

EMBEDDINGS OF QUADRATIC SPACES

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ABSTRACT. We introduce the concept of an *embedding* of a quadratic space in an associative algebra. Using Clifford Algebras we derive some fundamental properties that any embedding should satisfy. Conversely, there is a simple description of the Clifford Algebra and the corresponding Spin groups in terms of the algebra in which the quadratic space is embedded.

Though Clifford Algebras have been studied in detail, they may not always be easy to work with. Sometimes it may be useful to switch to a more concrete embedding to study low dimensional Spin and Epin (or Elementary Spin) groups.

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1 INTRODUCTION

In this paper, we examine the concept of an *embedding* of a quadratic space and analyze its connection to Spin groups.

The special case of the Hyperbolic space has been studied in [CV] using what are called Suslin Matrices. This paper generalizes the results of [CV] to any quadratic space, while at the same time giving a simpler “basis-free” route to the main theorem (using Lemmas 4.3 - 4.6). These lemmas, as we shall see, help characterize a norm function and capture the Spin representation (Theorem 4.7).

By a quadratic space (V, q) , we mean a free module V , over a commutative ring, equipped with a quadratic form q . Without further ado,

DEFINITION 1.1. Let (V, q) be a quadratic space and A be an associative algebra. We will say that (V, q) is embedded in A if $V \subseteq A$ and

$$q(v) = v\alpha(v) = \alpha(v)v$$

where α is an isometry of (V, q) .

Familiar examples of embeddings are given by the Composition algebras, Clifford Algebras and Suslin matrices. Using Clifford Algebras we derive some fundamental properties that any embedding should satisfy. Conversely, we can describe the Clifford Algebra and the corresponding Spin groups in terms of the algebra in which the quadratic space is embedded. It turns out that when there is an involution of A that acts trivially on the underlying quadratic space (V, q) , the Spin group acts faithfully.

The connection between the Clifford Algebra $\text{Cl}(V, q)$ and a general embedding $V \subseteq A$ is made by interpreting Cl as a subalgebra of $M_2(A)$. This is achieved with the help of a structure theorem, about the graded ideals in Clifford algebras (Theorem 2.7), that holds for any non-degenerate quadratic space.

Though Clifford Algebras have been studied in detail, they may not always be easy to work with. Sometimes it can be useful to switch to a more concrete embedding (as in the case of Suslin Matrices) to study low dimensional Spin and Epin (or Elementary Spin) groups. For instance, one can easily compute using Suslin Matrices, the exceptional isomorphisms $\text{Spin}(H(R^3)) \cong SL_4(R)$ (Theorem 7.1, [CV]) and $\text{Epin}(H(R^3)) \cong E_4(R)$ (Theorem 8.4, [CV]). We conclude this paper with a brief description of Suslin matrices and use them to give an explicit set of generators of the Clifford algebras of some quadratic spaces.

1.2 NOTATION

Let R be a commutative ring and V be a free R -module with basis $\{v_1, \dots, v_n\}$. In addition, V is equipped with a quadratic form q . Then there is a bilinear form associated to (V, q) , given by $\langle v, w \rangle = q(v+w) - q(v) - q(w)$, for $v, w \in V$.

We say that (V, q) is *non-singular* if the matrix corresponding to the bilinear form, $B = (\langle v_i, v_j \rangle)$ is invertible.

A quadratic space (V, q) is said to be *non-degenerate* when $\langle x, v \rangle = 0$ holds for all $v \in V$, if and only if $x = 0$.

Suppose (V, q) is degenerate. Then there exists a $x \in V$, $x \neq 0$ such that $\langle x, v_i \rangle = 0$ for $i \in \{1, \dots, n\}$. Writing $x = (x_1, \dots, x_n)$, this is equivalent to saying that there is a non-trivial solution to the matrix equation $xB = 0$, where $B = (\langle v_i, v_j \rangle)$. When R is a commutative ring, the matrix equation $xB = 0$ has a non-trivial solution if and only if the determinant of $B = (\langle v_i, v_j \rangle)$ is a zero divisor ([MD], Corollary I.30.). In other words, (V, q) is non-degenerate if and only if $\det(\langle v_i, v_j \rangle)$ is a non-zero divisor.

