A Step Towards Twist Conjecture

Jingyin Huang and Piotr Przytycki

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Abstract. Under the assumption that a defining graph of a Coxeter group admits only twists in $\mathbb{Z}_2$ and is of type FC, we prove Mühlherr's Twist Conjecture.

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1 Introduction

A *Coxeter generating set* S of a group W is a set such that (W,S) is a Coxeter system. This means that S generates W subject only to relations of the form $s^2=1$ for $s\in S$ and $(st)^{m_{st}}=1$, where $m_{st}=m_{ts}\geq 2$ for $s\neq t\in S$ (possibly there is no relation between s and t, and then we put by convention $m_{st}=\infty$). An S-*reflection* (or a *reflection*, if the dependence on S does not need to be emphasised) is an element of W conjugate to some element of S. We say that S is *reflection-compatible* with another Coxeter generating set S' if every S-reflection is an S'-reflection. Furthermore, S is *angle-compatible* with S' if for every $s,t\in S$ with $\langle s,t\rangle$ finite, the set $\{s,t\}$ is conjugate to some $\{s',t'\}\subset S'$. (Setting $s=t$ shows that angle-compatible implies reflection-compatible.)
Mühlherr's Twist Conjecture predicts that angle-compatible Coxeter generating sets of a Coxeter group differ by a sequence of elementary twists. We postpone the definition of an elementary twist to give a brief historical background. For an exhaustive 2006 state of affairs, see [11].
The Isomorphism Problem for Coxeter groups asks for an algorithm to determine if Coxeter systems $(W,S),(W',S')$ defined by m_{st},m'_{st} give rise to isomorphic groups W and W'. Hence listing all Coxeter generating sets S of W' solves the Isomorphism Problem. The articles of Howlett and Mühlherr [7], and Marquis and Mühlherr [9] reduce the question of listing all such sets S to the problem of listing all S angle-compatible with S'. In this way the

Twist Conjecture describes a possible solution to the Isomorphism Problem for Coxeter groups.

The first substantial work on the Twist Conjecture is the one by Charney and Davis [4], where they prove that if a group acts effectively, properly, and cocompactly on a contractible manifold, then all its Coxeter generating sets are conjugate. Caprace and Mühlherr [2] proved that for all $m_{st} < \infty$, a Coxeter generating set S angle-compatible with S' is conjugate to S'. This is what was predicted by the Twist Conjecture, since S with all $m_{st} < \infty$ does not admit any elementary twist. Building on that, Caprace and Przytycki [3] proved that an arbitrary S not admitting any elementary twist, and angle-compatible with S', is in fact conjugate to S'. This should be considered as the "base case" of the Twist Conjecture.

In a foundational article [12] Mühlherr and Weidmann verified the Twist Conjecture in the case where all $m_{st} \geq 3$. In that case there occur twists in $\mathbb{Z}_2$ as well as in dihedral groups. There is a more recent contribution of Weigel [15] which improves the result of [12]. Ratcliffe and Tschantz proved the Twist Conjecture for chordal Coxeter groups [13]. The Twist Conjecture is also known for the right-angled Coxeter groups (before the conjecture was stated), where the proof is implicit in the work of Laurence [8] and is explicit in [10]. In these papers the assumptions on m_{st} seem an artefact of the proposed proof. In our paper, we propose the following "step one" of a systematic approach towards Twist Conjecture. Our first assumption below is natural from the point of view of the statement of the conjecture, since it says that the occurring elementary twists are as simple as possible. Our second assumption is that S is of *type* FC meaning that for any $T \subseteq S$ with m_{tr} finite for all $t, r \in T$, we have that $\langle T \rangle$ is finite. This assumption seems less natural from the point of view of the conjecture statement, but plays a role already in our proof of the "base case". More precisely, [3, §3–7] resolve (implicitly) the "base case" under FC assumption, and [3, §8] is devoted to removing FC assumption.

Main Theorem. *Let S be a Coxeter generating set angle-compatible with S'. Suppose that S admits only twists in $\mathbb{Z}_2$, and is of type* FC. *Then S' is obtained from S by a sequence of elementary twists and a conjugation.*

We finally define an elementary twist. Let (W, S) be a Coxeter system. Given a subset $J \subseteq S$, we denote by W_J the subgroup of W generated by J. We call J *spherical* if W_J is finite. If J is spherical, let w_J denote the longest element of W_J. We say that two elements $s \neq t \in S$ are *adjacent* if $\{s, t\}$ is spherical. This gives rise to a graph whose vertices are S and whose edges (labelled by m_{st}) correspond to adjacent pairs of S. This graph is called the *defining graph* of S. Occasionally, when all m_{st} are finite, we will use another graph, whose vertices are still S, but (labelled) edges correspond to pairs of non-commuting elements of S. This graph is called the *Coxeter–Dynkin diagram* of S. Whenever we talk about adjacency of elements of S, we always mean adjacency in the defining graph unless otherwise specified.

Given a subset $J \subseteq S$, we denote by $J^\perp$ the set of those elements of $S \setminus J$ that

commute with J. A subset $J \subseteq S$ is *irreducible* if it is not contained in $K \cup K^{\perp}$ for some non-empty proper subset $K \subset J$.
Let $J \subseteq S$ be an irreducible spherical subset. We say that $C \subseteq S \setminus (J \cup J^{\perp})$ is a *component*, if the subgraph induced on C in the defining graph of S is a connected component of the subgraph induced on $S \setminus (J \cup J^{\perp})$. Assume that we have a nontrivial partition $S \setminus (J \cup J^{\perp}) = A \sqcup B$, where each component C is contained entirely in A or in B. In other words, for all $a \in A$ and $b \in B$, we have that a and b are non-adjacent. We then say that J *weakly separates* S. In the language of groups, this means that W splits as an amalgamated product over $W_{J \cup J^{\perp}}$. Note that A and B are in general not uniquely determined by J. We then consider the map $\tau \colon S \to W$ defined by

$$\tau(s) = \begin{cases} s & \text{for } s \in A \cup J \cup J^{\perp}, \\ w_J s w_J^{-1} & \text{for } s \in B, \end{cases}$$

which is called an *elementary twist in* $\langle J \rangle$ (see [1, Def 4.4]).
Coxeter generating sets S and S' of W are *twist equivalent* if S' can be obtained from S by a finite sequence of elementary twists and a conjugation. We say that S is k*-rigid* if for each weakly separating $J \subset S$ we have $|J| < k$. Thus 1-rigid means that there are no elementary twists (this was called *twist-rigid* in [3]). Our Main Theorem states that if a Coxeter generating set S is 2-rigid, of type FC, and angle-compatible to S', then it is twist equivalent to S'. Since twists in $\mathbb{Z}_2$ do not change the defining graph, it follows that S and S' have the same defining graphs. Note that right-angled Coxeter groups are 2-rigid.

Organisation. In Section 2 we recall some basic properties of the Davis complex and geometric sets of reflections. In Section 3 we recall the notions of bases and markings from [3]. In Section 4 we extend in two different ways a marking compatibility result from [3]. Section 5 contains a technical result required for the definition of complexity in Section 6. We prove the Main Theorem in Section 7.

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2 Preliminaries

2.1 Davis complex

Let $\mathbb{A}$ be the *Davis complex* of a Coxeter system (W, S). The 1-skeleton of $\mathbb{A}$ is the Cayley graph of (W, S) with vertex set W and a single edge spanned on $\{w, ws\}$ for each $w \in W, s \in S$. Higher dimensional cells of $\mathbb{A}$ are spanned on left cosets in W of remaining finite W_J. The left action of W on itself extends to the action on $\mathbb{A}$. Note that $\mathbb{A}$ carries a natural CAT(0) metric; however this metric will not play a role in the article.

A *chamber* is a vertex of $\mathbb{A}$. Collections of chambers corresponding to cosets wW_J are called J*-residues* of $\mathbb{A}$. A *gallery* is an edge-path in $\mathbb{A}$. For two chambers $c_1, c_2 \in \mathbb{A}$, we define their *gallery distance*, denoted by $d(c_1, c_2)$, to be the length of a shortest gallery from c_1 to c_2 (this coincides with the word-metric w.r.t. S).
Let $r \in W$ be an S-reflection. The fixed point set of the action of r on $\mathbb{A}$ is called its *wall* $\mathcal{W}_r$. The wall $\mathcal{W}_r$ determines r uniquely. Moreover, $\mathcal{W}_r$ separates $\mathbb{A}$ into two connected components, which are called *half-spaces (for r)*. If a non-empty $K \subset \mathbb{A}$ is contained in a single half-space (this happens for example if K is connected and disjoint from $\mathcal{W}_r$), then $\Phi(\mathcal{W}_r, K)$ denotes this half-space. An edge of $\mathbb{A}$ crossed by $\mathcal{W}_r$ is *dual* to $\mathcal{W}_r$. A chamber is *incident* to $\mathcal{W}_r$ if it is an endpoint of an edge dual to $\mathcal{W}_r$. The *distance* of a chamber c to $\mathcal{W}_r$, denoted by $d(c, \mathcal{W}_r)$, is the minimal gallery distance from c to a chamber incident to $\mathcal{W}_r$.
The following fact is standard, see eg. [14, Thm 2.9].

