On a Construction Due to Khoshkam and Skandalis

S. Sundar[1]

Received: August 7, 2017
Revised: November 19, 2018

Communicated by Andreas Thom

Abstract. In this paper, we consider the Wiener-Hopf algebra, denoted $\mathcal{W}(A,P,G,\alpha)$, associated to an action of a discrete subsemigroup P of a group G on a C^*-algebra A. We show that $\mathcal{W}(A,P,G,\alpha)$ can be represented as a groupoid crossed product. As an application, we show that when $P=\mathbb{F}_n^+$, the free semigroup on n generators, the K-theory of $\mathcal{W}(A,P,G,\alpha)$ and the K-theory of A coincides.

2010 Mathematics Subject Classification: Primary 22A22; Secondary 54H20, 43A65, 46L55
Keywords and Phrases: Wiener-Hopf C*-algebras, Semigroups, Groupoid dynamical systems

1 Introduction

Semigroup crossed products have been studied by several authors since the late 80's. We refer to the papers [9], [10] and the references therein for some of the earlier work on semigroup crossed products. We also refer the reader to [3] for Exel's work on semigroup crossed products. For more recent advances in semigroup C^*-algebras, the reader is referred to [6], [7].
In this paper, we analyse the semigroup crossed product considered by Khoskham and Skandalis in [4] for a general discrete subsemigroup of a group from a groupoid perspective. In [4], actions of $\mathbb{N}$ and $\mathbb{R}_+$ are considered. Even

[1]This work was initiated during the author's visit to the University of Orleans during May-June 2015. I thank Prof. Renault for providing me with a visit and for discussing with me for a month at Orleans. I thank Prof. Skandalis for spending a day at Orleans to discuss with me and Prof. Renault. I thank him especially for explaining the idea behind the proof of Theorem 6.4. I thank Prof. Philippon, IMJ Paris, for the financial support (IRSES-MODULI) which made this visit possible.

though the construction due to Khoskham and Skandalis can be carried out to topological semigroups, we restrict ourselves to the discrete case. For, we believe that the discrete case alone is interesting for its own sake and might be interesting to a wider audience. We plan to pursue the topological case elsewhere. We now give an overview of the problem considered and the results obtained.

Let $P \subset G$ be a subsemigroup of a discrete group G containing the identity e. Let A be a C^*-algebra and let $\alpha : P \to End(A)$ be a left action of P on A by endomorphisms. For the introduction, let us assume, for simplicity, that A is unital and the action of P on A is by unital endomorphisms. Consider the Hilbert A-module $\mathcal{E} := A \otimes \ell^2(P)$. The Hilbert module structure of $\mathcal{E}$ is that of the external tensor product where A has the usual Hilbert A-module structure. For $x \in A$ and $a \in P$, define the operators $\pi(x)$ and V_a by the equation:

$$\pi(x)(y \otimes \delta_b) := \alpha_b(x)y \otimes \delta_b,$$
$$V_a(y \otimes \delta_b) := y \otimes \delta_{ba}.$$

Note that V_a is an isometry for each $a \in P$ and $V_aV_b = V_{ba}$. Moreover the covariance condition $\pi(x)V_a = V_a\pi(\alpha_a(x))$ is satisfied for $x \in A$ and $a \in P$. It is natural to define the "reduced crossed product" $A \rtimes_{red} P$ as the C^*-algebra generated by $\{\pi(x) : x \in A\}$ and $\{V_a : a \in P\}$. When $A = \mathbb{C}$, the C^*-algebra $\mathbb{C} \rtimes_{red} P$ is called the reduced C^*-algebra of the semigroup P denoted by $C^*_{red}(P)$. The C^*-algebra $C^*_{red}(P)$ is studied in detail by Xin Li in [6] and in [7].

There is a related C^*-algebra called the Wiener-Hopf algebra which is much easier to describe from a groupoid perspective. For $g \in G$, let W_g be the operator on $\mathcal{E}$ defined by

$$W_g(y \otimes \delta_b) := \begin{cases} y \otimes \delta_{bg} & \text{if } bg \in P, \\ \\ 0 & \text{if } bg \notin P. \end{cases}$$

The C^*-algebra generated by $\{\pi(x) : x \in A\}$ and $\{W_g : g \in G\}$ is called the Wiener-Hopf algebra associated to (A, P, G, α) and we denote it by $\mathcal{W}(A, P, G, \alpha)$. Clearly $A \rtimes_{red} P$ is contained in $\mathcal{W}(A, P, G, \alpha)$. Up to the author's knowledge, it is not known if $A \rtimes_{red} P = \mathcal{W}(A, P, G, \alpha)$. However the equality $A \rtimes_{red} P = \mathcal{W}(A, P, G, \alpha)$ holds if either P is right Ore i.e. $PP^{-1} = G$ or if the pair (P, G) is quasi-lattice ordered in the sense of Nica ([12]).

In this paper, we represent $\mathcal{W}(A, P, G, \alpha)$ as a groupoid crossed product. The idea of representing the Wiener-Hopf algebra as a groupoid crossed product dates back to [8]. In fact, we prove that $\mathcal{W}(A, P, G, \alpha)$ is isomorphic to the reduced crossed product of the form $\mathcal{D} \rtimes \mathcal{G}$ where $\mathcal{G}$ is the Wiener-Hopf groupoid associated to (P, G). This is the content of Theorem 4.3. We imitate the construction of Khoskham and Skandalis carried out in [4] for the semigroup $\mathbb{N}$ to construct the bundle $\mathcal{D}$. This forms the content of Sections 3 and 4.

In [2], crossed products by automorphic actions of semigroups are considered. A remarkable result that the inclusion $A \ni x \to A \rtimes_{red} P$ is a KK-equivalence is proved if (P, G) is quasi-lattice ordered and if G satisfies Baum-Connes conjecture with coefficients. Here we prove, for endomorphic actions, a very modest result that when $P = \mathbb{F}_n^+$, the inclusion $A \ni x \to x \in A \rtimes_{red} \mathbb{F}_n^+$ is a KK-equivalence. We prove this KK-equivalence by explicitly constructing an inverse in $KK(A \rtimes_{red} \mathbb{F}_n^+, A)$. This KK-equivalence relies on the amenability of the Wiener-groupoid of the pair $(\mathbb{F}_n^+, \mathbb{F}_n)$ and the fact that $\mathcal{D} \rtimes \mathcal{G}$ can be expressed in terms of generators and relations which we achieve in Section 5.
Throughout this paper, we assume that the C^*-algebras bearing the name A are separable and we consider only discrete groups which we assume to be countable.

2 Preliminaries

In this section, we recall the essentials regarding $C_0(X)$-algebras and groupoid crossed products. This is to make the paper easier to read and also to fix notations. The reader is referred to [11] and [5] for more on groupoid crossed products.
Let G be a discrete group and X be a right G-space. For $f \in C_0(X)$ and $g \in G$, let $R_g(f) \in C_0(X)$ be defined by $R_g(f)(x) = f(xg)$ for $x \in X$. A $C_0(X)$-G algebra is a C^*-algebra A together with a group homomorphism $\alpha : G \to Aut(A)$ and a $*$-algebra homomorphism $\rho : C_0(X) \to M(A)$, where $M(A)$ is the multiplier algebra of A, such that

(1) for $f \in C_0(X)$ and $a \in A$, $\rho(f)a = a\rho(f)$,

(2) the representation ρ is non-degenerate i.e. $\overline{\rho(C_0(X))A} = A$, and

(3) the homomorphism ρ is G-equivariant i.e. for $f \in C_0(X), g \in G$ and $a \in A$, $\alpha_g(\rho(f)a) = \rho(R_g(f))\alpha_g(a)$.

Usually we omit the symbol ρ and simply write $\rho(f)a$ as fa for $f \in C_0(X)$ and $a \in A$. A $C_0(X)$-G algebra is sometimes called an (X, G)-algebra. Moreover if G is the trivial group, then $C_0(X)$-G algebras are referred as $C_0(X)$-algebras.
Let A be an (X, G)-algebra. For $x \in X$, define

$$\begin{aligned} C_0(X \backslash \{x\}) &:= \{f \in C_0(X) : f(x) = 0\}, \text{ and} \\ I_x &:= \overline{C_0(X \backslash \{x\})A}. \end{aligned}$$

Then I_x is a closed two sided ideal of A. We denote the quotient A/I_x by A_x.
Similarly for a closed subset $F \subset X$, we let

$$\begin{aligned} C_0(X \backslash F) &:= \{f \in C_0(X) : f(x) = 0 \text{ for } x \in F\}, \text{ and} \\ I_F &:= \overline{C_0(X \backslash F)A}. \end{aligned}$$

Let $\mathcal{A} := \coprod_{x \in X} A_x$. The bundle $\mathcal{A}$ over X is an upper semicontinuous bundle of C^*-algebras where the topology on $\mathcal{A}$ is determined by the family of sections $X \ni x \to a + I_x \in A_x$ for $a \in A$. Let $\Gamma_0(X, \mathcal{A})$ be the algebra of continuous sections vanishing at infinity. The map

$$A \ni a \to (X \ni x \to a + I_x \in A_x) \in \Gamma_0(X, \mathcal{A})$$

is a $C_0(X)$-algebra isomorphism. We refer the reader to Appendix C of [16] for more details, in particular to Theorems C.25 and C.26 of [16].

The transformation groupoid $X \rtimes G$ acts on the bundle $\mathcal{A}$. First note that if $g \in G$, then α_g maps $I_{x.g}$ onto I_x. Thus α_g descends to an isomorphism from $A/I_{x.g} \to A/I_x$ which we denote by $\alpha_{(x,g)}$. Moreover $\alpha_{(x,g)}\alpha_{(x.g,h)} = \alpha_{(x,gh)}$. Thus $\alpha := \{\alpha_{(x,g)}\}_{(x,g) \in X \rtimes G}$ defines an action of the groupoid $X \rtimes G$ on the bundle $\mathcal{A}$.

Now we recall the definition of groupoid crossed products. Let $\mathcal{G}$ be an r-discrete groupoid and let $p : \mathcal{A} \to \mathcal{G}^{(0)}$ be an upper semicontinuous bundle. As usual, we denote the unit space of $\mathcal{G}$ by $\mathcal{G}^{(0)}$ and the range and source maps by r and s respectively. For $x \in \mathcal{G}^{(0)}$, we denote the fibre $p^{-1}(x)$ by A_x. Let $\alpha := (\alpha_\gamma)_{\gamma \in \mathcal{G}}$ be an action of $\mathcal{G}$ on $\mathcal{A}$. Denote the C^*-algebra of continuous sections of $\mathcal{A}$ vanishing at infinity by $\Gamma_0(\mathcal{G}^{(0)}, \mathcal{A})$. Let

$$\Gamma_c(\mathcal{G}, r^*\mathcal{A}) := \{f : \mathcal{G} \to \mathcal{A} : \ f \text{ is cont., supp}(f) \text{ compact}, \ f(\gamma) \in A_{r(\gamma)}\}.$$

The vector space $\Gamma_c(\mathcal{G}, r^*\mathcal{A})$ has a $*$-algebra structure where the multiplication and the involution are defined by

$$f * g(\gamma) := \sum_{r(\gamma_1) = r(\gamma)} f(\gamma_1)\alpha_{\gamma_1}(g(\gamma_1^{-1}\gamma))$$
$$f^*(\gamma) := \alpha_\gamma(f(\gamma^{-1}))^*$$

for $f, g \in \Gamma_c(\mathcal{G}, r^*\mathcal{A})$.

For $f \in \Gamma_c(\mathcal{G}^{(0)}, \mathcal{A})$, let $\widehat{f} : \mathcal{G} \to \mathcal{A}$ be defined by

$$\widehat{f}(\gamma) := \begin{cases} f(\gamma) & \text{if } \gamma \in \mathcal{G}^{(0)}, \\ \\ 0_{r(\gamma)} & \text{if } \gamma \notin \mathcal{G}^{(0)}. \end{cases}$$

As $\mathcal{G}^{(0)}$ is a clopen subset of $\mathcal{G}$, it follows that $\widehat{f} \in \Gamma_c(\mathcal{G}, r^*\mathcal{A})$. Also the inclusion $\Gamma_c(\mathcal{G}^{(0)}, \mathcal{A}) \ni f \to \widehat{f} \in \Gamma_c(\mathcal{G}, r^*\mathcal{A})$ is a $*$-algebra homomorphism. We will consider $\Gamma_c(\mathcal{G}^{(0)}, \mathcal{A})$ as a $*$-subalgebra of $\Gamma_c(\mathcal{G}, r^*\mathcal{A})$.

A $*$-representation $\pi : \Gamma_c(\mathcal{G}, r^*\mathcal{A}) \to B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is said to be bounded if $||\pi(f)|| \leq ||f||_\infty := \sup_{x \in \mathcal{G}^{(0)}} ||f(x)||$ for every $f \in \Gamma_c(\mathcal{G}^0, \mathcal{A})$. For $f \in \Gamma_c(\mathcal{G}, r^*\mathcal{A})$, let

$$||f||_u := \sup\{||\pi(f)|| : \pi \text{ is a bounded representation of } \Gamma_c(\mathcal{G}, r^*\mathcal{A})\}.$$

Then $||.||_u$ is a C^*-norm on $\Gamma_c(\mathcal{G}, r^*\mathcal{A})$ and the completion of $\Gamma_c(\mathcal{G}, r^*\mathcal{A})$ is called the full crossed product and is denoted by $\mathcal{A} \rtimes \mathcal{G}$.
Now we recall the definition of the reduced crossed product. For $x \in \mathcal{G}^{(0)}$, let B_x be a C^*-algebra, $\mathcal{E}_x$ a Hilbert B_x module and $\pi_x : A_x \to \mathcal{L}_{B_x}(\mathcal{E}_x)$ a representation of the fibre A_x. Consider the Hilbert module B_x-module $L^2(\mathcal{G}^{(x)}, \mathcal{E}_x)$ where $\mathcal{G}^{(x)} = r^{-1}(x)$. For $f \in \Gamma_c(\mathcal{G}, r^*\mathcal{A})$, let $\Lambda_{x,\pi_x}(f)$ be the operator on $L^2(\mathcal{G}^{(x)}, \mathcal{E}_x)$ defined by the formula

$$(\Lambda_{x,\pi_x}(f)\xi)(\gamma) = \sum_{r(\gamma_1)=r(\gamma)=x} \pi_x\big(\alpha_\gamma(f(\gamma^{-1}\gamma_1))\big)\xi(\gamma_1)$$

for $\xi \in L^2(\mathcal{G}^{(x)}, \mathcal{E}_x)$ and $\gamma \in \mathcal{G}^{(x)}$. Then Λ_{x,π_x} is a bounded representation of $\Gamma_c(\mathcal{G}, r^*\mathcal{A})$ on the Hilbert module $L^2(\mathcal{G}^{(x)}, \mathcal{E}_x)$. We call the representation Λ_{x,π_x} the representation induced by π_x.
Let I be a non-empty set and $\tau : I \to \mathcal{G}^{(0)}$ be a map. For $i \in I$, let $\pi_i : A_{\tau(i)} \to \mathcal{L}_{B_i}(\mathcal{E}_i)$ be a representation. We say that the collection $\{\pi_i : i \in I\}$ is faithful if the map

$$\Gamma_0(\mathcal{G}^{(0)}, \mathcal{A}) \ni f \to \bigoplus_{i\in I} \pi_i(f(\tau(i))) \in \bigoplus_{i\in I} \mathcal{L}_{B_i}(\mathcal{E}_i)$$

is faithful.
Let $\{\pi_i : i \in I\}$ be a faithful family. For $f \in \Gamma_c(\mathcal{G}, r^*\mathcal{A})$, let

$$||f||_{red} := \sup_{i\in I} ||\Lambda_{\tau(i),\pi_i}(f)||.$$

Then $||\ ||_{red}$ is a norm on $\Gamma_c(\mathcal{G}, r^*\mathcal{A})$ and is independent of the map $\tau : I \to \mathcal{G}^{(0)}$ and also of the choice of the representations $\{\pi_i : i \in I\}$. The completion of $\Gamma_c(\mathcal{G}, r^*\mathcal{A})$ with respect to the reduced norm $||\ ||_{red}$ is called the reduced crossed product and is denoted $\mathcal{A} \rtimes_{red} \mathcal{G}$.

