

WEIGHTED KHOVANOV-LAUDA-ROUQUIER ALGEBRAS

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ABSTRACT. In this paper, we define a generalization of Khovanov-Lauda-Rouquier algebras which we call *Weighted Khovanov-Lauda-Rouquier algebras*. We show that these algebras carry many of the same structures as the original Khovanov-Lauda-Rouquier algebras, including induction and restriction functors which induce a twisted bialgebra structure on their Grothendieck groups.

We also define natural *steadied quotients* of these algebras, which in an important special cases give categorical actions of an associated Lie algebra. These include the algebras categorifying tensor products and Fock spaces defined by the author and Stroppel in [Web17a, SW]. For symmetric Cartan matrices, weighted KLR algebras also have a natural geometric interpretation as convolution algebras, generalizing that for the original KLR algebras by Varagnolo and Vasserot [VV11]; this result has positivity consequences important in the theory of crystal bases. In this case, we can also relate the Grothendieck group and its bialgebra structure to the Hall algebra of the associated quiver.

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1 INTRODUCTION

In this paper, we introduce a generalization of Khovanov-Lauda-Rouquier algebras [KL09, Rou], which we call WEIGHTED KHOVANOV-LAUDA-ROUQUIER ALGEBRAS. The original KLR algebras are finite dimensional algebras associated to a quiver, or more generally a symmetrizable Cartan datum. To define the weighted generalization of these algebras, one must choose in addition a WEIGHTING on the graph Γ underlying the Cartan datum; this is simply an assignment of a real number ϑ_e to each oriented edge of Γ .

This extra datum allows us to modify the relations of the KLR algebra in a way which is simple, but will probably initially look strange even to experts in the subject. The essential paradigm shift is that instead of beginning with idempotents indexed by sequences of nodes from the Dynkin diagram Γ , one should assign an idempotent to a sequence enriched with a position on the real number line for each element of the sequence, remembering the distance between points. We call such an object a **LOADING**. The elements of our algebra will be linear combinations of diagrams much like those of the KLR algebra, but unlike the original relations, interesting relations can occur when strands come within a fixed distance of each other; we call this phenomenon “action at a distance.”

If there is a single node and no loops, then there are no changes and we arrive at the nilHecke algebra exactly as in the KLR case. Let us consider the next easiest case, where Γ is a A_2 Dynkin diagram. As in the original KLR algebra (in Rouquier’s presentation from [Rou, §3.2], or as described in [Web17a, CL15]), one must choose a polynomial $Q_{12}(u, v) = au + bv$ that describes the interaction of these two strands via the relation

$$\begin{array}{c} \text{strand 1} \\ \text{strand 2} \end{array} \text{ crossing} = a \begin{array}{c} \bullet \\ | \\ \text{strand 1} \end{array} + b \begin{array}{c} | \\ \bullet \\ | \\ \text{strand 2} \end{array}$$

If the weighting on the unique edge e is $k < 0$, then we will see this relation not when a strand labeled 1 crosses one labeled 2 and then crosses back, but when it passes the line k units left of the strand labeled 2 and crosses back. In order to aid with visualizing this, we draw a dashed line k units left of each strand labeled 2. We will refer to these dashed lines as **GHOSTS** throughout the paper; in general, we must draw one for each pair consisting of a strand labeled with some node k , and an edge whose head is k . In this case, we will arrive at the relation:

$$\begin{array}{c} \text{strand 1} \\ \text{strand 2} \end{array} \text{ crossing with ghosts} = a \begin{array}{c} \bullet \\ | \\ \text{strand 1} \end{array} + b \begin{array}{c} | \\ \bullet \\ | \\ \text{strand 2} \end{array}$$

This case produces no interesting new algebras: we can recover the original KLR relations by shifting all strands with label 2 to the left by k units. In general, we can always find such a fix when Γ is a tree. However, when the graph Γ has cycles, interesting new algebras can appear. For example, for the

Jordan quiver and the dimension vector (n) , we arrive at the smash product $\mathbb{k}[S_n] \# \mathbb{k}[x_1, \dots, x_n]$.

Many properties of the original KLR algebras carry over: the weighted KLR algebra has a permutation type basis and a faithful representation on a sum of polynomials. Its category of representations is endowed with monoidal and co-monoidal structures given by induction and restriction, generalizing those structures for the KLR algebra. Furthermore, its Grothendieck group has a twisted bialgebra structure (or alternatively, Hopf structure for a particular braided monoidal category) induced by these functors.

This definition was motivated in large part by the desire to unify generalizations of the KLR algebras that have appeared in the author's previous work. In order to develop these, we associate to a quiver Γ and dominant weight λ a new quiver Γ_λ , which we call its CRAWLEY-BOEVEY QUIVER (see Section 3). These quivers appear naturally in the theory of Nakajima quiver varieties. The weighted KLR algebras attached to any weighting have a natural quotient we call their STEADIED QUOTIENT (see Section 2.6); these generalize the cyclotomic quotients of usual KLR algebras and always carry a categorical representation of the Kac-Moody algebra \mathfrak{g} (see Theorem 3.1).

These allow us to interpret the tensor product algebras T^λ and \tilde{T}^λ defined in [Web17a, §2] and the (extended) quiver Schur algebras A, A^λ and \tilde{A}^λ from [SW, §2 & 4] in terms of a single construction.

THEOREM A. *For each Cartan datum, and list of dominant weights $\lambda = (\lambda_1, \dots, \lambda_\ell)$, there is a weighting on the Crawley-Boevey quiver of $\lambda = \lambda_1 + \dots + \lambda_\ell$ whose weighted KLR algebra W_ν^ϑ is isomorphic to $\tilde{T}_{\lambda-\nu}^\lambda \otimes_{\mathbb{k}} \mathbb{k}[t]$. The steadied quotient $W_\nu^\vartheta(c)$ of this KLR algebra is isomorphic to $T_{\lambda-\nu}^\lambda \otimes_{\mathbb{k}} \mathbb{k}[t]$.*

For Γ a cycle, the weighted KLR algebra W_ν^ϑ is either Morita equivalent to the original KLR algebra or to the quiver Schur algebra A_ν , depending on whether the sum of weights on an oriented cycle is zero or not. In this case, there is also a weighting on the Crawley-Boevey quiver for λ and a fixed set of loadings whose weighted KLR algebra is Morita equivalent to $\tilde{A}_{\lambda-\nu}^\lambda \otimes_{\mathbb{k}} \mathbb{k}[t]$ with steadied quotient Morita equivalent to $A_{\lambda-\nu}^\lambda \otimes_{\mathbb{k}} \mathbb{k}[t]$.

Another significant motivation is that more general steadied quotients in the affine case are equivalent to category \mathcal{O} for a rational Cherednik algebra of the group $G(r, 1, \ell)$, as we prove in [Web17c]. Numerous constructions from this paper, including steadied quotients and canonical deformations play a key role in that work.

While this construction is purely algebraic in nature, it has a geometric inspiration: for a quiver Γ with vertex set I and a dimension vector $d: \Gamma \rightarrow \mathbb{Z}_{\geq 0}$, an integral weighting ϑ will define a \mathbb{C}^* -action on

$$E_\Gamma = \bigoplus_{i \rightarrow j} \text{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$$

by letting $t \cdot (f_e) = (t^{\partial_e} f_e)$. Varagnolo and Vasserot [VV11] have given an interpretation of some KLR algebras as Ext-algebras of complexes of constructible sheaves on the moduli stack E_ν/G_ν of representations of the quiver Γ which appeared in work of Lusztig [Lus91]; we can generalize this construction to give an analogous constructible complex Y of sheaves which is well-behaved with respect to the \mathbb{C}^* -action.

THEOREM B. *The weighted KLR W_ν^∂ associated to a quiver Γ with integral weighting is the Ext algebra $\text{Ext}_{E_\nu/G_\nu}(Y, Y)$. If $\text{char}(\mathbb{k}) = 0$ then Y is semi-simple.*

The map sending the class of a projective module $[P]$ to an appropriate Frobenius trace of $Y \otimes_{W_\nu^\partial} P$ on the \mathbb{F}_p points of E_ν is a bialgebra map from $K_q^0(W_\nu^\partial)$ to the Hall algebra of the quiver Γ .

This theorem has important positivity consequences; it is a key step in matching the bases defined by projective objects with their canonical bases in the sense of Lusztig (see [Web15, §6] and [Web17c, §4.7]). It will also play an important role in understanding generalizations of category \mathcal{O} in forthcoming work on the representation theory of quantizations of quiver varieties [Web17b].

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2 BASIC PROPERTIES

2.1 WEIGHTED ALGEBRAS DEFINED

Consider a graph Γ with vertex set I and oriented edge set Ω ; we allow these edges to have multiplicities $c_e, c_{\bar{e}} \in \mathbb{Z}_{\geq 0}$ for $e \in \Omega$. Let $h, t: \Omega \cup \bar{\Omega} \rightarrow I$ be the head and tail maps. We assume these multiplicities are symmetrizable, in the sense that there exist d_i such that $d_{h(e)}c_e = d_{t(e)}c_{\bar{e}}$.

There are two important examples to keep in mind:

- If C is a symmetrizable generalized Cartan matrix, then we have the associated Dynkin diagram Γ , with the multiplicities c_e given by the negative of the entries $-c_{ij}$ of the Cartan matrix. More generally, if \mathfrak{g} has no loops, then there is an associated symmetrizable Kac-Moody algebra.
- We can also take *any* locally finite graph Γ with all $c_e = c_{\bar{e}} = 1$.

Throughout, we will let a WEIGHTING on a quiver mean simply a map $\vartheta: \Omega \rightarrow \mathbb{R}$; that is an attachment of a real number to each edge. By convention, we extend this function to $\bar{\Omega}$ by $\vartheta_{\bar{e}} = -\vartheta_e$. Note that we can also think of this an \mathbb{R} -valued 1-cocycle on the underlying CW complex of Γ .

Fix a commutative ring \mathbb{k} . For each edge, we choose a polynomial $Q_e(u, v) \in \mathbb{k}[u, v]$ which is homogeneous of degree $d_{h(e)}c_e = d_{t(e)}c_{\bar{e}}$ when u is given degree $d_{h(e)}$ and v degree $d_{t(e)}$. We will always assume that Q_e has coefficients before the pure monomials in u and v which are units, and set $Q_{\bar{e}}(u, v) = Q_e(v, u)$. In particular, if (Γ, c_*) arises from a symmetrizable Cartan matrix, the polynomials $Q_{ij} = Q_e$ satisfy the properties we desire to define a KLR algebra (as in [Web17a, §2]). Furthermore, we assume that if e is a loop of degree 0, then $Q_e(u, v) = (u - v)P_e(u, v)$ for some symmetric polynomial $P_e(u, v)$.

DEFINITION 2.1. A LOADING \mathbf{i} is a function from \mathbb{R} to $I \cup \{0\}$ which is only non-zero at finitely many points. We can also think a loading as choosing a finite subset of the real line and labeling its elements with simple roots. A loading is called GENERIC if there is no real number such that $\mathbf{i}(a) = t(e)$, $\mathbf{i}(a - \vartheta_e) = h(e)$ for some edge $e \in \Omega$, or such that $\mathbf{i}(a - \vartheta_e) = h(e)$, $\mathbf{i}(a - \vartheta_{e'}) = h(e')$ and $\vartheta_e \neq \vartheta_{e'}$.

If we think of our loading as a set of labeled points, we can visualize this as adding a “ghost” of each point labeled $h(e)$ for each edge $e \in \Omega$ which is ϑ_e units to the right of the point, and require that none of these coincide with each other or with points of the loading when it can be avoided. We let $|\mathbf{i}| = \sum_{r \in \mathbb{R}} \mathbf{i}(a)$, and let d be the number of points in \mathbf{i} .

Remark 2.2. The reader familiar with KLR algebras will be used to thinking of \mathbf{i} as a sequence of simple roots which has an order, but no distance information. From now on, the distance between these elements will be essential, in a way that will be clear momentarily. We can always obtain a simple ordered list of nodes $\dot{\mathbf{i}}$ by forgetting the positions of the points; we call this the UNLOADING of \mathbf{i} .

Assume for now that

- (†) Γ is a graph such that no two edges of the same weight have matching tail and head, and there are no cyclically oriented bigons with opposite weights.

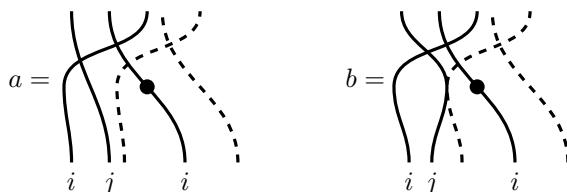
We now define the WEIGHTED KLR ALGEBRA W_B^ϑ on a finite set of loadings B .

DEFINITION 2.3. A WEIGHTED KLR (*wKLR*) DIAGRAM is a collection of finitely many oriented smooth curves in $\mathbb{R} \times [0, 1]$ with each oriented in the negative direction, with a labeling of each curve by an element $i \in I$. That is, each curve’s projection to the y -coordinate must be a diffeomorphism to $[0, 1]$. Each curve must have one endpoint on $y = 0$ and one on $y = 1$, at distinct points from the other curves. Curves are allowed to carry a finite number of dots.

