Categorically Morita Equivalent
Compact Quantum Groups

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Abstract. We give a dynamical characterization of categorical Morita equivalence between compact quantum groups. More precisely, by a Tannaka–Krein type duality, a unital C*-algebra endowed with commuting actions of two compact quantum groups corresponds to a bimodule category over their representation categories. We show that this bimodule category is invertible if and only if the actions are free, with finite dimensional fixed point algebras, which are in duality as Frobenius algebras in an appropriate sense. This extends the well-known characterization of monoidal equivalence in terms of bi-Hopf–Galois objects.

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Introduction

The Tannaka–Krein duality principle, which roughly says that a quantum group is characterized by its representation category viewed as a concrete category of vector spaces, has played fundamental role in the development of various approaches to quantum groups. In mathematical physics, the attempts by the Leningrad school to find an algebraic structure behind the solutions of Yang–Baxter equations (R-matrices) led to the famous Drinfeld–Jimbo quantized universal enveloping algebras, where R-matrices are regarded as intertwiners of tensor products of representations of Hopf algebras. In the operator algebraic framework, Woronowicz's Tannaka–Krein duality theorem has been

used to construct many examples of compact quantum groups beyond the q-deformations, see, e.g., [BCS10].

One natural question arising from this principle is the following: which categorical concepts for representation categories of quantum groups admit Hopf algebraic formulations? For example, the most fundamental question of when one has an equivalence $\operatorname{Rep} G_1 \cong \operatorname{Rep} G_2$ as abstract monoidal categories, has a very satisfactory answer due to Schauenburg [Sch04] building on an earlier work of Ulbrich [Ulb87]. It says that the representation categories are monoidally equivalent precisely when there is a G_1-G_2-Hopf–Galois object, which is an algebra with commuting coactions of the function algebras of G_1 and G_2, which are separately 'free' and 'transitive' (or 'ergodic').

In the operator algebraic context, the C*-analogue of this characterization [BdRV06] has been fruitfully used by many authors to deduce analytic properties of one compact or discrete quantum group from another, starting from the work of Vaes and Vergnioux [VV07], where they showed that exactness of the reduced function algebra of a compact quantum group is invariant under monoidal equivalence. More recently, induction of central multipliers along bi-Hopf–Galois objects was used to show that free quantum groups have the Haagerup property and the weak amenability [Fre13, dCFY14]. Another interesting development is the introduction of central property (T) by Arano [Ara18], which suggests that there is a close connection between harmonic analysis on the representation categories of the q-deformations of compact Lie groups and the classical theory of unitary representations of complex semisimple Lie groups.

These works have led to a study of analytic properties of C*-tensor categories, which also has roots in Popa's earlier work on approximation properties of standard invariants of subfactors [Pop99]. Indeed, as has been shown by Popa and Vaes [PV15] and the authors [NY16], 'central' approximation properties of quantum groups considered in [dCFY14, Ara18] can be formulated at the purely categorical level. As one of the applications, this has allowed one to unify property (T) for (quantum) groups and property (T) for subfactors.

One crucial insight from the subfactor theory is that there is a more interesting equivalence relation on tensor categories beyond the mere equivalence. It corresponds to exchanging a subfactor for its dual inclusion, and in the case of fusion categories, a relevant notion was introduced by Müger [Müg03] under the name of *weak monoidal Morita equivalence*, which is now also called *categorical Morita equivalence*. Namely, two fusion categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are called weakly monoidally Morita equivalent if one of them is monoidally equivalent to the category of bimodules over a Frobenius algebra in the other [Lon94, Yam04]. In a more symmetric form this can be formulated in terms of 2-categories, or as existence of an invertible $\mathcal{C}_1$-$\mathcal{C}_2$-module category [ENO10], see Section 3 for precise definitions. Yet another characterization is that the Drinfeld centers of $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent as braided monoidal categories [Sch01, ENO11]. Most of these admit straightforward generalizations to the setting of infinite C*-tensor categories, although a characterization of categorical Morita equiva-

lence in terms of the Drinfeld center seems to remain as an interesting problem. Popa's work on subfactors implies that sensible analytic properties should be invariant under categorical Morita equivalence. For central property (T), this is indeed the case [NY16, NY18]. It is therefore natural to expect that categorical Morita equivalence should be useful in studying analytic properties of quantum groups.

The goal of the present paper is to give an algebraic characterization of categorical Morita equivalence for representation categories of compact quantum groups. By the Tannaka–Krein type duality for quantum group actions [Ost03, dCY13b, Nes14], bimodule categories over representation categories correspond to C*-algebras with commuting actions of the quantum groups. Namely, given commuting actions of compact quantum groups G_1 and G_2 on a unital C*-algebra A, we can consider the category $\mathcal{D}_A$ of equivariant finitely generated right Hilbert A-modules. Therefore the precise question we are going to answer is the following: under what conditions is the category $\mathcal{D}_A$ invertible as a $(\mathrm{Rep}\, G_2)$-$(\mathrm{Rep}\, G_1)$-module category?

Our main result (Theorem 3.7) states that $\mathcal{D}_A$ is invertible if and only if the actions are separately free, have finite dimensional fixed point algebras A^{G_1} and A^{G_2}, and that these algebras sit nicely in A so that the equivariant A-modules $A^{G_1} \otimes A$ and $A^{G_2} \otimes A$ are isomorphic in a way that respects the actions of A^{G_1} and A^{G_2}, which we call the G_1-G_2-*Morita–Galois condition*. When the actions are ergodic, so that the fixed point subalgebras are trivial, we recover the bi-Hopf–Galois condition.

Finally, let us note that 2-categories have close connection to the theory of quantum groupoids, and a result of De Commer and Timmermann gives a characterization of categorical Morita equivalence of compact quantum groups in terms of what they call *partial compact quantum groups* [dCT15]. Their construction gives a 'zigzag' of the so-called linking and co-linking quantum groupoids between G_1 and G_2, as opposed to our one-step construction. They also do not give any characterization of the 'off-diagonal' parts of the co-linking groupoids, and the overall construction involving Hayashi's canonical partial quantum groups associated with the representation categories of G_1 and G_2 seems to be more involved than ours. At the same time their construction works beyond our setting of compact quantum groups and allows one to capture categorical Morita equivalence of several (partial) compact quantum groups at once. It would be an interesting problem to find a characterization of the 'off-diagonal' parts of their co-linking groupoids and compare it with our results, but we do not attempt to go in this direction in the present paper.

The paper consists of four sections and an appendix. Section 1 is a recollection of basic conventions and results that we use freely throughout the paper.

Section 2 is also of preliminary nature. Here we discuss Frobenius algebras in C*-tensor categories and modules over them, and compare such algebras in the category of finite dimensional Hilbert spaces to finite dimensional C*-algebras with prescribed faithful states (Frobenius C*-algebras). A large part of this material is surely known to experts, but for the lack of a comprehensive

reference we provide proofs of many results.

Section 3 is the main part of the paper. Here we formulate and prove our main result indicated above. We also provide a one-sided variant (Theorem 3.24) which starts from a single quantum group G and its action, and then produces another categorically Morita equivalent quantum group, generalizing the notion of G-Hopf–Galois objects.

In Section 4 we discuss relative tensor products of invertible bimodule categories which correspond to the transitivity of Morita equivalence, as well as give a few examples.

In Appendix we discuss a correspondence between module categories and Frobenius algebras. In the purely algebraic setting the existence of such a correspondences was established by Ostrik [Ost03]. Its adaption to the C*-setting is formulated in [AdC15], but we believe certain points concerning unitarity deserve further explanation.

1   Preliminaries

1.1   Quantum groups and tensor categories

For a detailed discussion of C*-tensor categories and compact quantum groups we refer the reader to [NT13]. Let us just recall a few basic definitions and facts.

A *C*-category* is a category $\mathcal{C}$ where the morphisms sets $\mathcal{C}(U, V)$ are complex Banach spaces endowed with complex conjugate involution $\mathcal{C}(U, V) \to \mathcal{C}(V, U)$, $T \mapsto T^*$, satisfying the C*-identity (so that every endomorphism ring $\mathcal{C}(X) = \mathcal{C}(X, X)$ becomes a C*-algebra) and having the property $T^*T \geq 0$ in $\mathcal{C}(X)$ for any $T \in \mathcal{C}(X, Y)$. The most basic example of such a category is Hilb_f, the category of finite dimensional Hilbert spaces. We tacitly assume that $\mathcal{C}$ is closed under finite direct sums and subobjects, which means that any idempotent in the endomorphism ring $\mathcal{C}(U)$ comes from a direct summand of U.

A *unitary functor*, or a *C*-functor*, $F \colon \mathcal{C} \to \mathcal{C}'$ between C*-categories is a $\mathbb{C}$-linear functor from $\mathcal{C}$ to $\mathcal{C}'$ satisfying $F(T^*) = F(T)^*$.

A *C*-tensor category* is a C*-category endowed with a unitary bifunctor $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a distinguished object $\mathbb{1} \in \mathcal{C}$, and natural unitary isomorphisms

$$\mathbb{1} \otimes U \cong U \cong U \otimes \mathbb{1}, \qquad \Phi \colon (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

satisfying certain compatibility conditions. Unless said otherwise, we always assume that $\mathcal{C}$ is *strict*, that is, the above isomorphisms are the identity mor-

phisms, but thanks to a C*-analogue of Mac Lane's coherence theorem this does not lead to loss of generality. We also assume that the unit $\mathbb{1}$ is simple.

A C^*-2-*category* on a set I of '0-cells' is given by a collection of C*-categories $\mathcal{C}_{st}$ for $s, t \in I$ together with bilinear unitary bifunctors $\mathcal{C}_{rs} \times \mathcal{C}_{st} \to \mathcal{C}_{rt}$ and unit objects $\mathbb{1}_s \in \mathcal{C}_{ss}$. The axioms which this structure satisfies are analogous to those of strict C*-tensor categories. In other words, the main difference from the latter categories is that the tensor product $X \otimes Y$ is defined not for all objects, but only when $X \in \mathcal{C}_{rs}$ and $Y \in \mathcal{C}_{st}$, and then $X \otimes Y \in \mathcal{C}_{rt}$. Again, it is possible to consider a non-strict version, C^*-*bicategories*, but we do not do this as there is no essential loss of generality in considering only C*-2-categories.

A *unitary tensor functor*, a *unitary monoidal functor*, or a C^*-*tensor functor*, $\mathcal{C} \to \mathcal{C}'$ between C*-tensor categories is a pair consisting of a unitary functor $F \colon \mathcal{C} \to \mathcal{C}'$, such that $F(\mathbb{1}_{\mathcal{C}}) \cong \mathbb{1}_{\mathcal{C}'}$, and a collection F_2 of natural unitary isomorphisms $F(U) \otimes F(V) \to F(U \otimes V)$ such that $F_2(F_2 \otimes \iota) = F_2(\iota \otimes F_2) \colon F(U) \otimes F(V) \otimes F(W) \to F(U \otimes V \otimes W)$. If $F(\mathbb{1}_{\mathcal{C}}) = \mathbb{1}_{\mathcal{C}'}$, $F(U \otimes V) = F(U) \otimes F(V)$ and $F_2 \colon F(U) \otimes F(V) \to F(U) \otimes F(V)$ are the identity morphisms, then we say that we have a strict tensor functor.

A C*-tensor category $\mathcal{C}$ is called *rigid*, if every object U has a conjugate object, that is, there exist an object $\bar{U}$ and morphisms $R \colon \mathbb{1} \to \bar{U} \otimes U$ and $\bar{R} \colon \mathbb{1} \to U \otimes \bar{U}$ solving the conjugate equations

$$(R^* \otimes \iota_{\bar{U}})(\iota_{\bar{U}} \otimes \bar{R}) = \iota_{\bar{U}}, \qquad (\bar{R}^* \otimes \iota_U)(\iota_U \otimes R) = \iota_U.$$

The minimum $d(U)$ of the numbers $\|R\| \|\bar{R}\|$ over all solutions is called the *intrinsic dimension* of U. A solution $(R, \bar{R})$ is called *standard* if $\|R\| = \|\bar{R}\| = d(U)^{1/2}$. Any standard solution $(R_U, \bar{R}_U)$ defines a trace on $\mathcal{C}(U)$ by

$$\mathrm{Tr}_U(T) = R_U^*(\iota \otimes T)R_U,$$

which is independent of any choices and is also equal to $\bar{R}_U^*(T \otimes \iota)\bar{R}_U$ (*sphericity*). The normalized trace $d(U)^{-1} \mathrm{Tr}_U$ is denoted by tr_U. More generally, we have *partial categorical traces*

$$\mathrm{Tr}_U \otimes \iota \colon \mathcal{C}(U \otimes V, U \otimes W) \to \mathcal{C}(V, W), \quad T \mapsto (R_U^* \otimes \iota)(\iota \otimes T)(R_U \otimes \iota),$$

and similarly $\iota \otimes \mathrm{Tr}_U \colon \mathcal{C}(V \otimes U, W \otimes U) \to \mathcal{C}(V, W)$. Once standard solutions are fixed, we can define a $*$-preserving anti-multiplicative map $\mathcal{C}(U, V) \to \mathcal{C}(\bar{V}, \bar{U})$, $T \mapsto T^\vee$, such that

$$(\iota \otimes T)R_U = (T^\vee \otimes \iota)R_V, \qquad (T \otimes \iota)\bar{R}_U = (\iota \otimes T^\vee)\bar{R}_V.$$

Rigidity can be similarly formulated for C*-2-categories. Briefly, in the above notation a dual of $X \in \mathcal{C}_{st}$ is given by an object $\bar{X} \in \mathcal{C}_{ts}$ and morphisms $R \colon \mathbb{1}_t \to \bar{X} \otimes X$, $\bar{R} \colon \mathbb{1}_s \to X \otimes \bar{X}$ satisfying the conjugate equations of the same form. The dimension $d(X)$ and standard solutions $(R_X, \bar{R}_X)$ make sense, and the functional $\mathrm{Tr}_X(T) = R_X^*(\iota \otimes T)R_X^*$ on $\mathcal{C}_{st}(X)$ is tracial and satisfies the sphericity condition.

An example of a rigid C*-tensor category is the representation category of a *compact quantum group*. A compact quantum group G is represented by a unital C*-algebra $C(G)$ equipped with a unital $*$-homomorphism $\Delta\colon C(G) \to C(G) \otimes C(G)$ satisfying the coassociativity $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ and the cancellation property, meaning that $(C(G) \otimes 1)\Delta(C(G))$ and $(1 \otimes C(G))\Delta(C(G))$ are dense in $C(G) \otimes C(G)$. There is a unique state h satisfying $(h \otimes \iota)\Delta = h(\cdot)1$ and $(\iota \otimes h)\Delta = h(\cdot)1$ called the *Haar state*. If h is faithful, then G is called a *reduced* quantum group. Throughout the whole paper we only consider such quantum groups.

A finite dimensional *unitary representation* of G is a unitary element $U \in B(H_U) \otimes C(G)$, where H_U is a finite dimensional Hilbert space, such that $(\iota \otimes \Delta)(U) = U_{12}U_{13}$. The tensor product of two representations U and V is defined by $U \oplus V = U_{13}V_{23}$. This turns the category $\operatorname{Rep} G$ of finite dimensional unitary representations of G into a rigid C*-tensor category.

The duality in the category $\operatorname{Rep} G$ can be described as follows. Take the *regular algebra* $\mathbb{C}[G]$ of G, which is the dense $*$-subalgebra of $C(G)$ spanned by the matrix coefficients of finite dimensional representations. It is a Hopf $*$-algebra, with the antipode characterized by $(\iota \otimes S)(U) = U^*$ for any unitary representation U. Consider the dual space $\mathcal{U}(G) = \mathbb{C}[G]^*$. It has the structure of a $*$-algebra, defined by duality from the Hopf $*$-algebra structure on $\mathbb{C}[G]$. Every finite dimensional unitary representation U of G defines a $*$-representation π_U of $\mathcal{U}(G)$ on H_U by $\pi_U(\omega) = (\iota \otimes \omega)(U)$. We often omit π_U in expressions and write $\omega\xi$ instead of $\pi_U(\omega)\xi$. There is a canonical positive element $\rho \in \mathcal{U}(G)$, called the *Woronowicz character*, characterized by

$$(\iota \otimes S^2)(U) = (\pi_U(\rho) \otimes 1)U(\pi_U(\rho^{-1}) \otimes 1), \qquad \operatorname{Tr}\pi_U(\rho) = \operatorname{Tr}\pi_U(\rho^{-1})$$

for any finite dimensional unitary representation U.

The (non-unitary) *contragredient* representation of U is given by $U^c = (j \otimes \iota)(U^*) \in B(\bar{H}_U) \otimes \mathbb{C}[G]$, where j denotes the canonical $*$-anti-isomorphism $B(H_U) \cong B(\bar{H}_U)$ defined by $j(T)\bar{\xi} = \overline{T^*\xi}$. Its unitarization, the *conjugate* unitary representation $\bar{U}$, is given by

$$\bar{U} = (j(\pi_U(\rho))^{1/2} \otimes 1)U^c(j(\pi_U(\rho))^{-1/2} \otimes 1).$$

Although S does not satisfy $S^2 = \iota$ nor $S(x^*) = S(x)^*$ (which are in fact equivalent) and is not bounded on $C(G)$ in general, the *unitary antipode* R, which is characterized by $(j \otimes R)(U) = \bar{U}$, does satisfy these properties.

Finally, standard solutions of the conjugate equations can be defined by

$$R_U(1) = \sum_i \bar{\xi}_i \otimes \rho^{-1/2}\xi_i \quad \text{and} \quad \bar{R}_U(1) = \sum_i \rho^{1/2}\xi_i \otimes \bar{\xi}_i, \qquad (1.1)$$

where $(\xi_i)_i$ is an orthonormal basis in H_U. Note that for this choice of standard solutions we have $T^\vee = j(T)$. The above expressions for standard solutions imply that the dimension $d(U)$ coincides with the quantum dimension $\dim_q U = \operatorname{Tr}\pi_U(\rho) = \operatorname{Tr}\pi_U(\rho^{-1})$. We also have $j(\pi_U(\rho)) = \pi_{\bar{U}}(\rho^{-1})$ on $\bar{H}_U = H_{\bar{U}}$.

For a unitary representation U, it will often be convenient to view H_U either as a unitary right comodule over $\mathbb{C}[G]$ by letting $\delta_U(\xi) = U(\xi \otimes 1)$, or as a unitary left comodule by letting $\delta_U(\xi) = U_{21}^*(1 \otimes \xi)$. This should not cause any confusion, as for a fixed compact quantum group we always use only one point of view depending on whether we consider right or left comodule algebras, and that will always be clearly stated. Note that if we consider the spaces H_U as right comodules, the tensor product of representations of G corresponds to the tensor product of right comodules, while if we consider H_U as left comodules, it corresponds to the opposite tensor product of left comodules.

1.2 Tannaka–Krein duality for quantum group actions

A *left action* of a compact quantum group G on a unital C*-algebra A is represented by an injective unital $*$-homomorphism $\alpha \colon A \to C(G) \otimes A$ such that $(\Delta \otimes \iota)\alpha = (\iota \otimes \alpha)\alpha$, and that $(C(G) \otimes 1)\alpha(A)$ is dense in $C(G) \otimes A$. In this case we say that A is a left G-C*-algebra, and also write $G \curvearrowright A$ to express this situation. Given such an algebra, we have a distinguished subalgebra $\mathcal{A} \subset A$, called the *regular subalgebra*, spanned by the elements a (the *regular elements*) such that $\alpha(a)$ lies in the algebraic tensor product of $\mathbb{C}[G]$ and A. More concretely, $\mathcal{A}$ is the linear span of elements $(h(\cdot\, x) \otimes \iota)\alpha(a)$, where $x \in \mathbb{C}[G]$ and $a \in A$. It is a dense unital $*$-subalgebra of A, and the restriction of α to $\mathcal{A}$ turns it into a left $\mathbb{C}[G]$-comodule algebra in the purely algebraic sense. Consider the fixed point C*-algebra

$$B = A^G = \{a \in A \mid \alpha(a) = 1 \otimes a\}.$$

Denote by $\mathrm{Corr}(B)$ the C*-tensor category of C*-correspondences over B, that is, the category of right Hilbert B-modules X equipped with a unital $*$-homomorphism from B into the C*-algebra of adjointable maps on X. This category is not rigid and generally it has nonsimple unit. We will mostly be interested in the case when B is finite dimensional, and instead of $\mathrm{Corr}(B)$ we will work with its full subcategory $\mathrm{Bimod}\text{-}B$ of finite dimensional correspondences. Define a functor $F \colon \mathrm{Rep}\, G \to \mathrm{Corr}(B)$ by

$$F(U) = (H_U \otimes A)^G = (H_U \otimes \mathcal{A})^G.$$

Here, according to our convention, we view H_U as a left $\mathbb{C}[G]$-comodule, since $\mathcal{A}$ is a left comodule algebra, and then $F(U)$ is the space of invariant vectors in the tensor product of comodules H_U and $\mathcal{A}$. Note that if we did consider H_U as a right comodule, then we could write $F(U)$ as the cotensor product $H_U \square_G \mathcal{A}$. The B-valued inner product on $(H_U \otimes \mathcal{A})^G$ is obtained by restricting the obvious A-valued inner product on $H_U \otimes A$: $\langle \xi \otimes a, \zeta \otimes b \rangle_A = (\zeta, \xi)a^*b$.[1] We then have natural isometries

$$F_2 \colon F(U) \otimes_B F(V) \to F(U \otimes V), \quad x \otimes y \mapsto x_{13}y_{23}. \tag{1.2}$$

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[1] Our convention is that inner products on Hilbert spaces are linear in the first variables, while those on right Hilbert modules over C*-algebras are linear in the second variables.

The pair (F, F_2) is called the *spectral functor* defined by the action α. In general the isometries F_2 are not unitary, so it is not a tensor functor but only a *weak*, or *lax*, tensor functor.

