Functoriality Properties of the Dual Group

Friedrich Knop

Received: July 29, 2017
Revised: August 23, 2018

Communicated by Dan Ciubotaru

Abstract. Let G be a connected reductive group. Previously, it was shown that for any G-variety X one can define the dual group $G_X^\vee$ which admits a natural homomorphism with finite kernel to the Langlands dual group $G^\vee$ of G. Here, we prove that the dual group is functorial in the following sense: if there is a dominant G-morphism $X \to Y$ or an injective G-morphism $Y \to X$ then there is a unique homomorphism with finite kernel $G_Y^\vee \to G_X^\vee$ which is compatible with the homomorphisms to $G^\vee$.

2010 Mathematics Subject Classification: 17B22, 14L30, 11F70
Keywords and Phrases: Spherical variety, Langlands dual group, root system, algebraic group, reductive group

1. Introduction

Let G be a connected reductive group defined over an algebraically closed field k of characteristic zero. To any G-variety X one can attach a finite reflection group $W(X)$ (its "little Weyl group") which, loosely speaking, determines the large scale geometry of X (see Brion [Bri90] and [Kno94]).

While it is known that $W(X)$ is a subgroup of the Weyl group of G, it is, in general, not true that it is the Weyl group of some subgroup of G. But surprisingly, the Langlands dual group $G^\vee$ of G does contain such a subgroup. At least in the case when X is spherical, this was first hinted at in work of Gaitsgory and Nadler, [GN10], who constructed a reductive subgroup of $G^\vee$ whose Weyl group is most likely equal to $W(X)$. Later Sakellaridis and Venkatesh, [SV17], refined (at least for X spherical) the description of a hypothetical subgroup with Weyl group $W(X)$. In particular, they worked out precisely how it should embed into $G^\vee$. They also replaced the subgroup by a particular finite cover $G_X^\vee$, the *dual group of* X, which carries more information about X.

In [KS17], it was shown that the Sakellaridis-Venkatesh construction does indeed work, i.e., that there is a homomorphism $\varphi_X : G_X^\vee \to G^\vee$ as predicted in [SV17]. The approach of [KS17] is purely combinatorial.

In the present paper we investigate the question whether the assignment $X \mapsto (G_X^\vee, \varphi_X)$ can be turned into a functor. To this end, we are going to normalize the homomorphism φ_X in such a way that it becomes unique up to conjugation by an element of the maximal torus of $G_X^\vee$. The main result of the present paper is:

Theorem 1.1. *Let X and Y be two G-varieties. Assume that there is either a dominant G-morphism $f : X \to Y$ or a generically injective G-morphism $Y \to X$. Then there exists a unique homomorphism (necessarily with finite kernel) $\eta : G_Y^\vee \to G_X^\vee$ such that $\varphi_Y = \varphi_X \circ \eta$.*

In the body of the paper, we prove a more precise version of the theorem (see Theorems 2.7 and 2.8).

The proof of Theorem 1.1 proceeds in several steps: first we treat the case of a dominant morphism. First, the theorem is reduced to the case when both X and Y are homogeneous with Y being of rank 1 and f being proper. Then we use a classification (due to Akhiezer [Ahi83] and Panyushev [Pan95]) to check the assertion case-by-case. To this end, we determine, given a spherical G-variety G/H of rank 1, the Luna data of G/P where P runs through all maximal parabolic subgroups of H. This might be of independent interest since the morphisms $G/P \to G/H$ are in a sense minimal among all dominant G-morphisms. The case of injective morphisms will finally follow from the dominant one.

As opposed to [KS17] we are going to argue much more geometrically than combinatorially. This is is due to the fact that the the weak spherical data used in [KS17] do not possess sufficient functorial properties.

2. The dual group and distinguished homomorphisms

Let G be a connected reductive group defined over an algebraically closed ground field k of characteristic 0. Let $B \subseteq G$ be a Borel subgroup and $T \subseteq B$ a maximal torus. Let $\Lambda := \Xi(B)$ be the weight lattice, $\Phi \subset \Lambda$ the root system of G, and $S \subseteq \Phi$ the set of simple roots with respect to B.

We recall the dual group $G_X^\vee$ of a G-variety X. A rational function $f \in k(X)$ is B-semiinvariant with character $\chi_f \in \Lambda$ if $f(b^{-1}x) = \chi_f(b)f(x)$ for all $b \in B$ and $x \in X$ where both sides are defined. All characters χ_f form a subgroup $\Xi = \Xi(X)$ of Λ, the *weight lattice of* X. The rank of $\Xi(X)$ is called the *rank of* X and is denoted by $\operatorname{rk} X$.

Now consider a discrete valuation $v : k(X) \to \mathbb{Q} \cup \{\infty\}$. It is called *central* if it is G-invariant and restricts to the trivial valuation on the field $k(X)^B$ of rational B-invariants. Then $v(f)$ depends, for any B-semiinvariant f, only on its character χ_f. Thus we get a map

$$\varrho : \mathcal{Z}(X) \to \mathcal{N}(X) := \operatorname{Hom}(\Xi, \mathbb{Q}) \tag{1}$$

where $\mathcal{Z}(X)$ is the set of all central valuations. It was proven in [LV83] that ϱ is injective. Hence we may and will identify $\mathcal{Z}(X)$ with a subset of the $\mathbb{Q}$-vector space $\mathcal{N}(X)$.

One can show that $\mathcal{Z}(X)$ is a finitely generated convex cone which is not contained in a hyperplane. Let

$$\Sigma = \Sigma(X) = \{\sigma_1, \ldots, \sigma_s\} \subseteq \Xi_{\mathbb{Q}} := \Xi \otimes \mathbb{Q} \tag{2}$$

be a minimal set of outward normal vectors (so-called *spherical roots of* X) such that

$$\mathcal{Z}(X) = \{a \in \mathcal{N}(X) \mid a(\sigma_1) \leq 0, \ldots, a(\sigma_s) \leq 0\}. \tag{3}$$

The σ_i are only unique up to positive factors and there are several normalizations possible. The one which we are adopting uses the fact that each σ_i lies in the intersection $\Xi_{\mathbb{Q}} \cap \mathbb{Q}S$. Thus we can and will normalize σ_i is such a way that it is primitive in the root lattice $\mathbb{Z}S$. Therefore, every σ_i is a linear combination $\sum_{\alpha \in S} n_\alpha \alpha$ with integral coprime coefficients which one can show to be non-negative. The *support* $|\sigma_i|$ of σ_i is the set $\{\alpha \in S \mid n_\alpha > 0\}$. More generally, we put $|\Sigma_0| = \cup_{\sigma \in \Sigma_0} |\sigma| \subseteq S$ for any subset $\Sigma_0 \subseteq \mathbb{Z}S$.
A third invariant of X is a certain set $S^p = S^p(X) \subseteq S$ of simple roots. It consists of all $\alpha \in S$ (called *parabolic for* X) such that $P_\alpha x = Bx$ for generic $x \in X$. Here $P_\alpha \subseteq G$ is the minimal parabolic subgroup corresponding to α. In other words, the parabolic subgroup $Q(X)$ corresponding to S^p is the stabilizer of a generic B-orbit.
The coefficients n_α are always non-negative. In fact much more is true. One can show that the triple $(|\sigma|, \sigma, S^p \cap |\sigma|)$ will always appear in Table 1. The items correspond to spherical varieties of rank 1 (listed in Table 3) which will be explained in more detail in Section 4.

TABLE 1.