Furthermore, for every edge with $h(e) = i$ we add a “ghost” of each strand labeled i shifted ϑ_e units to the right (or left if ϑ_e is negative). We require that there are no tangencies or triple intersection points between any combination of strands and ghosts, and no dots on intersection points. Note that by our assumption (\dagger), at a generic horizontal slice of the diagram, no two ghosts, two strands, or pair of ghost and strand coincide, except for those strands and ghosts that coincide because of edges of weight 0.

We’ll consider these diagrams up to isotopy which preserves all these conditions.

For example, if we have an edge $i \rightarrow j$, then the diagram a is a wKLR diagram, whereas b is not since it has a tangency between a strand and a ghost:



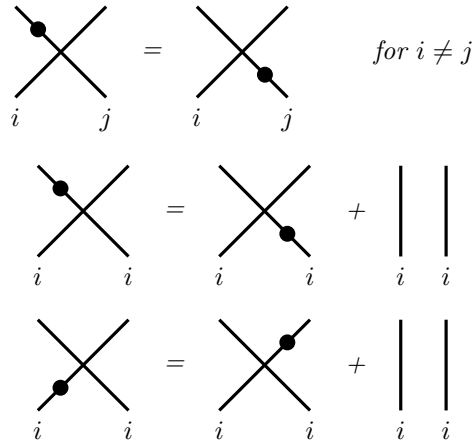
Reading along the lines $y = 0, 1$, we obtain loadings, which we call the TOP and BOTTOM of the diagram. There is a notion of COMPOSITION ab of wKLR diagrams a and b : this is given by stacking a on top of b and attempting to join the bottom of a and top of b . If the loadings from the bottom of a and top of b don’t match, then the composition is not defined and by convention is 0, which is not a wKLR diagram, just a formal symbol. This composition rule makes the formal span of all wKLR diagrams over \mathbb{k} into an algebra \tilde{W}^ϑ .

For any finite set B of loadings, we let \tilde{W}_B^ϑ be the subalgebra where we fix the top and bottom of the diagram to lie in the set B . For each loading $\mathbf{i} \in B$, we have a straight line diagram $e_{\mathbf{i}}$ where every horizontal slice is \mathbf{i} , and there are no dots.

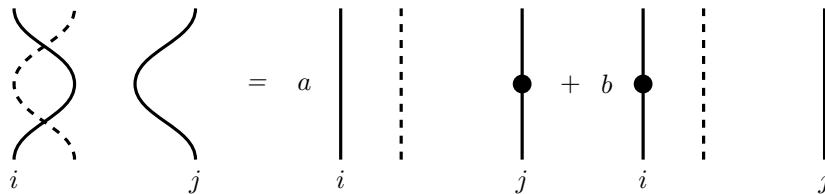
DEFINITION 2.4. The WEIGHTED KLR ALGEBRA W_B^ϑ is the quotient of \tilde{W}_B^ϑ by relations similar to the original KLR relations, but with interactions between differently labelled strands turned into relations between strands and ghosts of others. If there is a loop of weight 0 at i (there can be at most one), we let $P_i(u, v)$ be the polynomial $Q_e(u, v)/(u - v)$ attached to this loop earlier; if there is no such loop, we let $P_i(u, v) = 0$.

We give the list of local relations below. Some care must be used when understanding what it means to apply these relations locally. In each case, the LHS and RHS have a dominant term which are related to each other via an isotopy through a disallowed diagram with a tangency, triple point or a dot on a crossing. You can only apply the relations if this isotopy avoids tangencies, triple points and dots on crossings everywhere else in the diagram; one can always choose isotopy representatives sufficiently generic for this to hold.

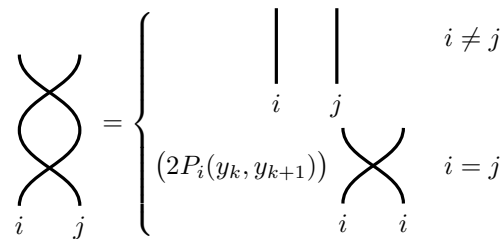
1. The relations for passing dots through crossings are exactly as in the KLR algebra.



2. Fix an edge e . If the m th strand has label $t(e)$ and we undo a bigon formed by the m th strand and the ghost of the n th coming from the edge e (assuming e is not a loop with $\vartheta_e = 0$; note that this requires the n th strand to have label $h(e)$), then we separate the strands and multiply by $Q_e(y_n, y_m)$. This is a bit harder to draw in complete generality, but for example, if there is an edge $e: i \rightarrow j$ with $\vartheta_e < 0$ and $Q_e(u, v) = au + bv$, then we have



3. If we undo a bigon formed by the k th strand and the $k+1$ st strand, we simply separate the strands if they have different labels. If they are both labelled with i , then the result is a single crossing of the strands times $2P_i(y_k, y_{k+1})$.



4. strands can move through triple points without effect, except

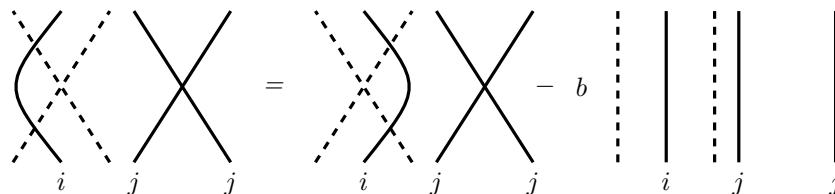
(a) when a ghost for an edge $e: i \rightarrow j$ which is ϑ_e to the right of the m th strand (which is labelled j) passes through a crossing of the n th and $n + 1$ st strands and these both have label i . In this case the diagrams where the strand is at the left differs from the one where it is at the right by

$$\partial_{n,n+1}Q_e(y_m, y_n) = \frac{Q_e(y_m, y_n) - Q_e(y_m, y_{n+1})}{y_n - y_{n+1}}.$$

(b) the m th strand (which is labelled i) passes through the ghosts attached to $e: i \rightarrow j$ attached to the n th and $n + 1$ st strands, which are both labelled j . In this case the diagrams where the strand is at the left differs from the one where it is at the right by

$$\partial_{n,n+1}Q_e(y_n, y_m) = \frac{Q_e(y_n, y_m) - Q_e(y_{n+1}, y_m)}{y_n - y_{n+1}}.$$

As before, we will not try to draw a completely general picture, but given an example when there is an edge $e: i \rightarrow j$, $\vartheta_e < 0$ and $Q_e(u, v) = au + bv$, then we have



(c) the triple point involves the m th, $m + 1$ st and $m + 2$ nd strands, all labelled i and there is a loop of weight 0 joining i to itself. In this case the diagrams where the strand is at the left differs from the one where it is at the right by

$$\begin{aligned} & \left(P_i(y_k, y_{k+1})P_i(y_{k+1}, y_{k+2}) + P_i(y_k, y_{k+2})P_i(y_{k+1}, y_k) \right. \\ & \quad \left. - P_i(y_k, y_{k+2})P_i(y_{k+1}, y_{k+2}) \right) \psi_k - \left(P_i(y_k, y_{k+1})P_i(y_{k+1}, y_{k+2}) \right. \\ & \quad \left. + P_i(y_k, y_{k+2})P_i(y_{k+2}, y_{k+1}) - P_i(y_k, y_{k+2})P_i(y_k, y_{k+1}) \right) \psi_{k+1} \end{aligned}$$

PROPOSITION 2.5. If we reverse the orientation of an edge $e \mapsto e'$, and set $\vartheta'_{e'} = -\vartheta_e$ and $Q'_{e'}(u, v) = Q_e(v, u)$, then $W^{\vartheta'} \cong W^\vartheta$ via the obvious isomorphism leaving strands unchanged.

By analogy with the geometry of Section 4, we call this isomorphism FOURIER TRANSFORM.

DEFINITION 2.6. If Γ is an arbitrary choice of graph with multiplicities, ϑ_e and Q_e associated polynomials, then the WEIGHTED KLR ALGEBRA W_B^ϑ for a set of loadings B is the weight KLR algebra for the graph where we replace all bigons

where the weights match (perhaps after reversing the orientation and negating the weight) with single edges of that weight, with $Q_{\text{new}} = \prod Q_{\text{old}}$. Proposition 2.5 shows that this does not depend on how one chooses to reverse orientations.

We note that this algebra has a natural anti-automorphism where a^* is the reflection of a diagram a through a horizontal line.

Of course, many readers used to more categorical language will prefer to think that there is a category where the objects are loadings, and the morphism spaces are the spaces $e_i W_B^\vartheta e_j$ described above. We will freely switch between these two formalisms throughout the paper.

2.2 A PERMUTATION TYPE BASIS

PROPOSITION 2.7. *This algebra W_B^ϑ acts faithfully on a sum of polynomial rings $\oplus_{B\mathbf{k}} \mathbb{k}[y_1, \dots, y_d]$, one for each loading, via the rule*

- when a strand passes from right of a ghost to left, we take the identity.
- when the j th strand has label $t(e)$ and passes from left of the ghost for e of the k th strand (which thus has label $h(e)$) to right of it, we multiply by $Q_e(y_k, y_j)$.
- when the j and $j + 1$ strands cross and have the different labels, we just apply the permutation s_j .
- when the j and $j + 1$ strands cross and have the same label i , we act with the Demazure operator $\partial_{j,j+1} = \frac{s_j - 1}{y_{j+1} - y_j}$ if there is no loop of weight 0 at i and if there is such a loop e , we act by $Q_e(y_j, y_{j+1}) \cdot \partial_{j,j+1} = P_e(y_j, y_{j+1}) \cdot (1 - s_j)$.

Proof. The confirmation of the relations is an easy modification of the proof of Khovanov and Lauda [KL09]. The relations (1) follow from the usual Leibnitz rule for Demazure operators:

$$\partial_{j,j+1}(fg) = \frac{f^{s_i} g^{s_i} - fg}{y_{j+1} - y_j} = f^{s_i} \partial_{j,j+1}(g) + \partial_{j,j+1}(f)g. \quad (2.1)$$

The relation (2) is simply follows from the fact that one of the crossings introduces a factor of $Q_e(y_k, y_j)$, and the other a factor of 1. The relation (3) is just $s_k^2 = 1$ if $i \neq j$, and if $i = j$, then for the no loop case, this is just $\partial_k^2 = 0$ and in the case where there is a loop, we have

$$\begin{aligned} P_i(y_k, y_{k+1})(1 - s_k)P_i(y_k, y_{k+1})(1 - s_k) &= P_i(y_k, y_{k+1})^2(1 - s_k)^2 \\ &= 2P_i(y_k, y_{k+1})^2(1 - s_k). \end{aligned}$$

The relations (4a) and (4b) follows immediately from (2.1).

The only really different relation to check is (4c); in this case, we use the notation $P_{ij} = P_e(y_{k+i-1}, y_{k+k-1})$. The action we check is

$$\begin{aligned}
 \text{Diagram} &= P_{12} \circ (s_k - 1) \circ P_{23} \circ (s_{k+1} - 1) \circ P_{12} \circ (s_k - 1) \\
 &= P_{12}P_{13}P_{23}s_k s_{k+1} s_k - P_{12}P_{13}P_{23}s_k s_{k+1} - P_{12}P_{13}P_{23}s_{k+1} s_k \\
 &\quad + P_{12}P_{13}P_{23}s_{k+1} + (P_{12}P_{23} + P_{21}P_{13})s_k - (P_{12}P_{23} + P_{21}P_{13})P_{12}
 \end{aligned}$$

Comparing with the mirror image, we arrive at the desired relations. □

Fix a pair of loadings \mathbf{i}, \mathbf{j} . For each permutation π such that the order of labels appearing in the loadings \mathbf{i}, \mathbf{j} differ by π , we fix an diagram b_π which wires together \mathbf{i} and \mathbf{j} according to that permutation.

Note that now even for a transposition of adjacent elements, this is not uniquely determined, since we may have a ghost that passes between both the pairs of elements which we wire in opposite order, and the element depends on whether we cross our strands to the left or right of this ghost; we let ψ_k denote the diagram in which we cross to the left of all possible ghosts. Obviously, these generate the algebra together with the dots y_i .

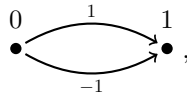
THEOREM 2.8. *The space $e_{\mathbf{i}}W^\vartheta e_{\mathbf{j}}$ is a free module over $\mathbb{k}[y_1, \dots, y_m]$, and the diagrams b_π are a free basis.*

Proof. Proof that these span is much like that of [Web17a, 3.9]. If the strands of a diagram ever cross each other twice, or cross a ghost twice, we can use the relations to rewrite them as a sum of diagrams with fewer crossings between pairs of strands or strands and ghosts using the relations of Definition 2.4(2-4). Thus, we need only consider diagrams that we could have chosen for b_π . Furthermore, we can use the triple-point moves to show that the difference between any two such diagrams for π has fewer crossings by Definition 2.4(4). Thus, the b_π 's must span and we need only show they are linearly independent. On the rational functions in the polynomial representation, the element b_π acts as a product of operators which are of the form s_i times a rational function plus a rational function times 1. The operator s_i commutes past multiplying by a rational function just by acting on it (the smash product rule); thus the product of these terms is π times a rational function, plus of a sum of shorter elements of S_n times rational functions. Thus, the linear independence over $\mathbb{k}[y_1, \dots, y_m]$ of the action of the elements of S_n guarantees the linear independence of the b_π 's. □

2.3 DEPENDENCE ON CHOICE OF LOADINGS

DEFINITION 2.9. *Call two loadings \mathbf{i}, \mathbf{i}' EQUIVALENT if for every edge $e : i \rightarrow j$, and each pair of integers (f, g) the ghost of the f th strand labeled with $h(e)$ is either to the left of the g th strand labeled $t(e)$ in both \mathbf{i}, \mathbf{i}' or to the right in both.*

Example 2.10. Let Γ be the Kronecker quiver



with the two edges are given weights 1 and -1 . For $\nu = \alpha_0 + \alpha_1$, a loading is determined the x -coordinates x_0 and x_1 of the points labeled with 0 and 1. There are 3 equivalence classes of loadings determined by the inequalities

$$\begin{array}{ccc}
 x_0 < x_1 - 1 & x_1 - 1 < x_0 < x_1 + 1 & x_1 + 1 < x_0. \\
 \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} & \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} & \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \\
 \begin{array}{c} | \\ | \\ | \end{array} & \begin{array}{c} | \\ | \\ | \end{array} & \begin{array}{c} | \\ | \\ | \end{array} \\
 0 & 1 & 0
 \end{array} \quad (2.2)$$

PROPOSITION 2.11. *In the algebra on any set B of loadings containing equivalent loadings $\mathbf{i}, \mathbf{i}' \in B$, the projective modules $W^\vartheta e_{\mathbf{i}}$ and $W^\vartheta e_{\mathbf{i}'}$ are isomorphic. That is, the original algebra is Morita equivalent to that with either loading excluded.*

In terms of the category of loadings mentioned earlier, these loadings are isomorphic.