3 Wiener-Hopf algebras

Throughout this section, let G be a discrete countable group and $P \subset G$ a subsemigroup containing the identity element e. For a C^*-algebra A, let $End(A)$ be the set of $*$-algebra homomorphisms on A. A map $\alpha : P \to End(A)$ is called a left action of P on A if

(1) for $a, b \in P$, $\alpha_a\alpha_b = \alpha_{ab}$, and

(2) for $x \in A$, $\alpha_e(x) = x$.

If A is unital and if α_a is unital for every $a \in P$, then we call the action α unital.
Consider the Hilbert space $\ell^2(P)$. Let $\{\delta_a : a \in P\}$ be the standard orthonormal basis of $\ell^2(P)$. For $g \in G$, let w_g be the partial isometry on $\ell^2(P)$ defined

as

$$w_g(\delta_b) = \begin{cases} \delta_{bg} & \text{if } bg \in P, \\ \\ 0 & \text{if } bg \notin P. \end{cases} \tag{3.1}$$

Let A be a C^*-algebra and $\alpha : P \to End(A)$ be a left action. Consider the Hilbert A-module $\mathcal{E} := A \otimes \ell^2(P)$. For $g \in G$ and $x \in A$, let W_g and $\pi(x)$ be the operators on $\mathcal{E}$ defined by

$$W_g := 1 \otimes w_g,$$
$$\pi(x)(y \otimes \delta_b) := \alpha_b(x)y \otimes \delta_b.$$

Observe the following.

(1) For $g \in G$, W_g is a partial isometry. Denote the final projection $W_gW_g^*$ by E_g. Then $\{E_g : g \in G\}$ is a commuting family of projections.

(2) For $g \in G$, $W_g^* = W_{g^{-1}}$. Also $W_gW_h = W_{hg}E_{h^{-1}}$ for $g, h \in G$. By taking adjoints, we also have $W_gW_h = E_gW_{hg}$ for $g, h \in G$.

(3) For $a \in P$, let $V_a := W_a$. Then V_a is an isometry for $a \in P$. Moreover the map $P \ni a \to V_a \in \mathcal{L}_A(\mathcal{E})$ is an anti-homomorphism i.e. $V_aV_b = V_{ba}$ and $V_e = Id$.

(4) Observe that $\pi(x)V_a = V_a\pi(\alpha_a(x))$ for $a \in P$ and $x \in A$. Also note that $\pi(x)$ commutes with E_g for every $x \in A$ and $g \in G$.

Let $\mathcal{W}(A, P, G, \alpha)$ be the C^*-algebra generated by $\{\pi(x)W_g : x \in A, g \in G\}$. We call the C^*-algebra $\mathcal{W}(A, P, G, \alpha)$ as the Wiener-Hopf algebra associated to the quadruple (A, P, G, α).

Remark 3.1 *Before proceeding further, let us note that $\mathcal{W}(A, P, G, \alpha)$ can alternatively be defined as follows. Let $\rho : A \to B(\mathcal{H})$ be a faithful representation of A on a Hilbert space $\mathcal{H}$. Consider the Hilbert space $\widetilde{\mathcal{H}} := \mathcal{H} \otimes \ell^2(P)$. For $g \in G$, let $\widetilde{W}_g = 1 \otimes w_g$. For $x \in \mathcal{H}$, let $\widetilde{\rho}(x) \in B(\widetilde{\mathcal{H}})$ be defined by $\widetilde{\rho}(x)(\xi \otimes \delta_a) = \rho(\alpha_a(x))\xi \otimes \delta_a$. Then $\mathcal{W}(A, P, G, \alpha)$ is isomorphic to the C^*-algebra generated by $\{\widetilde{\rho}(x)\widetilde{W}_g : x \in A, g \in G\}$. We omit the proof of this fact.*
We only indicate that this follows by Rieffel's induction. Note that by Rieffel's induction, we obtain a faithful representation of $\mathcal{W}(A, P, G, \alpha)$ on the Hilbert space $(A \otimes \ell^2(P)) \otimes_A \mathcal{H}$ and the map $(A \otimes \ell^2(P)) \otimes_A \mathcal{H} \ni (x \otimes \delta_a) \otimes \xi \to \rho(x)\xi \otimes \delta_a \in \widetilde{\mathcal{H}}$ is an isometric embedding. The remaining details are left to the reader.

Remark 3.2 *When $A = \mathbb{C}$, the C^*-algebra $\mathcal{W}(A, P, G, \alpha)$ is the usual Wiener-Hopf algebra, denoted $\mathcal{W}(P, G)$, whose study from the groupoid perspective was initiated in [8]. Our aim in this article is to perform a similar analysis for $\mathcal{W}(A, P, G, \alpha)$.*

From now on, we will drop the symbol π and simply write x in place of $\pi(x)$. Recall the C^*-algebra $A \rtimes_{red} P$ defined in the introduction i.e. $A \rtimes_{red} P$ is the C^*-algebra generated by $\{xV_{a_1}^*V_{b_1}V_{a_2}^* \cdots V_{a_n}^*V_{b_n} : x \in A, a_i, b_i \in P, n \in \mathbb{N}\}$. First we prove that $A \rtimes_{red} P \subset \mathcal{W}(A,P,G,\alpha)$. This is the content of the next lemma.

Lemma 3.3 *For $T \in \mathcal{W}(A,P,G,\alpha)$ and $g \in G$, $TW_g, W_gT \in \mathcal{W}(A,P,G,\alpha)$. Consequently $A \rtimes_{red} P \subset \mathcal{W}(A,P,G,\alpha)$.*

Proof. Observe that $\mathcal{W}(A,P,G,\alpha)$ is generated by $\{xW_h : x \geq 0, h \in G\}$ and that $\{W_g : g \in G\}$ is $*$-closed. Thus it suffices to prove that for $g,h \in G$ and $x \in A$ positive, $xW_hW_g, W_gxW_h \in \mathcal{W}(A,P,G,\alpha)$.
Let $x \in A$ be positive and $g,h \in G$ be given. Write $x = y^3$ with y positive. Now note that

$$\begin{aligned}(yW_h)(yW_h)^*yW_{gh} &= yE_hy^2W_{gh}\\ &= y^3E_hW_{gh} \ \ \text{(since } E_h \text{ commutes with } y)\\ &= xW_hW_g \ \ \text{(since } W_hW_g = E_hW_{gh}).\end{aligned}$$

Hence $xW_hW_g \in \mathcal{W}(A,P,G,\alpha)$. Note that $W_gxW_h = (yW_{g^{-1}})^*(y^2W_h) \in \mathcal{W}(A,P,G,\alpha)$. Thus $\mathcal{W}(A,P,G,\alpha)$ is closed under right and left multiplication by $\{W_g : g \in G\}$. Now the inclusion $A \rtimes_{red} P \subset \mathcal{W}(A,P,G,\alpha)$ is immediate. □

When $A = \mathbb{C}$, $\mathbb{C} \rtimes_{red} P$ is the reduced C^*-algebra of the semigroup P, denoted $C^*_{red}(P)$ (See [6], [7]). Up to the author's knowledge, even for the case $A = \mathbb{C}$, it is not known whether the equality $C^*_{red}(P) = \mathcal{W}(P,G)$ holds or not. We discuss two situations where the equality $A \rtimes_{red} P = \mathcal{W}(A,P,G,\alpha)$ does hold.
Nica's quasi-lattice ordered semigroups: Recall from [12], the pair (P,G) is called quasi-lattice ordered if $P \cap P^{-1} = \{e\}$ and the following holds: Let $g \in G$. If $Pg \cap P$ is non-empty, then there exists $a \in P$ such that $Pg \cap P = Pa$. Let (P,G) be quasi-lattice ordered and let $\alpha : P \to End(A)$ be an action of P on a C^*-algebra A. First observe that for $g \in G$, $W_g \neq 0$ if and only if $Pg \cap P \neq \emptyset$. If $Pg \cap P = Pa$ then $W_g = V_aV_b^*$ where $b = ag^{-1}$.
Thus $\mathcal{W}(A,P,G,\alpha)$ is the C^*-algebra generated by $\{xV_aV_b^* : x \in A, a, b \in P\}$ which is contained in $A \rtimes_{red} P$. Consequently $\mathcal{W}(A,P,G,\alpha) = A \rtimes_{red} P$. Note that the linear span of $\{V_axV_b^* : x \in A, a, b \in P\}$ is a dense $*$-subalgebra of $\mathcal{W}(A,P,G,\alpha)$. To see this, note that if $Pa \cap Pb = \emptyset$, then $V_b^*V_a = 0$. If $Pa \cap Pb = Pc$ then $V_b^*V_a = V_tV_s^*$ where $s,t \in P$ are such that $sa = tb = c$. Let $a_1, b_1, a_2, b_2 \in P$ and $x, y \in A$. Then $(V_{a_1}xV_{b_1}^*)(V_{a_2}yV_{b_2}^*) = 0$ if $Pb_1 \cap Pa_2 = \emptyset$. If $Pb_1 \cap Pa_2 = Pc$ then $(V_{a_1}xV_{b_1}^*)(V_{a_2}yV_{b_2}^*) = V_{sa_1}\alpha_s(x)\alpha_t(y)V_{tb_2}^*$ where $s,t \in P$ are such that $sb_1 = ta_2 = c$. We also note that when $P = \mathbb{N}$ and $G = \mathbb{Z}$, $\mathcal{W}(A,\mathbb{N},\mathbb{Z},\alpha)$ agrees with the Toeplitz algebra $\mathcal{T}_\alpha$ considered in [4].

Remark 3.4 *Nica in [12] considers left regular representations whereas we consider the right regular one. We consider here a "right" variant of Nica's definition of a quasi lattice ordered pair.*

Ore semigroups: A subsemigroup $P \subset G$ is called a right Ore subsemigroup of G if $PP^{-1} = G$. Let P be a right Ore subsemigroup of G and let $\alpha : P \to End(A)$ be an action of P on a C^*-algebra A. First note that if $g = ab^{-1}$ with $a, b \in P$ then $W_g = V_b^* V_a$. Hence $\mathcal{W}(A, P, G, \alpha)$ is the C^*-algebra generated by $\{xV_a^* V_b : a, b \in P\} \subset A \rtimes_{red} P$. This implies that $\mathcal{W}(A, P, G, \alpha) = A \rtimes_{red} P$.
We consider now a unitisation procedure. Let A be a C^*-algebra and $\alpha : P \to End(A)$ be a left action. Let $A^+ := A \oplus \mathbb{C}$ be the unitisation of A. For $a \in P$, let $\alpha_a^+ : A^+ \to A^+$ be defined by $\alpha_a^+(x, \lambda) = (\alpha_a(x), \lambda)$. Then $\alpha^+ : P \to End(A^+)$ is a unital left action. We call (A^+, P, G, α^+) as the unitisation of (A, P, G, α).

Lemma 3.5 *Let A be a C^*-algebra and $\alpha : P \to End(A)$ be a left action. We have the following short exact sequence*

$$0 \to \mathcal{W}(A, P, G, \alpha) \to \mathcal{W}(A^+, P, G, \alpha^+) \to \mathcal{W}(P, G) \to 0.$$

Also the above short exact sequence is split-exact.

Proof. Let ρ be a faithful representation of A^+ on a Hilbert space $\mathcal{H}$. Then by Remark 3.1, it follows that $\mathcal{W}(A^+, P, G, \alpha^+)$ is the C^*-algebra generated by $\{\widetilde{\rho}(x) : x \in A\}$ and $\{\widetilde{W}_g : g \in G\}$ where for $x \in A$, $\widetilde{\rho}(x)$ and $\widetilde{W}_g$ are defined as in Remark 3.1. Since ρ is faithful on A, it follows by Remark 3.1 that $\mathcal{W}(A, P, G, \alpha)$ is the C^*-algebra generated by $\{\widetilde{\rho}(x)\widetilde{W}_g : x \in A, g \in G\}$. Clearly $\mathcal{W}(A, P, G, \alpha)$ is an ideal in $\mathcal{W}(A^+, P, G, \alpha^+)$.
Now consider the Hilbert A^+-module $\mathcal{E}^+ := A^+ \otimes \ell^2(P)$. For $g \in G$, let $W_g = 1 \otimes w_g$. For $x \in A$, let $\pi^+(x)$ be the operator on $\mathcal{E}^+$ defined by

$$\pi^+(x)((y, \lambda) \otimes \delta_a) = (\alpha_a(x)y + \lambda\alpha_a(x), 0) \otimes \delta_a.$$

Then the argument in the preceeding paragraph implies that $\mathcal{W}(A, P, G, \alpha)$ is the C^*-algebra generated by $\{\pi^+(x)W_g : x \in A, g \in G\}$ and is an ideal in $\mathcal{W}(A^+, P, G, \alpha)$. By definition, $\mathcal{W}(A^+, P, G, \alpha)$ is the C^*-algebra generated by $\{\pi^+(x) : x \in A\}$ and $\{W_g : g \in G\}$.
Denote the map $A^+ \ni (x, \lambda) \to \lambda \in \mathbb{C}$ by ϵ. Clearly the internal tensor product $\mathcal{E}^+ \otimes_\epsilon \mathbb{C} \cong \ell^2(P)$. Denote the map $\mathcal{L}_{A^+}(\mathcal{E}^+) \ni T \to T \otimes 1 \in B(\mathcal{E}^+ \otimes_\epsilon \mathbb{C}) \cong B(\ell^2(P))$ by $\widetilde{\epsilon}$. Then $\widetilde{\epsilon}$ vanishes on $\mathcal{W}(A, P, G, \alpha)$ and the C^*-algebra $\mathcal{W}(A^+, P, G, \alpha^+)$ is mapped onto the Wiener-Hopf algebra $\mathcal{W}(P, G)$. Let $\sigma : \mathcal{W}(P, G) \to \mathcal{W}(A^+, P, G, \alpha^+)$ be the $*$-homomorphism such that $\sigma(w_g) = W_g$ for every $g \in G$. Observe that $\widetilde{\epsilon} \circ \sigma = id$.
We claim that the kernel of the map $\widetilde{\epsilon} : \mathcal{W}(A^+, P, G, \alpha^+) \to \mathcal{W}(P, G)$ is $\mathcal{W}(A, P, G, \alpha)$. With our notations $\mathcal{W}(A^+, P, G, \alpha^+)$ is generated by $\{\pi^+(x) : x \in A\}$ and $\sigma(\mathcal{W}(P, G))$. Observe that $T - \sigma(\widetilde{\epsilon}(T)) \in \mathcal{W}(A, P, G, \alpha)$ for $T \in \mathcal{W}(A^+, P, G, \alpha^+)$ and $\widetilde{\epsilon}$ is one-one on $\sigma(\mathcal{W}(P, G))$. Thus, as a vector space, $\mathcal{W}(A^+, P, G, \alpha^+)$ is the direct sum of $\mathcal{W}(A, P, G, \alpha)$ and $\sigma(\mathcal{W}(P, G))$. Also $\widetilde{\epsilon}$ vanishes on $\mathcal{W}(A, P, G, \alpha)$. Now it is immediate that the kernel of $\widetilde{\epsilon}$ is $\mathcal{W}(A, P, G, \alpha)$. This completes the proof. □
We end this section by discussing the Wiener-Hopf groupoid considered in [8]. What follows regarding the Wiener-Hopf groupoid can be found either in [8]

or in [13] in one form or another. But for the sake of completeness, we include proofs. The Wiener-Hopf groupoid will play a prominent role in our analysis of the algebra $\mathcal{W}(A,P,G,\alpha)$.

