The zeta Functions of Moduli Stacks of G-Zips and Moduli Stacks of Truncated Barsotti–Tate Groups

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Abstract. We study stacks of truncated Barsotti–Tate groups and the G-zips defined by Pink, Wedhorn & Ziegler. The latter occur naturally when studying truncated Barsotti–Tate groups of height 1 with additional structure. By studying objects over finite fields and their automorphisms we determine the zeta functions of these stacks. These zeta functions can be expressed in terms of the Weyl group of the reductive group G and its action on the root system. The main ingredients are the classification of G-zips over algebraically closed fields and their automorphism groups by Pink, Wedhorn & Ziegler, and the study of truncated Barsotti-Tate groups and their automorphism groups by Gabber & Vasiu.

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1 Introduction

Throughout this article, let p be a prime number. Over a field k of characteristic p, the truncated Barsotti–Tate groups of level 1 (henceforth BT_1) were first classified in [8]. The main examples of BT_1 come from p-kernels $A[p]$ of abelian

varieties A over k. As such, these results (independently obtained) were used in [12] to define a stratification on the moduli space of polarised abelian varieties. In [9] the first step was made towards generalising this relation to Shimura varieties of PEL type, by classifying Barsotti–Tate groups of level 1 with the action of a fixed semisimple $\mathbb{F}_p$-algebra and/or a polarisation. The classification of these BT_1 with extra structure over an algebraically closed field $\bar{k}$ turned out to be related to the Weyl group of an associated reductive group over $\bar{k}$. These BT_1 with extra structure were then generalised in [11] to so-called *F-zips*, that generalise the linear algebra objects that arise when looking at the Dieudonné modules corresponding to BT_1. Over an algebraically closed field the classification of these F-zips is also related to the Weyl group of a certain reductive group that depends on the chosen extra structure. In [14] and [13] this was again generalised to so-called $\hat{G}$-*zips*, taking the (not necessarily connected) reductive group $\hat{G}$ as the primordial object.[1] For certain choices of $\hat{G}$ these $\hat{G}$-zips correspond to F-zips with some additional structure. Again their classification over an algebraically closed field is expressed in terms of the Weyl group of $\hat{G}$.

These classifications suggest two possible directions for further research. First, one could try to study $\hat{G}$-zips over non-algebraically closed fields; the first step would then be to understand the classification over finite fields. Another direction would be to study BT_n for general n, either over finite fields or over algebraically closed fields. One may approach both these problems by looking at their moduli stacks. For a reductive group $\hat{G}$ over a finite field k, a cocharacter $\chi\colon \mathbb{G}_{\mathrm{m},k'} \to \hat{G}_{k'}$ defined over some finite extension k' of k, and a subgroup scheme $\Theta \subset \pi_0(\hat{G}_{k'})$ one can consider the stack $\hat{G}\text{-}\mathsf{Zip}_{k'}^{\chi,\Theta}$ of $\hat{G}$-zips of type (χ,Θ) (see Section 4); it is an algebraic stack of finite type over k'. Similarly, for two nonnegative integers $h \geq d$ one can consider the stack $\mathsf{BT}_n^{h,d}$ of truncated Barsotti–Tate groups of level n, height h and dimension d; this is an algebraic stack of finite type over $\mathbb{F}_p$ (see [19, Prop. 1.8]). One way to study these stacks is via their *zeta function*. For an algebraic stack of finite type X over a finite field $\mathbb{F}_q$, and an integer $v \geq 1$, the $\mathbb{F}_{q^v}$-point count of X is defined as

$$\#\mathsf{X}(\mathbb{F}_{q^v}) = \sum_{x \in [\mathsf{X}(\mathbb{F}_{q^v})]} \frac{1}{\#\mathrm{Aut}(x)},$$

where $[\mathcal{C}]$ denotes the set of isomorphism classes of a category $\mathcal{C}$. The zeta

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[1]Here we follow the notation of [14] and [13] in writing $\hat{G}$ for the reductive group, and G for its identity component.

function of X is defined to be the element of $\mathbb{Q}[\![t]\!]$ given by

$$Z(\mathsf{X},t) = \exp\left(\sum_{v\geq 1} \frac{q^v}{v} \#\mathsf{X}(\mathbb{F}_{q^v})\right).$$

By definition the zeta function encodes information about the point counts of X. Furthermore, the zeta function is related to the cohomology of ℓ-adic sheaves on X (see [1] and [17]). As a power series in t, it defines a meromorphic function that is defined everywhere (as a holomorphic map $\mathbb{C} \to \mathbb{P}^1(\mathbb{C})$), but it is not necessarily rational; the reason for this is that for stacks, contrary to schemes, the ℓ-adic cohomology algebra is in general not finite dimensional (see [17, 7.1]).

The aim of this article is to calculate the zeta functions of stacks of the form $\hat{G}\text{-}\mathsf{Zip}_{k'}^{\chi,\Theta}$ and $\mathsf{BT}_n^{h,d}$. The results are stated below. In the statement of Theorem 1.1, the finite set $\Xi^{\chi,\Theta}$ classifies the set of isomorphism classes in $\hat{G}\text{-}\mathsf{Zip}^{\chi,\Theta}(\bar{\mathbb{F}}_q)$; this classification turns out to be related to the Weyl group of $\hat{G}$ (see Proposition 4.5). For $\xi \in \Xi^{\chi,\Theta}$, let $a(\xi)$ be the dimension of the automorphism group of the corresponding object in $\hat{G}\text{-}\mathsf{Zip}^{\chi,\Theta}(\bar{\mathbb{F}}_q)$, and let $b(\xi)$ be the minimal integer b such that this object has a model over $\mathbb{F}_{q^b}$. It turns out that $\Xi^{\chi,\Theta}$ has a natural action of $\Gamma := \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$, and that the functions a and b are Γ-invariant. In the statement of Theorem 1.2 the notation is the same, applied to the group $\hat{G} = \mathtt{GL}_{h,\mathbb{F}_p}$ (with suitable χ; as a subgroup of $\pi_0(\hat{G})_{k'}$ the group Θ is necessarily trivial for connected $\hat{G}$).

THEOREM 1.1. *Let q_0 be a power of p, and let $\hat{G}$ be a reductive group over $\mathbb{F}_{q_0}$. Let q be a power of q_0, let $\chi\colon \mathbb{G}_{\mathrm{m},\mathbb{F}_q} \to \hat{G}_{\mathbb{F}_q}$ be a cocharacter, and let Θ be a subgroup scheme of the group scheme $\pi_0(\mathtt{Cent}_{\hat{G}_{\mathbb{F}_q}}(\chi))$. Let $\Xi^{\chi,\Theta}$ and Γ be as in Section 4 and let $a,b\colon \Gamma\backslash\Xi^{\chi,\Theta} \to \mathbb{Z}_{\geq 0}$ be as in Notation 5.6. Then*

$$Z(\hat{G}\text{-}\mathsf{Zip}_{\mathbb{F}_q}^{\chi,\Theta},t) = \prod_{\bar{\xi}\in\Gamma\backslash\Xi^{\chi,\Theta}} \frac{1}{1-(q^{-a(\bar{\xi})}t)^{b(\bar{\xi})}}.$$

THEOREM 1.2. *Let $h,n > 0$ and $0 \leq d \leq h$ be integers. Let Ξ and $a\colon \Xi \to \mathbb{Z}_{\geq 0}$ be as in Notation 6.1. Then*

$$Z(\mathsf{BT}_n^{h,d},t) = \prod_{\xi\in\Xi} \frac{1}{1-p^{-a(\xi)}t}.$$

In particular the zeta function of the stack $\mathsf{BT}_n^{h,d}$ does not depend on n.

As we will see later on, for split groups we have $b(\bar{\xi}) = 1$ for all $\bar{\xi} \in \Gamma\backslash\Xi^{\chi,\Theta} = \Xi^{\chi,\Theta}$. In particular, the zeta function of $\mathsf{BT}_n^{h,d}$ as determined in Theorem 1.2

coincides with that of $\mathtt{GL}_h\text{-}\mathtt{Zip}^\chi$ as determined in Theorem 1.1 (for a suitable χ).

All the terminology used in the statements above will be introduced in due time. For now let us note that the functions a and b can also be expressed in terms of the action of the Weyl group of $\hat{G}$ on the root system, and are readily calculated for a given $(\hat{G}, \chi, \Theta)$ (see Example 4.7). Furthermore, [13, §8] shows how to construct isomorphisms (on categories of k-points for perfect k) between moduli stacks of G-zips, and moduli stacks of F-zips and BT_1 with additional structure. One can use this and Theorem 1.1 to calculate the zeta functions of the latter.

We will spend some time developing theory about nonconnected algebraic groups, and much of the discussion would be simplified considerably when only considering connected $\hat{G}$. However, we choose to tackle the problem in this generality because the nonconnected case is interesting in its own right: $\hat{G}$-zips for nonconnected $\hat{G}$ appear, for instance, when studying F-zips with symmetric bilinear forms (see [13, §8.5]), which in turn appear when considering reductions of Shimura varieties attached to orthogonal groups.

2   The zeta function of quotient stacks

Throughout this section we let k be a finite field of characteristic p. In this section we study the point counts and zeta functions of categories associated to quotient stacks. The main results (Propositions 2.14 and 2.19) are quite technical in nature, but we need them in this form in order to prove Theorems 1.1 and 1.2.

Let G be a smooth algebraic group over k. Let X be a *variety* over k, by which we mean a reduced k-scheme of finite type. Suppose X has a left action of G. Recall that the *quotient stack* $[G\backslash X]$ is defined as follows: If S is a k-scheme, then the objects of the category $[G\backslash X](S)$ are pairs (T, f), where T is a left G-torsor over S in the étale topology, and $f\colon T \to X_S$ is a G_S-equivariant morphism of S-schemes. A morphism $(T, f) \to (T', f')$ in $[G\backslash X](S)$ is an isomorphism of G-torsors $\varphi\colon T \xrightarrow{\sim} T'$ such that $f = f'\varphi$. In order to calculate point counts and zeta functions we first need to set up a bit of notation.

NOTATION 2.1. Suppose G is a smooth algebraic group over k, and let z be a cocycle in $\mathrm{Z}^1(k,G)$. Recall that this means that z is a continuous map $z\colon \mathrm{Gal}(\bar{k}/k) \to G(\bar{k})$ (where the right hand side has the discrete topology) for which the following equation is satisfied for all $\gamma, \gamma' \in \mathrm{Gal}(\bar{k}/k)$:

$$z(\gamma\gamma') = z(\gamma) \cdot {}^{\gamma}z(\gamma'). \tag{2.2}$$

Let X be a k-variety with a left action of G, and let z be a cocycle in $\mathrm{Z}^1(k,G)$. We define the twisted algebraic space X_z as follows: Let $X_{z,\bar{k}}$ be isomorphic to $X_{\bar{k}}$ as $\bar{k}$-algebraic spaces with a $G_{\bar{k}}$-action via an isomorphism $\varphi_z\colon X_{z,\bar{k}} \xrightarrow{\sim} X_{\bar{k}}$. We define the Galois action on $X_z(\bar{k})$ by taking

$$^{\gamma}x := \varphi_z^{-1}(z(\gamma) \cdot {}^{\gamma}\varphi_z(x))$$

for all $x \in X_{z,\bar{k}}(\bar{k})$ and all $\gamma \in \mathrm{Gal}(\bar{k}/k)$; this defines an algebraic space X_z over k. Its isomorphism class only depends on the class of z in $\mathrm{H}^1(k,G)$. Two cases deserve special mention:

- We let G act on itself on the left by defining $g \cdot x := xg^{-1}$. Then G_z is a left G-torsor, and $\mathrm{H}^1(k,G)$ classifies the left G-torsors in this way.