Properties of spectral functors can be axiomatized and this way we get a one-to-one correspondence between the isomorphism classes of unital left G-C*-algebras and natural unitary monoidal isomorphism classes of weak unitary tensor functors [PR08,Nes14]. We will only need to know how an action $\alpha \colon A \to C(G) \otimes A$ can be reconstructed from the corresponding spectral functor (F, F_2). Consider the set $\mathrm{Irr}(G)$ of equivalence classes of irreducible representations of G and choose representatives $U_i \in B(H_i) \otimes C(G)$ for $i \in \mathrm{Irr}(G)$. As a G-space, $\mathcal{A}$ can be identified with

$$\bigoplus_{i \in \mathrm{Irr}(G)} \bar{H}_i \otimes F(U_i),$$

endowed with a left action of G given by

$$\alpha(\bar{\xi} \otimes x) = (U_i^c)_{21}^*(1 \otimes \bar{\xi} \otimes x).$$

For $\bar{\xi} \otimes x \in \bar{H}_i \otimes F(U_i)$ and $\bar{\zeta} \otimes y \in \bar{H}_j \otimes F(U_j)$, their product is given by

$$(\bar{\xi} \otimes x)(\bar{\zeta} \otimes y) = \sum_k \overline{w_k^*(\xi \otimes \zeta)} \otimes F(w_k^*)F_2(x \otimes y),$$

where $w_k \in \mathrm{Mor}(U_{l_k}, U_i \oplus U_j)$ are isometries defining a decomposition of $U_i \oplus U_j$ into irreducible representations. The involution is given by

$$(\bar{\xi} \otimes x)^* = \overline{(\iota \otimes \xi^*)R_i(1)} \otimes S_x^* F(\bar{R}_i)(1),$$

where $R_i \colon \mathbb{1} \to U_{\bar{i}} \oplus U_i$ and $\bar{R}_i \colon \mathbb{1} \to U_i \oplus U_{\bar{i}}$ form a solution of the conjugate equations and $S_x \colon F(U_{\bar{i}}) \to F(U_i \oplus U_{\bar{i}})$ is the map $y \mapsto F_2(x \otimes y)$.

A related categorical characterization of actions of G is in terms of module categories. Recall first that, given a C*-tensor category $\mathcal{C}$, a *right $\mathcal{C}$-module category* is a C*-category $\mathcal{D}$ together with a unitary bifunctor $\otimes \colon \mathcal{D} \times \mathcal{C} \to \mathcal{D}$ and natural unitary isomorphisms $X \otimes \mathbb{1} \cong X$ and $X \otimes (U \otimes V) \cong (X \otimes U) \otimes V$ satisfying certain compatibility conditions. For a strict C*-tensor category $\mathcal{C}$, a module category is called strict if these isomorphisms are just the identity morphisms, and unless explicitly stated otherwise we will assume that we deal with such module categories, which again does not lead to loss of generality. An equivalent way to define a right $\mathcal{C}$-module structure on a C*-category $\mathcal{D}$ is by saying that we have a unitary tensor functor from $\mathcal{C}^{\otimes \mathrm{op}}$ into the C*-tensor category $\mathrm{End}(\mathcal{D})$ of unitary endofunctors of $\mathcal{D}$, which has uniformly bounded natural transformations as its morphisms.

A *unitary $\mathcal{C}$-module functor* between right $\mathcal{C}$-module categories $\mathcal{D}$ and $\mathcal{D}'$ is given by a pair (F, θ) consisting of a unitary functor $F \colon \mathcal{D} \to \mathcal{D}'$ and a collection θ of natural unitary isomorphisms $\theta_{X,U} \colon F(X) \otimes U \to F(X \otimes U)$ satisfying some compatibility conditions, which in the case of strict module categories become

$\theta_{X \otimes U, V}(\theta_{X,U} \otimes \iota) = \theta_{X, U \otimes V} \colon F(X) \otimes U \otimes V \to F(X \otimes U \otimes V)$. We denote by $\mathrm{End}_{\mathcal{C}}(\mathcal{D})$ the C*-tensor category of unitary $\mathcal{C}$-module endofunctors of $\mathcal{D}$.

Returning to an action $\alpha \colon A \to C(G) \otimes A$, consider the category $\mathcal{D}_A$ of G-equivariant finitely generated right Hilbert A-modules. We will sometimes denote this category more suggestively by $\mathrm{Mod}_G\text{-}A$. Thus, the objects of $\mathcal{D}_A$ are right Hilbert A-modules X equipped with isometries $\delta_X \colon X \to C(G) \otimes X$, where we consider $C(G) \otimes X$ as a right Hilbert $(C(G) \otimes A)$-module, satisfying the following properties: $\delta_X(X)(C(G) \otimes 1)$ is dense in $C(G) \otimes X$, $(\Delta \otimes \iota)\delta_X = (\iota \otimes \delta_X)\delta_X$, $\delta_X(\xi a) = \delta_X(\xi)\alpha(a)$, and $\langle \delta_X(x), \delta_X(y) \rangle_{C(G) \otimes A} = \alpha(\langle x, y \rangle_A)$. For any such module (X, δ_X), we denote by $\mathcal{X} \subset X$ the dense subspace spanned by the vectors x such that $\delta_X(x)$ lies in the algebraic tensor product of $\mathbb{C}[G]$ and X, or more concretely, the subspace spanned by the vectors $(h(\cdot\, a) \otimes \iota)\delta_X(x)$, where $a \in \mathbb{C}[G]$ and $x \in X$. Then $\mathcal{X}$ is a left $\mathbb{C}[G]$-comodule and a right $\mathcal{A}$-module.

The category $\mathcal{D}_A$ is a right $(\mathrm{Rep}\,G)$-module category, the effect of the action of $U \in \mathrm{Rep}\,G$ on $X \in \mathcal{D}_A$ is the equivariant Hilbert module $H_U \otimes X$. Note once again that since we consider a left action of G on A, we view H_U as a left comodule. This indeed gives us a right action of $\mathrm{Rep}\,G$, since the tensor product of left comodules H_U corresponds to the opposite tensor product in $\mathrm{Rep}\,G$. The category $\mathcal{D}_A$ has a distinguished object represented by the algebra A itself.

This way we get a one-to-one correspondence between the isomorphism classes of left G-C*-algebras and the unitary isomorphism classes of pairs $(\mathcal{D}, X)$, where $\mathcal{D}$ is a right $(\mathrm{Rep}\,G)$-module category and X is a generating object in $\mathcal{D}$, meaning that any other object is a subobject of $X \otimes U$ for some $U \in \mathrm{Rep}\,G$ [dCY13b, Nes14]. If we chose another generating object, then we get a G-equivariantly Morita equivalent C*-algebra. Therefore we also get a one-to-one correspondence between the G-equivariant Morita equivalence classes of left G-C*-algebras and the unitary equivalence classes of singly generated right $(\mathrm{Rep}\,G)$-module categories. For finite quantum groups and their actions on finite dimensional algebras, which can then be considered as algebra objects in $\mathrm{Rep}\,G$, similar results were already obtained by Ostrik [Ost03].

The relation between the above two categorical descriptions can be described as follows. Assume we have a pair $(\mathcal{D}, X)$ as above. Then we can consider the unital C*-algebra $B = \mathcal{D}(X)$ and define a weak unitary tensor functor $\mathrm{Rep}\,G \to \mathrm{Corr}(B)$ by letting

$$F(U) = \mathcal{D}(X, X \otimes U),$$

with the B-valued inner product $\langle S, T \rangle_B = S^*T$, and the tensor structure

$$F_2 \colon F(U) \otimes_B F(V) \to F(U \oplus V), \quad S \otimes T \mapsto (S \otimes \iota)T.$$

Thus, for example, the formula for involution on $\mathcal{A} = \bigoplus_i \bar{H}_i \otimes \mathcal{D}(X, X \otimes U_i)$ becomes

$$(\bar{\xi} \otimes T)^* = \overline{(\iota \otimes \xi^*)R_i(1)} \otimes (T^* \otimes \iota_{\bar{\imath}})(\iota_X \otimes \bar{R}_i)$$

for $\xi \in H_i$ and $T \in \mathcal{D}(X, X \otimes U_i)$.

Of course, everything above makes sense also for right actions $\alpha \colon A \to A \otimes C(G)$ and left module categories. Briefly, given such an action and letting $B = A^G$, the corresponding spectral functor is defined by

$$F \colon (\operatorname{Rep} G)^{\otimes \operatorname{op}} \to \operatorname{Corr}(B), \quad F(U) = (H_U \otimes A)^G,$$
$$F_2 \colon F(U) \otimes_B F(V) \to F(V \oplus U), \quad X \otimes Y \mapsto X_{23} Y_{13}.$$

The dense subalgebra $\mathcal{A} \subset A$ is reconstructed from (F, F_2) by the same formula as before, $\mathcal{A} = \bigoplus_i \bar{H}_i \otimes F(U_i)$, endowed with the right action of G given by

$$\alpha(\bar{\xi} \otimes x) = (U_i^c)_{13}(\bar{\xi} \otimes x \otimes 1).$$

The product is defined similarly to the case of left actions. The involution is given by
$$(\bar{\xi} \otimes x)^* = \overline{(\xi^* \otimes \iota)\bar{R}_i(1)} \otimes S_x^* F(R_i)(1)$$

for $\bar{\xi} \otimes x \in \bar{H}_i \otimes F(U_i)$, where $S_x(y) = F_2(x \otimes y)$.

The left $(\operatorname{Rep} G)$-module category $\mathcal{D}_A$ is defined in the same way as before, but equivariant right Hilbert A-modules are now right $C(G)$-comodules. The spectral functor and the pair $(\mathcal{D}_A, X)$, where $X = A \in \mathcal{D}_A$, are related by $F(U) = \mathcal{D}_A(X, U \otimes X)$.

Remark 1.1. In [dCY13b], a right G-C*-algebra corresponding to a left module category $\mathcal{D}$ and $X \in \mathcal{D}$ was constructed as the completion of $\tilde{\mathcal{A}} = \bigoplus_i \mathcal{D}(U_i \otimes X, X) \otimes H_i$, where H_i has the coaction $\xi \mapsto U_i(\xi \otimes 1)$. This approach is of course equivalent to the one above, with the isomorphism $\tilde{\mathcal{A}} \to \mathcal{A}$ given by

$$\mathcal{D}(\bar{U}_i \otimes X, X) \otimes \bar{H}_i \to \bar{H}_i \otimes \mathcal{D}(X, U_i \otimes X),$$
$$S \otimes \bar{\xi} \mapsto (\iota \otimes \xi^*) R_i(1) \otimes (\iota_i \otimes S)(\bar{R}_i \otimes \iota_X).$$

In particular, when the weak tensor functor F is actually the fiber functor of G, $\tilde{\mathcal{A}}$ can be identified with $\mathbb{C}[G]$ on the nose, while the above map gives a right G-C*-algebra isomorphism $\mathbb{C}[G] \cong \mathcal{A}$.

Remark 1.2. Our correspondence between left actions and right module categories instead of left ones is more of a convention than a necessity. Given any right $(\operatorname{Rep} G)$-module category $\mathcal{D}$ we can reverse the directions of arrows in $\mathcal{D}$ to get a category $\mathcal{D}^{\operatorname{op}}$, and then define a left action of $\operatorname{Rep} G$ on $\mathcal{D}^{\operatorname{op}}$ using a contravariant functor $U \mapsto \bar{U}$. At the level of C*-algebras this corresponds to passing from a left action $\alpha \colon A \to C(G) \otimes A$ to the right action $a \mapsto (\iota \otimes R)(\alpha(a)_{21})$ on A^{op}, where R is the unitary antipode on $C(G)$. Concretely, the anti-isomorphism of the algebras corresponding to $(\mathcal{D}, X)$ and $(\mathcal{D}^{\operatorname{op}}, X^{\operatorname{op}})$ is given by

$$\bar{H}_i \otimes \mathcal{D}(X, X \otimes U_i) \to \bar{H}_i \otimes \mathcal{D}^{\operatorname{op}}(X^{\operatorname{op}}, U_i \otimes X^{\operatorname{op}}) = \bar{H}_i \otimes \mathcal{D}(X \otimes \bar{U}_i, X),$$
$$\bar{\xi} \otimes S \mapsto (\iota \otimes \xi^*) R_i(1) \otimes (\iota_X \otimes \bar{R}_i^*)(S \otimes \iota_{\bar{\imath}}).$$

1.3 Free actions

A left action $\alpha\colon A \to C(G) \otimes A$ of a compact quantum group on a unital C^*-algebra is called *free* [Ell00], if $(1 \otimes A)\alpha(A)$ is dense in $C(G) \otimes A$. By now there are many equivalent characterizations of freeness [dCY13a, BdCH17]. In particular, freeness is equivalent to any of the following conditions:

- the regular subalgebra $\mathcal{A} \subset A$ is a *Hopf–Galois extension* of $B = A^G$, that is, the *Galois map*

$$\Gamma\colon \mathcal{A} \otimes_B \mathcal{A} \to \mathbb{C}[G] \otimes \mathcal{A}, \quad a \otimes b \mapsto \alpha(a)(1 \otimes b),$$

   is bijective;

- for any $U \in \operatorname{Rep} G$, the *localized Galois map*

$$\Gamma_U\colon A_U \otimes_B A \to \mathbb{C}[G]_U \otimes A, \quad a \otimes b \mapsto \alpha(a)(1 \otimes b),$$

   is a unitary map of right Hilbert A-modules, where $\mathbb{C}[G]_U$ is the span of matrix coefficients of U, and $A_U = \{a \in A \mid \alpha(a) \in \mathbb{C}[G]_U \otimes A\}$ is the spectral subspace of A corresponding to U; here A_U has the structure of a right Hilbert B-module induced by the unique G-invariant conditional expectation $E\colon A \to B$, and $\mathbb{C}[G]_U$ is equipped with the scalar product using the Haar state;

- the spectral functor $\operatorname{Rep} G \to \operatorname{Corr}(B)$ is monoidal, that is, the natural isometries

$$(H_U \otimes A)^G \otimes_B (H_V \otimes A)^G \to (H_{U \oplus V} \otimes A)^G$$

   given by (1.2) are unitary.

Note that in the purely algebraic setting the equivalence of these conditions was established earlier by Ulbrich in the ergodic case [Ulb87] and by Schauenburg [Sch04, Section 2.5] in general.
Freeness for right actions can be characterized similarly, this time the Galois map being given by

$$\mathcal{A} \otimes_B \mathcal{A} \to \mathcal{A} \otimes \mathbb{C}[G], \quad a \otimes b \mapsto \alpha(a)(b \otimes 1).$$

Yet another characterization of freeness for left actions is as follows, which in the purely algebraic setting is due to Schneider [Sch90]. Needless to say, there is also a similar characterization for right actions.

PROPOSITION 1.3. *An action $\alpha\colon A \to C(G) \otimes A$ of a compact quantum group on a unital C^*-algebra A is free if and only if, for any G-equivariant right Hilbert A-module X, the map $X^G \otimes_{A^G} A \to X$, $x \otimes a \mapsto xa$, is a unitary isomorphism.*

Proof. We follow the idea of [Sch90], but there are several simplifications due to the cosemisimplicity of $\mathbb{C}[G]$. Since the map in the formulation is isometric, the only question is when it has dense image. Let us denote the subspace of regular vectors in X by $\mathcal{X}$, and look at the product map $\mu\colon X^G \otimes_B \mathcal{A} \to \mathcal{X}$, where $B = A^G$.

First suppose that the action is free. Using the inverse of the Galois map, we can consider the map

$$\nu\colon \mathcal{X} \to \mathcal{X} \otimes_B \mathcal{A}, \quad x \mapsto x_{(1)}\Gamma^{-1}(S^{-1}(x_{(0)}) \otimes 1)_1 \otimes \Gamma^{-1}(S^{-1}(x_{(0)}) \otimes 1)_2.$$

A standard computation shows that the image of this map is in $(\mathcal{X} \otimes_B \mathcal{A})^G$, where G acts only on the first factor of $\mathcal{X} \otimes_B \mathcal{A}$. Since the G-isotypic decomposition $\mathcal{X} = \bigoplus_{i \in \mathrm{Irr}(G)} X_i$ is compatible with the action of B, we have $(\mathcal{X} \otimes_B \mathcal{A})^G = X^G \otimes_B \mathcal{A}$. Then, using that the product map $\mathcal{A} \otimes_B \mathcal{A} \to \mathcal{A}$ equals $(\varepsilon \otimes \iota)\Gamma$, it is easy to check that ν, considered as a map $\mathcal{X} \to X^G \otimes_B \mathcal{A}$, is the inverse of μ.

Conversely, assume the map in the formulation is unitary for any X, or equivalently, the map $\mu\colon X^G \otimes_B \mathcal{A} \to \mathcal{X}$ is an isomorphism. Take $U \in \mathrm{Rep}\, G$. Consider the equivariant right Hilbert A-module $X = \mathbb{C}[G]_{\bar{U}} \otimes A$, where the inner product on $\mathbb{C}[G]_{\bar{U}}$ is defined by the Haar state. Then

$$X^G = \{S(a_{(0)}) \otimes a_{(1)} \mid a \in A_U\},$$

so $X^G \cong A_U$ as a right B-module. By assumption, the map

$$X^G \otimes_B \mathcal{A} \to \mathcal{X} = \mathbb{C}[G]_{\bar{U}} \otimes \mathcal{A}, \quad (S(a_{(0)}) \otimes a_{(1)}) \otimes b \mapsto S(a_{(0)}) \otimes a_{(1)}b,$$

is an isomorphism. But this shows that the map $A_U \otimes_B \mathcal{A} \to \mathbb{C}[G]_U \otimes \mathcal{A}$, $a \otimes b \mapsto a_{(0)} \otimes a_{(1)}b$, is an isomorphism. Hence the localized Galois map Γ_U is an isomorphism. $\square$

If an action is free, then it follows from [dCY13a, Corollary 4.2(2)] that, for any G-equivariant finitely generated right Hilbert A-module X, the module X^G is finitely generated over A^G. Therefore the 'only if' part of the above proposition implies that the functor $X \mapsto X^G$ defines an equivalence of the categories $\mathrm{Mod}_G\text{-}A$ and $\mathrm{Mod}\text{-}A^G$. Without the freeness assumption this is not even well-defined as a functor into the category of finitely generated modules. However, if the fixed point algebra is finite dimensional, the functor is well-defined and we get the following characterization of freeness.

PROPOSITION 1.4. *Let $\alpha\colon A \to C(G) \otimes A$ be an action of a compact quantum group on a unital C^*-algebra A. Assume that A^G is finite dimensional. Then the action is free if and only if $Y^G \neq 0$ for any nonzero $Y \in \mathcal{D}_A$.*

Proof. The 'only if' direction follows from the previous proposition and does not require finite dimensionality of $B = A^G$.

As for the converse, first, we claim that the finite dimensionality assumption on A^G implies that $\mathcal{D}_A$ is semisimple. Indeed, since any module in $\mathcal{D}_A$ is a

direct summand of $H_U \otimes A$ for some $U \in \operatorname{Rep} G$, it suffices to show that the endomorphism algebra $(B(H_U) \otimes A)^G$ of $H_U \otimes A \in \mathcal{D}_A$ is finite dimensional. But this is true, since any faithful G-invariant state φ on $B(H_U)$ defines a conditional expectation $\varphi \otimes \iota \colon (B(H_U) \otimes A)^G \to A^G$ of finite probabilistic index.

Let X be the object of $\mathcal{D}_A$ represented by A itself. Then the space $\mathcal{D}_A(X, Y) \cong Y^G$ is finite dimensional. In particular, Y^G is finitely generated over B for any $Y \in \mathcal{D}_A$. Assume that the action is not free. Then by the proof of the previous proposition, there exists $Y \in \mathcal{D}_A$ of the form $Y = \mathbb{C}[G]_U \otimes A$ such that the isometric map $Y^G \otimes_B A \to Y$, $x \otimes a \mapsto xa$, is not surjective. Since Y^G is finitely generated over B, the module $Y^G \otimes_B A$ is finitely generated over A, hence this map is a morphism in $\mathcal{D}_A$. Its image, the proper submodule $Y^G A \subset Y$, has a nonzero orthogonal complement Z. Clearly, $Z^G = 0$. $\qquad\square$

Remark 1.5. An equivalent way of formulating the above proposition is as follows: if a $(\operatorname{Rep} G)$-module category $\mathcal{D}$ is semisimple, then the action of G on the C*-algebra corresponding to a generating object $X \in \mathcal{D}$ is free if and only if every simple object of $\mathcal{D}$ is a subobject of X.

The following observation is useful for checking freeness in concrete examples.

PROPOSITION 1.6. *Assume* $\alpha \colon B \to C(G) \otimes B$ *is an action of a compact quantum group* G *on a unital C*-algebra* B, *and* $A \subset B$ *is an invariant C*-subalgebra containing the unit of* B *such that the action of* G *on* A *is free. Then the action of* G *on* B *is also free.*

Proof. Since the Galois map $\mathcal{B} \otimes_{B^G} \mathcal{B} \to \mathbb{C}[G] \otimes \mathcal{B}$, $b \otimes c \mapsto \alpha(b)(1 \otimes c)$, is always injective, we only have to check surjectivity. By the freeness of the action on A the image of this map contains $\mathbb{C}[G] \otimes 1$, hence it also contains $\mathbb{C}[G] \otimes \mathcal{B}$. $\qquad\square$

1.4 Conventions

We often fix representatives $(U_i)_i$ of isomorphism classes of simple objects in a rigid C*-tensor category, and then use the subscript i instead of U_i, so that we write d_i, $(R_i, \bar{R}_i)$ instead of $d(U_i)$, $(R_{U_i}, \bar{R}_{U_i})$, etc.

In order to simplify various expressions, we often omit the symbols $\otimes$ and $\oplus$ for tensor products of objects in tensor categories and module categories, as opposed to this preliminary section. We still write $\otimes$ for tensor products of morphisms and vector spaces.

When X and Y are objects in a rigid C*-tensor category (or in a rigid C*-2-category) and standard solutions for the corresponding conjugate equations are fixed, we take $\left((\iota_{\bar{Y}} \otimes R_X \otimes \iota_Y) R_X, (\iota_X \otimes \bar{R}_Y \otimes \iota_{\bar{X}}) \bar{R}_X \right)$ as a standard solution for $XY (= X \otimes Y)$. We also normalize the choice of standard solutions in $\operatorname{Rep} G$ as in (1.1). Thus, for $\xi \in H_i$, we have

$$(\xi^* \otimes \iota) \bar{R}_i(1) = \overline{\rho^{1/2}\xi}, \qquad\qquad (\iota \otimes \xi^*) R_i(1) = \overline{\rho^{-1/2}\xi}.$$

Recall once again that all compact quantum groups in this paper are assumed to be in the reduced form.

When we write formulas for linear maps on subspaces of vector spaces, such as $(H_U \otimes A)^G \subset H_U \otimes A$, we often consider only elementary tensors. By this we do not mean that the subspaces are spanned by such tensors, but that our formulas have obvious extensions to all the required tensors.

We use the Einstein summation convention, that is, if an index occurs once in an upper and once in a lower position in an expression, then we have a sum over this index.

2   Frobenius algebras and categories of modules

In this section we collect a number of results on algebra objects in C*-tensor categories and the corresponding categories of modules.