Let $\mathcal{C}(G)$ denote the set of all subsets of G. We identify $\mathcal{C}(G)$ with the product $\{0,1\}^G$ by identifying a subset of G with its characteristic function. We endow $\{0,1\}^G$ and thus $\mathcal{C}(G)$ with the product topology. Endowed with this product topology, $\mathcal{C}(G)$ is a compact Hausdorff space. Also G acts on $\mathcal{C}(G)$ by translation. The action of G on $\mathcal{C}(G)$ is given by $\mathcal{C}(G)\times G \ni (A,g) \to Ag \in \mathcal{C}(G)$. We denote the closure of $\{P^{-1}a : a \in P\}$ in $\mathcal{C}(G)$ by Ω and we call it the Wiener-Hopf compactification of the pair (P,G). Note that Ω is compact. Let

$$\widetilde{\Omega} := \{Ag : A \in \Omega, g \in G\}.$$

Then $\widetilde{\Omega}$ is G-invariant.

The basic facts concerning the spaces $\widetilde{\Omega}$ and Ω are summarised in the next lemma.

Lemma 3.6 *With the foregoing notations, we have the following.*

1. *Let $A \in \widetilde{\Omega}$ and $g \in G$. Then $Ag \in \Omega$ if and only if $g^{-1} \in A$.*
2. *The set Ω is a compact open subset of $\widetilde{\Omega}$.*
3. *The space $\widetilde{\Omega}$ is locally compact.*
4. *The C^*-subalgebra generated by $\{1_{\Omega g} : g \in G\}$ is dense in $C_0(\widetilde{\Omega})$.*

Proof. Observe that if $B \in \Omega$ then, it follows from the definition that, $e \in B$ and $P^{-1}B \subset B$. Now let $A \in \widetilde{\Omega}$ and $g \in G$ be given. Suppose $Ag \in \Omega$. Since $e \in Ag$, it follows that $g^{-1} \in A$. Now assume $g^{-1} \in A$. Write $A = Bh$ with $B \in \Omega$ and $h \in G$. Let (a_n) be a sequence in P such that $P^{-1}a_n \to B$. Now $g^{-1}h^{-1} \in B$. Thus there exists $N \in \mathbb{N}$ such that $g^{-1}h^{-1}a_n^{-1} \in P^{-1}$ for $n \geq N$. For $n \geq N$, let $b_n \in P$ be such that $b_n = a_n hg$. Then $P^{-1}b_n \to Bhg = Ag$. Thus $Ag \in \Omega$. This proves (1).

Observe that (1) implies that $\Omega = \{A \in \widetilde{\Omega} : 1_A(e) = 1\}$. Thus Ω is open in $\widetilde{\Omega}$. By definition, Ω is compact. This proves (2). The local compactness of $\widetilde{\Omega}$ follows from (2) and by the equality $\widetilde{\Omega} = \bigcup_{g\in G}\Omega g$.

Note that by (1), for $A \in \widetilde{\Omega}$, $1_{\Omega g}(A) = 1_A(g)$. The last assertion is then an immediate consequence of the Stone-Weierstrass theorem. This completes the proof $\Box$.

Consider the transformation groupoid $\widetilde{\Omega} \rtimes G$. The Wiener-Hopf groupoid ([8]), let us denote by $\mathcal{G}$, is defined as the restriction of $\widetilde{\Omega} \rtimes G$ onto Ω. That is

$$\begin{aligned}\mathcal{G} :&= \{(A,g) \in \widetilde{\Omega} \rtimes G : A \in \Omega, Ag \in \Omega\} \\ &= \{(A,g) : A \in \Omega, g^{-1} \in A\}\end{aligned}$$

where the groupoid multiplication and the inversion are defined as

$$(A,g)(Ag,h) = (A,gh),$$
$$(A,g)^{-1} = (Ag,g^{-1}).$$

The topology on $\mathcal{G}$ is the subspace topology inherited from the product topology on $\widetilde{\Omega} \times G$. Note that $\mathcal{G}$ is an r-discrete groupoid. Observe that for an open subset $U \subset \widetilde{\Omega}$ and $g \in G$,

$$r((U \times \{g\}) \cap \mathcal{G}) = \{A \in U \cap \Omega : 1_A(g^{-1}) = 1\}.$$

Thus the range map r is an open map. Clearly r restricted to $(U \times \{g\}) \cap \mathcal{G}$ is 1-1. Hence r is a local homeomorphism.

4 A groupoid crossed product picture

Let A be a C^*-algebra and $\alpha : P \to End(A)$ be a left action. Let Ω be the Wiener-Hopf compactification of (P,G) and $\widetilde{\Omega} = \bigcup_{g \in G} \Omega g$. Denote the Wiener-Hopf groupoid $\widetilde{\Omega} \rtimes G|_{\Omega}$ by $\mathcal{G}$. In this section, we show that the Wiener-Hopf algebra $\mathcal{W}(A,P,G,\alpha)$ is isomorphic to a reduced crossed product of the form $\mathcal{D} \rtimes_{red} \mathcal{G}$. To that end, we imitate the construction due to Khoshkam and Skandalis carried out in [4].

Let $A^+ := A \oplus \mathbb{C}$ be the unitisation of A. Let $\ell^\infty(G,A^+)$ be the C^*-algebra of bounded functions on G taking values in A^+. The group G acts on $\ell^\infty(G,A^+)$ by translation. Let us denote the action of G on $\ell^\infty(G,A^+)$ by β. The action β is given by the formula: for $g \in G$, $f \in \ell^\infty(G,A^+)$ and $s \in G$, $\beta_g(f)(s) = f(sg)$.

For $g \in G$, let $j_g : A \to \ell^\infty(G,A^+)$ be the $*$-homomorphism defined as follows: for $x \in A$ and $h \in G$,

$$j_g(x)(h) := \begin{cases} \alpha_{hg^{-1}}(x) & \text{if } h \in Pg, \\ \\ 0 & \text{if } h \notin Pg. \end{cases}$$

Note that for $s \in G$, $g \in G$ and $x \in A$, $\beta_s(j_g(x)) = j_{gs^{-1}}(x)$.

For $\phi \in C_0(\widetilde{\Omega})$, let $\widehat{\phi} \in \ell^\infty(G,A^+)$ be defined by $\widehat{\phi}(g) = \phi(P^{-1}g)$ for $g \in G$. Note that the map $C_0(\widetilde{\Omega}) \ni \phi \to \widehat{\phi} \in \ell^\infty(G,A^+)$ is G-equivariant. Since $\{P^{-1}a : a \in P\}$ is dense in Ω, it follows that $\{P^{-1}g : g \in G\}$ is dense in $\widetilde{\Omega}$. Thus the $*$-algebra homomorphism $C_0(\widetilde{\Omega}) \ni \phi \to \widehat{\phi} \in \ell^\infty(G,A^+)$ is 1-1. Henceforth we will identify $C_0(\widetilde{\Omega})$ as a $*$-subalgebra of $\ell^\infty(G,A^+)$ and will simply denote $\widehat{\phi}$ as ϕ for $\phi \in C_0(\widetilde{\Omega})$.

Let $\widetilde{D}_0$ be the C^*-subalgebra of $\ell^\infty(G,A^+)$ generated by $\{j_g(x) : x \in A, g \in G\}$ and $C_0(\widetilde{\Omega})$. Denote the C^*-algebra generated by $\{j_g(x) : x \in A, g \in G\}$ by $\widetilde{D}_1$ and the C^*-algebra generated by $\{\phi j_g(x) : \phi \in C_0(\widetilde{\Omega}), g \in G, x \in A\}$ by $\widetilde{D}$. Observe that $\widetilde{D} \subset \ell^\infty(G,A)$. Moroever the commutative algebra C^*-algebra

$C_0(\widetilde{\Omega})$ is contained in the center of $\widetilde{D}_0$. Also $\widetilde{D}$ is an ideal in $\widetilde{D}_0$. Thus $\widetilde{D}$ is a $C_0(\widetilde{\Omega})$-algebra.

The action of $C_0(\widetilde{\Omega})$ is given by left multiplication. To see that the action is non-degenerate, note that by (1) of Lemma 3.6 $1_{\Omega g} j_g(x) = j_g(x)$ for $g \in G$ and $x \in A$. Hence $1_{\Omega g}\phi j_g(x) = \phi j_g(x)$ for $\phi \in C_0(\widetilde{\Omega})$, $g \in G$ and $x \in A$. Thus the action of $C_0(\widetilde{\Omega})$ on $\widetilde{D}$ is non-degenerate. Moreover the equality $1_{\Omega g} j_g(x) = j_g(x)$ implies that $\widetilde{D}_1$ is contained in $\widetilde{D}$.

Note that $\widetilde{D}$, $\widetilde{D}_0$ and $\widetilde{D}_1$ are G-invariant under the translation action β. Thus $\widetilde{D}$ is a $(\widetilde{\Omega}, G)$-algebra. Denote the corresponding upper-semicontinuous bundle on which the transformation groupoid $\widetilde{\Omega} \rtimes G$ acts by $\widetilde{\mathcal{D}}$. Denote the action of $\widetilde{\Omega} \rtimes G$ on $\widetilde{\mathcal{D}}$ by $\widetilde{\beta} := (\beta_{(X,g)})$. Recall that the fibre $\widetilde{D}_X$ at a point $X \in \widetilde{\Omega}$ is given by $\widetilde{D}/I_X$ where $I_X = C_0(\widetilde{\Omega}\backslash\{X\})\widetilde{D}$.

Let $\mathcal{D} := \coprod_{X\in\Omega} \widetilde{D}/I_X$ be the restriction of $\widetilde{\mathcal{D}}$ onto the subset Ω. Then the action $\widetilde{\beta}$ restricts to an action $\beta := (\beta_{(X,g)})$ of the Wiener-Hopf groupoid $\mathcal{G} := \widetilde{\Omega} \rtimes G|_{\Omega}$. Recall that the map $\beta_{(X,g)} : \widetilde{D}/I_{X.g} \to \widetilde{D}/I_X$ is given by

$$\beta_{(X,g)}(d + I_{X.g}) = \beta_g(d) + I_X$$

for $d \in \widetilde{D}$ and $(X, g) \in \mathcal{G}$. We use the same letter β to denote the action of G on $\ell^\infty(G, A^+)$ and the action of the Wiener-Hopf groupoid on $\mathcal{D}$. We call the groupoid dynamical system $(\mathcal{D}, \mathcal{G}, \beta)$, the Wiener-Hopf groupoid dynamical system associated to the quadruple (A, P, G, α).

Remark 4.1 *Note that the map*

$$\widetilde{D} \ni d \to (\Omega \ni X \to d + I_X \in \widetilde{D}/I_X) \in \Gamma(\Omega, \mathcal{D})$$

descends to an isomorphism between $\widetilde{D}/I_\Omega$ and $\Gamma(\Omega, \mathcal{D})$ where I_Ω is the closure of $C_0(\widetilde{\Omega}\backslash\Omega)\widetilde{D}$. Since Ω is compact, we will simply denote $\Gamma_0(\Omega, \mathcal{D})$ by $\Gamma(\Omega, \mathcal{D})$.

Remark 4.2 *Let A be a unital C^*-algebra and let $\alpha : P \to End(A)$ be a unital action. Denote the multiplicative unit of A by 1_A. Define $\delta : A^+ \to A$ by $\delta(a, \lambda) = a + \lambda 1_A$. Then δ is a $*$-homomorphism. Let $\widetilde{\delta} : \ell^\infty(G, A^+) \to \ell^\infty(G, A)$ be defined by $\widetilde{\delta}(\phi) = \delta \circ \phi$.*

1. *The map $\widetilde{\delta}$ restricted to $\widetilde{D}$ is injective. For, $\widetilde{\delta}$ is injective on $\ell^\infty(G, A)$ and $\widetilde{D}$ is contained in $\ell^\infty(G, A)$.*

2. *For $\phi \in C_0(\widetilde{\Omega})$, let $\overline{\phi} \in \ell^\infty(G, A)$ be defined by $\overline{\phi}(g) = \phi(P^{-1}g)1_A$ for $g \in G$. In otherwords, $\overline{\phi} = \widetilde{\delta}(\widehat{\phi})$. Note that the map $C_0(\widetilde{\Omega}) \ni \phi \to \overline{\phi} \in \ell^\infty(G, A)$ is 1-1.*

3. *Note that, by Lemma 3.6, for $g \in G$, $\overline{1_{\Omega g}} = \widetilde{\delta}(j_g(1_A))$. Thus by Part (4) of Lemma 3.6, it follows that $\widetilde{\delta}(\widetilde{D})$ is generated by $\{\widetilde{\delta}(j_g(x)) : g \in G, x \in A\}$ i.e. $\widetilde{\delta}(\widetilde{D}) = \widetilde{\delta}(\widetilde{D}_1)$.*

Thus in the unital case, we will suppress the notation $\widetilde{\delta}$ and simply denote $\widetilde{\delta}(j_g(x))$ by $j_g(x)$ and $\widetilde{\delta}(\widetilde{D})$ by $\widetilde{D}$. Moreover we will denote $j_g(1_A) = \overline{1_{\Omega g}}$ by $1_{\Omega g}$ and consider $C_0(\widetilde{\Omega})$ as a C^*-subalgebra of $\widetilde{D}$.
In short, in the unital case, it is not necessary to pass to the unitisation A^+.

Now we can state our main theorem.

Theorem 4.3 *With the foregoing notations, the C^*-algebra $\mathcal{W}(A,P,G,\alpha)$ is isomorphic to the reduced groupoid crossed product $\mathcal{D} \rtimes_{red} \mathcal{G}$.*

First we fix some notations. For $g \in G$ and $d \in \widetilde{D}$, let $W_{d,g} \in \Gamma_c(\mathcal{G}, r^*\mathcal{D})$ be defined by

$$W_{d,g}(X,h) := \begin{cases} d + I_X & \text{if } h = g, \\ \\ 0 + I_X & \text{if } h \neq g \end{cases}$$

for $(X,h) \in \mathcal{G}$. Note that the linear span of $\{W_{d,g} : d \in \widetilde{D}, g \in G\}$ is $\Gamma_c(\mathcal{G}, r^*\mathcal{D})$. Observe that for $g \in G$, the map $\widetilde{D} \ni d \to W_{d,g} \in \Gamma_c(\mathcal{G}, r^*\mathcal{D})$ is linear. With this notation, the subalgebra $\Gamma(\Omega, \mathcal{D}) \subset \Gamma_c(\mathcal{G}, r^*\mathcal{D})$ is $\{W_{d,e} : d \in \widetilde{D}\}$.
For $a \in P$, let $\widetilde{\pi}_a : \widetilde{D} \to A$ be the map defined by $\widetilde{\pi}_a(d) = d(a)$. Note that $\widetilde{\pi}_a$ vanishes on $C_0(\widetilde{\Omega}\backslash\{P^{-1}a\})\widetilde{D}$ and descends to a $*$-homomorphism from $\widetilde{D}/I_{P^{-1}a}$ to A. We denote the resulting map by π_a.
Claim: The map $\Gamma(\Omega, \mathcal{D}) \ni f \to \bigoplus_{a \in P} \pi_a(f(P^{-1}a)) \in \bigoplus_{a \in P} A$ is injective. By Remark 4.1, it is enough to show that the kernel of the map $\widetilde{D} \ni d \to (d(a))_{a \in P} \in \bigoplus_{a \in P} A$ is I_Ω. Let $d \in \widetilde{D}$ be such that $d(a) = 0$ for every $a \in P$. Let $\epsilon > 0$ be given. Choose $\phi \in C_0(\widetilde{\Omega})$ such that $||d - \phi d|| \leq \epsilon$. Since $d(a) = 0$ for every $a \in P$, it follows that $1_\Omega d = 0$. Now $\phi d = \phi(1 - 1_\Omega)d$. Clearly $\phi(1 - 1_\Omega) \in C_0(\widetilde{\Omega}\backslash\Omega)$. Thus $\phi d \in I_\Omega$. Since I_Ω is a closed ideal, it follows that $d \in I_\Omega$. This proves the claim. Let us isolate the just proved fact in a remark for later purposes.