THEOREM 2.1. *Let $\mathcal{R}$ be a residue and let $x \in \mathcal{R}$ and $y \in W$ be chambers. Then there is a chamber $x' \in \mathcal{R}$ on a minimal length gallery from y to x such that $\Phi(\mathcal{W}_r, y) = \Phi(\mathcal{W}_r, x')$ for any reflection r fixing $\mathcal{R}$.*

2.2 GEOMETRIC SET OF REFLECTIONS

Let (W, S) be a Coxeter system. Let $\mathbb{A}_{\mathrm{ref}}$ be the Davis complex for (W, S) ("ref" stands for "reference complex"). For each reflection r, let $\mathcal{Y}_r$ be its wall in $\mathbb{A}_{\mathrm{ref}}$. Note that this notation differs from the one in Subsection 2.1.
Suppose that S is ANGLE-COMPATIBLE with another Coxeter generating set S'. Let $\mathbb{A}_{\mathrm{amb}}$ be the Davis complex for (W, S') ("amb" stands for "ambient complex"). For each reflection r, let $\mathcal{W}_r$ be its wall in $\mathbb{A}_{\mathrm{amb}}$. Let $P \subseteq S$.

DEFINITION 2.2. Let $\{\Phi_p\}_{p\in P}$ be a collection of half-spaces of $\mathbb{A}_{\mathrm{amb}}$ for $p \in P$. The collection $\{\Phi_p\}_{p\in P}$ is 2*-geometric* if for any pair $p, r \in P$, the set $\Phi_p \cap \Phi_r \cap \mathbb{A}^{(0)}_{\mathrm{amb}}$ is a fundamental domain for the action of $\langle p, r\rangle$ on $\mathbb{A}^{(0)}_{\mathrm{amb}}$.
The collection $\{\Phi_p\}_{p\in P}$ is *geometric* if additionally $F = \bigcap_{p\in P} \Phi_p \cap \mathbb{A}^{(0)}_{\mathrm{amb}}$ is non-empty.
The set P is 2*-geometric* if there exists a 2-geometric collection of half-spaces $\{\Phi_p\}_{p\in P}$.

THEOREM 2.3 ([2, Thm 4.2]). *If $\{\Phi_p\}_{p\in P}$ is 2-geometric, then after possibly replacing each Φ_p by opposite half-space, the collection $\{\Phi_p\}_{p\in P}$ is geometric.*

Theorem 2.3 justifies calling 2-geometric P *geometric* for simplicity. In fact, by [5] (see also [6, Thm 1.2] and [2, Fact 1.6]), we have:

PROPOSITION 2.4. *If P is geometric, then F is a fundamental domain for the action of $\langle P\rangle$ on $\mathbb{A}^{(0)}_{\mathrm{amb}}$, and for each $p \in P$ there is a chamber in F incident to $\mathcal{W}_p$. In particular, if $P = S$, then S is conjugate to S'.*

Note that since S is angle-compatible with S', every 2-element subset of S is geometric. However, this does not mean that S is 2-geometric. Nevertheless, for S spherical, it is easy to inductively choose 2-geometric Φ_s, and by Theorem 2.3 and Proposition 2.4 we obtain the following.

Lemma 2.5. *If S is spherical, then it is conjugate to S'.*

Corollary 2.6. *Let $J \subset S$ be spherical. Then J is conjugate to a spherical $J' \subset S'$. In particular, J is geometric, and if it is irreducible, there exist exactly 2 fundamental domains F for the action of $\langle J \rangle$ on $\mathbb{A}^{(0)}_{\mathrm{amb}}$ as in Proposition 2.4.*

Proof. Let $P \subset S$ be maximal spherical containing J. Then $\langle P \rangle$ is a maximal finite subgroup of W. By [1, Thm 1.9], we have that $\langle P \rangle$ is conjugate to $\langle P' \rangle$ for a maximal spherical $P' \subset S'$. Thus we can assume without loss of generality that $P = S$ and $P' = S'$. It now suffices to apply Lemma 2.5. □

2.3 Decomposition lemma

For $J \subseteq S$ let J^∞ be the set of those elements of $S \setminus J$ that are not adjacent to any element of J.

Lemma 2.7. *Let S be 2-rigid. Let $s, t \in S$ be adjacent and non-commuting, and let $r \in S$ be neither adjacent to s nor to t. Suppose that t and r are in distinct components of $S \setminus (s \cup s^\perp)$, and that s and r are in distinct components of $S \setminus (t \cup t^\perp)$. Let $J = \{s, t\}$. Then $S = J \cup J^\perp \cup J^\infty$.*

Proof. Suppose by contradiction that the collection of vertices of $S \setminus (J \cup J^\perp)$ that are adjacent to s or to t is non-empty. Since S is 2-rigid, there is a shortest edge-path γ in the subgraph induced on $S \setminus (J \cup J^\perp)$ that connects r to a vertex $p \in S \setminus (J \cup J^\perp)$ adjacent to s or t. We assume without loss of generality that p is adjacent to t. Since r and t are in distinct components of $S \setminus (s \cup s^\perp)$, there is a vertex p' of γ in $s^\perp$. If $p \neq p'$, then the subpath $\gamma' \subseteq \gamma$ from r to p' is a shorter path from r to a vertex adjacent to s or t, which is a contradiction. If $p = p'$, then since r and s are in distinct components of $S \setminus (t \cup t^\perp)$, there exists a vertex p'' of $\gamma' = \gamma$ in $t^\perp$. If $p'' \neq p$, then we can reach a contradiction as before. If $p'' = p$, then $p \in J^\perp$, which is impossible by our choice of γ. □

3 Bases and markings

Henceforth, in the entire article we assume that S is irreducible, non-spherical, and of type FC. (The reducible case easily follows from the irreducible.)

In this section we recall, in simplified form, several central notions from [3]. Let $W, S, \mathbb{A}_{\mathrm{ref}}, \mathcal{Y}_r$ (and later $S', \mathbb{A}_{\mathrm{amb}}, \mathcal{W}_r$) be as in Section 2.2. The aim is to introduce several natural choices for half-spaces for $s \in S$ in $\mathbb{A}_{\mathrm{amb}}$, which will be done in Definition 3.6. Let c_0 be the identity chamber of $\mathbb{A}_{\mathrm{ref}}$.

3.1 Bases

Definition 3.1. A *base* is a pair (s, w) with *core* $s \in S$ and $w \in W$ satisfying

(i) $w = j_1 \cdots j_n$ where j_i are pairwise distinct elements from $S \setminus \{s\}$,

(ii) $d(w.c_0, \mathcal{Y}_s) = n$,

(iii) the *support* $J = \{s, j_1, \ldots, j_n\}$ is spherical.

Note that in the language of [3, Def 3.1 and 3.6] our *base* would be called a *simple base with spherical support*. Indeed, Condition (ii) from [3, Def 3.1] saying that every wall that separates $w.c_0$ from c_0 intersects $\mathcal{Y}_s$ follows immediately from our Condition (iii); simplicity from [3, Def 3.6] is our Condition (i). Note that our Condition (ii) implies that J is irreducible.
In [3, Lem 3.7] and the paragraph preceding it, we established the following.

Remark 3.2. (i) If $J \subset S$ is irreducible spherical and $s \in J$, then there exists a base with support J and core s. Namely, it suffices to order the elements of $J \setminus \{s\}$ into a sequence (j_i) so that for every $1 \leq i \leq n$ the set $\{s, j_1, \ldots, j_i\}$ is irreducible. Then $(s, j_1 \ldots j_n)$ is a base.

(ii) The core s and support J determine the base (s, w) uniquely. Hence we sometimes write a base as (s, J), or even just J if the core is fixed. When $J = \{s\}$, we often write s instead of $\{s\}$ for simplicity.

Lemma 3.3. *Let $J \subset S$ be irreducible spherical, and let F be a fundamental domain for $\langle J \rangle$ in $\mathbb{A}^{(0)}_{\mathrm{amb}}$ guaranteed by Corollary 2.6. Let $s \in J$ and define $w \in W$ via $(s, w) = (s, J)$. Then we have $\Phi(\mathcal{W}_s, F) = \Phi(\mathcal{W}_s, w.F)$.*

Proof. First suppose $S = S'$. If $c_0 \in F$, then by Definition 3.1(ii) we have $\Phi(\mathcal{W}_s, c_0) = \Phi(\mathcal{W}_s, w.c_0)$, as desired. Otherwise, we have $w_J.c_0 \in F$. The half-spaces $\Phi(\mathcal{W}_s, w_J.c_0)$ and $\Phi(\mathcal{W}_s, ww_J.c_0)$ are opposite to $\Phi(\mathcal{W}_s, c_0)$ and $\Phi(\mathcal{W}_s, w.c_0)$, so they coincide as well.
If $S \neq S'$, then by Corollary 2.6 we have $gJg^{-1} = J'$, where J' is a spherical subset of S'. Then (gsg^{-1}, gwg^{-1}) is a base for S', and by the previous paragraph we have $\Phi(\mathcal{W}_{gsg^{-1}}, g.F) = \Phi(\mathcal{W}_{gsg^{-1}}, gw.F)$. Translating by g^{-1} we obtain the statement in the lemma. □

3.2 Markings

Definition 3.4. A *marking* is a pair $\mu = ((s, J), m)$, where (s, J) is a base and where the *marker* $m \in S$ is not adjacent to some element of J. The *core* and the *support* of the marking μ are the core and the support of its base.