Unless otherwise stated, all quadratic spaces in the paper are assumed to be non-degenerate. All modules considered are free-modules over a commutative ring R .

1.3 GENERAL REFERENCES

For general literature on Clifford algebras and Spin groups over a commutative ring, the reader is referred to [B2], [K]. To learn more about Suslin Matrices, see [RJ] and Chapter III.7, [L].

2 PRELIMINARIES ON CLIFFORD ALGEBRAS

Perhaps the most important example of an embedding of a quadratic space is given by the Clifford algebra. Given any quadratic space (V, q) , its Clifford algebra $\text{Cl}(V, q)$ (or simply, Cl) is the “freest” algebra generated by V subject to the condition $x^2 = q(x)$ for all $x \in V$.

More precisely, the Clifford algebra $\text{Cl}(V, q)$ is the quotient of the tensor algebra

$$T(V) = R \oplus V \oplus V^{\otimes 2} \oplus \dots \oplus V^{\otimes n} \oplus \dots$$

by the two sided ideal $I(V, q)$ generated by all $x \otimes x - q(x)$ with $x \in V$.

2.1 BASIC PROPERTIES OF THE CLIFFORD ALGEBRA $\text{Cl}(V, q)$:

- *Z_2 Grading of Cl* : The Clifford algebra $\text{Cl}(V, q)$ is an associative algebra (with unity) over R with a linear map $i : V \rightarrow \text{Cl}(V, q)$ such that $i(x)^2 = q(x)$. The terms $x \otimes x$ and $q(x)$ appearing in the generators of $I(V, q)$ have degrees 0 and 2 in the grading of $T(V)$. By grading $T(V)$ modulo 2 by even and odd degrees, it follows that the Clifford algebra has a Z_2 -grading $\text{Cl} = \text{Cl}_0 \oplus \text{Cl}_1$ such that $V \subseteq \text{Cl}_1$ and $\text{Cl}_i \text{Cl}_j \subseteq \text{Cl}_{i+j}$ ($i, j \pmod 2$).
- *Universal Property* : The Clifford algebra has the following universal property. Given any associative algebra A over R and any linear map $j : V \rightarrow A$ such that

$$j(x)^2 = q(x) \text{ for all } x \in V,$$

there is a unique algebra homomorphism $f : \text{Cl}(V, q) \rightarrow A$ such that $f \circ i = j$.

- *Basis of Cl* : The elements of V generate the Clifford algebra. Furthermore, the following result implies that if $\text{rank}(V) = n$, then $\text{rank}(\text{Cl}) = 2^n$.

THEOREM 2.2. (*Poincaré-Birkhoff-Witt*)

Let $\{v_1, \dots, v_n\}$ be a basis of (V, q) . Then $\{v_1^{e_1} \dots v_n^{e_n} : e_i = 0, 1\}$ is a basis of $\text{Cl}(V, q)$.

For a simple proof, see Theorem IV. 1.5.1, [K].

The above theorem implies in particular that the map $i : V \rightarrow \text{Cl}(V, q)$ is injective. Since $i(v)^2 = q(v)$ for $v \in V$, it is clear that $i(V) \subseteq \text{Cl}$ is an embedding. We will refer to this as the Clifford embedding of (V, q) .

2.3 STRUCTURE OF CLIFFORD ALGEBRAS

To analyze general embeddings, we need one more result about Clifford algebras, and this will be stated in Theorem 2.7. The theorem says that a graded homomorphism from Cl is injective whenever its restriction on R is injective. The reader may assume that and jump ahead to the next section.

For any Z_2 -graded-algebra $A = A_0 + A_1$, the elements in $h(A) = A_0 \cup A_1$ will be called the *homogeneous* elements of A . If $a \in h(A)$, we write $\partial(A) = i$ if $a \in A_i$, ($i = 0, 1$).

The graded tensor product of two algebras $A \hat{\otimes} B$ is defined as :

$$(a \hat{\otimes} b)(a' \hat{\otimes} b') = (-1)^{\partial(b)\partial(a')} aa' \hat{\otimes} bb'$$

for all homogeneous elements $a, a' \in A$ and $b, b' \in B$.

One can use the universal property of Clifford Algebras to compute the Clifford algebra of an orthogonal sum of quadratic spaces, (see [K], Theorem IV.1.3.1.).

