

THE MINIMAL EXACT CROSSED PRODUCT

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ABSTRACT. Given a locally compact group G , we study the smallest exact crossed-product functor $(A, G, \alpha) \mapsto A \rtimes_{\mathcal{E}} G$ on the category of G - C^* -dynamical systems. As an outcome, we show that the smallest exact crossed-product functor is automatically Morita compatible, and hence coincides with the functor $\rtimes_{\mathcal{E}}$ as introduced by Baum, Guentner, and Willett in their reformulation of the Baum-Connes conjecture (see [2]). We show that the corresponding group algebra $C_{\mathcal{E}}^*(G)$ always coincides with the reduced group algebra, thus showing that the new formulation of the Baum-Connes conjecture coincides with the classical one in the case of trivial coefficients.

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1 INTRODUCTION

The construction of crossed products $(A, G, \alpha) \mapsto A \rtimes_{\alpha} G$ provides a major source of examples in C^* -algebra theory and plays an important rôle in many applications of C^* -algebras in other fields of mathematics, such as group representation theory and topology. Classically, there were two crossed products attached to a given action $\alpha : G \rightarrow \text{Aut}(A)$ of a locally compact group G on a C^* -algebra A : the maximal crossed product $A \rtimes_{\alpha, \max} G$, which is universal for covariant representations (π, u) of the underlying dynamical system (A, G, α) , and the reduced crossed product $A \rtimes_{\alpha, r} G$, which can be defined as the image of $A \rtimes_{\alpha, \max} G$ under the regular covariant representation of the system. Both crossed products are completions of the algebraic crossed product $A \rtimes_{\text{alg}} G = C_c(G, A)$ by C^* -norms $\|\cdot\|_{\max}$ and $\|\cdot\|_r$, respectively and the fact that the identity map on $C_c(G, A)$ induces a quotient from $A \rtimes_{\max} G$ onto $A \rtimes_r G$ means that $\|f\|_{\max} \geq \|f\|_r$ for all $f \in C_c(G, A)$.

More recently, the study of exotic crossed-product functors $(A, G, \alpha) \rightarrow A \rtimes_{\alpha, \mu} G$ came into the focus of research. Here $A \rtimes_{\alpha, \mu} G$ is a completion of $C_c(G, A)$ with respect to a C^* -norm $\|\cdot\|_\mu$ satisfying

$$\|f\|_{\max} \geq \|f\|_\mu \geq \|f\|_r$$

for all $f \in C_c(G, A)$. The identity on $C_c(G, A)$ then induces surjective $*$ -homomorphisms

$$A \rtimes_{\max} G \rightarrow A \rtimes_\mu G \rightarrow A \rtimes_r G$$

for every G -algebra A .

The interest in exotic crossed products is motivated in a good part by the failure of the classical Baum-Connes conjecture, which predicted that a certain assembly map

$$\text{as}^r : K_*^{\text{top}}(G; A) \rightarrow K_*(A \rtimes_r G) \quad (1)$$

for the K -theory of the reduced crossed product should always be an isomorphism. However, it was shown by Higson, Lafforgue, and Skandalis in [15] that the conjecture fails for certain groups discovered by Gromov [14] (see [18] for a concrete construction). This failure is due to the fact that these groups are not exact in the sense that the sequence of reduced crossed products

$$0 \longrightarrow I \rtimes_r G \longrightarrow A \rtimes_r G \longrightarrow (A/I) \rtimes_r G \longrightarrow 0 \quad (2)$$

for a G -invariant ideal I of A may fail to be exact in general, even in a way that is detectable by K -theory. This led to the idea that one should replace the reduced crossed product by the smallest exact crossed-product functor which is compatible with Morita equivalences (at least in some weak sense – see Section 4 below for the precise definition). Indeed, it has been shown in [2] that for every locally compact group G a smallest exact Morita compatible functor $A \mapsto A \rtimes_{\mathcal{E}} G$ always exists; moreover, if we replace the reduced crossed product by $A \rtimes_{\mathcal{E}} G$ in (1) getting a new assembly map

$$\text{as}^{\mathcal{E}} : K_*^{\text{top}}(G; A) \rightarrow K_*(A \rtimes_{\mathcal{E}} G), \quad (3)$$

then the known counterexamples for the Baum-Connes conjecture disappear, some counterexamples become confirming examples, and the known confirming examples remain as such. Note that for exact groups, i.e., groups for which (2) is always exact, we have $A \rtimes_{\mathcal{E}} G = A \rtimes_r G$, and hence the new conjecture coincides with the old one for those groups.

The smallest exact Morita compatible crossed-product functor $\rtimes_{\mathcal{E}}$ has been studied further in [8, 9], where it has been shown (among other things), that for second countable G , its restriction to the category of separable G - C^* -algebras enjoys other good functorial properties: It is functorial for G -equivariant correspondences and it allows a descent in equivariant KK -theory. On the other hand, in many respects our understanding of the functor $\rtimes_{\mathcal{E}}$ has been very limited. Important questions are (among others):

- Q1. What is the group algebra $C_{\mathcal{E}}^*(G) := \mathbb{C} \rtimes_{\mathcal{E}} G$? Do we always have $C_{\mathcal{E}}^*(G) = C_r^*(G)$, the reduced group algebra?
- Q2. Is the smallest exact *Morita compatible* crossed-product functor $\rtimes_{\mathcal{E}}$ identical to the smallest exact crossed-product functor?
- Q3. Can we give concrete descriptions or constructions of the functor $\rtimes_{\mathcal{E}}$?
- Q4. How can we relate the smallest exact Morita compatible functor $\rtimes_{\mathcal{E}_G}$ for a group G to the same functor $\rtimes_{\mathcal{E}_H}$ for a closed subgroup H of G ?

Note that a positive answer to Question Q1 would imply that the new Baum-Connes conjecture coincides with the classical one in the case of the trivial coefficient algebra $A = \mathbb{C}$, which would fit with the fact that so far there are no known counterexamples for the classical Baum-Connes conjecture in this case. In this paper we will give positive answers to Questions Q1 and Q2 and give at least partial answers to Questions Q3 and Q4. Given any fixed crossed-product functor \rtimes_{μ} for a group G (which will be the reduced crossed-product functor in our main applications), we start in Section 2 with the construction of a crossed-product functor $\rtimes_{\mathcal{E}(\mu)}$ which is the smallest *half-exact* crossed-product functor (i.e., the analogue of sequence (3.8) is exact at the middle term) that dominates \rtimes_{μ} in the sense that $\|f\|_{\mathcal{E}(\mu)} \geq \|f\|_{\mu}$ for all $f \in C_c(G, A)$. We show (see Proposition 2.6) that the corresponding group algebra $C_{\mathcal{E}(\mu)}^*(G) = \mathbb{C} \rtimes_{\mathcal{E}(\mu)} G$ always coincides with the group algebra $C_{\mu}^*(G) := \mathbb{C} \rtimes_{\mu} G$. In particular, $C_{\mathcal{E}(r)}^*(G) = C_r^*(G)$.

In Section 3, building on ideas developed around Archbold's and Batty's property C (see [1] and the treatment of this property in [7, Chapter 9]) and work of Matsumura [17], we prove

THEOREM 1.1 (see Theorem 3.5). *Let \rtimes_{μ} be a crossed-product functor for the locally compact group G . Then the following are equivalent:*

1. \rtimes_{μ} is half-exact;
2. for every G -algebra A there is a canonical $*$ -homomorphism

$$A_c^{**} \rtimes_{\mu} G \rightarrow (A \rtimes_{\mu} G)^{**}$$

where A_c^{**} denotes the G -continuous part of the double dual A^{**} of A ;

3. \rtimes_{μ} is exact.

This theorem not only gives a new characterization of exact groups (when applied to the reduced crossed-product functor), it also shows that the functor $\rtimes_{\mathcal{E}(r)}$ of Section 2 is indeed the smallest exact exotic crossed-product functor. In Section 4 we then show that $\rtimes_{\mathcal{E}(r)}$ is Morita compatible, which gives a positive answer to Question Q2 (i.e., $\rtimes_{\mathcal{E}(r)} = \rtimes_{\mathcal{E}}$), and, since $C_{\mathcal{E}}^*(G) = C_{\mathcal{E}(r)}^*(G) = C_r^*(G)$, also to Question Q1.

In Section 5 we study certain equivariant lifting properties for G -algebras, which give rise to more concrete descriptions of the smallest exact functor $\rtimes_{\mathcal{E}}$. In particular, we say that a G -algebra C has the *weak equivariant lifting property* (WELP), if for any diagram of equivariant maps of the form

$$\begin{array}{ccc} & & B \\ & \nearrow \tilde{\sigma} & \downarrow \pi \\ C & \xrightarrow{\sigma} & B/J \end{array}$$

with σ a $*$ -homomorphism and π a quotient map, the dashed arrow can always be filled in with a G -equivariant ccp map $\tilde{\sigma}$. If G is discrete, it is not difficult to see that for any G -algebra A , there always exists a short exact sequence $0 \rightarrow I \rightarrow C \xrightarrow{\pi} A \rightarrow 0$ with C satisfying (WELP), and it then follows that

$$A \rtimes_{\mathcal{E}} G = \frac{C \rtimes_r G}{I \rtimes_r G}.$$

If G is discrete and A is unital, then C can always be chosen to be the maximal group algebra $C^*(F_{N \times G})$ of a free group generated by a set $N \times G$ with G -action induced by the translation action on the second factor on the generating set $N \times G$. It follows in particular that for all G -algebras satisfying (WELP) we have $C \rtimes_{\mathcal{E}} G = C \rtimes_r G$. These include all equivariantly projective G -algebras C as defined in [19].

Finally, in Section 6 we show that for an *open* subgroup H of G , the minimal exact functor $\rtimes_{\mathcal{E}_H}$ is always the *restriction* (see Section 6 for the definition) of the minimal exact functor $\rtimes_{\mathcal{E}_G}$ for G . This implies in particular, that validity of the reformulated Baum-Connes conjecture for a locally compact group G passes to all open subgroups of G . Note that an analogue of this result for *closed normal* subgroups of G has been obtained in [8].

CONVENTIONS: The phrase *G -algebra* will always mean a C^* -algebra equipped with a continuous action by $*$ -automorphisms of a locally compact group G . If A is a C^* -algebra equipped with a not-necessarily-continuous action

$$\alpha : G \rightarrow \text{Aut}(A)$$

of a locally compact group G , then the *continuous part* of A is defined to be

$$A_c := \{a \in A \mid g \mapsto \alpha_g(a) \text{ is norm continuous}\},$$

and is a G -algebra with the restricted action. For a locally compact group G and a G -algebra A , $C_c(G, A)$ denotes the collection of all compactly supported continuous functions from G to A , equipped with the usual $*$ -algebra operations. The reduced and maximal completions of $C_c(G, A)$ will be denoted $A \rtimes_r G$ and $A \rtimes_{\max} G$. A general crossed product functor as in [8, Definition 3.2] will be denoted \rtimes_{μ} , and the associated completion of $C_c(G, A)$ by $A \rtimes_{\mu} G$. For most of

the paper, there should be no ambiguity about which particular action a given C^* -algebra is equipped with; as such, we will not label crossed products with the name of the action unless it seems necessary to avoid confusion. Finally, if $g \in G$, then we denote the canonical associated unitary in the multiplier algebra $\mathcal{M}(A \rtimes_{\mu} G)$ of a crossed product by δ_g .

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2 HALF EXACT CROSSED PRODUCTS

Throughout this section, G denotes a locally compact group, and \rtimes_{μ} a fixed crossed-product functor for G as in [8, Definition 3.2]. At this point, we do not assume that \rtimes_{μ} has any other properties beyond those in this basic definition; however we will need to specialize to crossed product functors satisfying more stringent conditions later, and will make clear when this comes up. Our goal is to define a new crossed-product functor $\rtimes_{\mathcal{E}(\mu)}$, which should be thought of as the ‘best exact approximation to \rtimes_{μ} ’, and indeed in Section 3 we will prove that it is the smallest exact crossed product that is larger than \rtimes_{μ} . We will spend most of this section proving some basic properties of $\rtimes_{\mathcal{E}(\mu)}$.

The reader unfamiliar with exotic crossed products is encouraged to just assume that $\rtimes_{\mu} = \rtimes_r$ throughout, which is certainly the most important special case. Nonetheless, it seemed worthwhile to work in general as this causes no extra difficulties, and as it clarifies the ‘formal’ nature of the constructions and proofs; by ‘formal’ we mean that they rely on general C^* -algebra theory and functorial properties of \rtimes_{μ} , and have nothing to do with the specific construction underlying the definition of \rtimes_r .

We first need some ancillary notation.

DEFINITION 2.1. Let $B \subseteq A$ be an equivariant inclusion of C^* -algebras. Then $B \rtimes_{\mu, A} G$ denotes the completion of $C_c(G, B)$ for the norm it inherits as a subalgebra of $A \rtimes_{\mu} G$.