Our marking satisfies (but is not equivalent to the marking defined by) [3, Def 3.8]. To see that, note that by [3, Rem 3.12], we have that $w\mathcal{Y}_m$ is disjoint from $\mathcal{Y}_s$.

Remark 3.5. Let (s, J) be a base and $m \in S \setminus (J \cup J^\perp)$. If $J \cup \{m\}$ is not spherical, then since S is of type FC, the pair $((s, J), m)$ is a marking. In particular, since S is irreducible non-spherical, we have that for each $s \in S$ there exists a marking with core s, since we can start with $J \subset S$ maximal irreducible spherical containing s. Similarly, for each $s \in I \subset S$ with I irreducible spherical, there exists a marking with core s and support containing I.

The following picks up the geometry of the walls $\mathcal{W}_s$ for $s \in S$ inside the ambient complex for S'.

Definition 3.6. Let $\mu = ((s, w), m)$ be a marking. Since $w\mathcal{Y}_m$ is disjoint from $\mathcal{Y}_s$, the element $wmw^{-1}s$ is of infinite order, and hence also $w\mathcal{W}_m$ is disjoint from $\mathcal{W}_s$. We define $\Phi^\mu_s = \Phi(\mathcal{W}_s, w\mathcal{W}_m)$, which is the half-space for s in $\mathbb{A}_{\mathrm{amb}}$ containing $w\mathcal{W}_m$.

The next result is essentially [3, Prop 5.2]. Except for Lemma 2.5 this is the only place where we use angle-compatibility (instead of reflection-compatibility). Note that our markings are particular markings of [3], but the proof of [3, Prop 5.2] only uses such markings if S is of type FC.

Proposition 3.7. *Suppose that $P \subseteq S$ is irreducible and non-spherical. Let $p_1, p_2 \in P$. Suppose that for each $i = 1, 2$, any marking μ with core p_i and support and marker in P gives the same $\Phi_{p_i} = \Phi^\mu_{p_i}$. Then the pair $\{\Phi_{p_1}, \Phi_{p_2}\}$ is geometric.*

We summarise Proposition 3.7, Theorem 2.3, and Proposition 2.4 in the following.

Corollary 3.8. *If for each $s \in S$ any marking μ with core s gives rise to the same Φ^μ_s, then S is conjugate to S'.*

Also note that since S is of type FC, by [3, Lem 4.2 and Thm 4.5] a 1-rigid subset $P \subseteq S$ satisfies the hypothesis of Proposition 3.7.

Corollary 3.9. *If $P \subseteq S$ is 1-rigid, then it is geometric.*

4 Compatibility of markings

Let $S, S', W, \mathbb{A}_{\mathrm{ref}}$ and $\mathbb{A}_{\mathrm{amb}}$ be as in Section 3. The following trivially coincides with [3, Def 4.1].

Definition 4.1. Let $((s, J), m), ((s, J'), m')$ be markings with common core. We say that they are related by *move*

(M1) if $J = J'$, and the markers m and m' are adjacent;

(M2) if there is $j \in S$ such that $J = J' \cup \{j\}$ and moreover m equals m' and is adjacent to j.

We will write $((s, J), m) \sim ((s, J'), m')$ if there is a finite sequence of moves of type M1 or M2 that brings $((s, J), m)$ to $((s, J'), m')$.

The following is a special case of [3, Lem 4.2].

LEMMA 4.2. *If markings μ and μ' with common core s are related by move* M1 *or* M2*, then $\Phi_s^\mu = \Phi_s^{\mu'}$.*

The goal of this section is to provide two generalisations of [3, Thm 4.5].

PROPOSITION 4.3. *Let $I \subset S$ be irreducible spherical. Suppose that no irreducible spherical $I' \supsetneq I$ weakly separates S. Let $\mu_1 = (J_1, m_1)$ and $\mu_2 = (J_2, m_2)$ be markings with common core $s \in I$ and such that $I \subseteq J_1, J_2$. Moreover, for $i = 1, 2$, define $K_i = J_i \setminus (I \cup I^\perp)$ when $I \subsetneq J_i$, and $K_i = \{m_i\}$ when $J_i = I$. Suppose that K_1 and K_2 are in the same component C of $S \setminus (I \cup I^\perp)$. Then $\mu_1 \sim \mu_2$. Consequently $\Phi_s^{\mu_1} = \Phi_s^{\mu_2}$.*

Henceforth we will frequently use the FC assumption to say that $J \subset S$ is spherical if and only if it induces a clique in the defining graph. We will not mention this each time explicitly to be able to focus on the main line of reasoning.

Proof. We follow the proof of Wojtaszczyk [3, App C], and argue by contradiction. Let I be maximal irreducible spherical satisfying the hypothesis of the proposition but with $\mu_1 \not\sim \mu_2$.
The *I-distance* between μ_1 and μ_2 is the length of a shortest edge-path in (the subgraph induced on) C between a vertex of K_1 and a vertex of K_2. (Such a path exists by our hypotheses.) Among pairs μ_1, μ_2 as above choose a pair with minimal I-distance.
If the I-distance between μ_1 and μ_2 is 0, then first consider the case where one of J_i, say J_1, equals I. If also $J_2 = I$, then $\{m_1\} = K_1 = K_2 = \{m_2\}$ yielding $\mu_1 = \mu_2$, which is a contradiction. If $I \subsetneq J_2$, then $\{m_1\} = K_1 \subseteq K_2 \subset J_2$ and hence $J_1 \cup \{m_1\} \subset J_2$ is spherical, contradiction. It remains to consider the case where $I \subsetneq J_1, J_2$. Then $J_1 \cap J_2 \setminus (I \cup I^\perp) \neq \emptyset$ giving a contradiction with the maximality of I.
Now assume that the I-distance between μ_1 and μ_2 is 1. First consider the case where one of J_i, say J_1, equals I. If also $J_2 = I$, then m_1 and m_2 are adjacent. Thus μ_1 and μ_2 are related by move M1, which is a contradiction. If $I \subsetneq J_2$, then there exists $k_2 \in J_2 \setminus (I \cup I^\perp)$ such that k_2 and m_1 are adjacent. Thus μ_1 is related to $(I \cup \{k_2\}, m_1)$ by move M2. However, $(I \cup \{k_2\}, m_1) \sim \mu_2$ by the maximality of I, which is a contradiction. It remains to consider the second case where $I \subsetneq J_1, J_2$. Then there exist $k_i \in J_i \setminus (I \cup I^\perp)$ such that k_1 and k_2 are adjacent. Note that $I \cup \{k_1, k_2\}$ is spherical and irreducible. By Remark 3.5, there exists a marking ν with core s and support containing $I \cup \{k_1, k_2\}$. By the maximality of I, we have $\mu_1 \sim \nu \sim \mu_2$, which is a contradiction.
If the I-distance between μ_1 and μ_2 is ≥ 2, let γ be a shortest edge-path in C connecting a vertex $k_1 \in K_1$ to a vertex $k_2 \in K_2$. Let k be the vertex

on γ following k_1. If $I \cup \{k\}$ is spherical, then again by Remark 3.5, there exists a marking ν with core s and support containing $I \cup \{k\}$. Since we chose μ_1 and μ_2 to have minimal I-distance, we obtain $\mu_1 \sim \nu \sim \mu_2$, which is a contradiction. If $I \cup \{k\}$ is not spherical, then (I, k) is a marking, hence analogously $\mu_1 \sim (I, k) \sim \mu_2$, which is a contradiction. □

The following more technical proposition is used only in Case 4 of the proof of Lemma 5.3 and we recommend to skip it at a first reading.

Definition 4.4. Let $P \subseteq S$ be irreducible non-spherical. We say that P is 1-*rigid in* S if for any irreducible spherical $L \subset S$ with $L \cap P \neq \emptyset$, all elements of $P \setminus (L \cup L^{\perp})$ are in one component of $S \setminus (L \cup L^{\perp})$.

Proposition 4.5. *Let $P \subseteq S$ be 1-rigid in S. Then for any markings μ_1 and μ_2 with supports and markers in P and common core p, we have $\mu_1 \sim \mu_2$. Consequently $\Phi_p^{\mu_1} = \Phi_p^{\mu_2}$ and by Proposition 3.7, P is geometric.*

In the proof we need the following terminology.

Definition 4.6. Let P be 1-rigid in S. Note that $P \setminus (L \cup L^{\perp}) \neq \emptyset$ for any irreducible spherical $p \in L \subset S$. A marking $\mu = ((p, J), m)$ is (p, P)-*admissible* (or shortly *admissible* if p and P are fixed) if

1. $p \in P$, and
2. if $L \subset S$ is irreducible spherical such that $p \in L$ and $J \nsubseteq L$, then $J \setminus (L \cup L^{\perp})$ (which is non-empty) and $P \setminus (L \cup L^{\perp})$ are in the same component of $S \setminus (L \cup L^{\perp})$, and
3. if $L \subset S$ is irreducible spherical such that $J \subseteq L$, then m and $P \setminus (L \cup L^{\perp})$ are in the same component of $S \setminus (L \cup L^{\perp})$.