THEOREM 2.4. *The map $f : (V_1, q_1) \perp (V_2, q_2) \rightarrow \text{Cl}(V_1, q_1) \hat{\otimes} \text{Cl}(V_2, q_2)$ defined by $f(x_1 + x_2) = x_1 \hat{\otimes} 1 + 1 \hat{\otimes} x_2$ induces an isomorphism*

$$\text{Cl}(V_1 \perp V_2, q_1 \perp q_2) \cong \text{Cl}(V_1, q_1) \hat{\otimes} \text{Cl}(V_2, q_2).$$

The hyperbolic module $H(V) = V \oplus V^*$ is equipped with a quadratic form $q(x, f) = f(x)$. This quadratic space will be referred to as the hyperbolic space. We will return to them in the final section of the paper.

THEOREM 2.5. *Let (V, q) be a non-singular quadratic space where V is a free module with rank n . Then $(V, q) \perp (V, -q) \cong H(V)$ and*

$$\text{Cl}(V, q) \hat{\otimes} \text{Cl}(V, -q) \cong M_{2^n}(R)$$

Proof. For a proof that $(V, q) \perp (V, -q)$ and $H(V)$ are isomorphic as quadratic spaces, see [B1], Ch. 5, Lemma 2.2.

It is also known that $\text{Cl}(H(V)) \cong M_{2^n}(R)$ (Theorem 7.1.10, [HOM]). The result follows using Theorem 2.4. \square

An ideal $J \subseteq \text{Cl}$ is said to be graded if it is a direct sum of the intersections $J_i = J \cap \text{Cl}_i$. A homomorphism between two Z_2 - graded algebras is said to be graded if it preserves the grading. The kernel of a graded homomorphism is a graded ideal.

THEOREM 2.6. *Let (V, q) be a non-degenerate quadratic space. Let J be a graded ideal of $\text{Cl}(V, q)$ such that $J \cap R = \{0\}$. Then $J = \{0\}$.*

Proof. **CASE 1 :** (V, q) IS NON-SINGULAR.

Consider the graded homomorphism $\phi : \text{Cl}(V, q) \rightarrow \text{Cl}(V, q)/J$. By Theorem 2.5, the map ϕ extends to a graded homomorphism $\phi' : M_{2^n}(R) \rightarrow \text{Cl}(V, q)/J \hat{\otimes} \text{Cl}(V, -q)$. Now every ideal of $M_{2^n}(R)$ is of the form $M_{2^n}(I)$ for some ideal I in R . Since $J \cap R = \{0\}$, this is possible only if $I = \{0\}$.

CASE 2 : (V, q) IS NON-DEGENERATE.

When (V, q) is non-degenerate, $d = \det(\langle v_i, v_j \rangle)$ is a non-zero divisor in R (see Section 1.2). Then $V \otimes R[d^{-1}]$ is a non-singular quadratic space and so $J \otimes R[d^{-1}]$ is the zero ideal by Case 1. Therefore $J = \{0\}$. □

THEOREM 2.7. *Let (V, q) be a non-degenerate quadratic space. Let $\phi : \text{Cl}(V, q) \rightarrow A$ be a graded (algebra) homomorphism such that $\ker(\phi) \cap R = \{0\}$. Then ϕ is injective.*

Proof. The kernel of ϕ is a graded ideal in $\text{Cl}(V, q)$. The result follows immediately from Theorem 2.6. □

Observe that Theorem 2.7 allows the possibility of $d = \det(\langle v_i, v_j \rangle)$ being a zero divisor in A , even though d is a non-zero divisor in R .

3 BASIC PROPERTIES OF EMBEDDINGS

3.1 CONNECTING TWO DIFFERENT EMBEDDINGS :

Let Cl denote the Clifford algebra of (V, q) . To avoid any confusion, we will denote the copy of V in its Clifford algebra by V_{Cl} .

Let $(V, q) \subseteq A$ be an embedding with $q(v) = v\bar{v} = \bar{v}v$, for some isometry $v \rightarrow \bar{v}$, for $v \in V$.