Remark 4.4 *The map $\widetilde{D}/I_\Omega \ni d + I_\Omega \to (d(a))_{a \in P} \in \ell^\infty(P,A)$ is injective.*

By Lemma 3.6, it follows that $\mathcal{G}^X = r^{-1}(X) = \{(P^{-1}a, a^{-1}b) : b \in P\}$ for $X = P^{-1}a$. Recall from Section 1, the representation $\Lambda_{P^{-1}a,\pi_a}$ of $\Gamma_c(\mathcal{G}, r^*\mathcal{D})$ on the Hilbert A-module $\ell^2(\mathcal{G}^X, A) \cong \ell^2(a^{-1}P) \otimes A$ induced by the representation π_a. For $a \in P$, we will denote the representation $\Lambda_{P^{-1}a,\pi_a}$ simply by Λ_a.
A direct verification yields that for $d \in \widetilde{D}$, $g \in G$, the operator $\Lambda_a(W_{d,g})$ on the Hilbert A-module $\ell^2(a^{-1}P) \otimes A$ is given by the formula:

$$\Lambda_a(W_{d,g})(\delta_{a^{-1}b} \otimes y) = \begin{cases} \delta_{a^{-1}bg^{-1}} \otimes d(bg^{-1})y & \text{if } b \in Pg \\ \\ 0 & \text{if } b \notin Pg. \end{cases} \tag{4.2}$$

We leave the verification of the above expression to the reader.
For $a \in P$, let $U_a : \ell^2(P) \otimes A \to \ell^2(a^{-1}P) \otimes A$ be the unitary defined by the equation $U_a(\delta_b \otimes y) := \delta_{a^{-1}b} \otimes y$. For $d \in \ell^\infty(G, A^+)$, let $\pi(d)$ be the 'multiplication' operator on $\ell^2(P) \otimes A$ defined by

$$\pi(d)(\delta_b \otimes y) := \delta_b \otimes d(b)y.$$

Then π is a representation of $\ell^\infty(G, A^+)$ on the Hilbert A-module $\ell^2(P) \otimes A$. Then Eq.4.2 implies that for $a \in P$,

$$U_a^* \Lambda_a(W_{d,g}) U_a = \pi(d) W_g^* \tag{4.3}$$

for $d \in \widetilde{D}$ and $g \in G$. Here $W_g = w_g \otimes 1$ where w_g stands for the operator defined by Eq.3.1.

Proof of Theorem 4.3: Since the map $\Gamma(\Omega, \mathcal{D}) \ni f \to \pi_a(f(P^{-1}a)) \in \bigoplus_{a \in P} A$ is injective, the reduced norm on $\Gamma_c(\mathcal{G}, r^*\mathcal{D})$ is given by $||f||_{red} = \sup_{a \in P} ||\Lambda_a(f)||$. Also $\Gamma_c(\mathcal{G}, r^*\mathcal{A})$ is the linear span of $\{W_{d,g} : d \in \widetilde{D}, g \in G\}$. Thus if $f = \sum_i W_{d_i,g_i}$ is a finite sum, then by Eq.4.3, we have that $||f||_{red} = ||\sum_i \pi(d_i) W_{g_i}^*||$. This implies that the reduced crossed product $\mathcal{D} \rtimes_{red} \mathcal{G}$ is isomorphic to the C^*-subalgebra of $\mathcal{L}_A(\ell^2(P) \otimes A)$ generated by $\{\pi(d)W_g : d \in \widetilde{D}, g \in G\}$.

We claim that the C^*-algebra generated by $\{\pi(d)W_g : d \in \widetilde{D}, g \in G\}$ is $\mathcal{W}(A, P, G, \alpha)$. Let $\mathcal{T}$ stands for the C^*-algebra generated by $\{\pi(d)W_g : d \in \widetilde{D}, g \in G\}$. Note that for $x \in A$ and $g \in G$, $xW_g = \pi(j_e(x))W_g$. Hence $\mathcal{W}(A, P, G, \alpha) \subset \mathcal{T}$.

Observe that for $g \in G$ and $x \in A$, $\pi(1_{\Omega g}) = E_g := W_g W_g^*$ and $\pi(j_g(x)) = W_g x W_g^*$. Since $C_0(\widetilde{\Omega})$ is generated by $\{1_{\Omega g} : g \in G\}$, it follows that the linear span of products of the form $1_{\Omega g_1} 1_{\Omega g_2} \cdots 1_{\Omega g_m} j_{h_1}(x_1) \cdots j_{h_n}(x_n)$ forms a dense $*$-subalgebra of $\widetilde{D}$. Let $d := 1_{\Omega g_1} 1_{\Omega g_2} \cdots 1_{\Omega g_m} j_{h_1}(x_1) \cdots j_{h_n}(x_n) \in \widetilde{D}$ be such a product. Then

$$\pi(d) = E_{g_1} E_{g_2} \cdots E_{g_m} (W_{h_1} x_1 W_{h_1}^*) \cdots (W_{h_n} x_n W_{h_n}).$$

A repeated application of Lemma 3.3 implies that $\pi(d) \in \mathcal{W}(A, P, G, \alpha)$. Hence the image of $\widetilde{D}$ under π is contained in $\mathcal{W}(A, P, G, \alpha)$. Another application of Lemma 3.3 implies that $\{\pi(d)W_g : d \in \widetilde{D}, g \in G\} \subset \mathcal{W}(A, P, G, \alpha)$. As a consequence, it follows that $\mathcal{T} \subset \mathcal{W}(A, P, G, \alpha)$. The proof is now complete. $\Box$.

We end this section by recording the relations satsified by $\{W_{d,g} : d \in \widetilde{D}, g \in G\}$.

Lemma 4.5 *We have the following.*

(1) *For* $d \in \widetilde{D}$ *and* $g \in G$, $W_{d,g} = 0$ *if and only if* $1_{\Omega g^{-1}} 1_\Omega d = 0$. *Thus for every* $d \in \widetilde{D}$ *and* $g \in G$, $W_{d,g} = W_{1_\Omega 1_{\Omega g^{-1}} d, g}$.

(2) *For* $d \in \widetilde{D}$ *and* $g \in G$, $W_{d,g}^* = W_{\beta_g^{-1}(d^*), g^{-1}}$.

(3) *For* $d_1, d_2 \in \widetilde{D}$ *and* $g_1, g_2 \in G$, $W_{d_1,g_1} W_{d_2,g_2} = W_{d, g_1 g_2}$ *where* $d = 1_{\Omega g_1^{-1}} d_1 \beta_{g_1}(d_2)$.

(4) *The map* $\widetilde{D} \ni d \to W_{d,e} \in \mathcal{D} \rtimes \mathcal{G}$ *is a* $*$*-homomorphism whose kernel is* I_Ω.

Proof. Let $d \in \widetilde{D}$ and $g \in G$ be given. Observe that

$$1_{\Omega g^{-1}} 1_\Omega d + I_X = 1_{\Omega g^{-1}}(X) 1_\Omega(X)(d + I_X)$$

for every $X \in \widetilde{\Omega}$. Now $W_{d,g} = 0$ if and only if $d + I_X = 0$ whenever $X \in \Omega$ and $Xg \in \Omega$. In other words, $W_{d,g} = 0$ if and only if $1_{\Omega g^{-1}}(X) 1_\Omega(X)(d + I_X) = 0$ for every $X \in \widetilde{\Omega}$. Hence $W_{d,g} = 0$ if and only if $1_{\Omega g^{-1}} 1_\Omega d = 0$. This proves (1). The second assertion of (1) follows from the fact that for $g \in G$, the map $\widetilde{D} \ni d \to W_{d,g} \in \Gamma_c(\mathcal{G}, r^*\mathcal{D})$ is linear.
We leave the verification of (2) to the reader.
Let $d_1, d_2 \in \widetilde{D}$ and $g_1, g_2 \in G$ be given. Observe that for $(X, g) \in \mathcal{G}$,

$$\begin{aligned} W_{d_1,g_1} W_{d_2,g_2}(X, g) &= \sum_{(X,s) \in \mathcal{G}} W_{d_1,g_1}(X, s) \beta_{(X,s)}(W_{d_2,g_2}(Xs, s^{-1}g)) \\ &= \sum_{s \in G} 1_{\Omega s^{-1}}(X) W_{d_1,g_1}(X, s) \beta_{(X,s)}(W_{d_2,g_2}(Xs, s^{-1}g)). \end{aligned}$$

The above equation implies that $W_{d_1,g_1} W_{d_2,g_2}(X, g) = 0$ if $g \neq g_1 g_2$. Moreover the same equation implies that

$$\begin{aligned} W_{d_1,g_1} W_{d_2,g_2}(X, g_1 g_2) &= 1_{\Omega g_1^{-1}}(X)(d_1 + I_X)(\beta_{g_1}(d_2) + I_X) \\ &= 1_{\Omega g_1^{-1}} d_1 \beta_{g_1}(d_2) + I_X. \end{aligned}$$

This proves (3).
Using (1) and (3), note that that for $d_1, d_2 \in \widetilde{D}$,

$$\begin{aligned} W_{d_1,e} W_{d_2,e} &= W_{1_\Omega d_1 d_2, e} \\ &= W_{d_1 d_2, e}. \end{aligned}$$

Thus the map $\widetilde{D} \ni d \to W_{d,e} \in \mathcal{D} \rtimes \mathcal{G}$ is multiplicative. That it preserves the adjoint follows from (2). Now let $d \in \widetilde{D}$. Note that $W_{d,e} = 0$ if and only if $1_\Omega d = 0$ i.e. if and only if $d(a) = 0$ for every $a \in P$. By Remark 4.4, it follows that the kernel of the map $\widetilde{D} \ni d \to W_{d,e} \in \mathcal{D} \rtimes \mathcal{G}$ is I_Ω. This completes the proof. □.

5 Quasi-lattice ordered case

In this section, we show that the full groupoid crossed product $\mathcal{D} \rtimes \mathcal{G}$ admits a presentation in terms of generators and relations when (P, G) is quasi-lattice ordered and the action $\alpha : P \to End(A)$ is unital. Throughout this section, we assume that (P, G) is quasi-lattice ordered , A is a unital C^*-algebra and $\alpha : P \to End(A)$ is a unital action.

Definition 5.1 *Let B be a unital C^*-algebra. Denote the isometries of B by $\mathcal{V}(B)$. Let $\pi : A \to B$ be a $*$-homomorphism and $V : P \to \mathcal{V}(B)$ be an anti-homomorphism i.e. $V_e = 1$ and $V_aV_b = V_{ba}$ for $a, b \in P$. The pair (π, V) is called Nica-covariant if*

(1) for $x \in A$ and $a \in P$, $\pi(x)V_a = V_a\pi(\alpha_a(x))$,

(2) for $a, b \in P$,

$$E_aE_b := \begin{cases} E_c & \text{if } Pa \cap Pb = Pc, \\ \\ 0 & \text{if } Pa \cap Pb = \emptyset. \end{cases}$$

where $E_a := V_aV_a^*$ for $a \in P$. If (π, V) is Nica-covariant, we say (π, V) is a Nica-covariant representation of the triple (A, P, α).

It is convenient to introduce the following universal algebra. We must remark that this universal C^*-algebra was considered by Nica in [12] when $A = \mathbb{C}$.

Definition 5.2 *We let $A \rtimes P$ be the universal unital C^*-algebra generated by a unital copy of A and isometries $\{v_a : a \in P\}$ such that*

(C1) for $x \in A$ and $a \in P$, $xv_a = v_a\alpha_a(x)$,

(C2) for $a, b \in P$, $v_av_b = v_{ba}$, and

(C3) for $a, b \in P$,

$$e_ae_b := \begin{cases} e_c & \text{if } Pa \cap Pb = Pc, \\ \\ 0 & \text{if } Pa \cap Pb = \emptyset. \end{cases}$$

where $e_a := v_av_a^*$ for $a \in P$.

Let (π, V) be a -covariant representation of the triple (A, P, α) on a unital C^*-algebra B and assume that π is unital. Then there exists a unique $*$-homomorphism $\pi \rtimes V : A \rtimes P \to B$ such that $(\pi \rtimes V)(x) = \pi(x)$ and $(\pi \rtimes V)(v_a) = V_a$ for $a \in P$.

Remark 5.3 *Consider the pair (π, V) constructed in Section 3 on the Hilbert module A-module $A \otimes \ell^2(P)$. Recall that for $x \in A$ and $a \in P$, the operators $\pi(x)$ and V_a are given by*

$$\pi(x)(y \otimes \delta_b) := \alpha_b(x)(y) \otimes \delta_b,$$
$$V_a(y \otimes \delta_b) := y \otimes \delta_{ba}.$$

It is immediate that (π, V) is Nica-covariant. As a consequence, it follows that $A \rtimes P$ is non-zero. In particular, we obtain a unital $*$-homomorphism $\rho : A \rtimes P \to A \rtimes_{red} P$ such that $\rho(x) = \pi(x)$ and $\rho(v_a) = V_a$ for every $x \in A$ and $a \in P$. Let us call the map ρ as the regular representation of $A \rtimes P$ and the pair (π, V) the standard Nica-covariant pair of (A, P, α).

The following relation satisfied by the isometries $\{v_a : a \in P\}$ is due to Nica. We include the proof for completeness. Let $a, b \in P$ be given. Then

$$v_a^* v_b := \begin{cases} v_{ca^{-1}} v_{cb^{-1}}^* & \text{if} Pa \cap Pb = Pc, \\ \\ 0 & \text{if } Pa \cap Pb = \emptyset. \end{cases} \tag{5.4}$$

To see this observe that if $Pa \cap Pb = \emptyset$ then $e_a e_b = v_a v_a^* v_b v_b^* = 0$ and consequenty $v_a^* v_b = 0$. Now suppose $Pa \cap Pb = Pc$. Choose $s, t \in P$ such that $sa = tb = c$. Calculate as follows to observe that

$$\begin{aligned} v_a^* v_b &= v_a^* v_a v_a^* v_b v_b^* v_b \\ &= v_a^* e_a e_b v_b \\ &= v_a^* e_c v_b \\ &= v_a^* v_{sa} v_{tb}^* v_b \\ &= v_a^* v_a v_s v_t^* v_b^* v_b \\ &= v_{ca^{-1}} v_{cb^{-1}}^*. \end{aligned}$$

Let us introduce some notation which will be used throughout this section. Consider the C^*-algebra $A \rtimes P$. For $g \in G$, let

$$w_g := \begin{cases} 0 & \text{if } Pg \cap P = \emptyset, \\ \\ v_a v_{ag^{-1}}^* & \text{if } Pg \cap P = Pa. \end{cases} \tag{5.5}$$

Note that for $g \in G$, w_g is a product of two partial isometries with commuting range projections. Hence w_g is a partial isometry for each $g \in G$. Also note that $w_g^* = w_{g^{-1}}$. For $g \in G$, let $e_g := w_g w_g^*$. Note that by definition if $w_g \neq 0$ then $e_g = e_a$ for some $a \in P$. Thus $\{e_g : g \in G\}$ forms a commuting family of projections. The relations satisfied by $\{w_g : g \in G\}$ and $\{e_g : g \in G\}$ are summarised in the following lemma.

Lemma 5.4 *With the foregoing notations,*

(1) for $g, h \in G$,

$$e_g e_h := \begin{cases} 0 & \text{if } Pg \cap Ph \cap P = \emptyset, \\ \\ e_c & \text{if } Pg \cap Ph \cap P = Pc, \end{cases}$$

(2) for $g,h \in G$, $w_gw_h = e_gw_{hg}$.