A base (p, J) is (p, P)-*admissible* (or *admissible*) if it satisfies Conditions (1) and (2).

Note that markings with core p and supports and markers in P, such as μ_1, μ_2 in Proposition 4.5, are (p, P)-admissible, but not vice-versa. The class of (p, P)-admissible markings is a crucial ingredient in the proof of Proposition 4.5. In the remaining part of the section we fix P 1-rigid in S, and we fix $p \in P$.

Lemma 4.7. *Suppose that (p, J) is admissible. Let $\nu = ((p, J'), m)$ be a marking such that $J \subseteq J'$, $J' \setminus J \subset P$ and $m \in P$. Then ν is admissible.*

Note that such ν exists for each J. Namely, one can take $J' \supseteq J$ to be maximal irreducible spherical with $J' \setminus J \subset P$. Then take m inside $P \setminus (J' \cup J'^{\perp})$, which is non-empty since P is irreducible non-spherical.

Proof. Condition (1) is immediate. For Condition (2), let $L \subset S$ be irreducible spherical and such that $p \in L$ and $J' \nsubseteq L$. If $J \nsubseteq L$, then $\emptyset \neq J \setminus (L \cup L^{\perp}) \subseteq J' \setminus (L \cup L^{\perp})$. Since (p, J) is admissible, Condition (2) holds for such L and J'. If $J \subseteq L$, then $J' \setminus (L \cup L^{\perp}) \subseteq J' \setminus J \subset P$, hence Condition (2) holds for such L and J'. Condition (3) is immediate, since we have $m \in P$. □

Proof of Proposition 4.5. Note that both μ_1 and μ_2 are admissible. Hence to prove the proposition it suffices to show that for any two admissible markings μ_1, μ_2 with common core p, we have $\mu_1 \sim \mu_2$.
We argue by contradiction. Let $I \ni p$ be maximal irreducible spherical such that there are admissible markings $\mu_1 = (J_1, m_1)$ and $\mu_2 = (J_2, m_2)$ with $I \subseteq J_1, J_2$, and $\mu_1 \not\sim \mu_2$. We define K_1, K_2, and the I-distance between μ_1 and μ_2 as in the proof of Proposition 4.3. Since both μ_1 and μ_2 are admissible, their I-distance is finite (set $L = I$ in the definition of admissible marking). Among pairs μ_1, μ_2 as above choose a pair with minimal I-distance.
If the I-distance is 0, then either $\mu_1 = \mu_2$, or one of $J_i \cup \{m_i\}$ is spherical, or there is irreducible $I' \supsetneq I$ contained in both J_1 and J_2, contradiction. Suppose now that the I-distance is 1. There are two cases to consider.

Case 1: $J_1 = I$. If $J_2 = I$, then μ_1 and μ_2 are related by move M1, contradiction. Now we assume $I \subsetneq J_2$. Pick $k_2 \in K_2$ adjacent to m_1. Then $I' = I \cup \{k_2\}$ is spherical and irreducible. Moreover, $\mu_1 \sim (I', m_1)$ by move M2. We claim that (I', m_1) is admissible. Then $(I', m_1) \sim \mu_2$ by the maximality of I, which yields a contradiction. Now we prove the claim. For Condition (2), let $p \in L$ and $I' \not\subseteq L$. If $I \not\subseteq L$, it suffices to use Condition (2) in the admissibility of μ_1. Now suppose $I \subseteq L$. Then $I' \setminus (L \cup L^\perp) = \{k_2\}$. By Condition (2) in the admissibility of μ_2, we have that k_2 is in the same component of $S \setminus (L \cup L^\perp)$ as $P \setminus (L \cup L^\perp)$, as desired. Condition (3) follows immediately from Condition (3) in the admissibility of μ_1.

Case 2: $I \subsetneq J_1$ and $I \subsetneq J_2$. For $i = 1, 2$, pick $k_i \in K_i$ such that k_1 and k_2 are adjacent. Then $J = I \cup \{k_1, k_2\}$ is spherical and irreducible. It is easy to show that J is admissible following the argument from Case 1. Let ν be an admissible marking constructed from J as in Lemma 4.7. Then $\mu_1 \sim \nu \sim \mu_2$ by the maximality of I, which yields a contradiction.

Finally suppose that the I-distance d between μ_1 and μ_2 is ≥ 2. Let γ be a shortest edge-path in the subgraph induced on $S \setminus (I \cup I^\perp)$ starting at $k_1 \in K_1$ and ending at $k_2 \in K_2$. Let k be the vertex on γ following k_1. If $J = I \cup \{k\}$ is not spherical, then let $\nu = (I, k)$, otherwise let ν be defined from J as in Lemma 4.7. Since the I-distance between ν and μ_1, μ_2 is $< d$, to reach a contradiction it suffices to prove that ν is admissible.
Consider first the case where J is spherical. By Lemma 4.7, it suffices to prove that J is admissible. Let $p \in L$ and $J \not\subseteq L$. If $I \not\subseteq L$, then we use the admissibility of μ_1. Otherwise, we have $J \setminus (L \cup L^\perp) = \{k\}$. Since γ is a geodesic, $\gamma \cap L$ is empty, a vertex, or an edge. Moreover, $\gamma \cap L^\perp = \emptyset$, since $\gamma \cap I^\perp = \emptyset$ and $I \subseteq L$. Thus there is a subpath of γ from k to k_1 or k_2 outside $L \cup L^\perp$. Since μ_1, μ_2 were admissible, k is in the component of $S \setminus (L \cup L^\perp)$ containing $P \setminus (L \cup L^\perp)$, as desired.
If J is not spherical, then $\nu = (I, k)$. Condition (2) for ν follows from the admissibility of μ_1. For Condition (3), let $I \subseteq L$. As before, there is a subpath of γ from k to k_1 or k_2 outside $L \cup L^\perp$. Thus, again, since μ_1, μ_2 were admissible, k is in the component of $S \setminus (L \cup L^\perp)$ containing $P \setminus (L \cup L^\perp)$. □

5 GOOD VERTICES

In this section we introduce the notion of a good vertex t in an irreducible spherical $J \subset S$ w.r.t. $r \in S \setminus J$, and the related fundamental domain $E_{t,r}$ for the action of J on $\mathbb{A}^{(0)}_{\mathrm{amb}}$. Then in Proposition 5.2 we prove that $E_{t,r}$ does not depend on t. This will be crucial for the definition of the complexity of S w.r.t. S' in Section 6. Let $S, S', W, \mathbb{A}_{\mathrm{ref}}$ and $\mathbb{A}_{\mathrm{amb}}$ be as in Section 3. THROUGHOUT THE REMAINING PART OF THE ARTICLE, WE WILL ALSO ASSUME THAT S IS 2-RIGID.

DEFINITION 5.1. Let $J \subset S$ be irreducible spherical and $r \in S \setminus J$. A vertex $t \in J$ is *good with respect to* r, if t is adjacent to r, or $J \setminus (t \cup t^{\perp})$ is non-empty and in the same component of $S \setminus (t \cup t^{\perp})$ as r. Note that being good depends on J.
Let $I \subset S$ be spherical. J is *good* with respect to I if there exist non-adjacent $t \in J$ and $r \in I$ such that t is good with respect to r. Then let $E_{t,r}$ be that fundamental domain from Corollary 2.6 for the action of J on $\mathbb{A}^{(0)}_{\mathrm{amb}}$ that is contained in $\Phi(\mathcal{W}_t, \mathcal{W}_r)$.

PROPOSITION 5.2. *Let $J \subset S$ be irreducible spherical and $I \subset S$ be spherical. Suppose that we have pairs of non-adjacent vertices (t, r) and (t', r') in $J \times I$ such that t is good with respect to r, and t' is good with respect to r'. Then $E_{t,r} = E_{t',r'}$.*

The proof of Proposition 5.2 is the most technical part of the article, and we recommend to skip it at first reading. We need a preparatory lemma.

LEMMA 5.3. *Let $t, r \in S$ be non-adjacent. Let $J \subset S$ be irreducible spherical containing t. Let $j_0 \in J$ and let $\omega = (j_0, j_1, \dots)$ be the geodesic edge-path in the Coxeter–Dynkin diagram of J that starts at j_0 and ends at t (such a geodesic is unique since the Coxeter–Dynkin diagram of a spherical subset is a tree). Let j_n be the first vertex of ω not adjacent to r (possibly $j_n = j_0$ or $j_n = t$). Suppose that both $t, j_0 \in J$ are good w.r.t. r. Then we have $\Phi(j_n j_{n-1} \cdots j_1 \mathcal{W}_{j_0}, E_{t,r}) = \Phi(j_n j_{n-1} \cdots j_1 \mathcal{W}_{j_0}, \mathcal{W}_r)$.*

Proof. We write $E = E_{t,r}$ to shorten the notation.
We claim that for any non-commuting $j, j' \in J$ at least one of j, j' is good (w.r.t. r; we will skip repeating this in this proof). To justify the claim, if both j and j' are not good, then r and j are in distinct components of $S \setminus (j' \cup j'^{\perp})$, and r and j' are in distinct components of $S \setminus (j \cup j^{\perp})$. If $\{j, j'\} \subsetneq J$, then there is an element in $S \setminus (\{j, j'\} \cup \{j, j'\}^{\perp})$ adjacent to j and j', which contradicts Lemma 2.7. If $\{j, j'\} = J$, then one of j, j' equals t, which was assumed to be good, contradiction. This justifies the claim.
If $j_0 = t$, then there is nothing to prove. Otherwise, we induct on the length of ω and assume that the conclusion of the lemma holds for all good j_i distinct from j_0. By the claim either j_1 or j_2 is good. We look first at the situation where j_1 is good. There are four cases to consider.