Proof. The straight-line path from \mathbf{i} to \mathbf{i}' gives an isomorphism between these projectives. \square

In particular, if we simultaneously translate all points in a loading, we will obtain an equivalent one.

Consider the dominant cone $D_n = \{x_1 < \dots < x_n\} \subset \mathbb{R}^n$. For each $\nu = \sum v_i \alpha_i$, the set of loadings with $|\mathbf{i}| = \nu$ is naturally identified with the product of the dominant cones $D_{v_1} \times \dots \times D_{v_m} \subset \mathbb{R}^{v_1} \times \dots \times \mathbb{R}^{v_m}$ minus finitely many affine hyperplanes. It's clear from the definition that:

PROPOSITION 2.12. *The sets of equivalence classes are precisely the connected components of the complement in $D_{v_1} \times \dots \times D_{v_m}$ of affine hyperplanes associated to each edge $e: i \rightarrow j$ and $1 \leq m \leq v_i, 1 \leq n \leq v_j$:*

$$H_{e,m,n} = \{x_m^{(i)} - x_n^{(j)} = \vartheta_e\}.$$

In particular, there are only finitely many equivalence classes for each fixed ν .

DEFINITION 2.13. *Let $B(\nu)$ denote a fixed choice of a set of loadings containing one from each equivalence class with $|\mathbf{i}| = \nu$.*

From now on, when we say “the weighted KLR algebra” W_ν^ϑ we mean using that attached to the set $B(\nu)$ of loadings; this algebra is unique up to canonical isomorphism, and if we add any new generic loadings with $|\mathbf{i}| = \nu$ to this algebra, we will always obtain a Morita equivalent algebra. Generally, we will not carefully distinguish between equivalent loadings and will freely replace inconvenient loadings with equivalent ones.

In terms of the category of loadings, we have simply chosen a set of objects such that any object is isomorphic to one of the collection; this is almost the skeleton of the category, but we have not accounted for the fact that sometimes non-equivalent loadings will be isomorphic. Thus, the weighted KLR algebra can be thought of really as an equivalence class of linear categories, and from this perspective, it is manifestly well-defined.

For simplicity, we fix a real number $s > |\vartheta_e|$ for all e . Let B_s be the set of loadings where the points of the loading are spaced exactly s units apart and the first point is at $x = 0$. Such loadings are in canonical bijection with sequences of elements in I . For the Kronecker quiver weighted as in the example above, we must have $s > 1$, and only the first and third loadings of (2.2) are included in B_s .

PROPOSITION 2.14. *If the graph Γ has no loops, then the algebra $W_{B_s}^\vartheta$ is isomorphic to the original KLR algebra, with*

$$Q_{ij}(u, v) = \prod_{\substack{i=h(e) \\ j=t(e)}} Q_e(u, v).$$

In particular, if $\vartheta_e = 0$ for all e , we obtain the usual KLR algebra.

Proof. This isomorphism matches e_i to an idempotent in the KLR algebra for the corresponding sequence in I ; the dot y_k and crossing ψ_k correspond to the similarly named elements as well. Our condition on loadings forces that (after “pulling taut”) the j th strand crosses the k th if and only if it crosses all its ghosts; the relations induced between such crossings are exactly the original KLR relations. \square

This does not fully exhaust the cases where actually only obtain the original algebra. This is easier to see once we consider a symmetry of our definition. We can view the weighting ϑ as a 1-chain on Γ . If $\eta: I \rightarrow \mathbb{R}$ is a 0-chain, then we can consider the cohomologous 1-chain $(\vartheta + d\eta)_e = \vartheta_e + \eta_{h(e)} - \eta_{t(e)}$.

PROPOSITION 2.15. *The map $W_B^\vartheta \rightarrow W_B^{\vartheta+d\eta}$ moving each i -labelled strand η_i units right is an isomorphism.*

Proof. This map moves the ghost attached to an edge e to the right by $\eta_{t(e)}$, so this map maintains all crossings between strands of the same color and between ghosts and strands labelled with the tail of the associated edges. \square

COROLLARY 2.16. *If Γ is a tree, W_ν^ϑ is Morita equivalent to the original KLR algebra.*

Note that we say “Morita equivalent” here, since the set B_s may actually contain redundant loadings which are equivalent to each other (since equivalence is insensitive to the relative ordering of nodes with no edge connecting them).

2.4 INDUCTION AND RESTRICTION

For each decomposition $\nu = \nu' + \nu''$, we have a map $\iota_{\nu';\nu''} : W_{\nu'}^\vartheta \otimes W_{\nu''}^\vartheta \rightarrow W_\nu^\vartheta$, where we send a tensor product of diagrams $a \otimes b$ to the diagram where they are placed next to each other with s units of separation between them. Note that this map is not unital, but sends $1 \otimes 1$ to an idempotent $e_{\nu';\nu''}$. Up to the isomorphism induced by changing a loading in its equivalence class, this isomorphism is unchanged by adjusting the distance between the diagrams, as long as it is sufficiently large. This can be thought of as an induction operation on loadings themselves: $\iota_{\nu';\nu''}(e_i \otimes e_j) = e_{i \circ j}$.

DEFINITION 2.17. Define the functor of INDUCTION by

$$\text{Ind}_{\nu';\nu''}^\nu(M, N) = M \circ N := W_\nu^\vartheta \otimes_{W_{\nu'}^\vartheta \otimes W_{\nu''}^\vartheta} M \boxtimes N$$

and RESTRICTION by

$$\text{Res}_{\nu';\nu''}^\nu(L) := e_{\nu';\nu''} L.$$

PROPOSITION 2.18. The operation \circ makes the sum $\oplus_\nu W_\nu^\vartheta$ -mod into a monoidal category, and $\text{Res}_{*,*}$ makes this sum into a comonoidal category. The subcategory $\oplus_\nu R_\nu$ -mod is monoidally generated by $W_{\alpha_i}^\vartheta$ -mod.

Recall that the Grothendieck group $K^0(W_\nu^\vartheta)$ is the span of formal symbols corresponding to finitely generated projective W_ν^ϑ -modules subject to the relation that $[M \oplus N] = [M] + [N]$; we can think of the sum $K = \oplus_\nu K^0(W_\nu^\vartheta)$ as an abelian group graded by $\mathbb{Z}[I]$. Furthermore, we endow $\mathbb{Z}[I]$ with a pairing where

$$i \cdot j = 2d_i \delta_{ij} - d_i \left(\sum_{j \xrightarrow{e} i} c_e + \sum_{i \xrightarrow{e} j} c_{\bar{e}} \right) = 2d_j \delta_{ij} - d_j \left(\sum_{j \xrightarrow{e} i} c_{\bar{e}} + \sum_{i \xrightarrow{e} j} c_e \right).$$

We will sometimes view this as the symmetrization of the bilinear form

$$\langle j, i \rangle = d_i \delta_{ij} - \sum_{j \xrightarrow{e} i} d_i c_e \quad i \cdot j = \langle i, j \rangle + \langle j, i \rangle.$$

This allows us to define a twisted product structure on $A \otimes A$ for any $\mathbb{Z}[I]$ -graded algebra A by $q^{\text{deg}(b) \cdot \text{deg}(c)}(a \otimes b)(c \otimes d)$. As noted by Walker [Wal], we can think of this as the natural product in the braided monoidal category of $\mathbb{Z}[I]$ -graded vector spaces, where the braiding map on a tensor product of spaces V of pure degree μ and V' of degree μ' is the switch map $V \otimes V' \rightarrow V' \otimes V$ times $q^{\mu \cdot \mu'}$.

THEOREM 2.19. The Grothendieck group $K = \oplus_\nu K^0(W_\nu^\vartheta)$ endowed with

$$\text{the product } [M][N] = [M \circ N] \text{ and coproduct } \Delta([L]) = \sum_{\nu'+\nu''=\nu} [\text{Res}_{\nu';\nu''}^\nu(L)]$$

is a twisted bialgebra with a natural map $U_q^+(\mathfrak{g}_\Gamma) \rightarrow K$; in fact, it is a Hopf algebra in the braided category of $\mathbb{Z}[I]$ -graded vector spaces.

Proof. For a decomposition $\nu = \nu_1 + \nu_2 = \nu'_1 + \nu'_2$, we consider the restriction of W_ν^ϑ to $W_{\nu_1}^\vartheta \otimes W_{\nu_2}^\vartheta$ on the left and $W_{\nu'_1}^\vartheta \otimes W_{\nu'_2}^\vartheta$ on the right. We can filter W_ν^ϑ as a bimodule by the sum μ of the labels on the strands that pass from left to right, so the sum of the labels passing right to left is $\mu' = \nu'_1 - \nu_1 + \mu$. By the same argument as [KL09, 2.18], the successive quotients of this filtration are

$$(W_{\nu_1-\mu;\mu}^\vartheta \otimes W_{\mu';\nu_2-\mu'}^\vartheta) \otimes_{W_{\nu_1-\mu}^\vartheta \otimes W_\mu^\vartheta \otimes W_{\mu'}^\vartheta \otimes W_{\nu_2-\mu'}^\vartheta} (W_{\nu_1-\mu;\mu'}^\vartheta \otimes W_{\mu;\nu_2-\mu'}^\vartheta)$$

shifted upwards by the inner product $-\langle \mu, \mu' \rangle$. As noted in [KL09, 3.2], this suffices to prove that the coproduct Δ is an algebra map $K \rightarrow K \otimes K$ for the twisted product structure.

The counit ϵ just kills $K^0(W_\nu^\vartheta)$ for $\nu \neq 0$, and the antipode S , as in the work of Xiao [Xia97], can be constructed inductively by the formula

$$S([M]) = - \sum_{\substack{\nu=\nu'+\nu'' \\ \nu'' \neq 0}} (1 \otimes S)[\text{Res}_{\nu';\nu''}^\nu(M)] \quad \square$$

2.5 THE TWISTED ALGEBRA

There is a larger category \mathcal{P} whose objects are pairs $(\mathbf{i}; \vartheta)$ of loadings and weights. Morphisms $(\mathbf{i}_0; \vartheta_0)$ and $(\mathbf{i}_1; \vartheta_1)$ between two such pairs is very much like in the category of loadings for a fixed weight, but the distance from each ghosts to the strand it haunts is not a constant: instead at the horizontal slice $y = a$, the distance of a ghost for $e : i \rightarrow j$ from the corresponding j labeled strand is $a\vartheta_1(e) + (1-a)\vartheta_0(e)$. All the same local relations between morphisms apply without change.

PROPOSITION 2.20. *This category has a representation that associates a polynomial ring to each pair $(\mathbf{i}; \vartheta)$ with the action given by formulas as in Proposition 2.7. The morphism space between any two pairs in \mathcal{P} is spanned by a basis given by the product of monomials in the dots with a fixed stringing up of each permutation.*

Proof. We can define an action on a sum of polynomial rings by the same local rules as 2.7; since the same local relations are used, the same proof carries through. With this action in hand, we can use the same proof as Theorem 2.8. □

We will often be interested in considering the sum of all morphism spaces from loadings with one weighting ϑ to those with another ϑ' . This sum is naturally a bimodule $B^{\vartheta,\vartheta'}$ over W^ϑ and $W^{\vartheta'}$.

2.6 STEADIED QUOTIENTS

In this subsection, we define a natural quotient of W_ν^ϑ ; while the algebraic motivation for this definition may not be immediately apparent, we believe it

is well-motivated both by examples and by geometry. In fact, we recommend that the reader glance at the next section on examples before reading the definition below.