Here is the main definition of this section.

DEFINITION 2.2. Let A be a G -algebra, and let

$$0 \longrightarrow I \longrightarrow C \xrightarrow{\pi} A \longrightarrow 0$$

be an equivariant short exact sequence. Then we get a short exact sequence

$$0 \longrightarrow I \rtimes_{\mu, C} G \longrightarrow C \rtimes_{\mu} G \longrightarrow \frac{C \rtimes_{\mu} G}{I \rtimes_{\mu, C} G} \longrightarrow 0.$$

This gives rise to a (dense) $*$ -algebra inclusion

$$C_c(G, A) \hookrightarrow \frac{C \rtimes_{\mu} G}{I \rtimes_{\mu, C} G}.$$

The π -norm¹ on $C_c(G, A)$, denoted $\|\cdot\|_{\pi}$, is the norm induced by the above inclusion, and the corresponding completion is denoted $A \rtimes_{\pi} G$.

The $\mathcal{E}(\mu)$ -norm on $C_c(G, A)$ is defined by

$$\|a\|_{\mathcal{E}(\mu)} := \sup\{\|a\|_{\pi} \mid \pi : C \rightarrow A \text{ an equivariant surjection}\}$$

and the corresponding completion of $C_c(G, A)$ is denoted $A \rtimes_{\mathcal{E}(\mu)} G$.

Note that the supremum defining the $\mathcal{E}(\mu)$ -norm is over a non-empty set: indeed, it contains the π -norm associated to the identity function $\pi : A \rightarrow A$. Moreover, the supremum is finite as if we have an equivariant short exact sequence

$$0 \longrightarrow I \longrightarrow C \xrightarrow{\pi} A \longrightarrow 0$$

then exactness of the maximal crossed product gives rise to a quotient map

$$A \rtimes_{\max} G = \frac{C \rtimes_{\max} G}{I \rtimes_{\max} G} \rightarrow \frac{C \rtimes_{\mu} G}{I \rtimes_{\mu, C} G},$$

whence $\|a\|_{\pi} \leq \|a\|_{\max}$ for all $a \in C_c(G, A)$. On the other hand, functoriality of \rtimes_{μ} gives rise to a quotient map

$$\frac{C \rtimes_{\mu} G}{I \rtimes_{\mu, C} G} \rightarrow A \rtimes_{\mu} G$$

so that $\|a\|_{\mu} \leq \|a\|_{\pi}$ for all $a \in C_c(G, A)$. Moreover, every crossed-product norm is assumed to satisfy $\|a\|_r \leq \|a\|_{\mu}$, where $\|\cdot\|_r$ denotes the reduced norm. Hence we get the inequalities

$$\|a\|_r \leq \|a\|_{\mu} \leq \|a\|_{\mathcal{E}(\mu)} \leq \|a\|_{\max} \quad \text{for all } a \in C_c(G, A). \quad (4)$$

PROPOSITION 2.3. *Let $\phi : A \rightarrow B$ be an equivariant $*$ -homomorphism. Then the integrated form*

$$\phi \rtimes G : C_c(G, A) \rightarrow C_c(G, B); \quad a \mapsto \phi \circ a$$

extends (uniquely) to a $$ -homomorphism $A \rtimes_{\mathcal{E}(\mu)} G \rightarrow B \rtimes_{\mathcal{E}(\mu)} G$. In particular, $\rtimes_{\mathcal{E}(\mu)}$ is a crossed-product functor in the sense of [8, Definition 3.2].*

¹Of course, the π -norm also depends on the fixed crossed product \rtimes_{μ} , but the ‘parent’ crossed product should always be clear from context, so we do not include it in the notation.

Proof. Let

$$0 \longrightarrow I \longrightarrow C \xrightarrow{\pi} B \longrightarrow 0$$

be an arbitrary equivariant short exact sequence. Let $P = \{(c, a) \in C \oplus A \mid \pi(c) = \phi(a)\}$ be the pullback over the diagram

$$\begin{array}{ccc} P & \xrightarrow{\pi_A} & A \\ \downarrow \pi_C & & \downarrow \phi \\ C & \xrightarrow{\pi} & B \end{array}$$

(the maps labeled π_A and π_B are the restrictions of the coordinate projections from $C \oplus A$ to P). The direct sum G -action on $C \oplus A$ restricts to an action on P and we thus obtain a commutative diagram of equivariant short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & J & \longrightarrow & P & \xrightarrow{\pi_A} & A \longrightarrow 0, \\ & & \downarrow & & \downarrow \pi_C & & \downarrow \phi \\ 0 & \longrightarrow & I & \longrightarrow & C & \xrightarrow{\pi} & B \longrightarrow 0 \end{array}$$

where the map $J \rightarrow I$, which is an isomorphism, exists by commutativity. Taking crossed products thus induces a map

$$\frac{P \rtimes_{\mu} G}{J \rtimes_{\mu, P} G} \rightarrow \frac{C \rtimes_{\mu} G}{I \rtimes_{\mu, C} G}$$

that agrees with $\phi \rtimes G$ on $C_c(G, A)$. This implies that for all $a \in C_c(G, A)$ we have

$$\|\phi \rtimes G(a)\|_{\pi} \leq \|a\|_{\pi_A} \leq \|a\|_{\mathcal{E}(\mu)}.$$

Taking the supremum over all such π now gives that $\|\phi \rtimes G(a)\|_{\mathcal{E}(\mu)} \leq \|a\|_{\mathcal{E}(\mu)}$ and thus that $\phi \rtimes G$ extends as claimed. That $\rtimes_{\mathcal{E}(\mu)}$ is a crossed-product functor follows from functoriality of algebraic descent

$$(\phi : A \rightarrow B) \quad \mapsto \quad (\phi \rtimes G : C_c(G, A) \rightarrow C_c(G, B))$$

and the inequality in Equation (4). □

We now show that the supremum defining the $\mathcal{E}(\mu)$ -norm is always attained.

PROPOSITION 2.4. *Let G be a locally compact group, let \rtimes_{μ} be a crossed product for G , and let A be a G -algebra. Then there exists an equivariant quotient map $\pi : C \rightarrow A$ such that $\|a\|_{\mathcal{E}(\mu)} = \|a\|_{\pi}$ for all $a \in C_c(G, A)$. If A is unital, then C can be chosen to be unital as well.*

Proof. Let S be a set of equivariant quotient maps $\pi_s : C_s \rightarrow A$ such that for every $a \in C_c(G, A)$,

$$\|a\|_{\mathcal{E}(\mu)} = \sup_{s \in S} \|a\|_{\pi_s}.$$

Define

$$C_0 := \left\{ (c_s) \in \prod_{s \in S} C_s \mid \pi_s(c_s) = \pi_t(c_t) \text{ for all } s, t \in S \right\},$$

and let C be the continuous part of C_0 , i.e., for all $c = (c_s)_{s \in S} \in C$ the map

$$G \rightarrow C; \quad g \mapsto \gamma_g(c) := (\gamma_g^s(c_s))_{s \in S}$$

is continuous, where $\gamma^s : G \rightarrow \text{Aut}(C_s)$ denotes the action of G on C_s . For any fixed $t \in S$ let $\sigma_t : C_0 \rightarrow C_t$ denote the projection. We claim that its restriction to C is surjective. For each $f \in C_c(G)$ and $c = (c_s)_{s \in S} \in C_0$, we define

$$f * c := (f * c_s)_{s \in S} \quad \text{with } f * c_s := \int_G f(g) \gamma_g^s(c_s) dg.$$

We claim first that $f * c \in C$ for all $c \in C_0$. To see this, it suffices to show that if (g_j) is a net converging to the identity in G , then

$$\limsup_j \sup_s \|\gamma_{g_j}^s(f * c_s) - f * c_s\| = 0.$$

Note however that for any $s \in S$, if δ_g denotes the Dirac mass at g , then

$$\begin{aligned} \|\gamma_{g_j}^s(f * c_s) - f * c_s\| &= \|(\delta_{g_j} * f) * c_s - f * c_s\| \leq \|\delta_{g_j} * f - f\|_{L^1(G)} \|c_s\| \\ &\leq \|\delta_{g_j} * f - f\|_{L^1(G)} \|c\|; \end{aligned}$$

as f and $c = (c_s)_{s \in S}$ are fixed, this tends to zero as j tends to infinity at a rate independent of s as required. It follows that the image $\sigma_t(C) \subseteq C_t$ contains all elements of the form $\{f * c_t : f \in C_c(G), c_t \in C_t\}$; hence to show surjectivity of σ_t , it suffices to show that this set is dense in C_t . To see this, let \mathcal{V} be a neighbourhood base of $e \in G$ and for each $V \in \mathcal{V}$ let $f_V \in C_c(G)$ be a positive symmetric function with $\text{supp } f_V \subseteq V$ and $\int_G f_V(g) dg = 1$. Then $f_V * c_t$ converges in norm to c_t for any $c_t \in C_t$, $t \in S$. This proves the claim. For any fixed $t \in S$, there is thus a surjective quotient map

$$C \rightarrow A; \quad (c_s)_{s \in S} \mapsto \pi_t(c_t).$$

The definition of $C \subseteq C_0$ implies that this map does not depend on the choice of t , so we just denote it π .

We now have that C is a G -algebra equipped with an equivariant surjection $\pi : C \rightarrow A$, so it remains to show that $\|a\|_{\mathcal{E}(\mu)} = \|a\|_\pi$ for all $a \in C_c(G, A)$. For $s \in S$, recall that $\sigma_s : C \rightarrow C_s$ denotes the coordinate projection. Then we get an equivariant commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & C & \xrightarrow{\pi} & A \longrightarrow 0, \\ & & \downarrow & & \downarrow \sigma_s & & \parallel \\ 0 & \longrightarrow & I_s & \longrightarrow & C_s & \xrightarrow{\pi_s} & A \longrightarrow 0 \end{array}$$

where I and I_s are the kernels of π and π_s , respectively. This gives rise to a *-homomorphism

$$\frac{C \rtimes_{\mu} G}{I \rtimes_{\mu, C} G} \rightarrow \frac{C_s \rtimes_{\mu} G}{I_s \rtimes_{\mu, C_s} G}$$

that restricts to the identity on $C_c(G, A)$; as this *-homomorphism is contractive, this implies that $\|a\|_{\pi} \geq \|a\|_{\pi_s}$ for all $a \in C_c(G, A)$. As s was arbitrary, the choice of S then gives that $\|a\|_{\pi} \geq \|a\|_{\mathcal{E}(\mu)}$. By definition of $\|a\|_{\mathcal{E}(\mu)}$, this implies equality. Suppose now that A is unital. Let

$$0 \longrightarrow I \longrightarrow C \xrightarrow{\pi} A \longrightarrow 0 .$$

be a short exact sequence such that $\|\cdot\|_{\pi} = \|\cdot\|_{\mathcal{E}(\mu)}$ on $C_c(G, A)$ as above. Let \tilde{C} denote the unitization of C (even if C is already unital) with the extended G -action that necessarily fixes the unit. Let $\tilde{\pi} : \tilde{C} \rightarrow A$ denote the unique (equivariant) unital extension of π to \tilde{C} , and let J be the kernel of $\tilde{\pi}$. Noting that I is an ideal in J with quotient \mathbb{C} , and taking crossed products, we get a commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 & & . & (5) \\ & & \downarrow & & \downarrow & & \downarrow & & & \\ 0 & \longrightarrow & I \rtimes_{\mu, C} G & \longrightarrow & C \rtimes_{\mu} G & \longrightarrow & A \rtimes_{\pi} G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & & \\ 0 & \longrightarrow & J \rtimes_{\mu, \tilde{C}} G & \longrightarrow & \tilde{C} \rtimes_{\mu} G & \longrightarrow & A \rtimes_{\tilde{\pi}} G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & & \\ 0 & \longrightarrow & \mathbb{C} \rtimes_{\mu} G & \longrightarrow & \mathbb{C} \rtimes_{\mu} G & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & & \\ & & 0 & & 0 & & 0 & & & \end{array}$$

The middle column is exact since the unit inclusion $\sigma : \mathbb{C} \rightarrow \tilde{C}$, which is a G -homomorphism, induces a splitting homomorphism $\sigma \rtimes G : \mathbb{C} \rtimes_{\mu} G \rightarrow \tilde{C} \rtimes_{\mu} G$. Let now $E : \tilde{C} \rtimes_{\mu} G \rightarrow \tilde{C} \rtimes_{\mu} G$ be defined by

$$E := \text{Id} - (\sigma \rtimes G) \circ (\delta \rtimes G),$$

where $\delta : \tilde{C} \rightarrow \mathbb{C}$ denotes the canonical quotient map. Then E is a bounded linear idempotent operator with norm at most two. Note that the definition of E only needs functoriality of \rtimes_{μ} for *-homomorphisms. Moreover, it is straightforward to check that E restricts to a map $C_c(G, J) \rightarrow C_c(G, I)$, whence it takes $J \rtimes_{\mu, \tilde{C}} G$ onto $I \rtimes_{\mu, C} G$ (it has closed range as it is an idempotent), and acts as the identity on $I \rtimes_{\mu, \tilde{C}} G$. It follows from a diagram chase that if

$a \in J \rtimes_{\mu, \tilde{C}} G$ goes to zero under the quotient map to $\mathbb{C} \rtimes_{\mu} G$, then $E(a) = a$, and thus that $a \in I \rtimes_{\mu, C} G$. Hence the left hand vertical column is also exact. To complete the proof, note that we now have that the left two columns in diagram (5) above are exact, while the rows are all exact by definition. It follows from a diagram chase that the map $A \rtimes_{\pi} G \rightarrow A \rtimes_{\tilde{\pi}} G$ is an isomorphism, and thus that for any $a \in C_c(G, A)$ we have

$$\|a\|_{\tilde{\pi}} = \|a\|_{\pi} = \|a\|_{\mathcal{E}(\mu)},$$

and we are done. \square

Remark 2.5. Note that the algebra C constructed above depends strongly on A and we have no idea about its general structure as a C^* -algebra. For instance, it is not clear whether we can always find a C with the property in the proposition that is σ -unital if we assume that A is σ -unital.