2.1   Frobenius algebras in tensor categories

Recall that a *Frobenius algebra* over $\mathbb{C}$ is a finite dimensional algebra A together with a linear functional φ such that the pairing $A \times A \to \mathbb{C}$, $(a, b) \mapsto \varphi(ab)$ is nondegenerate. There are a number of other conditions equivalent to nondegeneracy of the pairing, see, e.g., [Koc04]. One of them is that the vector space A admits a (necessarily unique) coalgebra structure with counit φ and coproduct $\Delta \colon A \to A \otimes A$ such that Δ is an A-bimodule map. Explicitly, the coproduct is defined by

$$\Delta(y) = yx^i \otimes x_i \left(= \sum_i yx^i \otimes x_i\right), \tag{2.1}$$

where $(x_i)_i$ is a basis in A and $(x^i)_i$ is the dual basis, so that $\varphi(x_i x^j) = \delta_{ij}$.

By a *Frobenius C*-algebra* we mean a finite dimensional C*-algebra A together with a faithful positive linear functional φ. Define a scalar product on A by $(x, y) = \varphi(y^*x)$. Then the coproduct Δ defined by (2.1) coincides with the adjoint m^* of the product map $m \colon A \otimes A \to A$, while φ equals the adjoint of the map $v \colon \mathbb{C} \to A$, $v(1) = 1$. This justifies the following definition.

DEFINITION 2.1 (cf. [Müg03, Vic11, BKLR15]). An algebra object (A, m, v), with product $m \colon A \otimes A \to A$ and unit $v \colon \mathbb{1} \to A$, in a C*-tensor category $\mathcal{C}$ is called a *C*-Frobenius algebra* if $m^* \colon A \to A \otimes A$ is an A-bimodule morphism, that is,

$$(m \otimes \iota)(\iota \otimes m^*) = m^*m = (\iota \otimes m)(m^* \otimes \iota).$$

Since the unit v is uniquely determined, we will often write an algebra in $\mathcal{C}$ as a pair (A, m).

In a similar way, given a C*-Frobenius algebra (A, m), we say that a left A-module $(X, m_X \colon A \otimes X \to X)$ in $\mathcal{C}$ is *unitary* if $m_X^* \colon X \to A \otimes X$ is an A-module morphism:

$$(m \otimes \iota)(\iota \otimes m_X^*) = m_X^* m_X. \tag{2.2}$$

By the above discussion any Frobenius C*-algebra is a C*-Frobenius algebra in Hilb_f. It is known that the converse is also true. More precisely, we have the following.

LEMMA 2.2. *Let (A, m, v) be a C*-Frobenius algebra in Hilb_f. Then the algebra A admits a unique involution such that it becomes a C*-algebra and such that for the linear functional $\varphi = v^*$ on it we have $(x, y) = \varphi(y^*x)$ for all $x, y \in A$. Also, a left A-module X in Hilb_f is unitary if and only if the representation of A on the Hilbert space X is $*$-preserving.*

Proof. Since m^* is a coproduct with counit φ, the pair (A, φ) is a Frobenius algebra, so the pairing defined by φ is nondegenerate. Hence we can define an anti-linear operation $a \mapsto a^*$ on A such that $(x, y) = \varphi(y^*x)$ for all $x, y \in A$. For $a, b, c \in A$ we have

$$(c, ab) = (m^*(c), a \otimes b) = ((a^* \otimes 1)m^*(c), 1 \otimes b) = (m^*(a^*c), 1 \otimes b) = (a^*c, b).$$

This shows that the left regular representation of A is $*$-preserving, so the $*$-operation is an involution and A is a C*-algebra.

Next, consider a left A-module $X \in \mathrm{Hilb}_f$. By definition, the representation of A on X is $*$-preserving if, for all $a \in A$ and $x, y \in X$, we have $(ax, y) = (x, a^*y)$. The right hand side can be written as

$$(x, m_X(a^* \otimes y)) = ((m \otimes \iota)(a \otimes m_X^*(x)), 1 \otimes y)$$
$$= ((v^*m \otimes \iota)(\iota \otimes m_X^*)(a \otimes x), y),$$

so the representation is $*$-preserving if and only if

$$m_X = (v^*m \otimes \iota)(\iota \otimes m_X^*). \tag{2.3}$$

This condition is equivalent to (2.2) in any C*-tensor category. Indeed, identity (2.3) follows from (2.2) by multiplying the latter by $v^* \otimes \iota$ on the left. Conversely, starting with (2.3) and using that $m = (v^*m \otimes \iota)(\iota \otimes m^*)$ by the Frobenius condition, we compute:

$$(m \otimes \iota)(\iota \otimes m_X^*) = (v^*m \otimes \iota \otimes \iota)(\iota \otimes m^* \otimes \iota)(\iota \otimes m_X^*)$$
$$= (v^*m \otimes \iota \otimes \iota)(\iota \otimes \iota \otimes m_X^*)(\iota \otimes m_X^*)$$
$$= m_X^*(v^*m \otimes \iota)(\iota \otimes m_X^*) = m_X^*m_X.$$

This completes the proof of the lemma. □

For a C*-Frobenius algebra A in a C*-tensor category $\mathcal{C}$, we denote by $A\text{-Mod}_{\mathcal{C}}$, or simply by $A\text{-Mod}$, the category of left unitary A-modules in $\mathcal{C}$. It is easy to check that $A\text{-Mod}$ is a C*-category [NY16, p. 418] using condition (2.3) and arguments similar to the proof of the above proposition, where we in effect checked that the fact m_X is a morphism in $A\text{-Mod}$ implies that m_X^* is a morphism in $A\text{-Mod}$ as well. In a similar way we can introduce C*-categories

$\mathrm{Mod}_{\mathcal{C}}$-$A$ of unitary right A-modules and $\mathrm{Bimod}_{\mathcal{C}}$-$A$ of unitary A-bimodules in $\mathcal{C}$.

For abstract C*-tensor categories it is natural to consider unitary isomorphisms of C*-Frobenius algebras. But for Hilb_f there is a larger natural class of isomorphisms.

LEMMA 2.3. *Let (A, φ_A) and (B, φ_B) be Frobenius C*-algebras. Assume that $T\colon A \to B$ is an isomorphism of algebras. Consider the adjoint map $T^*\colon B \to A$ with respect to the scalar products defined by φ_A and φ_B. Then T is $*$-preserving if and only if $T^*T\colon A \to A$ is a left A-module map.*

Proof. The map T^*T is a left A-module map if and only if

$$(T^*T(ab), c) = (aT^*T(b), c) \tag{2.4}$$

for all $a, b, c \in A$. The left hand side of (2.4) equals $(T(a)T(b), T(c))$, while the right hand side equals

$$(T^*T(b), a^*c) = (T(b), T(a^*)T(c)).$$

We thus see that (2.4) holds for all $b, c \in A$ if and only if $T(a)^* = T(a^*)$.   □

Motivated by this we give the following definition.

DEFINITION 2.4. We say that an invertible morphism $T\colon A \to A'$ is an *isomorphism of C*-Frobenius algebras* (A, m) and (A', m') in a C*-tensor category $\mathcal{C}$ if

$$m'(T \otimes T) = Tm \quad \text{and} \quad m(\iota \otimes T^*T) = T^*Tm.$$

It is straightforward to check that compositions and inverses of isomorphisms are again isomorphisms. Furthermore, if (A, m) is a C*-Frobenius algebra and $T\colon A \to A'$ is any invertible morphism satisfying $m(\iota \otimes T^*T) = T^*Tm$, then by letting $m' = Tm(T^{-1} \otimes T^{-1})$ we get a C*-Frobenius algebra (A', m').

Remark 2.5. Instead of requiring T^*T to be a left A-module morphism in the above definition we could require T^*T to be a right A-module morphism. This would change the notion of an isomorphism, but the isomorphism classes of C*-Frobenius algebras would remain the same. Indeed, assume $T\colon A \to A'$ is an isomorphism according to Definition 2.4. Consider the polar decomposition $T = u|T|$, so that $|T|$ is a left A-module morphism. We have a linear isomorphism

$$\mathrm{End}_{A\text{-Mod}}(A) \cong \mathrm{End}_{\mathrm{Mod}\text{-}A}(A), \quad S \mapsto \pi(S) = m(Sv \otimes \iota),$$

which can be characterized by the identity

$$m(S \otimes \iota) = m(\iota \otimes \pi(S)),$$

since

$$m(\iota \otimes \pi(S)) = m(\iota \otimes m)(\iota \otimes Sv \otimes \iota) = m(m \otimes \iota)(\iota \otimes Sv \otimes \iota)$$
$$= m(Sm \otimes \iota)(\iota \otimes v \otimes \iota) = m(S \otimes \iota).$$

Therefore if we let $\tilde{T} = u\pi(|T|)$, then $\tilde{T}^*\tilde{T} \in \pi(|T|)^*\pi(|T|) \in \mathrm{End}_{\mathrm{Mod}\text{-}A}(A)$ and

$$m' = Tm(T^{-1} \otimes T^{-1}) = um(|T|^{-1}u^* \otimes u^*)$$
$$= um(u^* \otimes \pi(|T|)^{-1}u^*) = \tilde{T}m(\tilde{T}^{-1} \otimes \tilde{T}^{-1}),$$

proving our claim.

The following subclasses of C*-Frobenius algebras are of particular interest, see again [BKLR15] for details.

DEFINITION 2.6. A C*-Frobenius algebra (A, m, v) in $\mathcal{C}$ is called

- *irreducible*, if A is simple as a left, equivalently, as a right, A-module;

- *simple*, if A is simple as an A-bimodule;

- *special*, or a *Q-system* [Lon94], if mm^* is scalar;

- *standard*, if the pair (m^*v, m^*v) is a standard solution of the conjugate equations for A, that is, if $\|m^*v\|^2$ equals the intrinsic dimension $d(A)$ of $A \in \mathcal{C}$.

Remark 2.7.
(i) We always have a linear isomorphism $\mathrm{Mor}_{\mathcal{C}}(\mathbb{1}, A) \cong \mathrm{End}_{A\text{-Mod}}(A)$, $T \mapsto m(\iota \otimes T)$, with the inverse $S \mapsto Sv$. Therefore irreducibility is equivalent to the condition $\dim \mathrm{Mor}_{\mathcal{C}}(\mathbb{1}, A) = 1$.

(ii) As mm^* is an A-bimodule morphism, a simple C*-Frobenius algebra is automatically a Q-system. In particular, this is true for irreducible C*-Frobenius algebras. Irreducible Q-systems are also called ergodic in [AdC15].

(iii) If (A, m, v) is an algebra in $\mathcal{C}$ such that mm^* is scalar, then it is a Q-system, see [LR97, Section 6] or [BKLR15, Lemma 3.5]. Similarly, once we assume that A is a Q-system, a left A-module X is unitary if and only if $m_X m_X^*$ is scalar, and then it is the same scalar as mm^*, see [BKLR15, Lemma 3.23] and [NY16, Lemma 6.1].

(iv) Once mm^* is assumed to be scalar, it is natural to fix a normalization of the pair (m, v). For example, we may require this scalar to be 1. Another natural choice, made in [NY16], is to require v to be an isometry.

(v) In [BKLR15] Q-systems are required to be standard, but we do not do this.

LEMMA 2.8. *Any C*-Frobenius algebra is unitarily isomorphic to a direct sum of simple C*-Frobenius algebras.*

Proof. Note that the C*-algebra $\mathrm{End}_{\mathrm{Bimod}\text{-}A}(A)$ is abelian, since A is a unit object in the tensor category $\mathrm{Bimod}\text{-}A$. More directly, if $S, T \in \mathrm{End}_{\mathrm{Bimod}\text{-}A}(A)$, then

$$STm = Sm(\iota \otimes T) = m(S \otimes T) = Tm(S \otimes \iota) = TSm,$$

and multiplying on the right by $v \otimes \iota$ we get $ST = TS$.

For every minimal projection $z \in \mathrm{End}_{\mathrm{Bimod}\text{-}A}(A)$, the subobject zA of A defined by z becomes a C*-Frobenius algebra, with product defined by the restriction of m to $zA \otimes zA$, and A is the direct sum of these algebras. $\square$

2.2 Standard Q-systems

Assume (A, m) is a C*-Frobenius algebra. Then $mm^* \colon A \to A$ is an A-bimodule morphism, and as was observed in [BKLR15, Lemma 3.5], this morphism is invertible, so that the product $(mm^*)^{-1/2}m \colon A \otimes A \to A$ defines an isomorphic Q-system. We strengthen this observation as follows.

THEOREM 2.9. *Any C*-Frobenius algebra in a C*-tensor category $\mathcal{C}$ is isomorphic to a standard Q-system.*

In particular, since isomorphisms of irreducible Q-systems are unitary up to scalar factors, any irreducible Q-system is standard. This has been already observed by Müger in [Müg03, Remark 5.6(3)]. The general result holds for similar reasons, but the proof requires a bit more work.

First of all, by Lemma 2.8 it suffices to prove the theorem for simple Q-systems. Let (A, m, v) be such a Q-system. We may assume that v is an isometry and $mm^* = \lambda\iota$. Since the object A in $\mathcal{C}$ is self-dual, by passing to the subcategory generated by A we may assume that $\mathcal{C}$ is rigid. We can then construct a rigid C*-2-category $\mathcal{B}$ of modules in $\mathcal{C}$ with the set $\{1, 2\}$ of 0-cells in a standard way [Yam04, Müg03, NY18]. Concretely, we take $\mathcal{B}_{11} = \mathcal{C}$, $\mathcal{B}_{22} = \mathrm{Bimod}\text{-}A$, $\mathcal{B}_{12} = \mathrm{Mod}\text{-}A$, and $\mathcal{B}_{21} = A\text{-Mod}$. The tensor products are defined over A when possible, otherwise they are taken in $\mathcal{C}$. For a discussion of unitarity of the tensor product $\otimes_A$ and a proof of (C*-)rigidity of $\mathcal{B}$, see, e.g., [NY16, NY18]. We only want to make two additional remarks. First, the assumption of standardness made in the above cited papers did not play an essential role for the construction of $\mathcal{B}$, the only change is that $d(A)$ in various formulas has to be replaced by λ. Second, it is important to remember that given modules $X \in \mathrm{Mod}\text{-}A$ and $Y \in A\text{-Mod}$, the structure morphism $P_{X,Y} \colon XY \to X \otimes_A Y$ for the tensor product over A is normalized so that $P_{X,Y}P_{X,Y}^* = \lambda\iota$, which guarantees the unitarity of $\otimes_A$.

Proof of Theorem 2.9. Using the above notation, consider A as an object X in $\mathcal{B}_{12} = \mathrm{Mod}\text{-}A$. As a conjugate object $\bar{X}$ we can take A considered as an object in $\mathcal{B}_{21} = A\text{-Mod}$. We have a solution $(R, \bar{R})$ of the conjugate equations for X defined by

$$R = m^* \colon A = \mathbb{1}_2 \to \bar{X}X = A \otimes A, \quad \bar{R} = v \colon \mathbb{1}_\mathcal{C} = \mathbb{1}_1 \to X\bar{X} = A \otimes_A A = A.$$

We can find a positive invertible morphism $T \in \operatorname{End}(\bar{X}) = \operatorname{End}_{A\text{-}\mathrm{Mod}}(A)$ such that the morphisms $R' = (T \otimes \iota)R$ and $\bar{R}' = (\iota \otimes T^{-1})\bar{R}$ form a standard solution of the conjugate equations for X. Then the formula

$$R_A = \bar{R}_A = (\iota \otimes R' \otimes \iota)\bar{R}'$$

defines a standard solution of the conjugate equations for $X\bar{X} = A$. Note that the morphism $\bar{R} \colon \mathbb{1} \to A = A \otimes_A A$ lifts to the morphism $\lambda^{-1}m^*v \colon \mathbb{1} \to A \otimes A$, while $\iota \otimes R \otimes \iota \colon A \otimes_A A \to A \otimes_A A \otimes A \otimes_A A$ is induced by the morphism $\iota \otimes m^*v \otimes \iota \colon A \otimes A \to A \otimes A \otimes A \otimes A$. Hence we have

$$R_A = (m \otimes m)(\iota \otimes (T \otimes \iota)m^*v \otimes \iota)\lambda^{-1}(\iota \otimes T^{-1})m^*v$$
$$= (T \otimes \iota)(m \otimes \iota)(\iota \otimes m^*m)(\iota \otimes v \otimes \iota)\lambda^{-1}m^*T^{-1}v$$
$$= (T \otimes \iota)m^*m\lambda^{-1}m^*T^{-1}v = (T \otimes \iota)m^*T^{-1}v.$$

But this means that by letting $m' = m(T \otimes \iota) = T^{-1}m(T \otimes T)$ and $v' = T^{-1}v$ we get an isomorphic C*-Frobenius algebra structure on A with $m'^*v' = R_A = \bar{R}_A$, so this new C*-Frobenius algebra is standard. As it is simple, it is automatically a Q-system. □

One advantage of working with standard Q-systems is the following result.

PROPOSITION 2.10. *Any isomorphism of standard Q-systems is unitary up to a scalar factor.*

Proof. Assume $T \colon (A, m, v) \to (A', m', v')$ is an isomorphism of standard Q-systems. Using scalar isomorphisms we may replace these Q-systems by isomorphic ones and assume that m and m' are coisometries. Then $\|v\|^2 = \|v'\|^2 = d(A)$ by standardness. We want to show that T is unitary. By taking the polar decomposition of T and replacing (A', m', v') by a unitarily isomorphic Q-system we may further assume that T is a positive morphism, so that in particular $A' = A$ as objects. We then have to show that $T = \iota$.
As $m'(T \otimes T) = Tm$ and $Tm = m(\iota \otimes T)$, we have $m'(T \otimes \iota) = m$, and then

$$\operatorname{Tr}_A(T^2) = v'^*m'(T^2 \otimes \iota)m'^*v' = v'^*mm^*v' = d(A).$$

Similarly $\operatorname{Tr}_A(T^{-2}) = d(A)$. By the Cauchy–Schwarz inequality we conclude that T^2 is the identity morphism, hence T is the identity morphism as well. □

2.3 Canonical invariant states

Let us now consider the C*-Frobenius algebras in the representation categories of compact quantum groups.
By a straightforward refinement of Lemma 2.2, the C*-Frobenius algebras in $\operatorname{Rep} G$ correspond to the pairs (A, φ) consisting of a finite dimensional right G-C*-algebra A and a G-invariant faithful positive linear functional φ on A (for Q-systems such a correspondence is explicitly stated in [AdC15, Proposition 3.4]). Then Theorem 2.9 and Proposition 2.10 for $\mathcal{C} = \operatorname{Rep} G$ translate into the following.

THEOREM 2.11. *For any finite dimensional (left or right) G-C^*-algebra A there exists a unique G-invariant faithful state φ on A such that if we define a scalar product on A using φ, then for the product map $m\colon A \otimes A \to A$ we have $mm^* = (\dim_q A)\iota$.*

Proof. By passing from A to A^{op} if necessary, we may assume that A is a right G-C^*-algebra, see Remark 1.2. Then the existence of φ is equivalent to the statement that any C^*-Frobenius algebra in $\operatorname{Rep} G$ is isomorphic to a standard Q-system such that the unit $v\colon \mathbb{1} \to A$ is an isometry. If φ' is another state as in the formulation, then by Proposition 2.10 the identity map $A \to A$ must be unitary with respect to the scalar products defined by φ and φ', so $\varphi = \varphi'$. □

We call the state φ given by the above theorem the *canonical invariant state* on A, and will usually denote it by φ_A. Unless stated otherwise, we will also always assume that A is equipped with the scalar product defined by φ_A.

Assume A is a finite dimensional right G-C^*-algebra. If X is a finitely generated right Hilbert A-module, then we can turn it into a finite dimensional unitary representation of G with respect to the scalar product $(x, y) = \varphi_A(\langle y, x\rangle_A)$. Any covariant representation of the pair (A, G) on a finite dimensional Hilbert space arises this way. By Lemma 2.2 this allows us to identify the category $\mathcal{D}_A = \operatorname{Mod}_G\text{-}A$ of finitely generated right Hilbert A-modules with the category $\operatorname{Mod}_{\operatorname{Rep} G}\text{-}A$ of right unitary A-modules in $\operatorname{Rep} G$, so from now on we will only use the lighter notation $\operatorname{Mod}_G\text{-}A$. Of course, for this identification any faithful G-invariant state on A can be used, but φ_A is the most natural choice. Similarly, we identify $\operatorname{Bimod}_G\text{-}A$ with $\operatorname{Bimod}_{\operatorname{Rep} G}\text{-}A$ and $A\text{-}\operatorname{Mod}_G$ with $A\text{-}\operatorname{Mod}_{\operatorname{Rep} G}$.

The existence of canonical invariant states is not difficult to establish directly, without relying on Theorem 2.9. In order to see this, assume as above that A is a finite dimensional C^*-algebra with an action $\alpha\colon A \to A \otimes C(G)$ of G. Denote by $\lambda(a)$ the operator of multiplication on the left by $a \in A$. Consider the representation π_α of the algebra $\mathcal{U}(G)$ on the space A given by $\pi_\alpha(\omega)a = \omega \triangleright a = (\iota \otimes \omega)\alpha(a)$. Recall also that we denote by $\rho \in \mathcal{U}(G)$ the Woronowicz character.

PROPOSITION 2.12. *For any finite dimensional right G-C^*-algebra A, we have*

$$\varphi_A(a) = (\dim_q A)^{-1}\operatorname{Tr}(\lambda(a)\pi_\alpha(\rho)) \quad \text{for all} \quad a \in A.$$

Proof. It is well known that $(\dim_q A)^{-1}\operatorname{Tr}(\lambda(\cdot)\pi_\alpha(\rho))$ is an invariant state. Therefore we only have to show that if we define a scalar product on A using this state, then $mm^* = (\dim_q A)\iota$, or equivalently, if we define a scalar product using the positive linear functional $\operatorname{Tr}(\lambda(\cdot)\pi_\alpha(\rho))$, then $mm^* = \iota$.
Consider the one-parameter group of automorphisms β_t of A defined by $\beta_t(a) = \rho^{it} \triangleright a$. Since A is finite dimensional, there exists a positive invertible element $b \in A$ such that $\beta_t = \operatorname{Ad} b^{it}$. Then $\pi_\alpha(\rho) = \operatorname{Ad} b$. Since A is a direct sum of full matrix algebras, it is then enough to show the following: if $c \in \operatorname{Mat}_n(\mathbb{C})$ is

a positive invertible matrix and we define a scalar product on $\mathrm{Mat}_n(\mathbb{C})$ using the positive linear functional $a \mapsto \mathrm{Tr}(\lambda(a)(\mathrm{Ad}\, c)) = \mathrm{Tr}(ac)\,\mathrm{Tr}(c^{-1})$, then for the product m on $\mathrm{Mat}_n(\mathbb{C})$ we have $mm^* = \iota$. But this is a straightforward computation. □

Remark 2.13. The above expression for φ_A can be interpreted as follows. Consider the unique G-invariant conditional expectation $E \colon A \to A^G$. The elements of A^G act by left multiplication on A, and this way we can consider A^G as a subalgebra of the endomorphism ring of the object $A \in \mathrm{Rep}\,G$. Hence the normalized categorical trace tr_A on this endomorphism ring defines a tracial state on A^G. (To be more precise, in order to consider A as an object of $\mathrm{Rep}\,G$, we have to define a scalar product on A using an invariant faithful state. But the trace on A^G that we get this way is independent of any choices.) Then $\varphi_A = \mathrm{tr}_A E$.

