# Green Functions and Higher Deligne–Lusztig Characters

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Abstract. We give a generalisation of the character formula of Deligne–Lusztig representations from the finite field case to the truncated formal power series case. Motivated by this generalisation, we give a definition of Green functions for these local rings, and prove some basic properties along the lines of the finite field case, like a summation formula. As an application we show that the higher Deligne–Lusztig characters and Gérardin's characters agree at regular semisimple elements.

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## 1 Introduction and notations

The family of representations introduced in Deligne and Lusztig's work [DL76] plays a crucial role in the representation theory of reductive groups over finite fields, since 1976. To compute the characters of these representations, there are roughly two steps involved: The first one is a formula expressing these characters in terms of Green functions of smaller reductive groups, and the second one is to compute the Green functions using generalised Springer theory and character sheaves (Lusztig–Shoji algorithm).

In this paper we give a study on the generalisation of the above first step, from finite fields to discrete valuation rings. There are three perspectives. In Section 2, we prove a character formula for reductive groups over (quotients of) complete discrete valuation rings of positive characteristics, which expresses the characters of higher Deligne–Lusztig representations in terms of the traces of certain unipotent elements; see Theorem 2.1 and Theorem 3.3. This generalises the character formula from the finite field case in a natural way. In Section 3, motivated by this generalisation, we give a definition of Green functions in this ring setting; see Definition 3.1. Similar to the finite field case, these functions are defined on unipotent elements, and are independent of the choice of characters of the maximal torus (which are parameters of higher Deligne–Lusztig characters). We show that they enjoy some nice properties, such as a summation formula; see Corollary 3.4, Corollary 3.6, Corollary 3.7, and Proposition 3.8. In Section 4 we focus on regular semisimple elements. Using the Green functions we establish a formula of higher Deligne–Lusztig characters on regular semisimple elements; see Theorem 4.1. This formula is independent of an integer parameter involved, and as an immediate application, we deduce that, at regular semisimple elements, the values of higher Deligne–Lusztig characters are the same with that of Gérardin's characters (constructed in [Gér75]), thus obtaining a variation of a prediction of Lusztig [Lus04]; see Corollary 4.4 and Remark 4.5.

In the below we introduce some notations and describe some results with more details.

Let $\mathcal{O}$ be a complete discrete valuation ring with a finite residue field $\mathbb{F}_q$, let $\pi$ be a fixed uniformiser of $\mathcal{O}$, and let $\mathbb{G}$ be a connected reductive group over $\mathcal{O}_r := \mathcal{O}/\pi^r$ where $r$ is a fixed arbitrary positive integer. We want to study the complex smooth representations of reductive groups over $\mathcal{O}$, or equivalently, the complex representations of $\mathbb{G}(\mathcal{O}_r)$ (for all $r \in \mathbb{Z}_{>0}$). One approach for studying these representations for a general $r \geq 1$, on which this paper is based, is a geometric theory proposed by Lusztig in [Lus79] (with some missing proofs given in [Lus04] for $\mathrm{char}(\mathcal{O}) = 0$, which were generalised in [Sta09] for all characteristics using the Greenberg functor technique). This theory generalises the work in the $r = 1$ case in [DL76], so we call it the higher Deligne–Lusztig theory. In this paper we will be only working with $\mathrm{char}(\mathcal{O}) > 0$; we have $\mathcal{O} = \mathbb{F}_q[[\pi]]$.

We recall the basic settings in [Lus04]. Write $\mathbf{G} := \mathbb{G} \times_{\mathrm{Spec}\ \mathcal{O}_r} \mathrm{Spec}\ \mathcal{O}_r^{\mathrm{ur}}$, where $\mathcal{O}^{\mathrm{ur}} := \overline{\mathbb{F}}_q[[\pi]]$ is a maximal unramified extension of $\mathcal{O}$ and $\mathcal{O}_r^{\mathrm{ur}} := \mathcal{O}^{\mathrm{ur}}/\pi^r$. Let $\mathbf{B} = \mathbf{T} \ltimes \mathbf{U}$ be the Levi decomposition of a Borel subgroup of $\mathbf{G}$. By Weil restriction, we can view $\mathbf{G}(\mathcal{O}_r^{\mathrm{ur}})$, $\mathbf{B}(\mathcal{O}_r^{\mathrm{ur}})$, $\mathbf{T}(\mathcal{O}_r^{\mathrm{ur}})$, and $\mathbf{U}(\mathcal{O}_r^{\mathrm{ur}})$ as the $\overline{\mathbb{F}}_q$-points of some algebraic groups $G$, $B$, $T$, and $U$, respectively, over $\overline{\mathbb{F}}_q$. The Frobenius map on $\overline{\mathbb{F}}_q/\mathbb{F}_q$ induces a geometric Frobenius endomorphism on $G$, such that $G^F \cong \mathbb{G}(\mathcal{O}_r)$. We only consider the case that $F(T) \subseteq T$. Let $L$ be the Lang isogeny associated to $F$, namely, $L(g) = g^{-1}F(g)$ for all $g \in G$. We fix a rational prime integer $\ell \nmid q$; we will be working with the representations over $\overline{\mathbb{Q}}_\ell$.

The variety $L^{-1}(FU) \subseteq G$ admits a left $G^F$-action and a right $T^F$-action in a natural way. These actions induce a $(G^F, T^F)$-bimodule structure on the compactly supported $\ell$-adic cohomology groups $H_c^i(L^{-1}(FU),\overline{\mathbb{Q}}_\ell)$. Let $H_c^i(L^{-1}(FU),\overline{\mathbb{Q}}_\ell)_\theta$ be the isotypic component for a given $\theta\in\widehat{T^F}:=\mathrm{Hom}(T^F,\overline{\mathbb{Q}}_\ell^\times)$, then

$$R_{T,U}^\theta:=\sum_i(-1)^iH_c^i(L^{-1}(FU),\overline{\mathbb{Q}}_\ell)_\theta$$

is a virtual representation of $G^F$, referred to as a higher Deligne–Lusztig representation.