$\lvert\sigma\rvert$	σ	$S^p \cap \lvert\sigma\rvert$
A_1	α_1	$\varnothing$
$\mathsf{A}_n,\ n \geq 2$	$\alpha_1 + \ldots + \alpha_n$	$\{\alpha_2, \ldots, \alpha_{n-1}\}$
$\mathsf{B}_n,\ n \geq 2$	$\alpha_1 + \ldots + \alpha_n$	$\{\alpha_2, \ldots, \alpha_n\}$
$\mathsf{B}_n,\ n \geq 2$	$\alpha_1 + \ldots + \alpha_n$	$\{\alpha_2, \ldots, \alpha_{n-1}\}$
$\mathsf{C}_n,\ n \geq 3$	$\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n$	$\{\alpha_1, \alpha_3, \ldots, \alpha_n\}$
$\mathsf{C}_n,\ n \geq 3$	$\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n$	$\{\alpha_3, \ldots, \alpha_n\}$
F_4	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$	$\{\alpha_1, \alpha_2, \alpha_3\}$
G_2	$2\alpha_1 + \alpha_2$	$\{\alpha_2\}$
G_2	$\alpha_1 + \alpha_2$	$\{\alpha_1, \alpha_2\}$
D_2	$\alpha_1 + \alpha_2$	$\varnothing$
$\mathsf{D}_n,\ n \geq 3$	$2\alpha_1 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$	$\{\alpha_2, \ldots, \alpha_n\}$
B_3	$\alpha_1 + 2\alpha_2 + 3\alpha_3$	$\{\alpha_1, \alpha_2\}$

One unfortunate feature of the normalization of spherical roots is the possibility of $\Sigma \not\subseteq \Xi$. Therefore, we define the modified weight lattice of X as

$$\widetilde{\Xi} = \widetilde{\Xi}(X) := \Xi(X) + \mathbb{Z}\Sigma(X). \tag{4}$$

According to [KS17, Prop. 5.4], the triple $(\widetilde{\Xi}, \Sigma, S^p)$ is a *weak spherical datum*, i.e., satisfies:

- $\langle \widetilde{\Xi} \mid \alpha^\vee \rangle = 0$ for all $\alpha \in S^p$.
- $\langle \widetilde{\Xi} \mid \alpha^\vee - \beta^\vee \rangle = 0$ whenever $\sigma = \alpha + \beta \in \Sigma$ is of type D_2.
- $\langle \beta \mid \alpha^\vee \rangle \neq -1$ whenever $\alpha, \beta \in S$ with $\alpha, \alpha + \beta \in \Sigma$.

Looking at Table 1 one realizes that there are two types of spherical roots namely those which are also roots of G and those which are not. These types are separated by the middle horizontal line. Each non-root σ is the sum of two strongly orthogonal roots γ_1, γ_2 as can be seen by inspection of Table 2. The set $\{\gamma_1, \gamma_2\}$ can be made unique by requiring that

$$\gamma_1^\vee - \gamma_2^\vee = \delta_1^\vee - \delta_2^\vee \text{ with } \delta_1, \delta_2 \in S. \tag{5}$$

It then follows that the restrictions of $\gamma_1^\vee$ and $\gamma_2^\vee$ to $\widetilde{\Xi}$ coincide. Thus they

TABLE 2.

$\lvert\sigma\rvert$	γ_1, γ_2	$\gamma_1^\vee, \gamma_2^\vee$	$\delta_1^\vee, \delta_2^\vee$
D_2	$\alpha_1,\ \alpha_2$	$\alpha_1^\vee,\ \alpha_2^\vee$	$\alpha_1^\vee,\ \alpha_2^\vee$
$\mathsf{D}_{n\geq 3}$	$(\alpha_1 + \ldots + \alpha_{n-2}) + \alpha_{n-1},$ $(\alpha_1 + \ldots + \alpha_{n-2}) + \alpha_n$	$(\alpha_1^\vee + \ldots + \alpha_{n-2}^\vee) + \alpha_{n-1}^\vee,$ $(\alpha_1^\vee + \ldots + \alpha_{n-2}^\vee) + \alpha_n^\vee$	$\alpha_{n-1}^\vee,\ \alpha_n^\vee$
B_3	$\alpha_1 + \alpha_2 + 2\alpha_3,\ \alpha_2 + \alpha_3$	$\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee,\ 2\alpha_2^\vee + \alpha_3^\vee$	$\alpha_1^\vee,\ \alpha_2^\vee$

define an element of $\widetilde{\Xi}^\vee := \operatorname{Hom}(\widetilde{\Xi}, \mathbb{Z})$ which is denoted by $\sigma^\vee$. On the other hand, if $\sigma \in \Phi$ then the coroot $\sigma^\vee$ already has a meaning. Let $\Sigma^\vee := \{\sigma^\vee \mid \sigma \in \Sigma\}$. A fundamental fact about weak spherical data is the following

THEOREM 2.1 ([KS17, Thm. 7.1]). *Let $(\widetilde{\Xi}, \Sigma, S^p)$ be a weak spherical datum. Then $(\widetilde{\Xi}, \Sigma, \widetilde{\Xi}^\vee, \Sigma^\vee)$ is a based root datum.*

This theorem gives rise to the following definition.

DEFINITION 2.2. The dual group of a G-variety X is the connected *complex* reductive group $G_X^\vee$ whose based root datum is the dual root datum $(\widetilde{\Xi}^\vee, \Sigma^\vee, \widetilde{\Xi}, \Sigma)$.

REMARKS 2.3. *i)* The Weyl group of $G_X^\vee$ is, almost by definition, equal to the little Weyl group $W(X)$ of X. Observe that, due to our normalization, $\Sigma(X)$ and $W(X)$ determine each other unlike, e.g., the normalization used in [Kno96] where the set of spherical roots carries additionally information about the automorphism group of X.
ii) The normalization of the spherical roots by being primitive in $\mathbb{Z}S$ is forced on us by the requirement that $G_X^\vee$ should map to $G^\vee$ with finite kernel (see Theorem 2.5 below). This in turn forces the extension (4) of character groups. Note, however, that for the representation theoretic purposes of [SV17] this is the wrong lattice since it yields multiplicities which are too big.
iii) In the Langlands program, the most common approach is to define the dual group only over $\mathbb{C}$ and we follow this tradition. Working also simplifies

some definitions and arguments, most notably Definition 2.4 of a distinguished homomorphism in Lie algebraic terms. Nevertheless, it should be remarked that $G_X^\vee$ can be defined over $\mathbb{Z}$ and that distinguished homomorphism exist over $\mathbb{Z}[\frac{1}{2}]$ (see [KS17, Prop. 11.1]). Also our main Theorem 1.1 holds in that generality.

The dual group of G, i.e., the connected complex reductive group whose root datum is dual to that of G is denoted by $G^\vee$. It is equipped with a pinning, i.e., a choice of generating root vectors $e_{\alpha^\vee} \in \mathfrak{g}^\vee_{\alpha^\vee}$ with $\alpha \in S$.
It was proved in [KS17] that there exists an almost canonical homomorphism $\varphi : G_X^\vee \to G^\vee$ with finite kernel. To make this more precise, we define for each $\sigma \in \Sigma(X)$ a one-dimensional subspace $\mathfrak{g}^\vee_{\sigma^\vee}$ of $\mathfrak{g}^\vee$ as follows:

$$\mathfrak{g}^\vee_{\sigma^\vee} := \begin{cases} \mathfrak{g}^\vee_{\sigma^\vee} & \text{if } \sigma \in \Phi, \\ [\mathfrak{g}^\vee_{\beta^\vee}, e_{\delta_1^\vee} - e_{\delta_2^\vee}] & \text{if } \sigma \text{ is of type } \mathsf{D}_{n\geq 3}, \\ [\mathfrak{g}^\vee_{\beta^\vee}, 2e_{\delta_1^\vee} - e_{\delta_2^\vee}] & \text{if } \sigma \text{ is of type } \mathsf{B}_3, \\ \mathbb{C}(e_{\delta_1^\vee} - e_{\delta_2^\vee}) & \text{if } \sigma \text{ is of type } \mathsf{D}_2. \end{cases} \tag{6}$$

Here $\beta^\vee := \gamma_1^\vee - \delta_1^\vee = \gamma_2^\vee - \delta_2^\vee$ in case $\sigma \notin \Phi$. It is easy to check that $\beta^\vee \in \Phi^\vee$ unless σ is of type D_2 when $\beta^\vee = 0$. The definition implies that

$$\mathfrak{g}^\vee_{\sigma^\vee} \subseteq \mathfrak{g}^\vee_{\gamma_1^\vee} \oplus \mathfrak{g}^\vee_{\gamma_2^\vee} \subseteq \mathfrak{g}^\vee. \tag{7}$$

Next observe that the maximal tori $T^\vee \subseteq G^\vee$ and $A_X^\vee \subseteq G_X^\vee$ have the cocharacter group Λ and $\widetilde{\Xi}(X)$, respectively. Therefore, the inclusion $\widetilde{\Xi}(X) \hookrightarrow \Lambda$ induces a homomorphism $\varphi_A : A_X^\vee \to T^\vee$ with finite kernel.