Remark 2.14. Another consequence of the proposition is that φ_A satisfies the KMS condition with respect to $(\beta_t)_t$. Indeed, we have $\pi_\alpha(\rho)^z \lambda(a) \pi_\alpha(\rho)^{-z} = \lambda(\rho^z \rhd a)$ and

$$
\begin{aligned}
\varphi_A(ab) &= \dim_q(A)^{-1} \mathrm{Tr}(\lambda(ab)\pi_\alpha(\rho)) \\
&= \dim_q(A)^{-1} \mathrm{Tr}(\lambda(b)\pi_\alpha(\rho)\lambda(a)\pi_\alpha(\rho)^{-1}\pi_\alpha(\rho)) \\
&= \varphi_A(b\beta_{-i}(a)).
\end{aligned}
$$

3 INVERTIBLE BIMODULE CATEGORIES

3.1 INVERTIBLE BIMODULE CATEGORIES AND MORITA–GALOIS OBJECTS

The notion of an invertible bimodule category was introduced in [ENO10]. Since relative tensor product of module categories over infinite C*-tensor categories requires some discussion, we will adopt the following definition, which is equivalent to the unitary version of the one in [ENO10] for finite rigid C*-tensor categories. We will return to relative tensor products in Section 4.

DEFINITION 3.1. A nonzero $\mathcal{C}_1$-$\mathcal{C}_2$-module category $\mathcal{D}$ over rigid C*-tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$ is called *invertible* if there exists a rigid C*-2-category $\mathcal{C}$ with the set $\{1, 2\}$ of 0-cells such that $\mathcal{C}_{11}$ is unitarily monoidally equivalent to $\mathcal{C}_1$, $\mathcal{C}_{22}$ is unitarily monoidally equivalent to $\mathcal{C}_2$, and $\mathcal{C}_{12}$ is unitarily equivalent to $\mathcal{D}$ as a $\mathcal{C}_1$-$\mathcal{C}_2$-module category.

Invertible bimodule categories can be defined more intrinsically without mentioning 2-categories. For this we need to recall a few definitions.
Let $\mathcal{C}$ be a rigid C*-tensor category and $\mathcal{D}$ be a right $\mathcal{C}$-module category. Then $\mathcal{D}$ is called *indecomposable* if it is not equivalent to a direct sum of two nonzero module categories. If $\mathcal{D}$ is semisimple as a C*-category, then $\mathcal{D}$ indecomposable if and only if every nonzero object $X \in \mathcal{D}$ is generating, meaning that any other object is a subobject of XU for some $U \in \mathcal{C}$.

The action of $\mathcal{C}$ on $\mathcal{D}$ is called *proper*, or *cofinite* [AdC15], if for any $X, Y \in \mathcal{D}$ we have $\mathcal{D}(X, YU_i) = 0$ for all but finitely many i, where $(U_i)_i$ are representatives of the isomorphism classes of simple objects in $\mathcal{C}$. Note that if $\mathcal{D}$ is indecomposable this can be relaxed to $\mathcal{D}(X, YU_i) = 0$ for all but finitely many i, for *some* nonzero X, Y.

Finally, recall that we denote by $\mathrm{End}_{\mathcal{C}}(\mathcal{D})$ the C*-tensor category with objects the unitary $\mathcal{C}$-module endofunctors of $\mathcal{D}$ and morphisms the uniformly bounded natural transformations between such endofunctors. The purely algebraic counterpart of this category is also denoted by $\mathcal{C}_{\mathcal{D}}^*$.

THEOREM 3.2. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be rigid C*-tensor categories and $\mathcal{D}$ be a nonzero $\mathcal{C}_1$-$\mathcal{C}_2$-module category. Then $\mathcal{D}$ is invertible if and only if the following conditions are satisfied:*

(a) *$\mathcal{D}$ is semisimple as a C*-category;*

(b) *the action of $\mathcal{C}_2$ on $\mathcal{D}$ is proper;*

(c) *the functor $\mathcal{C}_1 \to \mathrm{End}_{\mathcal{C}_2}(\mathcal{D})$ defined by the action of $\mathcal{C}_1$ on $\mathcal{D}$ is an equivalence of C*-tensor categories.*

Furthermore, if these conditions are satisfied, then $\mathcal{D}$ is indecomposable as a left $\mathcal{C}_1$-module category and as a right $\mathcal{C}_2$-module category.

Proof. Assume first that $\mathcal{D}$ is invertible. By passing to equivalent categories we may assume that we have a rigid C*-2-category $\mathcal{C}$ with the set $\{1, 2\}$ of 0-cells such that $\mathcal{C}_{11} = \mathcal{C}_1$, $\mathcal{C}_{22} = \mathcal{C}_2$ and $\mathcal{D} = \mathcal{C}_{12}$. Condition (a) is satisfied as $\mathcal{C}$ is rigid. Condition (b) is also satisfied, since if $\mathcal{D}(X, YU_i) \neq 0$ then U_i appears in the decomposition of $\bar{Y}X \in \mathcal{C}_2$ into a direct sum of simple objects. It remains to check (c).

Let us fix a nonzero $X \in \mathcal{C}_{12}$, and let F be a $\mathcal{C}_2$-module endofunctor on $\mathcal{C}_{12}$. Putting $Y = F(X)$, for any object $Z \in \mathcal{C}_{12}$ the isometry $d(X)^{-1/2}F(\bar{R}_X \otimes \iota_Z)$ induces a realization of $F(Z)$ as a direct summand of $F(X\bar{X}Z) \cong Y\bar{X}Z$. Thus, F is a direct summand of $Y\bar{X} \in \mathcal{C}_1$ in $\mathrm{End}_{\mathcal{C}_2}(\mathcal{C}_{12})$. It follows that we just need to show

$$\mathcal{C}_1(U, U') \cong \mathrm{Mor}_{\mathrm{End}_{\mathcal{C}_2}(\mathcal{C}_{12})}(U, U') \text{ for all } U, U' \in \mathcal{C}_1.$$

Thus, suppose that $(\eta_Z \colon UZ \to U'Z)_Z$ is a natural transformation of $\mathcal{C}_2$-module functors. This means we have $\eta_{ZV} = \eta_Z \otimes \iota_V$ for $Z \in \mathcal{C}_{12}$ and $V \in \mathcal{C}_2$. We claim that $\eta_0 = (\iota \otimes \mathrm{tr}_X)(\eta_X) \in \mathcal{C}(U, U')$ satisfies $\eta_Z = \eta_0 \otimes \iota_Z$ for $Z \in \mathcal{C}_{12}$. Indeed, we have

$$(\iota \otimes \mathrm{tr}_X)(\eta_X) \otimes \iota_Z = \frac{1}{d(X)}(\iota_{U'} \otimes \bar{R}_X^* \otimes \iota_Z)(\eta_X \otimes \iota_{\bar{X}Z})(\iota_U \otimes \bar{R}_X \otimes \iota_Z)$$

but $\eta_X \otimes \iota_{\bar{X}Z} = \eta_{X\bar{X}Z}$ and the naturality of η implies the right hand side of the above identity is equal to $d(X)^{-1}\eta_Z(\iota_U \otimes \bar{R}_X^* \bar{R}_X \otimes \iota_Z) = \eta_Z$.

Finally, $\mathcal{D}$ is indecomposable as a right $\mathcal{C}_2$-module category, since for any nonzero objects $X, Y \in \mathcal{D}$, the object Y is a subobject of $X(\bar{X}Y)$. Similarly, $\mathcal{D}$ is indecomposable as a left $\mathcal{C}_1$-module category.

Conversely, suppose that a nonzero $\mathcal{C}_1$-$\mathcal{C}_2$-module category $\mathcal{D}$ satisfies conditions (a–c). By Theorem A.1, there is an irreducible Q-system A in $\mathcal{C}_2$ such that $\mathcal{D}$ is unitarily equivalent to A-$\mathrm{Mod}_{\mathcal{C}_2}$ as a $\mathcal{C}_2$-module category. We can therefore consider A-$\mathrm{Mod}_{\mathcal{C}_2}$ as a $\mathcal{C}_1$-$\mathcal{C}_2$-module category equivalent to $\mathcal{D}$. On the other hand, see Section 2.2, there is a rigid C*-2-category $(\mathcal{C}'_{ij})_{i,j}$ such that $\mathcal{C}'_{12} = A$-$\mathrm{Mod}_{\mathcal{C}_2}$ and $\mathcal{C}_2 = \mathcal{C}'_{22}$. By the first part of the proof, the action of $\mathcal{C}'_{11}$ on $\mathcal{C}'_{12}$ defines an equivalence between $\mathcal{C}'_{11}$ and $\mathrm{End}_{\mathcal{C}_2}(\mathcal{C}'_{12})$. The condition (c) implies then that $\mathcal{C}_1$ is unitarily monoidally equivalent to $\mathcal{C}'_{11}$ in a compatible way with respect to the $\mathcal{C}_1$-$\mathcal{C}_2$- and $\mathcal{C}'_{11}$-$\mathcal{C}_2$-module category structures on $\mathcal{C}'_{12}$. Thus, $\mathcal{D}$ is an invertible bimodule category. $\qquad\square$

From the above proof we also see that if $\mathcal{C}$ is a rigid C*-tensor category and $\mathcal{D}$ is a nonzero semisimple indecomposable right $\mathcal{C}$-module category such that the action of $\mathcal{C}$ on $\mathcal{D}$ is proper, then $\mathrm{End}_{\mathcal{C}}(\mathcal{D})$ is a rigid C*-tensor category and the $\mathrm{End}_{\mathcal{C}}(\mathcal{D})$-$\mathcal{C}$-module category $\mathcal{D}$ is invertible, so that the 2-category structure and rigidity automatically follow from the one-sided module structure.

We now turn to representation categories of compact quantum groups. Our goal is to find an algebraic characterization of invertibility of bimodule categories. Throughout the rest of this section G_1 and G_2 denote compact quantum groups, and A denotes a unital C*-algebra. We also fix representatives of irreducible classes $(U_i)_i$ and $(V_j)_j$ in $\mathrm{Rep}\, G_1$ and $\mathrm{Rep}\, G_2$ respectively. Following our conventions in Section 1.4, their underlying Hilbert spaces are denoted by H_i and H_j.

DEFINITION 3.3. Let G_1 and G_2 be reduced compact quantum groups. A G_1-G_2-*Morita–Galois object* is a unital C*-algebra A together with commuting free actions $\alpha_1 \colon A \to C(G_1) \otimes A$ and $\alpha_2 \colon A \to A \otimes C(G_2)$ such that there is a G_1-G_2-equivariant isomorphism

$$A^{G_1} \otimes \mathcal{A} \cong A^{G_2} \otimes \mathcal{A}$$

of $A^{G_1} \otimes A^{G_2}$-$\mathcal{A}$-modules.

A few comments are in order. The subspace

$$\mathcal{A} = \{a \in A \mid \alpha_1(a) \in \mathbb{C}[G_1] \otimes A, \ \alpha_2(a) \in A \otimes \mathbb{C}[G_2]\}$$

is the regular subalgebra of A with respect to the joint action of G_1 and G_2, and the tensor product is the algebraic one (we will later see that these assumptions force A^{G_1} and A^{G_2} to be finite dimensional). The left $A^{G_1} \otimes A^{G_2}$-module structure on $A^{G_1} \otimes \mathcal{A}$ is given by

$$(a \otimes b)(x \otimes y) = ax \otimes by,$$

while on $A^{G_2} \otimes \mathcal{A}$ by

$$(a \otimes b)(x \otimes y) = bx \otimes ay.$$

Since the actions of G_1 and G_2 commute, A^{G_1} is a right G_2-C*-algebra, in particular, a right $C(G_2)$-comodule, so $A^{G_1} \otimes \mathcal{A}$ is a right $C(G_2)$-comodule. Similarly, $A^{G_2} \otimes \mathcal{A}$ is a left $C(G_1)$-comodule. We therefore require the isomorphism $A^{G_1} \otimes \mathcal{A} \cong A^{G_2} \otimes \mathcal{A}$ to respect this comodule structures.

Existence of an isomorphism $A^{G_1} \otimes \mathcal{A} \cong A^{G_2} \otimes \mathcal{A}$ as in the above definition can be reformulated as a compatibility condition on Frobenius algebra structures on A^{G_1} and A^{G_2}. Namely, we have the following result.

PROPOSITION 3.4. *Assume we are given commuting actions $\alpha_1 \colon A \to C(G_1) \otimes A$ and $\alpha_2 \colon A \to A \otimes C(G_2)$. Then a G_1-G_2-equivariant isomorphism $A^{G_1} \otimes \mathcal{A} \cong A^{G_2} \otimes \mathcal{A}$ of $A^{G_1} \otimes A^{G_2}$-$\mathcal{A}$-modules exists if and only if the following conditions hold:*

  (a) *the fixed point algebras A^{G_1} and A^{G_2} are finite dimensional;*

  (b) *there exist a faithful G_2-invariant state ψ_1 on A^{G_1} and a faithful G_1-invariant state ψ_2 on A^{G_2} such that if $m_1^*(1) = x^i \otimes x_i$, where $m_1 \colon A^{G_1} \otimes A^{G_1} \to A^{G_1}$ is the product map and the adjoint is computed with respect to $\psi_1 \otimes \psi_1$ and ψ_1, and similarly $m_2^*(1) = y^j \otimes y_j$ for the product m_2 and the state ψ_2 on A^{G_2}, then we have*

$$x^i y^j x_i \otimes y_j = \lambda 1 \otimes 1 \quad and \quad y^j x^i y_j \otimes x_i = \lambda 1 \otimes 1$$

  for a nonzero scalar λ.

Furthermore, if these conditions are satisfied, then

  (i) *the map*

$$S \colon A^{G_1} \otimes \mathcal{A} \to A^{G_2} \otimes \mathcal{A}, \quad a \otimes b \mapsto y^j \otimes a y_j b,$$

  is a G_1-G_2-equivariant isomorphism of $A^{G_1} \otimes A^{G_2}$-$\mathcal{A}$-modules, with the inverse given by $e \otimes f \mapsto \lambda^{-1} x^i \otimes e x_i f$;

  (ii) *as the states ψ_1 and ψ_2 we can take the canonical invariant states, in which case $\lambda = \dim_q A^{G_1} = \dim_q A^{G_2}$, where we consider A^{G_1} as a G_2-module and A^{G_2} as a G_1-module;*

  (iii) *the relative commutants $(A^{G_1})' \cap A^{G_2}$ and $(A^{G_2})' \cap A^{G_1}$ are trivial; in particular, A^{G_1} is a simple G_2-algebra and A^{G_2} is a simple G_1-algebra.*

Remark 3.5. The identities in (b) can be equivalently expressed as

$$x^i y x_i = \lambda \psi_2(y) 1, \quad y^j x y_j = \lambda \psi_1(x) 1 \quad (x \in A^{G_1}, y \in A^{G_2}). \qquad (3.1)$$

Indeed, we may, and will now and in the proof below, assume that the vectors x_i form a basis. Then the vectors x^i are characterized by $\psi_1(x_i x^k) = \delta_{ik}$, see Section 2, and similarly for the y_j's. Then $1 \otimes 1 = \psi_2(y^j) 1 \otimes y_j$, so the first identity in (b) is equivalent to $x^i y^j x_i = \lambda \psi_2(y^j) 1$ for all j.

Proof of Proposition 3.4. Assume we are given a G_1-G_2-equivariant isomorphism $T \colon A^{G_1} \otimes \mathcal{A} \to A^{G_2} \otimes \mathcal{A}$ of $A^{G_1} \otimes A^{G_2}$-$\mathcal{A}$-modules. By the G_2-equivariance it maps $1 \otimes 1$ into a vector $z^k \otimes z_k \in A^{G_2} \otimes A^{G_2}$, and then T, being a morphism of A^{G_1}-$\mathcal{A}$-modules, must be given by

$$T(a \otimes b) = z^k \otimes a z_k b.$$

We may assume that the vectors z^k are linearly independent. It follows then that they form a basis in A^{G_2}. In particular, A^{G_2} is finite dimensional, and for similar reasons A^{G_1} is finite dimensional as well, which proves (a).

Using that T is a morphism of left A^{G_2}-modules, we also see that $z^k \otimes z_k$ must be a central vector in the A^{G_2}-bimodule $A^{G_2} \otimes A^{G_2}$. Generally speaking, if e_{st}^r are matrix units in A^{G_2}, any central vector in $A^{G_2} \otimes A^{G_2}$ has the form $\xi = \sum_{r,s,t} e_{st}^r v \otimes e_{ts}^r$ for a uniquely defined $v \in A^{G_2}$. Moreover, the slices $(\iota \otimes \omega)(\xi)$ for $\omega \in (A^{G_2})^*$ span A^{G_2} if and only if v is invertible.

Now, take any faithful G_1-invariant state ψ_2 on A^{G_2} and write $m_2^*(1) = y^j \otimes y_j$ with respect to ψ_2. By the above discussion, the map

$$S \colon A^{G_1} \otimes \mathcal{A} \to A^{G_2} \otimes \mathcal{A}, \quad a \otimes b \mapsto y^j \otimes a y_j b,$$

has the form $S(\zeta) = T(\zeta)(v \otimes 1)$ for some invertible element $v \in A^{G_2}$. Hence S is an isomorphism of $A^{G_1} \otimes A^{G_2}$-$\mathcal{A}$-modules. It is easy to see that this isomorphism is G_1-G_2-equivariant (note that the vector $m_2^*(1)$ is G_1-invariant, as m_2 is a G_1-equivariant map and the state ψ_2 is invariant).

Consider now the inverse map S^{-1}. By the same considerations as above, if we fix a faithful G_2-invariant state $\tilde{\psi}_1$ on A^{G_1} and write $m_1^*(1) = \tilde{x}^i \otimes \tilde{x}_i$ with respect to $\tilde{\psi}_1$, then S^{-1} has the form

$$S^{-1}(e \otimes f) = \tilde{x}^i u \otimes e \tilde{x}_i f \quad (e \in A^{G_2}, f \in \mathcal{A})$$

for an invertible element $u \in A^{G_1}$. Let ψ denote the linear functional $\tilde{\psi}_1(\cdot \, u^{-1})$ on A^{G_1}, and note that $\psi(\tilde{x}_i \tilde{x}^k u) = \delta_{ik}$. Then we have

$$\tilde{x}^i u \otimes y^j \tilde{x}_i y_j = S^{-1}(S(1 \otimes 1)) = 1 \otimes 1,$$
$$y^j \otimes \tilde{x}^i u y_j \tilde{x}_i = S(S^{-1}(1 \otimes 1)) = 1 \otimes 1.$$

As in Remark 3.5, this is equivalent to

$$\tilde{x}^i u y \tilde{x}_i = \psi_2(y) 1 \quad \text{and} \quad y^j x y_j = \psi(x) 1 \quad (x \in A^{G_1}, y \in A^{G_2}). \tag{3.2}$$

We may assume that the vectors y^j form an orthonormal basis in A^{G_2}. Then $y_j = y^{j*}$, and as $\psi(x) 1 = \sum_j y^j x y^{j*}$, we conclude that ψ is a G_2-invariant positive linear functional. As the pairing defined by ψ is nondegenerate, this functional is faithful. Put $\lambda = \psi(1)$ and $\psi_1 = \lambda^{-1} \psi$. If we use ψ_1 to define m_1^*, we get $m_1^*(1) = \lambda \tilde{x}^i u \otimes \tilde{x}_i$. Together with (3.2) this shows that condition (b) is satisfied. Note also that we have proved (i) along the way.

Conversely, if (a) and (b) are satisfied, then we get the required structure isomorphism for a G_1-G_2-Morita–Galois object, as described in (i).

Next, for (ii), from the above considerations we see that as the state ψ_2 we can take any faithful G_1-invariant state. If we take the canonical invariant state, then from $y^j y_j = \lambda 1$ we get $\lambda = \dim_q A^{G_2}$. But then the identity $x^i x_i = \lambda 1$ implies that $\dim_q A^{G_1} \leq \lambda = \dim_q A^{G_2}$. Similarly, if we start with the canonical invariant state on A^{G_1}, we get $\dim_q A^{G_1} \geq \dim_q A^{G_2}$. Therefore $\dim_q A^{G_1} = \dim_q A^{G_2}$ and if we take the canonical invariant state on one algebra, then we have to take the canonical invariant state on the other algebra as well.

Finally, for (iii), if $y \in (A^{G_1})' \cap A^{G_2}$, then

$$\lambda y = x^i x_i y = x^i y x_i = \lambda \psi_2(y) 1,$$

so y is scalar. In particular, there are no non-scalar G_1-invariant elements in the center of A^{G_2}, so A^{G_2} is a simple G_1-algebra. Similarly, $(A^{G_2})' \cap A^{G_1} = \mathbb{C}1$ and A^{G_1} is a simple G_2-algebra. $\qquad\blacksquare$

Remark 3.6. Another consequence of (3.1) is that, if we define A-valued inner products on $A^{G_i} \otimes \mathcal{A}$ by $\langle b \otimes a, b' \otimes a' \rangle_A = \psi_i(b^* b') a^* a'$, then the map S of (i) becomes a scalar multiple of a unitary. Indeed, for $a, a' \in \mathcal{A}$ and $b, b' \in A^{G_1}$, we have

$$\langle y^j \otimes b y_j a, y^k \otimes b' y_k a' \rangle_A = \psi_2(y^{j*} y^k) a^* y_j^* b^* b' y_k a'.$$

Since we can arrange $y^k = y_k^*$ as in the above proof, the right hand side equals

$$\psi_2(y_j y^k) a^* y^j b^* b' y_k a' = a^* y^j b^* b' y_j a' = \lambda \psi_1(b^* b') a^* a' = \lambda \langle b \otimes a, b' \otimes a' \rangle_A.$$

In particular, S and its inverse extend to isomorphisms of equivariant right Hilbert A-modules. Conversely, starting from a G_1-G_2-C*-algebra A, if we assume that the actions are free, A^{G_1} and A^{G_2} are finite dimensional, and that there is an isomorphism of equivariant $(A^{G_1} \otimes A^{G_2})$-$A$-correspondences $A^{G_1} \otimes A \to A^{G_2} \otimes A$, then taking the regular parts, we can verify the Morita–Galois conditions for A.

