise on measurable sets, and 0 is the measure identically zero.

Proof. Let μ be a signed vector measure. Then $(\mu_\kappa)_{\kappa\in C'}$ is the corresponding family of signed measures. By Jordan decomposition of signed measures, for any $\kappa \in C'$, there exist a unique decomposition $\mu_\kappa = \mu_\kappa^+ - \mu_\kappa^-$, with μ_κ^+ and μ_κ^- positive measures with at least one of the two finite, and μ_κ is finite iff $|\mu_\kappa|$ is finite. Hence if $(|\mu_k|)_{\kappa\in C'}$ is the image of a C-valued measure $|\mu|$, μ is finite iff $|\mu|$ is finite. In addition, suppose that there exists a $\tilde{\kappa} \in C'$ such that $\mu_{\tilde{\kappa}}^+$ is not finite. Then $+\infty \in \operatorname{Ran}\mu$, and therefore $-\infty \notin \operatorname{Ran}\mu$, i.e. for any $\kappa \in C'$, μ_κ^- is finite. It follows that if $(\mu_\kappa^-)_{\kappa\in C'}$ is the image of a C-valued measure, such measure is finite. An analogous statement holds with plus replaced by minus. By Lemma A.19, to prove the first part of the theorem it remains only to check that the families $(\mu_\kappa^+)_{\kappa\in C'}$ and $(\mu_\kappa^-)_{\kappa\in C'}$ are C-valued measures, i.e. that for any $b \in \Sigma$, the maps $\kappa \mapsto \mu_\kappa^\pm(b)$ are monoid morphisms. On one hand, we have by the fact that $\mu \in X_\infty$ and then Jordan decomposition that

$$\mu_{\kappa_1+\kappa_2}(b) = \mu_{\kappa_1}(b) + \mu_{\kappa_2}(b) = \mu_{\kappa_1}^+(b) + \mu_{\kappa_2}^+(b) - \bigl(\mu_{\kappa_1}^-(b) + \mu_{\kappa_2}^-(b)\bigr) \; ;$$

on the other hand, by Jordan decomposition we have also that

$$\mu_{\kappa_1+\kappa_2}(b) = \mu_{\kappa_1+\kappa_2}^+(b) - \mu_{\kappa_1+\kappa_2}^-(b) \; .$$

Now since the decomposition is unique, it follows that

$$\mu_{\kappa_1+\kappa_2}^\pm(b) = \mu_{\kappa_1}^\pm(b) + \mu_{\kappa_2}^\pm(b) \; ,$$

i.e. the map is a monoid morphism.
To prove the last part, let $\mu = \mu^+ - \mu^-$ be a vector signed measure with the respective decomposition. Then for any $b \in \Sigma$, we have that

$$X \ni \mu(b) = \mu^+(b) - \mu^-(b) \; ; \; \mu^+(b), \mu^-(b) \geq_C 0 \; .$$

If $(X, \leq_C)$ is a Riesz space, it then follows that $\mu^+ = \mu \vee_C 0$, $\mu^- = \mu \wedge_C 0$. ⊣

The complex vector measures are defined in an analogous fashion, and they are the sum of four C-valued measures. We quickly mention the basic definitions and results without proof, for they are equivalent to the ones for signed vector measures.

Definition A.22 (Complex vector measures). *Let (X, X', C) be a triple that satisfies (c1)-(c2), E a measurable space. A function $\mu \in \bigl((X_{\mathbb{C}})_\infty\bigr)^\Sigma$ is a complex vector measure on E iff it is countably additive and $\mu(\emptyset) = 0$.*

Lemma A.23. *Under the identifications $\mathbb{R} \ni \alpha \to \alpha + i0$, $+\infty \to \infty$, $-\infty \to \infty$; every signed vector measure is also a complex measure. Any complex linear combination of two signed measures is a complex measure.*