Denote $G$ by $G_r$, then for each $i\in[0,r]\cap\mathbb{Z}$, there is a morphism of algebraic groups $\rho_{r,i}\colon G_r\to G_i$, called the reduction map, induced by the reduction modulo $\pi^i$. The morphism $\rho_{r,i}$ is surjective by the smoothness of $\mathbf{G}$; we denote its kernel, a normal closed subgroup of $G$, by $G^i$. In particular, we have $G^0=G$ (not to be confused with the identity component $G^o$). Similar notation applies to the closed subgroups of $G$ (like $B$, $U$, and $T$).

The above objects lead to the representation

$$R_{T,U,b}^\theta:=\sum_i(-1)^iH_c^i(L^{-1}(FU^{b,r-b}),\overline{\mathbb{Q}}_\ell)_\theta,$$

where $b\in[0,r]\cap\mathbb{Z}$ and $U^{b,r-b}:=U^b(U^-)^{r-b}$ (here $U^-$ denotes the algebraic group corresponding to the opposite of $\mathbf{U}$). This construction was first studied in [Che18]; note that it naturally generalises the representations studied in [Lus79], [Lus04], [Sta09], [Che16], and [CS17] (as clearly $R_{T,U}^\theta=R_{T,U,r}^\theta$).

These $R_{T,U,b}^\theta$ are the basic objects in this paper. We prove that (see Theorem 2.1)

$$\begin{aligned}&\mathrm{Tr}(g,R_{T,U,b}^\theta)\\&=\frac{1}{|T^F|}\cdot\frac{1}{|(\mathrm{Stab}_G(s)^o)^F|}\cdot\sum_{\left\{\substack{h\in G^F\text{ s.t.}\\ s\in{}^h(T_1)^F}\right\}}\sum_{\tau\in{}^h(T^1)^F}\theta(s^h\cdot\tau^h)\\&\quad\cdot\mathrm{Tr}\left((u,\tau^{-1})\mid\sum_i(-1)^i\cdot H_c^i(\mathrm{Stab}_G(s)^o\cap L^{-1}({}^hFU^{r-b,b}),\overline{\mathbb{Q}}_\ell)\right),\end{aligned}$$

where $g=su\in G^F$ denotes the Jordan decomposition. Recall that $T^1$ is a unipotent group (and is trivial if $r=1$). This suggests a definition of two-variable Green functions $Q_{T,U,b}^G(-,-)$ defined on some unipotent elements (see Definition 3.1), which admit the following summation property (see Proposition 3.8)

$$\sum_{u\in(\mathcal{U}_G)^F}\sum_{\tau\in((T^1))^F}Q_{T,U,b}^G(u,\tau)=|G^F/T_1^F|.$$

These Green functions allow us to re-write the above character formula as (see Theorem 3.3)

$$\begin{aligned}\mathrm{Tr}(g, R^{\theta}_{T,U,b}) =& \frac{1}{|(\mathrm{Stab}_G(s)^o)^F|}\\ &\cdot \sum_{\left\{\substack{h\in G^F \text{ s.t.}\\ s\in {}^h(T_1)^F}\right\}} \sum_{\tau\in {}^h(T^1)^F} \theta(s^h\cdot\tau^h)\cdot Q^{\mathrm{Stab}_G(s)^o}_{{}^hT,{}^hU\cap\mathrm{Stab}_G(s)^o,b}(u,\tau^{-1}).\end{aligned}$$

Using this formula we can evaluate the values of $\mathrm{Tr}(-, R^{\theta}_{T,U,b})$ at regular semisimple elements easily: (See Theorem 4.1)

$$\mathrm{Tr}(s, R^{\theta}_{T,U,b}) = \sum_{w\in W(T)^F} ({}^w\theta)(s^c),$$

if $c\in G^F$ is an element conjugating $s$ into $T_1$. From this formula we can derive some independence properties: One of them links $R^{\theta}_{T,U}$ to certain irreducible representations constructed by Gérardin in [Gér75] (see Remark 4.3, Corollary 4.4, and Remark 4.5).

We record here some further notation and conventions. The set of roots of $\mathbf{T}$ will be denoted by $\Phi$; for $\alpha\in\Phi$, we let $\mathbf{U}_\alpha\subseteq\mathbf{U}$ be the corresponding root subgroup, and we let $U_\alpha\subseteq U$ be the corresponding algebraic group. There is an isomorphism of finite groups $N_G(T)/T\cong N_{G_1}(T_1)/T_1$ (see [CS17, Section 2]), thus we will use $W(T)$ to denote the Weyl group of $\mathbf{T}$. We will use the conjugation notation $a^b = {}^{b^{-1}}a = b^{-1}ab$ for suitable objects $a$ and $b$. For a character of a finite abelian group, we sometimes use the same notation for its representation space and itself.

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## 2 The character formula

We will work with the general $R^{\theta}_{T,U,b}$ (see Section 1) throughout this paper; a rewarding result for using this generality will be seen at Section 4.

We want to prove the following character formula:

Theorem 2.1. *Let $\mathfrak{R}^{\theta}_{T,U,b}$ be the character of $R^{\theta}_{T,U,b}$, then for $g\in G^F$ we have*

$$\begin{aligned}\mathfrak{R}^{\theta}_{T,U,b}(g) =& \frac{1}{|T^F|}\cdot\frac{1}{|(\mathrm{Stab}_G(s)^o)^F|}\cdot \sum_{\left\{\substack{h\in G^F \ s.t.\\ s\in {}^h(T_1)^F}\right\}} \sum_{\tau\in {}^h(T^1)^F} \theta(s^h\cdot\tau^h)\\ &\cdot \mathrm{Tr}\left((u,\tau^{-1}) \mid \sum_i (-1)^i\cdot H^i_c(\mathrm{Stab}_G(s)^o\cap L^{-1}({}^hFU^{r-b,b}),\overline{\mathbb{Q}}_\ell)\right),\end{aligned}$$