Proof. Let $g,h \in G$ be given. Suppose $Pg \cap Ph \cap P \neq \emptyset$. Then $Pg \cap P \neq \emptyset$ and $Ph \cap P \neq \emptyset$. Choose $a,b,c \in P$ be such that $Pg \cap P = Pa$ and $Ph \cap P = Pb$ and $Pg \cap Ph \cap P = Pa \cap Pb = Pc$. By definition, $e_g = e_a$ and $e_h = e_b$. Thus $e_ge_h = e_c$. Now suppose $Pg \cap Ph \cap P = \emptyset$. We claim that $e_ge_h = 0$. Suppose $e_ge_h \neq 0$. Then $e_g \neq 0$ and $e_h \neq 0$. Thus $Pg \cap P \neq \emptyset$ and $Ph \cap P \neq \emptyset$. Choose $a,b \in P$ such that $Pg \cap P = Pa$ and $Ph \cap P = Pb$. Note that by definition $e_g = e_a$ and $e_h = e_b$. Now $e_ge_h = e_ae_b \neq 0$ implies that $Pa \cap Pb = Pg \cap P \cap Ph$ is non-empty which is a contradiction. Thus if $Pg \cap Ph \cap P = \emptyset$ then $e_ge_h = 0$. This proves (1).

Let $g,h \in G$ be given. *Claim:* $w_gw_h = 0 \Leftrightarrow Phg \cap Pg \cap P = \emptyset \Leftrightarrow e_gw_{gh} = 0$. Note the following chain of equivalences.

$$\begin{aligned}
w_gw_h = 0 &\Leftrightarrow w_gw_hw_h^*w_g^* = w_ge_he_hw_g^* = 0\\
&\Leftrightarrow e_hw_{g^{-1}} = 0\\
&\Leftrightarrow e_hw_{g^{-1}}w_{g^{-1}}^*e_h = 0\\
&\Leftrightarrow e_he_{g^{-1}}e_h = e_he_{g^{-1}} = 0\\
&\Leftrightarrow Ph \cap Pg^{-1} \cap P = \emptyset \quad \text{(by (1))}\\
&\Leftrightarrow Phg \cap P \cap Pg = \emptyset\\
&\Leftrightarrow e_ge_{hg} = e_ge_{hg}e_g = e_gw_{hg}w_{hg}^*e_g = 0 \quad \text{(by (1))}\\
&\Leftrightarrow e_gw_{hg} = 0.
\end{aligned}$$

This proves the claim. Thus to prove (2), we can and will assume that $w_gw_h \neq 0$ and $e_gw_{hg} \neq 0$. This in particular implies that $Pg \cap P$, $Ph \cap P$, $Phg \cap P$ and $Phg \cap Pg \cap P$ are non-empty. Choose $a,b,c,d \in P$ such that $Pg \cap P = Pa$, $Ph \cap P = Pb$, $Phg \cap Pg \cap P = Pc$ and $Phg \cap P = Pd$. Note that $Pa \cap Pd = Pc$.

Let $r,s,t \in P$ be such that $rg = a$, $sh = b$ and $thg = d$. Now note that $Pr \cap Pb = Pag^{-1} \cap P \cap Ph = Pg^{-1} \cap P \cap Ph = Pcg^{-1}$. Hence $cg^{-1} \in P$. Also note that $Pa \cap Pd = Pg \cap P \cap Phg \cap P = Phg \cap Pg \cap P = Pc$.

Now compute as follows to observe that

$$\begin{aligned}
w_gw_h &= v_av_r^*v_bv_s^*\\
&= v_av_{cg^{-1}r^{-1}}v_{cg^{-1}b^{-1}}^*v_s^* \quad \text{(by Eq. 5.4)}\\
&= v_{cg^{-1}r^{-1}a}v_{cg^{-1}b^{-1}s}^*\\
&= v_{ca^{-1}a}v_{cg^{-1}h^{-1}}^*\\
&= v_cv_{c(hg)^{-1}}^*
\end{aligned}$$

and

$$\begin{aligned} e_g w_{hg} &= e_a v_d v_t^* \\ &= v_a v_a^* v_d v_t^* \\ &= v_a v_{ca^{-1}} v_{cd^{-1}}^* v_t^* \quad \text{(by Eq. 5.4)} \\ &= v_c v_{cd^{-1}t}^* \\ &= v_c v_{c(hg)^{-1}}^*. \end{aligned}$$

Hence $w_g w_h = e_g w_{hg}$. This proves (2) and the proof is complete. □.

Let $(\mathcal{D}, \mathcal{G}, \beta)$ be the Wiener-Hopf groupoid dynamical system associated to (A, P, G, α). Since we are dealing with the unital case, by Remark 4.2, the dynamical system $(\mathcal{D}, \mathcal{G}, \beta)$ is given as follows. For $g \in G$ and $x \in A$, let $j_g(x) \in \ell^\infty(G, A)$ be defined by

$$j_g(x)(h) := \begin{cases} \alpha_{hg^{-1}}(x) & \text{if } h \in Pg, \\ \\ 0 & \text{if } h \notin Pg. \end{cases}$$

Let $\widetilde{D}$ be the C^*-algebra generated by $\{j_g(x) : g \in G, x \in A\}$. The C^*-algebra $\widetilde{D}$ is invariant under the translation action β and contains $C_0(\widetilde{\Omega})$ as a C^*-subalgebra where we identify $C_0(\widetilde{\Omega})$ with the C^*-subalgebra generated by $\{j_g(1) : g \in G\}$. Thus $\widetilde{D}$ is a $(\widetilde{\Omega}, G)$ algebra.

Let $\mathcal{D} := \coprod_{X \in \Omega} \widetilde{D}/I_X$ be the corresponding upper semicontinuous bundle over Ω and let $(\beta_{(X,g)})_{(X,g)\in\mathcal{G}}$ be the action of the Wiener-Hopf groupoid $\mathcal{G} := \widetilde{\Omega} \rtimes G|_\Omega$ which is given by $\beta_{(X,g)} : \widetilde{D}/I_{Xg} \ni d + I_{Xg} \to \beta_g(d) + I_X \in \widetilde{D}/I_X$.

We claim that the linear span of $\{j_g(x) : g \in G, x \in A\}$ is a dense $*$-subalgebra of $\widetilde{D}$. To see this, let $g_1, g_2 \in G$ and $x_1, x_2 \in A$ be such that $j_{g_1}(x_1) j_{g_2}(x_2) \neq 0$. Then $Pg_1 \cap Pg_2 \neq \emptyset$. Since (P, G) is quasi-lattice ordered, it follows that there exists $g \in G$ such that $Pg_1 \cap Pg_2 = Pg$. Choose $a, b \in P$ such that $ag_1 = g$ and $bg_2 = g$. Then $j_{g_1}(x_1) j_{g_2}(x_2) = j_g(\alpha_a(x_1)\alpha_b(x_2))$.

Remark 5.5 *Note that by Lemma 4.5, it follows that for $d \in \widetilde{D}$ and $g \in G$,*

$$\begin{aligned} W_{1_\Omega,e} W_{d,g} &= W_{1_\Omega d,g} \quad (\textit{ by } (3), \textit{ Lemma } 4.5) \\ &= W_{d,g} \quad (\textit{ by } (1), \textit{ Lemma } 4.5) \end{aligned}$$

Thus $W_{1_\Omega,e}$ is the multiplicative identity of $\Gamma_c(\mathcal{G}, r^*\mathcal{D})$.

Proposition 5.6 *With the foregoing notations, there exists a unital $*$-homomorphism $\lambda : A \rtimes P \to \mathcal{D} \rtimes \mathcal{G}$ such that $\lambda(v_a) = W_{1_\Omega,a^{-1}}$ and $\lambda(x) = W_{j_e(x),e}$ for $a \in P$ and $x \in A$. Moreover λ is onto.*

Proof. Let $\pi : A \to \Gamma_c(\mathcal{G}, r^*\mathcal{D})$ be defined by $\pi(x) = W_{j_e(x),e}$. By Lemma 4.5 (Part (4)), it follows that π is a $*$-representation. Also note that π is unital. For $a \in P$, let $\widetilde{v}_a := W_{1_\Omega,a^{-1}}$ and let $\widetilde{e}_a := \widetilde{v}_a\widetilde{v}_a^*$. We verify that $(\pi, \widetilde{v})$ is Nica-covariant. Let $a \in P$ be given. By Lemma 4.5, it follows that

$$\begin{aligned}
\widetilde{v}_a^*\widetilde{v}_a &= W_{\beta_a(1_\Omega),a}W_{1_\Omega,a^{-1}}\\
&= W_{1_{\Omega a^{-1}},a}W_{1_\Omega,a^{-1}}\\
&= W_{1_{\Omega a^{-1}}\beta_a(1_\Omega),e}\\
&= W_{1_{\Omega a^{-1}},e}\\
&= W_{1_\Omega,e} \quad \text{(by (1) of Lemma 4.5)}
\end{aligned}$$

Thus $\{\widetilde{v}_a : a \in P\}$ is a collection of isometries. The fact that $(\pi, \widetilde{v})$ is Nica-covariant follows from repeated applications of Lemma 4.5.

Let $x \in A$ and $a \in P$ be given. First by definition, $j_a(\alpha_a(x)) = j_e(x)1_\Omega 1_{\Omega a}$. Now note that

$$\begin{aligned}
\pi(x)\widetilde{v}_a &= W_{j_e(x),e}W_{1_\Omega,a^{-1}}\\
&= W_{j_e(x)1_\Omega,a^{-1}}\\
&= W_{j_e(x)1_\Omega 1_{\Omega a},a^{-1}} \quad \text{(by (1) of Lemma 4.5)}\\
&= W_{j_a(\alpha_a(x)),a^{-1}}\\
&= W_{1_\Omega 1_{\Omega a} j_a(\alpha_a(x)),a^{-1}}\\
&= W_{1_\Omega 1_{\Omega a}\beta_a^{-1}(j_e(\alpha_a(x)),a^{-1}}\\
&= W_{1_\Omega,a^{-1}}W_{j_e(\alpha_a(x)),e}\\
&= \widetilde{v}_a\pi(\alpha_a(x)).
\end{aligned}$$

We leave it to the reader to verify that $\widetilde{v}_a\widetilde{v}_b = \widetilde{v}_{ba}$ for $a, b \in P$. Another application of Lemma 4.5 imply that $\widetilde{e}_a := \widetilde{v}_a\widetilde{v}_a^* = W_{1_{\Omega a},e} = W_{j_a(1),e}$ for every a. Now note that for $a, b \in P$, $j_a(1)j_b(1) = 0$ if $Pa \cap Pb = \emptyset$ and $j_a(1)j_b(1) = j_c(1)$ if $Pa \cap Pb = Pc$. Since the map $\widetilde{D} \ni d \to W_{d,e} \in \mathcal{D} \rtimes \mathcal{G}$ is a $*$-homomorphism, it follows that

$$\widetilde{e}_a\widetilde{e}_b := \begin{cases} \widetilde{e}_c & \text{if } Pa \cap Pb = Pc,\\ \\ 0 & \text{if } Pa \cap Pb = \emptyset.\end{cases}$$

Thus by the universal property of $A \rtimes P$, there exists a unique $*$-homomorphism $\lambda : A \rtimes P \to \mathcal{D} \rtimes \mathcal{G}$ such that $\lambda(x) = W_{j_e(x),e}$ for $x \in A$ and $\lambda(v_a) = W_{1_\Omega,a^{-1}}$ for $a \in P$.

Let $\mathcal{C}$ be the C^*-subalgebra of $\mathcal{D} \rtimes \mathcal{G}$ generated by $\{W_{j_e(x),e} : x \in A\}$ and $\{\widetilde{v}_a : a \in P\}$. To show that λ is surjective, it is enough to show that for each $d \in \widetilde{D}$ and $g \in G$, $W_{d,g} \in \mathcal{C}$.

Claim: $W_{d,e} \in \mathcal{C}$ for every $d \in \widetilde{D}$. Since the map $\widetilde{D} \ni d \to W_{d,e} \in \mathcal{D} \rtimes \mathcal{G}$ is a $*$-homomorphism, it is enough to show that $W_{j_g(x),e} = W_{1_\Omega j_g(x),e} \in \mathcal{C}$ for every

$g \in G$ and $x \in A$. Let $x \in A$ and $g \in G$ be such that $1_\Omega j_g(x) \neq 0$. Then, by definition, $Pg \cap P \neq \emptyset$. Since (P, G) is a quasi-lattice ordered pair, it follows that there exists $a \in P$ such that $Pg \cap P = Pa$. Let $b \in P$ be such that $bg = a$. Let $h \in G$. If $h \notin Pa = Pg \cap P$ then $1_\Omega(h) j_g(x)(h) = 0$. Suppose $h \in Pa$. Then

$$1_\Omega(h) j_g(x)(h) = \alpha_{hg^{-1}}(x) = \alpha_{ha^{-1}b}(x) = \alpha_{ha^{-1}}(\alpha_b(x)) = j_a(\alpha_b(x))(h).$$

Hence $1_\Omega j_g(x) = j_a(\alpha_b(x))$. Thus to prove the claim it is enough to show that $W_{j_a(x),e} \in \mathcal{C}$ for every $a \in P$ and $x \in A$. But observe using Lemma 4.5 that $W_{j_a(x),e} = \widetilde{v}_a W_{j_e(x),e} \widetilde{v}_a^*$ for $x \in A$ and $a \in P$. This proves the claim.
Let $g \in G$ be given. Observe that by (1) of Lemma 4.5, $W_{1_\Omega, g} \neq 0$ if and only if $1_\Omega 1_{\Omega g^{-1}} \neq 0$ i.e. if and only if $Pg^{-1} \cap P \neq \emptyset$. Let $a \in P$ be such that $Pg^{-1} \cap P = Pa$ and choose $b \in P$ such that $bg^{-1} = a$. Observe that $1_\Omega 1_{\Omega g^{-1}} = 1_{\Omega a}$.
Again by Lemma 4.5, note that

$$\begin{aligned}
\widetilde{v}_a \widetilde{v}_b^* &= W_{1_\Omega, a^{-1}} W_{1_{\Omega b^{-1}}, b} \\
&= W_{1_\Omega 1_{\Omega a} 1_{\Omega b^{-1} a}, a^{-1} b} \\
&= W_{1_{\Omega a}, a^{-1} b} \\
&= W_{1_\Omega 1_{\Omega g^{-1}}, g} \\
&= W_{1_\Omega, g} \quad \text{(by (1) of Lemma 4.5).}
\end{aligned}$$

This implies that $\{W_{1_\Omega, g} : g \in G\}$ is contained in $\mathcal{C}$. Now let $d \in \widetilde{D}$ and $g \in G$ be given. Note that $W_{d,g} = W_{d,e} W_{1_\Omega, g}$. As a consequence, it follows that $\{W_{d,g} : d \in \widetilde{D}, g \in G\}$ is contained in $\mathcal{C}$. Since the linear span of $\{W_{d,g} : d \in \widetilde{D}, g \in G\} = \Gamma_c(\mathcal{G}, r^*\mathcal{D})$ is dense in $\mathcal{D} \rtimes \mathcal{G}$, it follows that $\mathcal{C} = \mathcal{D} \rtimes \mathcal{G}$. This proves that λ is surjective. This completes the proof. $\square$.
The main aim of this section is to show that the map λ of the preceeding proposition is an isomorphism. We do this by constructing the inverse of λ.
For $g \in P$ and $x \in A$, let $i_g(x) \in \ell^\infty(P, A)$ be the restriction of $j_g(x)$ onto P. Let $g \in G$ and $x \in A$ be such that $i_g(x) \neq 0$. Then $Pg \cap P \neq \emptyset$. Let $a \in P$ be such that $Pg \cap P = Pa$ and let $b \in P$ be such that $bg = a$. Note that $i_g(x) = i_a(\alpha_b(x))$. Thus the C^*-algebra generated by $\{i_g(x) : g \in G, x \in A\}$ coincides with the C^*-subalgebra generated by $\{i_a(x) : x \in A, a \in P\}$. Let D denote the C^*-subalgebra generated by $\{i_a(x) : a \in P, x \in A\}$. Note by Remark 4.4 and (4) of Lemma 4.5, the map, call it σ, $\Gamma(\Omega, \mathcal{D}) \ni W_{d,e} \to (d(a))_{a \in P} \in \ell^\infty(P, A)$ is injective with range D.
Let $\mathcal{E}$ be the C^*-subalgebra of $A \rtimes P$ generated by the projections $\{e_a : a \in P\}$ and let $\mathcal{F}$ be the C^*-subalgebra of $A \rtimes P$ generated by $\{v_a x v_a^* : x \in A, a \in P\}$. We first prove that λ is 1-1 on $\mathcal{F}$. Let $\widetilde{\lambda} : \mathcal{F} \to D$ be defined by $\widetilde{\lambda} = \sigma \circ \lambda$. Note that for $a \in P$ and $x \in A$, $\lambda(v_a x v_a^*) = W_{j_a(x),e}$ and $\widetilde{\lambda}(v_a x v_a^*) = i_a(x)$. We show λ is injective on $\mathcal{F}$ by showing that $\widetilde{\lambda}$ is injective.