The following is our main result.

THEOREM 3.7. *Assume that we are given commuting actions* $\alpha_1 \colon A \to C(G_1) \otimes A$ *and* $\alpha_2 \colon A \to A \otimes C(G_2)$ *of reduced compact quantum groups* G_1 *and* G_2 *on a unital C*-algebra* A. *Consider the corresponding category* $\mathcal{D}_A$ *of finitely generated* G_1-G_2-equivariant right Hilbert A-modules. *Then the* $(\mathrm{Rep}\, G_2)$-$(\mathrm{Rep}\, G_1)$-*module category* $\mathcal{D}_A$ *is invertible if and only if* A *is a* G_1-G_2-*Morita–Galois object.*

By the Tannaka–Krein type correspondence for quantum group actions discussed in Section 1.2, we then get the following corollary.

Corollary 3.8. *For any reduced compact quantum groups G_1 and G_2, there is a bijective correspondence between the equivalence classes of invertible $(\operatorname{Rep} G_2)$-$(\operatorname{Rep} G_1)$-module categories and the G_1-G_2-equivariant Morita equivalence classes of G_1-G_2-Morita–Galois objects. We also have a bijective correspondence between the equivalence classes of pairs $(\mathcal{D}, X)$, consisting of an invertible $(\operatorname{Rep} G_2)$-$(\operatorname{Rep} G_1)$-category $\mathcal{D}$ and a nonzero object $X \in \mathcal{D}$, and the isomorphism classes of G_1-G_2-Morita–Galois objects.*

We divide the proof of the theorem into several parts.

3.2   From invertible bimodule categories to bi-Morita–Galois objects

Assume first that $\mathcal{D}_A$ is invertible. Consider the corresponding rigid C*-2-category $\mathcal{C}$ with the set $\{1, 2\}$ of 0-cells such that $\mathcal{C}_1 = \mathcal{C}_{11}$ is equivalent to $\operatorname{Rep} G_1$, $\mathcal{C}_2 = \mathcal{C}_{22}$ is equivalent to $\operatorname{Rep} G_2$, and $\mathcal{D}_A$ is equivalent to $\mathcal{C}_{21}$ as a $\mathcal{C}_2$-$\mathcal{C}_1$-module category. In order to simplify the exposition we are not going to distinguish between $\mathcal{C}_i$ and $\operatorname{Rep} G_i$, although to be pedantic we should either explicitly use our fixed unitary monoidal equivalences $\operatorname{Rep} G_i \to \mathcal{C}_i$ in all the formulas below or work with bicategories instead of 2-categories, that is, assume that $\mathcal{C}$ has nontrivial associativity morphisms.

Let $X \in \mathcal{C}_{21}$ be the object corresponding to A. From now on we will think of A as the result of the construction of a G_1-G_2-C*-algebra from the pair $(\mathcal{C}_{21}, X)$. We will see that the required isomorphism $A^{G_1} \otimes \mathcal{A} \cong A^{G_2} \otimes \mathcal{A}$ follows from the equality $(X\bar{X})X = X(\bar{X}X)$ in $\mathcal{C}_{21}$, while freeness of the actions follows from the indecomposability of $\mathcal{C}_{21}$ as a one-sided module category.

We start by establishing the freeness. The regular subalgebra $\mathcal{A} \subset A$ is

$$\bigoplus_{\substack{i \in \operatorname{Irr}(G_1) \\ j \in \operatorname{Irr}(G_2)}} \bar{H}_i \otimes \bar{H}_j \otimes \mathcal{C}_{21}(X, V_j X U_i).$$

Recall that $C(G_1)$ coacts on the left by $(U_i^c)_{21}^*$, while $C(G_2)$ coacts on the right by V_j^c.

By construction, we have

$$A^{G_2} = \bigoplus_{i \in \operatorname{Irr}(G_1)} \bar{H}_i \otimes \mathcal{C}_{21}(X, XU_i), \quad A^{G_1} = \bigoplus_{j \in \operatorname{Irr}(G_2)} \bar{H}_j \otimes \mathcal{C}_{21}(X, V_j X). \quad (3.3)$$

In other words, the fixed point algebras are the algebras corresponding to the object X in the category $\mathcal{C}_{21}$ regarded as a one-sided module category either over $\operatorname{Rep} G_1$ or over $\operatorname{Rep} G_2$. The joint fixed point subalgebra $(A^{G_1})^{G_2}$ is isomorphic to $\mathcal{C}_{21}(X) = \mathcal{C}_{21}(X, X)$, so G_1 and G_2 act jointly ergodically if and only if X is simple.

Consider a unitary equivalence $\mathcal{D}_A \to \mathcal{C}_{21}$ of bimodule categories provided by the Tannaka–Krein correspondence for actions. Up to a natural isomorphism, it

is described by the following properties, see [Nes14, Section 3]. For $U \in \operatorname{Rep} G_1$ and $V \in \operatorname{Rep} G_2$, we put

$$F(H_U \otimes H_V \otimes A) = VXU,$$

and take the morphism $VF(A)U \to F(H_U \otimes H_V \otimes A)$ required by the definition of a bimodule functor to be the identity. To describe the action of F on morphisms, consider a morphism $T \colon X \to V_j X U_i$ in $\mathcal{C}_{21}$. There is a unique morphism $\tilde{T} \colon A \to H_i \otimes H_j \otimes A$ in $\mathcal{D}_A$ mapping $1 \in A$ into

$$\sum_{\alpha,\beta} \xi_\alpha \otimes \zeta_\beta \otimes \bar{\xi}_\alpha \otimes \bar{\zeta}_\beta \otimes T \in H_i \otimes H_j \otimes \bar{H}_i \otimes \bar{H}_j \otimes \mathcal{C}_{21}(X, V_j X U_i) \subset H_i \otimes H_j \otimes \mathcal{A},$$

where $(\xi_\alpha)_\alpha$ and $(\zeta_\beta)_\beta$ are orthonormal bases in H_i and H_j respectively. Then we require $F(\tilde{T}) = T$.

LEMMA 3.9. *The strict* $(\operatorname{Rep} G_2)$-*module functor* $\mathcal{D}_A \to \mathcal{D}_{A^{G_1}}$, $Y \mapsto Y^{G_1}$, *and strict* $(\operatorname{Rep} G_1)$-*module functor* $\mathcal{D}_A \to \mathcal{D}_{A^{G_2}}$, $Y \mapsto Y^{G_2}$, *are equivalences of categories.*

Proof. We will only prove the first statement. Denote the functor $\mathcal{D}_A \to \mathcal{D}_{A^{G_1}}$, $Y \mapsto Y^{G_1}$, by E. As we already observed above, the G_2-C*-algebra A^{G_1} corresponds to the $(\operatorname{Rep} G_2)$-module category $\mathcal{C}_{21}$ and object X. It follows that similarly to the functor $F \colon \mathcal{D}_A \to \mathcal{C}_{21}$ we have a $(\operatorname{Rep} G_2)$-module functor $\tilde{F} \colon \mathcal{D}_{A^{G_1}} \to \mathcal{C}_{21}$, $\tilde{F}(H_V \otimes A^{G_1}) = VX$, defining an equivalence of categories. We obviously have $E\tilde{F} = F$ on the full subcategory of $\mathcal{D}_A$ consisting of the modules $H_V \otimes A$. Since X generates $\mathcal{C}_{21}$ as a $(\operatorname{Rep} G_2)$-module category and both F and $\tilde{F}$ are equivalences of categories, it follows that E is an equivalence of categories as well. □

LEMMA 3.10. *The actions of G_1 and G_2 on A are separately free.*

Proof. Let us only prove freeness of the action of G_2. By Proposition 1.4 it suffices to show that $Y^{G_2} \neq 0$ for any nonzero G_2-equivariant finitely generated right Hilbert A-module Y. Furthermore, the proof of that proposition respects the additional action of G_1 on A. In other words, if the action of G_2 is not free, then the proof shows that there exists a nonzero $Y \in \mathcal{D}_A$ such that $Y^{G_2} = 0$. But this contradicts the previous lemma. □

Let us now study the fixed point algebra A^{G_1} in more detail. Consider the object $X\bar{X} \in \operatorname{Rep} G_2$. It has the structure of a standard Q-system given by

$$m = d(X)^{1/2}(\iota_X \otimes R_X^* \otimes \iota_{\bar{X}}), \quad v = d(X)^{-1/2}\bar{R}_X.$$

In other words, if we use the picture of right unitary $C(G_2)$-comodules for $\operatorname{Rep} G_2$, we can view $X\bar{X}$ as a right G_2-C*-algebra with the scalar product

defined by the canonical invariant state. It can be reconstructed from the left $(\mathrm{Rep}\,G_2)$-module category $\mathrm{Mod}_{G_2}\text{-}X\bar{X}$, so we have a canonical isomorphism

$$X\bar{X} \cong \bigoplus_{j\in\mathrm{Irr}(G_2)} \bar{H}_j \otimes \mathrm{Mor}_{\mathrm{Mod}_{G_2}\text{-}X\bar{X}}(X\bar{X}, V_j X\bar{X}). \tag{3.4}$$

On the other hand, the functor $Y \mapsto Y\bar{X}$ defines a unitary strict $(\mathrm{Rep}\,G_2)$-module equivalence between $\mathcal{C}_{21}$ and $\mathrm{Mod}_{G_2}\text{-}X\bar{X}$, which is an observation going back to [Müg03, Proposition 4.5]. Therefore comparing (3.4) with (3.3) we get an isomorphism $\theta\colon X\bar{X} \to A^{G_1}$ of G_2-C*-algebras. If we as usual equip A^{G_1} with the scalar product defined by the canonical invariant state, θ becomes a unitary isomorphism of the standard Q-systems $X\bar{X}$ and A^{G_1} in $\mathrm{Rep}\,G_2$.
The particular isomorphism θ that we have defined has the following important property. As in the proof of Lemma 3.9, consider a $(\mathrm{Rep}\,G_2)$-module functor $\tilde{F}\colon \mathcal{D}_{A^{G_1}} \to \mathcal{C}_{21}$ allowing us to reconstruct A^{G_1} as in (3.3). Composing it with the functor $\mathcal{C}_{21} \to \mathrm{Rep}\,G_2$, $Y \mapsto Y\bar{X}$, we get a $(\mathrm{Rep}\,G_2)$-module functor $\tilde{\tilde{F}}\colon \mathcal{D}_{A^{G_1}} \to \mathrm{Rep}\,G_2$ such that $\tilde{\tilde{F}}(H_V \otimes A^{G_1}) = V X\bar{X}$.

LEMMA 3.11. *For any $V \in \mathrm{Rep}\,G_2$ and any morphism $S\colon H_V \otimes A^{G_1} \to A^{G_1}$ in $\mathcal{D}_{A^{G_1}}$, the following diagram commutes:*

$$\begin{array}{ccc} V X\bar{X} & \xrightarrow{\ \tilde{\tilde{F}}(S)\ } & X\bar{X} \\ {\scriptstyle \iota_V\otimes\theta}\downarrow & & \downarrow{\scriptstyle \theta} \\ H_V \otimes A^{G_1} & \xrightarrow{\ \ S\ \ } & A^{G_1} \end{array}$$

Proof. This is an immediate consequence of the definitions, making the following argument essentially tautological.
It is enough to consider $V = V_j$. Take a morphism $T\colon X \to V_j X$. Let $\tilde{T}\colon A^{G_1} \to H_j \otimes A^{G_1}$ be the morphism in $\mathcal{D}_{A^{G_1}}$ mapping $1 \in A^{G_1}$ into $\sum_\beta \zeta_\beta \otimes \bar{\zeta}_\beta \otimes T$, where $(\zeta_\beta)_\beta$ is an orthonormal basis in H_j. Then by definition we have $\tilde{\tilde{F}}(\tilde{T}) = T \otimes \iota_{\bar{X}}$. In terms of decomposition (3.4) this means that $\tilde{\tilde{F}}(\tilde{T})$ maps the unit of $X\bar{X}$ into $\sum_\beta \zeta_\beta \otimes \bar{\zeta}_\beta \otimes (T \otimes \iota_{\bar{X}}) \in H_j \otimes X\bar{X}$. Applying $\iota_j \otimes \theta$ to the last element we get $\sum_\beta \zeta_\beta \otimes \bar{\zeta}_\beta \otimes T = \tilde{T}(1)$. Therefore

$$\tilde{T}\theta = (\iota_j \otimes \theta)\tilde{\tilde{F}}(\tilde{T})\colon X\bar{X} \to H_j \otimes A^{G_1}.$$

Since any morphism $S\colon H_j \otimes A^{G_1} \to A^{G_1}$ in $\mathcal{D}_{A^{G_1}}$ has the form $\tilde{T}^*$ for some $T\colon X \to V_j X$, this proves the lemma. $\qquad\square$

Note that this lemma implies that θ extends to a natural isomorphism of the $(\mathrm{Rep}\,G_2)$-module functors $\tilde{\tilde{F}}$ and the forgetful functor $\mathcal{D}_{A^{G_1}} \to \mathrm{Rep}\,G_2$.

LEMMA 3.12. *Consider the multiplication map $m_1\colon A^{G_1} \otimes A \to A$. Then*

$$F(m_1)(\theta \otimes \iota_X) = d(X)^{1/2}(\iota_X \otimes R_X^*)\colon X\bar{X}X \to X.$$

Proof. As in the proof of Lemma 3.9, consider the functor $E\colon \mathcal{D}_A \to \mathcal{D}_{A^{G_1}}$. Since $\tilde{F}E = F$ on the modules $H_V \otimes A$, it suffices to show that

$$\tilde{F}(m_1)(\theta \otimes \iota_X) = d(X)^{1/2}(\iota_X \otimes R_X^*),$$

where now m_1 denotes the multiplication map on A^{G_1}. Applying the functor $Y \mapsto Y\bar{X}$ we have to check that

$$\tilde{\tilde{F}}(m_1)(\theta \otimes \iota_{X\bar{X}}) = d(X)^{1/2}(\iota_X \otimes R_X^* \otimes \iota_{\bar{X}}).$$

By the previous lemma the left hand side equals $\theta^* m_1(\theta \otimes \theta)$, which is exactly the right hand side, since θ is an isomorphism of Q-systems. $\qquad\square$

This lemma characterizes the unitary isomorphism θ. Indeed, any other isomorphism has the form θu, where u is a unitary automorphism of $X\bar{X}$. Then

$$(\iota_X \otimes R_X^*)(u \otimes \iota_{\bar{X}}) = \iota_X \otimes R_X^*,$$

which implies $u = \iota$.

Similar arguments apply to A^{G_2} and $\bar{X}X$. The main difference is that we have to use the picture of left unitary $C(G_1)$-comodules for $\operatorname{Rep} G_1$, and since the tensor product of left $C(G_1)$-comodules corresponds to the opposite tensor product of representations of $\operatorname{Rep} G_1$, we have to replace the product on A^{G_2} with the opposite one in order to get a C*-Frobenius algebra in $\operatorname{Rep} G_1$. As usual we equip A^{G_2} with the scalar product defined by the canonical invariant state, so $(a,b) = \varphi_{A^{G_2}}(b^*a)$ (where b^*a denotes the original product). Then $(A^{G_2})^{\operatorname{op}}$ becomes a standard Q-system in $\operatorname{Rep} G_1$ and we get the following result.

LEMMA 3.13. *Consider the standard Q-system $(\bar{X}X, m = d(X)^{1/2}(\iota_{\bar{X}} \otimes \bar{R}_X^* \otimes \iota_X), v = d(X)^{-1/2} R_X)$ in $\operatorname{Rep} G_1$. Then there exists a unique unitary isomorphism $\theta'\colon \bar{X}X \to (A^{G_2})^{\operatorname{op}}$ of standard Q-systems such that for the product map $m_2\colon A^{G_2} \otimes A \to A$ we have*

$$F(m_2)(\iota_X \otimes \theta') = d(X)^{1/2}(\bar{R}_X^* \otimes \iota_X)\colon X\bar{X}X \to X.$$

We are now ready to establish the key property of the algebra A.

LEMMA 3.14. *There is a G_1-G_2-equivariant isomorphism $A^{G_1} \otimes A \cong A^{G_2} \otimes A$ of $(A^{G_1} \otimes A^{G_2})$-$A$-modules.*

Proof. Consider the modules $X_1 = A^{G_1} \otimes A$ and $X_2 = A^{G_2} \otimes A$. They are A^{G_1}-$(A^{G_2})^{\operatorname{op}}$-modules in the category $\mathcal{D}_A$. Using the isomorphisms θ and θ' we can equivalently view X_1 and X_2 as $X\bar{X}$-$\bar{X}X$-modules in $\mathcal{D}_A$. Then the bimodule functor $F\colon \mathcal{D}_A \to \mathcal{C}_{21}$ allows us to introduce $X\bar{X}$-$\bar{X}X$-module structures on $F(X_1)$ and $F(X_2)$, hence also on $(\theta^* \otimes \iota_X)F(X_1)$ and $(\iota_X \otimes \theta'^*)F(X_2)$. Let us consider them in more detail.

We have $F(X_1) = A^{G_1}X$. The left A^{G_1}-module structure on $A^{G_1}X$ comes from the multiplication on A^{G_1}. Hence the left $X\bar{X}$-module structure on $(\theta^* \otimes$

$\iota_X)F(X_1) = X\bar{X}X$ also comes from the multiplication on $X\bar{X}$. On the other hand, the right $(A^{G_2})^{\mathrm{op}}$-structure on $A^{G_1}X$ is given by

$$\iota_{A^{G_1}} \otimes F(m_2) \colon A^{G_1}XA^{G_2} \to A^{G_1}X.$$

Using Lemma 3.13 we conclude that the right $\bar{X}X$-module structure on $(\theta^* \otimes \iota_X)F(X_1) = X\bar{X}X$ comes from the multiplication on $\bar{X}X$. Similar arguments apply to $(\iota_X \otimes \theta'^*)F(X_2)$.
We thus have the equalities

$$(\theta^* \otimes \iota_X)F(X_1) = X\bar{X}X = (\iota_X \otimes \theta'^*)F(X_2)$$

of $X\bar{X}$-$\bar{X}X$-modules. Hence the unique isomorphism $\pi \colon X_1 \to X_2$ in $\mathcal{D}_A$ such that

$$F(\pi) = (\iota_X \otimes \theta')(\theta^* \otimes \iota_X)$$

must be an isomorphism of $X\bar{X}$-$\bar{X}X$-modules, or equivalently, of A^{G_1}-$(A^{G_2})^{\mathrm{op}}$-modules. □

This finishes the proof of Theorem 3.7 in one direction.

Remark 3.15. In the above proof we used a functor $F \colon \mathcal{D}_A \to \mathcal{C}_{21}$. We could have equally well used the functor going in the opposite direction defined in [dCY13a]. Namely, we have a functor mapping $Y \in \mathcal{C}_{21}$ into a completion of

$$\mathcal{E}_Y = \bigoplus_{i,j} \bar{H}_i \otimes \bar{H}_j \otimes \mathcal{C}_{21}(X, V_jYU_i).$$

This functor has the obvious action on morphisms. However, its bimodule functor structure is a bit more difficult to describe. One minor advantage of using this functor is that we would be able to compute an isomorphism $A^{G_1} \otimes A \cong A^{G_2} \otimes A$ rather than merely prove its existence.

Before we turn to the proof of the theorem in the opposite direction, let us finish this section with the following observation.

PROPOSITION 3.16. *The canonical invariant state $\varphi_{A^{G_1}}$ on A^{G_1} is given by the composition of the G_2-invariant conditional expectation $A^{G_1} \to (A^{G_1})^{G_2} = \mathcal{C}_{21}(X)$ with the normalized categorical trace tr_X on $\mathcal{C}_{21}(X)$. Similarly for the canonical invariant state on A^{G_2}. In particular, there exists a unique G_1-G_2-invariant state φ on A such that its restrictions to A^{G_1} and A^{G_2} coincide with the canonical invariant states $\varphi_{A^{G_1}}$ and $\varphi_{A^{G_2}}$, respectively.*

We call this φ the *canonical invariant state* on A.

Proof. Take $S \in (A^{G_1})^{G_2} = \mathcal{C}_{21}(X)$. We have to show that $\varphi_{A^{G_1}}(S) = \mathrm{tr}_X(S)$. By Remark 2.13 and definition of the product in A^{G_1}, we have

$$\varphi_{A^{G_1}}(S) = (\dim_q A^{G_2})^{-1} \sum_j (\dim_q V_j)(\mathrm{Tr}\, S_j),$$

where S_j is the operator on the vector space $\mathcal{C}_{21}(X, V_j X)$ given by $T \mapsto (\iota \otimes S)T$. By the Frobenius reciprocity we can identify $\mathcal{C}_{21}(X, V_j X)$ with $\mathcal{C}_2(\bar{V}_j, X\bar{X})$. In this picture the operator S_j becomes $T \mapsto (S \otimes \iota)T$. But now $X\bar{X}$ is an object in $\operatorname{Rep} G_2$, and since it decomposes into a direct sum of copies of $\bar{V}_j$, we get

$$\sum_j (\dim_q V_j)(\operatorname{Tr} S_j) = \operatorname{Tr}_{X\bar{X}}(S \otimes \iota_{\bar{X}}) = \operatorname{Tr}_X(S)d(X) = \operatorname{tr}_X(S)d(X)^2.$$

This implies that

$$\varphi_{A^{G_1}}(S) = \operatorname{tr}_X(S) \quad \text{and} \quad \dim_q A^{G_1} = d(X)^2.$$

The statement for $\varphi_{A^{G_2}}$ is proved similarly. The last statement of the proposition is now obvious: the unique G_1-G_2-invariant state extending $\varphi_{A^{G_1}}$ and $\varphi_{A^{G_2}}$ is given by the composition of the unique G_1-G_2-invariant conditional expectation $A \to \mathcal{C}_{21}(X)$ with tr_X. $\square$

3.3 From bi-Morita–Galois objects to invertible bimodule categories

Conversely, assume that A is a G_1-G_2-Morita–Galois object. As above, we can consider A^{G_1} as a standard Q-system in $\operatorname{Rep} G_2$. Then we have an invertible $(\operatorname{Rep} G_2)$-$(\operatorname{Bimod}_{G_2}\text{-}A^{G_1})$-module category $\operatorname{Mod}_{G_2}\text{-}A^{G_1}$. We will show that the C*-tensor category $\operatorname{Bimod}_{G_2}\text{-}A^{G_1}$ is equivalent to $\operatorname{Rep} G_1$ and the bimodule category $\operatorname{Mod}_{G_2}\text{-}A^{G_1}$ is equivalent to $\mathcal{D}_A$ in a coherent way.