*where $g = su$ is the Jordan decomposition.*

*Proof.* By Broué's character formula on bimodule induction (see e.g. [DM91, Chapter 4]) we have

$$\mathfrak{R}^{\theta}_{T,U,b}(g) = \frac{1}{|T^F|} \cdot \sum_{t \in T^F} \theta(t^{-1}) \cdot \mathrm{Tr}\left((g,t) \mid \sum_i (-1)^i H^i_c(L^{-1}(FU^{r-b,b}), \overline{\mathbb{Q}}_\ell)\right).$$

So, applying Deligne and Lusztig's fixed point formula (see [DL76, 3]) to the RHS we see $\mathfrak{R}^{\theta}_{T,U,b}(g)$ equals to

$$\frac{1}{|T^F|} \cdot \sum_{t \in T^F} \theta(t^{-1}) \cdot \mathrm{Tr}\left((u,t'') \mid \sum_i (-1)^i H^i_c(L^{-1}(FU^{r-b,b})^{(s,t')}, \overline{\mathbb{Q}}_\ell)\right),$$

where $t = t't''$ is the natural decomposition via $T^F \cong (T_1)^F \times (T^1)^F$. We want to analyse the structure of the variety $L^{-1}(FU^{r-b,b})^{(s,t')}$.

If $h \in G^F$ conjugates $t'$ to $s^{-1}$ (i.e. $ht'h^{-1} = s^{-1}$), and $z \in \mathrm{Stab}_G(t')$ has its Lang image belonging to $FU^{r-b,b}$ (i.e. $L(z) \in FU^{r-b,b}$), then clearly $hz \in L^{-1}(FU^{r-b,b})^{(s,t')}$. So we have a well-defined multiplication morphism

$$\begin{array}{c}\{h \in G^F \mid ht' = s^{-1}h\} \times \{z \in \mathrm{Stab}_G(t')^o \mid L(z) \in FU^{r-b,b}\} \\ \phi\big\downarrow \\ L^{-1}(FU^{r-b,b})^{(s,t')},\end{array}$$

given by $\phi(h,z) = hz$.

We shall show that $\phi$ is surjective. Take $x \in L^{-1}(FU^{r-b,b})^{(s,t')}$, then by the rationality of $s$ and $t'$ we see that

$$sF(x)t' = F(x) = x \cdot L(x) = sxt' \cdot L(x),$$

which implies

$$L(x) \cdot t' = x^{-1}F(x)t' = x^{-1}s^{-1}(sF(x)t') = x^{-1}s^{-1}(sxt' \cdot L(x)) = t' \cdot L(x),$$

namely $L(x) \in \mathrm{Stab}_G(t')$. Now write $L(x) = x_1x_2$ with $x_1 \in G_1$ and $x_2 \in G^1$, via the product decomposition $G = G_1 \ltimes G^1$, then by taking the reduction map we see $x_1 \in \mathrm{Stab}_{G_1}(t')$, which also implies that $x_2 \in \mathrm{Stab}_{G^1}(t')$. Since $x_1 \in FU$ is unipotent, we have (see e.g. [DM91, 2.5])

$$x_1 \in \mathrm{Stab}_{G_1}(t')^o. \tag{1}$$

On the other hand, for a given $\tilde{x} \in \mathrm{Stab}_{G^1}(t')$, consider the unique Iwahori decomposition $\tilde{x} = \tilde{t}\tilde{u}$ (in the sense of [Sta09, 2.2]), where $\tilde{t} \in T^1$ and $\tilde{u} \in U^1(U^-)^1$. Clearly $\tilde{t}$ commutes with $t'$, so $\tilde{u}$ also commutes with $t'$. Write

$\tilde{u} = \prod u_\alpha$, where $\alpha$ runs over the roots and $u_\alpha \in (U_\alpha)^1$, then the commutativity between $t'$ and $\tilde{u}$ implies that, for each root $\alpha$, either $u_\alpha = 1$ or $\alpha(t') = 1$. Therefore $\mathrm{Stab}_{G^1}(t')$ is an affine space. (Moreover, the argument implies that $\mathrm{Stab}_G(t')^o$ is the Weil restriction of the base change of the connected reductive group $\mathrm{Stab}_{G_1}(t')^o$ from $\overline{\mathbb{F}}_q$ to $\mathcal{O}_r^{\mathrm{ur}}$; see also the argument in [DM91, 2.3].) In particular, we have

$$x_2 \in \mathrm{Stab}_{G^1}(t')^o. \tag{2}$$

It follows from (1) and (2) that

$$L(x) \in \mathrm{Stab}_G(t')^o,$$

so by the Lang–Steinberg theorem we deduce $L(x) = L(z)$ for some $z \in \mathrm{Stab}_G(t')^o$. Let $h := xz^{-1}$, then $h \in G^F$ and

$$ht'h^{-1} = xz^{-1}t'zt'^{-1}x^{-1}s^{-1} = s^{-1},$$

thus $\phi$ is surjective.

Meanwhile, note that $\phi(h,z) = \phi(h',z')$ if and only if $h^{-1}h' = zz'^{-1} \in (\mathrm{Stab}_G(t')^o)^F$, so, for a fixed set of representatives of $G^F/(\mathrm{Stab}_G(t')^o)^F$, $\phi$ induces an isomorphism

$$L^{-1}(FU^{r-b,b})^{(s,t')} \cong \coprod_{\left\{\substack{h\in G^F/(\mathrm{Stab}_G(t')^o)^F \\ \text{s.t. } ht'=s^{-1}h}\right\}} \{z \in \mathrm{Stab}_G(t')^o \mid L(z) \in FU^{r-b,b}\}_h,$$

where $\{z \in \mathrm{Stab}_G(t')^o \mid L(z) \in FU^{r-b,b}\}_h$ is a copy of $\{z \in \mathrm{Stab}_G(t')^o \mid L(z) \in FU^{r-b,b}\}$ on which the action of $(u,t'')$ is given by $z \mapsto (u^h)\cdot z \cdot t''$.