A CHARGE on the vertex set I is a map $c: I \rightarrow \mathbb{C}_+$ where

$$\mathbb{C}_+ = \{x \in \mathbb{C} \mid \text{either } \text{Im}(x) > 0 \text{ or } x \in \mathbb{R}_{>0}\}$$

We always extend c linearly to $\mathbb{Z}[I]$. Such a charge induces a preorder $>_c$ on $\mathbb{Z}_{>0}[I]$, using the argument of $c(\mathbf{d})$

DEFINITION 2.21. *We call an indecomposable W_ν^ϑ -module is called UNSTEADY if it is isomorphic to a summand of an induction $M_1 \circ M_2$ where $\text{wt}(M_1) >_c \text{wt}(M_2)$.*

In W_ν^ϑ , there is a natural 2-sided ideal I_c generated by all elements factoring through unsteady projectives (thought of as a map of left modules $W_\nu^\vartheta \rightarrow W_\nu^\vartheta$). Visually, this corresponds to diagrams where in the middle of the diagram, there is a horizontal slice whose the induced loading is $\mathbf{i}_1 \circ \mathbf{i}_2$ where $|\mathbf{i}_1| >_c |\mathbf{i}_2|$.

DEFINITION 2.22. *The STEADIED quotient $W_\nu^\vartheta(c)$ of W_ν^ϑ is the quotient W_ν^ϑ/I_c . We let $B^{\vartheta,\vartheta'}(c)$ denote the compatible quotient of the bimodule $B^{\vartheta,\vartheta'}$.*

2.7 CANONICAL DEFORMATIONS

The algebras W^ϑ have a canonical deformation. For each edge e with head j and tail i , we assign an alphabet of variables $z_{e,a,b}$ for integers $0 \leq a < d_i, 0 \leq b < d_j$ such that $ad_j + bd_i < d_jc_e = d_ic_{\bar{e}}$. We then consider the weighted KLR algebra over the ring $\mathbb{k}[\mathbf{z}_e]$ with Q -polynomials given by

$$\tilde{Q}_e(u, v) = Q_e(u, v) + \sum_{a,b} z_{e,a,b} u^a v^b.$$

This polynomial will be homogeneous if we endow $z_{e,a,b}$ with degree $d_jc_e - ad_j - bd_i = d_ic_{\bar{e}} - ad_j - bd_i$. Let $S = \mathbb{k}[\{z_{e,a,b}\}]$. In the case where each edge has multiplicity 1 ($c_e = 1$), then we only have one variable per edge and $\tilde{Q}_e(u, v) = Q_e(u, v) + z_e$.

PROPOSITION 2.23. *This deformation is free (and thus flat) over S .*

Proof. The proof of Theorem 2.8 works equally well over S , showing that the diagrams b_π give a free basis over $S[y_1, \dots, y_m]$. By multiplying by monomials, we easily obtain a free S -basis. \square

Fix a field K , and a non-zero homomorphism $\chi: S \rightarrow K$. Fix a finite subset M_i of K for each $i \in I$.

DEFINITION 2.24. *The graph Γ_{χ, M_\bullet} is the graph with underlying set $\bigcup_{i \in I} \{i\} \times M_i \subset \Gamma \times K$. For $q_1 \in M_i, q_2 \in M_j$, an edge $e: i \rightarrow j$ lifts to an edge \tilde{e} from (i, q_1) and (j, q_2) if and only if the polynomial satisfies $\chi(\tilde{Q}_e)(q_1, q_2) = 0$.*

Note that the natural map $\sqcup_{i \in I} M \rightarrow \Gamma$ is a graph homomorphism. We can naturally assign polynomials to this graph by

$$Q_{\check{e}}(u, v) := \chi(\tilde{Q}_{ij})(u + q_1, v + q_2).$$

Given a weighting ϑ of Γ , we also weight Γ_{χ, M_\bullet} with $\check{\vartheta}_{\check{e}} = \vartheta_e$.

Example 2.25. Assume Γ is an e -cycle, whose vertices we identify with $\mathbb{Z}/e\mathbb{Z} = \{0, \dots, e - 1\}$ with an edge $i \rightarrow i + 1$. If send z_e for the edge $e: e - 1 \rightarrow 0$ with -1 and set z_e for every other edge of this graph to 0, with K any characteristic 0 field, and take $M = \mathbb{Z}$. We thus find that we have an edge $(p, q) \rightarrow (p', q')$ if $p' \equiv p + 1 \pmod{e}$ and $q' - q = \delta_{p', 0}$. This is equivalent to $q'e + p' = qe + p + 1$. That is, the resulting graph Γ_{χ, M_\bullet} is isomorphic to \mathbb{Z} with an edge $i \rightarrow i + 1$, where we identify \mathbb{Z} and $\mathbb{Z} \times \mathbb{Z}/e\mathbb{Z}$ by division with remainder by e .

Example 2.26. Let Γ be any graph, and let K any field, with each z_e sent to 0. For any finite subset $M \subset K$, we can set $M_i = M$. The resulting graph is just $\Gamma \times M$, with the map to Γ being a trivial $\#M$ -fold covering.

Example 2.27. If Γ has a non-symmetric Cartan matrix, then for each pair $i, j \in I$, we let $e_{ij} = \gcd(c_{ij}, c_{ji})$, $f_{ij} = c_{ij}/\gcd(c_{ij}, c_{ji})$. Consider the polynomials $Q_e(u, v) = (u^{f_{ij}} - v^{f_{ji}})^{e_{ij}}$, let K be a field of characteristic coprime to each d_i , and let M_i be the $p_i = \text{lcm}(\{d_k\})/d_i$ th roots of unity in K . In this case, the graph structure is that ζ_1 and ζ_2 are connected by an edge if $\zeta_1^{f_{ij}} = \zeta_2^{f_{ji}}$. That is, each preimage of i is connected to preimages of j by $p_j/f_{ji} = \text{lcm}(\{d_k\})\gcd(c_{ij}, c_{ji})/d_j c_{ji}$ preimages, along edges with multiplicity e_{ij} .

Thus, Γ_{χ, M_\bullet} in this case is the standard branched cover of a non-symmetric Cartan matrix by a symmetric one.

We'd like to understand the specialization $W_\nu^\vartheta \otimes_S K$ at the homomorphism χ ; while we don't have a general description of this algebra, we can consider a natural completion of it.

Let $I_k \subset W_\nu \otimes_S K$ be the two-sided ideal in $W_\nu \otimes_S K$ generated by the products $\prod_{m \in M} (y_i - m)^k$ for each i . These are clearly nested, and have trivial intersection for reasons of degree; thus, we can consider the completion $\widehat{W_\nu^\vartheta \otimes_S K}$ at this system of ideals. Note, that this depends in a very strong way on M , but we will suppress this dependence from the notation. On the other hand, we can consider the weighted KLR algebra $\widehat{\check{W}^\vartheta}$ of the graph $\Gamma \times M$ over the field K , completed by the two-sided ideals generated by y_i^k for all i . This is the same completion applied before, but with $M = \{0\}$.

The completion $\widehat{W_\nu^\vartheta \otimes_S K}$ has a natural decomposition according to the topological generalized eigenvalues of the operators y_i . That this, we can decompose each quotient $W_\nu^\vartheta \otimes_S K/I_n$ according to these eigenvalues since it is finite dimensional, and take the inverse limit of this decomposition. Note that these generalized eigenvalues must lie in M , since the minimal polynomial of y_i on $W_\nu^\vartheta \otimes_S K/I_n$ divides $\prod_{m \in M} (y_i - m)^k$. This decomposes the idempotents e_i

corresponding to loadings as a sum of idempotents where we associate an additional choice of $m \in M$ to each point in the loading. Put another way, consider the ways of lifting the loading in Γ to one in $\Gamma \times M$. If \mathfrak{i} is such a loading, let $\epsilon_{\mathfrak{i}}$ denote the projection to its generalized eigenspace (which is an element of the algebra by abstract Jordan decomposition in each quotient).

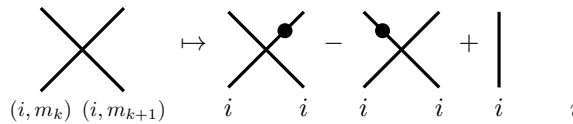
For any weighted KLR diagram for the graph $\Gamma \times M$, we have a “projection” where we apply the first projection to the labels of each strand; we can always isotope a KLR diagram so that this projection is a weighted KLR diagram as well (if we aren’t careful, we might introduce tangencies). Note that this result might not be independent of the isotopy.

PROPOSITION 2.28. *There is an isomorphism $\widehat{W}^{\vartheta} \cong W_{\nu}^{\vartheta} \otimes_S K$ such that:*

$$\epsilon_{\mathfrak{i}} \mapsto \epsilon_{\mathfrak{i}} \quad y_i \epsilon_{\mathfrak{i}} \mapsto (y_i - m_i) \epsilon_{\mathfrak{i}} \tag{2.3}$$

For diagrams, it is easier to describe this map locally. For most diagrams with a single crossing and no dots, we simply pass to the projection, times $\epsilon_{\mathfrak{i}}$, except in cases where:

- At $y = a$, in the projection of A , there is a crossing where the ℓ th strand (call its label i) crosses from left to right of a ghost haunting the k th strand for an edge $e: i \rightarrow j$ which doesn’t lift to an edge $\check{e}: (i, m_{\ell}) \rightarrow (j, m_k)$. In this case, we multiply the projection at $y = a$ by $\chi(\check{Q}_e)(y_{\ell} + m_{\ell}, y_k + m_k)^{-1}$. This exists because $\chi(\check{Q}_e)(y_{\ell} + m_{\ell}, y_k + m_k)$ is a power series with non-zero constant term by assumption, and thus invertible.
- At $y = a$, there is a crossing of two strands with labels (i, m_k) and (i, m_{k+1}) with $m_k \neq m_{k+1}$. We send the crossing to $y_{k+1} - y_k$ times the projection diagram plus the diagram with the crossing opened. That is:



Proof. Much like in [Web], we identify these algebras by giving an isomorphism between their completed polynomial representations.

The completion of the polynomial representation of $W_{\nu}^{\vartheta} \otimes_S K$ is a sum of completed polynomial rings $\oplus_{\mathfrak{i}} K[[y_1 - m_1, \dots, y_n - m_n]] \epsilon_{\mathfrak{i}}$, so we can use (2.3) as the definition of the isomorphism of this to the completed polynomial representation of \widehat{W}^{ϑ} .

Thus, we need only check that dotless diagrams act correctly. In all the cases where a diagram is sent to its projection, the match between the actions is clear. Now consider the case where there is a crossing where the ℓ th strand (call its label i) crosses from left to right of a ghost for the k th strand and an edge $e: i \rightarrow j$ which doesn’t lift to an edge $\check{e}: (i, m_{\ell}) \rightarrow (j, m_k)$; in this case,

the action of the projection is by multiplication by $\chi(\tilde{Q}_e)(y_\ell + m_\ell, y_k + m_k)$. Thus, $\chi(\tilde{Q}_e)(y_\ell + m_\ell, y_k + m_k)^{-1}$ times this diagram acts by the identity map, as does the diagram for $\Gamma \times M$.

Finally, consider the case where there is a crossing of two strands with labels (i, m_k) and (i, m_{k+1}) with $m_k \neq m_{k+1}$. The projection acts by the Demazure operator $\frac{s_j - 1}{y_j - y_{j+1}}$. Thus,

$$\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ i \quad i \end{array} - \begin{array}{c} \diagdown \quad \diagup \\ \bullet \\ i \quad i \end{array} + \begin{array}{c} | \\ i \end{array} \quad \begin{array}{c} | \\ i \end{array}$$

acts by the switch map s_j , as does the diagram for $\Gamma \times M$.

Thus we need only check that this map is invertible. The inverse applied to a diagram times $\epsilon_{\mathbf{i}}$ similarly goes to the “anti-projection” but times $\chi(\tilde{Q}_e)(y_\ell + m_\ell, y_k + m_k)$ where there is an appropriate crossing of a strand and a ghost, and when two like-colored strands with different m_k and m_{k+1} cross, the inverse map is given by

$$\epsilon_{\mathbf{i}} \begin{array}{c} \diagup \quad \diagdown \\ i \quad i \end{array} \mapsto (y_k - y_{k+1} + m_k - m_{k+1})^{-1} \left(\begin{array}{c} \diagdown \quad \diagup \\ (i, m_k) \quad (i, m_{k+1}) \end{array} - \begin{array}{c} | \\ (i, m_{k+1}) \end{array} \begin{array}{c} | \\ (i, m_k) \end{array} \right). \quad \square$$

Note that this also induces a map on the level of steadied quotients, since the loading $\hat{\mathbf{i}}$ is unsteady if and only if \mathbf{i} is, and the idempotent $\epsilon_{\mathbf{i}}$ is 0 in the steadied quotient if \mathbf{i} is.

3 RELATION TO PREVIOUS CONSTRUCTIONS

The motivation for the definition of weighted KLR algebras was to give a unifying framework to some seemingly disparate examples, as well as providing a language for new ones.

As Corollary 2.16 shows, we will encounter nothing new if we consider the weighted KLR algebras for a tree; in particular, for any Dynkin diagram, or extended Dynkin diagram of type other than \hat{A}_n , nothing interesting happens. On the other hand, there are some very interesting cases based on slightly less famous graphs.

3.1 THE CRAWLEY-BOEVEY TRICK AND CATEGORICAL ACTIONS

The most important case for us is the graph produced by “the Crawley-Boevey trick;” this was a construction which was originally designed with the aim of thinking of Nakajima’s quiver varieties, which were originally defined using auxiliary “shadow vertices,” as a space of usual representations of a pre-projective algebras.