Lemma 3.17. *Let Y be a G_1-G_2-equivariant A^{G_1}-A-correspondence, and Y' be an equivariant A^{G_2}-A-correspondence. We then have a G_1-G_2-equivariant unitary isomorphism*

$$Y^{G_1} \otimes Y' \cong Y'^{G_2} \otimes Y$$

of $A^{G_1} \otimes A^{G_2}$-A-correspondences.

Note that in this formulation the scalar product on Y^{G_1} is defined using the A^{G_1}-valued inner product and the canonical invariant state on A^{G_1}, and similarly for Y'^{G_2}.

Proof. Put $\lambda = \dim_q A^{G_1} = \dim_q A^{G_2}$, and let x_i, x^i, y_j, y^j be as in Proposition 3.4 (b), where we take the canonical invariant states. Consider the map

$$S_0 \colon Y^{G_1} \otimes Y'^{G_2} \otimes \mathcal{A} \to Y'^{G_2} \otimes Y, \quad \xi \otimes \eta \otimes a \mapsto \lambda^{-1/2} \eta y^j \otimes \xi y_j a.$$

Then by the A^{G_2}-centrality of $y^j \otimes y_j$, this descends to a map from $Y^{G_1} \otimes Y'^{G_2} \otimes_{A^{G_2}} \mathcal{A}$. Moreover, as the action of G_2 on A is free, Proposition 1.3 implies $Y'^{G_2} \otimes_{A^{G_2}} \mathcal{A} \cong \mathcal{Y}'$. Thus, S_0 induces a map

$$S \colon Y^{G_1} \otimes \mathcal{Y}' \to Y'^{G_2} \otimes \mathcal{Y}, \quad \xi \otimes \eta a \mapsto \lambda^{-1/2} \eta y^j \otimes \xi y_j a.$$

Similarly, $T(\eta \otimes \xi a) = \lambda^{-1/2} \xi x^i \otimes \eta x_i a$ is a well-defined map from $Y'^{G_2} \otimes \mathcal{Y}$ to $Y^{G_1} \otimes \mathcal{Y}'$.

By the above formulas, S and T are $A^{G_1} \otimes A^{G_2}$-$\mathcal{A}$-module morphisms. They are also equivariant with respect to the actions of G_1 and G_2, cf. the proof of Proposition 3.4. It remains to show that they are inverse to each other. When $\xi \in Y^{G_1}$, $\eta \in Y'^{G_2}$, and $a \in \mathcal{A}$, we have

$$ST(\eta \otimes \xi a) = \lambda^{-1} \eta y^j \otimes \xi x^i y_j x_i a.$$

Using (3.1), the right hand side is equal to $\varphi_{A^{G_2}}(y_j) \eta y^j \otimes \xi a = \eta \otimes \xi a$, which shows $ST = \iota$. A similar computation shows $TS = \iota$.

Finally, let us show that S is unitary with respect to the A-valued inner products. We have

$$\langle S(\xi' \otimes \eta' a'), S(\xi \otimes \eta a) \rangle_A = \lambda^{-1} \varphi_{A^{G_2}}(\langle \eta' y^k, \eta y^j \rangle_{A^{G_2}}) \langle \xi' y_k a', \xi y_j a \rangle_A.$$

Using

$$\varphi_{A^{G_2}}(\langle \eta' y^k, \eta y^j \rangle_{A^{G_2}}) y_j = \varphi_{A^{G_2}}(y^{k*} \langle \eta', \eta \rangle_{A^{G_2}} y^j) y_j = y^{k*} \langle \eta', \eta \rangle_{A^{G_2}},$$

we see that the above expression equals $\lambda^{-1} \langle \xi' y_k a', \xi y^{k*} \langle \eta', \eta \rangle_{A^{G_2}} a \rangle_A$. Using $y_k^* x y^{k*} = \lambda \varphi_{A^{G_1}}(x)$ for $x \in A^{G_1}$ and that the A^{G_1}-valued inner product on Y^{G_1} is the restriction of the A-valued one on Y, we arrive at $\varphi_{A^{G_1}}(\langle \xi', \xi \rangle_{A^{G_1}}) a'^* a$, which is the inner product of $\xi' \otimes \eta' a'$ and $\xi \otimes \eta a$. This shows the unitarity of S. $\qquad\square$

COROLLARY 3.18. *Any module $X \in \mathcal{D}_A$ embeds into $H_U \otimes A$ for some $U \in \mathrm{Rep}\, G_1$, as well as into $H_V \otimes A$ for some $V \in \mathrm{Rep}\, G_2$.*

Proof. Any $X \in \mathcal{D}_A$ embeds into $H_W \otimes H_V \otimes A$ for some $W \in \mathrm{Rep}\, G_1$ and $V \in \mathrm{Rep}\, G_2$. Moreover, H_V embeds into $A^{G_1} \otimes H_V$, and the above lemma implies $A^{G_1} \otimes H_V \otimes A \cong (H_V \otimes A)^{G_2} \otimes A$. Thus, X embeds into $H_W \otimes (H_V \otimes A)^{G_2} \otimes A$, which proves the first statement. The second is proved similarly. $\qquad\square$

Consider now the spectral functor

$$F \colon \mathrm{Rep}\, G_1 \to \mathrm{Bimod}\text{-}A^{G_1}, \quad U \mapsto (H_U \otimes A)^{G_1},$$

defined by the action of G_1 on A. Since the action is free, it is a unitary tensor functor, with the tensor structure given by

$$F_2 \colon (H_U \otimes A)^{G_1} \otimes_{A^{G_1}} (H_V \otimes A)^{G_1} \to (H_{UV} \otimes A)^{G_1}, \quad (\xi \otimes a) \otimes (\zeta \otimes b) \mapsto (\xi \otimes \zeta) \otimes ab.$$

Clearly we can view F as a unitary tensor functor $\mathrm{Rep}\, G_1 \to \mathrm{Bimod}_{G_2}\text{-}A^{G_1}$.

PROPOSITION 3.19. *The functor $F \colon \mathrm{Rep}\, G_1 \to \mathrm{Bimod}_{G_2}\text{-}A^{G_1}$ is an equivalence of categories.*

Proof. By Lemma 3.17, for any $V \in \operatorname{Rep} G_2$, we have a G_2-equivariant isomorphism

$$A^{G_1} \otimes H_V \otimes A^{G_1} \cong ((H_V \otimes A)^{G_2} \otimes A)^{G_1} = F((H_V \otimes A)^{G_2})$$

of A^{G_1}-bimodules. This shows that the functor F is dominant, that is, any object of $\operatorname{Bimod}_{G_2}\text{-}A^{G_1}$ is a subobject of the image of an object of $\operatorname{Rep} G_1$. Since F is also faithful, it remains to show that F is full. It suffices to check that the map

$$F \colon \operatorname{Mor}(\mathbb{1}, U) \to \operatorname{Mor}_{\operatorname{Bimod}_{G_2}\text{-}A^{G_1}}(A^{G_1}, (H_U \otimes A)^{G_1}) \tag{3.5}$$

is surjective for any $U \in \operatorname{Rep} G_1$. The morphism space on the right can be identified with the space of G_2-invariant A^{G_1}-central vectors in $(H_U \otimes A)^{G_1}$. Since

$$((H_U \otimes A)^{G_1})^{G_2} = (H_U \otimes A^{G_2})^{G_1},$$

this space coincides with

$$(H_U \otimes ((A^{G_1})' \cap A^{G_2}))^{G_1} = (H_U \otimes \mathbb{C}1)^{G_1} = H_U^{G_1} \otimes \mathbb{C}1,$$

where we used that $(A^{G_1})' \cap A^{G_2} = \mathbb{C}1$ by Proposition 3.4(iii). This shows that the map (3.5) is indeed surjective. $\qquad\square$

Consequently, we can view $\operatorname{Mod}_{G_2}\text{-}A^{G_1}$ as an invertible $(\operatorname{Rep} G_2)$-$(\operatorname{Rep} G_1)$-module category. Namely, for $X \in \operatorname{Mod}_{G_2}\text{-}A^{G_1}$ and $U \in \operatorname{Rep} G_1$, we have

$$XU = X \otimes_{A^{G_1}} (H_U \otimes A)^{G_1}.$$

In order to complete the proof of Theorem 3.7 it remains to establish the following.

LEMMA 3.20. *The $(\operatorname{Rep} G_2)$-$(\operatorname{Rep} G_1)$-module categories $\mathcal{D}_A$ and $\operatorname{Mod}_{G_2}\text{-}A^{G_1}$ are equivalent.*

Proof. By Proposition 1.3 we have an equivalence of C*-categories $E \colon \mathcal{D}_A \to \operatorname{Mod}_{G_2}\text{-}A^{G_1}$ given by $E(X) = X^{G_1}$. We want to enrich it to an equivalence of module categories. For this we have to define natural unitary isomorphisms

$$\theta_{V,X,U} \colon V(E(X)U) \to E(V(XU))$$

in $\operatorname{Mod}_{G_2}\text{-}A^{G_1}$, which is given by linear maps from $H_V \otimes X^{G_1} \otimes_{A^{G_1}} (H_U \otimes A)^{G_1}$ to $H_V \otimes (H_U \otimes X)^{G_1}$ for $U \in \operatorname{Rep} G_1$, $X \in \mathcal{D}_A$, $V \in \operatorname{Rep} G_2$. We define them by

$$\theta_{V,X,U}(\xi \otimes x \otimes (\zeta \otimes a)) = \xi \otimes (\zeta \otimes xa).$$

It is clear that this is a G_2-equivariant morphism of right A^{G_1}-modules. It is also easy to check that $\theta_{V,X,U}$ is isometric. In order to check that such morphisms are unitary it suffices to consider modules of the form $X = H_W \otimes A$ for $W \in \operatorname{Rep} G_2$, since any object in $\mathcal{D}_A$ is a subobject of such a module by Corollary 3.18. But for such modules the statement is obvious. It is then straightforward to check that (E, θ) is an equivalence of $(\operatorname{Rep} G_2)$-$(\operatorname{Rep} G_1)$-modules categories. $\qquad\square$

3.4 Fiber functors on categories of bimodules

In the previous sections we have developed an analogue of the bi-Hopf–Galois theory for categorically Morita equivalent compact quantum groups. We now turn to an analogue of the correspondence between fiber functors and Hopf–Galois objects.

DEFINITION 3.21. For a compact quantum group G and a finite dimensional simple right G-C*-algebra B, a *G-Morita–Galois object* for B is a unital C*-algebra A together a free action $\alpha\colon A \to A \otimes C(G)$, and a G-equivariant embedding $B \hookrightarrow A$ such that there is a G-equivariant isomorphism

$$A^G \otimes \mathcal{A} \cong B \otimes \mathcal{A}$$

of $A^G \otimes B$-$\mathcal{A}$-modules that maps $1 \otimes 1 \in A^G \otimes \mathcal{A}$ into an element of $B \otimes B \subset B \otimes \mathcal{A}$.

Similarly to Proposition 3.4, existence of an isomorphism as in the above definition can be reformulated as follows.

LEMMA 3.22. *Assume we are given a right action* $\alpha\colon A \to A \otimes C(G)$ *of a compact quantum group G on a unital C^*-algebra A and a finite dimensional invariant unital C^*-subalgebra $B \subset A$. Then a G-equivariant isomorphism $A^G \otimes \mathcal{A} \cong B \otimes \mathcal{A}$ of $A^G \otimes B$-$\mathcal{A}$-modules, mapping $1 \otimes 1$ into an element of $B \otimes B$, exists if and only if the following conditions hold:*

(a) *the fixed point algebra A^G is finite dimensional;*

(b) *there exist a faithful G-invariant state ψ_B on B and a faithful state ψ_{A^G} on A^G such that if $m_B^*(1) = x^i \otimes x_i$ with respect to ψ_B and $m_{A^G}^*(1) = y^j \otimes y_j$ with respect to ψ_{A^G}, then*

$$x^i y^j x_i \otimes y_j = \lambda 1 \otimes 1 \quad and \quad y^j x^i y_j \otimes x_i = \lambda 1 \otimes 1$$

for a nonzero scalar λ.

Furthermore, if these conditions are satisfied, then

(i) *the map $A^G \otimes \mathcal{A} \to B \otimes \mathcal{A}$, $a \otimes c \mapsto x^i \otimes ax_i c$, is a G-equivariant isomorphism of $A^G \otimes B$-$\mathcal{A}$-modules, with the inverse given by $e \otimes f \mapsto \lambda^{-1} y^j \otimes e y_j f$;*

(ii) *as the state ψ_B we can take the canonical G-invariant state φ_B on B, in which case $\lambda = \dim_q B$;*

(iii) *the relative commutants $(A^G)' \cap B$ and $B' \cap A^G$ are trivial; in particular, B is a simple G-C*-algebra.*

Let us also note that we have the following analogue of Lemma 3.17, with identical proof.

Lemma 3.23. *Let Y be an object of $\mathrm{Bimod}_G\text{-}B$ and Y' be a G-equivariant $A^G\text{-}A$-correspondence. Then we have a G-equivariant isomorphism*

$$Y \otimes Y' \cong Y'^G \otimes (Y \otimes_B A)$$

of $A^G \otimes B\text{-}A$-correspondences. In particular, for any $V \in \mathrm{Rep}\, G$, we have a G-equivariant isomorphism

$$B \otimes H_V \otimes A \cong (H_V \otimes A)^G \otimes A$$

of $B \otimes A^G\text{-}A$-correspondences.

We then have the following result.

Theorem 3.24. *For any reduced compact quantum group G and any finite dimensional simple right $G\text{-}C^*$-algebra B, there is a one-to-one correspondence between the isomorphism classes of G-Morita–Galois objects for B and the isomorphism classes of unitary fiber functors $\mathrm{Bimod}_G\text{-}B \to \mathrm{Hilb}_f$.*

Proof. Assume we are given a G-Morita–Galois object A for B as in Definition 3.21. We define a functor $F\colon \mathrm{Bimod}_G\text{-}B \to \mathrm{Hilb}_f$ by

$$F(X) = (\text{the space of } G\text{-invariant } B\text{-central vectors in } X \otimes_B A).$$

We will see later that the space $F(X)$ is finite dimensional. The Hilbert space structure is defined as follows. The space $X \otimes_B A$ is a right Hilbert A-module. If $\xi, \zeta \in F(X)$, then $\langle \xi, \zeta \rangle_A \in B' \cap A^G = \mathbb{C}1$, so we can define a scalar product by $(\zeta, \xi)1 = \langle \xi, \zeta \rangle_A$.
Next, we define a tensor structure on F by

$$F_2\colon F(X) \otimes F(Y) \to F(X \otimes_B Y), \quad F_2((\xi \otimes a) \otimes (\zeta \otimes c)) = (\xi \otimes \zeta) \otimes ca.$$

In order to check that F_2 is unitary it suffices to consider bimodules of the form $B \otimes H_V \otimes B$ for $V \in \mathrm{Rep}\, G$, since any $X \in \mathrm{Bimod}_G\text{-}B$ embeds isometrically into $B \otimes X \otimes B$. By Lemma 3.23 we have a G-equivariant isomorphism of $B\text{-}A$-modules

$$B \otimes H_V \otimes A \cong (H_V \otimes A)^G \otimes A, \tag{3.6}$$

so we can define a linear isomorphism

$$\begin{aligned} T_V\colon (H_V \otimes A)^G &\to F(B \otimes H_V \otimes B) \subset B \otimes H_V \otimes A, \\ \xi \otimes a &\mapsto (\dim_q B)^{-1/2} x^i \otimes \xi \otimes a x_i, \end{aligned} \tag{3.7}$$

which in particular shows that $F(X)$ is indeed finite dimensional for any $X \in \mathrm{Bimod}_G\text{-}B$. Note that the A-valued inner product on $B \otimes H_V \otimes A$ is given by

$$\langle b_1 \otimes \xi_1 \otimes a_1, b_2 \otimes \xi_2 \otimes a_2 \rangle_A = \varphi_B(b_1^* b_2)(\xi_2, \xi_1) a_1^* a_2.$$

Therefore if we define a scalar product on $(H_V \otimes A)^G$ in the standard way,

$$(\xi_1 \otimes a_1, \xi_2 \otimes a_2) = \varphi_{A^G}(\langle \xi_2 \otimes a_2, \xi_1 \otimes a_1 \rangle_A) = (\xi_1, \xi_2)\varphi_{A^G}(a_2^* a_1),$$

then T_V becomes unitary, since

$$\varphi_B(x^{i*}x^j)x_i^*ax_j = (\dim_q B)\varphi_{A^G}(a)1 \quad \text{for all} \ \ a \in A^G. \tag{3.8}$$

Thus, in order to show that the maps F_2 are unitary it suffices to check that, for all U and V,

$$T^{-1}_{H_U \otimes B \otimes H_V} F_2(T_U \otimes T_V) \colon (H_U \otimes A)^G \otimes (H_V \otimes A)^G \to (H_U \otimes B \otimes H_V \otimes A)^G,$$
$$(\xi \otimes a) \otimes (\zeta \otimes c) \mapsto (\dim_q B)^{-1/2}\xi \otimes x^i \otimes \zeta \otimes cx_i a,$$

is unitary. It is clear from (3.8) that this map is an isometry, so to prove that it is a unitary isomorphism it is enough to compare the dimensions of both sides. Using (3.6) again, we get isomorphisms

$$(H_U \otimes B \otimes H_V \otimes A)^G \cong (H_U \otimes (H_V \otimes A)^G \otimes A)^G \cong (H_U \otimes A)^G \otimes (H_V \otimes A)^G,$$

which completes the proof of unitarity of F_2. We have thus proved that (F, F_2) is a unitary tensor functor.

Let us show next that the spectral functor $(\operatorname{Rep} G)^{\otimes \mathrm{op}} \to \operatorname{Bimod} \text{-} A^G$ defined by the action of G on A can be reconstructed from F. Consider the dual C*-Frobenius algebra $B \otimes B \in \operatorname{Bimod}_G \text{-} B$ with product

$$(\dim_q B)^{1/2}\iota \otimes \varphi_B \otimes \iota \colon B \otimes B \otimes B = (B \otimes B) \otimes_B (B \otimes B) \to B \otimes B$$

and unit $(\dim_q B)^{-1/2}m_B^* \colon B \to B \otimes B$. By applying F we get a C*-Frobenius object in Hilb_f, that is, by Lemma 2.2, a Frobenius C*-algebra. It is easy to see that the unitary

$$T_{\mathbb{1}} \colon A^G \to F(B \otimes B) \subset B \otimes A, \quad a \mapsto (\dim_q B)^{-1/2}x^i \otimes ax_i,$$

is an isomorphism of $((A^G)^{\mathrm{op}}, \varphi_{A^G})$ with this Frobenius C*-algebra. Similarly, any B-bimodule $B \otimes H_U \otimes B$ is a $(B \otimes B)$-bimodule in $\operatorname{Bimod}_G \text{-} B$, so $F(B \otimes H_U \otimes B)$ becomes an $F(B \otimes B)$-bimodule, and using the isomorphisms $T_U \colon (H_U \otimes A)^G \to F(B \otimes H_U \otimes B)$ and $T_{\mathbb{1}} \colon (A^G)^{\mathrm{op}} \to F(B \otimes B)$ we recover the A^G-bimodule structure on $(H_U \otimes A)^G$. Finally, one can also easily check that the tensor structure of the spectral functor can be recovered from that F_2 and the maps $F(\iota \otimes \iota \otimes \varphi_B \otimes \iota \otimes \iota) \colon F(B \otimes H_V \otimes B \otimes H_U \otimes B) \to F(B \otimes H_V \otimes H_U \otimes B)$.

Assume now that we have another G-Morita–Galois object $\tilde{A}$ for B defining an isomorphic fiber functor $\tilde{F}$. Let $\eta \colon F \to \tilde{F}$ be such a unitary monoidal natural isomorphism. It follows from the above discussion that we then get an isomorphism $A^G \cong \tilde{A}^G$ intertwining the spectral functors $(\operatorname{Rep} G)^{\otimes \mathrm{op}} \to \operatorname{Bimod} \text{-} A^G$ and $(\operatorname{Rep} G)^{\otimes \mathrm{op}} \to \operatorname{Bimod} \text{-} \tilde{A}^G$. Hence we get a G-equivariant isomorphism $\theta \colon A \to \tilde{A}$. We claim that θ is the identity map on B, so that θ is an isomorphism of Morita–Galois objects for B.

In view of the way we obtained an isomorphism of the spectral functors, we
have commutative diagrams

$$
\begin{array}{ccc}
(H_U \otimes A)^G & \xrightarrow{\ \iota \otimes \theta\ } & (H_U \otimes \tilde{A})^G \\
\downarrow & & \downarrow \\
F(B \otimes H_U \otimes B) & \xrightarrow[\eta_{B \otimes H_U \otimes B}]{} & \tilde{F}(B \otimes H_U \otimes B),
\end{array}
$$

where the vertical arrows are the maps T defined by (3.7). In other words, for
any $\xi \otimes a \in (H_U \otimes A)^G$ we have

$$
\eta_{B \otimes H_U \otimes B}(x^i \otimes \xi \otimes a x_i) = x^i \otimes \xi \otimes \theta(a) x_i.
$$

Using that $\eta F_2 = \tilde{F}_2(\eta \otimes \eta)$ we then get that for any $\zeta \otimes c \in (H_V \otimes A)^G$ we
have

$$
x^i \otimes \xi \otimes x^j \otimes \zeta \otimes \theta(c) x_j \theta(a) x_i = x^i \otimes \xi \otimes x^j \otimes \zeta \otimes \theta(c x_j a) x_i
$$

in $B \otimes H_U \otimes B \otimes H_V \otimes \tilde{A}$. In the simplest case $U = V = \mathbb{1}$ this gives

$$
x^i \otimes x^j \otimes x_j x_i = x^i \otimes x^j \otimes \theta(x_j) x_i,
$$

and applying φ_B to the first leg we obtain $x^j \otimes x_j = x^j \otimes \theta(x_j)$. Hence
$\theta(x_j) = x_j$, so θ is the identity map on B.