For discrete G we shall see in Section 5 below that we can get much more concrete descriptions of algebras C and surjective morphisms $\pi : C \rightarrow A$ which attain the norm $\|\cdot\|_{\mathcal{E}(\mu)}$ for G .

Using the above result, we can compute $\mathbb{C} \rtimes_{\mathcal{E}(\mu)} G$.

PROPOSITION 2.6. *The canonical quotient map $\mathbb{C} \rtimes_{\mathcal{E}(\mu)} G \rightarrow \mathbb{C} \rtimes_{\mu} G$ is an isomorphism.*

Proof. Lemma 2.4 implies in particular that there is a unital G -algebra C and a G -invariant character $\pi : C \rightarrow \mathbb{C}$ such that $\|a\|_{\mathcal{E}(\mu)} = \|a\|_{\pi}$ for all $a \in C_c(G) = C_c(G, \mathbb{C})$.

Any unital equivariant surjection $\pi : C \rightarrow \mathbb{C}$ splits equivariantly by the unit inclusion $*$ -homomorphism $\mathbb{C} \rightarrow C$, which implies that the induced sequence

$$0 \longrightarrow I \rtimes_{\mu, C} G \longrightarrow C \rtimes_{\mu} G \longrightarrow \mathbb{C} \rtimes_{\mu} G \longrightarrow 0$$

is exact, and thus that $\|\cdot\|_{\mathcal{E}(\mu)} = \|\cdot\|_{\pi} = \|\cdot\|_{\mu}$ on $C_c(G)$. The result follows. \square

Our next aim is to show that $\mathcal{E}(\mu)$ is always half-exact as in the next definition, and is in fact minimal amongst all half-exact crossed-product functors dominating μ .

DEFINITION 2.7. A crossed-product functor $A \mapsto A \rtimes_{\mu} G$ is called *half-exact* if for every short exact sequence of G -algebras

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\rho} B \longrightarrow 0$$

the sequence

$$0 \longrightarrow I \rtimes_{\mu, A} G \longrightarrow A \rtimes_{\mu} G \xrightarrow{\rho \rtimes G} B \rtimes_{\mu} G \longrightarrow 0$$

is exact, where $I \rtimes_{\mu, A} G$ is as in Definition 2.1.

Remark 2.8. A half-exact functor is exact if and only if it has the *ideal property* of [9, Definition 3.2]: This means that if $I \subseteq A$ is a G -invariant ideal, then the induced map $I \rtimes_{\mu} G \rightarrow A \rtimes_{\mu} G$ is injective. The image of this homomorphism is $I \rtimes_{\mu, A} G$ so that $I \rtimes_{\mu, A} G \cong I \rtimes_{\mu} G$ canonically in this case.

PROPOSITION 2.9. *The functor $\rtimes_{\mathcal{E}(\mu)}$ is half-exact.*

Proof. Fix a G -invariant ideal I in a G -algebra A and write $\sigma : A \rightarrow A/I$ for the quotient map. Let

$$0 \longrightarrow J \longrightarrow C \xrightarrow{\pi} A \longrightarrow 0 .$$

be any equivariant short exact sequence such that $\|a\|_{\mathcal{E}(\mu)} = \|a\|_{\pi}$ for all $a \in C_c(G, A)$, as exists by Proposition 2.4. This fits into a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & J & \longrightarrow & J & \longrightarrow & 0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \pi^{-1}(I) & \longrightarrow & C & \xrightarrow{\sigma \circ \pi} & A/I \longrightarrow 0 \\
 & & \downarrow \pi_I & & \downarrow \pi & & \downarrow \\
 0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow{\sigma} & A/I \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

with all rows and columns exact, where π_I denotes the restriction of π to $\pi^{-1}(I)$. Taking crossed products gives a commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & J \rtimes_{\mu, C} G & \longrightarrow & J \rtimes_{\mu, C} G & \longrightarrow & 0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \pi^{-1}(I) \rtimes_{\mu, C} G & \longrightarrow & C \rtimes_{\mu} G & \longrightarrow & (A/I) \rtimes_{\sigma \circ \pi} G \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \frac{\pi^{-1}(I) \rtimes_{\mu, C} G}{J \rtimes_{\mu, C} G} & \longrightarrow & A \rtimes_{\mathcal{E}(\mu)} G & \longrightarrow & \frac{A \rtimes_{\mathcal{E}(\mu)} G}{(\pi^{-1}(I) \rtimes_{\mu, C} G) / (J \rtimes_{\mu, C} G)} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

where the first two columns and first two rows (at least) are exact. The canonical isomorphisms

$$\begin{aligned} (A/I) \rtimes_{\sigma \circ \pi} G &\cong \frac{C \rtimes_{\mu} G}{\pi^{-1}(I) \rtimes_{\mu, C} G} \cong \frac{(C \rtimes_{\mu} G) / (J \rtimes_{\mu, C} G)}{(\pi^{-1}(I) \rtimes_{\mu, C} G) / (J \rtimes_{\mu, C} G)} \\ &\cong \frac{A \rtimes_{\mathcal{E}(\mu)} G}{(\pi^{-1}(I) \rtimes_{\mu, C} G) / (J \rtimes_{\mu, C} G)} \end{aligned}$$

identify the bottom right term with $A \rtimes_{\sigma \circ \pi} G$, and a diagram chase shows that the map

$$\frac{\pi^{-1}(I) \rtimes_{\mu, C} G}{J \rtimes_{\mu, C} G} \rightarrow A \rtimes_{\mathcal{E}(\mu)} G$$

is injective. Thus in fact all the rows and columns in the above diagram are exact.

Now, it follows that $\frac{\pi^{-1}(I) \rtimes_{\mu, C} G}{J \rtimes_{\mu, C} G}$ identifies with the completion $I \rtimes_{\mathcal{E}(\mu), A} G$ of $C_c(G, I)$ inside $A \rtimes_{\mathcal{E}(\mu)} G$, and that we have a canonical identification

$$\frac{A \rtimes_{\mathcal{E}(\mu)} G}{I \rtimes_{\mathcal{E}(\mu), A} G} = (A/I) \rtimes_{\sigma \circ \pi} G.$$

From this, we see that for any $a \in C_c(G, A/I)$,

$$\|a\|_{(A \rtimes_{\mathcal{E}(\mu)} G) / (I \rtimes_{\mathcal{E}(\mu), A} G)} = \|a\|_{(A/I) \rtimes_{\sigma \circ \pi} G} \leq \|a\|_{\mathcal{E}(\mu)},$$

where the right hand inequality follows from the definition of the $\mathcal{E}(\mu)$ -norm on $C_c(G, A/I)$.

To see the opposite inequality observe that the fact that $\rtimes_{\mathcal{E}(\mu)}$ is a functor gives a quotient map

$$\frac{A \rtimes_{\mathcal{E}(\mu)} G}{I \rtimes_{\mathcal{E}(\mu), A} G} \rightarrow (A/I) \rtimes_{\mathcal{E}(\mu)} G,$$

so we are done. □

PROPOSITION 2.10. *The crossed product $\rtimes_{\mathcal{E}(\mu)}$ is minimal amongst all half-exact crossed products that dominate \rtimes_{μ} .*

Proof. It is shown in Proposition 2.9 that $\rtimes_{\mathcal{E}(\mu)}$ is half-exact, and it dominates \rtimes_{μ} by the inequalities in Equation (4).

Let \rtimes_{ν} be any other half exact crossed-product functor that dominates \rtimes_{μ} . Let A be a G -algebra, and let

$$0 \longrightarrow J \longrightarrow C \xrightarrow{\pi} A \longrightarrow 0$$

be a short exact sequence of G -algebras as in Proposition 2.4. We thus get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & J \rtimes_{\nu, C} G & \longrightarrow & C \rtimes_{\nu} G & \longrightarrow & A \rtimes_{\nu} G \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & J \rtimes_{\mu, C} G & \longrightarrow & C \rtimes_{\mu} G & \longrightarrow & A \rtimes_{\mathcal{E}(\mu)} G \longrightarrow 0 \end{array}$$

of short exact sequences. As the rows are exact, the dashed arrow can be filled in with a (necessarily surjective) $*$ -homomorphism which extends the identity on $C_c(G, A)$, and we are done. \square

Remark 2.11. It is shown in [5, Theorem A] that a locally compact² group G is exact if and only if the sequence

$$0 \longrightarrow C_0(G) \rtimes_r G \longrightarrow C_{ub}(G) \rtimes_r G \longrightarrow C(\partial G) \rtimes_r G \longrightarrow 0$$

is exact, where $C_{ub}(G)$ is the C^* -algebra of bounded left uniformly continuous functions endowed with the left translation G -action, that is, the continuous part of the translation G -action on $C_b(G) = M(C_0(G))$, and $C(\partial G) := C_{ub}(G)/C_0(G)$ is the quotient G -algebra. A simple chase with the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_0(G) \rtimes_{\mathcal{E}(r)} G & \longrightarrow & C_{ub}(G) \rtimes_{\mathcal{E}(r)} G & \longrightarrow & C(\partial G) \rtimes_{\mathcal{E}(r)} G \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_0(G) \rtimes_r G & \longrightarrow & C_{ub}(G) \rtimes_r G & \longrightarrow & C(\partial G) \rtimes_r G \longrightarrow 0 \end{array}$$

together with fact that $C_0(G) \rtimes_{\mathcal{E}(r)} G = C_0(G) \rtimes_r G$ shows that G is exact iff $C(\partial G) \rtimes_{\mathcal{E}(r)} G = C(\partial G) \rtimes_r G$, that is, the right vertical arrow in the above diagram is an isomorphism. For a general crossed-product functor \rtimes_{μ} , one can follow exactly the same idea and prove part of the analogous result: if $C(\partial G) \rtimes_{\mu} G = C(\partial G) \rtimes_{\mathcal{E}(\mu)} G$, then the sequence

$$0 \longrightarrow C_0(G) \rtimes_{\mu} G \longrightarrow C_{ub}(G) \rtimes_{\mu} G \longrightarrow C(\partial G) \rtimes_{\mu} G \longrightarrow 0$$

is exact. It is, however, not clear whether the exactness of a general \rtimes_{μ} can be detected by this sequence alone.

3 THE IDEAL PROPERTY AND EXACTNESS

Throughout this section, G again denotes a locally compact group, and \rtimes_{μ} a crossed product functor for G . We will mainly be interested in the case that $\rtimes_{\mu} = \rtimes_{\mathcal{E}(r)}$ or $\rtimes_{\mu} = \rtimes_r$, and the reader is encouraged to bear those two cases in mind; nonetheless, working in general is no more difficult, and seemed more conceptual, so we do this.

Our goal is to prove a necessary and sufficient condition for \rtimes_{μ} to be exact. As a consequence we shall see that every half-exact crossed-product functor in the sense of Definition 2.7 is exact and, in particular, that the functor $\rtimes_{\mathcal{E}(\mu)}$ as constructed in the previous section is always exact. It follows from this and

²The reference given assumes that G is second countable for this result; however, Kang Li has pointed out to us that one can use the structure theory of locally compact groups to deduce the general case from this.

Proposition 2.10 that if $\rtimes_{\mu} = \rtimes_r$, then $\rtimes_{\mathcal{E}(\mu)}$ is the minimal exact crossed-product functor.

The formulation and proof of our main theorem (see Theorem 3.5 below) is inspired in part by work of Matsumura [17], which is in turn inspired by the equivalence between property C' of Archbold and Batty [1] and exactness of the minimal tensor product as discussed in [7, Chapter 9].

We need some conventions, which are set up as follows.