Remark 5.7 *Observe the following.*

(1) For $g \in G$, $e_g \in \mathcal{E}$. For if $e_g \neq 0$ then $e_g = e_a$ where $a \in P$ is such that $Pg \cap P = Pa$.

(2) For $g \in G$ and $x \in A$, $w_g x w_g^* \in \mathcal{F}$. For, suppose $x \in A$ and $g \in G$ be such that $w_g \neq 0$. Choose $a, b \in P$ such that $Pg \cap P = Pa$ and $bg = a$. Then by definition $w_g x w_g^* = v_a v_b^* x v_b v_a^* = v_a \alpha_b(x) v_a^* \in \mathcal{F}$.

(3) We leave it to the reader to verify that $\lambda(w_g x w_g^*) = W_{j_g(x),e}$ and $\widetilde{\lambda}(w_g x w_g^*) = i_g(x)$ for $g \in G$ and $x \in A$.

The spectrum of $\mathcal{E}$ can be identified with the Wiener-Hopf compactification Ω of (P, G). This is essentially due to Nica ([12]).
Notation: As A is unital, we identify $\ell^\infty(P)$ as a $*$-subalgebra of $\ell^\infty(P, A)$. For a subset $Y \subset P$, let $1_Y \in \ell^\infty(P)$ denote the characteristic function of Y. Note that $\widetilde{\lambda}(e_a) = 1_{Pa}$ for $a \in P$. We also identify $C(\Omega)$ as a $*$-subalgebra of $\ell^\infty(P)$ via the map $C(\Omega) \ni \phi \to (\phi(P^{-1}a))_{a \in P} \in \ell^\infty(P)$. Under this map $1_{\Omega a}$ is mapped to 1_{Pa} for $a \in P$.

(1) For $a, b \in P$, we write $a \leq b$ if there exists $c \in P$ such that $b = ca$. Let A be a subset of P. The subset A is called directed if given $a, b \in A$ there exists $c \in A$ such that $c \geq a, b$ and is called hereditary if $b \in A$ and $a \leq b$ then $a \in A$.

(2) Let $\Omega_{\mathcal{N}}$ denote the set of all non-empty, directed and hereditary subsets of P. Topologise $\Omega_{\mathcal{N}}$ by considering $\Omega_{\mathcal{N}}$ as a subset of $\{0,1\}^P$ and endow $\Omega_{\mathcal{N}}$ with the subspace topology inherited from the product topology on $\{0,1\}^P$. For $a \in P$, let $[e, a] := \{x \in P : x \leq a\}$. Then $\{[e, a] : a \in P\}$ is dense in $\Omega_{\mathcal{N}}$. For $a \in P$, let $U_a := \{A \in \Omega_{\mathcal{N}} : a \in A\}$.

(3) Let Ω be the Wiener-Hopf compactification of (P, G). Let $A \in \Omega$ be given. Note that A contains the identity element e and $P^{-1}A \subset A$. Thus $A \cap P$ is non-empty and hereditary. We claim that $A \cap P$ is directed. Let $a, b \in A \cap P$ be given. Choose a sequence (a_n) in P such that $P^{-1}a_n \to A$. Then there exists $N \in \mathbb{N}$ such that $a, b \in P^{-1}a_n$ for $n \geq N$. In other words, $a_n \in Pa \cap Pb$ eventually. Since (P, G) is quasi-lattice ordered, it follows that there exists $c \in P$ such that $Pa \cap Pb = Pc$. Thus $a_n \in Pc$ eventually or $c \in P^{-1}a_n$ eventually. Consequently $c \geq a, b$ and $c \in A \cap P$. This proves that $A \cap P$ is hereditary. Thus the map $\Omega \ni A \to A \cap P \in \Omega_{\mathcal{N}}$ is well-defined.

(4) The map $\Omega \ni A \to A \cap P \in \Omega_{\mathcal{N}}$ is a homeomorphism. The continuity of the map $\Omega \ni A \to A \cap P \in \Omega_{\mathcal{N}}$ is clear. Let $A, B \in \Omega$ be such that $A \cap P = B \cap P$. Choose a sequence $a_n \in P$ such that $P^{-1}a_n \to A$. Consider an element $g \in A$. Then $a_n \in Pg$ eventually. Thus $Pg \cap P \neq \emptyset$. Since (P, G) is quasi-lattice ordered, it follows that there exists $a \in P$ such that $Pg \cap P = Pa$. Then $a_n \in Pa$ or equivalently $a \in P^{-1}a_n$ eventually. Thus $a \in A \cap P = B \cap P$. This implies that $a \in B$. Note

that $g \leq a$ and $P^{-1}B \subset B$. As a consequence it follows that $g \in B$. This shows that $A \subset B$. Similarly $B \subset A$. This proves that the map $\Omega \ni A \to A \cap P$ is 1-1. Note that $P^{-1}a \cap P = [e,a]$ for $a \in P$. Since $\{[e,a] : a \in P\}$ is dense in $\Omega_{\mathcal{N}}$ and Ω is compact, it follows that the map $\Omega \ni A \to A \cap P \in \Omega_{\mathcal{N}}$ is a homeomorphism. Henceforth we identify Ω and $\Omega_{\mathcal{N}}$ via this homeomorphism. Under the homeomorphism $\Omega \ni A \to A \cap P \in \Omega_{\mathcal{N}}$, by Lemma 3.6, the subset Ωa is mapped onto U_a for every $a \in P$.

(5) Let $\widehat{\mathcal{E}}$ be the character space of $\mathcal{E}$. For $\chi \in \widehat{\mathcal{E}}$, let $A_\chi := \{a \in P : \chi(e_a) = 1\}$. Then A_χ contains the identity element, is directed and hereditary. Note that the map $\widehat{\mathcal{E}} \ni \chi \to A_\chi \in \Omega_{\mathcal{N}} \cong \Omega$ is continuous and injective. Thus one obtains a $*$-homomorphism $\widetilde{\mu} : C(\Omega) \to C(\widehat{\mathcal{E}}) \cong \mathcal{E}$ defined by $\widetilde{\mu}(f)(\chi) = f(B_\chi)$ for $f \in C(\Omega)$ and $\chi \in \widehat{\mathcal{E}}$ where B_χ is the unique element in Ω such that $B_\chi \cap P = A_\chi$. We leave it to the reader to verify that $\widetilde{\mu}(1_{\Omega a}) = e_a$ for $a \in P$. Since $\widetilde{\lambda}(e_a) = 1_{Pa}$, it follows that $\widetilde{\lambda} : \mathcal{E} \to C(\Omega)$ is an isomorphism with $\widetilde{\mu}$ being its inverse.

Let $\mathcal{I} := \{Y \subset P : 1_Y \in C(\Omega) \subset \ell^\infty(P)\}$. Note that $\mathcal{I}$ is closed under finite unions, finite intersections and complements. Moreover for every $a \in P$, $Pa \in \mathcal{I}$. For $Y \in \mathcal{I}$, let $e_Y \in \mathcal{E}$ be such that $\widetilde{\lambda}(e_Y) = 1_Y$. For $Y \in \mathcal{I}$, note that e_Y is a projection. The collection $\mathcal{I}$ is called the set of constructible left ideals of P by Li in [7].

LEMMA 5.8 *Let $\widetilde{\mathcal{F}}$ be the linear span of $\{v_a x v_a^* : x \in A, a \in P\}$.*

(1) Then $\widetilde{\mathcal{F}}$ is a dense $*$-subalgebra of $\mathcal{F}$.

(2) The algebra $\mathcal{E}$ is contained in the center of $\mathcal{F}$.

(3) Let $z \in \widetilde{\mathcal{F}}$ be given. Then there exists $n \in \mathbb{N}$, $Y_1, Y_2, \cdots Y_n \in \mathcal{I}$, $a_1, a_2, \cdots a_n \in P$ and $x_1, x_2, \cdots x_n \in A$ such that $a_i \in Y_i$ for every i, $Y_i \cap Y_j = \emptyset$ if $i \neq j$ and $z = \sum_{i=1}^n e_{Y_i} v_{a_i} x_i v_{a_i}^*$.

(4) The map $\widetilde{\lambda} : \mathcal{F} \to D$ is isometric and onto.

Proof. Let $a, b \in P$ and $x, y \in A$ be given. Suppose $Pa \cap Pb = \emptyset$. The relation $e_a e_b = 0$ implies that $v_a^* v_b = 0$. Hence $(v_a x v_a^*)(v_b y v_b^*) = 0$. Now suppose $Pa \cap Pb \neq \emptyset$. Since (P, G) is quasi-lattice ordered, it follows that there exists $c \in P$ such that $Pa \cap Pb = Pc$. Choose $s, t \in P$ such that $sa = tb = c$. Now calculate as follows to find that

$$\begin{aligned}(v_a x v_a^*)(v_b y v_b^*) &= v_a x v_s v_t^* y v_b^* \quad \text{(by Eq. 5.4)} \\ &= v_a v_s \alpha_s(x) \alpha_t(y) v_t^* v_b^* \\ &= v_{sa} \alpha_s(x) \alpha_t(y) v_{tb}^* \\ &= v_c \alpha_s(x) \alpha_t(y) v_c^*.\end{aligned}$$

This proves that $\widetilde{\mathcal{F}}$ is closed under multiplication. Clearly $\widetilde{\mathcal{F}}$ is $*$-closed. Thus $\widetilde{\mathcal{F}}$ is dense in $\mathcal{F}$. This proves (1).
Let $a,b \in P$ and $x \in A$ be given. The calculation done to prove (1) implies that

$$e_a v_b x v_b^* := \begin{cases} v_c\alpha_{cb^{-1}}(x)v_c^* & \text{if } Pa\cap Pb = Pc, \\ \\ 0 & \text{if } Pa\cap Pb=\emptyset \end{cases} \tag{5.6}$$

and

$$v_b x v_b^* e_a := \begin{cases} v_c\alpha_{cb^{-1}}(x)v_c^* & \text{if } Pb\cap Pa = Pc, \\ \\ 0 & \text{if } Pb\cap Pa=\emptyset. \end{cases}$$

Hence $e_a v_b x v_b^* = v_b x v_b^* e_a$ for every $a,b \in P$ and $x \in A$. This proves (2).
Let $z := \sum_{i=1}^n v_{b_i} y_i v_{b_i}^* \in \widetilde{\mathcal{F}}$ be given with $b_i \in P$ and $y_i \in A$. For $B \subset \{1,2,\cdots n\}$, let $Z_B := \bigcap_{i\in B} Pb_i$. Since (P,G) is quasi-lattice ordered, it follows that if Z_B is non-empty, then there exists $a_B \in P$ such that $Z_B = Pa_B$. For $B \subset \{1,2,\cdots,n\}$, let $Y_B := Z_B \backslash \bigcup\limits_{B\subsetneq C} Z_C$. We use here the convention that empty intersection is the whole set, in this case, P and empty union is empty. We leave it to the reader to verify that $\{Y_B : B \subset \{1,2,\cdots n\}\}$ forms a disjoint collection of subsets of P and for $B \subset \{1,2,\cdots,n\}$, $Z_B = \bigcup\limits_{C\supset B} Y_C$. Also observe that if $Y_B \neq \emptyset$ then $a_B \in Y_B$. To see this, let $B \subset \{1,2,\cdots,n\}$ be such that $Y_B \neq \emptyset$. Note that $Y_B \subset Z_B = Pa_B$. Thus if $a_B \notin Y_B$ then there exists $C \supsetneq B$ such that $a_B \in Z_C = Pa_C$. But Z_C is closed under left multiplication by P. Thus $Z_B \subset Z_C$ and hence $Y_B = \emptyset$. Thus if $Y_B \neq \emptyset$ then $a_B \in Y_B$. Let $\mathcal{V} := \{B \subset \{1,2,\cdots,n\} : Y_B \neq \emptyset\}$.
Now calculate as follows to see that

$$\begin{aligned}
z &= \sum_{i=1}^n e_{Pb_i} v_{b_i} y_i v_{b_i}^* \\
&= \sum_{i=1}^n \Big(\sum_{B\in\mathcal{V}, i\in B} e_{Y_B}\Big) v_{b_i} y_i v_{b_i}^* \\
&= \sum_{B\in\mathcal{V}} e_{Y_B} e_{Z_B} \Big(\sum_{i\in B} e_{Z_B} v_{b_i} y_i v_{b_i}^*\Big) \\
&= \sum_{B\in\mathcal{V}} e_{Y_B} \Big(\sum_{i\in B} e_{a_B} v_{b_i} y_i v_{b_i}^*\Big) \\
&= \sum_{B\in\mathcal{V}} e_{Y_B} \Big(\sum_{i\in B} v_{a_B} \alpha_{a_B a_i^{-1}}(y_i) v_{a_B}^*\Big) \quad (\text{ By Eq. 5.6 and } Pa_B \subset Pa_i \text{ if } i \in B) \\
&= \sum_{B\in\mathcal{V}} e_{Y_B} v_{a_B} x_B v_{a_B}^*
\end{aligned}$$

where for $B \in \mathcal{V}$, $x_B := \sum_{i \in B} \alpha_{a_B a_i^{-1}}(y_i)$. This proves (3).

To prove (4), it is enough to show that $\widetilde{\lambda}$ is isometric on $\widetilde{\mathcal{F}}$. Let $z \in \widetilde{\mathcal{F}}$ be given. By (3) we can write $z = \sum_{i=1}^{n} e_{Y_i} v_{a_i} x_i v_{a_i}^*$ with Y_i disjoint, $a_i \in Y_i$ and $x_i \in A$. Since $\{e_{Y_i} : i = 1, 2 \cdots, n\}$ are orthogonal central projections in $\mathcal{F}$, it follows that

$$||z|| \leq \max_{i \in \{1,2,\cdots n\}} ||v_{a_i} x_i v_{a_i}^*|| \leq \max_{i \in \{1,2,\cdots,n\}} ||x_i||.$$

Note that $\widetilde{\lambda}(z) = \sum_{i=1}^{n} 1_{Y_i} i_{a_i}(x_i)$. Since $a_i \in Y_i$, $\widetilde{\lambda}(z)(a_i) = x_i$ for $i \in \{1, 2, \cdots, n\}$. Thus $||x_i|| \leq ||\widetilde{\lambda}(z)||$ for every $i \in \{1, 2, \cdots, n\}$. As a consequence, it follows that $||z|| \leq ||\widetilde{\lambda}(z)||$. But $\widetilde{\lambda} : \mathcal{F} \to D$ is a $*$-homomorphism and hence $||\widetilde{\lambda}(z)|| \leq ||z||$. This proves that $||\widetilde{\lambda}(z)|| = ||z||$ for every $z \in \widetilde{\mathcal{F}}$. Since $\widetilde{\mathcal{F}}$ is dense in $\mathcal{F}$, it follows that $\widetilde{\lambda}$ is isometric. Surjectivity of $\widetilde{\lambda}$ follows from the fact that $\widetilde{\lambda}(v_a x v_a^*) = i_a(x)$ for $x \in A$ and $a \in P$ and the observation that D is generated by $\{i_a(x) : a \in P, x \in A\}$. This completes the proof. $\square$

Let $\widetilde{\mu} : D \to \mathcal{F}$ be the inverse of $\widetilde{\lambda} : \mathcal{F} \to D$. Let us denote the map $\widetilde{D} \ni d \to (d(a))_{a \in P} \in D$ by res and let $\nu : \widetilde{D} \to \mathcal{F}$ be defined by $\nu = \widetilde{\mu} \circ res$. Note that $\nu(1_{\Omega g}) = e_g$ for every $g \in G$. In particular, $\nu(1_\Omega) = 1$.