DEFINITION 2.4. A homomorphism $\varphi : G_X^\vee \to G^\vee$ is called *distinguished* if $\mathrm{res}_{A_X^\vee} \varphi = \varphi_A$ and $\varphi(\mathfrak{g}^\vee_{X,\sigma^\vee}) = \mathfrak{g}^\vee_{\sigma^\vee}$ for all $\sigma \in \Sigma(X)$.

Here is an immediate consequence of the main result of [KS17]:

THEOREM 2.5. *Let X be a G-variety. Then:*

i) There exists a distinguished homomorphism $\varphi_X : G_X^\vee \to G^\vee$.
ii) Any other distinguished homomorphism is of the form $\varphi \circ \mathrm{Ad}(a)$ with $a \in A_X^\vee$.
iii) The kernel of φ_X is finite.
iv) The image $G_X^* := \varphi_X(G_X^\vee)$ is a well-defined subgroup of $G^\vee$, i.e., it is independent of the choice of φ_X.

Proof. [KS17, Thm. 7.7] shows the existence of an adapted homomorphism $\varphi : G_X^\vee \to G^\vee$ which means that $\mathfrak{g}^\vee_{X,\sigma^\vee}$ is mapped just diagonally into $\mathfrak{g}^\vee_{\gamma_1^\vee} \oplus \mathfrak{g}^\vee_{\gamma_2^\vee} \subseteq \mathfrak{g}^\vee$ in case $\sigma \notin \Phi$. More precisely, the image of φ is contained in the associated group $G_X^\wedge \subseteq G^\vee$ (see loc.cit. Def. 7.2 and Thm 7.3). Thus, there an element t of $T^\wedge_{\mathrm{ad}}$, the maximal torus of the adjoint group of $G_X^\wedge$, such that $\mathrm{Ad}(t) \circ \varphi$ is distinguished (cf. loc.cit Thm. 7.10). The other parts follow from the construction of φ_X. □

REMARKS 2.6. *i)* Let $L_X^\vee \subseteq G^\vee$ be the Levi subgroup corresponding to $S^p(X) \subseteq S$. The pinning of $G^\vee$ induces a pinning of $L_X^\vee$. This in turn gives rise to a canonical principal homomorphism $\psi : \mathrm{SL}(2,\mathbb{C}) \to L_X^\vee$. Then it was shown, [KS17, Prop. 9.10], that the images of φ_X and ψ commute with each other, i.e., they combine to a group homomorphism $G_X^\vee \times SL(2) \to G^\vee$. In fact, the normalization (6) for σ of type $\mathcal{D}_{n\geq 3}$ or B_3 is equivalent to this commutation property.
ii) Distinguished homomorphisms are invariant under certain automorphisms of G. More precisely, let E be a group of automorphisms of the based root datum of G. Then E acts canonically on $G^\vee$ by fixing the chosen pinning $\{e_{\alpha^\vee}\}$. We say that *E and X are compatible* if E fixes $\Xi(X)$, $\Sigma(X)$, and $S^p(X)$. Then (6) implies

$$ {}^s\mathfrak{g}^\vee_{\sigma^\vee} = \mathfrak{g}^\vee_{{}^s\sigma^\vee} \text{ for all } s \in E \text{ and } \sigma \in \Sigma(X). \tag{8}$$

This follows from (6) together with the observation that ${}^s\delta_i^\vee = \overline{\delta}_i^\vee$ in case σ and $\overline{\sigma} = {}^s\sigma$ are both of type B_3. Now (8) implies that E fixes G_X^*. Moreover, the E-action lifts uniquely to $G_X^\vee$ such that φ_X is E-equivariant. Observe, though, that E will in general *not* fix any pinning of $G_X^\vee$, i.e., the action may be *non-standard* in the sense of [KS17, §10].
A typical situation we have in mind is if G and X are defined over a subfield $k_0 \subseteq k$. Then the Galois group E of k_0 acts on the based root datum of G by means of the so-called $*$-action. Since X is defined over k_0 it is known (see [KK16]) that E and X are compatible.
iii) The normalization (6) also plays a role in the proof of Theorem 2.7 below. More precisely, it is needed to prove equation (18).

Now we come to homomorphisms between different dual groups. For this let X, Y be two G-varieties and let φ_X, φ_Y be distinguished homomorphisms. A homomorphism $\eta : G_Y^\vee \to G_X^\vee$ is called *distinguished* if $\varphi_Y = \varphi_X \circ \eta$. Since φ_X and φ_Y have finite kernel, η is unique with finite kernel if it exists. Here is the main result of the paper:

THEOREM 2.7. *Let $\varphi : X \to Y$ be a dominant G-morphism between two G-varieties. Then there exists a distinguished homomorphism $\eta : G_Y^\vee \to G_X^\vee$. This implies, in particular, that $G_Y^* \subseteq G_X^* \subseteq G^\vee$.*

There is an analogous statement for injective morphisms. It is an easy consequence of Theorem 2.7 (see the proof following Theorem 3.2).

THEOREM 2.8. *Let $\varphi : Y \to X$ be an injective G-morphism between two G-varieties (e.g., Y is a G-stable subvariety of X). Then there exists a distinguished homomorphism $\eta : G_Y^\vee \to G_X^\vee$ and therefore, in particular, $G_Y^* \subseteq G_X^* \subseteq G^\vee$.*

The proof of Theorem 2.7 will occupy the remainder of this paper.

REMARK 2.9. In principle, all statements can be formulated and should be valid in some form also over fields of positive characteristic p. However, the necessary changes would come at the expense of the readability of the paper so that we decided to treat the characteristic 0 case separately. The main problems in positive characteristic are: First, the list of spherical roots in Table 1 has to be extended by roots obtained by inseparable isogenies. In particular, the D_2-roots $\alpha_1 + p^n\alpha_2$ cause trouble. Secondly, the weight lattice $\Xi(X)$ may not be $W(X)$-stable, so has to be modified. Finally, our reasoning in Section 5 uses the classification of spherical varieties. This is more a matter of convenience but it would require considerable effort to work around it.

3. REDUCTION TO RANK ONE

We start the proof of Theorem 2.7 by a number of reduction steps. Let $G'_X := (G^*_X)'$ be the semisimple part of G^*_X. Observe that G'_X depends only on $\Sigma(X)$ and not on the lattice $\Xi(X)$. Since the valuation cone $\mathcal{Z}(X)$ is a birational invariant so is $\Sigma(X)$. Therefore we may later (tacitly) replace X and Y by suitable open dense subsets.

LEMMA 3.1. *Let $f : X \to Y$ be dominant or let $f : Y \to X$ be injective. Assume $G'_Y \subseteq G'_X$. Then there is exists a distinguished homomorphism $\eta : G^\vee_Y \to G^\vee_X$.*

Proof. We claim that $\Xi(Y) \subseteq \Xi(X)$ in both cases. This is clear if f is dominant since the pull-back of a B-semiinvariant is again a B-semiinvariant for the same character. For f injective let $p : \overline{X} \to X$ be the normalization and let $\overline{Y} \subseteq \overline{X}$ be a component of $p^{-1}(Y)$ mapping dominantly to Y. By [Kno91, Thm. 1.3 *b)*], every B-semiinvariant rational function on $\overline{Y}$ extends to a B-semiinvariant rational function on $\overline{X}$. Since the character remains unchanged we get $\Xi(Y) \subseteq \Xi(\overline{Y}) \subseteq \Xi(\overline{X}) = \Xi(X)$.
It is a general fact that if $H \subseteq G$ is reductive then the coroot lattice of H is contained in the coroot lattice of G (look at simply connected covers). Applying this to $G'_Y \subseteq G'_X$ we get $\mathbb{Z}\Sigma(Y) \subseteq \mathbb{Z}\Sigma(X)$ and therefore

$$\widetilde{\Xi}(Y) \subseteq \widetilde{\Xi}(X) \tag{9}$$

This inclusion induces a homomorphism of maximal tori $A^\vee_Y \to A^\vee_X$. Because G^*_X is generated by G'_X and $\varphi_X(A^\vee_X)$ (and similarly for Y) it follows that $G^*_Y \subseteq G^*_X$.
Finally, the coweight lattice of $G^{*\vee}_Y := \varphi_X^{-1}(G^*_Y)^0 \subseteq G^\vee_X$ is $\widetilde{\Xi}(Y)_\mathbb{Q} \cap \widetilde{\Xi}(X)$. By (9), it contains the coweight lattice $\widetilde{\Xi}(Y)$ of $G^\vee_Y$. Hence the inclusion $G^*_Y \hookrightarrow G^*_X$ lifts to an isogeny $G^\vee_Y \to G^{*\vee}_Y$ yielding the desired homomorphism $\eta : G^\vee_Y \to G^\vee_X$. □

The following comparison result will be crucial later on. It is a more precise version of Theorem 2.8 in case Y is of codimension 1.