It is also clear that isomorphic Morita–Galois objects define isomorphic fiber
functors. It remains to show that any unitary fiber functor is defined by a
Morita–Galois object. Assume we are given such a functor

$$
E \colon \mathrm{Bimod}_G\text{-}B \to \mathrm{Hilb}_f.
$$

By Woronowicz's Tannaka–Krein duality it defines a compact quantum group
G_1. Then $\mathrm{Mod}_G\text{-}B$ becomes an invertible $(\mathrm{Rep}\,G)$-$(\mathrm{Rep}\,G_1)$-module category
with generator B and we can consider the corresponding G_1-G-Morita–Galois
object A.

We claim that the canonical fiber functor $\mathrm{Rep}\,G_1 \to \mathrm{Hilb}_f$ is isomorphic to the
composition of the spectral functor $E_A \colon \mathrm{Rep}\,G_1 \to \mathrm{Bimod}_G\text{-}B$ for $G_1 \curvearrowright A$,
with the functor $F_A \colon \mathrm{Bimod}_G\text{-}B \to \mathrm{Hilb}_f$ corresponding to A as defined at
the beginning of the proof. We thus have to define a natural isomorphism

$$
H_U \to F((H_U \otimes A)^{G_1}) \subset (H_U \otimes A)^{G_1} \otimes_B A.
$$

As $(H_U \otimes A)^{G_1} \otimes_B A \cong H_U \otimes A$ by Proposition 1.3, it is straightforward to
check that $H_U \ni \xi \mapsto \xi \otimes 1 \in H_U \otimes A$ is the required isomorphism.

In other words, we have proved that the spectral functor associated with the
action of G_1 on A gives an autoequivalence E_A of $\mathrm{Rep}\,G_1 = \mathrm{Bimod}_G\text{-}B$ such
that $F_A E_A \cong E$. Now, in order to complete the proof of the theorem it would

be enough to show that E_A is isomorphic to the identity functor. It is indeed possible to do so, but let us instead finish the proof by giving a more formal argument, as follows.

Suppose that $\tilde{E} \colon \mathrm{Bimod}_G\text{-}B \to \mathrm{Hilb}_f$ is a unitary fiber functor, and let $\tilde{G}_1$ be the corresponding compact quantum group. Consider the bi-Hopf–Galois object $\tilde{A}$ corresponding to the pair of functors $\tilde{E}$ and E. In other words, $\tilde{A}$ is the Morita–Galois object defined by the category $\mathrm{Rep}\,G_1$, considered as a $(\mathrm{Rep}\,G_1)$-$(\mathrm{Rep}\,\tilde{G}_1)$-module category, and by the object $\mathbb{1} \in \mathrm{Rep}\,G_1$. Then by Proposition 4.4 below, the cotensor product $\tilde{A} \square_{G_1} \mathcal{A}$ is the regular subalgebra of a $\tilde{G}_1$-G-Morita–Galois object C, and $C^{\tilde{G}_1}$ is canonically isomorphic to $A^{G_1} = B$.

By definition we have $\mathrm{Rep}\,\tilde{G}_1 = \mathrm{Bimod}_G\text{-}B$. Under this identification, the spectral functor associated with $\tilde{G}_1 \curvearrowright \tilde{A}$, which corresponds to the monoidal equivalence $\mathrm{Rep}\,\tilde{G}_1 \to \mathrm{Rep}\,G_1$, is just the identity functor on $\mathrm{Bimod}_G\text{-}B$. Similarly, the spectral functor associated with $\tilde{G}_1 \curvearrowright C$ can be regarded as an autoequivalence E_C of $\mathrm{Bimod}_G\text{-}B$, which is naturally unitarily monoidally isomorphic to E_A by associativity of the cotensor product operation. We thus get $F_C E_A \cong F_C E_C$, but the latter is isomorphic to $\tilde{E}$ by the same observation as for A and E above. In particular, if we started with $\tilde{E} = E E_A$, we would get $F_C \cong E$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4 Categorical Morita equivalence

4.1 Weak monoidal Morita equivalence and tensor product of bimodule categories

Recall that two rigid C*-tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are called *unitarily weakly monoidally Morita equivalent* [Müg03] if there exists a rigid C*-2-category $\mathcal{C}$ with the set $\{1, 2\}$ of 0-cells such that $\mathcal{C}_{11}$ and $\mathcal{C}_{22}$ are unitarily monoidally equivalent to $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively, and $\mathcal{C}_{12} \neq 0$, or in other words, if there exists an invertible $\mathcal{C}_1$-$\mathcal{C}_2$-bimodule category. Using Frobenius algebras it is shown in [Müg03] that this is indeed an equivalence relation. In the fusion category case a more transparent proof is obtained using relative tensor product of bimodule categories. We now want to make sense of this in our infinite C*-setting.

In fact, we will show a bit more. By passing to equivalent categories we may assume that $\mathcal{C}_{11} = \mathcal{C}_1$ and $\mathcal{C}_{22} = \mathcal{C}_2$. Assume also that $\mathcal{C}_2$ is unitarily weakly monoidally Morita equivalent to a third rigid C*-tensor category $\mathcal{C}_3$, and let $(\mathcal{C}_{ij})_{i,j=2,3}$ be the corresponding rigid C*-2-category. Let us show then that the two 2-categories $(\mathcal{C}_{ij})_{i,j=1}^2$ and $(\mathcal{C}_{ij})_{i,j=2}^3$ can be 'combined' into a C*-2-category with 0-cells $\{1, 2, 3\}$. We thus need to define $\mathcal{C}_{13}$, $\mathcal{C}_{31}$ as bimodule categories over $\mathcal{C}_1$ and $\mathcal{C}_3$, and define the horizontal compositions $\mathcal{C}_{13} \times \mathcal{C}_{31} \to \mathcal{C}_1$, $\mathcal{C}_{13} \times \mathcal{C}_{32} \to \mathcal{C}_{12}$, etc. The idea is simple: using the duality morphisms we can express everything in terms of the categories that we already have.

Thus, we define $\mathcal{C}_{13}$ as the idempotent completion of the category with objects

XY for $X \in \mathcal{C}_{12}$ and $Y \in \mathcal{C}_{23}$, with respect to the morphism sets

$$\mathcal{C}_{13}(XY, X'Y') = \mathcal{C}_2(\bar{X}'X, Y'\bar{Y}).$$

In the following exposition, given a morphism $S \colon XY \to X'Y'$ in $\mathcal{C}_{13}$, let us denote by S_0 its representative in $\mathcal{C}_2(\bar{X}'X, Y'\bar{Y})$. The composition of morphisms in $\mathcal{C}_{13}$ is then defined by

$$(S \circ T)_0 = (\iota \otimes R_{Y'}^* \otimes \iota)(S_0 \otimes T_0)(\iota \otimes \bar{R}_X \otimes \iota)$$

for $T \in \mathcal{C}_{13}(XY, X'Y')$ and $S \in \mathcal{C}_{13}(X'Y', X''Y'')$, so that $(R_X^* \otimes \iota_{Y\bar{Y}})(\iota_{\bar{X}X} \otimes \bar{R}_Y)$ represents the identity morphism of XY. Moreover, $(X, Y) \mapsto XY$ is a bifunctor: for $S \in \mathcal{C}_{12}(X, X')$ and $T \in \mathcal{C}_{23}(Y, Y')$, the morphism $S \otimes T \colon XY \to X'Y'$ is represented by $(T_0 \otimes \iota)\bar{R}_Y R_{X'}^*(\iota \otimes S_0)$.

The left $\mathcal{C}_1$-module category structure is defined by $U(XY) = (UX)Y$ at the level of objects, and by

$$(S \otimes T)_0 = T_0((R_{U'}^*(\iota \otimes S)) \otimes \iota)$$

for $S \in \mathcal{C}_1(U, U')$ and $T \in \mathcal{C}_{13}(XY, X'Y')$, at the level of morphisms. The right $\mathcal{C}_3$-module category structure on $\mathcal{C}_{13}$ is defined in a similar way. The $\mathcal{C}_3$-$\mathcal{C}_1$-module category $\mathcal{C}_{31}$ is also defined in an analogous way as the idempotent completion of the category of objects ZW for $Z \in \mathcal{C}_{32}$ and $W \in \mathcal{C}_{21}$, with morphism sets

$$\mathcal{C}_{31}(ZW, Z'W') = \mathcal{C}_2(W\bar{W}', \bar{Z}Z').$$

The horizontal composition $\mathcal{C}_{13} \times \mathcal{C}_{31} \to \mathcal{C}_1$ is given by $(XY)(ZW) = X(YZ)W$ at the level of objects, and at the level of morphisms $S \otimes T \in \mathcal{C}_1(XYZW, X'Y'Z'W')$, for $S \in \mathcal{C}_{13}(XY, X'Y')$ and $T \in \mathcal{C}_{31}(WZ, W'Z')$, is given by

$$(((((\iota \otimes R_Y^*)(\iota_{X'} \otimes S_0 \otimes \iota)) \otimes ((\bar{R}_Z^* \otimes \iota)(\iota \otimes T_0 \otimes \iota_{W'})))(\bar{R}_{X'} \otimes \iota \otimes R_{W'}).$$

The horizontal composition $\mathcal{C}_{31} \times \mathcal{C}_{13} \to \mathcal{C}_3$ is defined in a similar way. Next let us describe $\mathcal{C}_{13} \times \mathcal{C}_{32} \to \mathcal{C}_{12}$. At the level of objects, it is given by $(XY)Z = X(YZ)$ for $X \in \mathcal{C}_{12}$, $Y \in \mathcal{C}_{23}$, and $Z \in \mathcal{C}_{32}$. At the level of morphisms, $S \otimes T \in \mathcal{C}_{12}(XYZ, X'Y'Z')$ for $S \in \mathcal{C}_{13}(XY, X'Y')$ and $T \in \mathcal{C}_{32}(Z, Z')$ is given by

$$(\iota_{X'Y'Z'} \otimes R_{YZ}^*)(\iota_{X'Y'} \otimes T \otimes \iota_{\bar{Z}\bar{Y}YZ})(\iota_{X'Y'} \otimes \bar{R}_Z \otimes \iota_{\bar{Y}YZ})$$
$$(\iota_{X'} \otimes S_0 \otimes \iota_{YZ})(\bar{R}_{X'} \otimes \iota_{XYZ}).$$

The remaining horizontal compositions are defined similarly.

Lemma 4.1. *The category $\mathcal{C}_{13}$ is a C^*-category with the norm $\|S\| = \|S \otimes \iota_{\bar{Y}}\|_{\mathcal{C}_{12}(XY\bar{Y}, X'Y'\bar{Y})}$ and the involution $(S^*)_0 = (R_{\bar{X}'X} \otimes \iota)(\iota \otimes S_0^* \otimes \iota)(\iota \otimes \bar{R}_{Y'\bar{Y}}) \in \mathcal{C}_2(\bar{X}X', Y\bar{Y}')$ for $S \in \mathcal{C}_{13}(XY, X'Y')$.*

Proof. Take any nonzero object $Z \in \mathcal{C}_{23}$. It is easy to check that we can define a faithful $*$-preserving functor $F_Z \colon \mathcal{C}_{13} \to \mathcal{C}_{12}$ by letting $F_Z(XY) = (XY)\bar{Z}$ on objects and $F_Z(S) = S \otimes \iota_{\bar{Z}}$ on morphisms. It follows that the $*$-operation in the formulation of the proposition is indeed an involution and that the C*-norm on morphisms in $\mathcal{C}_{12}$ defines a C*-norm on morphisms in $\mathcal{C}_{13}$. The latter norm is independent of the choice of Z, since any other object $Z' \in \mathcal{C}_{23}$ embeds into $Z(\bar{Z}Z')$. □

In a similar way we check that $\mathcal{C}_{31}$ is a C*-category. A straightforward verification shows then that $(\mathcal{C}_{ij})_{i,j=1}^3$ is a rigid C*-2-category.

We denote the invertible $\mathcal{C}_1$-$\mathcal{C}_3$-module category $\mathcal{C}_{13}$ by $\mathcal{C}_{12} \boxtimes_{\mathcal{C}_2} \mathcal{C}_{23}$. Note that using representatives $(V_j)_j$ of the isomorphism classes of simple objects in $\mathcal{C}_2$, we can write

$$\mathcal{C}_{13}(XY, X'Y') = \bigoplus_j \mathcal{C}_2(\bar{X}'X, V_j) \otimes \mathcal{C}_2(V_j, Y'\bar{Y})$$

$$\cong \bigoplus_j \mathcal{C}_{12}(X, X'V_j) \otimes \mathcal{C}_{23}(Y, \bar{V}_j Y'). \quad (4.1)$$

Remark 4.2. Consider the Deligne tensor product $\mathcal{C}_{12} \boxtimes \mathcal{C}_{23}$, which is the category with objects $X \boxtimes Y$ and morphisms

$$\mathrm{Mor}_{\mathcal{C}_{12} \boxtimes \mathcal{C}_{23}}(X \boxtimes Y, X' \boxtimes Y') = \mathcal{C}_{12}(X, X') \otimes \mathcal{C}_{23}(Y, Y').$$

The functor $T(X \boxtimes Y) = \bigoplus_j XV_j \boxtimes \bar{V}_j Y$ is an endofunctor of the ind-category of $\mathcal{C}_{12} \boxtimes \mathcal{C}_{23}$. Decomposition of the tensor products $V_j V_{j'}$ into simple objects induces the structure of a *monad* on T, that is, a natural transformation $T^2 \to T$ (together with $\mathrm{Id} \to T$). Formula (4.1) shows that the morphism sets in $\mathcal{C}_{13}$ are given by

$$\mathcal{C}_{13}(XY, X'Y') \cong \mathrm{Mor}_{\mathcal{C}_{12} \boxtimes \mathcal{C}_{23}}(X \boxtimes Y, T(X' \boxtimes Y'))$$

for $X, X' \in \mathcal{C}_{12}$ and $Y, Y' \in \mathcal{C}_{23}$. The right hand side of the above can be regarded as the set of T-module morphisms between the free T-modules $T(X \boxtimes Y)$ and $T(X' \boxtimes Y')$. Thus, $\mathcal{C}_{13}$ can be interpreted as the category of T-modules in $\mathcal{C}_{12} \boxtimes \mathcal{C}_{23}$, and XY is represented by $T(X \boxtimes Y)$.

Remark 4.3. By [Müg03, Proposition 4.5], or by Theorem 3.2, we may assume that $\mathcal{C}_{12} = \mathrm{Mod}_{\mathcal{C}_1}\text{-}Q$ and $\mathcal{C}_2 = \mathrm{Bimod}_{\mathcal{C}_1}\text{-}Q$ for a standard Q-system (Q, m, v) in $\mathcal{C}_1$, and $\mathcal{C}_{23} = \mathrm{Mod}_{\mathcal{C}_2}\text{-}Q'$ and $\mathcal{C}_3 = \mathrm{Bimod}_{\mathcal{C}_2}\text{-}Q'$ for a standard Q-system (Q', m', v') in $\mathcal{C}_2$. Then, denoting by $P_{Q',Q'} \colon Q' \otimes Q' \to Q' \otimes_Q Q'$ the structure morphism of the tensor product over Q, the morphisms $\tilde{m} = m' P_{Q',Q'}$ and $\tilde{v} = v'v$ define the structure of a standard Q-system on Q' as an object in $\mathcal{C}_1$. We claim that $\mathcal{C}_{13}$ is equivalent to $\mathrm{Mod}_{\mathcal{C}_1}\text{-}Q'$ as a $\mathcal{C}_1$-$\mathcal{C}_3$-module category in such a way that XY corresponds to $X \otimes_Q Y$ for $X \in \mathrm{Mod}_{\mathcal{C}_1}\text{-}Q$ and $Y \in \mathrm{Mod}_{\mathrm{Bimod}\text{-}Q}\text{-}Q'$ (note that the category $\mathrm{Bimod}_{\mathcal{C}_1}\text{-}Q'$ can be regarded as $\mathcal{C}_3 = $

$\text{Bimod}_{\text{Bimod-}Q}\text{-}Q'$, since any Q'-module is also a Q-module by the inclusion $v' \colon Q \to Q'$).

Indeed, $X \otimes_Q Y$ inherits the structure of a right Q'-module from Y, and by the Frobenius reciprocity we have

$$\text{Mor}_{\text{Mod}_{\mathcal{C}_1}\text{-}Q'}(X \otimes_Q Y, X' \otimes_Q Y') \cong \text{Mor}_{\text{Bimod-}Q}(\bar{X}' \otimes X, Y' \otimes_{Q'} \bar{Y}).$$

This shows that the subcategory of $\text{Mod}_{\mathcal{C}_1}\text{-}Q'$ generated by the objects of the form $X \otimes_Q Y$ is equivalent to $\mathcal{C}_{13}$. But this is the whole category $\text{Mod}_{\mathcal{C}_1}\text{-}Q'$, since any right Q'-module X in $\mathcal{C}_1$ is a submodule of $X \otimes_Q Q'$.

4.2   Cotensor product of bi-Morita–Galois objects

At the level of bi-Morita–Galois objects relative tensor product of bimodule categories corresponds to cotensor product. In the Hopf algebra setting this result has been already obtained in [Mom12], so we will only give a sketch of an alternative argument in our C*-setting. Note that this result does not need a characterization of algebras arising from invertible bimodule categories.

Proposition 4.4. *Let G_1, G_2 and G_3 be compact quantum groups, A be a G_1-G_2-Morita–Galois object and B be a G_2-G_3-Morita–Galois object. Consider the bimodule categories $\mathcal{D}_A$ and $\mathcal{D}_B$, and let $X \in \mathcal{D}_A$ and $Y \in \mathcal{D}_B$ be the generators corresponding to A and B, respectively. Then the G_1-G_3-Morita–Galois object corresponding to the invertible bimodule category $\mathcal{D}_B \boxtimes_{\text{Rep}\,G_2} \mathcal{D}_A$ and its generator YX is the completion of $\mathcal{A} \square_{G_2} \mathcal{B}$.*

Proof. We write $\mathcal{C}_n$ for $\text{Rep}\,G_n$ ($n = 1, 2, 3$), $\mathcal{C}_{32}$ for $\mathcal{D}_B$ and $\mathcal{C}_{21}$ for $\mathcal{D}_A$. Choose representatives $(U_i)_i$, $(V_j)_j$ and $(W_k)_k$ of the isomorphism classes of irreducible representations of G_1, G_2, and G_3 respectively. The regular subalgebra of the G_1-G_3-C*-algebra corresponding to $YX \in \mathcal{C}_{31} = \mathcal{D}_B \boxtimes_{\text{Rep}\,G_2} \mathcal{D}_A$ is given by

$$\bigoplus_{\substack{i \in \text{Irr}(G_1) \\ k \in \text{Irr}(G_3)}} \bar{H}_i \otimes \mathcal{C}_{31}(YX, W_k YX U_i) \otimes \bar{H}_k.$$

Similarly to (4.1), we have

$$\begin{aligned}
\mathcal{C}_{31}(YX, W_k YX U_i) &\cong \mathcal{C}_2(X\bar{U}_k\bar{X}, \bar{Y}W_k Y) \\
&\cong \bigoplus_{j \in \text{Irr}(G_2)} \mathcal{C}_2(X\bar{U}_i\bar{X}, V_j) \otimes \mathcal{C}_2(V_j, \bar{Y}W_k Y) \\
&\cong \bigoplus_{j \in \text{Irr}(G_2)} \mathcal{C}_{21}(X, V_j X U_i) \otimes \mathcal{C}_{32}(Y, W_k Y \bar{V}_j).
\end{aligned}$$

From this we see that the regular subalgebra is isomorphic to $\mathcal{A} \square_{G_2} \mathcal{B}$ as a left $\mathbb{C}[G_1]$-comodule and a right $\mathbb{C}[G_3]$-comodule. It is also not difficult to compare the products and involutions on the two algebras. $\square$

4.3   Categorical Morita equivalence and Brauer–Picard group

Similarly to [Nai07] we give the following definition.

Definition 4.5. Two compact quantum groups G_1 and G_2 are called *categorically Morita equivalent* if there is an invertible $(\mathrm{Rep}\,G_2)$-$(\mathrm{Rep}\,G_1)$-module category.

By Theorem 3.7, two compact quantum groups G_1 and G_2 are categorically Morita equivalent if and only if there exists a G_1-G_2-Morita–Galois object.

An invertible bimodule category implementing categorical Morita equivalence of G_1 and G_2 is by no means unique. This leads to a notion of the Brauer–Picard group [ENO10]. Namely, in our analytic setting, by the *Brauer–Picard group* of a rigid C*-tensor category $\mathcal{C}$ we mean the set $\mathrm{BrPic}(\mathcal{C})$ of equivalence classes of invertible $\mathcal{C}$-bimodule categories, with the group law defined by the relative tensor product $\boxtimes_{\mathcal{C}}$. For $\mathcal{C} = \mathrm{Rep}\,G$, we can equivalently define $\mathrm{BrPic}(\mathrm{Rep}\,G)$ as the set of equivariant Morita equivalence classes of G-G-Morita–Galois objects, with the group law defined by the cotensor product over G. We will discuss these notions for compact quantum groups in detail elsewhere, confining ourselves for the moment to a few examples and remarks.

Example 4.6. Any finite quantum group G is categorically Morita equivalent to its dual $\hat{G}$. This follows by considering a depth 2 subfactor $N \subset N \rtimes G$ and was already observed by Müger [Müg03, Corollary 6.16], but let us show this using Morita–Galois objects.

Consider the C*-algebra $A = C(G) \rtimes G$, where G acts on $C(G)$ by right translations. The action of G on $C(G)$ by left translations extends in the obvious way to an action on A, while $\hat{G}$ also acts on A by the dual action. These two actions commute and we claim that A is a G-$\hat{G}$-Morita–Galois object.