DEFINITION 3.1. Let A be a G -algebra with action $\alpha : G \rightarrow \text{Aut}(A)$. Let $\iota \rtimes u : A \rtimes_{\mu} G \rightarrow \mathcal{B}(H)$ be a faithful and nondegenerate representation of $A \rtimes_{\mu} G$ on some Hilbert space H and let $A'' := \iota(A)''$ and $(A \rtimes_{\mu} G)''$ denote the double commutants of A and $A \rtimes_{\mu} G$ respectively in $\mathcal{B}(H)$. We write $\iota'' : A'' \rightarrow (A \rtimes_{\mu} G)''$ for the canonical inclusion; note that this is the unique normal extension of ι to A'' .

Now let I be a large directed set such that every element $a \in A''$ can be obtained as a limit of a bounded net $(a_i)_{i \in I}$ over the directed set I in the strong* topology (for example, Kaplansky's density theorem implies that letting I be a neighbourhood base of $0 \in \mathcal{B}(H)$ for the strong* topology would work). Then

$$A^I := \{(a_i)_{i \in I} \mid (a_i)_{i \in I} \text{ is a strong* convergent net}\} \subseteq \prod_{i \in I} A$$

is a C^* -algebra, because multiplication and involution are strong*-continuous on bounded subsets of $\mathcal{B}(H)$. There is moreover a *-homomorphism

$$\rho : A^I \rightarrow A''; \quad \rho((a_i)) = \text{strong*} \lim a_i \in A'',$$

which is surjective by the choice of I . Since (ι, u) is a covariant representation it follows that for each $g \in G$ the automorphism α_g of $A \cong \iota(A)$ extends to A'' via the automorphism $\alpha_g'' := \text{Ad}u_g$ on A'' . Moreover, for each $g \in G$, we get a *-automorphism α_g^I of $\prod_{i \in I} A$ defined componentwise by $\alpha_g^I((a_i)) := (\alpha_g(a_i))$, and the fact that the underlying unitary representation $u : G \rightarrow \mathcal{U}(H)$ is strong* continuous implies that α_g^I preserves A^I ; we use the same notation for the restricted *-automorphism of A^I . We thus get homomorphisms

$$\alpha'' : G \rightarrow \text{Aut}(A''), \quad \alpha^I : G \rightarrow \text{Aut}(A^I), \quad (6)$$

neither of which is necessarily continuous for the point-norm topologies on the right hand side. The map ρ is equivariant for these (not-necessarily-continuous) actions by strong* continuity of u again. Finally, we denote by A_c^I and A_c'' the C^* -subalgebras of A^I and A'' consisting of continuous elements for the actions in line (6), and note that ρ restricts to an equivariant map

$$\rho : A_c^I \rightarrow A_c''$$

of G -algebras.

LEMMA 3.2. *With notation as in Definition 3.1, the map $\rho : A_c^I \rightarrow A_c''$ is surjective.*

Proof. We first claim that if $(a_i)_{i \in I}$ is a bounded net which converges to $a \in A''$ in the strong* topology, then for each compact subset $K \subseteq H$ and $\epsilon > 0$ there exists an $i_0 \in I$ such that

$$\forall \xi \in K, \forall i \geq i_0 : \|(a_i - a)\xi\|, \|(a_i - a)^*\xi\| < \epsilon.$$

Indeed, let $\epsilon > 0$ and let $R := \sup_i \|a_i\|$. Then there exist finitely many vectors $\xi_1, \dots, \xi_k \in H$ such that $K \subseteq \bigcup_{l=1}^k B_\delta(\xi_l)$ with $\delta = \frac{\epsilon}{4R}$. Choose $i_0 \in I$ such that for all $i \geq i_0$ and all $1 \leq l \leq k$ we have

$$\|(a_i - a)\xi_l\|, \|(a_i - a)^*\xi_l\| < \frac{\epsilon}{2}.$$

Then for all $\xi \in K$ there exists $l \in \{1, \dots, k\}$ such that $\|\xi - \xi_l\| \leq \frac{\epsilon}{4R}$ and then for all $i \geq i_0$ we get

$$\|(a_i - a)\xi\| \leq \|a_i(\xi - \xi_l)\| + \|(a_i - a)\xi_l\| + \|a(\xi_l - \xi)\| < \epsilon$$

and similarly $\|(a_i - a)^*\xi\| < \epsilon$ for all $i \geq i_0$, completing the proof of the claim. For $a \in A_c''$ and $f \in C_c(G)$ define

$$f * a := \int_G f(g)\alpha_g''(a)dg.$$

As in the proof of Lemma 2.4 it follows from the existence of an approximate identity for $L^1(G)$ in $C_c(G)$ that the collection

$$\{f * a \mid f \in C_c(G), a \in A_c''\}$$

is norm-dense in A_c'' . We now show that all such elements $f * a$ lie in the image of $\rho : A_c^I \rightarrow A_c''$, which will complete the proof.

Fix then $f \in C_c(G)$ and $a \in A''$. Since the representation $u : G \rightarrow \mathcal{U}(H)$ is strong* continuous, it follows that for all $\xi \in H$ the set $K_\xi := \{u_g\xi : g \in \text{supp } f \cup (\text{supp } f)^{-1}\}$ is compact in H . Hence, given an element $(a_i)_{i \in I} \in A^I$ with $a_i \xrightarrow{\text{strong}^*} a$, then for all $g \in \text{supp}(f)$ we have

$$\begin{aligned} \|f(g)\alpha_g(a_i)\xi - f(g)\alpha_g''(a)\xi\| &= \|f(g)u_g(a_i - a)u_g^*\xi\| \\ &\leq \|f\|_\infty \sup_{g \in \text{supp}(f)} \|(a_i - a)u_g^*\xi\| \end{aligned}$$

and by the claim at the start of the proof, the right hand side converges to zero, whence the left hand side converges to zero uniformly for g in the support of f . Thus it follows that for all $\xi \in H$ we have

$$(f * a_i)\xi = \int_G f(g)\alpha_g(a_i)\xi dg \rightarrow \int_G f(g)\alpha_g''(a)\xi dg = (f * a)\xi$$

and, similarly, $(f * a_i)^*\xi \rightarrow (f * a)^*\xi$. Hence $\rho((f * a_i)_i) = f * a$. Now, as in the proof of Lemma 2.4 we see that $(f * a_i)_i$ is a G -continuous element of $\prod_{i \in I} A$ (and hence of A^I), which completes the proof. \square

LEMMA 3.3. *With notation as in Definition 3.1, the *-homomorphism*

$$(\iota'' \circ \rho) \rtimes u : C_c(G, A_c^I) \rightarrow (A \rtimes_\mu G)''$$

*extends to a *-homomorphism*

$$A_c^I \rtimes_\mu G \rightarrow (A \rtimes_\mu G)''.$$

Proof. It will suffice to show that if $f \in C_c(G, A_c^I)$, then

$$\|\iota'' \circ \rho \circ f\|_{(A \rtimes_\mu G)''} \leq \|f\|_{A_c^I \rtimes_\mu G}.$$

Write $f : G \rightarrow A_c^I \subseteq \prod_{i \in I} A$ as a net (f_i) of functions $f_i : G \rightarrow A$; note that the net $(f_i)_i$ is equicontinuous, uniformly bounded, and all the f_i have support in some fixed compact subset of G . Computing, we get

$$\|\iota'' \circ \rho \circ f\|_{(A \rtimes_\mu G)''} = \left\| \int_G \text{strong}^* \text{-} \lim_i f_i(g) u_g dg \right\|_{B(H)}.$$

Write

$$f_\infty(g) = \text{strong}^* \text{-} \lim_i f_i(g), \tag{7}$$

so $f_\infty : G \rightarrow A_c''$ is a (norm) continuous and compactly supported function. We first claim that

$$\int_G \text{strong}^* \text{-} \lim_i f_i(g) u_g dg = \text{strong}^* \text{-} \lim_i \int_G f_i(g) u_g dg.$$

Fix $\epsilon > 0$. Using uniform boundedness, uniform compact support, and equicontinuity of the net (f_i) , there is a finite subset $\{g_1, \dots, g_N\}$ of G and scalars $\{t_1, \dots, t_N\}$ such that

$$\left\| \int_G f_i(g) u_g dg - \sum_{k=1}^N t_k f_i(g_k) u_{g_k} \right\| < \epsilon,$$

for all i , and similarly for the limit function $f_\infty \in C_c(G, A_c'')$ as in line (7) above we have

$$\left\| \int_G f_\infty(g) u_g dg - \sum_{k=1}^N t_k f_\infty(g_k) u_{g_k} \right\| < \epsilon. \tag{8}$$

Hence, using that strong* limits do not increase norms,

$$\left\| \text{strong}^* \text{-} \lim_i \left(\int_G f_i(g) u_g dg - \sum_{k=1}^N t_k f_i(g_k) u_{g_k} \right) \right\| \leq \epsilon,$$

and so using that strong* limits commute with finite linear combinations

$$\left\| \text{strong}^* \text{-} \lim_i \int_G f_i(g) u_g dg - \sum_{k=1}^N t_k f_\infty(g_k) u_{g_k} \right\| \leq \epsilon.$$

Combining this with the inequality in line (8), we get that

$$\left\| \int_G f_\infty(g)u_g dg - \text{strong}^*\text{-}\lim_i \int_G f_i(g)u_g dg \right\| < 2\epsilon ;$$

as ϵ was arbitrary, this completes the proof of the claim.

Now, using the claim,

$$\begin{aligned} \|\iota'' \circ \rho \circ f\|_{(A \rtimes_\mu G)''} &= \left\| \text{strong}^*\text{-}\lim_i \int_G f_i(g)u_g dg \right\|_{\mathcal{B}(H)} \\ &\leq \sup_i \left\| \int_G f_i(g)u_g dg \right\|_{\mathcal{B}(H)}, \end{aligned} \tag{9}$$

where the inequality follows again as strong* limits do not increase norms. On the other hand, the evaluations $\text{ev}_i : A_c^I \rightarrow A$ induce norm-decreasing *-homomorphisms $\text{ev}_i \rtimes G : A_c^I \rtimes_\mu G \rightarrow A \rtimes_\mu G$ for all $i \in I$, and therefore

$$\|f\|_{A_c^I \rtimes_\mu G} \geq \sup_i \|f_i\|_{A \rtimes_\mu G} = \left\| \left(\int_G f_i(g)u_g dg \right) \right\|_{\prod_{i \in I} \mathcal{B}(H)}, \tag{10}$$

where the equality follows as the representation of $A \rtimes_\mu G$ into $\mathcal{B}(H)$ is faithful. The norm on the right in line (9) is the norm of the net $(\int_G f_i(g)u_g dg)_{i \in I}$ in the product $\prod_{i \in I} \mathcal{B}(H)$, whence combining line (9) with line (10) gives that

$$\|\iota \circ \rho \circ f\|_{(A \rtimes_\mu G)''} \leq \left\| \left(\int_G f_i(g)u_g dg \right) \right\|_{\prod_i \mathcal{B}(H)} \leq \|f\|_{A_c^I \rtimes_\mu G}$$

as desired. □

LEMMA 3.4. *Suppose that \rtimes_μ is a half-exact crossed-product functor in the sense of Definition 2.7. Then with notation as in Definition 3.1, the *-homomorphism*

$$\iota'' \rtimes u : C_c(G, A_c'') \rightarrow (A \rtimes_\mu G)''$$

*extends to a *-homomorphism*

$$A_c'' \rtimes_\mu G \rightarrow (A \rtimes_\mu G)''.$$

Proof. Write J for the kernel of the *-homomorphism $\rho : A_c^I \rightarrow A_c''$. Lemma 3.2 says that ρ is surjective, so we have a short exact sequence

$$0 \longrightarrow J \longrightarrow A_c^I \xrightarrow{\rho} A_c'' \longrightarrow 0.$$

Note that the *-homomorphism $A_c^I \rtimes_\mu G \rightarrow (A \rtimes_\mu G)''$ of Lemma 3.3 contains $C_c(G, J)$ in its kernel, and so in the notation of Definition 2.1 it induces a *-homomorphism

$$\frac{A_c^I \rtimes_\mu G}{J \rtimes_{\mu, A_c^I} G} \rightarrow (A \rtimes_\mu G)''.$$

Since \rtimes_μ is half-exact, this translates to a $*$ -homomorphism

$$(A_c^I/J) \rtimes_\mu G \rightarrow (A \rtimes_\mu G)''$$

and using the canonical isomorphism $A_c^I/J = A_c''$ this gives the desired homomorphism

$$A_c'' \rtimes_\mu G \rightarrow (A \rtimes_\mu G)'',$$

so we are done. \square

Look now at the special case of Definition 3.1 where $\iota \rtimes u : A \rtimes_\mu G \rightarrow \mathcal{B}(H)$ is the universal representation of $A \rtimes_\mu G$. Then $(A \rtimes_\mu G)''$ is the enveloping von Neumann algebra of $A \rtimes_\mu G$, which identifies with the double dual $(A \rtimes_\mu G)^{**}$. Let $\iota^{**} : A^{**} \rightarrow (A \rtimes_\mu G)^{**}$ denote the normal extension of the representation $\iota : A \rightarrow (A \rtimes_\mu G)^{**} \subseteq \mathcal{B}(H)$, and abusing notation, use ι^{**} also for the restriction of this map to the continuous part A_c^{**} . Then (ι^{**}, u) is a covariant representation of the C^* -dynamical system (A_c^{**}, G) into $(A \rtimes_\mu G)^{**}$ and therefore integrates to a $*$ -homomorphism

$$\iota^{**} \rtimes u : C_c(G, A_c^{**}) \rightarrow (A \rtimes_\mu G)^{**}. \quad (11)$$

Notice that the map $\iota^{**} : A^{**} \rightarrow (A \rtimes_\mu G)^{**}$ is injective: indeed, if H_A is the universal representation for A , then we have a sequence of canonical maps

$$A_c^{**} \xrightarrow{\iota^{**}} (A \rtimes_\mu G)^{**} \longrightarrow (A \rtimes_r G)^{**} \longrightarrow \mathcal{B}(H_A \otimes L^2(G)),$$

where the last map is the normal extension of the regular representation associated to the universal representation of A , and whose composition is easily seen to be injective. A similar reasoning shows that the integrated form (11) is injective, but we shall not use this fact. Thus ι^{**} is a special case of the map ι'' from Definition 3.1.