THEOREM 3.2. *Let X be a normal G-variety and let $Y \subset X$ be a G-invariant irreducible subvariety of codimension 1. Then $\Sigma(Y) \subseteq \Sigma(X)$ and therefore $G'_Y \subseteq G'_X$. Moreover, if the valuation $v := v_Y$ induced by Y is non-central then $\mathcal{N}(Y) = \mathcal{N}(X)$. Otherwise, $\mathcal{N}(Y) = \mathcal{N}(X)/\mathbb{Q}v$ and*

$$\Sigma(Y) = \{\sigma \in \Sigma(X) \mid v(\sigma) = 0\}. \tag{10}$$

Proof. This is essentially proved in [Kno93]. Assume first that v is central, i.e., that the restriction of v to $k(X)^B$ is trivial (that's automatic if X is spherical). Then there is a surjective homomorphism

$$\mathcal{N}(X) \twoheadrightarrow \mathcal{N}(Y) \tag{11}$$

with kernel $\mathbb{Q}v$ such that $\mathcal{Z}(Y)$ is the image of $\mathcal{Z}(X)$ (loc.cit. Satz 7.5.2 with $v_0 = o$). Thus, the preimage of $\mathcal{Z}(Y)$ is the cone $\mathcal{Z}(X) + \mathbb{Q}v$. Because of $v \in \mathcal{Z}(X)$, this cone is defined by the inequalities $\sigma \leq 0$ with $\sigma \in \Sigma(X)$ and $v(\sigma) = 0$. This proves (10).
Assume now that v is not central and let v_0 be the restriction of v to $k(X)^B$. Let $\mathcal{Z}_{v_0}$ be the set of G-invariant valuations whose restriction of $k(X)^B$ is a multiple of v_0. Then $\mathcal{Z}_{v_0}$ can be identified with a convex cone in some $\mathbb{Q}$-vector space $\mathcal{N}_{v_0}$. Moreover, $\mathcal{N}(X)$ is a hyperplane of $\mathcal{N}_{v_0}$ such that $\mathcal{Z}_{v_0} \cap \mathcal{N}(X) = \mathcal{Z}(X)$ (see the exact sequence in loc.cit. §5 where $\mathcal{N}_{v_0}$ is corresponds to $\operatorname{Hom}(\mathcal{Q}_{v_0}(K), \mathbb{Q})$).
There is a surjective homomorphism (loc.cit. Satz 7.5.2)

$$\mathcal{N}_{v_0} \twoheadrightarrow \mathcal{N}(Y) \tag{12}$$

with kernel $\mathbb{Q}v$ such that $\mathcal{Z}(Y)$ is the image of $\mathcal{Z}_{v_0}$. Since by assumption $v \notin \mathcal{N}(X)$ we have $\mathcal{N}(X) \xrightarrow{\sim} \mathcal{N}(Y)$, as asserted.
It is a non-trivial fact (loc.cit. Satz 9.2.2) that as a cone $\mathcal{Z}_{v_0}$ is generated by $\mathcal{Z}(X)$ along with one extremal non-central valuation v_e, i.e.,

$$\mathcal{Z}_{v_0} = \mathcal{Z}(X) + \mathbb{Q}_{\geq 0} v_e. \tag{13}$$

Let $v = v_1 + cv_e$ with $v_1 \in \mathcal{Z}(X)$ and $c > 0$. Then the preimage of $\mathcal{Z}(Y)$ in $\mathcal{N}_{v_0}$ equals

$$\mathcal{Z}_{v_0} + \mathbb{Q}v = \mathcal{Z}(X) + \mathbb{Q}_{\geq 0} v_e + \mathbb{Q}v = \mathcal{Z}(X) + \mathbb{Q}v_1 + \mathbb{Q}v_e. \tag{14}$$

This shows that

$$\mathcal{Z}(Y) = (\mathcal{Z}_{v_0} + \mathbb{Q}v) \cap \mathcal{N}(X) = \mathcal{Z}(X) + \mathbb{Q}v_1 \tag{15}$$

is defined by the inequalities $\sigma \leq 0$ with $\sigma \in \Sigma(X)$ and $v_1(\sigma) = 0$. In particular $\Sigma(Y) \subseteq \Sigma(X)$. □

At this point we already have a

Proof of Theorem 2.8 assuming Theorem 2.7. We may assume that Y is a subvariety of X. It suffices to construct a normal G-variety $\overline{X}$, a birational G-morphism $\pi : \overline{X} \to X$, and a G-stable subvariety $\overline{Y} \subset \overline{X}$ of codimension 1 which maps dominantly to Y. In fact, in this case we have $G'_Y \subseteq G'_{\overline{Y}} \subseteq G'_{\overline{X}} = G'_X$ by Theorem 2.7 and Theorem 3.2. Then Lemma 3.1 yields a distinguished homomorphism $G^\vee_Y \to G^\vee_X$.
To construct $\overline{X}$ let $p : X_1 \to X$ be the normalization of X and let $Y_1 \subseteq X_1$ be a component of $p^{-1}(Y)$ which maps surjectively to Y. Next, let $X_2 \to X_1$ be the blow up of X_1 in Y_1 and let $Y_2 \subset X_2$ be a component of the exceptional divisor. Finally, the normalization $p_2 : \overline{X} \to X_2$ with $\overline{Y} \subset \overline{X}$ a component of $p_2^{-1}(Y_2)$ meets all requirements. □

For the next step, recall that a homogeneous variety G/H is *parabolically induced* if there is a proper parabolic subgroup $Q \subset G$ with $Q_u \subseteq H \subseteq Q$. It is *cuspidal* if is not parabolically induced and if H does not contain a simple factor of G.

Lemma 3.3. *Assume $G'_Y \subseteq G'_X$ in the following situation:*

- *G is of adjoint type,*
- *$Y = G/H$ is homogeneous, spherical and cuspidal of rank 1, and H is connected.*
- *$X = G/P$ where $P \subset H$ is a maximal parabolic subgroup.*

Then $G'_Y \subseteq G'_X$ for all G-varieties X, Y and all dominant G-morphisms $X \to Y$.

Proof. We will prove the assertion by induction on $\dim X + \dim G$. For this let $f : X \to Y$ be an arbitrary dominant G-morphism.
Reduction to $\operatorname{rk} Y = \#\Sigma(Y) = 1$*:* Assume $\operatorname{rk} Y \geq 2$. Every $\tau \in \Sigma(Y)$ is a simple coroot of G'_Y and therefore induces a semisimple rank-1-subgroup $G'_Y(\tau) \subseteq G'_Y$. Since the subgroups of this form generate G'_Y it suffices to prove $G'_Y(\tau) \subseteq G'_X$ for all τ.
If $\Sigma(Y) = \varnothing$ then $G'_Y = 1$ and there is nothing to prove. So fix $\tau \in \Sigma(Y)$. Then τ defines a codimension-1-face $\mathcal{F}$ of the valuation cone $\mathcal{Z}(Y)$. Since $\dim \mathcal{F} = \operatorname{rk} Y - 1 \geq 1$ there is a non-trivial valuation v in the relative interior of $\mathcal{F}$. Let $Y \hookrightarrow \overline{Y} = Y \cup Y_0$ be the smooth equivariant embedding where Y_0 is an irreducible divisor such that v_{Y_0} is a rational multiple of v. Then $\operatorname{rk} Y_0 = \operatorname{rk} Y - 1$ and $\Sigma(Y_0) = \{\tau\}$ by Theorem 3.2. By [Kno93, Kor. 3.2] there exists a lift of v to a (possibly non-central) equivariant valuation $\overline{v}$ of X. This gives rise to a similar embedding $X \hookrightarrow \overline{X} = X \cup X_0$ such that f extends to a morphism $\overline{X} \to \overline{Y}$ which maps X_0 dominantly to Y_0. Theorem 3.2 implies that $\Sigma(X_0) \subseteq \Sigma(X)$. Hence we have