Note that

$$\{z \in \mathrm{Stab}_G(t')^o \mid L(z) \in FU^{r-b,b}\} = \mathrm{Stab}_G(t')^o \cap L^{-1}(\mathrm{Stab}_G(t')^o \cap FU^{r-b,b})$$

is the corresponding higher Deligne–Lusztig variety of $\mathrm{Stab}_G(t')^o$; let us denote it by $L_{t'}$. Therefore $\mathfrak{R}^\theta_{T,U,b}(g)$ equals to

$$\begin{aligned}
&\frac{1}{|T^F|}\cdot\sum_{t\in T^F}\theta(t^{-1})\cdot\sum_{\left\{\substack{h\in G^F/(\mathrm{Stab}_G(t')^o)^F \\ \text{s.t. } ht'=s^{-1}h}\right\}}\mathrm{Tr}\left((u^h,t'')\mid H_c^*(L_{t'})\right)\\
&=\frac{1}{|T^F|}\cdot\sum_{t\in T^F}\theta(t^{-1})\cdot\sum_{\left\{\substack{h\in G^F \text{ s.t.} \\ ht'=s^{-1}h}\right\}}\frac{1}{|(\mathrm{Stab}_G(t')^o)^F|}\cdot\mathrm{Tr}\left((u^h,t'')\mid H_c^*(L_{t'})\right)\\
&=\frac{1}{|T^F|}\cdot\frac{1}{|(\mathrm{Stab}_G(s)^o)^F|}\cdot\sum_{t\in T^F}\sum_{\left\{\substack{h\in G^F \text{ s.t.} \\ ht'=s^{-1}h}\right\}}\theta(t^{-1})\cdot\mathrm{Tr}\left((u^h,t'')\mid H_c^*(L_{t'})\right)\\
&=\frac{1}{|T^F|}\frac{1}{|(\mathrm{Stab}_G(s)^o)^F|}\sum_{\left\{\substack{h\in G^F \text{ s.t.} \\ s\in {}^h(T_1)^F}\right\}}\sum_{\tau\in {}^h(T^1)^F}\theta((s\tau)^h)\mathrm{Tr}\left((u,\tau^{-1})\mid H_c^*({}^hL_{t'})\right),
\end{aligned}$$

where $H_c^*(-) := \sum_i (-1)^i H_c^i(-, \overline{\mathbb{Q}}_\ell)$ and in the last summation we put $t' = (s^h)^{-1}$. Since $\mathrm{Stab}_G(t') = \mathrm{Stab}_G(t'^{-1})$, we complete the proof by expressing $L_{t'}$ in terms of $s$. □

Remark 2.2. Let $s \in G_1$ be a semisimple element, then $\mathrm{Stab}_{G_1}(s)^o$ is a connected reductive group; from the proof of Theorem 2.1 we see that an analogue of this property also holds for $\mathrm{Stab}_G(s)^o$: If $s \in T_1$ is a semisimple element, and if $H$ is the $\overline{\mathbb{F}}_q$-Weil restriction of $\mathrm{Stab}_{G_1}(s)^o \times_{\mathrm{Spec}\ \overline{\mathbb{F}}_q} \mathrm{Spec}\ \mathcal{O}_r^{\mathrm{ur}}$, then $H = \mathrm{Stab}_G(s)^o$.

Remark 2.3. In general, if $s' \in G$ is a semisimple element, then it is contained in a maximal torus, thus can be conjugated to be some $s \in T_1 \subseteq G_1$ (though, there are lots of semisimple elements not in $G_1$). However, in this general situation it can happen that the property in Remark 2.2 fails for $\mathrm{Stab}_G(s')^o$: Suppose $s'^h = s$, then the Weil restriction of $(\mathrm{Stab}_G(s')^o)_1 \times_{\mathrm{Spec}\ \overline{\mathbb{F}}_q} \mathrm{Spec}\ \mathcal{O}_r^{\mathrm{ur}}$ is $(\mathrm{Stab}_G(s)^o)^{\rho_{r,1}(h)}$, which may not be equal to $(\mathrm{Stab}_G(s)^o)^h = \mathrm{Stab}_G(s')^o$.

## 3 Green functions

We want to give a Green function theoretic interpretation of Theorem 2.1, in a way similar to [DL76, 4.2]; we start with the following definition.

Definition 3.1. Suppose that $G$ is a closed subgroup of some linear algebraic group $\widetilde{G}$ and that the Frobenius endomorphism $F$ is also defined on $\widetilde{G}$. Then for $h \in \widetilde{G}^F$ and a quadruple $(G, T, U, b)$, the function

$$Q^{G^h}_{T^h, U^h, b} \colon (\mathcal{U}_{G^h})^F \times ((T^1)^h)^F \to \overline{\mathbb{Q}}_\ell$$

defined by

$$(u, \tau) \mapsto \frac{1}{|T^F|} \mathrm{Tr}\left((u, \tau) \mid \sum_i (-1)^i \cdot H_c^i(L^{-1}(FU^{r-b,b})^h, \overline{\mathbb{Q}}_\ell)\right)$$

is called a Green function on $(G^h)^F$. Here $\mathcal{U}_{G^h}$ is the variety of unipotent elements in $G^h$.

The above definition actually includes a wider class of groups than the Weil restrictions of reductive groups over $\mathcal{O}_r$, as it may happen that $G^h$ does not equal to the Weil restriction of $(G^h)_1 \times_{\mathrm{Spec}\ \overline{\mathbb{F}}_q} \mathrm{Spec}\ \mathcal{O}_r^{\mathrm{ur}}$ (see Remark 2.3).

Remark 3.2. Recall that, if $r = 1$ and $\theta$ a general character of $T^F$, then the values of $\mathfrak{R}^\theta_{T,U}$ at unipotent elements only depend on the dimension of $\theta$; however, in the general $r \geq 1$ case, the values of $\mathfrak{R}^\theta_{T,U}$ at unipotent elements also depend on the values of $\theta$ at non-trivial unipotent elements in $T^F$ (see [Lus04, Section 3]). So, instead of using $R^1_{T,U}$ to define Green functions as in Deligne–Lusztig's original work [DL76], we used the above average form.