- We let G act on itself on the left by inner automorphisms. The twist is denoted $G_{\mathrm{in}(z)}$, and this is again an algebraic group. If X is a k-variety with a left G-action, then X_z naturally has a left $G_{\mathrm{in}(z)}$-action.

REMARK 2.3. Since the algebraic space X_z is in particular an algebraic stack, we have a notion of the point count $\#X_z(k')$ for any finite extension k' of k. Since the objects of $X_z(\bar{k})$ have no nontrivial automorphisms, we can regard $X_z(k')$ as a set, and its point count as the cardinality of this set.

This terminology enables us to formulate the following proposition.

PROPOSITION 2.4. *Let k' be a finite extension of k. Let G be a smooth algebraic group over k, and let X be a k-variety equipped with a left action of G. Then*

$$\#[G\backslash X](k') = \sum_{z\in \mathrm{H}^1(k',G)} \frac{\#X_z(k')}{\#G_{\mathrm{in}(z)}(k')}.$$

Proof. It suffices to show this for $k' = k$. Let T be a left G-torsor over k, and let $z \in \mathrm{Z}^1(k,G)$ be such that $T \cong G_z$. Then the automorphism group scheme of T as a left G-torsor is $G_{\mathrm{in}(z)}$, which acts by right multiplication on G_z. As such, we may consider T as a $(G, G_{\mathrm{in}(z)})$-bitorsor. If we look at the left G-action, we

can define a variety T_z as in Notation 2.1. This naturally has the structure of a $(G_{\text{in}(z)}, G_{\text{in}(z)})$-bitorsor; in fact, a straightforward calculation using Notation 2.1 shows that it is a trivial bitorsor. If $f\colon T \to X_k$ is a (left) G-equivariant map, then the map $f_{\bar{k}}\colon T_{\bar{k}} \to X_{\bar{k}}$ is defined over k when considered as a map $T_{z,\bar{k}} \to X_{z,\bar{k}}$, and we denote the resulting map $T_z \to X_z$ by f_z; it is (left) $G_{\text{in}(z)}$-equivariant. This gives a one-to-one correspondence between $\text{Hom}_G(T, X)$ and $\text{Hom}_{G_{\text{in}(z)}}(T_z, X_z)$. Let t_0 be an element of $T_z(k)$, which exists since T_z is a trivial $G_{\text{in}(z)}$-torsor. We may identify the sets $\text{Hom}_{G_{\text{in}(z)}}(T_z, X_z)$ and $X_z(k)$ by identifying a map with its image of t_0, and two maps $f_z, f_z' \in \text{Hom}_{G_{\text{in}(z)}}(T_z, X_z)$ correspond to isomorphic objects $(T, f), (T, f')$ in $[G\backslash X](k)$ if and only if $f_z(t_0)$ and $f_z'(t_0)$ are in the same $G_{\text{in}(z)}(k)$-orbit in $X_z(k)$. On the other hand, the automorphism group of (T, f) is identified with $\text{Stab}_{G_{\text{in}(z)}(k)}(f_z(t_0))$. From the orbit-stabiliser formula we find

$$\sum_{\substack{(T', f') \in [[G\backslash X](k)], \\ T' \cong T}} \frac{1}{\#\text{Aut}(T', f')} = \sum_{x \in G_{\text{in}(z)}(k)\backslash X_z(k)} \frac{1}{\#\text{Stab}_{G_{\text{in}(z)}(k)}(x)}$$

$$= \frac{\#X_z(k)}{\#G_{\text{in}(z)}(k)}.$$

Summing over all cohomology classes in $\text{H}^1(k, G)$ now proves the proposition. $\square$

While Proposition 2.4 gives a direct formula for the point count of a quotient stack over a given field extension k' of k, it is not as useful in a context where k' varies, as it is a priori unclear how $\text{H}^1(k', G)$ varies with it. In Propositions 2.14 and 2.19 we give formulas for the point counts $[G\backslash X](k')$ that do not involve determining the cohomology set $\text{H}^1(k', G)$, under some (quite technical) conditions on G and X. We first set up some notation.

NOTATION 2.5. As before let G be a smooth algebraic group over k, and let $\gamma \in \text{Gal}(\bar{k}/k)$ be the $\#k$-th power Frobenius. We let $G(\bar{k})$ act on itself on the left by defining

$$g \cdot x := gx(^\gamma g)^{-1}. \tag{2.6}$$

Its set of orbits is denoted $\text{Conj}_k(G)$.

LEMMA 2.7. *Let G be a smooth algebraic group over k. Let $\gamma \in \text{Gal}(\bar{k}/k)$ be the $\#k$-th power Frobenius. Then the map*

$$\begin{array}{rcl} \text{Z}^1(k, G) & \to & G(\bar{k}) \\ z & \mapsto & z(\gamma) \end{array}$$

is a bijection, and it induces a bijection $\mathrm{H}^1(k,G) \xrightarrow{\sim} \mathrm{Conj}_k(G)$.

Proof. Let Γ be the Galois group $\mathrm{Gal}(\bar{k}/k)$. Since $\langle\gamma\rangle \subset \Gamma$ is a dense subgroup, the map is certainly injective. To show that it is surjective, fix a $g \in G(\bar{k})$, and define a map $z \colon \langle\gamma\rangle \to G(\bar{k})$ by

$$z(\gamma^n) = \begin{cases} g \cdot ({}^{\gamma}g) \cdots ({}^{\gamma^{n-1}}g), & \text{if } n \geq 0; \\ ({}^{\gamma^{-1}}g^{-1}) \cdots ({}^{\gamma^{n}}g^{-1}) & \text{if } n < 0. \end{cases}$$

This satisfies the cocycle condition (2.2) on $\langle\gamma\rangle$. Let e be the unit element of $G(\bar{k})$. To show that we can extend z continuously to Γ, we claim that there is an integer n such that $z(\gamma^N) = e$ for all $N \in n\mathbb{Z}$. To see this, let k' be a finite extension of k such that $g \in G(k')$. Then from the definition of the map z we see that z maps $\langle\gamma\rangle$ to $G(k')$. The latter is a finite group, and hence there must be two nonnegative integers $m < m'$ such that $z(\gamma^m) = z(\gamma^{m'})$. Set $n = m' - m$. From the definition of z we see that

$$z(\gamma^{m'}) = z(\gamma^m) \cdot ({}^{\gamma^m}g) \cdots ({}^{\gamma^{m'-1}}g),$$

hence $({}^{\gamma^m}g) \cdots ({}^{\gamma^{m'-1}}g) = e$; but the left hand side of this is equal to ${}^{\gamma^m}z(\gamma^n)$, hence $z(\gamma^n) = e$. The cocycle condition (2.2) now tells us that $z(\gamma^N) = e$ for every multiple N of n; furthermore, we see that for general $f \in \mathbb{Z}$ the value $z(\gamma^f)$ only depends on $\bar{f} \in \mathbb{Z}/n\mathbb{Z}$. Hence we can extend z to all of Γ via the composite map

$$\Gamma \twoheadrightarrow \Gamma/n\Gamma \xrightarrow{\sim} \langle\gamma\rangle/\langle\gamma^n\rangle \xrightarrow{z} G(\bar{k}),$$

and this is an element of $\mathrm{Z}^1(k,G)$ that sends γ to g; hence the map in the lemma is surjective, as was to be shown. This map is also $G(\bar{k})$-equivariant with respect to the actions that give rise to the quotients $\mathrm{H}^1(k,G)$ and $\mathrm{Conj}_k(G)$, which proves the second statement of the lemma. $\qquad\square$

Recall that the *classifying stack* of an algebraic group G is defined to be $\mathsf{B}(G) := [G\backslash *]$, where $* = \mathrm{Spec}(k)$ (with the trivial G-action).

Lemma 2.8. *Let G be a finite étale group scheme over k. Then for every finite extension k' of k we have $\#\mathsf{B}(G)(k') = 1$.*

Proof. It suffices to show this for $k = k'$. The category $\mathsf{B}(G)(k)$ is the category of G-torsors over k; its objects are classified by $\mathrm{H}^1(k,G)$. Let $\gamma \in \mathrm{Gal}(\bar{k}/k)$ be the $\#k$-th power Frobenius, and let $z \in \mathrm{H}^1(k,G)$. Then the automorphism

group (as an abstract group) of the torsor G_z is equal to $G_{\mathrm{in}(z)}(k)$, which equals

$$
\begin{aligned}
G_{\mathrm{in}(z)}(k) &\cong \left\{ g \in G(\bar{k}) : g = z(\gamma) \cdot {}^{\gamma}g \cdot z(\gamma)^{-1} \right\} \\
&= \left\{ g \in G(\bar{k}) : z(\gamma) = g \cdot z(\gamma) \cdot ({}^{\gamma}g)^{-1} \right\} \\
&= \mathrm{Stab}_{G(\bar{k})}(z(\gamma)),
\end{aligned}
$$

where the action of $G(\bar{k})$ on itself in the last line is the one in (2.6). For every orbit $C \in \mathrm{Conj}_k(G)$ choose an element $x_C \in C$; then the orbit-stabiliser formula and Lemma 2.7 yield

$$
\begin{aligned}
\sum_{z \in \mathrm{H}^1(k,G)} \frac{1}{\#\mathrm{Aut}(G_z)} &= \sum_{C \in \mathrm{Conj}_k(G)} \frac{1}{\#\mathrm{Stab}_{G(\bar{k})}(x_C)} \\
&= \sum_{C \in \mathrm{Conj}_k(G)} \frac{\#C}{\#G(\bar{k})} \\
&= 1. \qquad \qquad \square
\end{aligned}
$$

LEMMA 2.9. *Let $1 \to A \to B \to C \to 1$ be a short exact sequence of smooth algebraic groups over k. Suppose that A is connected.*

1. *The natural map $\mathrm{H}^1(k,B) \to \mathrm{H}^1(k,C)$ is bijective.*

2. *For $z \in \mathrm{H}^1(k,B) = \mathrm{H}^1(k,C)$, let A_z be the twist of A induced by the image of z under the natural map $\mathrm{H}^1(k,B) \to \mathrm{H}^1(k, \mathrm{Aut}(A_{\bar{k}}))$. Then*

$$
\#B_{\mathrm{in}(z)}(k) = \#A_z(k) \cdot \#C_{\mathrm{in}(z)}(k).
$$

Proof. The short exact sequence of algebraic groups over k

$$
1 \to A \to B \to C \to 1
$$

induces an exact sequence of pointed cohomology sets

$$
1 \to A(k) \to B(k) \to C(k) \to \mathrm{H}^1(k,A) \to \mathrm{H}^1(k,B) \to \mathrm{H}^1(k,C).
$$

From Lang's theorem we know that $\mathrm{H}^1(k,A)$ is trivial. By [16, III.2.4.2 Cor. 2] the last map is surjective, so by exactness it is bijective, which proves the first statement. Furthermore for a $z \in \mathrm{H}^1(k,B)$ the inclusion map $A_z(\bar{k}) \to B_{\mathrm{in}(z)}(\bar{k})$ is Galois-equivariant, and the quotient of $B_{\mathrm{in}(z)}(\bar{k})$ by the image of this map is isomorphic to $C_{\mathrm{in}(z)}(\bar{k})$. This shows that we get a twisted short exact sequence

$$
1 \to A_z \to B_{\mathrm{in}(z)} \to C_{\mathrm{in}(z)} \to 1.
$$

Since A_z is connected, we find $\mathrm{H}^1(k, A_z) = 1$, and then a long exact sequence analogous to the one above proves the second statement. $\square$

DEFINITION 2.10. Let X be an algebraic stack over a field k. Let $k' \subset k''$ be two field extensions of k, and let $x \in \mathsf{X}(k'')$. Then a *model of x over k'* is an object $y \in \mathsf{X}(k')$ such that $y_{k''} \cong x$.