$$G'_Y(\tau) = G'_{Y_0} \text{ and } G'_{X_0} \subseteq G'_X. \tag{16}$$

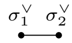
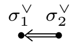
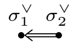
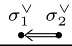
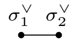
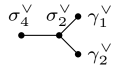
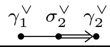
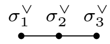
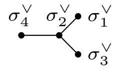
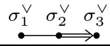
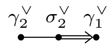
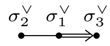
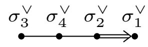
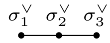
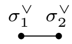
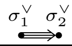
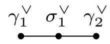
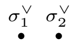
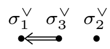
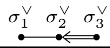
By induction we have $G'_{Y_0} \subseteq G'_{X_0}$ which proves the assertion.
Reduction to G semisimple: Let $Z = Z(G)^0$ be the connected center of G. If Z acts trivially on X then one can replace G by the semisimple group G/Z. Otherwise, consider the morphism $X_0 := X'/Z \to Y_0 := Y'/Z$ where $X' \subseteq X$ and $Y' \subseteq Y$ are non-empty, open, and G-stable such that the Z-orbit spaces exist (these exist by [Ros56, Thm. 2]). Because of $\Sigma(X_0) = \Sigma(X)$ and $\Sigma(Y_0) = \Sigma(Y)$ by [Kno93, Satz 8.1.4] we have $G'_Y \subseteq G'_X$ if and only if $G'_{Y_0} \subseteq G'_{X_0}$. The latter holds by induction.
Reduction to X and Y homogeneous: Let $Y_0 \subseteq Y$ be a general orbit. Then $\Sigma(Y_0) = \Sigma(Y)$ by [Kno90, Satz 6.5.4]. Let $X_0 \subseteq X$ be a general orbit in the preimage of Y_0 in X. Then X_0 is also a general orbit of X and therefore $\Sigma(X_0) = \Sigma(X)$. This proves the assertion by induction unless $X = X_0$ and $Y = Y_0$.
Reduction to f proper: We may assume that X and Y are homogeneous. If f is not proper choose a normal equivariant embedding $X \hookrightarrow \overline{X}$ such that f

extends to a proper morphism $\overline{X} \to Y$. Let X_0 be a component of $\overline{X} \setminus X$. By blowing up X in X_0 and normalizing, if necessary, we may assume that X_0 is a G-invariant irreducible divisor. Then $\Sigma(X_0) \subseteq \Sigma(X)$ by Theorem 3.2 and therefore $G'_{X_0} \subseteq G'_X$. The assertion follows by applying the induction hypotheses to $X_0 \to Y$.
Because of the last steps we may assume that $X = G/P$, $Y = G/H$ with $P^0 \subseteq H^0$ parabolic and $\operatorname{rk} Y = 1$.
Reduction to P and H connected: Follows from the fact that $W(X)$, hence $\Sigma(X)$, hence G'_X is invariant under étale maps (see [Kno90, Satz 6.5.3]).
Reduction to $P \subset H$ maximal parabolic: Assume that there is a parabolic Q with $P \subset Q \subset H$ and put $Z := G/Q$. We may assume P to be maximal parabolic in Q. By induction on the morphism $Z \to Y$ it suffices to prove $G'_Z \subseteq G'_X$ for the morphism $X \to Z$. This is indeed implied by the first reduction step unless $\operatorname{rk} Z = 1$.
Reduction to H cuspidal: Suppose there is a parabolic subgroup $Q = LQ_u \subset G$ with $Q_u \subseteq H \subseteq Q$. Then $Q_u \subseteq H_u$ and $H_u \subseteq P_u$ (since P is parabolic in H). This shows that P is also induced by Q. The $L = Q/Q_u$-varieties $X_0 = Q/P = L/(P \cap L)$ and $Y_0 = Q/H = L/(H \cap L)$ have $\Sigma(X_0) = \Sigma(X)$ and $\Sigma(Y_0) = \Sigma(X)$ (see, e.g., [KK16] Prop. 8.2). Then we conclude by induction. If H contains a simple factor G_1 of G then there are decompositions $G = G_1 \cdot G_2$ and $H = G_1 \cdot H_2$. A maximal parabolic subgroup of H is either of the form $P_1 \cdot H_1$ (in which case $\Sigma(X) = \Sigma(Y)$) or $G_1 \cdot P_2$ (in which case G_1 acts trivially on both X and Y and we may replace G by G/G_1).
Reduction to H spherical: The only cuspidal homogeneous rank-1-varieties which are not spherical are of the form G/H where $G = \mathrm{SL}(2)$ and H is finite ([Pan95]). By previous reduction steps we may assume that H is connected (hence trivial) and contains a proper parabolic subgroup. So this case does not occur.
This finishes the reduction of a general dominant morphism to the situation in the Lemma. □

4. The rank-1-case

Using Lemma 3.3, the proof of Theorem 2.7 is now reduced to the cases where G is of adjoint type, $Y = G/H$ is homogeneous, spherical and cuspidal of rank 1, with H connected, and $X = G/P$ where $P \subset H$ is a maximal parabolic subgroup.
The classification of all possible pairs (G, H) is due to Akhiezer [Ahi83] (see also Brion's simplification [Bri89]) and is reproduced in Table 3 below. In the case B'_n, the group P_n denotes a maximal parabolic subgroup of $\mathrm{SO}(2n)$ whose Levi part is $GL(n)$. In C'_n, the group $B_2 \subseteq \mathrm{Sp}(2)$ is a Borel subgroup. Finally U_3 in case G'_2 is a 3-dimensional unipotent group. The two columns on the right will be used in the final step of the proof of Theorem 2.7.
We have $\Sigma(G/H) = \{\tau\}$ and we need to compute $\Sigma = \Sigma(G/P)$ for all maximal parabolic subgroups $P \subset H$. This is done in Section 5. All varieties G/P turn

Table 3.

	G	H	$\tau^\wedge$	$\Sigma^\wedge$
A_1	PGL(2)	$\mathbf{G}_m$		
$\mathsf{A}_{n\geq 2}$	PGL$(n+1)$	$\mathbf{G}_m\mathrm{SL}(n)$	(1) $\sigma_1^\vee{+}\sigma_2^\vee$	
$\mathsf{B}_{n\geq 2}$	SO$(2n+1)$	SO$(2n)$	(1) $2\sigma_1^\vee{+}\sigma_2^\vee$	
			(2) $\sigma_1^\vee$	
$\mathsf{B}'_{n\geq 2}$	SO$(2n+1)$	P_n	(1) $2\sigma_1^\vee{+}\sigma_2^\vee$	
			(2) $2\sigma_1^\vee{+}\sigma_2^\vee$	
$\mathsf{C}_{n\geq 3}$	PSp$(2n)$	Sp(2)Sp$(2n{-}2)$	(1) $\sigma_1^\vee$	
			(2) $\sigma_1^\vee{+}\sigma_2^\vee$	
			(3) $\gamma_1^\vee{+}2\sigma_2^\vee{+}\gamma_2^\vee{+}\sigma_3^\vee$	
			(4) $\gamma_1^\vee{+}2\sigma_2^\vee{+}2\gamma_2^\vee$	
$\mathsf{C}'_{n\geq 3}$	PSp$(2n)$	B_2Sp$(2n-2)$	(1) $\sigma_1^\vee{+}\sigma_2^\vee{+}\sigma_3^\vee$	
			(2) $\sigma_1^\vee{+}2\sigma_2^\vee{+}\sigma_3^\vee{+}\sigma_4^\vee$	
			(3) $\sigma_1^\vee{+}2\sigma_2^\vee{+}2\sigma_3^\vee$	
F_4	F_4	Spin(9)	(1) $\gamma_2^\vee{+}2\sigma_2^\vee{+}2\gamma_1^\vee$	
			(2) $\sigma_2^\vee{+}2\sigma_3^\vee{+}2\sigma_1^\vee$	
			(3) $\sigma_3^\vee{+}2\sigma_4^\vee{+}2\sigma_2^\vee{+}2\sigma_1^\vee$	
			(4) $\sigma_2^\vee{+}\sigma_1^\vee{+}\sigma_3^\vee$	
G_2	G_2	SL(3)	(1) $\sigma_1^\vee + \sigma_2^\vee$	
G'_2	G_2	$\mathbf{G}_m\mathrm{SL}(2)U_3$	(1) $\sigma_1^\vee{+}3\sigma_2^\vee$	
$\mathsf{D}_{n\geq 2}$	PSO$(2n)$	SO$(2n-1)$	(1) $\{\gamma_1^\vee{+}\sigma_1^\vee, \sigma_1^\vee{+}\gamma_2^\vee\}$	
			(2) $\{\sigma_1^\vee, \sigma_2^\vee\}$	
B''_3	SO(7)	G_2	(1) $\{\sigma_1^\vee{+}\sigma_3^\vee, \sigma_2^\vee\}$	
			(2) $\{\sigma_1^\vee{+}\sigma_2^\vee{+}\sigma_3^\vee, 2\sigma_2^\vee{+}\sigma_3^\vee\}$	