Let $\phi : V_{\text{Cl}} \rightarrow M_2(A)$ defined by $v_{\text{Cl}} \rightarrow \begin{pmatrix} 0 & v \\ \bar{v} & 0 \end{pmatrix}$. As $\phi^2(v_{\text{Cl}}) = q(v)$ for all $v \in V$, the map ϕ extends to an R -algebra homomorphism $\phi : \text{Cl} \rightarrow M_2(A)$. This is in fact a graded homomorphism, where the even and odd elements of $M_2(A)$ correspond to matrices of the form $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ and $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$.

Let $\ker(\phi)$ denote the kernel of ϕ . Since ϕ restricts to an injective map on V_{Cl} , we have $\ker(\phi) \cap R = \{0\}$. Therefore it follows from Theorem 2.7 that ϕ is injective.

From here on, we will identify Cl with its image in $M_2(A)$.

THEOREM 3.2. *Let (V, q) be a quadratic space embedded in an algebra A . Let $v, w \in V$. Then $vwv \in V$ and $\overline{vwv} = \bar{v} \cdot \bar{w} \cdot \bar{v}$.*

Proof. We will first prove the theorem for the Clifford embedding. Let $z_1, z_2 \in V_{\text{Cl}}$. Then

$$\langle z_1, z_2 \rangle := (z_1 + z_2)^2 - z_1^2 - z_2^2 = z_1z_2 + z_2z_1$$

is an element in R . Multiplying by z_1 we get

$$z_1 \langle z_1, z_2 \rangle = z_1^2 z_2 + z_1 z_2 z_1.$$

Since $z_1^2 = q(z_1)$, it follows that $z_1 z_2 z_1 \in V_{\text{Cl}}$. For any embedding $V \subseteq A$, there is a map $\phi : \text{Cl} \rightarrow M_2(A)$ given by $v_{\text{Cl}} \rightarrow \begin{pmatrix} 0 & v \\ \bar{v} & 0 \end{pmatrix}$.

Take $z_1 \rightarrow \begin{pmatrix} 0 & v \\ \bar{v} & 0 \end{pmatrix}$ and $z_2 \rightarrow \begin{pmatrix} 0 & \bar{w} \\ w & 0 \end{pmatrix}$. Then $z_1 z_2 z_1 \rightarrow \begin{pmatrix} 0 & v w v \\ \bar{v} w \bar{v} & 0 \end{pmatrix}$. \square

Given an embedding, one can also treat (V, q) as a non-associative algebra, with its multiplication given by $v \bullet w = v w v$ for $v, w \in V$. Under this multiplication, (V, q) becomes a Quadratic Jordan algebra. By the above theorem, $\overline{v \bullet w} = \bar{v} \bullet \bar{w}$. Then the bijection $v \rightarrow \bar{v}$ in (V, q) is not only an isometry, but also an algebra automorphism of (V, q) . In addition, if $1 \in V$, then $(1+v)w(1+v) \in V$, and so $vw + wv \in V$. A general theory of Quadratic Jordan algebras can be found in [J, M].

3.3 THE STANDARD INVOLUTION

The map $v_{\text{Cl}} \rightarrow -v_{\text{Cl}}$ can be viewed as an inclusion of V_{Cl} in the opposite algebra of Cl . By the universal property of the Clifford algebra, this map extends to an involution $*$ of Cl . This is called the standard involution on Cl . In terms of the algebra $M_2(A)$, we have $\begin{pmatrix} 0 & v \\ \bar{v} & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & -v \\ -\bar{v} & 0 \end{pmatrix}$. We will see below that under certain circumstances, the standard involution on Cl restricts to an involution of A .

One might also wonder if the map $v \rightarrow -v$ or the isometry $\alpha(v) = \bar{v}$ (for $v \in V$) extends to an involution of A . We will keep returning to similar questions in the paper. For now it is not even clear what values, the order of α can take.

In the Clifford embedding, the isometry of V is simply the identity map. For Composition algebras and (as we'll see later) Suslin matrices, the isometry corresponding to the embedding has order 2. This is not an accident and we will now show that this is true whenever $1_A \in V$ and $\overline{1_A} = 1_A$.

Suppose $1_A \in V$ and $\overline{1_A} = 1_A$. Then we have the following nice implications :

- Let $v \in V$. Since $q(1+v) = (1+v)(\overline{1+v})$ is a scalar, so is $v + \bar{v}$.
- We have

$$\alpha^2(x)\alpha(x) = \alpha(x)x, \quad \alpha^2(x+1)\alpha(x+1) = \alpha(x+1)(x+1).$$

Using the fact that α is linear and cancelling terms, it follows that $\alpha^2(x) = x$ for all $x \in V$, i.e. α has order 2.