LEMMA 2.11. *Let G be a smooth algebraic group over k, and let X be a variety over k. Then there is a bijection $[G\backslash X](\bar{k}) \xrightarrow{\sim} G(\bar{k})\backslash X(\bar{k})$ with the following property: let k' be a finite extension of k, and let ξ be an element of $G(\bar{k})\backslash X(\bar{k})$, corresponding to a $(T, f) \in [G\backslash X](\bar{k})$. Then (T, f) has a model over k' if and only if ξ is fixed under the action of $\mathrm{Gal}(\bar{k}/k')$ on $G(\bar{k})\backslash X(\bar{k})$.*

Proof. Over $\bar{k}$ every torsor is trivial, and a G-equivariant map $f \colon G_{\bar{k}} \to X_{\bar{k}}$ is determined by its image of the unit element $e \in G(\bar{k})$. Furthermore, two maps $f, f' \colon G_{\bar{k}} \to X_{\bar{k}}$ yield isomorphic elements $(G_{\bar{k}}, f), (G_{\bar{k}}, f')$ of $[G\backslash X](\bar{k})$ if and only if $f(e)$ and $f'(e)$ lie in the same $G(\bar{k})$-orbit. Since $f(G(\bar{k}))$ is a $G(\bar{k})$-orbit in $X(\bar{k})$, we get a bijection:

$$\Phi \colon [[G\backslash X](\bar{k})] \quad \xrightarrow{\sim} \quad G(\bar{k})\backslash X(\bar{k}) \qquad (2.12)$$
$$(G_{\bar{k}}, f) \quad \mapsto \quad f(G(\bar{k})).$$

Now suppose (T, f) is an element of $[G\backslash X](k')$. Then $f \colon T(\bar{k}) \to X(\bar{k})$ is $\mathrm{Gal}(\bar{k}/k')$-equivariant. Hence $\xi := f(T(\bar{k}))$ is an element of $G(\bar{k})\backslash X(\bar{k})$ that is invariant under the action of $\mathrm{Gal}(\bar{k}/k')$. On the other hand, suppose a $\xi \in G(\bar{k})\backslash X(\bar{k})$ is $\mathrm{Gal}(\bar{k}/k')$-invariant. Let $\gamma \in \mathrm{Gal}(\bar{k}/k')$ be the $\#k'$-th power Frobenius. Let $x \in \xi$; then there exists a $g \in G(\bar{k})$ such that $g \cdot \gamma(x) = x$. Let $z \in \mathrm{Z}^1(k', G)$ be the unique cocycle such that $z(\gamma) = g$ as in Lemma 2.7. Then the G-equivariant map

$$G_{\bar{k}} \to X_{\bar{k}}$$
$$g \mapsto g \cdot x$$

descends to a G-equivariant map of k'-varieties $f \colon G_z \to X_{k'}$ (where we identify $G_{z,\bar{k}}$ with $G_{\bar{k}}$ via φ_z as in Notation 2.1), and $\Phi(G_z, f) = \xi$. $\square$

REMARK 2.13. Let ξ be a $G(\bar{k})$-orbit in $X(\bar{k})$, and let x be an element of ξ. Then the automorphism group of the object of $[G\backslash X](\bar{k})$ corresponding to ξ by Lemma 2.11 is isomorphic to $\mathrm{Stab}_{G_{\bar{k}}}(x)$. In particular its isomorphism class does not depend on the choice of x in ξ. We write $\mathtt{A}(\xi)$ for the algebraic group $\mathtt{Stab}_{G_{\bar{k}}}(x)$ over $\bar{k}$.

While in general the point count $\#G_{\mathrm{in}(z)}(k)$ depends on the choice of the cocycle $z \in \mathrm{H}^1(k, G)$, reduced unipotent groups are always isomorphic (as varieties) to affine space. Under suitable conditions on X and G this allows us to simplify the expression in Proposition 2.4.

PROPOSITION 2.14. *Let G be an algebraic group over k. Let X be a k-variety with an action of G, such that for every $\xi \in G(\bar{k}) \backslash X(\bar{k})$ the identity component of the algebraic group $\mathtt{A}(\xi)^{\mathrm{red}}$ is unipotent. Define $a(\xi) := \dim(\mathtt{A}(\xi))$, and $Y := G(\bar{k}) \backslash X(\bar{k})$.*

1. *Let k' be a finite field extension of k. Then*

$$\#[G \backslash X](k') = \sum_{\xi \in Y^{\mathrm{Gal}(\bar{k}/k')}} (\#k')^{-a(\xi)}.$$

2. *Write $k = \mathbb{F}_q$ and suppose that $Y := G(\bar{k}) \backslash X(\bar{k})$ is finite. Let $\Gamma := \mathrm{Gal}(\bar{\mathbb{F}}_q / \mathbb{F}_q)$, and for $\xi \in Y$, let $b(\xi)$ be the cardinality of the orbit $\Gamma \cdot \xi$ in Y. Then $a, b \colon Y \to \mathbb{Z}_{\geq 0}$ are Γ-invariant, and*

$$Z(\mathsf{X}, t) = \prod_{\bar{\xi} \in \Gamma \backslash Y} (1 - (q^{-a(\bar{\xi})} t)^{b(\bar{\xi})})^{-1}.$$

Proof. 1. As before it suffices to show this for $k = k'$. Let Φ be as in (2.12). We may then define the full subcategory $\mathsf{S}(\xi)$ of $[G \backslash X](k)$, the isomorphism classes of whose objects form the set

$$\left\{ x \in [[G \backslash X](k)] : x_{\bar{k}} = \Phi^{-1}(\xi) \right\}.$$

By Lemma 2.11 this category is nonempty if and only if $\xi \in (G(\bar{k}) \backslash X(\bar{k}))^{\mathrm{Gal}(\bar{k}/k)}$. Suppose this is true for ξ, and let x_0 be an object of $\mathsf{S}(\xi)$. Then the algebraic group $\mathtt{Aut}(x_0)$ is a k-form of $\mathtt{A}(\xi)$. By [6, Thm. III.2.5.1] $\mathsf{S}(\xi)$ is equivalent to the category $\mathsf{B}(\mathtt{Aut}(x_0))(k)$; its elements are classified by $\mathrm{H}^1(k, \mathtt{Aut}(x_0)) = \mathrm{H}^1(k, \mathtt{Aut}(x_0)^{\mathrm{red}})$. Write $L := \mathtt{Aut}(x_0)^{\mathrm{red}}$; we now find for the point count

$$\#\mathsf{S}(\xi) := \sum_{x \in [\mathsf{S}(\xi)]} \frac{1}{\#\mathrm{Aut}(x)} = \sum_{z \in \mathrm{H}^1(k,L)} \frac{1}{\#L_{\mathrm{in}(z)}(k)}. \tag{2.15}$$

Let L^0 be the identity component of L; this is a connected unipotent group of dimension $\dim(\mathtt{A}(\xi))$. Let $\pi_0(L)$ be the component group of L. By Lemma 2.9, applied to the short exact sequence

$$1 \to L^0 \to L \to \pi_0(L) \to 1,$$

we see that the natural map $\mathrm{H}^1(k, L) \to \mathrm{H}^1(k, \pi_0(L))$ is a bijection. On the other hand, let $z \in \mathrm{H}^1(k, L)$; then the same lemma tells us that

$$\#L_{\mathrm{in}(z)}(k) = (\#L_{\mathrm{in}(z)}^0(k)) \cdot (\#\pi_0(L_{\mathrm{in}(z)})(k)). \qquad (2.16)$$

By [15, Thm. 5] we get an equality

$$\#L_{\mathrm{in}(z)}^0(k) = (\#k)^{a(\xi)} \qquad (2.17)$$

which does not depend on the choice of z. Furthermore, if we identify $\mathrm{H}^1(k, L)$ and $\mathrm{H}^1(k, \pi_0(L))$ as above, we find $\pi_0(L_{\mathrm{in}(z)}) \cong \pi_0(L)_{\mathrm{in}(z)}$. Applying Lemma 2.8 to the finite étale group scheme $\pi_0(L)$ yields

$$\sum_{z \in \mathrm{H}^1(k, \pi_0(L))} \frac{1}{\#\pi_0(L)_{\mathrm{in}(z)}(k)} = \#\mathsf{B}(\pi_0(L)) = 1. \qquad (2.18)$$

Combining (2.15), (2.16), (2.17), and (2.18) now gives us

$$\begin{aligned}
\#\mathsf{S}(\xi) &= \sum_{z \in \mathrm{H}^1(k, L)} \frac{1}{\#L_{\mathrm{in}(z)}(k)} \\
&= \sum_{z \in \mathrm{H}^1(k, \pi_0(L))} \frac{1}{\#\pi_0(L)_{\mathrm{in}(z)}(k) \cdot (\#k)^{a(\xi)}} \\
&= (\#k)^{-a(\xi)}.
\end{aligned}$$

Summing over all $\xi \in (G(\bar{k}) \backslash X(\bar{k}))^{\mathrm{Gal}(\bar{k}/k)}$ now proves the statement.

2. From the definition it is clear that b is Γ-invariant. To see that a is Γ-invariant, note that $a(\xi) = \dim(G) - \dim(\xi)$ (remember that ξ is a G-orbit in X), and note that $\dim(\gamma \cdot \xi) = \dim(\xi)$ for all $\gamma \in \Gamma$. For a $\xi \in Y$ we have that the object in $[G \backslash X](\bar{k})$ has a model over $\mathbb{F}_{q^v}$ if and only if ξ is fixed under the action of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_{q^v})$; this happens if and

only if $b(\xi) \mid v$. As such we find

$$
\begin{aligned}
Z([G\backslash X], t) &= \exp\left(\sum_{v \geq 1} \frac{t^v}{v} \#[G\backslash X](\mathbb{F}_{q^v})\right) \\
&= \exp\left(\sum_{v \geq 1} \frac{t^v}{v} \sum_{\xi \in Y^{\mathrm{Gal}(\overline{\mathbb{F}}_q / \mathbb{F}_{q^v})}} q^{-a(\xi)v}\right) \\
&= \exp\left(\sum_{v \geq 1} \sum_{\substack{\xi \in Y: \\ b(\xi) \mid v}} \frac{(q^{-a(\xi)}t)^v}{v}\right) \\
&= \exp\left(\sum_{\xi \in Y} \sum_{w \geq 1} \frac{(q^{-a(\xi)}t)^{b(\xi)w}}{b(\xi)w}\right) \\
&= \prod_{\xi \in Y}\left(\exp\left(\sum_{w \geq 1} \frac{(q^{-a(\xi)}t)^{b(\xi)w}}{w}\right)\right)^{\frac{1}{b(\xi)}} \\
&= \prod_{\xi \in Y}(1 - (q^{-a(\xi)}t)^{b(\xi)})^{-\frac{1}{b(\xi)}} \\
&= \prod_{\bar{\xi} \in \Gamma\backslash Y}(1 - (q^{-a(\bar{\xi})}t)^{b(\bar{\xi})})^{-1}. \quad \square
\end{aligned}
$$