Now we define a map $\mu : \Gamma_c(\mathcal{G}, r^*\mathcal{D}) \to A \rtimes P$ as follows: Let $f \in \Gamma_c(\mathcal{G}, r^*\mathcal{D})$. Write $f = \sum_{d \in \widetilde{D}, g \in G} W_{d_g, g}$ where $d_g = 0$ except for finitely many g. We let $\mu(f) := \sum_{g \in G} \nu(d_g) w_{g^{-1}}$. We claim that μ is well-defined. Let $d_1, d_2, \cdots, d_n \in \widetilde{D}$ and $g_1, g_2, \cdots, g_n \in G$ (with g_i's distinct) be such that $\sum_{i=1}^{n} W_{d_i, g_i} = 0$. To show that μ is well-defined, it is enough to verify that $\sum_{i=1}^{n} \nu(d_i) w_{g_i}^* = 0$.

Let $i \in \{1, 2, \cdots, n\}$ be given. Let $X \in \Omega \cap \Omega g_i^{-1}$ be given. Then $(X, g_i) \in \mathcal{G}$. Now $\sum_{i=1}^{n} W_{d_i, g_i}(X, g_i) = 0$ implies that $d_i + I_X = 0$. Hence $d_i + I_X = 0$ for every $X \in \Omega \cap \Omega g_i^{-1}$. Now for $X \in \widetilde{\Omega}$, $1_\Omega 1_{\Omega g_i^{-1}} d_i + I_X = 1_\Omega(X) 1_{\Omega g_i^{-1}}(X)(d_i + I_X)$. Thus $1_\Omega 1_{\Omega g_i^{-1}} d_i + I_X = 0$ for every $X \in \widetilde{\Omega}$. This implies that $d_i 1_\Omega 1_{\Omega g_i^{-1}} = 0$. Now observe that

$$\begin{aligned} \nu(d_i) w_{g_i^{-1}} &= \nu(d_i) e_{g_i^{-1}} w_{g_i^{-1}} \\ &= \nu(d_i) \nu(1_\Omega) \nu(1_{\Omega g_i^{-1}}) w_{g_i^{-1}} \\ &= \nu(d_i 1_\Omega 1_{\Omega g_i^{-1}}) w_{g_i^{-1}} \\ &= 0. \end{aligned}$$

As a result, the sum $\sum_{i=1}^{n} \nu(d_i) w_{g_i^{-1}}$ vanishes. This proves that μ is well-defined.

Next we verify that μ is multiplicative. We leave it to the reader to convince himself that it is enough to prove that $\mu(W_{d_1, g_1} W_{d_2, g_2}) = \mu(W_{d_1, g_1}) \mu(W_{d_2, g_2})$ for $d_1, d_2 \in \widetilde{D}$ and $g_1, g_2 \in G$.

Claim: For $g \in G$ and $d \in \widetilde{D}$, $w_g\nu(d)w_g^* = \nu(\beta_g^{-1}(d)1_{\Omega g})$.
Since the linear span of $\{j_h(x) : h \in G, x \in A\}$ is dense in $\widetilde{D}$, it is enough to prove the claim when $d = j_h(x)$ for some $h \in G$ and $x \in A$. Let $h \in G$ and $x \in A$ be given and let $d := j_h(x)$. Note that $\nu(j_h(x)) = w_hxw_h^*$. Now calculate as follows to observe that

$$\begin{aligned} w_gw_hxw_h^*w_g^* &= e_gw_{hg}xw_{hg}^*e_g \quad \text{(by Lemma 5.4)} \\ &= e_g\nu(j_{hg}(x)) \\ &= e_g\nu(\beta_g^{-1}(d)) \\ &= \nu(1_{\Omega g})\nu(\beta_g^{-1}(d)) \\ &= \nu(\beta_g^{-1}(d)1_{\Omega g}) \end{aligned}$$

This proves the claim. Now let $g_1, g_2 \in G$ and $d_1, d_2 \in \widetilde{D}$ be given. By Lemma 4.5, it follows that $W_{d_1,g_1}W_{d_2,g_2} = W_{d,g_1g_2}$ where $d = 1_{\Omega g_1^{-1}}d_1\beta_{g_1}(d_2)$. Thus by definition, we have $\mu(W_{d_1,g_1}W_{d_2,g_2}) = \nu(d)w_{g_1g_2}^*$. Now calculate as follows to observe that

$$\begin{aligned} \mu(W_{d_1,g_1})\mu(W_{d_2,g_2}) &= \nu(d_1)w_{g_1}^*\nu(d_2)w_{g_2}^* \\ &= \nu(d_1)w_{g_1}^*e_{g_1}\nu(d_2)w_{g_2}^* \\ &= \nu(d_1)w_{g_1}^*\nu(d_2)w_{g_1}w_{g_1}^*w_{g_2}^* \quad \text{(Since } e_g \text{ commutes with } \mathcal{F}) \\ &= \nu(d_1)w_{g_1^{-1}}\nu(d_2)w_{g_1^{-1}}^*e_{g_1^{-1}}w_{g_1g_2}^* \quad \text{(by Lemma 5.4)} \\ &= \nu(d_1)\nu(\beta_{g_1}(d_2)1_{\Omega g_1^{-1}})\nu(1_{\Omega g_1^{-1}})w_{g_1g_2}^* \\ &= \nu(d_1\beta_{g_1}(d_2)1_{\Omega g_1^{-1}})w_{g_1g_2}^* \\ &= \nu(d)w_{g_1g_2}^*. \end{aligned}$$

Hence $\mu(W_{d_1,g_1}W_{d_2,g_2}) = \mu(W_{d_1,g_1})\mu(W_{d_2,g_2})$. This shows that μ is multiplicative. A similar computation yields that μ is $*$-preserving. Thus $\mu : \Gamma_c(\mathcal{G}, r^*\mathcal{D}) \to A \rtimes P$ is a $*$-homomorphism. Note that by defintion $\mu(W_{d,e}) = \nu(d)$ for every $d \in \widetilde{D}$. Moreover the surjection ν and the surjection $\widetilde{D} \ni d \to W_{d,e} \in \Gamma(\Omega, \mathcal{D})$ have the same kernel. Thus μ restricted to $\Gamma(\Omega, \mathcal{D})$ is bounded. Hence the $*$-homomorphism μ extends to a $*$-homomorphism from the full crossed product $\mathcal{D} \rtimes \mathcal{G}$ to $A \rtimes P$ which we still denote by μ.

Theorem 5.9 *Let $\lambda : A \rtimes P \to \mathcal{D} \rtimes \mathcal{G}$ be the $*$-homomorphism constructed in Proposition 5.6. Then λ is an isomorphism.*

Proof. Let $\mu : \mathcal{D} \rtimes \mathcal{G} \to A \rtimes P$ be the $*$-homomorphism such that $\mu(W_{d,g}) = \nu(d)w_g^*$. The existence of such a map is shown in the paragraphs preceeding this theorem. By definition, for $a \in P$, $\mu(\widetilde{v}_a) = \nu(1_\Omega)w_a = v_a$ and for $x \in A$, $\mu(W_{j_e(x),e}) = \nu(j_e(x))w_e = x$. Thus $\mu \circ \lambda = id$. This proves that λ is 1-1. The surjectivity of λ is already proven in Proposition 5.6. Hence λ is an isomorphism. This completes the proof.

6 A K-group computation

For the rest of this paper, let $\mathbb{F}_n$ be the free group on n generators and we denote its generators by $a_1, a_2, \cdots, a_n$. Denote the semigroup generated by $a_1, a_2, \cdots a_n$ by $\mathbb{F}_n^+$ i.e. $\mathbb{F}_n^+$ consists of words in $a_1, a_2, \cdots, a_n$. We reserve the letters $a_1, a_2, \cdots, a_n$ to denote the canonical generators of $\mathbb{F}_n$ and $\mathbb{F}_n^+$. It is due to Nica, (Example 4, Page 23 of [12]), that $(\mathbb{F}_n^+, \mathbb{F}_n)$ is a quasi-lattice ordered pair. We should remark that we consider the right variant of Nica's definition of a quasi-lattice ordered pair. But we can apply Nica's proof by considering the pair $(Q := (\mathbb{F}_n^+)^{-1}, \mathbb{F}_n)$. Note Q is the free semigroup on the generators $a_1^{-1}, a_2^{-1}, \cdots a_n^{-1}$. Thus it follows that if $gQ \cap Q \neq \emptyset$ for $g \in \mathbb{F}_n$ then there exists $a \in Q$ such that $gQ \cap Q = aQ$. Taking inverses imply that $(\mathbb{F}_n^+, \mathbb{F}_n)$ is quasi-lattice ordered in our sense.
Let $\mathcal{G}$ be the Wiener-Hopf groupoid associated to the pair $(\mathbb{F}_n^+, \mathbb{F}_n)$. We need the fact that $\mathcal{G}$ is amenable. To see this, let $P := \mathbb{F}_n^+$ and $Q := P^{-1}$. For $a \in Q$, let $V_a : \ell^2(Q) \to \ell^2(Q)$ be defined by $V_a(\delta_b) = \delta_{ab}$ for $b \in Q$. Here $\{\delta_b : b \in Q\}$ denotes the canonical orthonormal basis of $\ell^2(Q)$. The unitary $\ell^2(P) \ni \delta_a \to \delta_{a^{-1}} \in \ell^2(Q)$ induces an isomorphism between the Wiener-Hopf algebra $\mathcal{W}(\mathbb{F}_n^+, \mathbb{F}_n)$ and the C^*-algebra generated by $\{V_a : a \in P\}$, called the C^*-algebra associated to the left regular representation of Q and let us denote it by $C^*_{\ell,r}(Q)$. By Corollary 8.3 of [7], it follows that the $C^*_{\ell,r}(Q)$ is nuclear. Thus the C^*-algebra $\mathcal{W}(\mathbb{F}_n^+, \mathbb{F}_n)$ is nuclear. But Theorem 4.3 implies that $\mathcal{W}(\mathbb{F}_n^+, \mathbb{F}_n)$ is isomorphic to $C^*_{red}(\mathcal{G})$. Now the amenability of $\mathcal{G}$ follows by appealing to Theorem 5.6.18 of [1]. The following statement is an immediate consequence of Theorem 4.3 and Theorem 5.9

Corollary 6.1 *Let A be a unital C^*-algebra and $\alpha : \mathbb{F}_n^+ \to End(A)$ be a unital left action. The regular representation $\rho : A \rtimes \mathbb{F}_n^+ \to A \rtimes_{red} \mathbb{F}_n^+$ is an isomorphism.*

Proof. Let $(\mathcal{D}, \mathcal{G}, \beta)$ be the Wiener-Hopf groupoid dynamical system associated to $(A, \mathbb{F}_n^+, \mathbb{F}_n, \alpha)$. Since the Wiener-Hopf groupoid $\mathcal{G}$ is amenable, it follows from Theorem 1 of [15] that the natural map from $\mathcal{D} \rtimes \mathcal{G}$ to the reduced crossed product $\mathcal{D} \rtimes_{red} \mathcal{G}$ is an isomorphism. The proof is complete by applying Theorem 4.3 and Theorem 5.9. □
Let A be a C^*-algebra and $\alpha : \mathbb{F}_n^+ \to End(A)$ be a left action. For $i \in \{1, 2, \cdots, n\}$, let $\alpha_i := \alpha_{a_i}$. Note that α is completely determined by the n endomorphisms $\alpha_1, \alpha_2, \cdots, \alpha_n$. Conversely let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be endomorphisms of A. Then there exists a unique action $\alpha : \mathbb{F}_n^+ \to End(A)$ such that $\alpha_{a_i} = \alpha_i$. To see this, let $w := a_{i_1} a_{i_2} \cdots a_{i_k}$ be a word in $\{a_1, a_2, \cdots, a_n\}$. As any word in $\{a_1, a_2, \cdots, a_n\}$ is a reduced word [See [14]], it follows that the expression $a_{i_1} a_{i_2} \cdots a_{i_k}$ representing w is unique. Now set $\alpha_w = \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k}$. Then $\alpha : \mathbb{F}_n^+ \to End(A)$ is the required action.
Let $\mathcal{H}$ be a Hilbert space and let $V : \mathbb{F}_n^+ \to B(\mathcal{H})$ be an anti-homomorphism such that for each $a \in \mathbb{F}_n^+$, V_a is an isometry. For $i \in \{1, 2, \cdots, n\}$, let $V_i := V_{a_i}$. Then the n-isometries $V_1, V_2, \cdots, V_n$ determine V. Conversely, let

$V_1, V_2, \cdots, V_n$ be isometries on $\mathcal{H}$. Then there exists a unique anti-homorphism $V : \mathbb{F}_n^+ \to B(\mathcal{H})$ such that for $i \in \{1, 2, \cdots, n\}$, $V_{a_i} = V_i$. We leave the details to the reader.

Lemma 6.2 *Let A be a C^*-algebra and $\alpha : \mathbb{F}_n^+ \to End(A)$ be a left action. Let π be a representation of A on a Hilbert space $\mathcal{H}$ and $V : \mathbb{F}_n^+ \to B(\mathcal{H})$ be an anti-homomorphism of isometries. For $i = 1, 2, \cdots, n$, let $\alpha_i := \alpha_{a_i}$ and $V_i := V_{a_i}$. Then the following are equivalent.*

(1) The pair (π, V) is Nica-covariant.

(2) For $i \in \{1, 2, \cdots, n\}$ and $x \in A$, $\pi(x)V_i = V_i\alpha_i(x)$. For $i, j \in \{1, 2, \cdots, n\}$ with $i \neq j$, $V_i^* V_j = 0$.

Proof. Note that for $i \neq j$, $\mathbb{F}_n^+ a_i \cap \mathbb{F}_n^+ a_j = \emptyset$. Thus clearly (1) implies (2). Now assume that (2) holds. For $w \in \mathbb{F}_n^+$, let $E_w := V_w V_w^*$. We leave it to the reader to verify that $\pi(x)V_w = V_w\alpha_w(x)$ for $x \in A$ and $w \in \mathbb{F}_n^+$.
Observe that for $w_1, w_2 \in \mathbb{F}_n^+$,

$$\mathbb{F}_n^+ w_1 \cap \mathbb{F}_n^+ w_2 = \begin{cases} \mathbb{F}_n^+ w_2 & \text{if } w_1 \leq w_2, \\ \mathbb{F}_n^+ w_1 & \text{if } w_2 \leq w_1, \\ \emptyset & \text{else.} \end{cases}$$

Let $w_1, w_2 \in \mathbb{F}_n^+$ be given. Suppose $w_1 \leq w_2$. Let $w \in \mathbb{F}_n^+$ be such that $ww_1 = w_2$. Note that $E_{w_2} = V_{ww_1}V_{ww_1}^* = V_{w_1}(V_w V_w^*)V_{w_1} \leq V_{w_1}V_{w_1}^* = E_{w_1}$. Thus $E_{w_1}E_{w_2} = E_{w_2}$ if $w_1 \leq w_2$. Thus if $w_1, w_2, w \in \mathbb{F}_n^+$ are such that $\mathbb{F}_n^+ w_1 \cap \mathbb{F}_n^+ w_2 = \mathbb{F}_n^+ w$ then $E_{w_1}E_{w_2} = E_w$. Now let $w_1, w_2 \in \mathbb{F}_n^+$ be such that $\mathbb{F}_n^+ w_1 \cap \mathbb{F}_n^+ w_2 = \emptyset$. Write $w_1 = a_{i_1}a_{i_2}\cdots a_{i_n}$ and $w_2 = a_{j_1}a_{j_2}\cdots a_{j_m}$ with $i_k, j_\ell \in \{1, 2, \cdots, n\}$. We claim that $E_{w_1}E_{w_2} = 0$. Without loss of generality, we can assume that $n \leq m$. Let $k \in \{0, 1, \cdots, n-1\}$ be the least integer for which $i_{n-k} \neq j_{m-k}$. Such an integer exists, otherwise $w_1 \leq w_2$ which contradicts the assumption that the intersection $\mathbb{F}_n^+ w_1 \cap \mathbb{F}_n^+ w_2$ is empty.
Now use the fact that V_i is an isometry to observe that

$$\begin{aligned} V_{w_1}^* V_{w_2} &= V_{i_1}^* V_{i_2}^* \cdots V_{i_{n-k}}^* V_{j_{m-k}} V_{j_{m-k-1}} \cdots V_{j_1} \\ &= 0 \ \ \big(\text{Since } V_i^* V_j = 0 \text{ if } i \neq j \text{ and } i_{n-k} \neq j_{m-k} \ \big). \end{aligned}$$

Thus $E_{w_1}E_{w_2} = 0$ if $\mathbb{F}_n^+ w_1 \cap \mathbb{F}_n^+ w_2 = \emptyset$. This proves that (π, V) is Nica-covariant. This completes the proof. □

Let A be a unital C^*-algebra and $\alpha : \mathbb{F}_n^+ \to End(A)$ be a unital left action. We show that the inclusion $A \ni x \to x \in A \rtimes \mathbb{F}_n^+$ is a KK-equivalence. Let (π, V) be the standard Nica-covariant pair for $(A, \mathbb{F}_n^+, \alpha)$. For $i = 1, 2, \cdots, n$, let $\alpha_i := \alpha_{a_i}$.