THEOREM 3.5. *Let \rtimes_μ be a crossed-product functor for the locally compact group G . Then the following are equivalent:*

1. \rtimes_μ is half-exact;
2. for every G -algebra A and every faithful representation $\iota \rtimes \mu : A \rtimes_\mu G \rightarrow \mathcal{B}(H)$, with notation as in Definition 3.1 we have a $*$ -homomorphism

$$\iota'' \rtimes u : A_c'' \rtimes_\mu G \rightarrow (A \rtimes_\mu G)'';$$

3. for every G -algebra A , the map of line (11) extends to a $*$ -homomorphism

$$\iota^{**} \rtimes u : A_c^{**} \rtimes_\mu G \rightarrow (A \rtimes_\mu G)^{**};$$

4. \rtimes_μ is exact.

Proof. The implication (1) \Rightarrow (2) follows from Lemma 3.4, and (3) is a special case of (2).

Suppose now that (3) holds and let $0 \rightarrow J \rightarrow C \rightarrow A \rightarrow 0$ be any short exact sequence of G -algebras. We need to show that this sequence descends to a short exact sequence

$$0 \longrightarrow J \rtimes_{\mu} G \longrightarrow C \rtimes_{\mu} G \longrightarrow A \rtimes_{\mu} G \longrightarrow 0 .$$

It is clear that the map $C \rtimes_{\mu} G \rightarrow A \rtimes_{\mu} G$ is surjective. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & J \rtimes_{\mu} G & \longrightarrow & C \rtimes_{\mu} G & \longrightarrow & A \rtimes_{\mu} G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & J_c^{**} \rtimes_{\mu} G & \longrightarrow & C_c^{**} \rtimes_{\mu} G & \longrightarrow & A_c^{**} \rtimes_{\mu} G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & (J \rtimes_{\mu} G)^{**} & \longrightarrow & (C \rtimes_{\mu} G)^{**} & \longrightarrow & (A \rtimes_{\mu} G)^{**} & \longrightarrow & 0
 \end{array} \tag{12}$$

where the top three vertical arrows are induced by functoriality of \rtimes_{μ} , and the bottom three vertical arrows are as in assumption (3); note that the vertical compositions are just the canonical inclusions of each algebra into its double dual. Since the inclusion $J \rtimes_{\mu} G \rightarrow (J \rtimes_{\mu} G)^{**}$ is injective (and similarly for C and A), it follows that the upper vertical arrows are all injective. Since C_c^{**} decomposes as the direct sum of the G -algebras J_c^{**} and $(C/J)_c^{**} = A_c^{**}$, it follows that the map $J_c^{**} \rtimes_{\mu} G \rightarrow C_c^{**} \rtimes_{\mu} G$ is split injective. Hence the upper left square

$$\begin{array}{ccc}
 J \rtimes_{\mu} G & \longrightarrow & C \rtimes_{\mu} G \\
 \downarrow & & \downarrow \\
 J_c^{**} \rtimes_{\mu} G & \longrightarrow & C_c^{**} \rtimes_{\mu} G
 \end{array}$$

of diagram (12) implies injectivity of $J \rtimes_{\mu} G \rightarrow C \rtimes_{\mu} G$.

Suppose now that $x \in C \rtimes_{\mu} G$ goes to 0 in $A \rtimes_{\mu} G$. Then its image in $C_c^{**} \rtimes_{\mu} G$ is mapped to $0 \in A_c^{**} \rtimes_{\mu} G$, and hence must lie in $J_c^{**} \rtimes_{\mu} G$ by exactness of the middle horizontal sequence. Therefore, x lies in the intersection $(J \rtimes_{\mu} G)^{**} \cap C \rtimes_{\mu} G$ inside $(C \rtimes_{\mu} G)^{**}$. By [7, Lemma 9.2.6] this intersection equals $J \rtimes_{\mu} G$, and we are done.

The implication (4) \Rightarrow (1) is trivial. □

Remark 3.6. Property (3) in Theorem 3.5 is a direct analogue of *property C'* of Archbold and Batty [7, Definition 2.2]. Hence the equivalence of (3) and (4) is an analogue for crossed-product functors of the fact that property *C'* for a C^* -algebra B is equivalent to exactness of the functor $A \mapsto A \otimes B$ (that is, the exactness of B); see [7, Proposition 9.2.7] for a proof of this. The proof of Theorem 3.5 above is (indirectly) inspired by the proof of the cited proposition.

We were directly inspired by work of Matsumura [17], who proved (4) implies (3) for the special case when G is discrete and \rtimes_{μ} is the reduced crossed product.

Remark 3.7. Inspection of the proof of Theorem 3.5 shows that if the map $\iota^{**} \rtimes u : A_c^{**} \rtimes_r G \rightarrow (A \rtimes_r G)^{**}$ exists, then any short exact sequence of the form $0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$ descends to a short exact sequence

$$0 \longrightarrow I \rtimes_r G \longrightarrow A \rtimes_r G \longrightarrow B \rtimes_r G \longrightarrow 0,$$

or in other words, that A is an exact G -algebra in the sense of [20, Definition 1.2]. In particular, if $A = C_{ub}(G)$, then using Remark 2.11, existence of the map $\iota^{**} \rtimes u : C_{ub}(G)_c^{**} \rtimes_r G \rightarrow (C_{ub}(G) \rtimes_r G)^{**}$ is equivalent to exactness of G .

As an immediate corollary of this discussion and Theorem 3.5, we get the following characterization of exact groups.

COROLLARY 3.8. *Let G be a locally compact group. Then the following are equivalent:*

1. for every G -algebra A and every faithful representation $\iota \rtimes \mu : A \rtimes_r G \rightarrow \mathcal{B}(H)$, with notation as in Definition 3.1 we have a $*$ -homomorphism

$$\iota'' \rtimes u : A_c'' \rtimes_r G \rightarrow (A \rtimes_r G)'';$$

2. for every G -algebra A , the map of line (11) extends to a $*$ -homomorphism

$$\iota^{**} \rtimes u : A_c^{**} \rtimes_r G \rightarrow (A \rtimes_r G)^{**};$$

3. for the G -algebra $A = C_{ub}(G)$, the map of line (11) extends to a $*$ -homomorphism

$$\iota^{**} \rtimes u : C_{ub}(G)_c^{**} \rtimes_r G \rightarrow (C_{ub}(G) \rtimes_r G)^{**};$$

4. G is exact. □

Finally, we have the following immediate corollary of Theorem 3.5, Proposition 2.9, and Proposition 2.10.

COROLLARY 3.9. *For a given crossed product \rtimes_{μ} , $\rtimes_{\mathcal{E}(\mu)}$ is the minimal exact crossed-product functor that dominates \rtimes_{μ} . In particular, $\rtimes_{\mathcal{E}(r)}$ is the minimal exact crossed-product functor among all crossed-product functors for G . □*

4 MORITA COMPATIBILITY

Throughout this section, G denotes a locally compact group, and \rtimes_{μ} a crossed product functor. As before, the reader is encouraged to assume that $\rtimes_{\mu} = \rtimes_r$, which is the most important special case, but the general case causes no extra difficulties.

Our goal is to show that *Morita compatibility* as defined in [2, Definition 3.3] (see also Definition 4.1 below) passes from \rtimes_{μ} to $\rtimes_{\mathcal{E}(\mu)}$, as long as the input \rtimes_{μ} has the ideal property. In particular, $\rtimes_{\mathcal{E}(r)}$ is Morita compatible. From this, it follows readily that $\rtimes_{\mathcal{E}(r)}$ agrees with the *minimal exact Morita compatible functor* $\rtimes_{\mathcal{E}}$ of [2, Theorem 3.13] on the category of all G -algebras, and, if G is second countable, with the *minimal exact correspondence functor* $\rtimes_{\mathcal{E}_{\text{ortt}}}$ of [9, Corollary 8.8] on the category of separable G -algebras ([9, Corollary 8.13]). To state the definition of Morita compatibility, we need some notation. Let H be a Hilbert space equipped with a G -action u , and let Adu denote the induced action by conjugation on the compact operators $\mathcal{K} = \mathcal{K}(H)$. Let A be a G -algebra, and equip $A \otimes \mathcal{K}$ with the tensor product action. Consider the $*$ -homomorphism defined on the level of pre-completed crossed products and algebraic tensor products by the formula

$$\Psi_{\text{alg}} : C_c(G, A) \odot \mathcal{K} \rightarrow C_c(G, A \otimes \mathcal{K}), \quad \Psi(a \otimes k)(g) := a(g) \otimes ku_g^*.$$

Completing to the maximal crossed products and spatial tensor product, we get a $*$ -homomorphism

$$\Psi_{\text{max}} : (A \rtimes_{\text{max}} G) \otimes \mathcal{K} \rightarrow (A \otimes \mathcal{K}) \rtimes_{\text{max}} G. \quad (13)$$

which is well-known to be a $*$ -isomorphism. An explicit inverse to Ψ_{max} is constructed as follows. Consider the $*$ -homomorphism

$$\pi : A \otimes \mathcal{K} \rightarrow \mathcal{M}((A \rtimes_{\text{max}} G) \otimes \mathcal{K}); \quad a \otimes k \mapsto \iota(a) \otimes k,$$

where $\iota : A \rightarrow \mathcal{M}(A \rtimes_{\text{max}} G)$ denotes the canonical inclusion, and the unitary representation

$$v : G \rightarrow \mathcal{M}((A \rtimes_{\text{max}} G) \otimes \mathcal{K}); \quad g \mapsto \delta_g \otimes u_g.$$

Then the pair (π, v) is readily checked to be covariant. The integrated form gives a $*$ -homomorphism

$$\Phi_{\text{max}} : (A \otimes \mathcal{K}) \rtimes_{\text{max}} G \rightarrow (A \rtimes_{\text{max}} G) \otimes \mathcal{K}, \quad (14)$$

which one can check is the inverse to Ψ_{max} .

Now, let \rtimes_{μ} be an arbitrary crossed-product functor. Then postcomposing Φ_{max} and Ψ_{max} with the canonical quotient maps from maximal to μ -crossed products gives $*$ -homomorphisms

$$\begin{aligned} \Psi_{\text{max}, \mu} &: (A \rtimes_{\text{max}} G) \otimes \mathcal{K} \rightarrow (A \otimes \mathcal{K}) \rtimes_{\mu} G \\ \Phi_{\text{max}, \mu} &: (A \otimes \mathcal{K}) \rtimes_{\text{max}} G \rightarrow (A \rtimes_{\mu} G) \otimes \mathcal{K}. \end{aligned} \quad (15)$$

DEFINITION 4.1. Let \rtimes_{μ} be a crossed-product functor, and let H be a Hilbert space equipped with a G -action u . The functor \rtimes_{μ} is *u -Morita compatible* if for any G -algebra A the $*$ -homomorphism

$$\Psi_{\text{max}, \mu} : (A \rtimes_{\text{max}} G) \otimes \mathcal{K} \rightarrow (A \otimes \mathcal{K}) \rtimes_{\mu} G.$$

from line (15) above descends to a $*$ -isomorphism

$$\Psi_\mu : (A \rtimes_\mu G) \otimes \mathcal{K} \rightarrow (A \otimes \mathcal{K}) \rtimes_\mu G$$

Following [2, Definition 3.3], \rtimes_μ is *Morita compatible* if it is u -Morita compatible with u the tensor product of the left regular and trivial representations on $L^2(G) \otimes \ell^2(\mathbb{N})$.

Note that the maximal and reduced crossed products are u -Morita compatible for any u .