Using the above Green functions, Theorem 2.1 can be re-written as:

Theorem 3.3. *Let $\mathfrak{R}^\theta_{T,U,b}$ be the character of $R^\theta_{T,U,b}$, then for $g \in G^F$ we have*

$$\mathfrak{R}^\theta_{T,U,b}(g) = \frac{1}{|(\mathrm{Stab}_G(s)^o)^F|} \cdot \sum_{\left\{\substack{h\in G^F \ s.t. \\ s\in {}^h(T_1)^F}\right\}} \sum_{\tau \in {}^h(T^1)^F} \theta(s^h \cdot \tau^h) \cdot Q^{\mathrm{Stab}_G(s)^o}_{{}^hT, {}^hU \cap \mathrm{Stab}_G(s)^o, b}(u, \tau^{-1}),$$

*where $g = su$ is the Jordan decomposition.*

In the $r = 1$ case, the Green functions are $\mathbb{Z}$-valued (see e.g. [Car93, 7.6]). Based on some computations in special cases, it seems this is also true for $Q^G_{T,U,b}$ for any $r$ and $b$; in any case, we have the following weaker version:

Corollary 3.4. *The function $\sum_{\tau\in(T^1)^F} Q^G_{T,U,b}(-,\tau)$ on $(\mathcal{U}_G)^F$ is $\mathbb{Z}$-valued.*

*Proof.* Let $u \in G^F$ be a unipotent element. By Theorem 3.3 we have

$$\mathfrak{R}^1_{T,U,b}(u) = \frac{1}{|G^F|} \sum_{h\in G^F} \sum_{\tau\in {}^h(T^1)^F} Q^G_{{}^hT, {}^hU, b}(u, \tau^{-1}).$$

For $u$ and $\tau$ in the above expression, consider the commutative diagram

$$\begin{array}{ccc} L^{-1}(FU^{r-b,b})^{h^{-1}} & \xrightarrow{(u,\tau^{-1})} & L^{-1}(FU^{r-b,b})^{h^{-1}} \\ \Big\downarrow{\scriptstyle \mathrm{conj}\ h} & & \Big\downarrow{\scriptstyle \mathrm{conj}\ h} \\ L^{-1}(FU^{r-b,b}) & \xrightarrow{(u^h,(\tau^{-1})^h)} & L^{-1}(FU^{r-b,b}), \end{array}$$

in which all arrows are isomorphisms, hence

$$Q^G_{{}^hT, {}^hU, b}(u, \tau^{-1}) = Q^G_{T,U,b}(u^h, (\tau^{-1})^h);$$

moreover, by definition we can view $Q^G_{T,U,b}$ as a restriction of a class function of $G^F \times (T^1)^F$, thus $Q^G_{T,U,b}(u^h, (\tau^{-1})^h) = Q^G_{T,U,b}(u, (\tau^{-1})^h)$. Therefore we get

$$\begin{aligned} \mathfrak{R}^1_{T,U,b}(u) &= \frac{1}{|G^F|} \sum_{h\in G^F} \sum_{\tau\in {}^h(T^1)^F} Q^G_{T,U,b}(u, (\tau^{-1})^h) \\ &= \frac{1}{|G^F|} \sum_{h\in G^F} \sum_{\tau\in (T^1)^F} Q^G_{T,U,b}(u, \tau^{-1}), \end{aligned}$$

which equals to $\sum_{\tau\in(T^1)^F} Q^G_{T,U,b}(u,\tau^{-1}) = \sum_{\tau\in(T^1)^F} Q^G_{T,U,b}(u,\tau)$.

In particular, $\sum_{\tau\in(T^1)^F} Q^G_{T,U,b}(-,\tau)$ has its values being algebraic integers. Meanwhile, by basic properties of Lefschetz numbers (see [Lus78, 1.2]) we know this sum also takes values in $\mathbb{Q}$, so it must takes values in $\mathbb{Z}$. □

Definition 3.5. Let $p := \mathrm{char}(\mathbb{F}_q)$. A class function $f\colon G^F \to \overline{\mathbb{Q}}_\ell$ is called $p$-constant if $f(g) = f(s)$ for all $g \in G^F$, where $g = su$ is the Jordan decomposition with $s$ semisimple and $u$ unipotent.

The notion of $p$-constant functions is very useful in the representations of Lie type groups. (For example, see [BM89, Section 2] for an application on the relations between $\ell$-blocks and Lusztig series.) The following corollary generalises a property in the $r = 1$ case (see [DM87, 3.8]).

Corollary 3.6. *Let $f\colon G^F \to \overline{\mathbb{Q}}_\ell$ be a $p$-constant class function, then $\mathfrak{R}_{T,U,b}^{\theta\cdot\mathrm{Res}^{G^F}_{T^F} f} = \mathfrak{R}^\theta_{T,U,b} \cdot f$.*

*Proof.* Let $g \in G^F$ and let $g = su$ be the Jordan decomposition. By Theorem 3.3 we see that $\mathfrak{R}_{T,U,b}^{\theta\cdot\mathrm{Res}^{G^F}_{T^F} f}(g)$ equals to

$$\frac{1}{|(\mathrm{Stab}_G(s)^o)^F|}\cdot \sum_{\left\{\substack{h\in G^F \text{ s.t.}\\ s\in {}^h(T_1)^F}\right\}} \sum_{\tau\in {}^h(T^1)^F} \theta((s\tau)^h)\cdot f((s\tau)^h) \cdot Q^{\mathrm{Stab}_G(s)^o}_{{}^hT,{}^hU\cap \mathrm{Stab}_G(s)^o,b}(u,\tau^{-1}).$$

Note that $f((s\tau)^h) = f(s\tau) = f(s) = f(g)$, which completes the proof. □

There is also an inner product version of the above corollary (see [DL76, 7.11] or [Car93, 7.6.3] for the $r = 1$ case).