Consider the Hilbert A-modules $\mathcal{E}^{(0)} = A \otimes \ell^2(\mathbb{F}_n^+)$ and $\mathcal{E}^{(1)} = A \otimes \ell^2(\mathbb{F}_n^+ \backslash \{1\})$. Here 1 denote the identity element of $\mathbb{F}_n^+$. Note that for $x \in A$ and $a \in \mathbb{F}_n^+$, $\pi(x)$ and V_a leaves $\mathcal{E}^{(1)}$ invariant. For $x \in A$ and $a \in \mathbb{F}_n^+$, denote the restriction of $\pi(x)$ and V_a on $\ell^2(\mathbb{F}_n^+ \backslash \{1\})$ by $\widetilde{\pi}(x)$ and $\widetilde{V}_a$. Note that $(\widetilde{\pi}, \widetilde{V})$ is Nica-covariant. The only thing that needs verification is that for $i \neq j$, the range of $\widetilde{V}_{a_i}$ is orthogonal to that of V_{a_j}. This follows from the fact that the range of V_{a_i} is $A \otimes \ell^2((\mathbb{F}_n^+ \backslash \{1\})a_i)$. Let us denote the map $\pi \rtimes V : A \rtimes \mathbb{F}_n^+ \to \mathcal{L}_A(\mathcal{E}^{(0)})$ by $\lambda^{(0)}$ and the map $\widetilde{\pi} \rtimes \widetilde{V} : A \rtimes \mathbb{F}_n^+ \to \mathcal{L}_A(\mathcal{E}^{(1)})$ by $\lambda^{(1)}$ respectively. Denote the orthogonal projection from $\ell^2(\mathbb{F}_n^+)$ onto $\ell^2(\mathbb{F}_n^+ \backslash \{1\})$ by Q and Let $P : \mathcal{E}^{(0)} \to \mathcal{E}^{(1)}$ be defined by $P := 1 \otimes Q$.

Notations: Let $\{\delta_x : x \in \mathbb{F}_n^+\}$ be the standard orthonormal basis for $\ell^2(\mathbb{F}_n^+)$ and let $\{e_{x,y} : x, y \in \mathbb{F}_n^+\}$ be the standard 'matrix units' with respect to the orthonormal basis $\{\delta_x : x \in \mathbb{F}_n^+\}$. For $i \in \{1, 2, \cdots, n\}$, let $v_i : \ell^2(\mathbb{F}_n^+) \to \ell^2(\mathbb{F}_n^+)$ be given by $v_i(\delta_a) = \delta_{aa_i}$. Here $\{\delta_a : a \in \mathbb{F}_n^+\}$ stands for the standard orthonormal basis. Let p be the orthogonal projection from $\ell^2(\mathbb{F}_n^+)$ onto the one dimensional space subspace $\ell^2(\{1\})$ and set $q := 1 - p$.

Lemma 6.3 *The triple* $\left(\mathcal{E} := \mathcal{E}^{(0)} \oplus \mathcal{E}^{(1)}, \lambda := \lambda^{(0)} \oplus \lambda^{(1)}, F := \begin{bmatrix} 0 & P^* \\ P & 0 \end{bmatrix}\right)$ *is a Kasparov* $A \rtimes \mathbb{F}_n^+$*-*A *bimodule.*

Proof. Note that $PP^* = 1$ and $P^*P = 1 - 1 \otimes p$. Thus $P^*P - 1$ and $PP^* - 1$ are compact. Note that we have assumed that A is unital.
Note that for $x \in A$ and $a, b \in \mathbb{F}_n^+ \backslash \{1\}$,

$$\begin{aligned} P\pi(x) - \widetilde{\pi}(x)P &= 0, \\ PV_a - \widetilde{V}_a P &= 1 \otimes e_{a,1}, \text{ and} \\ PV_b^* - \widetilde{V}_b^* P &= -1 \otimes e_{1,b}. \end{aligned}$$

Also observe that $PV_1 - \widetilde{V}_1 P = 0$ and $PV_1^* - \widetilde{V}_1^* P = 0$. Now the fact that the linear span of $\{v_a x v_b^* : a, b \in \mathbb{F}_n^+, x \in A\}$ is dense in $A \rtimes \mathbb{F}_n^+$, it follows that $P\lambda^{(0)}(T) - \lambda^{(1)}(T)P$ is compact for every $T \in A \rtimes \mathbb{F}_n^+$. The proof is now complete. ☐

Let us denote the Kasparov element defined in Lemma 6.3 by $[d]$. Let us denote the inclusion $A \ni x \to x \in A \rtimes \mathbb{F}_n^+$ by j and denote the corresponding KK-element representing j in $KK(A, A \rtimes \mathbb{F}_n^+)$ by $[j]$. We use the notation $\sharp$ to denote the Kasparov product. We adapt the proof of Theorem 2.3 of [4].

Theorem 6.4 *The* KK*-elements* $[d]$ *and* $[j]$ *are inverses of each other with respect to the Kasparov product.*

Proof. For this proof, let us denote $A \rtimes \mathbb{F}_n^+$ by $\mathcal{T}$. Note the decomposition $\mathcal{E}^{(0)} = A \oplus \mathcal{E}^{(1)}$ as Hilbert A-modules. With respect to this decomposition, observe that, the pull-back of $[d]$ by the homomorphism j i.e. the Kasparov product $[j]\sharp[d]$ is isomorphic to the direct sum of $(A, m, 0)$ and $\mathcal{D} := \Big(\mathcal{E}^{(1)} \oplus$

$\mathcal{E}^{(1)}, \widetilde{\pi} \oplus \widetilde{\pi}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Big)$ where $m : A \to A = M(A)$ is the usual multiplication representation. Observe that $\mathcal{D}$ is degenerate. Thus the product $[j]\sharp[d] = [1_A]$. Note that the Kasparov product $[d]\sharp[j]$, the push-forward of $[d]$ by the homomorphism j, is given by $(\mathcal{E} \otimes_A \mathcal{T}, \sigma := \lambda \otimes 1, G := F \otimes 1)$. The map

$$(\ell^2(\Gamma) \otimes A) \otimes_A \mathcal{T} \ni (\delta_a \otimes x) \otimes y \to \delta_a \otimes xy \in \ell^2(\Gamma) \otimes \mathcal{T}$$

is unitary (The surjectivity follows from the assumption that the action α is unital and hence π is unital). Here Γ stands for either $\mathbb{F}_n^+$ or $\mathbb{F}_n^+\backslash\{1\}$.
We identify this way the Hilbert $\mathcal{T}$-modules, $\mathcal{E}^{(0)} \otimes_A \mathcal{T}$ with $\mathcal{F}^{(0)} := \ell^2(\mathbb{F}_n^+) \otimes \mathcal{T}$ and $\mathcal{E}^{(1)} \otimes_A \mathcal{T}$ with $\mathcal{F}^{(1)} := \ell^2(\mathbb{F}_n^+\backslash\{1\}) \otimes \mathcal{T}$. With this identification, the representation $\sigma = \sigma^{(0)} \oplus \sigma^{(1)}$ of $\mathcal{T}$ is given by the formulas: for $x \in A$ and $a \in \mathbb{F}_n^+$,

$$\sigma^{(i)}(x)(\delta_c \otimes y) := \delta_c \otimes \alpha_c(x)y$$
$$\sigma^{(i)}(v_a)(\delta_c \otimes y) := \delta_{ca} \otimes y$$

Let Q be the orthogonal projection of $\ell^{(2)}(\mathbb{F}_n^+)$ onto $\ell^{(2}(\mathbb{F}_n^+\backslash\{1\})$ and let $R : \mathcal{F}^{(0)} \to \mathcal{F}^{(1)}$ be given by $R := Q \otimes 1$. The operator $G = F \otimes 1$ is then given by $\begin{bmatrix} 0 & R^* \\ R & 0 \end{bmatrix}$.
For $i = 1, 2, \cdots, n$ and $t \in [0, \frac{\pi}{2}]$, let $w_i^{(t)} \in \mathcal{L}_{\mathcal{T}}(\ell^2(\mathbb{F}_n^+) \otimes \mathcal{T})$ be defined by

$$w_i^{(t)} := \cos(t)(v_i p \otimes 1) + \sin(t)(p \otimes V_i) + v_i(1-p) \otimes 1$$

where $V_i = V_{a_i}$ and p denotes the orthognal projection of $\ell^2(\mathbb{F}_n^+)$ onto the one-dimensional subspace spanned by $\{\delta_1\}$. Observe that $\{w_i^t : i = 1, 2, \cdots, n\}$ is a collection of isometries with orthogonal range projections. For $t \in [0, \frac{\pi}{2}]$, let $w^t : \mathbb{F}_n^+ \to \mathcal{L}_{\mathcal{T}}(\mathcal{F}^{(0)})$ be the unique anti-homomorphism of isometries such that $w_{a_i}^t = w_i^t$.
For $t \in [0, \frac{\pi}{2}]$, let $\pi^t : A \to \mathcal{L}_{\mathcal{T}}(\mathcal{F}^{(0)})$ be defined by $\pi^t(x) = \sigma^{(0)}(x)$ for $x \in A$. Note that for $t \in [0, \frac{\pi}{2}]$, $x \in A$ and $i \in \{1, 2, \cdots, n\}$, $\pi^t(x)w_i^t = w_i^t \pi^t(\alpha_i(x))$. By Lemma 6.2, it follows that the pair (π^t, w^t) is Nica-covariant. Let $\tau^{(t)} : \mathcal{T} \to \mathcal{L}_{\mathcal{T}}(\mathcal{F}^{(0)})$ be the $*$-homomorphism such that $\tau^{(t)}(x) = \pi^t(x)$ and $\tau^{(t)}(v_a) = w_a^t$. Observe that $\tau^{(0)} = \sigma^{(0)}$. Also note that for $t \in [0, \frac{\pi}{2}]$, $\tau^{(t)}(v_{a_i}) - \sigma^{(0)}(v_{a_i})$ is compact for every $i = 1, 2, \cdots, n$ and $\tau^{(t)}(x) = \sigma^{(0)}(x)$ for $x \in A$. Since $\mathcal{T}$ is generated by A and the isometries $\{v_{a_i} : i = 1, 2, \cdots, n\}$, it follows that for every $T \in \mathcal{T}$, $\tau^{(t)}(T) - \sigma^{(0)}(T)$ is compact.
Hence in $KK(\mathcal{T}, \mathcal{T})$, $[d]\sharp[j] = (\mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)}, \tau^{(t)} \oplus \sigma^{(1)}, G)$ for every $t \in [0, \frac{\pi}{2}]$. Observe that the decomposition $\mathcal{F}^{(0)} = \mathcal{T} \oplus \mathcal{F}^{(1)}$ is left invariant by $\tau^{\frac{\pi}{2}}$. With respect to this decomposition $\tau^{\frac{\pi}{2}} = m \oplus \sigma^{(1)}$ where $m : \mathcal{T} \to \mathcal{T}$ is the identity map. Thus the product $[d]\sharp[j] = (\mathcal{T}, m, 0) \oplus \mathcal{D}$ where $\mathcal{D} = (\mathcal{F}^{(1)} \oplus \mathcal{F}^{(1)}, \sigma^{(1)} \oplus \sigma^{(1)}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$. Note that $\mathcal{D}$ is degenerate. As a consequence, it follows that $[d]\sharp[j] = [1_{\mathcal{T}}]$ in $KK(\mathcal{T}, \mathcal{T})$. This completes the proof. □

Remark 6.5 *Let A be a separable C^*-algebra and $\alpha : \mathbb{F}_n^+ \to End(A)$ be a left action. We do not assume that A is unital or α is unital. The KK-equivalence of the inclusion $A \ni x \to x \in A \rtimes_{red} \mathbb{F}_n^+$, can be deduced from the unital case by passing to the unitisation and by appealing to the split-exact sequence in Theorem 3.5.*

References

[1] Nathanial P. Brown and Narutaka Ozawa, *C^*-algebras and finite dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.

[2] Joachim Cuntz, Siegfried Echterhoff, and Xin Li, *On the K-theory of crossed products by automorphic semigroup actions*, Q.J. Math. 64 (2013), no. 3, 747-784.

[3] Ruy Exel, *A new look at the crossed-product of a C^*-algebra by an endomorphism*, Ergodic Theory Dynam. Systems. 23 (2003), no. 6, 1733-1750.

[4] M. Khoshkam and G. Skandalis, *Toeplitz algebras associated with endomorphisms and Pimsner-Voiculescu exact sequences*, Pacific J. Math. 181 (1997), no. 2, 315-331.

[5] Mahmood Khoshkam and Georges Skandalis, *Crossed products of C^*-algebras by groupoids and inverse semigroups*, J. Operator Theory 51 (2004), no. 2, 255-279.

[6] Xin Li, *Semigroup C^*-algebras and amenability of semigroups*, J. Funct. Anal. 262 (2012), no. 10, 4302-4340.

[7] Xin Li, *Nuclearity of semigroup C^*-algebras and the connection to amenability*, Adv. Math. 244 (2013), 626-662.

[8] Paul S. Muhly and Jean N. Renault, *C^*-algebras of multivariable Wiener-Hopf operators*, Trans. Amer. Math. Soc. 274 (1982), no. 1, 1-44.

[9] Gerard J. Murphy, *Crossed products of C^*-algebras by semigroups of automorphisms*, Proc. London Math. Soc. (3) 68 (1994), no. 2, 423-448.

[10] Gerard J. Murphy, *Crossed products of C^*-algebras by endomorphisms*, Integral Equations and Operator Theory 24 (1996), no. 3, 298-319.

[11] Paul S. Muhly and Dana P. Williams, *Renault's equivalence theorem for groupoid crossed products*, NYJM Monographs, vol. 3, 2008.

[12] Alexandru Nica, *C^*-algebras generated by isometries and Wiener-Hopf operators*, J. Operator Theory 27 (1992), no. 1, 17-52.

[13] Alexandru Nica, *On a groupoid construction for actions of certain inverse semigroups*, Internat. J. Math. 5 (1994), no. 3, 349-372.

[14] Jean-Pierre Serre, *Trees*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[15] Aidan Sims and Dana P. Williams, *Amenability for Fell bundles over groupoids*, Illinois J. Math. 57 (2013), no. 2, 429-444.

[16] Dana P. Williams, *Crossed products of C^*-algebras*, Mathematical Surveys and Monographs, vol. 134, American Mathematical Society, Providence RI, 2007.

S. Sundar
Institute of Mathematical Sciences
CIT Campus
4th Cross Street
Taramani, Chennai 600113
India
sundarsobers@gmail.com