Given a graph Γ and a function $w: I \rightarrow \mathbb{Z}_{\geq 0}$, we can define a new graph Γ_w where we take the original graph Γ , add a new vertex 0 and string in w_i edges from 0 to i . More formally, Γ_w has vertex set $I \cup \{0\}$ and edge set $\Omega \cup \{e_i^1, \dots, e_i^{(w_i)}\}_{i \in I}$ with $t(e_i^{(k)}) = 0, h(e_i^{(k)}) = i$. We call the original edges of Γ OLD EDGES, and the edges $e_i^{(*)}$ NEW EDGES. For simplicity, we always choose $c_{e_i^{(w_i)}} = c_{e_i^{(w_i)}} = 1$ and $Q_{e_i^{(k)}}(u, v) = u - v$.

As we noted, this graph has previously appeared in the literature on Nakajima quiver varieties, since

- there's a canonical bijection between representations of Γ_w with $V_0 \cong k$ and representations of Γ together with a choice of map $\mathbb{C}^{w_i} \rightarrow V_i$, and
- similarly, representations of the preprojective algebra of Γ_w with $V_0 \cong k$ are in canonical bijection with elements of the vector space Nakajima denotes \mathbf{M} subject to the moment map conditions [Nak94, (2.5)], and
- this representation is stable in the sense of Craw for the character which is the product of the determinants of the action on V_i 's for $i \in I$, and the $-\sum_{i \in I} \dim V_i$ -power of the determinant on V_0 if and only if it is stable as in [Nak94, 3.5].

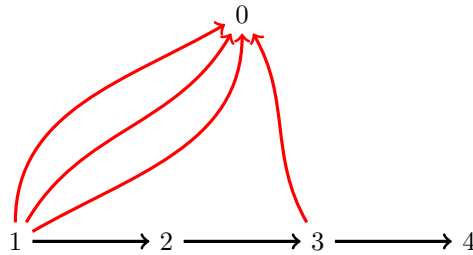


Figure 1: The Crawley-Boevey quiver of $3\omega_1 + \omega_3$ for \mathfrak{sl}_5 .

This observation carries over into the algebras attached to these quivers. Given a highest weight λ of the Kac-Moody Lie algebra \mathfrak{g} associated to Γ , we let $\Gamma_\lambda = \Gamma_w$ where $w(i) = \lambda(\alpha_i^\vee)$.

For any weighting ϑ , call the REDUCED quotient \bar{W}_ν^ϑ of the algebra W_ν^ϑ for Γ_w with weight $\tilde{\nu} = \nu + \alpha_0$ by the ideal generated by all dots on the 0-labelled strand. Consider the charge c which assigns $c(i) = -1 + i$ for all old vertices and $i + \sum d_i$ to 0, and the reduced steadied quotient $\bar{W}_\nu^\vartheta(c)$. When we relate this construction to the geometry of quiver representations, this will correspond to only acting by change of basis on the old vertices.

Since the single strand with label 0 in each diagram of \bar{W}_ν^ϑ plays a special role, we will represent its ghosts using red ribbons like $\{\color{red}\parallel\}$; this is suggestive of a relationship to the tensor product algebras of [Web17a, §3] which we will discuss shortly.

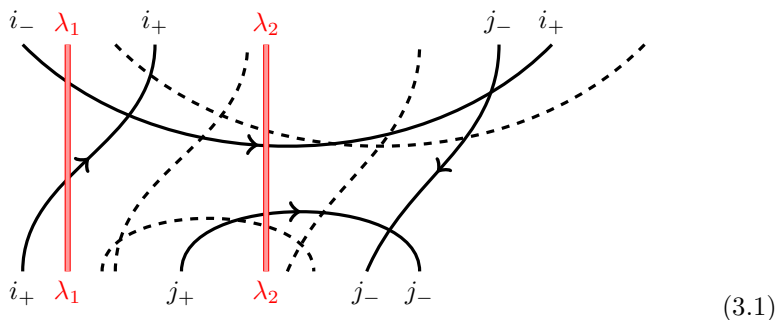
Assume that Γ has no loops. Recall that there is a 2-category \mathcal{U} , defined using the ring k and the polynomials Q_{ij} , which categorifies the universal enveloping algebra of the associated Kac-Moody algebra \mathfrak{g} . We use the conventions established in our previous papers [Web18, Web17a] for this category which (modulo minor conventional differences) is that defined by Cautis and Lauda [CL15] building on work of Rouquier [Rou] and Khovanov and Lauda [KL10].

THEOREM 3.1. *There is a categorical action of the Kac-Moody Lie algebra \mathfrak{g} on the categories $\bigoplus_{\nu} \bar{W}_{\nu}^{\vartheta}(c)\text{-pmod}$, with \mathcal{F}_i given by the induction functor $M \mapsto M \circ W_{\alpha_i}^{\vartheta}$, and \mathcal{E}_i by its left adjoint.*

This is in principle the same proof as [Web17a, 3.26] but it is a bit more work to describe the obvious module over \mathcal{U} .

Consider the category \mathcal{Y}'_y whose objects are SIGNED LOADINGS, that is, loadings where each point is marked with a $+$ or $-$, which we can also represent as either an upward or downward arrow. We'll use i_{\pm} to represent the label of a point in a signed loading.

We let a BLANK DOUBLE WEIGHTED KLR DIAGRAM be a collection of curves which are decorated with dots which are oriented and match the up and down arrows on the source loading at $y = 0$ and the target at $y = 1$, and are generic in the same sense as weighted KLR diagrams. These strands have ghosts positioned ϑ_e units right of each strand (regardless of orientation) labelled with the head of e ; for purposes of weight labeling we also need to include ghosts for the opposite orientation, that is ghosts (which we will draw as dotted lines \cdots) ϑ_e units left of each strand labelled with $t(e)$. The diagrams are the same as those used in the 2-category \mathcal{U} , except for the genericity conditions imposed by ghosts. Here is an example of such a diagram:



Some care is necessary when labeling the regions of the plane. We let a DOUBLE WEIGHTED KLR DIAGRAM be a blank DWKLRD with a labeling of each region of the plane minus strands and ghosts labeled by a weight of \mathfrak{g} . Rather than using the rules of [KL10] or [Web17a], these must be consistent with the rules¹

¹When the Cartan matrix is not invertible, we should be a bit careful about precisely what fundamental weights mean, but this is actually a red herring. What we really want to assign to regions are functions $I \rightarrow \mathbb{Z}$, but it has been conventionally handy to write these

that

$$\mu \begin{array}{c} \color{red}{\downarrow} \\ \color{red}{\downarrow} \\ \color{red}{\downarrow} \\ \lambda \end{array} \mu + \lambda \qquad \mu \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ i \end{array} \mu - 2\omega_i$$

and for ghosts corresponding to an edge $e: i \rightarrow j$:

$$\mu \begin{array}{c} \vdots \\ \downarrow \\ \vdots \\ e \end{array} \mu + c_e \omega_i \qquad \mu \begin{array}{c} \vdots \\ \downarrow \\ \vdots \\ e \end{array} \mu + c_e \omega_j$$

As in [Web17a], we let \mathcal{L} denote the label of the leftmost region, and similarly for \mathcal{R} and the rightmost. We refine the scalars $t_{ij} = Q_{ij}(1, 0)$ as follows: for an edge e and node i , we let

$$t_{i;e} = \begin{cases} Q_e(1, 0) & \text{if } i = h(e), \\ 1 & \text{otherwise.} \end{cases} \qquad u_{i;e} = \begin{cases} Q_e(0, 1) & \text{if } i = h(e), \\ 1 & \text{otherwise.} \end{cases}$$

We let \mathcal{Y}_θ be the 2-category with:

- objects given by weights of \mathfrak{g} .
- 1-morphisms $\lambda \rightarrow \mu$ given by loadings with label $\mathcal{L} = \lambda, \mathcal{R} = \mu$. Composition is the horizontal composition of loadings.
- 2-morphisms $\mathbf{i} \rightarrow \mathbf{j}$ given by double weighted KLR diagrams with \mathbf{i} as bottom and \mathbf{j} as top, modulo the relations [Web17a, (2.2-3)] and
 - the bigon relation [Web17a, (2.4a-b)] is replaced by

$$\lambda \begin{array}{c} \curvearrowright \\ \curvearrowleft \\ i \quad j \end{array} = \begin{array}{c} \downarrow \\ \uparrow \\ i \quad j \end{array} \lambda \qquad \lambda \begin{array}{c} \curvearrowleft \\ \curvearrowright \\ i \quad j \end{array} = \begin{array}{c} \uparrow \\ \downarrow \\ i \quad j \end{array} \lambda \qquad (3.2a)$$

$$\lambda \begin{array}{c} \curvearrowright \\ \curvearrowleft \\ i \quad e \end{array} = u_{i;e} \begin{array}{c} \downarrow \\ \uparrow \\ i \quad e \end{array} \lambda \qquad \lambda \begin{array}{c} \curvearrowleft \\ \curvearrowright \\ e \quad j \end{array} = u_{j;e} \begin{array}{c} \uparrow \\ \downarrow \\ e \quad j \end{array} \lambda \qquad (3.2b)$$

functions in the form $\alpha_i^\vee(\mu)$ for some weight μ . Thus, pedants should consider ω_i to be the characteristic function of $i \in I$.

$$\lambda \begin{array}{c} \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} = t_{i,e} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \lambda \quad \lambda \begin{array}{c} \text{---} \\ \diagup \quad \diagdown \\ \text{---} \end{array} = t_{j,e} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \lambda \quad (3.2c)$$

- the KLR relations [Web17a, (2.5a-g)] replaced with the weighted KLR relations of Definition 2.4.

In both cases, we ignore the dotted ghosts; these are only necessary to label the plane so that \mathfrak{sl}_2 relations function correctly.

Note that if the loadings have each pair of points at least s units apart, both these changes in relations become irrelevant, and we recover the relations of the original category \mathcal{U} .

Note that \mathcal{Y}_ϑ has a pair of commuting left and right actions of \mathcal{U} , given by placing diagrams in \mathcal{U} (drawn on loadings with points more than s units apart) to the far left or far right of a diagram in \mathcal{Y}_ϑ .

The morphism spaces in \mathcal{Y}_ϑ have a natural spanning set analogous to that for \mathcal{U} described by Khovanov and Lauda, which we'll denote Z_ϑ . Each vector in Z_ϑ is indexed by matching of the points of the two loadings such that points in the different loadings have the same sign or in the same loading have different signs. The diagram is gotten by choosing a way of stringing together the matched points, placing an arbitrary number of dots at a fixed point on each strand, and then multiplying at the right by a monomial in the bubbles (which are far enough apart to avoid any interaction with ghosts).

LEMMA 3.2. *The set Z_ϑ is a basis.*

Proof. The proof that these relations span is very similar to that of Theorem 2.8: one can use the relations of Definition 2.4 to remove any bigons, and show any two choices of the vectors in Z_ϑ are the same, modulo diagrams with fewer crossings.

Assume we have a non-trivial linear combination of diagrams in Z_ϑ . This must be gotten as a sum of the relations in the category as described earlier. Now, attach the morphism that pulls all strands to the far right and separates them at least s units from each other from each other to the top and bottom of the diagram. The result of is a linear combination of morphisms in \mathcal{U} . Since every relation in \mathcal{Y}_ϑ remains a relation when a red line is dragged through it, or its ends are pulled further apart, the relations that we used to write this linear combination remain relations in \mathcal{U} . That is, the sum of diagrams we arrive at in \mathcal{U} is 0 as well. However, we know by [Web17a, 2.32] that the analogous spanning set to Z_ϑ in \mathcal{U} is a basis, so when written in terms of these elements, it must be a trivial linear combination.

Consider a diagram of Z_ϑ with a maximal number of crossings among those that appear in the linear combination. The diagram corresponding to the same

matching (with some new dots) appears in our new linear combination, and no other diagram from the proposed basis could cancel it out. Thus, it must have trivial coefficient in the original linear combination, contradicting the assumption that it did not.

Thus, the set Z_{ϑ} is a basis; in particular, if we consider usual loadings as signed loadings with all signs negative, we get an injection of the weighted KLR algebra into the morphism space in \mathcal{Y} . \square

Now, we apply a similar principle to have we have use many times in [Web17a]; we call a signed loading UNSTEADY like in the unsigned case if it is horizontal composition of a purely black loading with one containing all the red strands. We let $D\bar{W}^{\vartheta}(c)$ be the quotient of the algebra spanned by double weighted KLR diagrams with $\mathcal{L} = 0$ by the relations of the category \mathcal{Y}_{ϑ} and the ideal generated by all unsteady signed loadings.

LEMMA 3.3. *The natural map of algebras $\bar{W}^{\vartheta}(c) \rightarrow D\bar{W}^{\vartheta}(c)$ is a Morita equivalence.*

Proof. First, we must show that the morphism space in the quotient $D\bar{W}^{\vartheta}(c)$ between two usual loadings is the reduced steadied quotient of the weighted KLR algebra. This follows from a similar argument to [Web17a, 3.23]. As in [Web17a], we call a signed loading DOWNWARD if all its points have negative sign. Consider any diagram with downward top and bottom, and an unsteady loading at $y = 1/2$. As in that proof, we can isotope the strands coming from the unsteading part of the loading so that they meet the line $y = 1/2$ again before meeting any part of the rest of the loading. Now isotope the diagram again, so that all but one of the resulting cups is pushed below $y = 1/2$. Now we see that our diagram is unsteadied by a loading beginning with a $\pm i$ and then a $\mp i$. Now, we can run the argument of [Web17a, 3.23] to finish the proof. This shows that the map is injective.