Case 1: both j_0 and j_1 are not adjacent to r. Since j_1 is good, j_0 it is in the same component of $S \setminus (j_1 \cup j_1^\perp)$ as r. Thus by Proposition 4.3 applied with $I = \{j_1\}$ and the assumption that S is 2-rigid we have $(j_1, r) \sim ((j_1, j_0), r)$. Analogously $(j_0, r) \sim ((j_0, j_1), r)$. Let $\Sigma \subset \mathbb{A}_{\mathrm{amb}}$ be the union of the two sectors of the form $\Phi_{j_0} \cap \Phi_{j_1}$ for $\{\Phi_{j_0}, \Phi_{j_1}\}$ geometric. Denoting $\Phi_{j_1} = \Phi(\mathcal{W}_{j_1}, \mathcal{W}_r)$, from $(j_1, r) \sim ((j_1, j_0), r)$ we obtain $\Phi_{j_1} = \Phi(\mathcal{W}_{j_1}, j_0\mathcal{W}_r)$, and so $j_0\Phi_{j_1} \supset \mathcal{W}_r$. Consequently $\mathcal{W}_r \subset \Phi_{j_1} \cap j_0\Phi_{j_1} \subset \Sigma \cup j_0\Sigma$. Analogously $(j_0, r) \sim ((j_0, j_1), r)$ implies $\mathcal{W}_r \subset \Sigma \cup j_1\Sigma$, and so $\mathcal{W}_r \subset \Sigma$. By induction assumption, $\Phi(\mathcal{W}_{j_1}, E) = \Phi(\mathcal{W}_{j_1}, \mathcal{W}_r)$, thus E and $\mathcal{W}_r$ are in the same sector of Σ, and it follows that $\Phi(\mathcal{W}_{j_0}, E) = \Phi(\mathcal{W}_{j_0}, \mathcal{W}_r)$.

Case 2: j_1 is adjacent to r, but j_0 is not adjacent to r. Then $n = 0$. Let j_m be the first vertex of ω distinct from j_0 not adjacent to r.

First, we claim $\Phi(j_0\mathcal{W}_{j_1}, E) = \Phi(j_0\mathcal{W}_{j_1}, \mathcal{W}_r)$. Indeed, since (j_1, j_0) and $(j_1, j_2 \cdots j_m)$ are bases, by two applications of Lemma 3.3 we have

$$\Phi(\mathcal{W}_{j_1}, j_0.E) = \Phi(\mathcal{W}_{j_1}, E) = \Phi(\mathcal{W}_{j_1}, j_2 \cdots j_m.E),$$

which equals $\Phi(\mathcal{W}_{j_1}, j_2 \cdots j_m\mathcal{W}_r)$ by induction. Furthermore, $((j_1, j_2 \cdots j_m), r)$ is a marking and j_2 is adjacent to j_0. Thus by Proposition 4.3 and the fact that S is 2-rigid, we obtain

$$((j_1, j_2 \cdots j_m), r) \sim ((j_1, j_0), r),$$

and the claim follows.

Let Φ_{j_0}, Φ_{j_1} be the half-spaces for j_0, j_1 containing E and let $\Lambda = \Phi_{j_0} \cap \Phi_{j_1}$. Since $\mathcal{W}_r$ intersects $\mathcal{W}_{j_1}$, by the claim we have that $\mathcal{W}_r$ intersects Λ. It follows that $\Phi(\mathcal{W}_{j_0}, E) = \Phi(\mathcal{W}_{j_0}, \mathcal{W}_r)$.

Case 3: j_0 is adjacent to r, but j_1 is not adjacent to r. By induction, we have $\Phi(\mathcal{W}_{j_1}, E) = \Phi(\mathcal{W}_{j_1}, \mathcal{W}_r)$. We need to show $\Phi(j_1\mathcal{W}_{j_0}, E) = \Phi(j_1\mathcal{W}_{j_0}, \mathcal{W}_r)$. To do this, it suffices to reverse the argument in the previous paragraph.

Case 4: both j_0 and j_1 are adjacent to r. Let $P = \{j_0, j_1, \ldots, j_n, r\}$. We claim that P is geometric. Indeed, by Proposition 4.5, to justify the claim it suffices to prove that P is 1-rigid in S. We have that P is irreducible and non-spherical. Now let $L \subset S$ be irreducible spherical with $L \cap P \neq \emptyset$. Since S is 2-rigid, it suffices to consider $L = \{l\}$ a singleton in P. Note that in P the only two non-adjacent elements are r and j_n. Thus the cases $l = r, j_n$ are clear. It remains to consider the case $l \in K = P \setminus \{r, j_n\}$. Since K is irreducible and $|K| \geq 2$, we have $K \setminus (l \cup l^\perp) \neq \emptyset$. Consequently, $\{l\}$ does not weakly separate P, verifying the claim.

By Theorem 2.3 and Proposition 2.4, there are half-spaces $\{\Phi_{j_0}, \Phi_{j_1}, \cdots, \Phi_{j_n}, \Phi_r\}$ whose intersection contains a vertex x incident to $\mathcal{W}_r$. Thus by induction we have

$$\Phi(j_n j_{n-1} \cdots j_2\mathcal{W}_{j_1}, E) = \Phi(j_n j_{n-1} \cdots j_2\mathcal{W}_{j_1}, \mathcal{W}_r) = \Phi(j_n j_{n-1} \cdots j_2\mathcal{W}_{j_1}, x).$$

Let F and F_{ant} (resp. V and V_{ant}) be the two fundamental domains for $\{j_0, j_1, \ldots, j_n\}$ (resp. $\{j_1, \ldots, j_n\}$) from Corollary 2.6. Assume without loss of generality $F \subset V$. Then x and E are both inside F or F_{ant}, say F, otherwise they would be separated by $j_n j_{n-1} \cdots j_2 \mathcal{W}_{j_1}$. It follows that both x and E are in V. In particular, $\Phi(j_n j_{n-1} \cdots j_1 \mathcal{W}_{j_0}, E) = \Phi(j_n j_{n-1} \cdots j_1 \mathcal{W}_{j_0}, x)$, which equals $\Phi(j_n j_{n-1} \cdots j_1 \mathcal{W}_{j_0}, \mathcal{W}_r)$, as desired.
Now we turn to the situation where j_1 is not good, hence j_2 is good. Since j_1 is not good, it is not adjacent to r, and furthermore r is not adjacent to j_0, nor to j_2. Since j_2 is good and S is 2-rigid, by Proposition 4.3 we obtain $\Phi(\mathcal{W}_{j_2}, \mathcal{W}_r) = \Phi(\mathcal{W}_{j_2}, j_1 \mathcal{W}_r)$. By induction, we have $\Phi(\mathcal{W}_{j_2}, \mathcal{W}_r) = \Phi(\mathcal{W}_{j_2}, E)$. Thus Lemma 5.4 below gives $\Phi(\mathcal{W}_{j_1}, j_2 \mathcal{W}_r) = \Phi(\mathcal{W}_{j_1}, E)$. Since S is 2-rigid, by Proposition 4.3 we have $\Phi(\mathcal{W}_{j_1}, j_2 \mathcal{W}_r) = \Phi(\mathcal{W}_{j_1}, j_0 j_2 \mathcal{W}_r) = \Phi(\mathcal{W}_{j_1}, j_0 \mathcal{W}_r)$, and finally $\Phi(\mathcal{W}_{j_0}, \mathcal{W}_r) = \Phi(\mathcal{W}_{j_0}, j_1 \mathcal{W}_r)$, since j_0 is good. Applying Lemma 5.4 with j_0 in place of j_2 we obtain $\Phi(\mathcal{W}_{j_0}, \mathcal{W}_r) = \Phi(\mathcal{W}_{j_0}, E)$, as desired.