Remark 4.2. One can separate checking u -Morita compatibility into two questions as follows.

- (i) Does $\Psi_{\max, \mu}$ descend to a $*$ -homomorphism

$$\Psi_\mu : (A \rtimes_\mu G) \otimes \mathcal{K} \rightarrow (A \otimes \mathcal{K}) \rtimes_\mu G ?$$

- (ii) Does $\Phi_{\max, \mu}$ descend to a $*$ -homomorphism

$$\Phi_\mu : (A \otimes \mathcal{K}) \rtimes_\mu G \rightarrow (A \rtimes_\mu G) \otimes \mathcal{K} ?$$

The crossed product \rtimes_μ is u -Morita compatible if and only if the answer to both of these questions is ‘yes’, and in that case the descended $*$ -homomorphisms Φ_μ and Ψ_μ will automatically be mutually inverse (as they are mutually inverse on dense subalgebras).

One can at least always answer question (i) positively in the presence of the ideal property: recall this means that \rtimes_μ takes an equivariant inclusion $I \subseteq A$ of an ideal to an injective map $I \rtimes_\mu G \rightarrow A \rtimes_\mu G$.

LEMMA 4.3. *Let H be a Hilbert space equipped with a G -action u , let A be a G -algebra, and let \rtimes_μ be a crossed product with the ideal property. Then the $*$ -homomorphism*

$$\Psi_{\max, \mu} : (A \rtimes_{\max} G) \otimes \mathcal{K} \rightarrow (A \otimes \mathcal{K}) \rtimes_\mu G$$

of line (15) above descends to a $*$ -homomorphism

$$\Psi_\mu : (A \rtimes_\mu G) \otimes \mathcal{K} \rightarrow (A \otimes \mathcal{K}) \rtimes_\mu G.$$

Proof. We have an equivariant $*$ -homomorphism

$$A \rightarrow \mathcal{M}((A \otimes \mathcal{K}) \rtimes_\mu G); \quad a \mapsto a \otimes 1$$

and a unitary representation

$$G \rightarrow \mathcal{M}((A \otimes \mathcal{K}) \rtimes_\mu G); \quad g \mapsto \delta_g.$$

These form a covariant pair for (A, G) , which integrates to a $*$ -homomorphism

$$C_c(G, A) \rightarrow \mathcal{M}((A \otimes \mathcal{K}) \rtimes_{\mu} G).$$

As \rtimes_{μ} has the ideal property, [9, Lemma 3.3] implies that the integrated form extends to a $*$ -homomorphism

$$\Psi_A : A \rtimes_{\mu} G \rightarrow \mathcal{M}((A \otimes \mathcal{K}) \rtimes_{\mu} G).$$

On the other hand, we have a $*$ -homomorphism

$$\mathcal{K} \rightarrow \mathcal{M}((A \rtimes_{\max} G) \otimes \mathcal{K}); \quad k \mapsto 1 \otimes k.$$

Postcomposing this with the map induced on multipliers by Ψ_{\max} gives a $*$ -homomorphism

$$\mathcal{K} \rightarrow \mathcal{M}((A \otimes \mathcal{K}) \rtimes_{\max} G)$$

and postcomposing again with the map on multipliers induced by the canonical quotient map $(A \otimes \mathcal{K}) \rtimes_{\max} G \rightarrow (A \otimes \mathcal{K}) \rtimes_{\mu} G$ gives a $*$ -homomorphism

$$\Psi_{\mathcal{K}} : \mathcal{K} \rightarrow \mathcal{M}((A \otimes \mathcal{K}) \rtimes_{\mu} G).$$

The images of Ψ_A and $\Psi_{\mathcal{K}}$ commute, so they combine to give a $*$ -homomorphism

$$\Psi_A \odot \Psi_{\mathcal{K}} : (A \rtimes_{\mu} G) \odot \mathcal{K} \rightarrow \mathcal{M}((A \otimes \mathcal{K}) \rtimes_{\mu} G)$$

on the algebraic tensor product. Checking on generators, this $*$ -homomorphism actually takes image in $(A \otimes \mathcal{K}) \rtimes_{\mu} G$, not just in the multiplier algebra. It moreover extends to the spatial tensor product by nuclearity of \mathcal{K} , giving us

$$\Psi_{\mu} : (A \rtimes_{\mu} G) \otimes \mathcal{K} \rightarrow (A \otimes \mathcal{K}) \rtimes_{\mu} G,$$

and one checks on generators that this is exactly the desired map. \square

PROPOSITION 4.4. *Fix a unitary G -representation u , and let \rtimes_{μ} be a u -Morita compatible crossed product with the ideal property. Then the crossed product $\rtimes_{\mathcal{E}(\mu)}$ is u -Morita compatible. In particular, if \rtimes_{μ} has the ideal property and is Morita compatible, then so is $\rtimes_{\mathcal{E}(\mu)}$.*

Proof. Let A be a G -algebra. The crossed product $\rtimes_{\mathcal{E}(\mu)}$ is exact by Corollary 3.9, whence in particular has the ideal property. Hence Lemma 4.3 implies that we have a $*$ -homomorphism

$$\Psi_{\mathcal{E}(\mu)} : (A \rtimes_{\mathcal{E}(\mu)} G) \otimes \mathcal{K} \rightarrow (A \otimes \mathcal{K}) \rtimes_{\mathcal{E}(\mu)} G.$$

To complete the proof, it suffices as in Remark 4.2 to show that the $*$ -homomorphism $\Phi_{\max, \mathcal{E}(\mu)}$ descends to a $*$ -homomorphism

$$\Phi_{\mathcal{E}(\mu)} : (A \otimes \mathcal{K}) \rtimes_{\mathcal{E}(\mu)} G \rightarrow (A \rtimes_{\mathcal{E}(\mu)} G) \otimes \mathcal{K}.$$

Let

$$0 \longrightarrow I \longrightarrow C \xrightarrow{\pi} A \longrightarrow 0$$

be an equivariant short exact sequence such that $A \rtimes_{\pi} G = A \rtimes_{\mathcal{E}(\mu)} G$ (this exists by Proposition 2.4). Tensoring the above sequence by the G -algebra \mathcal{K} (equipped as always with the action $\text{Ad}u$, with the tensor products given the tensor product action) and taking crossed products gives a short exact sequence

$$0 \longrightarrow (I \otimes \mathcal{K}) \rtimes_{\mu} G \longrightarrow (C \otimes \mathcal{K}) \rtimes_{\mu} G \longrightarrow \frac{(C \otimes \mathcal{K}) \rtimes_{\mu} G}{(I \otimes \mathcal{K}) \rtimes_{\mu} G} \longrightarrow 0 ;$$

here we have used the ideal property for \rtimes_{μ} to identify the closure of $C_c(G, I \otimes \mathcal{K})$ inside $(C \otimes \mathcal{K}) \rtimes_{\mu} G$ with $(I \otimes \mathcal{K}) \rtimes_{\mu} G$. The definition of $(A \otimes \mathcal{K}) \rtimes_{\mathcal{E}(\mu)} G$ implies that the identity map on $C_c(G, A)$ extends to a (surjective) $*$ -homomorphism

$$(A \otimes \mathcal{K}) \rtimes_{\mathcal{E}(\mu)} G \rightarrow \frac{(C \otimes \mathcal{K}) \rtimes_{\mu} G}{(I \otimes \mathcal{K}) \rtimes_{\mu} G}. \tag{16}$$

On the other hand, the assumption that \rtimes_{μ} is u -Morita compatible gives isomorphisms

$$\Phi_{\mu} : (C \otimes \mathcal{K}) \rtimes_{\mu} G \rightarrow (C \rtimes_{\mu} G) \otimes \mathcal{K} \quad \text{and} \quad \Phi_{\mu} : (I \otimes \mathcal{K}) \rtimes_{\mu} G \rightarrow (I \rtimes_{\mu} G) \otimes \mathcal{K},$$

which together with exactness of the functor $B \mapsto B \otimes \mathcal{K}$, the definition of the π -norm, and the choice of $\pi : C \rightarrow A$, give rise to a map

$$\frac{(C \otimes \mathcal{K}) \rtimes_{\mu} G}{(I \otimes \mathcal{K}) \rtimes_{\mu} G} \rightarrow \left(\frac{C \rtimes_{\mu} G}{I \rtimes_{\mu} G} \right) \otimes \mathcal{K} = (A \rtimes_{\pi} G) \otimes \mathcal{K} = (A \rtimes_{\mathcal{E}(\mu)} G) \otimes \mathcal{K}. \tag{17}$$

Composing the maps from lines (16) and (17) thus gives a $*$ -homomorphism

$$(A \otimes \mathcal{K}) \rtimes_{\mathcal{E}(\mu)} G \rightarrow (A \rtimes_{\mathcal{E}(\mu)} G) \otimes \mathcal{K};$$

checking on generators, one sees that this is the desired $*$ -homomorphism $\Phi_{\mathcal{E}(\mu)}$, so we are done. □

The next corollary is immediate from [9, Proposition 8.10].

COROLLARY 4.5. *Suppose that \rtimes_{μ} is a Morita compatible crossed product functor with the ideal property. Then the restriction of $\rtimes_{\mathcal{E}(\mu)}$ to the category of σ -unital G -algebras is strongly Morita compatible in the sense of [9, Definition 4.7].* □

The following corollary is immediate from Proposition 4.4 and Corollary 3.9 above.

COROLLARY 4.6. *The crossed product $\rtimes_{\mathcal{E}(r)}$ is the same as the minimal exact Morita compatible functor $\rtimes_{\mathcal{E}}$ of [2, Theorem 3.13].* □

The following corollary is immediate from Corollary 4.6 and [9, Corollary 8.14].

COROLLARY 4.7. *Let G be second countable. Then, on the category of separable G -algebras, the crossed-product functor $\rtimes_{\mathcal{E}(r)}$ agrees with the minimal exact correspondence functor $\rtimes_{\mathcal{E}_{\text{ottr}}}$ of [9, Corollary 8.8]. \square*

Note that this answers [2, Questions 8.5, (iv) and (v)]. Indeed, question (iv) asks whether $\mathbb{C} \rtimes_{\mathcal{E}} G$ equals $C_r^*(G)$, and Proposition 2.6 and Corollary 4.6 imply that this is always true. On the other hand, question (v) asks whether $\rtimes_{\mathcal{E}}$ can be a KLQ functor for non-exact G ; as a KLQ functor is uniquely determined by what it does on \mathbb{C} and as $\rtimes_r \neq \rtimes_{\mathcal{E}(r)}$ for a non-exact group G , the answer ‘always’ to question (iv) shows that the answer to question (v) is ‘never’.

To finish this section, we discuss which group algebras (i.e., C^* -algebra completions of $C_c(G)$) can appear as $\mathbb{C} \rtimes_{\mu} G$ for certain types of crossed product functors \rtimes_{μ} . As discussed in [8, Section 2], every group algebra can be viewed as the group C^* -algebra $C_E^*(G)$ associated to a G -invariant weak*-closed G -invariant subspace E of the Fourier-Stieltjes algebra $B(G)$ containing the Fourier algebra $A(G)$.

The Brown-Guentner construction [6] (see also the discussion in [8, Section 3]) shows that any such $C_E^*(G)$ arises as the group algebra associated to an exact crossed product $\rtimes_{E_{BG}}$. On the other hand, the Kaliszewski-Landstad-Quigg construction [16] (see also the discussion in [8, Section 3] again) shows that if E is an ideal in $B(G)$, then $C_E^*(G)$ arises as the group C^* -algebra of a functor $\rtimes_{E_{KLQ}}$ that is u -Morita compatible for any u , and that has the ideal property; conversely [9, Corollary 5.7]³ shows that if $C_E^*(G)$ arises as the group algebra of any functor with these properties, then E must be an ideal.

However, Brown-Guentner crossed products are generally not Morita compatible, and Kaliszewski-Landstad-Quigg crossed products are generally not exact (see the discussion in [8, Section 4]). Thus it was not previously clear which group algebras can arise as $\mathbb{C} \rtimes_{\mu} G$ for a functor that is both exact and Morita compatible; this is a natural question, as such functors seem to behave best with respect to the Baum-Connes assembly map.

COROLLARY 4.8. *Let G be a locally compact group. Then a group algebra $C_E^*(G)$ is of the form $\mathbb{C} \rtimes_{\mu} G$ for a functor \rtimes_{μ} that is exact and u -Morita compatible for any u if and only if E is an ideal in $B(G)$.*

Proof. Take the Kaliszewski-Landstad-Quigg functor \rtimes_{μ} associated to E and consider $\rtimes_{\mathcal{E}(\mu)}$. This is an exact Morita compatible crossed-product functor by Theorems 3.5 and Proposition 4.4; and it has the same group algebra $C_E^*(G)$ as the original functor \rtimes_{μ} by Proposition 2.6. The converse follows from [9, Corollary 5.7] again, noting that the conditions in the statement give what is needed for the proof of that corollary to work. \square

³[9, Corollary 5.7] is written for correspondence functors, but inspection of the proof shows that it holds in this slightly more general setting.