- Now suppose $A \subseteq \text{Cl}$, i.e., Cl contains all elements $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ with $a \in A$; Then we will show that the standard involution restricts to A .

First notice that the even part of the Clifford algebra Cl_0 is closed under the standard involution, and its image in $M_2(A)$ consists of matrices of the form $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$. We will simply write (x, y) instead of $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$.

Let $(a, a)^* = (x, y)$. For A to be closed under the involution, we need $x = y$.

Let $e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We have $e^* = -e$ as $e \in V_{M_2(A)}$. Since $e(a, a)e = (a, a)$, we have $e^*(a, a)^*e^* = (a, a)^*$. Therefore

$$(y, x) = e \cdot (a, a)^* \cdot e = (x, y)$$

and so the standard involution on Cl restricts to A .

Conversely, given an involution of A , one might ask if it can be extended to the standard involution on Cl. To explore this possibility, let us write $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as a 2×2 matrix and analyze its conjugate in terms of its blocks. The table below illustrates, for a few examples, how the action of an involution $*$ on A can be extended to the standard involution in Cl, which is seen as a sub-algebra of $M_2(A)$.

Note that in order to show that an involution corresponds to the standard involution of the Clifford algebra, it is enough to check that its action on the elements of V_{Cl} is multiplication by -1 . In other words, if $z = \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}$, then we need $z^* = -z$. This is clearly the case for both the involutions defined in the table below.

Standard Involution on $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$			
Form 1:	$v^* = u \cdot v$	$u^2 = 1, u \in R$	$M^* = \begin{pmatrix} d^* & -ub^* \\ -uc^* & a^* \end{pmatrix}$
Form 2 :	$v^* = u \cdot \bar{v}$	$u^2 = 1, u \in R$	$M^* = \begin{pmatrix} a^* & -uc^* \\ -ub^* & d^* \end{pmatrix}$

Since the involution acts trivially on scalar matrices, notice that if $1_A \in V$, and $v^* = uv$ or $v^* = u\bar{v}$, then it follows that $u = 1$.

4 THE Spin REPRESENTATION: WHEN $v^* = v$ FOR ALL $v \in V$

Motivated by the discussion in the previous section, we will now analyze embeddings $(V, q) \subseteq A$ with the following conditions:

1. $1_A \in V_A$ and $\overline{1_A} = 1_A$.
2. There is an involution $*$ of A that restricts to the identity map on V , i.e, $v^* = v$ for all $v \in V$.

We will continue to identify Cl as a sub-algebra of $M_2(A)$. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A)$. Then $M^* = \begin{pmatrix} d^* & -b^* \\ -c^* & a^* \end{pmatrix}$ gives us the standard involution on Cl.

In particular, $\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}^* = \begin{pmatrix} g_2^* & 0 \\ 0 & g_1^* \end{pmatrix}$. For convenience, we will sometimes write (g_1, g_2) instead of $\begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$.

4.1 Spin GROUP

The following groups are relevant to our discussion :

$$U^0(V) := \{x \in \text{Cl}_0 \mid xx^* = 1\}$$

$$\text{Spin}(V) := \{x \in U^0(R) \mid xV_{\text{Cl}_1}x^{-1} = V\}.$$

Notice that the action of the Spin group on V is an isometry of V .

4.2 THE Spin REPRESENTATION :

We will define a group $SG(A) \subset A$ and show that it is isomorphic to the Spin group, when $v^* = \bar{v}$ for all $v \in V$.

Let $(g_1, g_2) \in \text{Spin}(V)$. Since $(g_1, g_2)^* = (g_2^*, g_1^*)$, we have $g_2 = g_1^{*-1}$.

Now take any $(g, g^{*-1}) \in \text{Spin}(V)$ and $v \in V$. By definition, there exists an element $w \in V$ such that $(g, g^{*-1}) \begin{pmatrix} 0 & v \\ \bar{v} & 0 \end{pmatrix} (g^{-1}, g^*) = \begin{pmatrix} 0 & w \\ \bar{w} & 0 \end{pmatrix}$, i.e.,

$$\begin{pmatrix} 0 & gvg^* \\ g^{*-1}\bar{v}g^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & w \\ \bar{w} & 0 \end{pmatrix}.$$

Let $g \bullet v = gvg^*$ for $g \in A$. Then $g \bullet v \in V$ when $(g, g^{*-1}) \in \text{Spin}(V)$.