□

Lemma 5.4. *Let $j_1, j_2 \in S$ be adjacent and non-commuting. Suppose that $r \in S$ is not adjacent to j_2 and $\Phi(\mathcal{W}_{j_2}, \mathcal{W}_r) = \Phi(\mathcal{W}_{j_2}, j_1 \mathcal{W}_r)$. Let F be a fundamental domain for $\langle j_1, j_2 \rangle$ in $\mathbb{A}^{(0)}_{\mathrm{amb}}$ from Corollary 2.6. Then we have $\Phi(\mathcal{W}_{j_2}, \mathcal{W}_r) = \Phi(\mathcal{W}_{j_2}, F)$ if and only if $\Phi(\mathcal{W}_{j_1}, j_2 \mathcal{W}_r) = \Phi(\mathcal{W}_{j_1}, F)$.*

Proof. Denote $\Phi_{j_2} = \Phi(\mathcal{W}_{j_2}, \mathcal{W}_r)$, and choose Φ_{j_1} so that Φ_{j_2} and Φ_{j_1} are geometric. Let $\Lambda = \Phi_{j_2} \cap \Phi_{j_1}$. Since $\Phi(\mathcal{W}_{j_2}, \mathcal{W}_r) = \Phi(\mathcal{W}_{j_2}, j_1 \mathcal{W}_r)$, as in Case 1 of the proof of Lemma 5.3 we obtain $\mathcal{W}_r \subset \Lambda \cup j_1 \Lambda$. Note that $\Lambda \cup j_1 \Lambda$ is contained entirely in one of the half-spaces for $\mathcal{W}_{j_2}$, and in one of the half-spaces for $j_2 \mathcal{W}_{j_1}$. Thus $\Phi(\mathcal{W}_{j_2}, \mathcal{W}_r) = \Phi(\mathcal{W}_{j_2}, F)$ if and only if $F \subset \Lambda$ if and only if $\Phi(j_2 \mathcal{W}_{j_1}, \mathcal{W}_r) = \Phi(j_2 \mathcal{W}_{j_1}, F)$. By Lemma 3.3 the latter is equivalent to $\Phi(\mathcal{W}_{j_1}, j_2 \mathcal{W}_r) = \Phi(\mathcal{W}_{j_1}, F)$. □

We are finally ready for the following.

Proof of Proposition 5.2. We prove the proposition by induction on the distance between t and t' in the Coxeter–Dynkin diagram of J. If $t = t'$, then since $\mathcal{W}_r \cap \mathcal{W}_{r'} \neq \emptyset$, the proposition is clear. If $r = r'$, then we apply Lemma 5.3 with $j_0 = t'$, where $n = 0$. By Lemma 5.3, we have $\Phi(\mathcal{W}_{t'}, E_{t,r}) = \Phi(\mathcal{W}_{t'}, \mathcal{W}_r)$ and thus $E_{t',r} = E_{t,r}$, as desired.
Now we assume $t \neq t'$ and $r \neq r'$. If t and r' are non-adjacent, then t is good with respect to r' (since r and r' are adjacent). Thus we can pass from (t, r) to (t', r') via (t, r') by the previous discussion. The case where t' and r are non-adjacent is analogous. Thus it remains to consider the case where t and r' are adjacent, and t' and r are adjacent.
We first look at the case where t and t' do not commute. We consider $P = \{t, t', r, r'\}$. Note that the defining graph of P is a square, thus P is 1-rigid. Hence P is geometric by Corollary 3.9. Let $F \subset \mathbb{A}^{(0)}_{\mathrm{amb}}$ be the fundamental domain for $\langle P \rangle \curvearrowright \mathbb{A}^{(0)}_{\mathrm{amb}}$ from Proposition 2.4. Let $V \subset \mathbb{A}^{(0)}_{\mathrm{amb}}$ be the

fundamental domain for $\langle t, t' \rangle$ that contains F. Since t and t' do not commute, V is the only fundamental domain for $\langle t, t' \rangle$ contained in $\Phi(\mathcal{W}_t, \mathcal{W}_r)$ and the only one in $\Phi(\mathcal{W}_{t'}, \mathcal{W}_{r'})$. Thus $E_{t,r} \subset V$ and $E_{t',r'} \subset V$. It follows that $E_{t,r} = E_{t',r'}$.

Now we deal with the general situation. We consider the geodesic edge-path $(t_i)_{i=0}^n$ from $t_0 = t$ to $t_n = t'$ in the Coxeter–Dynkin diagram of J (which is a tree). Let i' be minimal such that $t_{i'}$ is not adjacent to r' and i maximal such that t_i is not adjacent to r. Then $t_{i'}$ is good respect to r' (since r' and $t_{i'-1}$ are adjacent) and t_i is good with respect to r (since r and t_{i+1} are adjacent). Note that $i' \geq 1$ and $i \leq n-1$. If $i' \leq n-1$, then by the induction assumption we can pass from (t, r) to (t', r') via $(t_{i'}, r')$. The case $i \geq 1$ is analogous. Thus in the remaining part of the proof we assume $i' = n$ and $i = 0$, in other words, t_i is adjacent to both r and r' for each $1 \leq i \leq n-1$.

Let $P = \{t_0, \ldots, t_n, r, r'\}$. Note that the defining graph of P is a join of a 4-cycle (whose consecutive vertices are t, r', r, t') and a complete graph (whose vertices are $t_1, \ldots, t_{n-1}$). Since (t_i) was an edge-path in the Coxeter–Dynkin diagram, it is easy to prove that the defining graph of P is 1-rigid. Thus P is geometric by Corollary 3.9. Let $F \subset \mathbb{A}^{(0)}_{\mathrm{amb}}$ be the fundamental domain for $\langle P \rangle \curvearrowright \mathbb{A}^{(0)}_{\mathrm{amb}}$ from Proposition 2.4. Let $V \subset \mathbb{A}^{(0)}_{\mathrm{amb}}$ be the fundamental domain for $\langle t_0, \ldots, t_n \rangle$ that contains F. Since $\{t_0, \ldots, t_n\}$ is irreducible, V is the only fundamental domain for $\langle t_0, \ldots, t_n \rangle$ contained in $\Phi(\mathcal{W}_t, \mathcal{W}_r)$ and the only one in $\Phi(\mathcal{W}_{t'}, \mathcal{W}_{r'})$. Thus $E_{t,r} \subset V$ and $E_{t',r'} \subset V$. Hence $E_{t,r} = E_{t',r'}$. □

6 COMPLEXITY

In this section, we introduce the complexity of the Coxeter generating set S w.r.t. S'. We keep the setup from Section 5. To start, we need to describe particular subsets of pairs of maximal spherical residues.

DEFINITION 6.1. Let $J \subset S$ be a maximal spherical subset. By Corollary 2.6, W_J stabilises a unique maximal cell $\sigma_J \subset \mathbb{A}_{\mathrm{amb}}$. Let C_J be the collection of vertices in σ_J and let D_J be the elements of C_J incident to each $\mathcal{W}_j$ for $j \in J$.

When J is irreducible, then by Corollary 2.6, it is easy to see that D_J consists of two antipodal vertices. In general, let $J = J_1 \sqcup \cdots \sqcup J_k$ be the decomposition of J into maximal irreducible subsets. Let $\sigma_J = \sigma_1 \times \cdots \times \sigma_k$ be the induced product decomposition of the associated cell. Then D_J is a product of pairs of antipodal vertices $\{u_i, v_i\}$ for each σ_i. Let $\pi_i \colon D_J \to \{u_i, v_i\}$ be the coordinate projections.

DEFINITION 6.2. For each ordered pair (J, I) of maximal spherical subsets of S, we define the following subset $E_{J,I} \subseteq D_J$. First, for each i, consider the following $E^i_{J,I} \subseteq D_J$. If J_i is not good with respect to I, then we take $E^i_{J,I} = D_J$. If J_i is good, then let t and r be as in Definition 5.1. Then we take $E^i_{J,I} = C_J \cap E_{t,r}$ (which is contained in D_J and equal $\pi_i^{-1}(u_i)$ or $\pi_i^{-1}(v_i)$).

Note that $E^i_{J,I}$ does not depend on t and r by Proposition 5.2. We define $E_{J,I} = E^1_{J,I} \cap \cdots \cap E^k_{J,I}$.

Definition 6.3. We define the *complexity* of S, denoted $\mathcal{K}(S)$, to be the ordered pair of numbers

$$\big(\mathcal{K}_1(S), \mathcal{K}_2(S)\big) = \Big(\sum_{J \neq I} d(C_J, C_I), \sum_{J \neq I} d(E_{J,I}, E_{I,J}) \Big),$$

where J and I range over all maximal spherical subsets of S, and $E_{J,I}$ is defined in Definition 6.2. Note that the distance d is computed in $\mathbb{A}^{(1)}_{\mathrm{amb}}$ and so we have $\mathcal{K}_1(S') = \mathcal{K}_2(S') = 0$, since $c_0 \in C_J, c_0 \in E_{J,I}$ for all maximal spherical subsets $J, I \subset S'$.
For two Coxeter generating sets S and S_τ, we define $\mathcal{K}(S_\tau) < \mathcal{K}(S)$ if $\mathcal{K}_1(S_\tau) < \mathcal{K}_1(S)$, or $\mathcal{K}_1(S_\tau) = \mathcal{K}_1(S)$ and $\mathcal{K}_2(S_\tau) < \mathcal{K}_2(S)$.

7 Proof of the main theorem

We keep the setup from Section 5. Note that since S is 2-rigid, an elementary twist does not change its defining graph. Thus Main Theorem reduces to the following.