PROPOSITION 2.19. *Let G be a smooth algebraic group over k with a unipotent identity component. Let X be a variety over k isomorphic to $\mathbb{A}^n_k$ for some nonnegative integer n. Suppose that the action of G on X factors through a connected group $\tilde{G}$. Let k' be a finite field extension of k. Then*

$$
\#[G\backslash X](k') = (\#k')^{\dim(V)-\dim(G)}.
$$

If $k = \mathbb{F}_q$, then

$$
Z([G\backslash X], t) = (1 - q^{\dim(V)-\dim(G)}t)^{-1}.
$$

Proof. As for the first statement, it suffices to prove this for $k' = k$. Lang's theorem tells us that $\mathrm{H}^1(k, \tilde{G}) = 1$. Since the action of G on X factors through $\tilde{G}$, we find that $X_z \cong X$ for all $z \in \mathrm{H}^1(k, G)$. If we denote the identity component of G by G^0 and its component group by $\pi_0(G)$, and apply Lemma 2.9 to the short exact sequence

$$
1 \to G^0 \to G \to \pi_0(G) \to 1,
$$

we get the following from Proposition 2.4 and Lemma 2.8:

$$
\begin{aligned}
\#[G\backslash X](k) &= \sum_{z \in \mathrm{H}^1(k,G)} \frac{\#X_z(k)}{\#G_{\mathrm{in}(z)}(k)} \\
&= \sum_{z \in \mathrm{H}^1(k,\pi_0(G))} \frac{\#X(k)}{\#G_z^0(k) \cdot \#\pi_0(G)_{\mathrm{in}(z)}(k)} \\
&= (\#k)^{\dim(X)-\dim(G)} \cdot \sum_{z \in \mathrm{H}^1(k,\pi_0(G))} \frac{1}{\#\pi_0(G)_{\mathrm{in}(z)}(k)} \\
&= (\#k)^{\dim(X)-\dim(G)} \cdot \#\mathsf{B}(\pi_0(G))(k) \\
&= (\#k)^{\dim(X)-\dim(G)}.
\end{aligned}
$$

The statement on the zeta function is then a straightforward calculation. $\qquad\square$

3 Weyl groups and Levi decompositions

In this section we briefly review some relevant facts about Weyl groups and Levi decompositions, in particular those of nonconnected reductive groups.

3.1 The Weyl group of a connected reductive group

Let G be a connected reductive algebraic group over a field k. For any pair (T,B) of a Borel subgroup $B \subset G_{\bar{k}}$ and a maximal torus $T \subset B$, let $\Phi_{T,B}$ be the based root system of G with respect to (T,B), and let $W_{T,B}$ be the Weyl group of this based root system, i.e. the Coxeter group generated by the set $S_{T,B}$ of simple reflections. As an abstract group $W_{T,B}$ is isomorphic to $\mathrm{Norm}_{G(\bar{k})}(T(\bar{k}))/T(\bar{k})$. If (T',B') is another choice of a Borel subgroup and a maximal torus, then there exists a $g \in G(\bar{k})$ such that $(T',B') = (gTg^{-1}, gBg^{-1})$. Furthermore, such a g is unique up to right multiplication by $T(\bar{k})$, which gives us a unique isomorphism $\Phi_{T,B} \xrightarrow{\sim} \Phi_{T',B'}$. As such, we can simply talk about *the* based root system Φ of G, with corresponding Coxeter system (W,S). By these canonical identifications Φ, W and S come equipped with an action of $\mathrm{Gal}(\bar{k}/k)$.

The set of parabolic subgroups of $G_{\bar{k}}$ containing B is classified by the power set of S, by associating to $I \subset S$ the parabolic subgroup $P = L \cdot B$, where L is the reductive group with maximal torus T whose root system is Φ_I, the root subsystem of Φ generated by the roots whose associated reflections lie in I. We call I the *type* of P. Let $U := \mathrm{R}_{\mathrm{u}}P$ be the unipotent radical of P; then $P = L \ltimes U$ is the Levi decomposition of P with respect to T (see Subsection 3.3).

For every subset $I \subset S$, let W_I be the subgroup of W generated by I; it is the Weyl group of the root system Φ_I, with I as its set of simple reflections.

For $w \in W$, define the *length $\ell(w)$* of w to be the minimal integer such that there exist $s_1, s_2, \ldots, s_{\ell(w)} \in S$ such that $w = s_1 s_2 \cdots s_{\ell(w)}$. Since $\mathrm{Gal}(\bar{k}/k)$ acts on W by permuting S, the length is Galois invariant. Let $I, J \subset S$; then every (left, double, right) coset $W_I w, W_I w W_J$ or $w W_J$ has a unique element of minimal length, and we denote the subsets of W of elements of minimal length in their (left, double, right) cosets by ${}^I W, {}^I W^J$, and W^J.

PROPOSITION 3.1. (See [3, Prop. 4.18]) *Let $I, J \subset S$. Let $x \in {}^I W^J$, and set $I_x = J \cap x^{-1} I x \subset W$. Then for every $w \in W_I x W_J$ there exist unique $w_I \in W_I$, $w_J \in {}^{I_x} W_J$ such that $w = w_I x w_J$. Furthermore $\ell(w) = \ell(w_I) + \ell(x) + \ell(w_J)$.* □

LEMMA 3.2. (See [14, Prop. 2.8]) *Let $I, J \subset S$. Every element $w \in {}^I W$ can uniquely be written as $x w_J$ for some $x \in {}^I W^J$ and $w_J \in {}^{I_x} W_J$.* □

LEMMA 3.3. (See [14, Lem. 2.13]) *Let $I, J \subset S$. Let $w \in {}^I W$ and write $w = x w_J$ with $x \in {}^I W^J$, $w_J \in W_J$. Then*

$$\ell(x) = \#\Big\{ \alpha \in \Phi^+ \backslash \Phi_J : w\alpha \in \Phi^- \backslash \Phi_I \Big\}. \qquad \square$$

3.2   The Weyl group of a nonconnected reductive group

Now let us drop the assumption that our group is connected. Let $\hat{G}$ be a reductive algebraic group and write G for its connected component. Let B be a Borel subgroup of $G_{\bar{k}}$, and let T be a maximal torus of B. Define the following groups:

$$W = \mathrm{Norm}_{G(\bar{k})}(T)/T(\bar{k});$$
$$\hat{W} = \mathrm{Norm}_{\hat{G}(\bar{k})}(T)/T(\bar{k});$$
$$\Omega = (\mathrm{Norm}_{\hat{G}(\bar{k})}(T) \cap \mathrm{Norm}_{\hat{G}(\bar{k})}(B))/T(\bar{k}).$$

LEMMA 3.4.   *1. One has $\hat{W} = W \rtimes \Omega$.*

2. The composite map $\Omega \hookrightarrow G(\bar{k})/T(\bar{k}) \twoheadrightarrow \pi_0(G)(\bar{k})$ is an isomorphism of groups.

Proof.   1. First note that W is a normal subgroup of $\hat{W}$, since it consists of the elements of $\hat{W}$ that have a representative in $G(\bar{k})$, and G is a normal subgroup of $\hat{G}$. Furthermore, $\hat{W}$ acts on the set X of Borel subgroups

of $G_{\bar{k}}$ containing T. The stabiliser of B under this action is Ω, whereas W acts simply transitively on X; hence $\Omega \cap W = 1$ and $W\Omega = \hat{W}$, and together this proves $\hat{W} = W \rtimes \Omega$.

2. By the previous point, we see that

$$\Omega \cong \hat{W}/W \cong \mathrm{Norm}_{\hat{G}(\bar{k})}(T)/\mathrm{Norm}_{G(\bar{k})}(T),$$

so it is enough to show that every connected component of $\hat{G}_{\bar{k}}$ has an element that normalises T. Let $x \in \hat{G}(\bar{k})$; then xTx^{-1} is another maximal torus of $G_{\bar{k}}$, so there exists a $g \in G(\bar{k})$ such that $xTx^{-1} = gTg^{-1}$. From this we find that $T = (g^{-1}x)T(g^{-1}x)^{-1}$, and $g^{-1}x$ is in the same connected component as x. $\qquad\Box$

We call $\hat{W}$ the Weyl group of $\hat{G}$ with respect to (T, B). Again, choosing a different (T, B) leads to a canonical isomorphism, so we may as well talk about *the* Weyl group of $\hat{G}$. The two statements of Lemma 3.4 are then to be understood as isomorphisms of groups with an action of $\mathrm{Gal}(\bar{k}/k)$. Note that we can regard W as the Weyl group of the connected reductive group G; as such we can apply the results of the previous subsection to it. Let $S \subset W$ be the generating set of simple reflections.

Now let us define an extension of the length function to a suitable subset of $\hat{W}$. First, let I and J be subsets of the set S of simple reflections in W, and consider the set ${}^{I}\hat{W} := {}^{I}W\Omega$. Define a subset ${}^{I}\hat{W}^{J}$ of ${}^{I}\hat{W}$ as follows: every element $w \in {}^{I}\hat{W}$ can uniquely be written as $w = w'\omega$, with $w' \in {}^{I}W$ and $\omega \in \Omega$. We rewrite this as $w = \omega w''$, with $w'' = \omega^{-1}w'\omega \in {}^{\omega^{-1}I\omega}W$; then per definition $w \in {}^{I}\hat{W}^{J}$ if and only if $w'' \in {}^{\omega^{-1}I\omega}W^{J}$. Note that the set ${}^{I}W^{J}$ is a subset of the set ${}^{I}\hat{W}^{J}$.

Now let $w \in {}^{I}\hat{W}$; write $w = \omega w''$ with $\omega \in \Omega$ and $w'' \in {}^{\omega^{-1}I\omega}W$ as above. Since w'' is an element of ${}^{\omega^{-1}I\omega}W$, we can uniquely write $w'' = yw_{J}$ by Lemma 3.2, with $y \in {}^{\omega^{-1}I\omega}W^{J}$ and $w_{J} \in {}^{I_{\omega y}}W_{J}$. Then define the *extended length function* $\ell_{I,J} \colon {}^{I}\hat{W} \to \mathbb{Z}_{\geq 0}$ by

$$\ell_{I,J}(w) := \#\left\{\alpha \in \Phi^{+}\backslash\Phi_{J} : \omega y\alpha \in \Phi^{-}\backslash\Phi_{I}\right\} + \ell(w_{J}). \qquad (3.5)$$

REMARK 3.6.   1. By Proposition 3.1 and Lemma 3.3 the map $\ell_{I,J} \colon {}^{I}\hat{W} \to \mathbb{Z}_{\geq 0}$ extends the length function $\ell \colon {}^{I}W \to \mathbb{Z}_{\geq 0}$.

2. Analogously to Proposition 3.1 we see that every $w \in {}^{I}\hat{W}$ can be uniquely written as xw_{J} with $x \in {}^{I}\hat{W}^{J}$, $w_{J} \in {}^{I_{x}}W_{J}$, and $\ell_{I,J}(w) = \ell_{I,J}(x) + \ell(w_{J})$.

3. In general $\ell_{I,J}$ depends on J. It also depends on I, in the sense that if $I, I' \subset S$, then $\ell_{I,J}(w)$ and $\ell_{I',J}(w)$ for $w \in {}^I\hat{W} \cap {}^{I'}\hat{W} = {}^{I \cap I'}\hat{W}$ need not coincide. As an example, consider over any field the group $G = \mathtt{SL}_2$. Let $\Omega = \langle \omega \rangle$ be cyclic of order 2, and let $\hat{G} = G \rtimes \Omega$ be the extension given by $\omega g \omega^{-1} = g^{\mathrm{T},-1}$. Then ω acts as -1 on the root system, and S has only one element. A straightforward calculation shows $\ell_{\varnothing,\varnothing}(\omega) = 1$, whereas $\ell_{\varnothing,S}(\omega) = \ell_{S,S}(\omega) = \ell_{S,\varnothing}(\omega) = 0$.