Now, in order to prove Morita equivalence, we need only prove that the idempotent for any signed loading \mathbf{i} factors through downward loadings in this quotient. This is closely modeled on [Web17a, 3.24]. We induct on the number of positive signs in \mathbf{i} , as well as the length of the minimal permutation sending all positive signs to the left and negative to the right. If this permutation is the identity, then the left-most point carries a positive sign, and without changing the isomorphism type, we can pull it to the far left, so this loading is trivial in \mathcal{Y}_{ϑ} . Thus, we must have a pair of consecutive points where the leftward one carries a $-$ and the rightward one carries a $+$. We can move the rightward one to the left through any ghosts or strands with different labels using the relations (3.2a-3.2c). If they carry the same label, then by the relation [Web17a, (2.3c)], e_i factors through loadings where these points have switched (lowering the length of the permutation) plus some number where they have been removed (lowering the number of $+$'s). By induction, this map is a Morita equivalence. \square

Proof of Theorem 3.1. The category of modules over $D\bar{W}^\vartheta(c)$ manifestly carries a categorical \mathfrak{g} -action since it carries an action of \mathcal{Y}_ϑ which contains \mathcal{U} as a subcategory. The bimodules defining this action can be given exactly as in [Web17a, (3.6)]. \square

Exactly as in [Web17a, 5.7], we have that:

PROPOSITION 3.4. *The functor of tensor product with $B^{\vartheta, \vartheta'}(c)$ commutes naturally with the action of \mathcal{U} .*

3.2 RELATIONS TO TENSOR PRODUCT ALGEBRAS

Fix a list of highest weights $\underline{\lambda} = (\lambda_1, \dots, \lambda_\ell)$. Choose any sequence of real numbers $\varpi_1 < \dots < \varpi_\ell$, and consider the weighting on Γ_λ where all old edges have degree 0, and there are $\alpha_i^\vee(\lambda_j)$ new edges with weight ϖ_j connecting 0 to i . We denote these edges $e_{i,j,1}, \dots, e_{i,j,\alpha_i^\vee(\lambda_j)}$. Recall that in [Web17a, §3], the author defined algebras $T^\underline{\lambda}$ and $\tilde{T}^\underline{\lambda}$ attached to the list $\underline{\lambda}$.

THEOREM 3.5. *The algebra $\tilde{T}_{\lambda-\nu}^\underline{\lambda}$ is the reduced quotient \bar{W}_ν^ϖ of W_ν^ϖ . The map replaces the ghosts of the 0-labelled strand with red strands, decorated by the weights λ_1 through λ_ℓ if the 0-labelled strand has no dots on it, and sends the diagram to 0 if there are any dots on the 0-labelled strand.*

Proof. All relations between black strands satisfy the KLR relations in both cases. When we undo a bigon between the i -labelled k th strand and the p th 0-labelled ghost (from the left) where the m th strand is 0-labelled, we multiply by $(y_k - y_m)^{\alpha_i^\vee(\lambda_p)}$, which becomes $y_k^{\alpha_i^\vee(\lambda_p)}$ after setting $y_m = 0$. Similarly, if a ghost passes through a crossing of the k th and $k + 1$ st strands, the correction term is the opened crossing times

$$\partial_{k,k+1}((y_k - y_m)^{\alpha_i^\vee(\lambda_p)}) = \frac{(y_k - y_m)^{\alpha_i^\vee(\lambda_p)} - (y_{k+1} - y_m)^{\alpha_i^\vee(\lambda_p)}}{y_k - y_{k+1}},$$

which becomes $y_k^{\alpha_i^\vee(\lambda_p)} + y_k^{\alpha_i^\vee(\lambda_p)-1}y_{k+1} + \dots + y_{k+1}^{\alpha_i^\vee(\lambda_p)}$ after setting $y_m = 0$, which is exactly the relation expected from [Web17a, (3.2)]. Finally, in all other triple points, there is no correction term in either set of relations. This confirms all the relations of $\tilde{T}^\underline{\lambda}$.

Thus, turning all ghosts into red strands gives a surjective map $\bar{W}_\nu^\varpi \rightarrow \tilde{T}_{\lambda-\nu}^\underline{\lambda}$. Note that this map sends basis vectors to basis vectors for the diagram bases of these algebras, and thus is an isomorphism. \square

THEOREM 3.6. *The tensor product algebra $T_{\lambda-\nu}^\underline{\lambda}$ is the reduced steadied quotient of the weighted algebra $W_\nu^\varpi(c)$ for Γ_w . Similarly, the bimodule $B^{\vartheta, \vartheta'}(c)$ for two different tensor product weightings is exactly \mathfrak{B}^σ , where σ is the positive braid lift of the permutation sending the total order on new edges by weight in ϑ to that induced by weight in ϑ' .*

Proof. Note that if $\nu' + \nu'' = \tilde{\nu}$, then $\nu' >_c \nu''$ if and only if the 0-component of ν' is 0 and that of ν'' is 1. Thus, the unsteady ideal is generated by diagrams where a block of strands all labeled with old vertices are “much further” left than the 0-labelled strands. This obviously corresponds to the violating ideal as defined in [Web17a, §3], so we have the desired isomorphism. \square

In this case, we can apply the canonical deformation discussed in Section 2.7, which gives algebras like those appearing in [Web17a, §3.5]. Let us take this deformation for the weighted KLR algebra of Crawley-Boevey quiver, and set all coefficients $z_{e,a,b} = 0$ for e an old edge (one from the original quiver). We're left with the parameter $z_{e,0,0}$ for each new edge; we'll abbreviate $z_{i,j,k} = -z_{e_{i,j,k},0,0}$. This results in a deformation of the algebra T^Δ , where the number of parameters $\{z_{i,j,k}\}$ is the number of new edges, that is, $\rho^\vee(\lambda)$.

We can easily describe how the relations of T^Δ deform in this case. For each $i \in \Gamma$ and $j \in [1, \ell]$, let $p_{i,j}(u) = (u - z_{i,j,1}) \cdots (u - z_{i,j,\alpha_i^\vee(\lambda_j)})$. The relations [Web17a, (4.1a,4.2)] thus deform to:

$$\begin{array}{c} \text{Diagram 1} \end{array} - \begin{array}{c} \text{Diagram 2} \end{array} = \sum_{p=1}^{\lambda_j} \sum_{a+b=p-1} e_{\ell-p}(-z_{i,j,*}) \cdot \left(\begin{array}{c} \text{Diagram 3} \end{array} \right). \quad (3.3a)$$

The RHS can alternately be written as $(p_{i,j}(y_{r+1}) - p_{i,j}(y_r))/(y_{r+1} - y_r)$.

$$\begin{array}{c} \text{Diagram 4} \end{array} = p_{i,j} \left(\begin{array}{c} \text{Diagram 5} \end{array} \right) \quad (3.3b)$$

$$\begin{array}{c} \text{Diagram 6} \end{array} = p_{i,j} \left(\begin{array}{c} \text{Diagram 7} \end{array} \right)$$

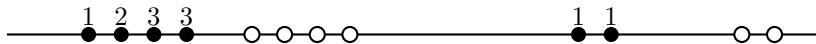
3.3 RELATION TO QUIVER SCHUR ALGEBRAS

When Γ is a cycle with n vertices, then we have some particularly interesting behavior. The choice of weightings (up to equivalence) is 1-dimensional, since $H^1(\Gamma; \mathbb{R}) \cong \mathbb{R}$. Weightings are distinguished by the sum of the weights over

an oriented cycle. We can identify $\Gamma = \mathbb{Z}/n\mathbb{Z}$, with an edge $i \rightarrow i + 1$; we let $\vartheta_e = k$, a constant.

Choose $0 < \epsilon \ll |k| \ll s$. For each vector composition $\hat{\mu} = \mu^{(1)}, \dots, \mu^{(m)}$, we associate the following loading $\mathbf{j}(\hat{\mu})$: take the residue sequence (as defined in [SW, (3)]) for this sequence, and for each entry of the j th block of the residue sequence p_1, \dots , add a points at $js + \ell\epsilon$ labeled with p_ℓ (so, we assume that $\epsilon < |k|/\ell_{max}$). Thus, for each piece of the vector composition, we have a cluster of points in the loading whose labels sum to that piece, and the clusters are very far apart. Now take the idempotent mapping the loading to itself which on the like-labelled strands of each piece of the loading does the idempotent which acts on polynomials by projecting to symmetric polynomials. Note that within each block, rearranging strands will result in isomorphic idempotents.

Example 3.7. If $\hat{\mu} = (1, 1, 2), (2, 0, 0)$ and $k > 0$, the loading is



where we represent ghosts by hollow circles.

There are some obvious idempotents acting on each of these loadings $\mathbf{j}(\hat{\mu})$; let $e'_{\mathbf{j}(\hat{\mu})}$ be the idempotent that acts on $\mathbf{j}(\hat{\mu})$ by applying the idempotent e_n projecting to symmetric polynomials to the like-labelled points in each cluster. Let e_{QS} be the sum of the idempotents $e'_{\mathbf{j}(\hat{\mu})}$.

THEOREM 3.8. *The algebra $e_{QS}W_v^\vartheta e_{QS}$ is isomorphic to the quiver Schur algebra $A_{\mathbf{d}}$ defined in [SW].*

Proof. This isomorphism sends the split of [SW] to the analogous splitting of the idempotents we described without crossing any like-labelled strands, and the merge to merging with crossing all pairs of like-labelled strands from the two merging pieces. These are shown in Figure 2. It's easily checked that these act exactly as in [SW, 3.4]; in fact this is already shown in [SW, (23)]. Thus, $A_{\mathbf{d}}$ injects into this space, and the graded dimensions of the two algebras coincide, since the dimensions of the summands going between vector compositions $\hat{\mu}$ and $\hat{\mu}'$ both count double cosets for the subgroups of S_m corresponding to the vector compositions. \square

More generally, there are algebras, defined in [SW, §4], which mix together features of the quiver Schur algebras above with those of the tensor product algebras. These arise from the Crawley-Boevey quiver $\Gamma_{\mathbf{w}}$ for the n -cycle and some dimension vector \mathbf{w} . As before, choose a weighting ϑ , and let k be the sum of the weights on the cycle.

For each pair of new edges e_1, e_2 , one can consider all the closed paths which leave the CB vertex using e_1 and arrive using e_2 . If these connect to the same vertex in the cycle, there's a unique such path which isn't self-intersecting (just the bigon), and otherwise, there are two which go around the cycle in opposite directions. We call a choice of ϑ WELL-SEPARATED for a dimension vector \mathbf{d} if

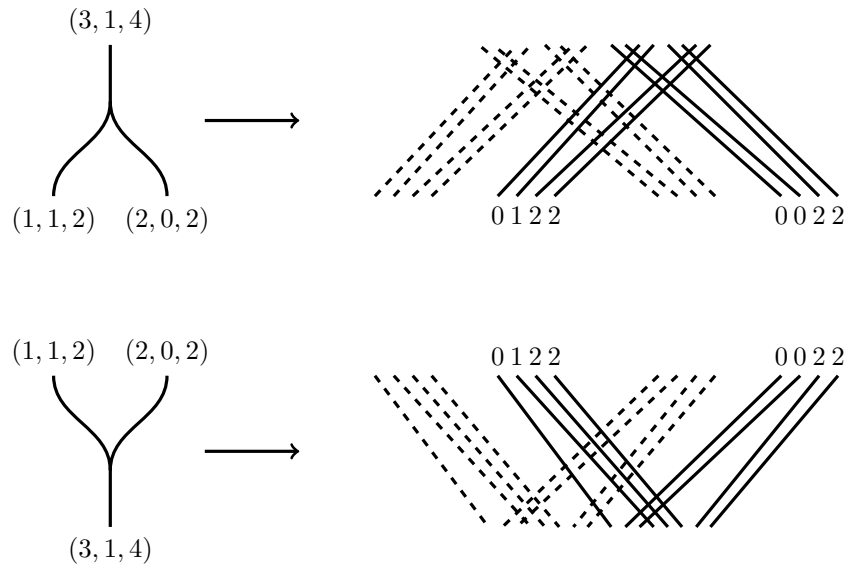


Figure 2: The comparison map with quiver Schur algebras

for any pair of new edges, the absolute value of the weight assigned to any non-self-intersection loop which starts with one and ends with the other is greater than $k(\sum_{i \in I} d_i)$.

In a well-separated weighting, we can order the new edges according to their weight unambiguously, since the weight of the two non-intersecting paths have the same sign (otherwise, we might have one positive, and one negative). We can consider the new edges in increasing order. Each one connects to a node in the cycle, to which we have associated a fundamental weight. Thus, we obtain a list of fundamental weights $\underline{\lambda} = \{\lambda_1, \dots, \lambda_\ell\}$, where ℓ is the total number of new edges, usually called the LEVEL in this context. Furthermore, to each list $\hat{\mu} = (\hat{\mu}(0), \hat{\mu}(1), \dots, \hat{\mu}(\ell))$ of vector compositions, we can associate a loading as follows: we place a copy of the loading for $\hat{\mu}(i)$ and its idempotent $e'_{\hat{\mu}(i)}$ (as constructed above) shifted by the position b_i of the i th red strand. That is, we place it on the real line just right of the i th red strand.

Let $e_{QS;\underline{\lambda}}$ be the sum of idempotents attached to these loadings.