Corollary 3.7. *Let $R$ be the representation space of a $p$-constant virtual character of $G^F$, then $\langle R, R^\theta_{T,U,b}\rangle_{G^F} = \langle \mathrm{Res}^{G^F}_{T^F} R, \theta\rangle_{T^F}$.*

*Proof.* By Corollary 3.6 and the Hom–tensor adjunction we see

$$\langle R, R^\theta_{T,U,b}\rangle_{G^F} = \langle R_{T,U,b}^{\theta^{-1}\cdot\mathrm{Res}^{G^F}_{T^F}\chi_R}, 1\rangle_{G^F}, \tag{3}$$

where $\chi_R$ means the character of $R$. As $R^{(-)}_{T,U,b}$ is the induction (from the virtual representations of $T^F$ to the virtual representations of $G^F$) provided by the alternating sum of bimodules $\sum_i (-1)^i H^i_c(L^{-1}(FU^{b,r-b}), \overline{\mathbb{Q}}_\ell)$, there is a restriction functor ${}^*R^{(-)}_{T,U,b}$ adjoint to $R^{(-)}_{T,U,b}$ (see [DM91, Chapter 4] for more details); we get

$$(3) = \langle \theta^{-1}\otimes \mathrm{Res}^{G^F}_{T^F}\chi_R, {}^*R^1_{T,U,b}\rangle_{T^F}.$$

Consider the natural Lang surjection $L\colon L^{-1}(FU^{b,r-b}) \to FU^{b,r-b}$; as $L$ is finite separable with fibres isomorphic to $G^F$, we have

$$G^F\backslash L^{-1}(FU^{b,r-b}) \cong FU^{b,r-b},$$

thus

$$ {}^*R^1_{T,U,b} = \sum_i (-1)^i H^i_c(G^F \backslash L^{-1}(FU^{b,r-b}), \overline{\mathbb{Q}}_\ell) = \sum_i (-1)^i H^i_c(FU^{b,r-b}, \overline{\mathbb{Q}}_\ell). $$

Sine $FU^{b,r-b}$ is an affine space, $\sum_i (-1)^i H^i_c(FU^{b,r-b}, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell$; meanwhile, as the action of $T^F$ on the affine space $FU^{b,r-b}$ extends to the action of the connected algebraic group $T = FT$, we see that ${}^*R^1_{T,U,b} = \overline{\mathbb{Q}}_\ell$ is actually the trivial representation (by [DL76, 6.4]), from which the assertion follows. □

There is a "Green (function) integration formula":

PROPOSITION 3.8. *Let $\widetilde{G}$ be as in Definition 3.1. We have*

$$ \sum_{u \in (\mathcal{U}_{G^x})^F} \sum_{\tau \in ((T^1)^x)^F} Q^{G^x}_{T^x, U^x, b}(u, \tau) = |G^F / T_1^F| $$

*for every $x \in \widetilde{G}^F$. (Compare the $r = 1$ case in [Car93, 7.6.1].)*

*Proof.* We use an induction argument on $i := |\dim G / \dim T|$. If $i = 1$, then $G = T$ (note that $\mathcal{U}_T = T^1$), hence

$$ \begin{aligned} &\sum_{u \in (\mathcal{U}_{G^x})^F} \sum_{\tau \in ((T^1)^x)^F} Q^{G^x}_{T^x, U^x, b}(u, \tau) \\ &= \frac{1}{|T^F|} \sum_{u \in ((T^1)^x)^F} \sum_{\tau \in ((T^1)^x)^F} \mathrm{Tr}((u, \tau) \mid \overline{\mathbb{Q}}_\ell[(T^x)^F]) \\ &= |T^F / T_1^F|, \end{aligned} $$

as desired. Suppose now the assertion is true for every $i \leq n$, and suppose $\dim G / \dim T = n + 1$. From the proof of Corollary 3.7 we see that $\langle \sum_i (-1)^i H^i_c(L^{-1}(FU)^x, \overline{\mathbb{Q}}_\ell), 1 \rangle_{(G^x)^F} = 1$, that is: (Use Theorem 3.3)

$$ \begin{aligned} &\frac{1}{|G^F|} \sum_{g = su \in (G^x)^F} \frac{1}{|(\mathrm{Stab}_{G^x}(s)^o)^F|} \\ &\quad \times \sum_{\left\{ \substack{h \in (G^x)^F \text{ s.t.} \\ s \in {}^h((T_1)^x)^F} \right\}} \sum_{\tau \in {}^h((T^1)^x)^F} Q^{\mathrm{Stab}_{G^x}(s)^o}_{{}^h(T^x), {}^h(U^x) \cap \mathrm{Stab}_{G^x}(s)^o, b}(u, \tau^{-1}) = 1, \end{aligned} \tag{4} $$

where $g = su$ denotes the Jordan decomposition.

To proceed, we need to show that $u \in \mathrm{Stab}_{G^x}(s)^o$ for the Jordan decomposition in the above summation, which is well-known when $r = 1$. As $h \in (G^x)^F$, there is $a \in G^F$ such that $h = x^{-1} a x$; put $y := h x^{-1} = x^{-1} a$. Note that $G^{xy} = G^{x x^{-1} a} = G$. Meanwhile, since $s \in ({}^h(T_1)^x)^F$, we have

$$ s^y = x h^{-1} s h x^{-1} \in x h^{-1} ({}^h(T_1)^x)^F h x^{-1} = (T_1)^F. $$

So $\rho_{r,1}(u^y) \in \mathrm{Stab}_G(s^y)^o$ (use e.g. Remark 2.2), thus $u^y \in \mathrm{Stab}_G(s^y)^o$, so $u \in \mathrm{Stab}_{G^x}(s)^o$.