3.3   Levi decomposition of nonconnected groups

Let P be a connected smooth linear algebraic group over a field k. A *Levi subgroup* of P is the image of a section of the map $P \twoheadrightarrow P/\mathrm{R}_\mathrm{u}P$, i.e. a subgroup $L \subset P$ such that $P = L \ltimes \mathrm{R}_\mathrm{u}P$. In characteristic p, such a Levi subgroup need not always exist, nor need it be unique. However, if P is a parabolic subgroup of a connected reductive algebraic group, then for every maximal torus $T \subset P$ there exists a unique Levi subgroup of P containing T (see [4, Prop. 1.17]). The following proposition generalises this result to the non-connected case.

Proposition 3.7. *Let $\hat{G}$ be a reductive group over a field k, and let $\hat{P}$ be a subgroup of $\hat{G}$ whose identity component P is a parabolic subgroup of G. Let T be a maximal torus of P. Then there exists a unique Levi subgroup of $\hat{P}$ containing T, i.e. a subgroup $\hat{L} \subset \hat{P}$ such that $\hat{P} = \hat{L} \ltimes \mathrm{R}_\mathrm{u}P$.*

Proof. Let L be the Levi subgroup of P containing T. Then any $\hat{L}$ satisfying the conditions of the proposition necessarily has L as its identity component, hence $\hat{L} \subset \mathtt{Norm}_{\hat{P}}(L)$. On the other hand we know that $\mathtt{Norm}_P(L) = L$, so the only possibility is $\hat{L} = \mathtt{Norm}_{\hat{P}}(L)$, and we have to check that $\pi_0(\mathtt{Norm}_{\hat{P}}(L)) = \pi_0(\hat{P})$, i.e. that every connected component in $\hat{P}_{\bar{k}}$ has an element normalising L. Let $x \in \hat{P}(\bar{k})$. Then xTx^{-1} is another maximal torus of $P_{\bar{k}}$, so there exists a $y \in P(\bar{k})$ such that $xTx^{-1} = yTy^{-1}$. Then $y^{-1}x$ is in the same connected component as x, and $(y^{-1}x)T(y^{-1}x)^{-1} = T$. Since L is the unique Levi subgroup of P containing T, and $(y^{-1}x)L(y^{-1}x)^{-1}$ is another Levi subgroup of P, we see that $y^{-1}x$ normalises L, which completes the proof. $\qquad\square$

4 G-zips

In this section we give the definition of G-zips from [13] along with their classification and their connection to BT_1. We will need the discussion on Weyl groups from Subsection 3.2. As before, we denote the component group of a nonconnected algebraic group A by $\pi_0(A)$.

Let q_0 be a power of p. Let $\hat{G}$ be a reductive group over $\mathbb{F}_{q_0}$, and write G for its identity component. Let q be a power of q_0, and let $\chi\colon \mathbb{G}_{\mathrm{m},\mathbb{F}_q} \to G_{\mathbb{F}_q}$ be a cocharacter of $G_{\mathbb{F}_q}$. Let $L = \mathtt{Cent}_{G_{\mathbb{F}_q}}(\chi)$, and let $U_+ \subset G_{\mathbb{F}_q}$ be the unipotent subgroup defined by the property that $\mathrm{Lie}(U_+) \subset \mathrm{Lie}(G_{\mathbb{F}_q})$ is the direct sum of the weight spaces of positive weight; define U_- similarly. Note that L is connected (see [4, Prop. 0.34]). This defines parabolic subgroups $P_\pm = L \ltimes U_\pm$ of $G_{\mathbb{F}_q}$. Now take an $\mathbb{F}_q$-subgroup scheme Θ of $\pi_0(\mathtt{Cent}_{\hat{G}_{\mathbb{F}_q}}(\chi))$, and let $\hat{L}$ be the inverse image of Θ under the canonical map $\mathtt{Cent}_{\hat{G}_{\mathbb{F}_q}}(\chi) \to \pi_0(\mathtt{Cent}_{\hat{G}_{\mathbb{F}_q}}(\chi))$; then $\hat{L}$ has L as its identity component and $\pi_0(\hat{L}) = \Theta$. We may regard Θ as a subgroup of $\pi_0(\hat{G})$ via the inclusion

$$\pi_0(\mathtt{Cent}_{\hat{G}_{\mathbb{F}_q}}(\chi)) = \mathtt{Cent}_{\hat{G}_{\mathbb{F}_q}}(\chi)/L \hookrightarrow \pi_0(\hat{G}_{\mathbb{F}_q}).$$

We may then define the algebraic subgroups $\hat{P}_\pm := \hat{L} \ltimes U_\pm$ of $\hat{G}_{\mathbb{F}_q}$, whose identity components $P_\pm$ are equal to $L \ltimes U_\pm$. Let $\gamma \in \mathrm{Gal}(\bar{\mathbb{F}}_{q_0}/\mathbb{F}_{q_0})$ be the q_0-th power Frobenius. Then $\hat{G}$ and $\hat{G}_\gamma$ are canonically isomorphic; as such we can regard $\hat{P}_{\pm,\gamma}$, $\hat{L}_{\pm,\gamma}$, etc. as subgroups of $\hat{G}$. They correspond to the parabolic and Levi subgroups associated to the cocharacter $\varphi \circ \chi$ of $\hat{G}_k$ and the subgroup $\varphi(\Theta)$ of $\pi_0(\hat{G})$, where $\varphi\colon \hat{G} \to \hat{G}$ is the relative q_0-th Frobenius isogeny.

DEFINITION 4.1. Let A be an algebraic group over a field k, and let B be a subgroup of A. Let T be an A-torsor over some k-scheme S. A B-*subtorsor of T* is an S-subscheme Y of T, together with an action of B_S, such that Y is a B-torsor over S and such that the inclusion map $Y \hookrightarrow T$ is equivariant under the action of B_S.

DEFINITION 4.2. Let S be a scheme over $\mathbb{F}_q$. A $\hat{G}$-*zip of type (χ, Θ) over S* is a tuple $\mathcal{Y} = (Y, Y_+, Y_-, \upsilon)$ consisting of:

- A right-$\hat{G}_{\mathbb{F}_q}$-torsor Y over S;

- A right-$\hat{P}_+$-subtorsor Y_+ of Y;

- A right-$\hat{P}_{-,\gamma}$-subtorsor Y_- of Y;

- An isomorphism $\upsilon\colon Y_{+,\gamma}/U_{+,\gamma} \xrightarrow{\sim} Y_-/U_{-,\gamma}$ of right-$\hat{L}_\gamma$-torsors.

Together with the obvious notions of pullbacks and morphisms we get a fibred category $\hat{G}\text{-}\mathsf{Zip}_{\mathbb{F}_q}^{\chi;\Theta}$ over $\mathbb{F}_q$. If $\hat{G}$ is connected there is no choice for Θ, and we will omit it from the notation.

Proposition 4.3. (See [13, Prop. 3.2 & 3.11]) *The fibred category $\hat{G}$-$\mathsf{Zip}_{\mathbb{F}_q}^{\chi,\Theta}$ is a smooth algebraic stack of finite type over $\mathbb{F}_q$.*   □

Now let $q_0, q, \hat{G}, \chi, \Theta, \hat{L}, U_\pm$ and $\hat{P}_\pm$ be as above. As in subsection 3.2 let $\hat{W} = W \rtimes \Omega$ be the Weyl group of $\hat{G}$. Let $I \subset S$ be the type of P_+ and let J be the type of $P_{-,\gamma}$. If $w_0 \in W$ is the unique longest word, then $J = \gamma(w_0 I w_0^{-1}) = w_0 \gamma(I) w_0^{-1}$. Let $w_1 \in {}^J W^{\gamma(I)}$ be the element of minimal length in $W_J w_0 W_{\gamma(I)}$, and let $w_2 = \gamma^{-1}(w_1)$; then we may write this relation as $J = \gamma(w_2 I w_2^{-1}) = w_1 \gamma(I) w_1^{-1}$.

The group Θ can be considered as a subgroup of $\Omega \cong \pi_0(\hat{G})$. Let $\hat{\psi}$ be the automorphism of $\hat{W}$ given by $\hat{\psi} = \mathrm{inn}(w_1) \circ \gamma = \gamma \circ \mathrm{inn}(w_2)$, and let Θ act on $\hat{W}$ by

$$\theta \cdot w := \theta w \hat{\psi}(\theta)^{-1}.$$

Lemma 4.4. *The subset ${}^I\hat{W} \subset \hat{W}$ is invariant under the Θ-action.*

Proof. Since $\hat{L}$ normalises the parabolic subgroup P_+ of $G_{\mathbb{F}_q}$, the subset $I \subset S$ is stable under the action of Θ by conjugation; hence for each $\theta \in \Theta$ one has $\theta({}^I W)\theta^{-1} = {}^I W$, so

$$\theta({}^I\hat{W})\hat{\psi}(\theta)^{-1} = (\theta({}^I W)\theta^{-1}) \cdot (\theta \Omega \hat{\psi}(\theta)^{-1}) = {}^I W \Omega = {}^I\hat{W}.   \qquad \square$$

Let us write $\Xi^{\chi,\Theta} := \Theta \backslash {}^I\hat{W}$.

Proposition 4.5. (See [13, Rem. 3.21]) *There is a natural bijection between the sets $\Xi^{\chi,\Theta}$ and $[\hat{G}\text{-}\mathsf{Zip}_{\mathbb{F}_q}^{\chi,\Theta}(\bar{\mathbb{F}}_q)]$.*   □

This bijection can be described as follows. Choose a Borel subgroup B of $G_{\bar{\mathbb{F}}_q}$ contained in $P_{-,\gamma}$, and let T be a maximal torus of B. Let $\gamma \in G(\bar{\mathbb{F}}_q)$ be such that $(\gamma B \gamma^{-1})_\gamma = B$ and $(\gamma T \gamma^{-1})_\gamma = T$. For every $w \in \hat{W} = \mathrm{Norm}_{\hat{G}(\bar{\mathbb{F}}_q)}(T)/T(\bar{\mathbb{F}}_q)$, choose a lift $\dot{w}$ to $\mathrm{Norm}_{\hat{G}(\bar{\mathbb{F}}_q)}(T)$, and set $g = \gamma \dot{w}_2$. Then $\xi \in \Xi^{\chi,\Theta}$ corresponds to the $\hat{G}$-zip $\mathcal{Y}_w = (\hat{G}, \hat{P}_+, g\dot{w}\hat{P}_{-,\gamma}, g\dot{w}\cdot)$ for any representative $w \in {}^I\hat{W}$ of ξ; its isomorphism class does not depend on the choice of the representatives w and $\dot{w}$. Note that this description differs from the one given in [13, Rem. 3.21], as that description seems to be wrong. Since there it is assumed that $B \subset P_{-,K}$ rather than that $B \subset P_{-,\gamma,K}$, the choice of (B, T, g) presented there will not be a frame for the connected zip datum $(G_K, P_{+,K}, P_{-,\gamma,K}, \varphi: L_K \to L_{\gamma,K})$. Also, the choice for g given there needs to be modified to account for the fact that $P_{+,K}$ and $P_{-,\gamma,K}$ might not have a common maximal torus.

The rest of this subsection is dedicated to the extended length functions $\ell_{I,J}$ defined in Subsection 3.2. We need Lemma 4.6 in order to show a result on the dimension of the automorphism group of a $\hat{G}$-zip that extends [13, Prop. 3.34(a)] to the nonconnected case (see Proposition 5.7.2).