THEOREM 3.9. *If we choose ϑ well-separated, then the subalgebra $e_{QS;\underline{\lambda}} \bar{W}_\nu^\vartheta e_{QS;\underline{\lambda}}$ of the reduced quotient is the extended quiver Schur algebra $\tilde{A}_{\mathbf{d}}^{\underline{\lambda}}$ associated to $\underline{\lambda}$, and the subalgebra $e_{QS;\underline{\lambda}} \bar{W}_\nu^\vartheta(c) e_{QS;\underline{\lambda}}$ of reduced steadied quotient is isomorphic to $A_{\mathbf{d}}^{\underline{\lambda}}$, and thus isomorphic to a cyclotomic q -Schur algebra.*

Proof. The first isomorphism is exactly as in Theorem 3.8; we simply note that the action of these operators on the appropriate symmetric polynomials exactly match those of $\tilde{A}_{\mathbf{d}}^{\lambda}$.

The steadied quotient exactly kills all idempotents where $\hat{\mu}(0) \neq 0$, and thus coincides with the cyclotomic quotient. \square

In fact, both these inclusions of subalgebras induce Morita equivalences, but we omit a proof of this fact; the construction of a cellular basis in [Web17c, §3] shows that no simple representation is killed by this idempotent. It is more natural to consider this in the context of a general weighting of an affine quiver, which is probably the most interesting and powerful application of the theory developed here; we develop this further in [Web17c].

4 THE GEOMETRY OF QUIVERS

4.1 LOADED FLAG SPACES

Throughout this section, we assume that Γ is a multiplicity-free quiver; that is, we assume that $c_e = 1$ for all edges, though we do allow multiple edges between the same pair of vertices. Furthermore, we assume that $Q_e(u, v) = u - v$ for every oriented edge. If $\nu = \sum d_i \alpha_i$, we let $V_i = \mathbb{C}^{d_i}$, $V = \oplus_i V_i$ and let

$$E_{\nu} = \bigoplus_{e \in \Omega} \text{Hom}(V_{t(e)}, V_{h(e)}).$$

This vector space has a natural action of $G_{\nu} = \prod_{i \in I} GL(V_i)$ by pre- and post-composition.

The vector $\mathbf{d} = (d_i)_{i \in \Gamma}$ is called the DIMENSION VECTOR, and we will freely identify \mathbb{Z}^{Γ} with the root lattice $X(\Gamma)$ by sending $\mathbf{d} \mapsto \nu = \sum d_i \alpha_i$.

Let \mathbf{i} be a loading.

DEFINITION 4.1. *We let an \mathbf{i} -loaded flag on V be a flag of I -homogeneous subspaces $F_a \subset V$ for each real number a such that $F_b \subset F_a$ for $b \leq a$, and $\dim F_a = \sum_{b \leq a} \mathbf{i}(b)$. Even though this filtration is indexed by real numbers, only finitely many different spaces appear; the dimension vector can only change at points in the support of the loading, by adding the simple root labeling that point to the dimension vector. Let $\text{Fl}_{\mathbf{i}}$ denote the space of \mathbf{i} -loaded flags.*

The relationship of these flags to the loadings we discussed earlier (justifying the name) is as follows: we can imagine the space F_a as being attached to the dots left of $x = a$. We read from left to right, and each time we pass a dot with label i , we increase the size of the space in the flag in V_i .

Each loaded flag F_{\bullet} has a corresponding UNLOADING, which is the complete flag of spaces appearing as F_a for $a \in \mathbb{R}$, indexed by dimension as usual.

DEFINITION 4.2. *For \mathbf{i} a loading with $|\mathbf{i}| = \nu$, let*

$$X_{\mathbf{i}} = \{(f, F_{\bullet}) \in E_{\nu} \times \text{Fl}_{\mathbf{i}} \mid f_e(F_a) \subset F_{a-\vartheta_e}\}$$

be the space of \mathbf{i} -loaded flags and compatible representations. Let $p: X_{\mathbf{i}} \rightarrow E_{\nu}$ be the map forgetting the flags, and let

$$Z = \bigsqcup_{\mathbf{i}, \mathbf{j} \in B(\nu)} X_{\mathbf{i}} \times_{E_{\nu}} X_{\mathbf{j}}.$$

We can also interpret compatibility visually in terms of loadings: rather than require that F_a be preserved by f_e , we require that the piece of V_i corresponding to a dot at $x = a - \vartheta_e$ can only be hit under the map f_e by the pieces corresponding to dots right of the corresponding ghost, that with $x \geq a$. Put differently, the piece of the filtration F_a corresponding to dots left of a must land under f_e in the span of pieces for dots whose ghosts are left of $x = a$.

Example 4.3. For $\mathbf{j}(\hat{\mu})$ with $k > 0$, as defined in Section 3.3, the map f_e for each edge $e: i \rightarrow i + 1$ must send the F_{j_s} space associated to the first j parts of the vector composition to the space $F_{(j-1)_s}$ for the $j - 1$ pieces, since we have specifically set things up so that a dot in the j th piece is to the left of the ghosts attached to the j th piece, and those to the right, and right of the ghosts for the $j - 1$ st piece, and those to the left. Note that this is closely related to the flag spaces considered in [SW], where arbitrary strongly preserved flags were considered, but the flags we consider here come with a refinement to complete flags. While this may seem extraneous, it makes the convolution algebras much easier to deal with.

If $k < 0$, then the picture is quite different. Now, each dot for the j th piece is right of the dots in the j th piece (and those to the left), so our conditions just say that $f_e(F_{j_s}) \subset F_{j_s}$, so this flag is weakly preserved.

Example 4.4. If we consider a Crawley-Boevey quiver $\Gamma_{\mathbf{w}}$, with the weight on all old edges being 0, then the result is that the flag F_a must be preserved in the usual sense by all the maps associated to old edges. Furthermore, the map f_e along a new edge is constrained to be 0 on F_{ϑ_e} . That is, we are only allowed to use one of the new edges on pieces of the flag corresponding to dots coming right of the corresponding red line (in the usual pictures discussed in Section 3.2).

If $\vartheta_e = 0$, then these are simply QUIVER FLAG VARIETIES, as used by Lusztig [Lus91], and considered by many other authors since. In particular, we can define a collection of objects in the G_{ν} -equivariant derived category of E_{ν} generalizing those considered by Lusztig, by considering the pushforwards

$$Y_{\mathbf{i}} := p_* \mathbb{k}_{X_{\mathbf{i}}}[\mathbf{u}(\mathbf{i})]$$

where

$$\begin{aligned} \mathbf{u}(\mathbf{i}) &= \dim X_{\mathbf{i}}/G_{\nu} \\ &= \#\{(e, a, b) \mid \mathbf{i}(a) = t(e), \mathbf{i}(b) = h(e), a - b \geq \vartheta_e\} - \sum_{i \in I} \frac{|\mathbf{i}|_i(|\mathbf{i}|_i + 1)}{2}. \end{aligned}$$

Since p is proper, if \mathbb{k} is characteristic 0, then these sheaves will be a sum of shifts of simple perverse sheaves; this can fail when the characteristic is positive and small. In favorable cases, where we obtain parity vanishing results, the summands of these sheaves will be parity sheaves in the sense of Juteau, Mautner and Williamson [JMW14]. This is the case when Γ is of finite or affine type A, but seems to be unknown in general; see [Mak15] for a more detailed discussion of parity sheaves on $E_{\mathbf{v}}$.

In the case of a tensor product weighting, these spaces and sheaves have been studied by Li [Li14]. In the affine case, closely related spaces were considered in [SW]; as long as the weights on new edges are well separated, the sheaves $Y_{\mathbf{i}}$ have the same simple summands as the pushforwards from the spaces $\mathcal{Q}(\hat{\mu})$. This definition of the spaces $X_{\mathbf{i}}$ has motivated in large part by the desire to unify these examples and put them in a more general context.

4.2 AN EXT-ALGEBRA CALCULATION

Consider the tautological line bundle \mathcal{L}_i given by the quotient of the i -dimensional space of the flag by the $i - 1$ st. The cohomology ring $H_{G_{\mathbf{v}}}^*(\text{Fl}_{\mathbf{i}})$ is a polynomial ring, in variables that can be identified with the equivariant Chern classes \mathcal{L}_k .

Given two loadings \mathbf{i} and \mathbf{j} and a permutation σ , we have a natural correspondence

$$\overset{\circ}{X}_{\mathbf{i};\mathbf{j}}^{\tau} = \{(f, \{F_{\bullet}\}, f', \{F'_{\bullet}\}) \in X_{\mathbf{i}} \times X_{\mathbf{j}} \mid r(V_*, V'_*) = \tau\} \quad X_{\mathbf{i};\mathbf{j}}^{\tau} = \overline{\overset{\circ}{X}_{\mathbf{i};\mathbf{j}}^{\tau}}$$

where $r(-, -)$ is the usual relative position between the unloadings of these flags. This space is non-empty if and only if the unloadings of \mathbf{i} and \mathbf{j} are permuted to each other by τ .

We let $H_*^{BM, G^d}(-)$ denote the equivariant Borel-Moore homology of a space with coefficients in \mathbb{k} , as discussed in [VV11, §1]; for any proper map $p: X \rightarrow Y$, the Borel-Moore homology $H^{BM}(X \times_Y X)$ carries a convolution algebra structure, defined in [CG97, 2.7]; in [CG97, 8.6], it's proven that this is isomorphic to the Ext algebra $\text{Ext}^{\bullet}(p_*\mathbb{k}_X, p_*\mathbb{k}_X)$, and this result is easily extended to the equivariant case.

THEOREM 4.5. *We have isomorphisms of dg-algebras*

$$\text{Ext}_{G_{\mathbf{v}}}^{\bullet} \left(\bigoplus_{\mathbf{i} \in B(\nu)} Y_{\mathbf{i}}, \bigoplus_{\mathbf{i} \in B(\nu)} Y_{\mathbf{i}} \right) \cong H_*^{BM, G_{\nu}}(Z) \cong W_{\nu}^{\vartheta}$$

where the RHS has trivial differential. The right hand isomorphism sends

$$e_{\mathbf{i}} b_1 e_{\mathbf{j}} \mapsto [X_{\mathbf{i};\mathbf{j}}^1] \quad e_{\mathbf{i}} \psi_k e_{\mathbf{j}} \mapsto [X_{\mathbf{i};\mathbf{j}}^{S_k}] \quad y_k \rightarrow c_1(\mathcal{L}_k).$$

This map intertwines Verdier duality and the duality $a \mapsto a^*$ on W_{ν}^{ϑ} .

Recall that replacing an object by another in which precisely the same indecomposable summands occur preserves the graded Morita class of the Ext-algebra. Thus, if we replace $\oplus Y_{\mathbf{i}}$ by the sum of all IC-sheaves whose shifts appear as summands of $Y_{\mathbf{i}}$ for some \mathbf{i} , we obtain that:

COROLLARY 4.6. *Assume $\text{char}(\mathbb{k}) = 0$. The algebra W^ϑ is graded Morita equivalent to a non-negatively graded algebra. That is, there is a projective generator G of W^ϑ -mod with no negative degree endomorphisms. We can choose this generator so that if P is a graded projective that occurs as a summand in $W^\vartheta e_{\mathbf{i}}$ for some \mathbf{i} and no shift of P does, then P is a summand of G .*

Note that it is easy to find examples where this fails if \mathbb{k} has characteristic p . Such an example for $\widehat{\mathfrak{sl}}_2$ is discussed in [Web15, 5.7]; Williamson [Wil14] has shown that examples exist for KLR algebras in finite type A for any prime p . As we see in [Web15, Web17c], this property is key for proving a relationship between categorifications and canonical bases, along the same lines as [VV11].

We now turn to the proof of Theorem 4.5, which we will prove via a series of lemmata. As we noted in the proof of [SW, 3.11], we have an equivariant map

$$X_{\mathbf{i}} \times_{E_\nu} X_{\mathbf{j}} \rightarrow \text{Fl}_{\mathbf{i}} \times \text{Fl}_{\mathbf{j}},$$

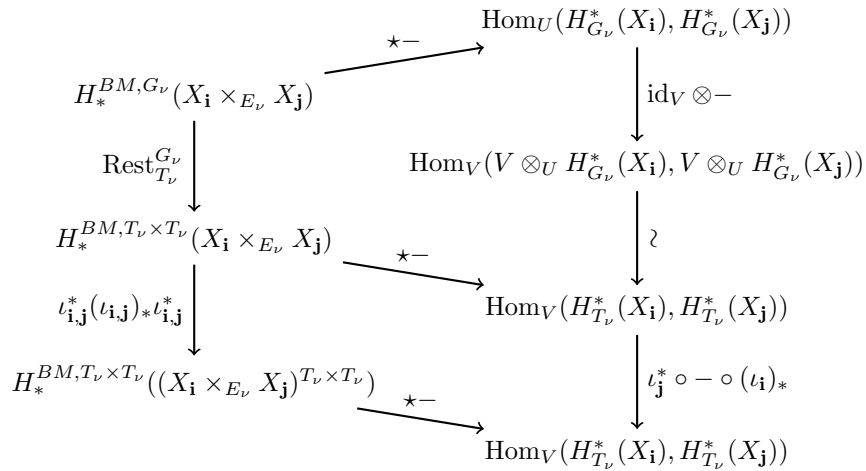
projecting to the second factor. This map is an affine bundle over each G_ν -orbit. These orbits are in turn homotopic to G_ν/T_ν , letting T_ν be a maximal torus in G_ν . Thus $X_{\mathbf{i}} \times_{E_\nu} X_{\mathbf{j}}$ is a union of finitely many spaces each with even and equivariantly formal Borel-Moore homology, so the same is true of $X_{\mathbf{i}} \times_{E_\nu} X_{\mathbf{j}}$.