Theorem 7.1. *Let S be angle-compatible with S'. Suppose that S is 2-rigid and of type* FC*. Assume moreover that S has minimal complexity among all Coxeter generating sets twist-equivalent to S. Then S is conjugate to S'.*

The proof will take the remaining part of the article, and we divide it into several steps. For $\mu = ((s, w), m)$ a marking with support J, we define $K_\mu = J \setminus (s \cup s^\perp)$ if $J \neq \{s\}$, and $K_\mu = \{m\}$ otherwise.
By Corollary 3.8, to prove Theorem 7.1 it suffices to show that for any markings μ and μ' with common core $s \in S$, we have $\Phi^\mu_s = \Phi^{\mu'}_s$. Note that for each component A of $S \setminus (s \cup s^\perp)$, there exists a marking μ with $K_\mu \subseteq A$. By Proposition 4.3 and the fact that S is 2-rigid, if $K_{\mu'} \subseteq A$, then $\Phi^\mu_s = \Phi^{\mu'}_s$. Thus each component A of $S \setminus (s \cup s^\perp)$ determines a half-space $\Phi_A := \Phi^\mu_s$ for s. Two components A_1 and A_2 of $S \setminus (s \cup s^\perp)$ are *compatible* if $\Phi_{A_1} = \Phi_{A_2}$. We will show that all the components of $S \setminus (s \cup s^\perp)$ are compatible. Fixing $s \in S$, we shall divide these components into several classes and conduct a case analysis.

7.1 Big components are compatible

Definition 7.2. A component A of $S \setminus (s \cup s^\perp)$ is *big* if there is $a \in A$ not adjacent to s. Otherwise A is *small*.

Lemma 7.3. *Any two big components are compatible.*

Proof. We argue by contradiction and assume that the big components of $S \setminus (s \cup s^\perp)$ can be divided into two non-empty families $\{A_k\}$ and $\{B_k\}$ such that

all Φ_{A_k} coincide (call that half-space Φ_A) and are distinct from all Φ_{B_k}, which also coincide (call that half-space Φ_B). Let B be the union of all the B_k. Let τ be the elementary twist that sends each element $b \in B$ to sbs and fixes other elements of S. For a contradiction, we will prove $\mathcal{K}_1(\tau(S)) < \mathcal{K}_1(S)$.
Let $J \subset S$ be maximal spherical. J is *twisted* if it contains an element of B and $s \notin J$. A twisted J exists, since we can take any maximal spherical J containing $b \in B$ not adjacent to s. Note that if J is twisted, then for each $j \in J$ we have $\mathcal{W}_{\tau(j)} = s\mathcal{W}_j$, and hence $C_{\tau(J)} = s.C_J$. Moreover, there is an element $b \in J \setminus \{s\}$ not adjacent to s, since otherwise $J \cup \{s\}$ would be spherical contradicting the maximality of J. Then $\Phi(\mathcal{W}_s, C_J) = \Phi(\mathcal{W}_s, \mathcal{W}_b) = \Phi_B$.
Consider now maximal spherical $I \subset S$ that is not twisted. If $s \in I$, then $C_{\tau(I)} = s.C_I = C_I$. If $s \notin I$, then $I \cap B = \emptyset$, and we also have $C_{\tau(I)} = C_I$. As before, there exists such I with $s \notin I$. Moreover, then there is $a \in I \setminus \{s\}$ not adjacent to s, and $\Phi(\mathcal{W}_s, C_I) = \Phi(\mathcal{W}_s, \mathcal{W}_a) = \Phi_A$.
Let $J, I \subset S$ be maximal spherical. If both J and I are twisted or both are not twisted, then $d(C_J, C_I) = d(C_{\tau(J)}, C_{\tau(I)})$. Now suppose that J is twisted and I is not twisted. If $s \in I$, we still have $d(C_J, C_I) = d(C_{\tau(J)}, C_{\tau(I)})$. If $s \notin I$, then since $\Phi_B \neq \Phi_A$, we have $\Phi(\mathcal{W}_s, C_J) \neq \Phi(\mathcal{W}_s, C_I)$. Hence a minimal length gallery β from a chamber in C_J to a chamber in C_I has an edge dual to $\mathcal{W}_s$. Removing this edge from β and reflecting $\beta \cap \Phi(\mathcal{W}_s, C_J)$ by s, we obtain a shorter gallery from a chamber in $s.C_J$ to a chamber in C_I. Thus $d(C_{\tau(J)}, C_{\tau(I)}) = d(s.C_J, C_I) < d(C_J, C_I)$. Consequently $\mathcal{K}_1(\tau(S)) < \mathcal{K}_1(S)$. □

7.2 Exposed components

Definition 7.4. A small component A is *exposed* if there is $t \in A$ and r inside a different component of $S \setminus (s \cup s^\perp)$ such that s and r are in distinct components of $S \setminus (t \cup t^\perp)$.

Lemma 7.5. *If there exists an exposed component, then all components are compatible.*

Proof. Let t and r be as in Definition 7.4. Note that r is adjacent to neither s nor t. By Lemma 2.7, none of the elements of $S \setminus (\{s,t\} \cup \{s,t\}^\perp)$ is adjacent to s or t. It follows that there is only one small component of $S \setminus (s \cup s^\perp)$, and this small component equals $\{t\}$.
Observe that a maximal spherical subset $J \subset S$ contains s if and only if it contains t. Indeed, if say $s \in J$, then each element of $J \setminus \{s\}$ is adjacent to s. Hence $J \subseteq \{s,t\} \cup \{s,t\}^\perp$ by Lemma 2.7. If $t \notin J$, then $J \cup \{t\}$ is spherical, which contradicts the maximality of J. We say that J is *exposed* if $\{s,t\} \subseteq J$. Let $\mathcal{W}_{\{s,t\}}$ be the union of all the walls in $\mathbb{A}_{\mathrm{amb}}$ for the reflections in the dihedral group $\langle s,t \rangle$. Since S is 2-rigid, the graph induced on $S \setminus (\{s,t\} \cup \{s,t\}^\perp)$ is connected. Thus all the walls $\mathcal{W}_r$ for $r \in S \setminus (\{s,t\} \cup \{s,t\}^\perp)$ lie in the same connected component Λ of $\mathbb{A}_{\mathrm{amb}} \setminus \mathcal{W}_{\{s,t\}}$. Consequently, all D_J for J not exposed lie in Λ. Let $\Sigma \subset \mathbb{A}_{\mathrm{amb}}$ be the union of the two sectors of the

form $\Phi_s \cap \Phi_t$ for $\{\Phi_s, \Phi_t\}$ geometric. Assume first $\Lambda \subset \Sigma$. Then $\Phi(\mathcal{W}_s, \Lambda) = \Phi(\mathcal{W}_s, t\Lambda)$, hence $\Phi(\mathcal{W}_s, \mathcal{W}_r) = \Phi(\mathcal{W}_s, t\mathcal{W}_r)$. These half-spaces correspond to markings $\mu = ((s,t), r)$ with $K_\mu = \{t\}$ and $\mu' = (s, r)$ with $K_{\mu'} = \{r\}$. Consequently, the unique small component $\{t\}$ of $S \setminus (s \cup s^\perp)$ is compatible with a big component. In view of Lemma 7.3, all the components are compatible. It remains to consider the case $\Lambda \not\subset \Sigma$.

Let τ_s (resp. τ_t) be the elementary twist that sends t to sts (resp. s to tst) and fixes other elements of S. For any $w \in \langle s, t\rangle$, composing appropriately τ_s and τ_t (while keeping the notation s, t for the images of s, t under the twist), we obtain $\tau = \tau_w$ sending s to wsw^{-1}, t to wtw^{-1} and fixing other elements of S. We will justify the following.

1. $\mathcal{W}_{\tau(s)} = w\mathcal{W}_s$ and $\mathcal{W}_{\tau(t)} = w\mathcal{W}_t$;
2. if J is maximal spherical that is exposed (resp. not exposed), then $D_{\tau(J)} = w.D_J$ (resp. $D_{\tau(J)} = D_J$);
3. if J and I are both maximal spherical and exposed (resp. not exposed), then $E_{\tau(J),\tau(I)} = w.E_{J,I}$ (resp. $E_{\tau(J),\tau(I)} = E_{J,I}$);
4. if J is maximal spherical that is exposed and I is maximal spherical that is not exposed, then $E_{\tau(J),\tau(I)} = w.E_{J,I}$ and $E_{\tau(I),\tau(J)} = E_{I,J}$.

Here (1) is immediate and implies (2), while (3) follows from (2) and Definition 6.2 (note that an elementary twist does change the defining graph, so it does not change the good subsets of J and I). Now we prove (4). Note that for each $j \in J$, we have $\mathcal{W}_j \cap \mathcal{W}_{\tau(j)} \neq \emptyset$. Moreover, τ fixes each element of I. Thus for non-adjacent $i \in I$ and $j \in J$, the walls $\mathcal{W}_j$ and $\mathcal{W}_{\tau(j)}$ are in the same half-space for $i = \tau(i)$. Hence it follows from Definition 6.2 that $E_{\tau(I),\tau(J)} = E_{I,J}$. It remains to verify the first equality of (4). Note that the elements of $J \setminus \{s, t\}$ are fixed by τ, and $\{s, t\} \subset J$ is maximal irreducible that is not good in view of Definition 7.4 and Lemma 2.7. Thus $E_{\tau(J),\tau(I)} = D_{\tau(J)} = w.D_J = w.E_{J,I}$, finishing the proof of (4).