Let A^\times denote the invertible elements of A . Consider the set

$$G(A) = \{g \in A^\times \mid g \bullet v \in V \forall v \in V\}.$$

Since the action \bullet is bijective, it is easy to see that $G(A)$ is a group and is closed under the involution $*$. One has the homomorphism

$$\chi : \text{Spin}(V) \rightarrow G(A)$$

given by $(g, g^{*-1}) \rightarrow g$.

Next, we will use the quadratic form q on V to define a ‘norm’ on $G(A)$. The Spin group will turn out to be isomorphic to the subgroup of $G(A)$ whose elements have unit norm.

Let us begin with three simple lemmas which help us show that

$$q(g \bullet v) = q(gg^*)q(v), \quad \text{for } g \in G(A).$$

LEMMA 4.3. *Let $v \in V$ such that $\{v, 1\}$ are linearly independent. Suppose there exists an element $v' \in V$ such that $v + v'$ and vv' are scalars. Then $v' = \bar{v}$.*

Proof. Since $v + \bar{v} \in R$, it follows that $\bar{v} = v' + r$ for some $r \in R$. Since $v\bar{v} = q(v)$, it follows that $v(v' + r) \in R$, implying $rv \in R$. Therefore $r = 0$ and $v' = \bar{v}$. \square

LEMMA 4.4. *Suppose $v_1, v_2 \in V$ and $q(v_2) = v_2\bar{v}_2 = 1$. Then $\bar{v}_1 + v_2v_1v_2 \in v_2R$.*

Proof. Since $(\bar{v}_1 + v_2)(v_1 + \bar{v}_2) \in R$, it follows that

$$\langle \bar{v}_1, v_2 \rangle = \bar{v}_1 \cdot \bar{v}_2 + v_2v_1 \in R.$$

Multiplying by v_2 on the right, this implies that

$$\bar{v}_1 + v_2v_1v_2 \in v_2R. \quad \square$$

LEMMA 4.5. *Let $g \in G(A)$. If $q(gg^*) = 1$, then $q(g^*g) = 1$.*

Proof. Let $X = g^*g$.

If $X \in R$, then $gg^* = g^*g$ and we are done. Suppose $X \notin R$. We will show that $X + X^{-1} \in R$ and infer from Lemma 4.3 that $X^{-1} = \bar{X}$. Now,

$$X^{-1} = g^{-1}g^{*-1} = g^* \bullet (g^{*-1}g^{-1})^2.$$

Since $q(gg^*) = 1$, we have $q(g^{*-1}g^{-1}) = q(\overline{gg^*}) = 1$. Therefore (using Lemma 4.4),

$$\begin{aligned} X + X^{-1} &= g^* \bullet (1 + (g^{*-1}g^{-1})^2) \\ &= g^* \bullet (rg^{*-1}g^{-1}), \quad \text{for some } r \in R. \\ &= r \in R \end{aligned} \quad \square$$

LEMMA 4.6. *Let $g \in G(A)$. For all $v \in V$, we have*

$$q(g \bullet v) = q(gg^*)q(v).$$

Proof. CASE 1 : $q(gg^*) = 1$.

Let $w = g \bullet v$ and $w' = g^{*-1} \bullet \bar{v}$. Since $w \cdot w' = v\bar{v}$, it is enough to prove that $\bar{w} = w'$.

Let us assume for now that $\{w, 1\}$ are linearly independent. We will first show that $w + w' \in R$ and use it to prove that $w' = \bar{w}$.