out to be spherical, even wonderful, a fact for which we don't have a conceptual argument.
For every spherical root σ define its set $\sigma^\wedge$ of associated roots as

$$(17)\qquad \sigma^\wedge = \begin{cases} \{\sigma^\vee\} & \text{if } \sigma\in\Phi,\\ \{\gamma_1^\vee,\gamma_2^\vee\} & \text{otherwise (with } \gamma_i^\vee \text{ as in Table 2).}\end{cases}$$

Put $\Sigma^\wedge := \cup_{\sigma\in\Sigma}\sigma^\wedge$. It was shown in [KS17] that $\Sigma^\wedge$ is the basis of a maximal rank subgroup $G_X^\wedge \subseteq G^\vee$. Moreover, the root system of $G_X^\vee$ is obtained from that of $G_X^\wedge$ by a process called "folding". Let $\Phi_X^\wedge$ be the set of roots of $G_X^\wedge$.
From Table 4 one can read off $\Sigma^\wedge$ and $\tau^\wedge$ as a linear combination of $\Sigma^\wedge$. The result is recorded in the two right hand columns of Table 3. As an example, consider case $\mathsf{C}_n(4)$. Here $\sigma_1=\gamma_1+\gamma_2$ with $\gamma_1=\alpha_1$ and $\gamma_2=\alpha_n$. Since σ_2 is a root we have $\Sigma^\wedge=\{\gamma_1^\vee,\sigma_2^\vee,\gamma_2^\vee\}$ which is a basis of a root system of type B_3. Moreover, one verifies $\tau^\wedge=\alpha_1^\vee+2\alpha_2^\vee+\ldots+2\alpha_{n-1}^\vee+2\alpha_n^\vee=\gamma_1^\vee+2\sigma_2^\vee+2\gamma_2^\vee$.
Now it is easy to finish the proof of Theorem 2.7.
First, we consider the case $\Sigma^\wedge=\Sigma^\vee$ (recognizable by the non-appearance of $\gamma_i^\vee$'s). Here one checks that $\tau^\wedge\subseteq\Phi_X^\wedge$ which implies $G_Y^\vee\subseteq G_X^\wedge=G_X^\vee$.
Next assume that $\Sigma^\wedge\ne\Sigma^\vee$ but $\tau^\wedge=\{\tau^\vee\}$. Here, one checks that $\tau^\vee$ is actually the highest root of $\Phi_X^\wedge$. Since all simple roots of $G_X^\wedge$ restrict to simple roots of $G_X^\vee$, there is no other root of $G_X^\wedge$ which has the same restriction as $\tau^\wedge$. This implies $\mathfrak{g}_{X,\tau}^\vee=\mathfrak{g}_{X,\tau}^\wedge=\mathfrak{g}_\tau^\vee$ and therefore $G_Y^\vee\subseteq G_X^\vee$.
The only case remaining is that of $\mathsf{D}_n(1)$ depending on a parameter $\nu\in\{1,\ldots,n-2\}$. It suffices to prove

$$(18)\qquad \mathfrak{g}_{\tau^\vee}^\vee=[\mathfrak{g}_{\sigma_1^\vee}^\vee,\mathfrak{g}_{\sigma_2^\vee}^\vee]$$

since then $\mathfrak{g}_{\tau^\vee}^\vee\subseteq\mathfrak{g}_X^\vee$ and therefore $G_Y^\vee\subseteq G_X^\vee$.
Using the standard basis ε_i for the weight lattice of D_n and the normalization (6) we have

$$(19)\qquad \mathfrak{g}_{\tau^\vee}^\vee=[\mathfrak{g}_{\varepsilon_1-\varepsilon_{n-1}}^\vee,E] \text{ with } E:=e_{\varepsilon_{n-1}-\varepsilon_n}-e_{\varepsilon_{n-1}+\varepsilon_n}.$$

If $\nu=n-2$ then $\mathfrak{g}_{\sigma_2^\vee}^\vee=\mathbb{C}E$ and $\sigma_1^\vee=\varepsilon_1-\varepsilon_{\nu+1}=\varepsilon_1-\varepsilon_{n-1}$ which proves (18). Otherwise, we have

$$(20)\qquad \mathfrak{g}_{\sigma_2^\vee}^\vee=[\mathfrak{g}_{\varepsilon_{\nu+1}-\varepsilon_{n-1}}^\vee,E]$$

and therefore

$$(21)\qquad [\mathfrak{g}_{\sigma_1^\vee}^\vee,\mathfrak{g}_{\sigma_2^\vee}^\vee]=[\mathfrak{g}_{\varepsilon_1-\varepsilon_{\nu+1}}^\vee,[\mathfrak{g}_{\varepsilon_{\nu+1}-\varepsilon_{n-1}}^\vee,E]]=[\mathfrak{g}_{\varepsilon_1-\varepsilon_{n-1}}^\vee,E]=\mathfrak{g}_{\tau^\vee}^\vee.$$

Theorem 2.7 is proved. □

5. Appendix: Maximal parabolics in rank-1-subgroups

In the following, we use the classification of spherical varieties using Luna diagrams due to Luna [Lun01], Losev [Los09], and Bravi-Pezzini [BP16]. A very good introduction to this topic can be found in [BL11].
Table 4 below lists the Luna diagrams of all cuspidal rank-1-varieties $Y=G/H$ (G adjoint, H connected). For each such diagram we list a number of further

Luna diagrams. We claim that these classify all varieties $X = G/P$ with $P \subset H$ maximal parabolic.

Along with the diagram of X we are also giving the complete generalized Cartan matrix so that the "decorations" of the diagrams by arrow heads "$<$" or "$>$" are not needed. The rows of the Cartan matrix are labelled by the spherical roots $\sigma_i \in \Sigma := \Sigma(X)$. The columns correspond to the colors, i.e., to the B-invariant irreducible divisors D_j of X. They also correspond to the circles (filled or empty) in the Luna diagram. The index j of D_j means that D_j is attached to the simple root α_j. The entries of the Cartan matrix are the numbers $v_{D_j}(f_{\sigma_i}) \in \mathbb{Z}$ where $f_{\sigma_i} \in k(X)$ is a B-semiinvariant for the character σ_i.

The claim can be verified in several easy steps:

1. First, one checks that all diagrams and Cartan matrices satisfy Luna's axioms. Thus, each belongs to a unique spherical (even wonderful) variety $X = G/P$.

2. Let $\mathcal{D}_0$ be the set of colors which are printed in boldface. The corresponding columns sum up to 0 which shows that $\mathcal{D}_0$ is distinguished in the sense of [BL11, 2.3]. Therefore, $\mathcal{D}_0$ defines a G-morphism $X \to Y' = G/H'$ with $P \subseteq H' \subseteq G$ and H'/P is connected.