LEMMA 4.6. *The length function* $\ell_{I,J}\colon {}^I\hat{W} \to \mathbb{Z}_{\geq 0}$ *is invariant under the semilinear conjugation action of* Θ.

Proof. Let $w \in {}^I\hat{W}$, let $\theta \in \Theta$, and let $\tilde{w} = \theta w \hat{\psi}(\theta)^{-1}$. Let $w = \omega y w_J$ be the decomposition as in subsection 3.2. A straightforward computation shows $\tilde{w} = \tilde{\omega}\tilde{w}'' = \tilde{\omega}\tilde{y}\tilde{w}_J$ with

$$
\begin{aligned}
\tilde{\omega} &= \theta\omega\hat{\psi}(\theta)^{-1} \in \Omega; \\
\tilde{w}'' &= \hat{\psi}(\theta)w''\hat{\psi}(\theta)^{-1} \in {}^{\tilde{\omega}^{-1}I\tilde{\omega}}W; \\
\tilde{y} &= \hat{\psi}(\theta)y\hat{\psi}(\theta)^{-1} \in {}^{\tilde{\omega}^{-1}I\tilde{\omega}}W^J; \\
\tilde{w}_J &= \hat{\psi}(\theta)w_J\hat{\psi}(\theta)^{-1} \in {}^{I_{\tilde{\omega}\tilde{y}}}W_J,
\end{aligned}
$$

since conjugation by $\hat{\psi}(\theta)$ fixes J. Furthermore, $\hat{\psi}(\Theta)$ fixes Φ_J (as a subset of Φ) and Θ fixes Φ_I, and Ω fixes Φ^+ and Φ^-, hence

$$
\begin{aligned}
\ell_{I,J}(\tilde{w}) &= \#\Big\{\alpha \in \Phi^+\backslash\Phi_J : \tilde{\omega}\tilde{y}\alpha \in \Phi^-\backslash\Phi_I\Big\} + \ell(\tilde{w}_J) \\
&= \#\Big\{\alpha \in \Phi^+\backslash\Phi_J : \theta\omega y\hat{\psi}(\theta)^{-1}\alpha \in \Phi^-\backslash\Phi_I\Big\} + \ell(\tilde{w}_J) \\
&= \#\Big\{\alpha \in \Phi^+\backslash\Phi_J : \omega y\alpha \in \Phi^-\backslash\Phi_I\Big\} + \ell(w_J) \\
&= \ell_{I,J}(w). \qquad\qquad \square
\end{aligned}
$$

EXAMPLE 4.7. Let p be an odd prime, let V be the $\mathbb{F}_p$-vector space $\mathbb{F}_p^4$, and let ψ be the symmetric nondegenerate bilinear form on V given by the matrix

$$
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
$$

Let $\hat{G}$ be the algebraic group $\mathtt{O}(V,\psi)$ over $\mathbb{F}_p$; it has two connected components. The Weyl group W of its identity component $G = \mathtt{SO}(V,\psi)$ is of the form $W \cong \{\pm 1\}^2$ (with trivial Galois action), and its root system is of the form $\Psi \cong \{r_1, r_2, -r_1, -r_2\}$, where the i-th factor of W acts on $\{r_i, -r_i\}$. The set of generators of W is $S = \{(-1,1),(1,-1)\}$. Furthermore, $\#\Omega = 2$, and the

nontrivial element σ of Ω permutes the two factors of W (as well as e_1 and e_2); hence $\hat{W} \cong \{\pm 1\}^2 \rtimes S_2$.

Let $\chi \colon \mathbb{G}_m \to G$ be the cocharacter that sends t to $\operatorname{diag}(t, t, t^{-1}, t^{-1})$. Its associated Levi factor L is isomorphic to $\mathtt{GL}_2$; the isomorphism is given by the injection $\mathtt{GL}_2 \hookrightarrow \hat{G}$ that sends a $g \in \mathtt{GL}_2$ to $\operatorname{diag}(g, g^{-1,\mathrm{T}})$. The associated parabolic subgroup P_+ is the product of L with the subgroup $B \subset \hat{G}$ of upper triangular orthogonal matrices. The type of P_+ is a singleton subset of S; without loss of generality we may choose the isomorphism $W \cong \{\pm 1\}^2$ in such a way that P^+ has type $I = \{(-1, 1)\}$. Recall that J denotes the type of the parabolic subgroup $P_{-,\gamma}$ of G. Since W is abelian and has trivial Galois action, the formula $J = w_0 \gamma(I) w_0^{-1}$ shows us that $J = I$. Furthermore, since $\mathtt{Cent}_{\hat{G}}(\chi)$ is connected, the group Θ has to be trivial.

An element of $\hat{W}$ is of the form (a, b, c), with $a, b \in \{\pm 1\}$ and $c \in S_2 = \{1, \sigma\}$; then ${}^I \hat{W}$ is the subset of $\hat{W}$ consisting of elements for which $a = 1$. Also, note that $\Phi^+ \setminus \Phi_J = \{e_2\}$, $\Phi^- \setminus \Phi_I = \{-e_2\}$, so to calculate the length function $\ell_{I,J}$ as in (3.5) we only need to determine $\ell(w_J)$ and whether ωy sends e_2 to $-e_2$ or not. If we use the terminology ω, w'', y, w_J from subsection 3.2, we get the following results:

	w			
	$(1,1,1)$	$(1,-1,1)$	$(1,1,\sigma)$	$(1,-1,\sigma)$
ω	$(1,1,1)$	$(1,1,1)$	$(1,1,\sigma)$	$(1,1,\sigma)$
w''	$(1,1,1)$	$(1,-1,1)$	$(1,1,1)$	$(-1,1,1)$
y	$(1,1,1)$	$(1,-1,1)$	$(1,1,1)$	$(1,1,1)$
w_J	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	$(-1,1,1)$
$\omega y e_2 = -e_2$?	no	yes	no	no
$\ell(w_J)$	0	0	0	1
$\ell_{I,J}(\hat{w})$	0	1	0	1

5 Zeta functions of stacks of G-zips

We fix q_0, G, q, χ and Θ as in Section 4. The aim of this section is to calculate the point counts and the zeta function of the stack $\hat{G}\text{-}\mathsf{Zip}_{\mathbb{F}_q}^{\chi,\Theta}$. Before proving Theorem 1.1 we first need to introduce some auxiliary results. Let φ be as in Section 4, and let $\mathrm{r}_\pm \colon \hat{P}_\pm \to \hat{L}$ denote the natural projection. Then to the triple $(\hat{G}, \chi, \Theta)$ we can associate the reduced algebraic group over $\mathbb{F}_q$ whose set

of $\bar{\mathbb{F}}_q$-points is defined as

$$E(\bar{\mathbb{F}}_q) = \left\{ (y_+, y_-) \in \hat{P}_+(\bar{\mathbb{F}}_q) \times \hat{P}_-(\bar{\mathbb{F}}_q) : \varphi(\mathrm{r}_+(y_+)) = \mathrm{r}_-(y_-) \right\}.$$

Then E acts on $\hat{G}_{\mathbb{F}_q}$ by $(y_+, y_-) \cdot g' = y_+ g' y_-^{-1}$, and this action allows us to represent stacks of $\hat{G}$-zips as quotient stacks:

PROPOSITION 5.1. (See [13, Prop. 3.11]) *There is an isomorphism* $\hat{G}$-$\mathsf{Zip}_{\mathbb{F}_q}^{\chi, \Theta} \cong [E \backslash \hat{G}_{\mathbb{F}_q}]$ *of* $\mathbb{F}_q$-*stacks.* $\qquad\square$

The next step is to connect the quotient stack $[E \backslash \hat{G}_{\mathbb{F}_q}]$ to the Weyl group of G. To make the discussion more explicit, we define the Weyl group using a maximal torus T and a Borel subgroup B of G satsifying some nice properties.

LEMMA 5.2. *Let* $B \subset P_{-,\gamma}$ *be a Borel subgroup defined over* $\mathbb{F}_q$ *containing* L_γ, *and let* $T \subset B$ *be a maximal torus defined over* $\mathbb{F}_q$. *Then there exists an element* $g \in G(\mathbb{F}_q)$ *such that:*

- gBg^{-1} *is a Borel subgroup of* P_+ *containing* L;

- $\varphi(gTg^{-1}) = T$.

Proof. Let $B' \subset P_+$ be a Borel subgroup of G containing L. Consider the algebraic subset

$$X = \left\{ g \in G(\bar{\mathbb{F}}_q) : gBg^{-1} = B', \varphi(gTg^{-1}) = T \right\}$$

of $G(\bar{\mathbb{F}}_q)$. Since $\mathsf{Norm}_G(B) \cap \mathsf{Norm}_G(T) = T$, we see that X forms a T-torsor over $\mathbb{F}_q$. By Lang's theorem such a torsor is trivial, hence X has a rational point. $\qquad\square$

For the rest of this section we fix B, T, g as above, and we use T and B to define the Weyl group of $\hat{G}$.

LEMMA 5.3. *Choose, for every* $w \in \hat{W} = \mathrm{Norm}_{\hat{G}(\bar{\mathbb{F}}_q)}(T(\bar{\mathbb{F}}_q))/T(\bar{\mathbb{F}}_q)$, *a lift* $\dot{w}$ *of* w *to the group* $\mathrm{Norm}_{\hat{G}(\bar{\mathbb{F}}_q)}(T(\bar{\mathbb{F}}_q))$. *Then the map*

$$\Xi^{\chi, \Theta} \to E(\bar{\mathbb{F}}_q) \backslash \hat{G}(\bar{\mathbb{F}}_q)$$
$$\Theta \cdot w \mapsto E(\bar{\mathbb{F}}_q) \cdot g\dot{w}$$

is well-defined, and it is an isomorphism of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$-*sets that does not depend on the choices of* w *and* $\dot{w}$.

Proof. In [14, Thm. 10.10] it is proven that this map is a well-defined bijection independent of the choices of w and $\dot{w}$ (applied to the zip datum from [13, Def. 3.6] and the frame (B, T, g) from Lemma 5.2). Furthermore, if τ is an element of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$, then the fact that T and g are defined over $\mathbb{F}_q$ implies that $\tau(\dot{w})$ is a lift of $\tau(w)$ to $\mathrm{Norm}_{\hat{G}}(T)$; this shows that the map is Galois-equivariant. $\square$

REMARK 5.4. Together with the identification $[[E\backslash \hat{G}_{\mathbb{F}_q}](\bar{\mathbb{F}}_q)] \cong E(\bar{\mathbb{F}}_q)\backslash \hat{G}(\bar{\mathbb{F}}_q)$ from Lemma 2.11.1 the isomorphism above gives the natural bijection in Proposition 4.5.

The following proposition gives an explicit formula for the orbits of $\hat{G}$ under the action by E. It is proven in the case that $\hat{G}$ is connected in [14, Thm. 7.5c & Thm. 8.1], applied to the zip datum from [13, Def. 3.6]. While the proof is long (it requires most of sections 3–8 of [14]), a lot of it carries over essentially unchanged to the nonconnected case. The few modifications that are needed for the proof are discussed in Remark 5.10.

PROPOSITION 5.5. *Let $w \in {}^I\hat{W}$, and let $\dot{w}$ be a lift of w to $\mathrm{Norm}_{\hat{G}(\bar{\mathbb{F}}_q)}(T(\bar{\mathbb{F}}_q))$. Then the orbit $E_{\bar{\mathbb{F}}_q} \cdot (g\dot{w}) \subset \hat{G}_{\bar{\mathbb{F}}_q}$ has dimension $\dim(P_+) + \ell_{I,J}(w)$. The reduced stabiliser $\mathtt{Stab}_{E_{\bar{\mathbb{F}}_q}}(g\dot{w})^{\mathrm{red}}$ has a unipotent identity component.* $\square$

We are now in a position to define the functions a and b in the statement of Theorem 1.1.