Let $X = g^*g$. Since $q(gg^*) = 1$, it follows from Lemma 4.5 that $q(X) = 1$. We have

$$\begin{aligned} w &= gv g^* \\ &= g^{*-1}(XvX)g^{-1} \\ &= g^{*-1} \bullet (XvX). \end{aligned}$$

Since $q(X) = 1$, we know (from Lemma 4.4) that $\bar{v} + XvX = rX$ for some $r \in R$. Therefore

$$\begin{aligned} w' + w &= g^{*-1} \bullet (\bar{v} + XvX) \\ &= g^{*-1} \bullet rg^*g \\ &= r \in R \end{aligned}$$

Since $w' + w$ and ww' are scalars, it follows from Lemma 4.3 that

$$w' = \bar{w}.$$

Now suppose $\{w, 1\}$ are linearly dependent. Then we can write $w = (w_0 + w) - w_0$, where $\{w_0, 1\}$ are linearly independent. As \bullet is a linear action it follows that $w' = \bar{w}$.

CASE 2 : $q(gg^*) = a$.

Clearly a is invertible since $g \in G(A)$.

Suppose there is an $x \in R$ such that $x^2 = a^{-1}$. Take $h = xg$. Then $q(h \bullet v) = x^2 \cdot q(g \bullet v)$ and $q(hh^*) = 1$. The result follows immediately from Case 1.

Now suppose $x^2 = a^{-1}$ has no solutions in R . Then one has the identity $q(g \bullet v) = q(gg^*)q(v)$ over the ring $\frac{R[x]}{(x^2 - a^{-1})}$; Since each term of the equation lies in R , the result follows in this case too. \square

We can now describe the Spin group as a group inside A . Let R^\times denote the group of invertible elements in R .

THEOREM 4.7. *Define $d : G(A) \rightarrow R^\times$ as $d(g) = q(gg^*)$.*

Then d is a group homomorphism and

$$\ker(d) = SG(A) \cong \text{Spin}(V).$$

Proof. As a consequence of Lemma 4.6 we have, for $g, h \in G(A)$,

$$d(gh) = q(ghh^*g^*) = q(gg^*)q(hh^*) = d(g)d(h).$$

Thus d is a group homomorphism and

$$\ker(d) = SG(A) \cong \text{Spin}(V)$$

in the case when the identity map on V can be lifted to an involution of A . \square

5 THE SUSLIN EMBEDDING

Let R be any commutative ring and $H(R^n) := R^n \oplus R^{n*}$. By fixing a basis of R^n , one can then write the quadratic form on $H(R^n)$ as

$$q(v, w) = v \cdot w^T = a_1b_1 + \cdots + a_nb_n.$$

for $v = (a_1, \dots, a_n)$, $w = (b_1, \dots, b_n)$. This quadratic space $(H(R^n), q)$ is referred to as the hyperbolic space. We will now define Suslin matrices which give an embedding of the hyperbolic space into the ring of matrices $M_{2n-1}(R)$.

The Suslin matrix $S_n(v, w)$ of size $2^n \times 2^n$ is constructed from two vectors v, w in R^{n+1} as follows :

Let $v = (a_0, v_1)$, $w = (b_0, w_1)$ where v_1, w_1 are vectors in R^n . Define

$$S_1(v, w) = \begin{pmatrix} a_0 & v_1 \\ -w_1 & b_0 \end{pmatrix} \quad \overline{S_1(v, w)} = \begin{pmatrix} b_0 & -v_1 \\ w_1 & a_0 \end{pmatrix}$$

$$S_n(v, w) = \begin{pmatrix} a_0 I_{2^{n-1}} & S_{n-1}(v_1, w_1) \\ -S_{n-1}(v_1, w_1) & b_0 I_{2^{n-1}} \end{pmatrix}$$

and

$$\overline{S_n(v, w)} = \begin{pmatrix} b_0 I_{2^{n-1}} & -S_{n-1}(v_1, w_1) \\ S_{n-1}(v_1, w_1) & a_0 I_{2^{n-1}} \end{pmatrix}$$

It easily follows that $S_n = S_n(v, w)$ satisfies the following properties:

1. $S_n \overline{S_n} = \overline{S_n} S_n = (v \cdot w^T) I_{2^n}$,
2. $\det S_n = (v \cdot w^T)^{2^{n-1}}$, for $n \geq 1$.

In his paper [S], A. Suslin then describes a sequence of matrices $J_n \in M_{2^n}(R)$ such that $J J^T = I$,

$$J_{n-1} S_{n-1}^T J_{n-1}^T = \begin{cases} S_{n-1} & \text{for } n \text{ odd,} \\ \overline{S_{n-1}} & \text{for } n \text{ even.} \end{cases} \tag{1}$$

Clearly $M^* = J M^T J^T$ is an involution of $M_{2^n}(R)$ (as $J J^T = 1$). Thus there are two types of involution for the Suslin embedding, depending on the parity of $\text{rank}(V) = n$.