3. Next one uses [BL11, 2.3] to verify that the spherical systems of Y and Y' coincide which then implies that H' is conjugate to H. To do this one shows that τ (whose coordinates in terms of the σ_i are provided in the leftmost column) generates the orthogonal complement of the boldface columns. One also has to observe that the colors not in $\mathcal{D}_0$ correspond to the colors of Y.

4. That P is parabolic in H is equivalent to $G/P \to G/H$ being proper which is equivalent to no G-invariant valuation of G/P restricting to the trivial valuation of G/H. This in turn translates into τ being a linear combination of the σ_i with strictly positive coefficients. This is clear from looking at the leftmost column.

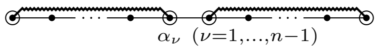
5. The submatrix given by the boldface entries is always a square matrix of defect 1. Hence the columns of every proper subset of $\mathcal{D}_0$ are linear independent which shows that such a subset in not distinguished. This means that P is maximal proper subgroup of H.

6. The preceding steps show that P is a maximal parabolic in H. To see that all of them are listed one checks that the number of items in the table equals the number of G-conjugacy classes of maximal parabolics of H. To do this one can consult Table 3 for H. In most cases this number equals the number of maximal parabolics of H. Only in the cases B_n and G_2 there is an element of $N_G(H)$ acting as an outer automorphism on H. This results in two non-conjugate maximal parabolics of H being conjugate in G resulting in one item less.

Table 4.

A_n 

$\tau = \alpha_1 + \ldots + \alpha_n$

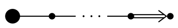
(1) 

α_ν $(\nu{=}1,\ldots,n{-}1)$

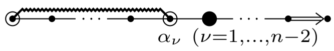
$\sigma_1 = \alpha_1 + \ldots + \alpha_\nu$

$\sigma_2 = \alpha_{\nu+1} + \ldots + \alpha_n$

		D_1	$\boldsymbol{D_\nu}$	$\boldsymbol{D_{\nu+1}}$	D_n
1	σ_1	1	**1**	**−1**	0
1	σ_2	0	**−1**	**1**	1

B_n 

$\tau = \alpha_1 + \ldots + \alpha_n$

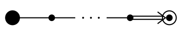
(1) 

α_ν $(\nu{=}1,\ldots,n{-}2)$

$\sigma_1 = \alpha_1 + \ldots + \alpha_\nu$

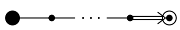
$\sigma_2 = \alpha_{\nu+1} + \ldots + \alpha_n$

		D_1	$\boldsymbol{D_\nu}$	$\boldsymbol{D_{\nu+1}}$
1	σ_1	1	**1**	**−1**
1	σ_2	0	**−1**	**1**

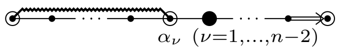
(2) 

$\sigma_1 = \alpha_1 + \ldots + \alpha_n$

		D_1	$\boldsymbol{D_n}$
1	σ_1	1	**0**

B'_n 

$\tau = \alpha_1 + \ldots + \alpha_n$

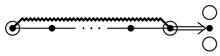
(1) 

α_ν $(\nu{=}1,\ldots,n{-}2)$

$\sigma_1 = \alpha_1 + \ldots + \alpha_\nu$

$\sigma_2 = \alpha_{\nu+1} + \ldots + \alpha_n$

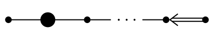
		D_1	$\boldsymbol{D_\nu}$	$\boldsymbol{D_{\nu+1}}$	D_n
1	σ_1	1	**1**	**−1**	0
1	σ_2	0	**−1**	**1**	0

(2) 

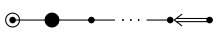
$\sigma_1 = \alpha_1 + \ldots + \alpha_{n-1}$

$\sigma_2 = \alpha_n$

		D_1	$\boldsymbol{D_{n-1}}$	$\boldsymbol{D_n^+}$	D_n^-
1	σ_1	1	**1**	**−1**	−1
1	σ_2	0	**−1**	**1**	1

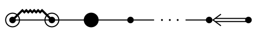
C_n 

$\tau = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n$

(1) 

$\sigma_1 = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n$

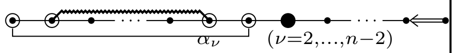
		$\boldsymbol{D_1}$	D_2
1	σ_1	**0**	1

(2) 

$\sigma_1 = \alpha_1 + \alpha_2$

$\sigma_2 = \alpha_2 + 2\alpha_3 + \ldots + 2\alpha_{n-1} + \alpha_n$

		$\boldsymbol{D_1}$	D_2	$\boldsymbol{D_3}$
1	σ_1	**1**	1	**−1**
1	σ_2	**−1**	0	**1**

(3) 

α_ν $(\nu{=}2,\ldots,n{-}2)$

$\sigma_1 = \alpha_1 + \alpha_{\nu+1}$

$\sigma_2 = \alpha_2 + \ldots + \alpha_\nu$

$\sigma_3 = \alpha_{\nu+1} + 2\alpha_{\nu+2} + \ldots + 2\alpha_{n-1} + \alpha_n$

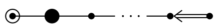
		$\boldsymbol{D_1}$	D_2	$\boldsymbol{D_\nu}$	$\boldsymbol{D_{\nu+2}}$
1	σ_1	**2**	−1	**−1**	**−1**
2	σ_2	**−1**	1	**1**	**0**
1	σ_3	**0**	0	**−1**	**1**

(4) 

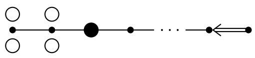
$\sigma_1 = \alpha_1 + \alpha_n$

$\sigma_2 = \alpha_2 + \ldots + \alpha_{n-1}$

		$\boldsymbol{D_1}$	D_2	$\boldsymbol{D_{n-1}}$
1	σ_1	**2**	−1	**−2**
2	σ_2	**−1**	1	**1**

C'_n 

$\tau = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n$

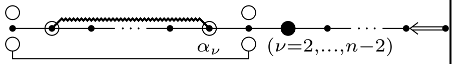
(1) 

$\sigma_1 = \alpha_1$

$\sigma_2 = \alpha_2$

$\sigma_3 = \alpha_2 + 2\alpha_3 + \ldots + 2\alpha_{n-1} + \alpha_n$

		$\boldsymbol{D_1^+}$	D_1^-	D_2^+	$\boldsymbol{D_2^-}$	$\boldsymbol{D_3}$
1	σ_1	**1**	1	0	**−1**	**0**
1	σ_2	**0**	−1	1	**1**	**−1**
1	σ_3	**−1**	0	0	**0**	**1**

(2) 

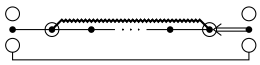
$\sigma_1 = \alpha_1$

$\sigma_2 = \alpha_2 + \ldots + \alpha_\nu$

$\sigma_3 = \alpha_{\nu+1}$

$\sigma_4 = \alpha_{\nu+1} + 2\alpha_{\nu+2} + \ldots + 2\alpha_{n-1} + \alpha_n$

		D_1^+	$\boldsymbol{D_1^-}$	D_2	$\boldsymbol{D_\nu}$	$\boldsymbol{D_{\nu+1}^+}$	$\boldsymbol{D_{\nu+2}}$
1	σ_1	1	**1**	−1	**0**	**−1**	**0**
2	σ_2	0	**−1**	1	**1**	**0**	**0**
1	σ_3	−1	**1**	0	**−1**	**1**	**−1**
1	σ_4	0	**0**	0	**−1**	**0**	**1**

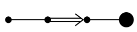
(3) 

$\sigma_1 = \alpha_1$

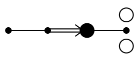
$\sigma_2 = \alpha_2 + \ldots + \alpha_{n-1}$

$\sigma_3 = \alpha_n$

		D_1^+	$\boldsymbol{D_1^-}$	D_2	$\boldsymbol{D_{n-1}}$	$\boldsymbol{D_n^+}$
1	σ_1	1	**1**	−1	**0**	**−1**
2	σ_2	0	**−1**	1	**1**	**0**
1	σ_3	−1	**1**	0	**−2**	**1**

F_4 

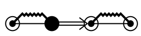
$\tau = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$

(1) 