Therefore (4) becomes

$$\frac{1}{|G^F|} \sum_{\substack{s\in(G^x)^F \\ \text{semisimple}}} \frac{1}{|(\mathrm{Stab}_{G^x}(s)^o)^F|} \sum_{\substack{u\in(\mathrm{Stab}_{G^x}(s)^o)^F \\ \text{unipotent}}} \times \sum_{\left\{\substack{h\in(G^x)^F \text{ s.t.} \\ s\in {}^h((T_1)^x)^F}\right\}} \sum_{\tau\in {}^h((T^1)^x)^F} Q^{\mathrm{Stab}_{G^x}(s)^o}_{{}^h(T^x), {}^h(U^x)\cap \mathrm{Stab}_{G^x}(s)^o, b}(u,\tau^{-1}) = 1. \tag{5}$$

There are two cases of $s$, depending on whether $s \in Z^x$, where $Z$ denotes the center of $G$. As $\mathrm{Stab}_{G^x}(s)^o = G^x$ if and only if $s \in Z^x$, by our induction assumption we can re-write (5) as (note that $h \in (G^x)^F$)

$$\frac{|Z_1^F|}{|G^F|} \cdot \sum_{u\in(\mathcal{U}_{G^x})^F} \sum_{\tau\in((T^1)^x)^F} Q^{G^x}_{T^x,U^x,b}(u,\tau^{-1}) + \frac{1}{|G^F|} \sum_{\substack{s\in(G^x)^F\setminus(Z^x)^F \\ \text{semisimple}}} \sum_{\left\{\substack{h\in(G^x)^F \text{ s.t.} \\ s\in {}^h((T_1)^x)^F}\right\}} 1/|T_1^F| = 1.$$

Therefore

$$\frac{|Z_1^F|}{|G^F|} \cdot \sum_{u\in(\mathcal{U}_{G^x})^F} \sum_{\tau\in((T^1)^x)^F} Q^{G^x}_{T^x,U^x,b}(u,\tau^{-1}) - \frac{1}{|G^F|} \sum_{s\in(Z^x)_1^F} \sum_{\left\{\substack{h\in(G^x)^F \text{ s.t.} \\ s\in {}^h((T_1)^x)^F}\right\}} 1/|T_1^F| + \frac{1}{|G^F|} \sum_{\substack{s\in(G^x)^F \\ \text{semisimple}}} \sum_{\left\{\substack{h\in(G^x)^F \text{ s.t.} \\ s\in {}^h((T_1)^x)^F}\right\}} 1/|T_1^F| = 1. \tag{6}$$

We can simplify the sums in the above equality in the following way: First, for the second sum, as $Z^x$ is the centre of $G^x$, the condition "$s \in {}^h((T_1)^x)^F$" is equivalent to "$s \in ((T_1)^x)^F$", thus the sum equals to $-(1/|G^F|) \cdot |Z_1^F| \cdot |(G^x)^F| \cdot (1/|T_1^F|) = -|Z_1^F|/|T_1^F|$. Second, note that, if $s \in {}^h((T_1)^x)^F$, then $s$ is automatically semisimple, thus in the third sum we can first count all elements $h \in (G^x)^F$, then count all elements in $({}^h(T_1)^x)^F$, so the sum equals to $(1/|G^F|) \cdot |(G^x)^F| \cdot |({}^h(T_1)^x)^F| \cdot (1/|T_1^F|) = 1$. Therefore (6) can be written as

$$\frac{|Z_1^F|}{|G^F|} \cdot \sum_{u\in(\mathcal{U}_{G^x})^F} \sum_{\tau\in((T^1)^x)^F} Q^{G^x}_{T^x,U^x,b}(u,\tau^{-1}) - \frac{|Z_1^F|}{|T_1^F|} + 1 = 1,$$

from which the proposition follows. □

## 4 Regular semisimple elements

In this section we focus on the values of higher Deligne–Lusztig characters at regular semisimple elements. Recall that a semisimple element is called regular, if its centraliser is of minimal dimension; in our situation, this is equivalent to the saying that, a semisimple element $s$ is regular if and only if $\mathrm{Stab}_G(s)^o$ is isomorphic to $T$.

Theorem 4.1. *Let $s \in G^F$ be a regular semisimple element. We have: $\mathfrak{R}^{\theta}_{T,U,b}(s) = 0$ if the conjugacy class of $s$ in $G^F$ does not intersect $T_1$, and*

$$\mathfrak{R}^{\theta}_{T,U,b}(s) = \sum_{w \in W(T)^F} ({}^{w}\theta)(s^c)$$

*if $s^c \in T_1$ for some $c \in G^F$ (note that this can happen even if $s \in G \backslash G_1$). Here $W(T) := N(T)/T$ is isomorphic to the Weyl group $W(T_1) := N_{G_1}(T_1)/T_1$; see [CS17, Section 2].*

*Proof.* From the character formula we only need to deal with the case that the intersection of $T_1^F$ and the conjugacy class of $s$ (in $G^F$) is non-empty.

By regularity, the only conjugation of $T$ containing $s^c$ is $T$ (as otherwise $\mathrm{Stab}_G(s^c)^o$ is not isomorphic to $T$), thus any connected commutative subgroup containing $s^c$ is a subgroup of $T$. So the formula in Theorem 3.3 can be simplified as

$$\mathfrak{R}^{\theta}_{T,U,b}(s) = \mathfrak{R}^{\theta}_{T,U,b}(s^c) = \frac{1}{|T^F|} \cdot \sum_{h \in N_G(T_1)^F} \sum_{\tau \in (T^1)^F} \theta(s^{c \cdot h} \cdot \tau^h) \cdot Q^T_{T,\{1\},b}(1, \tau^{-1}). \tag{7}$$

In this summation, note that the function

$$Q^T_{T,\{1\},b}(1, \tau^{-1}) = \frac{1}{|T^F|} \mathrm{Tr}\left((1, \tau^{-1}) \mid \overline{\mathbb{Q}}_\ell[T^F]\right)$$

on $(T^1)^F$ is the characteristic function at the identity element, and note that $N_G(T_1) = N_{G_1}(T_1)T^1$ by an Iwahori decomposition argument as in the proof of Theorem 2.1, thus

$$(7) = \sum_{w \in W(T)^F} ({}^{w}\theta)(s^c),$$

as desired. □

Corollary 4.2. *The values of $\mathfrak{R}^{\theta}_{T,U,b}$ at regular semisimple elements are independent of the choices of $U$ and $b$.*

*Proof.* This follows immediately from Theorem 4.1. □

Remark 4.3. In [Gér75], by purely algebraic methods, whenever $(G, T, \theta)$ satisfies certain conditions (namely, $\mathbb{G}$ is defined from an unramified split group with the derived subgroup being simply connected, $T$ is special in the sense of [Gér75, 3.3.9], and $\theta$ is regular and in general position in the sense of [Lus04]; see [CS17, Remark 3.4]), Gérardin constructed an irreducible representation $R(\theta)$ of $G^F$ of the form: If $r$ is even, then $R(\theta) = \mathrm{Ind}_{(TG^{r/2})^F}^{G^F}\widetilde{\theta}$, where $\widetilde{\theta}$ is the trivial lift of $\theta$; if $r$ is odd, then $R(\theta) = \mathrm{Ind}_{(TG^{(r-1)/2})^F}^{G^F}\widetilde{\theta}$, where $\widetilde{\theta}$ is some irreducible representation of $(TG^{(r-1)/2})^F$ of dimension $q^{\#\Phi/2}$. Lusztig suggested in [Lus04] that these algebraically constructed representations are likely to be the same as the geometrically constructed representations $R_{T,U}^{\theta}$; when $r$ is even, this was proved in [Che17] for $\mathrm{GL}_n$ and in [CS17] in general.