Proposition A.24. *Let (X, X', C) be a triple satisfying (c1)-(c2); and μ a complex vector measure on a measurable space E. Then there exist five C-valued measures $\mu_{\mathrm{R}}^+, \mu_{\mathrm{R}}^-, \mu_{\mathrm{I}}^+, \mu_{\mathrm{I}}^-, |\mu|$ such that:*

- $\mu = \mu_{\mathrm{R}}^+ - \mu_{\mathrm{R}}^- + i(\mu_{\mathrm{I}}^+ - \mu_{\mathrm{I}}^-)$*, and the decomposition is unique;*

- $|\mu| = \mu_{\mathrm{R}}^{+} + \mu_{\mathrm{R}}^{-} + \mu_{\mathrm{I}}^{+} + \mu_{\mathrm{I}}^{-}$;
- *μ is finite iff $|\mu|$ is finite, or equivalently if $\mu_{\mathrm{R}}^{+}, \mu_{\mathrm{R}}^{-}, \mu_{\mathrm{I}}^{+}, \mu_{\mathrm{I}}^{-}$ are all finite.*

COROLLARY A.25. *The integral with respect to a* finite *complex vector measure μ is a map $\int_{(\cdot)} \mathrm{d}\mu : \Sigma \to X_{\mathbb{C}}$ defined by*

$$\int_b \mathrm{d}\mu = \int_b \mathrm{d}\mu_{\mathrm{R}}^{+} - \int_b \mathrm{d}\mu_{\mathrm{R}}^{-} + i\Big(\int_b \mathrm{d}\mu_{\mathrm{I}}^{+} - \int_b \mathrm{d}\mu_{\mathrm{I}}^{-}\Big) .$$

A.4 A CONCRETE REALIZATION: DUALS OF C*-ALGEBRAS

In this subsection, we discuss explicitly the relevant class of triples satisfying the properties (c1)-(c6) that was used in this paper.

- Given a C*-algebra $\mathfrak{A}$, we denote by $\mathfrak{A}_{+}$ the set of elements with positive spectrum, and by $\mathfrak{A}_{*}$ the set of self-adjoint elements.
- If $\mathfrak{A}'_{*}$ is the continuous dual of the set of self-adjoint elements $\mathfrak{A}_{*}$ of a C*-algebra $\mathfrak{A}$, we denote by $\mathfrak{A}'_{+}$ the functionals that are positive when acting on $\mathfrak{A}_{+}$.

In order to verify conditions (c1)-(c6), we make use of the following classical result [see, *e.g.*, 79].

PROPOSITION A.26. *Let $\mathfrak{A}$ be a C*-algebra. Then:*

- *$\mathfrak{A}_{*}$ is a real Banach subspace of $\mathfrak{A}$ and $\mathfrak{A} = \mathfrak{A}_{*} + i\mathfrak{A}_{*}$.*
- *$\mathfrak{A}_{+}$ is a closed, pointed and generating convex cone of $\mathfrak{A}_{*}$.*
- *$\mathfrak{A}'_{+}$ is a pointed and generating convex cone of $\mathfrak{A}'_{*}$; in particular for any $\alpha \in \mathfrak{A}'_{*}$ there is a unique decomposition*

$$\alpha = \alpha^{+} - \alpha^{-} \ , \quad \textit{with } \alpha^{+}, \alpha^{-} \in \mathfrak{A}'_{+} \ .$$

- $(\mathfrak{A}'_{*})_{\mathbb{C}} = \mathfrak{A}'$.

By means of Proposition A.26, conditions (c1), (c3)-(c6) are immediately proved. Condition (c2) is proved using a remarkable result of Neeb [68, Lemma I.5]. In fact, if we call C'_1 the set of elements of C' with $\mathfrak{A}_{*}$-norm one, then $C'_1 - C'_1$ is a 0-neighbourhood of $\mathfrak{A}_{*}$. In § 3, the $\mathfrak{A}'_{+}$-valued measures played an important role; from the discussion above, it follows that the usage of all the results in this appendix, and especially Bochner's theorem, is justified.

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