5 LIFTING PROPERTIES AND ATTAINING NORMS

Throughout this section, G is a locally compact group (although we will need to assume that G is discrete for some results), and \rtimes_μ an associated crossed product.

Recall from Lemma 2.4 that the $\mathcal{E}(\mu)$ -norm is always attained via a fixed G -equivariant surjection $\pi : C \rightarrow A$. Our goal in this section is to get further information about possible choices for such surjections, at least in the case when G is discrete. The following definition, due to Phillips, Sørensen, and Thiel [19], will be very useful here.

DEFINITION 5.1. A G -algebra C is *equivariantly projective* if whenever B is a G -algebra, $J \subseteq B$ is a G -invariant ideal, and $\phi : C \rightarrow B/J$ is an equivariant $*$ -homomorphism, there is an equivariant $*$ -homomorphism $\tilde{\phi} : C \rightarrow B$ that lifts ϕ (in other words, if $\pi : B \rightarrow B/J$ is the quotient map, then the diagram

$$\begin{array}{ccc} & & B \\ & \nearrow \tilde{\phi} & \downarrow \pi \\ C & \xrightarrow{\phi} & B/J \end{array}$$

commutes).

We say that C is equivariantly projective in the unital category if it has the above property, but with all C^* -algebras and maps appearing above assumed unital.

PROPOSITION 5.2. Let \rtimes_μ be a crossed product. Let A be a G -algebra, and let C be an equivariantly projective G -algebra equipped with an equivariant quotient map $\pi : C \rightarrow A$. Then for any $a \in C_c(G, A)$, $\|a\|_\pi = \|a\|_{\mathcal{E}(\mu)}$. Moreover, if A is unital, the same conclusion follows if we assume that C is equivariantly projective in the unital category and equipped with a unital quotient map $\pi : C \rightarrow A$.

Proof. We need to show that if $\sigma : B \rightarrow A$ is any other equivariant quotient map, then for any $a \in C_c(G, A)$, $\|a\|_\sigma \leq \|a\|_\pi$. Consider the diagram

$$\begin{array}{ccc} & & B \\ & \nearrow \tilde{\pi} & \downarrow \sigma \\ C & \xrightarrow{\pi} & A \end{array}$$

Equivariant projectivity for C implies that the dashed arrow can be filled in by an equivariant $*$ -homomorphism $\tilde{\pi} : C \rightarrow B$. Letting I and J be the kernels of π and σ respectively, we get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & C & \xrightarrow{\pi} & A \longrightarrow 0 \\ & & & & \downarrow \tilde{\pi} & & \parallel \\ 0 & \longrightarrow & J & \longrightarrow & B & \xrightarrow{\sigma} & A \longrightarrow 0. \end{array} \tag{18}$$

Commutativity of this diagram gives that $\tilde{\pi}$ restricts to a map from I to J . Hence we get a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & I \rtimes_{\mu,C} G & \longrightarrow & C \rtimes_{\mu} G & \longrightarrow & \frac{C \rtimes_{\mu} G}{I \rtimes_{\mu,C} G} \longrightarrow 0 \\
 & & \downarrow \tilde{\pi}|_I \rtimes_r G & & \downarrow \tilde{\pi} \rtimes_r G & & \\
 0 & \longrightarrow & J \rtimes_{\mu,B} G & \longrightarrow & B \rtimes_{\mu} G & \longrightarrow & \frac{B \rtimes_{\mu} G}{J \rtimes_{\mu,B} G} \longrightarrow 0.
 \end{array}$$

Commutativity implies that $\tilde{\pi} \rtimes_r G$ induces a $*$ -homomorphism on quotients

$$\frac{C \rtimes_{\mu} G}{I \rtimes_{\mu,C} G} \rightarrow \frac{B \rtimes_{\mu} G}{J \rtimes_{\mu,B} G},$$

which by commutativity of the diagram in line (18) restricts to the identity on the $*$ -subalgebra $C_c(G, A)$ of both sides. This implies that

$$\|a\|_{\sigma} \leq \|a\|_{\pi}$$

for any $a \in C_c(G, A)$ and we are done in the general case.

The statement in the unital case follows from essentially the same argument as by Lemma 2.4 we may assume B to be unital as well. □

As it gives some interesting examples, we will also explore a weakening of equivariant projectivity. To explain the terminology, recall (see, e.g. [7, Definition 13.1.1]) that a C^* -algebra C has the *lifting property* (LP) if whenever $\phi : C \rightarrow B/J$ is a contractive completely positive (ccp) map into a quotient C^* -algebra, there exists a ccp lift $\psi : C \rightarrow B$.

DEFINITION 5.3. A G -algebra C has the *weak equivariant lifting property* (WELP) if whenever B is a G -algebra, $J \subseteq B$ is a G -invariant ideal, and $\phi : C \rightarrow B/J$ is an equivariant $*$ -homomorphism, then there is an equivariant ccp map $\tilde{\phi} : C \rightarrow B$ that lifts ϕ (in other words, if $\pi : B \rightarrow B/J$ is the quotient map, then the diagram

$$\begin{array}{ccc}
 & & B \\
 & \nearrow \tilde{\phi} & \downarrow \pi \\
 C & \xrightarrow{\phi} & B/J
 \end{array}$$

commutes).

We say that C has the *unital weak equivariant lifting property* (UWELP) if it has the above property, but with all maps and C^* -algebras assumed unital.

Remark 5.4. It would be natural to define a stronger *equivariant lifting property* (ELP): by analogy with the LP, one would here ask for equivariant ccp lifts of equivariant ccp maps ϕ as in the above, rather than just for equivariant $*$ -homomorphic ϕ . We do not know if any non-trivial examples of ELP G -algebras exist for non-compact G , however. This seems an interesting question.

Even for the trivial group the WELP is, a priori, weaker than the ordinary LP, although both are equivalent for separable C^* -algebras by an application of Stinespring's dilation theorem as in the proof of [7, Theorem 13.1.3].

The proof of the following result is the same as that of Proposition 5.2 – with equivariant ccp maps replacing equivariant $*$ -homomorphisms at appropriate points – and thus omitted.

PROPOSITION 5.5. *Let \rtimes_{μ} be a crossed product which is functorial for completely positive maps⁴. Let A be a G -algebra, and let C be a G -algebra with the WELP, and equipped with an equivariant quotient map $\pi : C \rightarrow A$. Then for any $a \in C_c(G, A)$, $\|a\|_{\pi} = \|a\|_{\mathcal{E}(\mu)}$.*

Moreover, if A is unital, the same conclusion follows if we assume that C has the UWELP and is equipped with a unital quotient map $\pi : C \rightarrow A$. \square

The following corollary gives a way to compute $\rtimes_{\mathcal{E}(\mu)}$ in some special cases. Note that \mathbb{C} is clearly equivariantly projective in the unital category, so this gives a slightly different approach to proving Proposition 2.6.

COROLLARY 5.6. *If \rtimes_{μ} is a crossed-product functor, and if A is equivariantly projective, or equivariantly projective in the unital category, then $A \rtimes_{\mu} G = A \rtimes_{\mathcal{E}(\mu)} G$.*

If \rtimes_{μ} is a crossed-product functor which is functorial for completely positive maps, and if A has either the WELP or the UWELP, then $A \rtimes_{\mu} G = A \rtimes_{\mathcal{E}(\mu)} G$.

Proof. In either case, Proposition 5.2 or Proposition 5.5 implies that the $\mathcal{E}(\mu)$ -norm on $C_c(G, A)$ equals the π -norm where $\pi : A \rightarrow A$ is the identity map; clearly this is just the \rtimes_{μ} -norm, however. \square

Having got through the above, it is maybe not clear that interesting equivariantly projective, or even WELP, G -algebras exist. Moreover, to get much use out of the above results, we would need to show that for any G -algebra A there exists an equivariant surjection $\pi : C \rightarrow A$, where C has the WELP. Unfortunately, we can only prove this in the discrete case, and must leave the general locally compact case as a question for now.

The following result is essentially a special case of [19, Proposition 2.4]; we nonetheless give a direct proof for the reader's convenience.

PROPOSITION 5.7. *Let G be a discrete group, let X be a set, and let C be the universal C^* -algebra generated by a set $\{c_{x,g} \mid (x,g) \in X \times G\}$ of positive contractions indexed by the set $X \times G$. Equip C with the G -action induced by the set action*

$$g : (x, h) \mapsto (x, gh)$$

and universality. Then C is equivariantly projective, and admits a surjective equivariant $$ -homomorphism onto any G -algebra generated by a set of positive contractions of cardinality at most that of X .*

⁴For example, the reduced crossed product. See [9, Theorem 4.9] for some equivalent conditions.

Proof. We first show that C is equivariantly projective. Let then $\phi : C \rightarrow B/J$ be an equivariant $*$ -homomorphism. For each $x \in X$, choose a positive contraction $b_x \in B$ that lifts $\phi(c_{x,e})$. Write β for the action of G on B , and let $\tilde{\phi} : C \rightarrow B$ be the $*$ -homomorphism uniquely defined by the map

$$c_{x,g} \mapsto \beta_g(b_x)$$

on generators. Clearly this is equivariant and lifts ϕ on the generators; as it is a $*$ -homomorphism it is thus equivariant and lifts ϕ on all of C .

Now, let A have a generating set S of positive contractions of cardinality at most that of X . Write α for the action of G on A . Choose a surjective map $f : X \rightarrow S$. Let now $\pi : C \rightarrow A$ be the $*$ -homomorphism uniquely determined by the map

$$c_{x,g} \mapsto \alpha_g(f(x)).$$

This is equivariant on generators, so everywhere, and is surjective as S generates A . □

Another interesting example (and the one that originally inspired this work) is as follows.

PROPOSITION 5.8. *Let G be a discrete group, let X be a set, let $F_{X \times G}$ be the free group on $X \times G$, and let $C := C_{\max}^*(F_{X \times G})$ be the maximal group C^* -algebra of $F_{X \times G}$. Equip C with the G -action induced by the set action*

$$g : (x, h) \mapsto (x, gh)$$

and universality. Then C has the UWELP, and admits a surjective unital equivariant $$ -homomorphism onto any unital G -algebra generated by a set of unitaries of cardinality at most that of X .*

Proof. The statement about the existence of a quotient map $C \rightarrow A$ follows from essentially the same construction as in the proof of Proposition 5.7: we leave the details to the reader. It remains to show that C has the UWELP, so let $\phi : C \rightarrow B/J$ be a unital equivariant $*$ -homomorphism. For each $(x, g) \in X \times G$, let $u_{x,g}$ be the corresponding generating unitary for C , and choose a contractive lift $b_x \in B$ of $\phi(u_{x,e}) \in B/J$. Now define

$$v_x := \begin{pmatrix} b_x & (1 - b_x b_x^*)^{1/2} \\ (1 - b_x^* b_x)^{1/2} & -b_x^* \end{pmatrix} \in M_2(B),$$

which is unitary. Let β denote the action on B . Universality implies that the map defined on generators by

$$u_{x,g} \mapsto \beta_g(v_x)$$

extends to an equivariant $*$ -homomorphism $C \rightarrow M_2(B)$. The top left corner of this $*$ -homomorphism is the desired ucp equivariant lift of ϕ . □

Remark 5.9. It is proved in [19, Proposition 2.4] that the Bernoulli shift G -actions on free products of the form $A = \ast_{g \in G} B$ are always G -equivariantly projective if the base C^* -algebra B is (non-equivariantly) projective. This result contains Proposition 5.7 as a special case by taking $B \cong C_0(0, 1]$, the universal C^* -algebra generated by a positive contraction, which is a projective C^* -algebra. Similarly, one can show that $A = \ast_{g \in G} B$ has the WELP for the Bernoulli shift G -action provided that B has the WELP for the trivial group action. An analogous version of the UWELP holds for unital free products (i.e., amalgamated over \mathbb{C}), generalising Proposition 5.8. To prove these assertions one can use Boca's result from [3, Theorem 3.1] on free products of ccp (or ucp) maps. Indeed, in the non-equivariant situation, this idea has already been used by Boca to prove that certain lifting properties are preserved by free products in [4].

6 RESTRICTION

Suppose that H is a closed subgroup of a locally compact group G and that \rtimes_{μ} is a crossed-product functor for G . Our goal in this section is to study the relationship between the minimal exact crossed products for H and G , with a view to applications to the (reformulated) Baum-Connes conjecture.