The map $\phi : H(R^n) \rightarrow M_{2^n}(R)$ defined by $\phi(v, w) = \begin{pmatrix} 0 & S_{n-1}(v, w) \\ S_{n-1}(v, w) & 0 \end{pmatrix}$ induces an R -algebra homomorphism $\phi : \text{Cl} \rightarrow M_{2^n}(R)$. In fact ϕ is an isomorphism (Section 3.1, [CV]); the elements $\phi(v, w)$ give a set of generators of the Clifford algebra.

REMARK 5.1. There cannot be two involutions $*_1, *_2$ of $M_{2^n}(R)$ (for a fixed n) such that $S^{*_1} = S$ and $S^{*_2} = \overline{S}$. Otherwise both involutions can be lifted to the standard involution as in Table 3.3. This is not possible as the two (lifted) involutions act differently on $\begin{pmatrix} S(v, w) & 0 \\ 0 & S(v, w) \end{pmatrix} \in \text{Cl}$, when $\overline{S(v, w)} \neq S(v, w)$.

REMARK 5.2. When $v \cdot w^T = 1$, the kernel of the map $R^n \rightarrow R$, defined by $w \rightarrow v \cdot w^T$, is a projective module. This projective module is not isomorphic to its dual when the row v has odd size > 3 . However, these projective modules are self-dual when n is even. (See [NRS]).

Perhaps this difference in *duality* in the odd and even cases can give a deeper explanation for the corresponding behavior of the Suslin Matrices, described above in Equation 1.

To learn more about the connection between Suslin Matrices and Clifford Algebras, see [CV]. Suslin matrices were first introduced by A. Suslin in his paper [S], in connection with unimodular rows and K-Theory. The recent work of A. Asok and J. Fasel (see [AF]) uses Suslin Matrices in the context of \mathbb{A}^1 -homotopy theory and Bott periodicity.

5.3 APPLICATIONS TO QUADRATIC SPACES

In a similar fashion, using Suslin Matrices, one can construct an explicit set of generators of the Clifford Algebra for other classes of quadratic spaces. Here are a few examples :

Quadratic Space (V, q)	Clifford Embedding	Cl
$(R^{2n}, \sum_{i=1}^n v_i w_i)$	$(v, w) \rightarrow \begin{pmatrix} 0 & S(v, w) \\ S(v, w) & 0 \end{pmatrix}$	$M_{2n}(R)$
$(R^{2n+1}, -v_0^2 + \sum_{i=1}^n v_i w_i)$	$(x, v, w) \rightarrow \begin{pmatrix} v_0 \lambda_1 & S(v, w) \\ S(v, w) & -v_0 \lambda_1 \end{pmatrix}$	$M_{2n}(R[\lambda_1])$
$(R^{2n+2}, -v_0^2 - w_0^2 + \sum_{i=1}^n v_i w_i)$	$(x, y, v, w) \rightarrow \begin{pmatrix} v_0 \lambda_1 + w_0 \lambda_2 & S(v, w) \\ S(v, w) & -v_0 \lambda_1 - w_0 \lambda_2 \end{pmatrix}$	$M_{2n}(R[\lambda_1, \lambda_2])$

(In the above table, we have $\lambda_1^2 = \lambda_2^2 = -1$ and $\lambda_1 \lambda_2 + \lambda_2 \lambda_1 = 0$.)

6 CONCLUSION

We have analyzed general embeddings of quadratic spaces by connecting them to their respective Clifford algebras. Given an embedding $(V, q) \subseteq A$, we know that $vvv \in V$ and so V becomes a Quadratic Jordan algebra. It would be very interesting to learn the conditions under which the identity map on $V \subseteq A$ (or more generally, maps of the type $v^* = uv, u \in R$) can be lifted to an involution of A . We know that this is not always the case for Suslin matrices (see Remark 5.1).

The classification and structure of Special Jordan algebras has been worked out in the 20th century (see [M] for a survey). But their relationship with the overlying associative algebras, in different embeddings, remains to be explored further.

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