However, even when $r$ is even, if one does not impose any restrictions on $\theta$, then, as can be seen from Lusztig's computations [Lus04, Section 3], it can happen that $R_{T,U}^{\theta}$ is not isomorphic to $\mathrm{Ind}_{(TG^{r/2})^F}^{G^F}\widetilde{\theta}$ for some $\theta$; in any case, we show that, even though the characters of these two representations may not be identical, they always agree at regular semisimple elements, without any conditions on $\theta$:

Corollary 4.4. *When $r$ is even, $\mathfrak{R}_{T,U,b}^{\theta}(s) = \mathrm{Tr}(s, \mathrm{Ind}_{(TG^{r/2})^F}^{G^F}\widetilde{\theta})$ for any regular semisimple element $s \in G^F$.*

*Proof.* Let $b = r/2$, then $R_{T,U,b}^{\theta} \cong \mathrm{Ind}_{(TG^{r/2})^F}^{G^F}\widetilde{\theta}$ according to [CS17, 3.3]. So the assertion follows from Corollary 4.2. □

Remark 4.5. If $(G, T, \theta)$ satisfies Gérardin's conditions mentioned in Remark 4.3, then the above equality also holds for $r$ odd, i.e. $\mathfrak{R}_{T,U}^{\theta}(s) = \mathrm{Tr}(s, R(\theta))$ for any regular semisimple element $s \in T^F$ and any $r$: Indeed, when $r$ is odd, according to [Gér75, 4.3.4], the character value of $R(\theta)$ at $s$ is $\frac{1}{|(TG^{(r-1)/2})^F|}\sum_{\{h\in G^F | s^h \in (TG^{(r-1)/2})^F\}} \theta(s^h) = \sum_{w\in W(T)^F} \theta(s^w)$.

Theorem 4.1 also implies an agreement at regular semisimple elements in another direction: It is easy to see that the image of a reduction map on a Deligne–Lusztig variety is a Deligne–Lusztig variety at a lower level (see e.g. [Che17, Lemma 3.3.3]). In particular, the image of $L^{-1}(FU)$ along $\rho_{r,1}$ is a classical Deligne–Lusztig variety; let $\mathfrak{R}_{T_1}^{\theta_1}$ be its associated Deligne–Lusztig character, where $\theta_1$ denotes the restriction of $\theta$ to $T_1^F$.

Corollary 4.6. *Let $s$ be a regular semisimple element of $G_1^F$, then $\mathfrak{R}_{T,U}^{\theta}(s) = \mathfrak{R}_{T_1}^{\theta_1}(s)$.*

*Proof.* This follows immediately from Theorem 4.1. □

Remark 4.7. Note that the assertions in Theorem 4.1, Corollary 4.2 Corollary 4.4, and Corollary 4.6 do not hold for every semisimple $s$, in general; for example, they do not hold for $s = 1$ for some $\theta$, as can be seen from [Lus04].

REFERENCES

[BM89] Michel Broué and Jean Michel. Blocs et séries de Lusztig dans un groupe réductif fini. *J. Reine Angew. Math.*, 395:56–67, 1989.

[Car93] Roger W. Carter. *Finite groups of Lie type*. Wiley Classics Library. John Wiley & Sons Ltd., Chichester, 1993. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.

[Che16] Zhe Chen. Generic character sheaves on reductive groups over a finite ring. Preprint 2016 arXiv 1604.02016.

[Che17] Zhe Chen. On generalised Deligne–Lusztig constructions. 2017. PhD Thesis. Durham University.

[Che18] Zhe Chen. On the inner products of some Deligne–Lusztig-type representations. *Accepted by International Mathematics Research Notices*, 2018. DOI 10.1093/imrn/rny195

[CS17] Zhe Chen and Alexander Stasinski. The algebraisation of higher Deligne–Lusztig representations. *Selecta Math. (N.S.)*, 23(4):2907–2926, 2017.

[DL76] Pierre Deligne and George Lusztig. Representations of reductive groups over finite fields. *Ann. of Math. (2)*, 103(1):103–161, 1976.

[DM87] François Digne and Jean Michel. Foncteurs de Lusztig et caractères des groupes linéaires et unitaires sur un corps fini. *J. Algebra*, 107(1):217–255, 1987.

[DM91] François Digne and Jean Michel. *Representations of finite groups of Lie type*, volume 21 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1991.

[Gér75] Paul Gérardin. *Construction de séries discrètes $p$-adiques*. Lecture Notes in Mathematics, Vol. 462. Springer-Verlag, Berlin-New York, 1975.

[Lus78] George Lusztig. *Representations of finite Chevalley groups*, volume 39 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, R.I., 1978. Expository lectures from the CBMS Regional Conference held at Madison, Wis., August 8–12, 1977.

[Lus79] George Lusztig. Some remarks on the supercuspidal representations of $p$-adic semisimple groups. In *Automorphic forms, representations and $L$-functions, Part 1*, pages 171–175. Amer. Math. Soc., 1979.

[Lus04] George Lusztig. Representations of reductive groups over finite rings. *Represent. Theory*, 8:1–14, 2004.

[Sta09] Alexander Stasinski. Unramified representations of reductive groups over finite rings. *Represent. Theory*, 13:636–656, 2009.

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