For an H -algebra (A, α) consider the induced G -algebra $(\text{Ind}_H^G(A, \alpha), \text{Ind } \alpha)$ in which

$$\text{Ind}_H^G(A, \alpha) := \left\{ F \in C_b(G, A) : \begin{array}{l} \alpha_h(F(gh)) = F(g) \ \forall g \in G, h \in H, \\ \text{and } (gH \mapsto \|F(g)\|) \in C_0(G/H) \end{array} \right\}.$$

The G -action on $\text{Ind}_H^G(A, \alpha)$ is given by $(\text{Ind } \alpha_g(F))(k) = F(g^{-1}k)$. Green's imprimitivity theorem (see [13, Theorem 17] or [12, Section 2.6]) provides a natural equivalence bimodule $X(A, \alpha)$ between $\text{Ind}_H^G(A, \alpha) \rtimes_{\text{Ind } \alpha, \max} G$ and $A \rtimes_{\alpha, \max} H$. Let

$$I_{\mu} = \ker \left(\text{Ind}_H^G(A, \alpha) \rtimes_{\max} G \rightarrow \text{Ind}_H^G(A, \alpha) \rtimes_{\mu} G \right).$$

By the Rieffel correspondence between ideals in $\text{Ind}_H^G(A, \alpha) \rtimes_{\max} G$ and ideals in $A \rtimes_{\max} H$ there is a unique ideal $I_{\mu|_H} \subseteq A \rtimes_{\max} H$ such that $X(A, \alpha)$ factors through an equivalence bimodule $X_{\mu}(A, \alpha)$ between $\text{Ind}_H^G(A, \alpha) \rtimes_{\mu} G$ and the quotient

$$A \rtimes_{\mu|_H} H := (A \rtimes_{\max} H) / I_{\mu|_H}. \quad (19)$$

The following definition is taken from [8]:

DEFINITION 6.1. Let \rtimes_{μ} be a crossed-product functor for G . Then the assignment $(A, \alpha) \mapsto A \rtimes_{\mu|_H} H$ with $A \rtimes_{\mu|_H} H$ constructed as above is called the *restriction* of \rtimes_{μ} to H .

In [8, Proposition 6.6] we also observed that for a second countable locally compact group G , the Baum-Connes assembly map

$$\mathrm{as}_{(A,H)}^{\mu|_H} : K_*^{\mathrm{top}}(H, A) \rightarrow K_*(A \rtimes_{\mu|_H} H)$$

is an isomorphism if and only if the assembly map

$$\mathrm{as}_{(\mathrm{Ind}_H^G A, G)}^{\mu} : K_*^{\mathrm{top}}(G, \mathrm{Ind}_H^G A) \rightarrow K_*(\mathrm{Ind}_H^G A \rtimes_{\mathrm{Ind} \alpha, \mu} G)$$

is an isomorphism. In particular, if G satisfies the analogue of the Baum-Connes conjecture for the \rtimes_{μ} -crossed product, then H satisfies the conjecture for the $\rtimes_{\mu|_H}$ -crossed product. Thus, it is interesting to study the question whether the restriction $\rtimes_{\mathcal{E}_G|_H}$ of the minimal exact crossed-product functor $\rtimes_{\mathcal{E}_G}$ for G to a closed subgroup H will always be the minimal exact functor $\rtimes_{\mathcal{E}_H}$ for H , since this would imply that the new conjecture of Baum, Guentner, and Willett (as explained in the introduction) passes to closed subgroups. In [8, Theorem 7.13] we showed this for the case where H is normal or cocompact in G .⁵ Below we shall give a proof of this fact if H is open in G . We need the following result, which follows by the same arguments as used in the proof of [8, Lemma 7.6]. In what follows, we denote by $i_B : B \rightarrow M(B \rtimes_{\mu} G)$ and $i_G : G \rightarrow M(B \rtimes_{\mu} G)$ the canonical embeddings into the multiplier algebra of a crossed product $B \rtimes_{\mu} G$.

LEMMA 6.2. *Let H be a closed subgroup of G , and let B be a G -algebra. Then the canonical mapping $i_B \rtimes i_G|_H : B \rtimes_{\max} H \rightarrow M(B \rtimes_{\max} G)$ factors to a well-defined $*$ -homomorphism*

$$i_B^{\mathcal{E}_G} \rtimes i_G^{\mathcal{E}_G}|_H : B \rtimes_{\mathcal{E}_H} H \rightarrow M(B \rtimes_{\mathcal{E}_G} G). \quad \square$$

Note that if H is open in G , the above homomorphism extends the canonical inclusion of $C_c(H, B)$ into $C_c(G, B)$, and hence it takes its image in $B \rtimes_{\mathcal{E}_G} G$. We are now ready for

THEOREM 6.3. *Suppose that H is an open subgroup of the locally compact group G . Then $\rtimes_{\mathcal{E}_G|_H} = \rtimes_{\mathcal{E}_H}$.*

Proof. For the proof we shall use a special form of Green's imprimitivity theorem in case where H is open in G . For this let χ_{eH} denote the characteristic function of the coset $\{eH\} \subseteq G/H$, viewed as a projection in the multiplier algebra $\mathcal{M}(\mathrm{Ind}_H^G A)$ in the canonical way, and let $p \in \mathcal{M}(\mathrm{Ind}_H^G A \rtimes_{\max} G)$ denote its image in the crossed product. Then it is shown in [12, Proposition 2.6.8] that p is a full projection such that $p(\mathrm{Ind}_H^G A \rtimes_{\max} G)p = A \rtimes_{\max} H$ and the resulting $\mathrm{Ind}_H^G A \rtimes_{\max} G$ - $A \rtimes_{\max} H$ equivalence bimodule $(\mathrm{Ind}_H^G A \rtimes_{\max} G)p$ is isomorphic to Green's equivalence bimodule $X(A, \alpha)$. This implies that the

⁵In fact, we showed this for the smallest exact *correspondence functor* $\rtimes_{\mathcal{E}_{\mathrm{corr}}}$ which coincides with $\rtimes_{\mathcal{E}}$ on the category of separable G -algebras if G is second countable. But the same arguments as used in the proof of [8, Theorem 7.13] apply directly to the smallest exact functors considered here.

image of p in $\mathcal{M}(\text{Ind}_H^G A \rtimes_{\mathcal{E}_G} G)$ (which we also denote by p) is a full projection such that $A \rtimes_{\mathcal{E}_G|_H} G = p(\text{Ind}_H^G A \rtimes_{\mathcal{E}_G} G)p$. It follows from [8, Theorem 6.3] that $\rtimes_{\mathcal{E}_G|_H}$ is an exact functor for H . Thus, by minimality of \mathcal{E}_H , the identity on $C_c(G, A)$ induces a quotient map $q : A \rtimes_{\mathcal{E}_G|_H} H \rightarrow A \rtimes_{\mathcal{E}_H} H$.

We now construct an inverse for q . Indeed, by Lemma 6.2 we have a canonical $*$ -homomorphism $\varphi : \text{Ind}_H^G A \rtimes_{\mathcal{E}_H} H \rightarrow \text{Ind}_H^G A \rtimes_{\mathcal{E}_G} G$ which extends the canonical inclusion of $C_c(H, \text{Ind}_H^G A)$ into $C_c(G, \text{Ind}_H^G A)$. As an H -algebra, $\text{Ind}_H^G A$ decomposes as a direct sum $A \oplus I$, where we identify A with the functions in $\text{Ind}_H^G A$ which live on the coset eH and I with the functions which vanish on eH . This implies a decomposition $\text{Ind}_H^G A \rtimes_{\mathcal{E}_H} H \cong (A \rtimes_{\mathcal{E}_H} H) \oplus (I \rtimes_{\mathcal{E}_H} H)$ and it is easily verified on functions in $C_c(H, A)$ that the homomorphism φ maps the summand $A \rtimes_{\mathcal{E}_H} H$ into the corner $p(\text{Ind}_H^G A \rtimes_{\mathcal{E}_G} G)p \cong A \rtimes_{\mathcal{E}_G|_H} H$, thus providing an inverse for q . \square

Remark 6.4. Using [8, Lemma 7.6] instead of Lemma 6.2 in the above proof, the same arguments as used above also show that the restriction of the minimal exact correspondence functor $\rtimes_{\mathcal{E}_G}$ of G to an open subgroup H of G coincides with the minimal exact correspondence functor $\rtimes_{\mathcal{E}_H}$ of H .

7 SOME QUESTIONS

There are still many important open questions about the smallest exact crossed-product functor $\rtimes_{\mathcal{E}}$. Here are some of them:

QUESTION 7.1. Is it true that the smallest exact crossed-product functor for G is automatically a correspondence functor?

Since exact functors automatically satisfy the ideal property, it follows from [9, Theorem 4.9] that being a correspondence functor is equivalent to any of the following assertions:

1. For each G -algebra A and G -equivariant projection $p \in \mathcal{M}(A)$, the canonical map $pAp \rtimes_{\mathcal{E}} G \rightarrow A \rtimes_{\mathcal{E}} G$ is injective.
2. For each G -algebra A and G -equivariant *full* projection $p \in \mathcal{M}(A)$, the canonical map $pAp \rtimes_{\mathcal{E}} G \rightarrow A \rtimes_{\mathcal{E}} G$ is injective.
3. For each (full) G -invariant hereditary subalgebra B of a G -algebra A , the canonical map $B \rtimes_{\mathcal{E}} G \rightarrow A \rtimes_{\mathcal{E}} G$ is injective.
4. $\rtimes_{\mathcal{E}}$ is *strongly Morita compatible* in the sense that for any G -equivariant Morita equivalence bimodule (X, γ) between two G -algebras A and B , the canonical $C_c(G, A)$ - $C_c(G, B)$ bimodule $C_c(G, X)$ completes to give an $A \rtimes_{\mathcal{E}} G$ - $B \rtimes_{\mathcal{E}} G$ equivalence bimodule.
5. The functor $\rtimes_{\mathcal{E}}$ is functorial for G -equivariant ccp maps.

Note that it is shown in Corollary 4.5 above that all this holds on the category of σ -unital G -algebras.

QUESTION 7.2. Is the smallest exact crossed-product functor $\rtimes_{\mathcal{E}}$ injective?

We say that a crossed-product functor \rtimes_{μ} is *injective*, if for every injective G -equivariant $*$ -homomorphism $\varphi : A \rightarrow B$ the descent $\varphi \rtimes_{\mu} G : B \rtimes_{\mu} G \rightarrow A \rtimes_{\mu} G$ is injective as well. The reduced crossed-product functor \rtimes_r is well known to be injective but lacks exactness in general. At some early point of the project we thought we could show that injectivity holds for $\rtimes_{\mathcal{E}}$, but our argument had a serious gap. We then thought we had an argument proving that $\rtimes_{\mathcal{E}}$ is not injective in general, but again found a gap in the proof. So right now, we have no clue about the correct answer to this question. Indeed, in the moment we do not know of any example of an exact and injective crossed-product functor for a non-exact group.

Note that injectivity of $\rtimes_{\mathcal{E}}$, if true, would imply some nice properties of this functor: It would be continuous for general inductive limits of G -algebras, i.e., we would get $(\lim_i A_i) \rtimes_{\mathcal{E}} G = \lim_i (A_i \rtimes_{\mathcal{E}} G)$ for every directed system of G -algebras $(A_i, \varphi_i)_{i \in I}$. It would also imply that the functor $\rtimes_{\mathcal{E}}$ preserves continuous fields of C^* -algebras in the sense that if G acts fibrewise on the section algebra A of a continuous field of C^* -algebras over a base space X with fibres A_x , $x \in X$, then $A \rtimes_{\mathcal{E}} G$ would be the section algebra of a continuous field of C^* -algebras with fibres $A_x \rtimes_{\mathcal{E}} G$.

QUESTION 7.3. Suppose that $G = N \times H$ is the product of two groups. Can we decompose the crossed product $A \rtimes_{\mathcal{E}_G} G = A \rtimes_{\mathcal{E}_G} (N \times H)$ as an iterated crossed product $(A \rtimes_{\mathcal{E}_N} N) \rtimes_{\mathcal{E}_H} H$? Does it hold for discrete groups N and H ?

A positive answer would give the first step for proving that for general closed normal subgroups $N \subseteq G$ we could write $A \rtimes_{\mathcal{E}_G} G$ as an iterated crossed product $(A \rtimes_{\mathcal{E}} N) \rtimes_{\mathcal{E}_{G/N}} G/N$, where in general the outer crossed product has to be viewed as a twisted crossed product compatible with the functor $\rtimes_{\mathcal{E}_{G/N}}$ for the quotient group G/N . Such a decomposition would give a major step for a proof that the new formulation of the Baum-Connes conjecture enjoys the same permanence properties as were shown for the classical conjecture in [10] and [11]. We refer to [8, Section 8] for a discussion of this problem.

QUESTION 7.4. Let H be a closed subgroup for G . Can we always show that the smallest exact crossed-product functor $\rtimes_{\mathcal{E}_G}$ restricts to the minimal exact crossed-product functor $\rtimes_{\mathcal{E}_H}$ for H ?

So far, we only know this if H is open in G (by Section 6) above, and for normal and cocompact subgroups H of G (by [8, Theorem 7.13]). A positive answer would imply that the validity of the new formulation of the Baum-Connes conjecture for a group G would pass to all closed subgroups of G .

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