Since the action G on $C(G)$ by left translations is free, the action of G on A is also free by Proposition 1.6. For similar reasons the action of $\hat{G}$ is free. Next, let u_{ij}^s, $s \in \mathrm{Irr}(G)$, $i, j = 1, \ldots, d_s$, be matrix coefficients of irreducible unitary representations of G. The dual basis with respect to the Haar state is given by $d_s u_{ij}^{s*}$. For any $\omega \in C(\hat{G}) = C^*G \subset A$, we have

$$\sum_{s,i,j} d_s u_{ij}^{s*} \omega u_{ij}^s = \sum_{s,i,j,k} d_s (u_{kj}^s, \omega_{(1)}) u_{ij}^{s*} u_{ik}^s \omega_{(2)} = \sum_{s,j} d_s (u_{jj}^s, \omega_{(1)}) \omega_{(2)}.$$

Up to normalization, the Haar state $h_{\hat{G}}$ on $C(\hat{G}) \cong \bigoplus_s \mathrm{Mat}_{d_s}(\mathbb{C})$ is given by $\bigoplus_s d_s \mathrm{Tr}$. Hence, up to a scalar factor, the above expression equals

$$h_{\hat{G}}(\omega_{(1)}) \omega_{(2)} = h_{\hat{G}}(\omega) 1.$$

Therefore the second identity in (3.1) is satisfied for $G_1 = G$ and $G_2 = \hat{G}$, as $A^G = C(\hat{G})$ and $A^{\hat{G}} = C(G)$. Since the roles of G and $\hat{G}$ are symmetric, the first identity there is satisfied as well. Hence A is indeed a G-$\hat{G}$-Morita–Galois object. Note also that the canonical invariant state on $A \cong M_{\dim C(G)}(\mathbb{C})$ is the unique tracial state.

Example 4.7. Assume G is a genuine compact group and $\pi\colon G \to \mathrm{PU}(H)$ is a projective unitary representation of G on a finite dimensional Hilbert space H. Consider the algebra $C(G)\otimes B(H)$ with two commuting actions of G: one action is given by left translations of G on $C(G)$, the other by the tensor product of the action by right translations on $C(G)$ and by $\mathrm{Ad}\,\pi$ on $B(H)$. These actions are free by Proposition 1.6 and both fixed point algebras are isomorphic to $B(H)$. Taking the unique tracial states on these algebras it is easy to check that identities (3.1) are satisfied. Therefore A is a G-G-Morita–Galois object. In particular, the categories $\mathrm{Rep}\,G$ and $\mathrm{Bimod}_G\text{-}B(H)$ are unitarily monoidally equivalent. Modulo unitarity, this, in fact, follows already from [Par76] (see also [vOZ98, Corollary 3.2]), since $B(H)$ is an Azumaya algebra in the symmetric monoidal category $\mathrm{Rep}\,G$.

This simple example has the following consequence: if G is a genuine compact connected group, then any compact quantum group categorically Morita equivalent to G is monoidally equivalent to G. Indeed, if G' is such a compact quantum group, then $\mathrm{Rep}\,G'$ is unitarily monoidally equivalent to $\mathrm{Bimod}_G\text{-}B$ for some simple G-C*-algebra B. Since G is connected, B must be a full matrix algebra $B(H)$, and the claim follows.

A Q-SYSTEMS AND PROPER MODULE CATEGORIES

The goal of this appendix is to prove the following correspondence between Q-systems and proper module categories.

THEOREM A.1. *Let $\mathcal{C}$ be a rigid C*-tensor category with simple unit, and let $\mathcal{D}$ be a nonzero indecomposable semisimple proper right $\mathcal{C}$-module category. Then there is an irreducible Q-system A in $\mathcal{C}$ such that $\mathcal{D}$ is unitarily equivalent to A-$\mathrm{Mod}_{\mathcal{C}}$ as a right $\mathcal{C}$-module category.*

This is an adaptation to the infinite C*-setting of a result of Ostrik [Ost03, Theorem 3.1]. It is certainly known to experts, see, e.g., [AdC15, Section 3], but the precise details seem to be somewhat elusive in the literature. The main point is to show that, if $\mathcal{D}$ is a proper right module category over $\mathcal{C}$, the unitary structure induces the structure of a Q-system on the internal endomorphism object $\underline{\mathrm{End}}(X)$ for any X, cf. [GS12, p. 625]. This would imply that $\mathcal{D}$ is a part of a rigid C*-bicategory which has $\mathcal{C}$ in one of its diagonal corners.

Fix representatives $(U_i)_i$ and $(X_a)_a$ of the isomorphism classes of simple objects in $\mathcal{C}$ and $\mathcal{D}$ respectively. For any $X \in \mathcal{D}$, we always consider $\mathcal{D}(X_a, X)$ and $\mathcal{D}(X, X_a)$ as Hilbert spaces equipped with the scalar products

$$(S,T)_{\mathcal{D}(X_a,X)}\iota_a = T^*S, \qquad\qquad (S,T)_{\mathcal{D}(X,X_a)}\iota_a = ST^*.$$

More generally, for $X, Y \in \mathcal{D}$, we consider $\mathcal{D}(X,Y)$ as a Hilbert space via the identification

$$\mathcal{D}(X,Y) \cong \bigoplus_a \mathcal{D}(X,X_a) \otimes \mathcal{D}(X_a,Y).$$

This way the functor $Y \mapsto \mathcal{D}(X, Y)$ is a C*-functor from $\mathcal{D}$ to Hilb_f for any X. The dual module category of $\mathcal{D}$ is given by the C*-category of right $\mathcal{C}$-module functors $\mathrm{Hom}_{\mathcal{C}}(\mathcal{D}, \mathcal{C})$. We have canonical pairings $\mathrm{Hom}_{\mathcal{C}}(\mathcal{D}, \mathcal{C}) \times \mathcal{D} \to \mathcal{C}$ and

$$\mathcal{D} \times \mathrm{Hom}_{\mathcal{C}}(\mathcal{D}, \mathcal{C}) \to \mathrm{End}_{\mathcal{C}}(\mathcal{D}), \quad (X, F) \mapsto (Y \mapsto XF(Y)).$$

Fix a simple object X in $\mathcal{D}$. We define a 'dual' of $X \in \mathcal{D}$ as an object in $\mathrm{Hom}_{\mathcal{C}}(\mathcal{D}, \mathcal{C})$ by

$$\bar{X}Y = \bigoplus_i \mathcal{D}(X, Y\bar{U}_i) \otimes U_i,$$

where we write $\bar{X}Y$ instead of $\bar{X}(Y)$ and make use of a unitary bifunctor $\mathrm{Hilb}_f \times \mathcal{C} \to \mathcal{C}$ characterized by
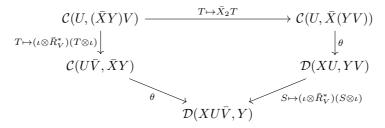
$$\mathcal{C}(H \otimes U, K \otimes V) \cong B(H, K) \otimes \mathcal{C}(U, V).$$

The object $\bar{X}Y$ is well-defined by the properness assumption on $\mathcal{D}$. The functor $\bar{X}$ is adjoint to the functor $\mathcal{C} \to \mathcal{D}$, $U \mapsto XU$, via the natural isomorphisms

$$\theta_{X,Y,U} \colon \mathcal{C}(U, \bar{X}Y) = \bigoplus_i \mathcal{D}(X, Y\bar{U}_i) \otimes \mathcal{C}(U, U_i) \to \mathcal{D}(XU, Y), \tag{A.1}$$

$$\mathcal{D}(X, Y\bar{U}_i) \otimes \mathcal{C}(U, U_i) \ni S \otimes T \mapsto d_i^{1/2}(\iota \otimes R_i^*)(S \otimes T),$$

for $U \in \mathcal{C}$. In other words, $\bar{X}Y$ is the internal Hom object $\underline{\mathrm{Hom}}(X, Y)$. Here and below we identify U with $\mathbb{C} \otimes U$, so that when H is a finite dimensional Hilbert space the space of morphisms $U \to H \otimes V$ equals $B(\mathbb{C}, H) \otimes \mathcal{C}(U, V) = H \otimes \mathcal{C}(U, V)$. The natural isomorphisms $(\bar{X}_2)_{Y,V} \colon (\bar{X}Y)V \to \bar{X}(YV)$ are characterized by commutativity of the diagrams

$$
\begin{array}{ccc}
\mathcal{C}(U, (\bar{X}Y)V) & \xrightarrow{T \mapsto \bar{X}_2 T} & \mathcal{C}(U, \bar{X}(YV)) \\
{\scriptstyle T \mapsto (\iota \otimes \bar{R}_V^*)(T \otimes \iota)} \downarrow & & \downarrow {\scriptstyle \theta} \\
\mathcal{C}(U\bar{V}, \bar{X}Y) & & \mathcal{D}(XU, YV) \\
& \searrow {\scriptstyle \theta} \quad \swarrow {\scriptstyle S \mapsto (\iota \otimes \bar{R}_V^*)(S \otimes \iota)} & \\
& \mathcal{D}(XU\bar{V}, Y) &
\end{array}
$$

for $U \in \mathcal{C}$. From this characterization we have $(\bar{X}_2)_{YU,V}((\bar{X}_2)_{Y,U} \otimes \iota_V) = (\bar{X}_2)_{Y,UV}$ as morphisms from $(\bar{X}Y)UV$ to $\bar{X}(YUV)$.

Now, for $Y \in \mathcal{D}$, let us denote by μ_Y the morphism $X\bar{X}Y \to Y$ which corresponds to $\iota_{\bar{X}Y}$ under isomorphism (A.1) for $U = \bar{X}Y$. Then the morphism $\bar{X}(\mu_X)(\bar{X}_2)_{X, \bar{X}X} \colon (\bar{X}X)(\bar{X}X) \to \bar{X}X$ defines an algebra structure on $\bar{X}X$ with unit $\iota_X \otimes \iota_{\mathbb{1}} \in \mathcal{D}(X) \otimes \mathcal{C}(\mathbb{1}) \subset \mathcal{C}(\mathbb{1}, \bar{X}X)$. Furthermore, the morphism $\bar{X}(\mu_Y)(\bar{X}_2)_{X, \bar{X}Y} \colon (\bar{X}X)(\bar{X}Y) \to \bar{X}Y$ defines a left $\bar{X}X$-module structure on $\bar{X}Y$. Then the functor $Y \mapsto \bar{X}Y$ extends to an equivalence between $\mathcal{D}$ and the category of left $\bar{X}X$-modules in $\mathcal{C}$ (without any compatibility with the *-structures for the moment) [Ost03; EGNO15, Section 7.9]. It remains to show

that $\bar{X}X$ is an irreducible Q-system and that the $\bar{X}X$-module structure on $\bar{X}Y$ is unitary.

LEMMA A.2. *The morphism* $(\bar{X}_2)_{Y,V}$ *is given by*

$$\sum_{i,j} \left(\frac{d_i}{d_j} \right)^{1/2} F_\beta \otimes v_\beta^* \colon \bigoplus_i \mathcal{D}(X, Y\bar{U}_i) \otimes U_i V \to \bigoplus_j \mathcal{D}(X, YV\bar{U}_j) \otimes U_j,$$

where $(v_\beta \colon U_j \to U_i V)_\beta$ *is an orthonormal basis of isometries, and* F_β *is the map*

$$\mathcal{D}(X, Y\bar{U}_i) \to \mathcal{D}(X, YV\bar{U}_j), \quad S \mapsto (\iota \otimes R_i^* \otimes \iota_{V\bar{U}_j})(S \otimes v_\beta \otimes \iota_{\bar{j}})(\iota \otimes \bar{R}_j).$$

Proof. It is enough to check the commutativity of the above diagram. Let

$$S \otimes T \in \mathcal{D}(X, Y\bar{U}_i) \otimes \mathcal{C}(U, U_i V) \subset \mathcal{C}(U, (\bar{X}Y)V).$$

Chasing this element along the arrows on the left, we obtain

$$d_i^{1/2}(\iota \otimes R_i^* \otimes \bar{R}_V^*)(S \otimes T \otimes \iota_{\bar{V}}) \in \mathcal{D}(XU\bar{V}, Y).$$

On the other hand, chasing the top and right arrows, we obtain

$$\sum_{j,\beta} d_i^{1/2}(\iota \otimes R_i^* \otimes \bar{R}_V^*)(S \otimes v_\beta \otimes R_j^*(\iota_{\bar{j}} \otimes v_\beta^* T) \otimes \iota_{\bar{V}})(\iota_X \otimes \bar{R}_j \otimes \iota_{U\bar{V}})$$

$$= \sum_{j,\beta} d_i^{1/2}(\iota \otimes R_i^* \otimes \bar{R}_V^*)(S \otimes v_\beta v_\beta^* T \otimes \iota_{\bar{V}}) = d_i^{1/2}(\iota \otimes R_i^* \otimes \bar{R}_V^*)(S \otimes T \otimes \iota_{\bar{V}}),$$

which proves the assertion. $\qquad\qquad\square$

We can now show that $\bar{X}_2$ is unitary thanks to the normalization of (A.1).

LEMMA A.3. *The morphism* $\bar{X}_2$ *is unitary, and its inverse is given by*

$$\sum_{i,j} \left(\frac{d_i}{d_j} \right)^{1/2} G_\beta \otimes v_\beta \colon \bigoplus_j \mathcal{D}(X, YV\bar{U}_j) \otimes U_j \to \bigoplus_i \mathcal{D}(X, Y\bar{U}_i) \otimes U_i V,$$

where G_β *is the map*

$$\mathcal{D}(X, YV\bar{U}_j) \to \mathcal{D}(X, Y\bar{U}_i), \quad T \mapsto (\iota_{Y\bar{U}_i} \otimes \bar{R}_j^*)(\iota_{Y\bar{U}_i} \otimes v_\beta^* \otimes \iota_{\bar{j}})(\iota_Y \otimes R_i \otimes \iota_{V\bar{U}_j})T.$$

Proof. Since F_β can be written as

$$F_\beta(S) = (\iota \otimes R_i^* \otimes \iota_{V\bar{U}_j})(\iota_{Y\bar{U}_i} \otimes v_\beta \otimes \iota_{\bar{j}})(\iota_{Y\bar{U}_i} \otimes \bar{R}_j)S,$$

the morphism $\bar{X}_2^*$ is indeed given by the formula in the formulation. It remains to show that $\bar{X}_2$ is an isometry.

By the above formula, the component of $\bar{X}_2^* \bar{X}_2$ for $\mathcal{D}(X, Y\bar{U}_i) \otimes U_i V \to \mathcal{D}(X, Y\bar{U}_{i'}) \otimes U_{i'} V$ is given by

$$\sum_{j,\beta,\gamma} \frac{(d_i d_{i'})^{1/2}}{d_j} H_{\beta,\gamma} \otimes w_\gamma v_\beta^*,$$

where $(w_\gamma \colon U_j \to U_{i'} V)_\gamma$ is an orthonormal basis, and $H_{\beta,\gamma}$ is the linear map

$$\mathcal{D}(X, Y\bar{U}_i) \to \mathcal{D}(X, Y\bar{U}_{i'}),$$
$$S \mapsto (\iota_{Y\bar{U}_{i'}} \otimes \bar{R}_j^*)(\iota_{Y\bar{U}_{i'}} \otimes w_\gamma^* \otimes \iota_{\bar{j}})$$
$$(\iota_Y \otimes R_{i'} R_i^* \otimes \iota_{V\bar{U}_j})(\iota_{Y\bar{U}_i} \otimes v_\beta \otimes \iota_{\bar{j}})(S \otimes \bar{R}_j).$$

Since $(\iota_{\bar{i}'} \otimes \bar{R}_j^*)(\iota_{\bar{i}'} \otimes w_\gamma^* \otimes \iota_{\bar{j}})(R_{i'} R_i^* \otimes \iota_{V\bar{U}_j})(\iota_{\bar{i}} \otimes v_\beta \otimes \iota_{\bar{j}})(\iota_{\bar{i}} \otimes \bar{R}_j)$ is a morphism from U_i to $U_{i'}$, the only nonzero terms are for $i' = i$. Moreover, it is easy to see that the family

$$\left(\left(\frac{d_i}{d_j}\right)^{1/2} (R_i^* \otimes \iota_{V\bar{U}_j})(\iota_{\bar{i}} \otimes v_\beta \otimes \iota_{\bar{j}})(\iota_{\bar{i}} \otimes \bar{R}_j) \right)_\beta$$

forms an orthonormal basis of isometries $\bar{U}_i \to V\bar{U}_j$. It follows that if (for $i' = i$) we take $(w_\gamma)_\gamma = (v_\beta)_\beta$, then $(d_i/d_j)H_{\beta,\beta}(S) = S$. Thus, we see that $\bar{X}_2^* \bar{X}_2$ indeed acts as the identity morphism on the direct summand $\mathcal{D}(X, Y\bar{U}_i) \otimes U_i V$. $\qquad \square$

Lemma A.4. *The linear isomorphism*

$$\mathcal{D}(X_a, X_b U) \to \mathcal{D}(X_a \bar{U}, X_b), \quad T \mapsto \frac{d(\bar{X} X_b)^{1/2}}{d(\bar{X} X_a)^{1/2}}(\iota_b \otimes \bar{R}_U^*)(T \otimes \iota_{\bar{U}}),$$

is unitary for any $U \in \mathcal{C}$.

Note that by the indecomposability assumption the object $\bar{X} X_a$ is nonzero for any a, so the formulation makes sense.

Proof. Put $\tilde{T} = (\iota_b \otimes \bar{R}_U^*)(T \otimes \iota_{\bar{U}})$. By definition of $(\tilde{S}, \tilde{T})$, we have

$$\bar{X}((\iota_b \otimes \bar{R}_U^*)(S \otimes \iota_{\bar{U}})(T^* \otimes \iota_{\bar{U}})(\iota_b \otimes \bar{R}_U)) = (\tilde{S}, \tilde{T})\iota_{\bar{X} X_b}.$$

Using the module functor structure on $\bar{X}$, the left hand side can be written as

$$(\iota_{\bar{X} X_b} \otimes \bar{R}_U^*)(\bar{X}_2)_{X_b, U\bar{U}}^{-1}(\bar{X}_2)_{X_b U, \bar{U}}(\bar{X}(ST^*) \otimes \iota_{\bar{U}})$$
$$(\bar{X}_2)_{X_b U, \bar{U}}^{-1}(\bar{X}_2)_{X_b, U\bar{U}}(\iota_{\bar{X} X_b} \otimes \bar{R}_U),$$

which is $(\iota_{\bar{X} X_b} \otimes \bar{R}_U^*)((\bar{X}_2)_{X_b, U}^{-1} \bar{X}(ST^*)(\bar{X}_2)_{X_b, U} \otimes \iota_{\bar{U}})(\iota_{\bar{X} X_b} \otimes \bar{R}_U)$ by the multiplicativity of $(\bar{X}_2)_{Y, U}$ in U. Thus, the scalar $(\tilde{S}, \tilde{T})$ can be extracted by

applying the categorical trace:

$$
\begin{aligned}
(\tilde{S}, \tilde{T}) &= \mathrm{tr}_{\bar{X}X_b}\left((\iota_{\bar{X}X_b} \otimes \bar{R}_U^*)((\bar{X}_2)_{X_b,U}^{-1} \bar{X}(ST^*)(\bar{X}_2)_{X_b,U} \otimes \iota_{\bar{U}})(\iota_{\bar{X}X_b} \otimes \bar{R}_U) \right) \\
&= \frac{1}{d(\bar{X}X_b)} \mathrm{Tr}_{(\bar{X}X_b)U}\left((\bar{X}_2)_{X_b,U}^{-1} \bar{X}(ST^*)(\bar{X}_2)_{X_b,U} \right) \\
&= \frac{1}{d(\bar{X}X_b)} \mathrm{Tr}_{\bar{X}X_a}(\bar{X}(T^*)\bar{X}(S)) = \frac{d(\bar{X}X_a)}{d(\bar{X}X_b)}(S,T),
\end{aligned}
$$

which proves the assertion. $\square$

PROPOSITION A.5. *We have $\mu_Y \mu_Y^* = d(\bar{X}X)\iota_Y$ for any $Y \in \mathcal{D}$.*

Proof. We may assume that $X = X_a$ and $Y = X_b$ for some a, b. We identify

$$
\mathcal{C}(\bar{X}_a X_b) = \bigoplus_{i,j} B(\mathcal{D}(X_a, X_b \bar{U}_i), \mathcal{D}(X_a, X_b \bar{U}_j)) \otimes \mathcal{C}(U_i, U_j)
$$

with

$$
\bigoplus_i \mathcal{D}(X_a, X_b \bar{U}_i) \otimes \overline{\mathcal{D}(X_a, X_b \bar{U}_i)} \otimes \mathcal{C}(U_i, U_j),
$$

so that $\iota_{\bar{X}_a X_b}$ is represented by $\sum_{i,\alpha} u_\alpha \otimes \bar{u}_\alpha \otimes \iota_i$, where $(u_\alpha \colon X_a \to X_b \bar{U}_i)_\alpha$ is an orthonormal basis of isometries. Then we have

$$
\begin{aligned}
\mu_b &= \theta_{X_a, X_b, \bar{X}_a X_b}\left(\sum_{i,\alpha} u_\alpha \otimes \bar{u}_\alpha \otimes \iota_i \right) \\
&= \sum_{i,\alpha} d_i^{1/2}(\iota_b \otimes R_i^*)(u_\alpha \otimes \iota_i)(\iota_X \otimes (u_\alpha^* \otimes \iota_i)p_i),
\end{aligned}
$$

where $p_i \colon \bar{X}_a X_b \to \mathcal{D}(X_a, X_b \bar{U}_i) \otimes U_i$ is the orthogonal projection onto the isotypic component for U_i. Hence $\mu_b^* \colon X_b \to X_a \bar{X}_a X_b$ is given by $\sum_{i,\alpha} d_i^{1/2} u_\alpha \otimes (u_\alpha^* \otimes \iota_{\bar{\imath}})(\iota_b \otimes R_i)$. We thus have

$$
\mu_b \mu_b^* = \sum_{i,\alpha,\alpha'} d_i(\iota_b \otimes R_i^*)(u_{\alpha'} u_\alpha^* \otimes \iota_i)(\iota_b \otimes R_i).
$$

By Lemma A.4, we have $(\iota_b \otimes R_i^*)(u_{\alpha'} u_\alpha^* \otimes \iota_i)(\iota_b \otimes R_i) = \delta_{\alpha,\alpha'} \frac{d(\bar{X}_a X_a)}{d(\bar{X}_a X_b)} \iota_b$. Hence

$$
\mu_b \mu_b^* = \sum_i \left(d_i \frac{d(\bar{X}_a X_a)}{d(\bar{X}_a X_b)} \dim \mathcal{D}(X_a, X_b \bar{U}_i) \right) \iota_b = d(\bar{X}_a X_a)\iota_b,
$$

which finishes the proof of the proposition. $\square$

It follows that $\bar{X}X$ is a standard Q-system in $\mathcal{C}$ and, for any $Y \in \mathcal{D}$, the $\bar{X}X$-module $\bar{X}Y$ satisfies the unitarity condition. The Q-system $\bar{X}X$ is irreducible, since $\mathcal{C}(\mathbb{1}, \bar{X}X)$ is one-dimensional. Thus Theorem A.1 is proved.

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