NOTATION 5.6. Let $\Gamma = \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$. We define functions $a, b\colon {}^I\hat{W} \to \mathbb{Z}_{\geq 0}$ on ${}^I\hat{W}$ as follows:

- $a(w) = \dim(G/P_+) - \ell_{I,J}(w)$;

- $b(w)$ is the cardinality of the Γ-orbit of $\Theta \cdot w$ in $\Xi^{\chi,\Theta}$, i.e.

$$b(w) = \#\left\{\xi \in \Xi^{\chi,\Theta} : \xi \in \Gamma \cdot (\Theta \cdot w)\right\}.$$

The fact that $a(w)$ is nonnegative for every $w \in {}^I\hat{W}$ is a consequence of the following proposition.

PROPOSITION 5.7. *For $\xi \in \Xi^{\chi,\Theta}$, let $\mathcal{Y}_\xi$ be the $\hat{G}$-zip over $\bar{\mathbb{F}}_q$ corresponding to ξ. Then one has $\dim(\mathtt{Aut}(\mathcal{Y}_\xi)) = a(\xi)$ and the identity component of the group scheme $\mathtt{Aut}(\mathcal{Y}_\xi)^{\mathrm{red}}$ is unipotent.*

Proof. Note that $\dim(E) = \dim(G)$. Let $w \in {}^I\hat{W}$ be such that $\xi = \Theta \cdot w$. By Remark 2.13 and Proposition 5.5 we have

$$
\begin{aligned}
\dim(\mathtt{Aut}(\mathcal{Y}_\xi)) &= \dim(\mathtt{Stab}_{E_{\hat{z}}}(g\dot{w})) \\
&= \dim(E) - \dim(E \cdot g\dot{w}) \\
&= \dim(G) - \dim(E \cdot g\dot{w}) \\
&= \dim(G) - \dim(P_+) - \ell_{I,J}(\xi) \\
&= a(\xi).
\end{aligned}
$$

By Proposition 5.5 the identity component of $\mathtt{Aut}(\mathcal{Y}_\xi)^{\mathrm{red}}$ is unipotent. $\qquad\square$

REMARK 5.8. The formula $\dim(\mathtt{Aut}(\mathcal{Y}_\xi)) = \dim(G/P) - \ell_{I,J}(\xi)$ from Proposition 5.7 apparently contradicts the proof of [13, Thm. 3.26]. There an extended length function $\ell\colon \hat{W} \to \mathbb{Z}_{\geq 0}$ is defined by $\ell(w\omega) = \ell(w)$ for $w \in W$, $\omega \in \Omega$. It is stated that the codimension of $E \cdot (g\dot{w})$ in $\hat{G}$ is equal to $\dim(G/P_+) - \ell(w)$. In other words, if this were correct, $\dim(\mathtt{Aut}(\mathcal{Y}_\xi))$ would be equal to $\dim(G/P^+) - \ell(w)$ rather than $\dim(G/P^+) - \ell_{I,J}(w)$. However, the proof seems to be incorrect (and the theorem itself as well). The dimension formula is based on [14, Thm. 5.11], but that result only treats the connected case. It fails in the nonconnected case, as there $\ell(w)$ and $\ell_{I,J}(w)$ do not generally coincide. One can construct a counterexample by taking $\hat{G}$ as in Remark 3.6.3, and taking the cocharacter $\chi\colon \mathbb{G}_{\mathrm{m}} \to G$ given by

$$
x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix} \right).
$$

Then a straightforward calculation shows that $\ell(\omega) = 0$ and $\ell_{I,J}(\omega) = 1$ do not coincide.

REMARK 5.9. In general $\mathtt{Aut}(\mathcal{Y}_\xi)$ will not be reduced; see [10, Rem. 3.1.7] for the first found instance of this phenomenon, or [13, Rem. 3.35] for the general case.

Proof of Theorem 1.1. By Proposition 5.1 we can consider $\hat{G}\text{-}\mathsf{Zip}_{\mathbb{F}_q}^{\chi,\Theta}$ as a quotient stack, and by Propositions 4.5 and 5.7 the assumptions of Proposition 2.14.2 are satisfied. Furthermore, in the notation of this proposition, we find $Y = \Xi^{\chi,\Theta}$, and $a, b\colon Y \to \mathbb{Z}_{\geq 0}$ are as in Notation 5.6 by Proposition 5.7. The theorem is now a direct consequence of Proposition 2.14.2. $\qquad\square$

REMARK 5.10. Although the proof of Proposition 5.5 over from the connected case without much difficulty, we feel compelled to make some comments about

what exactly changes in the non-connected case, since the proofs of these theorems require most of the material of [14]. The key change is that in [14, Section 4] we allow x to be an element of $^I\hat{W}^J$, rather than just $^IW^J$; however, one can keep working with the connected algebraic zip datum $\mathcal{Z}$, and define from there a connected algebraic zip datum $\mathcal{Z}_{\dot{x}}$ as in [14, Constr. 4.3]. There, one needs the Levi decomposition for non-connected parabolic groups; but this is handled in our Proposition 3.7. The use of non-connected groups does not give any problems in the proofs of most propositions and lemmas in [14, §4–8]. In [14, Prop. 4.8], the term $\ell(x)$ in the formula will now be replaced by $\ell_{I,J}(x)$. The only property of $\ell(x)$ that is used in the proof in the connected case is that if $x \in {}^IW^J$, then $\ell(x) = \#\{\alpha \in \Phi^+\backslash\Phi_J : x\alpha \in \Phi^-\backslash\Phi_I\}$. In our case, we have $x \in {}^I\hat{W}^J$, and $\ell_{I,J}\colon {}^I\hat{W}^J \to \mathbb{Z}_{\geq 0}$ is the extension of $\ell\colon {}^IW^J \to \mathbb{Z}_{\geq 0}$ that gives the correct formula. Furthermore, in the proof of [14, Prop. 4.12] the assumption $x \in {}^IW^J$ is used, to conclude that $x\Phi_J^+ \subset \Phi^+$. However, the same is true for $x \in {}^I\hat{W}^J$: write $x = \omega x'$ with $\omega \in \Omega$ and $x' \in {}^{\omega^{-1}I\omega}W^J$; then $x'\Phi_J^+ \subset \Phi^+$, and $\omega\Phi^+ = \Phi^+$, since Ω acts on the based root system. Finally, the proofs of both [14, Thm. 7.5c] and [14, Thm. 8.1] rest on an induction argument, where the authors use that an element $w \in {}^IW$ can uniquely be written as $w = xw_J$, with $x \in {}^IW^J$, $w_J \in {}^{I_x}W_J$, and $\ell(w) = \ell(x) + \ell(w_J)$. The analogous statement that we need to use is that any $w \in {}^I\hat{W}$ can uniquely be written as $w = xw_J$, with $x \in {}^I\hat{W}^J$, $w_J \in {}^{I_x}W_J$, and $\ell_{I,J}(w) = \ell_{I,J}(x) + \ell(w_J)$, see Remark 3.6.2. The proofs of the other lemmas, propositions and theorems work essentially unchanged.

6　Stacks of truncated Barsotti–Tate groups

The aim of this section is to prove Theorem 1.2. We fix integers $h > 0$ and $0 \leq d \leq h$, and we want to determine the zeta function of the stack $\mathsf{BT}_n^{h,d}$ over $\mathbb{F}_p$ for every integer $n \geq 1$. This turns out to be related to the theory of $\hat{G}$-zips and their moduli stacks. Our strategy will be to interpret the results of [18] and [5], which concern the set of BT_{n+1} over $\bar{k}$ extending a given BT_n, in a 'stacky' sense over a finite k. This allows us to invoke the results of Section 2.

NOTATION 6.1. For the rest of this section, let G be the reductive group $\mathtt{GL}_{h,\mathbb{F}_p}$. Let $\chi\colon \mathbb{G}_{\mathrm{m},\mathbb{F}_p} \to G$ be a cocharacter that induces the weights 0 with multiplicity d and weight 1 with multiplicity $h - d$ on the standard representation of G. Employing the notation of sections 3 and 4, we see that W is the permutation group on h elements (with trivial Galois action), $S = \{(1\ 2), (2\ 3), ..., (h-1\ h)\}$,

and $I = S \setminus \{(d \; d+1)\}$. Note that Θ has to be trivial, as we can regard it as a subgroup of $\Omega \cong \pi_0(G)$, which is trivial. Hence $\Xi := \Gamma \backslash \Xi^{\chi,\Theta}$ is equal to ${}^I W$, and the map $a \colon \Xi \to \mathbb{Z}_{\geq 0}$ from Notation 5.6 is given by $a(\xi) = \dim(G/P_+) - \ell(\xi) = d(h - d) - \ell(\xi)$.

For general n, let $\mathsf{D}_n^{h,d}$ be the stack over $\mathbb{F}_p$ of truncated Dieudonné crystals D of level n that are locally of rank h, for which the map $F \colon D \to D^{(p)}$ has rank d locally (see [7, Rem. 2.4.10]). Then Dieudonné theory (see [2, 3.3.6 & 3.3.10]) tells us that there is a morphism of stacks over $\mathbb{F}_p$

$$\mathbb{D}_n \colon \mathsf{BT}_n^{h,d} \to \mathsf{D}_n^{h,d}$$

that is an equivalence of categories over perfect fields; hence $Z(\mathsf{BT}_n^{h,d}, t) = Z(\mathsf{D}_n^{h,d}, t)$. As such, we are interested in the categories $\mathsf{D}_n^{h,d}(\mathbb{F}_q)$. An object in this category is a Dieudonné module of level n, i.e. a triple (D, F, V) where:

1. D is a free module of rank h over $\mathrm{W}_n(\mathbb{F}_q)$, the Witt vectors of length n over $\mathbb{F}_q$;

2. F is a σ-semilinear endomorphism of D of rank d, where σ is the automorphism of $\mathrm{W}_n(\mathbb{F}_q)$ lifting the automorphism Fr_p of $\mathbb{F}_q$;

3. V is a σ^{-1}-semilinear endomorphism of D satisfying $FV = VF = p$.

Now fix h and d, and choose a (non-truncated) Barsotti–Tate group $\mathcal{G}$ of height h and dimension d over $\mathbb{F}_p$. Let (D_n, F_n, V_n) be the Dieudonné module of $\mathcal{G}[p^n]$, and choose a basis for every D_n in a compatible manner (i.e. the basis of D_n is the image of the basis of D_{n+1} under the natural reduction map $D_{n+1}/p^n D_{n+1} \xrightarrow{\sim} D_n$). Then for every power q of p, every element in $\mathsf{D}_n^{h,d}(\mathbb{F}_q)$ is isomorphic to $D_{n,g} := (D_n \otimes_{\mathbb{Z}/p^n\mathbb{Z}} \mathrm{W}_n(\mathbb{F}_q), gF_n, V_n g^{-1})$ for some $g \in \mathrm{GL}_h(\mathrm{W}_n(\mathbb{F}_q))$ (See [18, 2.2.2]).

For a smooth affine group scheme G over $\mathrm{Spec}(\mathrm{W}(\mathbb{F}_p))$, let $\mathbb{W}_n(G)$ be the group scheme over $\mathrm{Spec}(\mathbb{F}_p)$ defined by $\mathbb{W}_n(G)(R) = G(\mathrm{W}_n(R))$ (see [18, 2.1.4]); it is again smooth and affine. For every n there is a natural reduction morphism $\mathbb{W}_{n+1}(G) \twoheadrightarrow \mathbb{W}_n(G)$.