$\sigma_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3$

$\sigma_2 = \alpha_4$

		$\boldsymbol{D_3}$	$\boldsymbol{D_4^+}$	D_4^-
1	σ_1	**2**	**−2**	−1
2	σ_2	**−1**	**1**	1

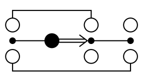
(2) 

$\sigma_1 = \alpha_1 + \alpha_2$

$\sigma_2 = \alpha_2 + \alpha_3$

$\sigma_3 = \alpha_3 + \alpha_4$

		$\boldsymbol{D_1}$	$\boldsymbol{D_2}$	$\boldsymbol{D_3}$	D_4
1	σ_1	**1**	**1**	**−2**	0
1	σ_2	**−1**	**1**	**0**	−1
2	σ_3	**0**	**−1**	**1**	1

(3) 

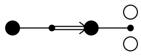
$\sigma_1 = \alpha_1$

$\sigma_2 = \alpha_2 + \alpha_3$

$\sigma_3 = \alpha_3$

$\sigma_4 = \alpha_4$

		$\boldsymbol{D_1^+}$	$\boldsymbol{D_1^-}$	$\boldsymbol{D_2}$	$\boldsymbol{D_3^-}$	D_4^+
1	σ_1	**1**	**1**	**−1**	**−1**	−1
2	σ_2	**0**	**−1**	**1**	**0**	0
1	σ_3	**1**	**−1**	**−1**	**1**	0
2	σ_4	**−1**	**1**	**0**	**0**	1

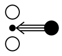
(4) 

$\sigma_1 = \alpha_1 + \alpha_2 + \alpha_3$
$\sigma_2 = \alpha_2 + 2\alpha_3 + \alpha_4$
$\sigma_3 = \alpha_4$

		$\boldsymbol{D_1}$	$\boldsymbol{D_3}$	$\boldsymbol{D_4^+}$	D_4^-
1	σ_1	**1**	**0**	**−1**	0
1	σ_2	**−1**	**1**	**0**	0
1	σ_3	**0**	**−1**	**1**	1

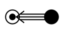
G_2 

$\tau = 2\alpha_1 + \alpha_2$

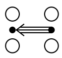
(1) 

$\sigma_1 = \alpha_1$
$\sigma_2 = \alpha_1 + \alpha_2$

		D_1^+	$\boldsymbol{D_1^-}$	$\boldsymbol{D_2}$
1	σ_1	1	**1**	**−1**
1	σ_2	0	**−1**	**1**

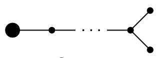
G_2' 

$\tau = \alpha_1 + \alpha_2$

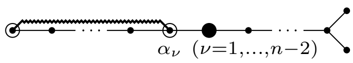
(1) 

$\sigma_1 = \alpha_1$
$\sigma_2 = \alpha_2$

		D_1^+	$\boldsymbol{D_1^-}$	$\boldsymbol{D_2^+}$	D_2^-
1	σ_1	1	**1**	**−1**	0
1	σ_2	−2	**−1**	**1**	1

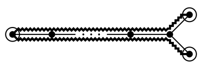
D_n 

$\tau = 2\alpha_1 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$

(1) 

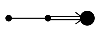
$\sigma_1 = \alpha_1 + \ldots + \alpha_\nu$
$\sigma_2 = 2\alpha_{\nu+1} + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$

		D_1	$\boldsymbol{D_\nu}$	$\boldsymbol{D_{\nu+1}}$
2	σ_1	1	**1**	**−1**
1	σ_2	0	**−2**	**2**

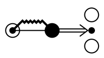
(2) 

$\sigma_1 = \alpha_1 + \ldots + \alpha_{n-2} + \alpha_{n-1}$
$\sigma_2 = \alpha_1 + \ldots + \alpha_{n-2} + \alpha_n$

		D_1	$\boldsymbol{D_{n-1}}$	$\boldsymbol{D_n}$
1	σ_1	1	**1**	**−1**
1	σ_2	1	**−1**	**1**

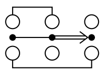
B_3'' 

$\tau = \alpha_1 + 2\alpha_2 + 3\alpha_3$

(1) 

$\sigma_1 = \alpha_1 + \alpha_2$
$\sigma_2 = \alpha_2 + a_3$
$\sigma_3 = \alpha_3$

		$\boldsymbol{D_1}$	$\boldsymbol{D_2}$	$\boldsymbol{D_3^+}$	D_3^-
1	σ_1	**1**	**1**	**−2**	0
1	σ_2	**−1**	**1**	**0**	0
2	σ_3	**0**	**−1**	**1**	1

(2) 

$\sigma_1 = \alpha_1$
$\sigma_2 = \alpha_2$
$\sigma_3 = \alpha_3$

		$\boldsymbol{D_1^+}$	$\boldsymbol{D_1^-}$	$\boldsymbol{D_2^-}$	D_3^+
1	σ_1	**1**	**1**	**−2**	−1
2	σ_2	**1**	**−2**	**1**	0
3	σ_3	**−1**	**1**	**0**	1

References

[Ahi83] Dmitry Ahiezer, *Equivariant completions of homogeneous algebraic varieties by homogeneous divisors*, Ann. Global Anal. Geom. 1 (1983), 49–78.

[BL11] Paolo Bravi and Domingo Luna, *An introduction to wonderful varieties with many examples of type* F_4, J. Algebra 329 (2011), 4–51, arxiv:0812.2340.

[BP16] Paolo Bravi and Guido Pezzini, *Primitive wonderful varieties*, Math. Z. 282 (2016), 1067–1096, arxiv:1106.3187.

[Bri89] Michel Brion, *On spherical varieties of rank one (after D. Ahiezer, A. Huckleberry, D. Snow)*, Group actions and invariant theory (Montreal, PQ, 1988), CMS Conf. Proc., vol. 10, Amer. Math. Soc., Providence, RI, 1989, pp. 31–41.

[Bri90] Michel Brion, *Vers une généralisation des espaces symétriques*, J. Algebra 134 (1990), 115–143.

[GN10] Dennis Gaitsgory and David Nadler, *Spherical varieties and Langlands duality*, Mosc. Math. J. 10 (2010), 65–137, 271, arxiv:math/0611323.

[Kno90] Friedrich Knop, *Weylgruppe und Momentabbildung*, Invent. Math. 99 (1990), 1–23 (German, with English summary).

[Kno91] Friedrich Knop, *The Luna-Vust theory of spherical embeddings*, Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), Manoj Prakashan, Madras, 1991, pp. 225–249.

[Kno93] Friedrich Knop, *Über Bewertungen, welche unter einer reduktiven Gruppe invariant sind*, Math. Ann. 295 (1993), 333–363.

[Kno94] Friedrich Knop, *The asymptotic behavior of invariant collective motion*, Invent. Math. 116 (1994), 309–328.

[Kno96] Friedrich Knop, *Automorphisms, root systems, and compactifications of homogeneous varieties*, J. Amer. Math. Soc. 9 (1996), 153–174.

[KK16] Friedrich Knop and Bernhard Krötz, *Reductive group actions*, Preprint (2016), 62 pp., arxiv:1604.01005.

[KS17] Friedrich Knop and Barbara Schalke, *The dual group of a spherical variety*, Preprint (2017), 30 pp., arxiv:1702.08264.

[Los09] Ivan Losev, *Uniqueness property for spherical homogeneous spaces*, Duke Math. J. 147 (2009), 315–343, arxiv:0904.2937.

[Lun01] Domingo Luna, *Variétés sphériques de type A*, Publ. Math. Inst. Hautes Études Sci. 94 (2001), 161–226.

[LV83] Domingo Luna and Thierry Vust, *Plongements d'espaces homogènes*, Comment. Math. Helv. 58 (1983), 186–245.

[Pan95] Dmitri Panyushev, *On homogeneous spaces of rank one*, Indag. Math. (N.S.) 6 (1995), 315–323.

[Ros56] Maxwell Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. 78 (1956), 401–443.

[SV17] Yiannis Sakellaridis and Akshay Venkatesh, *Periods and harmonic analysis on spherical varieties* (2017), 296 p., arxiv:1203.0039v4.

Friedrich Knop
Department Mathematik
FAU Erlangen-Nürnberg
Cauerstraße 11
D-91058 Erlangen
Germany
friedrich.knop@fau.de