Coming back to the case $\Lambda \not\subset \Sigma$, choose $\tau = \tau_w$ a composition of twists as above so that $w\Sigma$ contains Λ. We will reach a contradiction by showing $\mathcal{K}_1(\tau(S)) = \mathcal{K}_1(S)$ and $\mathcal{K}_2(\tau(S)) < \mathcal{K}_2(S)$. The equality follows from the fact that for any maximal spherical $J \subset S$, we have $C_{\tau(J)} = C_J$. Now we verify the inequality. Consider maximal spherical subsets $J, I \subset S$. If both J and I are exposed or both are not exposed, then by (3) we have $d(E_{\tau(J),\tau(I)}, E_{\tau(I),\tau(J)}) = d(E_{J,I}, E_{I,J})$.

Now we assume that J is exposed but I is not exposed. Let β be a shortest gallery from a chamber $y \in E_{I,J}$ to a chamber $x \in E_{J,I}$. By angle-compatibility, $\{s, t\}$ is conjugate to $\{s', t'\} \subset S'$. By Theorem 2.1, we can assume that β is a concatenation of galleries β' and β'', where β' is a minimal gallery from y to some chamber (call it x') in the $\{s', t'\}$-residue $\mathcal{R}$ containing x. Furthermore, $\beta' \subset \Lambda$. Note that $x \neq x'$ since $\Lambda \not\subseteq \Sigma$.

We have $x' = w.x$ or $x' = w.x_{\text{ant}}$, where x_{ant} is the chamber antipodal to x in $\mathcal{R}$. Note that $x_{\text{ant}} \in E_{J,I}$, since $\{s,t\}$ is an irreducible component of J that is not good with respect to I. Thus from (4) we deduce $x' \in E_{\tau(J),\tau(I)}$ and $y \in E_{\tau(I),\tau(J)}$. Consequently $d(E_{\tau(J),\tau(I)}, E_{\tau(I),\tau(J)}) < d(E_{J,I}, E_{I,J})$, giving $\mathcal{K}_2(\tau(S)) < \mathcal{K}_2(S)$. □

7.3 Non-exposed small components

To prove Theorem 7.1, it remains to consider the case where all components of $S \setminus (s \cup s^\perp)$ are big, or small and not exposed. We argue by contradiction and assume that the components of $S \setminus (s \cup s^\perp)$ can be divided into two non-empty families $\{A_k\}$ and $\{B_k\}$ such that all Φ_{A_k} coincide and are distinct from all Φ_{B_k}, which also coincide. Let A (resp. B) be the union of all B_k (resp. A_k). By Lemma 7.3, we can assume that all the big components (if they exist) are in A. Let τ be the elementary twist that sends each element $b \in B$ to sbs and fixes other elements of S.

Let $J \subset S$ be a maximal spherical subset. J is *twisted* if it contains an element of B. In that case, s is adjacent to each element in J since B is a union of small components. Consequently $J \cup \{s\}$ is spherical so $s \in J$ by the maximality of J.

In particular, τ preserves all C_J, and hence $\mathcal{K}_1(S) = \mathcal{K}_1(\tau(S))$. For a contradiction, we will prove $\mathcal{K}_2(\tau(S)) < \mathcal{K}_2(S)$.

Consider maximal spherical subsets J and I. If both of them are twisted or both are not-twisted, then we have

$$d(E_{\tau(J),\tau(I)}, E_{\tau(I),\tau(J)}) = d(E_{J,I}, E_{I,J}). \tag{7.1}$$

Now we assume that J is twisted and I is not twisted. If $I \subseteq \{s\} \cup \{s\}^\perp$, then (7.1) holds as well. It remains to discuss the case where $I \nsubseteq s \cup s^\perp$. We will prove $d(E_{\tau(J),\tau(I)}, E_{\tau(I),\tau(J)}) < d(E_{J,I}, E_{I,J})$, which implies $\mathcal{K}_2(\tau(S_0)) < \mathcal{K}_2(S_0)$ and finishes the proof of Theorem 7.1.

Case 1: I contains s. In that case, pick $r \in I \setminus (s \cup s^\perp)$. Let $I_1 \subseteq I$ be maximal irreducible containing r. Then $s \in I_1$, since s and r do not commute. Pick $t \in J \setminus (s \cup s^\perp)$. Let $J_1 \subseteq J$ be maximal irreducible containing t. Then $s \in J_1$. Since both t and r are adjacent to s, we have that $t \in J_1$ is good with respect to r, and $r \in I_1$ is good respect to t.

We first justify that $E_{J,I}$ and $E_{I,J}$ lie in distinct half-spaces for s. Otherwise, $\{r,s,t\}$ is geometric. In particular, we have $\Phi(\mathcal{W}_s, t\mathcal{W}_r) = \Phi(\mathcal{W}_s, r\mathcal{W}_t)$. These half-spaces correspond to markings $\mu = ((s,t),r)$ with $K_\mu = \{t\}$ and $\mu' = ((s,r),t)$ with $K_{\mu'} = \{r\}$. This contradicts the assumption that t and r belong to incompatible components.

We have $D_{\tau(J)} = s.D_J$. Note that τ fixes all the elements of I and $J \setminus J_1$, and hence $E_{\tau(J),\tau(I)} = s.E_{J,I}$ in view of

$$\Phi(s\mathcal{W}_t, \mathcal{W}_r) = \Phi(s\mathcal{W}_t, \mathcal{W}_r \cap \mathcal{W}_s) = s\Phi(\mathcal{W}_t, \mathcal{W}_r \cap \mathcal{W}_s) = s\Phi(\mathcal{W}_t, \mathcal{W}_r).$$

On the other hand, we have $E_{\tau(I),\tau(J)} = E_{I,J}$, since $\mathcal{W}_j \cap \mathcal{W}_{\tau(j)} \neq \emptyset$ for each $j \in J$, and hence $\mathcal{W}_j$ and $\mathcal{W}_{\tau(j)}$ are in the same half-space for $i = \tau(i) \in I$ not adjacent to j.

To conclude Case 1, pick a gallery β of minimal length from $x \in E_{J,I}$ to $y \in E_{I,J}$. Since chambers x and y lie in distinct half-spaces for s and x is incident to $\mathcal{W}_s$, we can assume that the first edge of β is dual to $\mathcal{W}_s$ (Theorem 2.1). Since $s.x \in s.E_{J,I} = E_{\tau(J),\tau(I)}$ and $y \in E_{I,J} = E_{\tau(I),\tau(J)}$, we have $d(E_{\tau(J),\tau(I)}, E_{\tau(I),\tau(J)}) < d(E_{J,I}, E_{I,J})$, as desired.

Case 2: *I contains an element not adjacent to s*. Let this element be r. Let t and J_1 be as in Case 1. Since t is inside a non-exposed small component, $t \in J_1$ is good with respect to r. In particular, J_1 is good with respect to I.

Let $\Sigma \subset \mathbb{A}_{\mathrm{amb}}$ be the union of the two sectors of the form $\Phi_s \cap \Phi_t$ for $\{\Phi_s, \Phi_t\}$ geometric. We first justify $\mathcal{W}_r \subset s\Sigma$. Indeed, note that $\mathcal{W}_r$ is disjoint from any wall in $\mathcal{W}_{\{s,t\}}$. Since s and r are in the same component of $S \setminus (t \cup t^{\perp})$, we have $(t, r) \sim ((t, s), r)$ by Proposition 4.3 and the fact that S is 2-rigid. Thus $\Phi(\mathcal{W}_t, \mathcal{W}_r) = \Phi(\mathcal{W}_t, s\mathcal{W}_r)$. It follows that $\mathcal{W}_r \subset \Sigma \cup s\Sigma$. Now recall that $t \in B$ and $r \in A$, thus $\Phi(\mathcal{W}_s, \mathcal{W}_r) \neq \Phi(\mathcal{W}_s, t\mathcal{W}_r)$ by the incompatibility of A and B. It follows that $\mathcal{W}_r \subset \Sigma$ is not possible, justifying $\mathcal{W}_r \subset s\Sigma$.

Let Λ be the sector of Σ satisfying $\mathcal{W}_r \subset s\Lambda$. It follows that $E_{J,I} \subset \Lambda$ and $E_{\tau(J),\tau(I)} \subset s\Lambda$. Consequently $E_{\tau(J),\tau(I)} = sE_{J,I}$. We also have $E_{\tau(I),\tau(J)} = E_{I,J}$ as in Case 1. Note that $E_{I,J}$ and $E_{J,I}$ are in distinct half-spaces for s. Now we can prove $d(E_{\tau(J),\tau(I)}, E_{\tau(I),\tau(J)}) < d(E_{J,I}, E_{I,J})$ in the same way as in Case 1.

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Jingyin Huang
Math. & Stats.
McGill University
805 Sherbrooke Street West
Montreal, Quebec H3A 0B9
Canada
and
Department of Mathematics
The Ohio State University
231 W. 18th Avenue
Columbus, Ohio, 43210
USA
huang.929@osu.edu

Piotr Przytycki
Math. & Stats.
McGill University
805 Sherbrooke Street West
Montreal, Quebec H3A 0B9
Canada
piotr.przytycki@mcgill.ca