PROPOSITION 6.2. *Let $\mathcal{D}_n := \mathbb{W}_n(\mathsf{GL}_h)$. Then there exists a smooth affine group scheme $\mathcal{H}$ over $\mathbb{Z}_p$ and for every n an action of $\mathcal{H}_n := \mathbb{W}_n(\mathcal{H})$ on $\mathcal{D}_n$, compatible with the reduction maps $\mathcal{H}_{n+1} \twoheadrightarrow \mathcal{H}_n$ and $\mathcal{D}_{n+1} \twoheadrightarrow \mathcal{D}_n$, such that for every power q of p, there exists for every $g, g' \in \mathcal{D}_n(\mathbb{F}_q)$ an isomorphism of $\mathbb{F}_q$-group varieties*

$$\varphi_{g,g'} \colon \mathrm{Transp}_{\mathcal{H}_{n,\mathbb{F}_q}}(g, g')^{\mathrm{red}} \xrightarrow{\sim} \mathrm{Isom}(D_{n,g}, D_{n,g'})^{\mathrm{red}}$$

that is compatible with compositions in the sense that for every $g, g', g'' \in \mathcal{D}_n(\mathbb{F}_q)$ the following diagram commutes, where the horizontal maps are the natural composition morphisms:

$$
\begin{array}{ccc}
\mathrm{Transp}_{\mathcal{H}_{n,\mathbb{F}_q}}(g',g'')^{\mathrm{red}} \times \mathrm{Transp}_{\mathcal{H}_{n,\mathbb{F}_q}}(g,g')^{\mathrm{red}} & \longrightarrow & \mathrm{Transp}_{\mathcal{H}_{n,\mathbb{F}_q}}(g,g'')^{\mathrm{red}} \\
\downarrow{\scriptstyle \varphi_{g',g''} \times \varphi_{g'\,g'}} & & \downarrow{\scriptstyle \varphi_{g,g''}} \\
\mathrm{Isom}(D_{n,g'}, D_{n,g''})^{\mathrm{red}} \times \mathrm{Isom}(D_{n,g}, D_{n,g'})^{\mathrm{red}} & \longrightarrow & \mathrm{Isom}(D_{n,g}, D_{n,g''})^{\mathrm{red}}
\end{array}
$$

Proof. The group $\mathcal{H}$ and the action $\mathcal{H}_n \times \mathcal{D}_n \to \mathcal{D}_n$ are defined in [18, 2.1.1 & 2.2] over an algebraically closed field k of characteristic p, but the definition still makes sense over $\mathbb{F}_p$. The isomorphism of groups $\varphi_{g,g}$ is given on k-points in [18, 2.4(b)]. The definition of the map there shows that it is algebraic and defined over $\mathbb{F}_p$. Since it is an isomorphism on $\bar{\mathbb{F}}_p$-points, it is an isomorphism of reduced group schemes over $\mathbb{F}_p$. Furthermore, a morphism $\mathrm{Transp}_{\mathcal{H}_{n,\mathbb{F}_q}}(g,g') \to \mathrm{Isom}(D_{n,g}, D_{n,g'})$ is given in the proof of [18, 2.2.1]. It is easily seen that this map is compatible with compositions in the sense of the diagram above, and that it is equivariant under the action of $\mathrm{Stab}_{\mathcal{H}_n}(g)(\bar{\mathbb{F}}_p) \cong \mathrm{Isom}(D_{n,g})(\bar{\mathbb{F}}_p)$. Since both varieties are torsors under this action, this must be an isomorphism as well. $\qquad\square$

COROLLARY 6.3. *For every power q of p the categories $\mathsf{D}_n^{h,d}(\mathbb{F}_q)$ and $[\mathcal{H}_n \backslash \mathcal{D}_n](\mathbb{F}_q)$ are equivalent.*

Proof. For every object $D \in \mathsf{D}_n^{h,d}(\mathbb{F}_q)$ choose a $g_D \in \mathcal{D}_n(\mathbb{F}_q)$ such that $D \cong D_{n,g_D}$. Define a functor

$$
E \colon \mathsf{D}_n^{h,d}(\mathbb{F}_q) \to [\mathcal{H}_n \backslash \mathcal{D}_n](\mathbb{F}_q)
$$

that sends a D to the pair $(\mathcal{H}_n, f_D)$, where $f_D \colon \mathcal{H}_n \to \mathcal{D}_n$ is given by $f_D(h) = h \cdot g_D$. We send an isomorphism from D to D' to the corresponding element of

$$
\mathrm{Isom}((\mathcal{H}_n, f_D), (\mathcal{H}_n, f_{D'})) = \mathrm{Transp}_{\mathcal{H}_n(\mathbb{F}_q)}(g_D, g_{D'}).
$$

From the description of $\mathcal{H}$ in [18] it is clear that each $\mathcal{H}_n$ is connected, hence every $\mathcal{H}_n$-torsor is trivial, and E is essentially surjective. By Proposition 6.2 it is also fully faithful, hence it is an equivalence of categories. $\qquad\square$

By [13, 9.18, 8.3 & 3.21] (and before by [8] and [9]) the set of isomorphism classes of Dieudonné modules of level 1 over an algebraically closed field of characteristic p are classified by Ξ as in Notation 6.1. For each $\xi \in \Xi$, let $\mathsf{D}_n^{h,d,\xi}$

be the substack of $\mathsf{D}_n^{h,d}$ consisting of truncated Barsotti–Tate groups of level n, locally of rank h, and with F locally of rank d, whose reduction to a BT_1 is of type ξ at all geometric points. Then over fields k of characteristic p one has $\mathsf{D}_n^{h,d}(k) = \bigsqcup_{\xi \in \Xi} \mathsf{D}_n^{h,d,\xi}(k)$ as categories, hence

$$Z(\mathsf{D}_n^{h,d}, t) = \prod_{\xi \in \Xi} Z(\mathsf{D}_n^{h,d,\xi}, t).$$

From Proposition 2.11.1, or directly from the description in [8, §5], each isomorphism class over $\bar{\mathbb{F}}_p$ has a model over $\mathbb{F}_p$. For every $\xi \in \Xi$ choose a $g_{1,\xi} \in \mathcal{D}_1(\mathbb{F}_p)$ such that the isomorphism class of $D_{1,g_{1,\xi}} \otimes \bar{\mathbb{F}}_p$ corresponds to ξ. For every n, let $\mathcal{D}_{n,\xi}$ be the preimage of $g_{1,\xi}$ under the reduction map $\mathcal{D}_n \twoheadrightarrow \mathcal{D}_1$. Let $\mathcal{H}_{n,\xi}$ be the preimage of $\mathtt{Stab}_{\mathcal{H}_1}(g_{1,\xi})$ in $\mathcal{H}_n$; then analogous to Corollary 6.3 for every power q of p we get an equivalence of categories (see [5, 3.2.3 Lem. 2(b)])

$$\mathsf{D}_n^{h,d,\xi}(\mathbb{F}_q) \cong [\mathcal{H}_{n,\xi} \backslash \mathcal{D}_{n,\xi}](\mathbb{F}_q).$$

Proof of Theorem 1.2. By the discussion above we see that

$$Z(\mathsf{BT}_n^{h,d}, t) = \prod_{\xi \in \Xi} Z([\mathcal{H}_{n,\xi} \backslash \mathcal{D}_{n,\xi}], t).$$

By [13, 9.18 & 8.3] there is an isomorphism of stacks over $\mathbb{F}_p$

$$\mathsf{D}_{1,p}^{h,d} \xrightarrow{\sim} G\text{-}\mathsf{Zip}_{\mathbb{F}_p}^{\chi,\Theta},$$

where G, χ, Θ are as in Notation 6.1. By Proposition 5.7, or earlier by [10, 2.1.2(i) & 2.2.6], the group scheme $\mathtt{Stab}_{\mathcal{H}_1}(g_{1,\xi})^{\mathrm{red}} \cong \mathtt{Aut}(D_{1,g_{1,\xi}})^{\mathrm{red}}$ has an identity component that is unipotent of dimension $a(w)$. The reduction morphism $\mathcal{H}_n \to \mathcal{H}_1$ is surjective and its kernel is unipotent of dimension $h^2(n-1)$, see [5, 3.1.1 & 3.1.3]. This implies that $\mathcal{H}_{n,\xi}$ has a unipotent identity component of dimension $h^2(n-1) + a(\xi)$. Now fix a $g_{n,\xi} \in \mathcal{D}_{n,\xi}(\mathbb{F}_p)$; then we can identify $\mathcal{D}_{n,\xi}$ with the affine group $X = \mathbb{W}_{n-1}(\mathtt{Mat}_{h \times h})$, by sending an $x \in X$ to $g_{n,\xi} + ps(x)$, where $s \colon \mathbb{W}_{n-1}(\mathtt{Mat}_{h \times h}) \xrightarrow{\sim} p\mathbb{W}_n(\mathtt{Mat}_{h \times h}) \subset \mathbb{W}_n(\mathtt{Mat}_{h \times h})$ is the canonical identification. Furthermore, the action of an element $h \in \mathcal{H}_{n,\xi}$ on $(g_{n,\xi} + ps(x)) \in \mathcal{D}_{n,\xi}$ is given by $f(h)(g_{n,\xi} + ps(x))f'(h)$ for some algebraic $f, f' \colon \mathcal{H}_{n,\xi} \to \mathbb{W}_n(\mathtt{GL}_h)$ (see [18, 2.2.1a]). From this we see that the induced action of $\mathcal{H}_{n,\xi}$ on the variety X is given by

$$h \cdot x = f(h)xf'(h) + \frac{1}{p}(f(h)g_{n,\xi}f'(h) - g_{n,\xi}),$$

which makes sense because $f(h)g_{n,\xi}f'(h)$ is equal to $g_{n,\xi}$ modulo p. If we regard X as $\mathbb{W}_{n-1}(\mathbb{G}_a^{h^2})$ via its canonical coordinates, this shows us that the action of

$\mathcal{H}_{n,\xi}$ on X factors through the canonical action of $\mathbb{W}_{n-1}(\mathbb{G}_a^{h^2}) \rtimes \mathbb{W}_{n-1}(\mathtt{GL}_{h^2})$ on $\mathbb{W}_{n-1}(\mathbb{G}_a^{h^2})$. This algebraic group is connected, so we can apply Proposition 2.19, from which we find

$$
\begin{aligned}
Z([\mathcal{H}_{n,\xi}\backslash\mathcal{D}_{n,\xi}],t) &= \tfrac{1}{1-p^{\dim(\mathcal{D}_{n,\xi})-\dim(\mathcal{H}_{n,\xi})}} \\
&= \tfrac{1}{1-p^{h^2(n-1)-(h^2(n-1)+a(\xi))}t} \\
&= \tfrac{1}{1-p^{-a(\xi)}t},
\end{aligned}
$$

which completes the proof. $\qquad\square$

REMARK 6.4. Since the zeta function $Z(\mathsf{BT}_n^{h,d},t)$ does not depend on n, one might be tempted to think that the stack $\mathsf{BT}^{h,d}$ of non-truncated Barsotti–Tate groups of height h and dimension d has the same zeta function. However, this stack is not of finite type. For instance, every Barsotti–Tate group $\mathcal{G}$ over $\mathbb{F}_q$ has a natural injection $\mathbb{Z}_p^\times \hookrightarrow \mathrm{Aut}(\mathcal{G})$, which shows us that its zeta function is not well-defined.

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