LEMMA 4.7. *The Ext-algebra $E = \text{Ext}_{G_\nu}^\bullet \left(\bigoplus_{\mathbf{i} \in B(\mathbf{d})} Y_{\mathbf{i}}, \bigoplus_{\mathbf{i} \in B(\mathbf{d})} Y_{\mathbf{i}} \right)$ is formal and acts faithfully on*

$$\bigoplus_{\mathbf{i} \in B(\nu)} H_{G_\nu}^*(X_{\mathbf{i}}) \cong \bigoplus_{\mathbf{i} \in B(\nu)} H_{G_\nu}^*(\text{Fl}_{\mathbf{i}}).$$

Proof. This has essentially the same proof as [SW, 4.7]; Let $U = H^*(BG_\nu)$ and $V = H^*(BT_\nu)$. The restriction functor $\text{Rest}_{T_\nu}^{G_\nu}$ on equivariant derived categories and the inclusion $\iota_{\mathbf{i}, \mathbf{j}} : (X_{\mathbf{i}} \times_{E_\nu} X_{\mathbf{j}})^{T_\nu \times T_\nu} \hookrightarrow X_{\mathbf{i}} \times_{E_\nu} X_{\mathbf{j}}$ induce a

commutative diagram



The composition of the two vertical lines are both injective, since V is a free module of finite rank over U and the Borel-Moore homology of every space that appears is even and equivariantly formal. Since the fixed points of T_ν are isolated, the algebra

$$E^T = \bigoplus_{i,j \in B(\nu)} H_*^{BM, T_\nu \times T_\nu}((X_i \times_{E_\nu} X_j)^{T_\nu \times T_\nu})$$

is just the matrix algebra on $V^T = \bigoplus_{i \in B(\nu)} H_{T_\nu}^*(X_i^{T_\nu})$. In particular, it is formal as a dg-algebra since $H_{T_\nu}^*(X_i)$ is formal. This implies that E is formal as a dg-algebra as well.

Furthermore, E^T acts faithfully on V^T . That is, the bottom rung of the ladder is injective. Thus, any class $a \in H_*^{BM, G_\nu}(X_i \times_{E_\nu} X_j)$ which the top action kills is also killed by the map from the northwest corner to the southeast. This map is injective, so we are done. \square

LEMMA 4.8. *The non-zero classes $[X_{i,j}^\sigma]$ are a basis of $H_*^{BM, G_\nu}(X_i \times_{E_\nu} X_j)$ over $H_{G_\nu}^*(\text{Fl}_i)$.*

Proof. Pick a total order on permutations refining Bruhat order; our inductive statement is that $[X_{i,j}^\sigma]$ for $\sigma \leq \tau$ is a basis of $H_*^{BM, G_\nu}(\cup_{\sigma \leq \tau} X_{i,j}^\sigma)$. If $\tau = 1$, then $X_{i,j}^1$ is an affine bundle over $\text{Fl}_i = \text{Fl}_j$, since the left and right flags must agree. Thus, its equivariant Borel-Moore homology is freely generated over $H^*(\text{Fl}_i)$ by $[X_{i,j}^1]$.

Now, by induction, let τ' be maximal w.r.t $\tau' \prec \tau$. Then we have long exact sequence

$$\dots \longrightarrow H_i^{BM, G_\nu}(\cup_{\sigma \leq \tau'} X_{i,j}^\sigma) \longrightarrow H_i^{BM, G_\nu}(\cup_{\sigma \leq \tau} X_{i,j}^\sigma) \longrightarrow H_i^{BM, G_\nu}(\mathring{X}_{i,j}^\tau) \longrightarrow \dots$$

The space $\hat{X}_{\mathbf{i};\mathbf{j}}^\tau$ is an affine bundle over the space in $\text{Fl}_{\mathbf{i}} \times \text{Fl}_{\mathbf{j}}$ with relative position τ , since being compatible with two fixed flags is a linear condition on matrix coefficients of quiver representations, and all fibers are conjugate under the action of G_ν . This space is, in turn, an affine bundle over $\text{Fl}_{\mathbf{i}}$ since the space of flags of relative position exactly τ to a fixed flag is an affine space. Thus, the equivariant Borel-Moore homology $H_i^{BM, G^a}(\hat{X}_{\mathbf{i};\mathbf{j}}^\tau)$ is free of rank 1 over $H^*(BG_{\mathbf{i}})$ if the unloading of \mathbf{i} is sent to the unloading of \mathbf{j} by τ , and rank 0 otherwise (since the space is empty). Furthermore, it is generated by the fundamental class of $\hat{X}_{\mathbf{i};\mathbf{j}}^\tau$, and in particular all lies in even degree. This shows that, by induction, all groups appearing in the above sequence vanish in odd degree, so the l.e.s. splits into a sum of s.e.s.

Thus, any subset of $H_i^{BM, G^a}(\cup_{\sigma \preceq \tau} X_{\mathbf{i};\mathbf{j}}^\sigma)$ consisting of a basis of $H_i^{BM, G^a}(\cup_{\sigma \preceq \tau'} X_{\mathbf{i};\mathbf{j}}^\sigma)$ and an element projecting to $[\hat{X}_{\mathbf{i};\mathbf{j}}^\tau]$ (if that space is non-empty) is a basis of $H_i^{BM, G^a}(\cup_{\sigma \preceq \tau} X_{\mathbf{i};\mathbf{j}}^\sigma)$. Since $[X_{\mathbf{i};\mathbf{j}}^\tau]$ projects to $[\hat{X}_{\mathbf{i};\mathbf{j}}^\tau]$ if that class is non-zero, and is itself 0 otherwise, induction yields the desired fact. \square

Proof of Theorem 4.5. First, the left hand isomorphism is an immediate consequence of [CG97, 8.6.7].

Now we wish to confirm that the action of the classes $[X_{\mathbf{i};\mathbf{j}}^1]$ and $[X_{\mathbf{i};\mathbf{j}}^{s_k}]$ act on

$$\bigoplus_{\mathbf{i} \in B(\nu)} H_{G_\nu}^*(X_{\mathbf{i}}) \cong \bigoplus_{\mathbf{i} \in B(\nu)} \mathbb{k}[y_1, \dots, y_d]$$

in the same way as $e_{\mathbf{i}}b_1e_{\mathbf{j}}$ and $e_{\mathbf{i}}\psi_k e_{\mathbf{j}}$.

- If going from \mathbf{i} to \mathbf{j} passes a strand from right of a ghost to left of it, then $X_{\mathbf{i};\mathbf{j}}^1 \cong X_{\mathbf{j}}$: any \mathbf{j} -loaded flag is easily modified to be a \mathbf{i} -loaded flag using reindexing. Thus, the desired convolution is just the pull-back map for the inclusion $X_{\mathbf{j}} \rightarrow X_{\mathbf{i}}$ in cohomology, which sends Chern classes to Chern classes, and induces the identity map on $\mathbb{C}[y_1, \dots, y_n]$.
- If going from \mathbf{i} to \mathbf{j} passes the j th strand from left of a ghost for e of the k th strand to right, then symmetrically $X_{\mathbf{i};\mathbf{j}}^1 \cong X_{\mathbf{i}}$. Thus, the desired convolution is the pushforward by the inclusion $X_{\mathbf{i}} \rightarrow X_{\mathbf{j}}$, which on the level of cohomology rings multiplies by the Euler class of the normal bundle for the inclusion, which is $\text{Hom}(\mathcal{L}_j, \mathcal{L}_k)$, whose Euler class is $y_k - y_j = Q_e(y_k, y_j)$.

This deals with all crossings of strands and ghosts. We now need only consider the case where no ghosts separate the k and $k + 1$ st strands, and we apply ψ_k .

- If k th and $k + 1$ st strands have different labels, then $X_{\mathbf{i};\mathbf{j}}^{s_k}$ is the graph of an isomorphism between the sets of loadings $X_{\mathbf{i}}$ and $X_{\mathbf{j}}$; there is a unique \mathbf{j} -loaded flag which agrees with a given \mathbf{i} -loaded flag at all jumps but the k th. The only effect of this isomorphism is that it reindexes the

line bundles of interest to us via the permutation s_k ; hence this is also the effect on Chern classes.

- If the k th and $k + 1$ st strands have the same labels, we can take $\mathbf{i} = \mathbf{j}$. Let W be the subvariety of $X_{\mathbf{i}}$ where all loops of weight 0 send the $k + 1$ st step of the flag to the $k - 1$ st, and let $\mathcal{L}_{k;k+1}$ be the rank 2 vector bundle on W given by the $k + 1$ st step of the flag modulo the $k - 1$ st. The space $X_{\mathbf{i};\mathbf{i}}^{s_k}$ is the projectivization over W of the vector bundle $\mathcal{L}_{k;k+1}$. Thus, if $i: W \rightarrow X_{\mathbf{i}}$ is the inclusion, and $p: X_{\mathbf{i};\mathbf{i}}^{s_k} \rightarrow W$ the projection, then $[X_{\mathbf{i};\mathbf{i}}^{s_k}] = i_* p_* p^* i^*$. The two pullbacks just act as the identity; the pushforward p_* acts as Demazure operator in the variables y_k and y_{k+1} , and the pushforward i_* multiplies by the Euler class of the normal bundle, which is 1 if there is no loop of degree 0, and $\prod_e Q_e(y_k, y_{k+1})$ where e ranges over such loops otherwise. Applying Definition 2.6, we see that this matches the action of Proposition 2.7.

This shows that we have an injective algebra map $a: e_{\mathbf{i}} W_{\nu}^{\vartheta} e_{\mathbf{j}} \rightarrow H_*^{BM, G_{\nu}}(X_{\mathbf{i}} \times_{E_{\nu}} X_{\mathbf{j}})$. Finally, we need to confirm that this map is surjective.

We let $e_{\mathbf{i}} x_{\sigma} e_{\mathbf{j}} = [X_{\mathbf{i};\mathbf{j}}^{\sigma}]$ if the word in simple roots attached to \mathbf{j} is the permutation by σ of that for \mathbf{i} and 0 otherwise.

Now, consider a factorization of b_{τ} into pieces where there is only one crossing of two strands or of a strand and a ghost. The image $a(b_{\tau})$ of this diagram is the convolution of all the classes attached to these diagrams, which are each of the form $[X_{-, -}^{s_k}]$ or $[X_{-, -}^1]$. That is, there is sequences $t_m \in \{1\} \cup \{s_1, \dots, s_n\}$ and $\mathbf{i}^{(m)}$ such that $b_{\tau} = e_{\mathbf{i}} b_{t_1} e_{\mathbf{i}^{(1)}} b_{t_2} \cdots e_{\mathbf{i}^{(\ell-1)}} b_{t_{\ell}} e_{\mathbf{j}}$. In particular we obtain a reduced decomposition $\tau = t_{k_1} \cdots t_{k_{\ell}}$. Now, consider an element $(f, F_{\bullet}, F'_{\bullet}) \in \check{X}_{\mathbf{i};\mathbf{j}}^{\tau}$. Consider the unique flag which has relative position $t_{k_1} \cdots t_{k_h}$ to the unloading of the left flag and $t_{k_{\ell}} \cdots t_{k_{n+1}}$ to the unloading of the right. Let F_{\bullet}^h be the unique $\mathbf{i}^{(h)}$ -loaded flag whose unloading is the complete flag we have just described.

LEMMA 4.9. *The $\mathbf{i}^{(h)}$ -loaded flag F_{\bullet}^h is compatible with the representation f_{\bullet} .*

Proof. Without loss of generality, we can assume that both F_{\bullet} and F'_{\bullet} are coordinate flags for a single basis, which is in bijection with the points in the loadings \mathbf{i} and \mathbf{j} ; we let w_i and w_j be the accompanying positions on the real line. By the compatibility with F_{\bullet} and F'_{\bullet} , the image $f_e(v_m)$ is in the span of v_k with $w_i(v_k) \leq w_i(v_m) - \vartheta_e$ and $w_j(v_k) \leq w_j(v_m) - \vartheta_e$.

One of the essential characteristics of b_{τ} is that up to isotopy, we can assume that the distance between any pair of strands monotonically increases or decreases, so we may assume that the distance between the weights associated to v_k and v_m in $\mathbf{i}^{(h)}$ are strictly between that for \mathbf{i} and \mathbf{j} . Thus, the same inequalities hold for every slice, and we are done. \square

Thus, we see that the map from the fiber product

$$q: X_{\mathbf{i};\mathbf{i}(1)}^{t_1} \times_{X_{\mathbf{i}(1)}} X_{\mathbf{i}(1);\mathbf{i}(2)}^{t_2} \times_{X_{\mathbf{i}(2)}} \cdots \times_{X_{\mathbf{i}(\ell-1)}} X_{\mathbf{i}(\ell-1);\mathbf{j}}^{t_\ell} \rightarrow X_{\mathbf{i};\mathbf{j}}^\tau$$

must map bijectively over $\overset{\circ}{X}_{\mathbf{i};\mathbf{j}}^\tau$; at each intermediate point, we have a single unique choice for the $\mathbf{i}^{(h)}$ -loaded flag compatible with f_* , which is, of course, F_*^h .

Thus, we have that

$$a(b_\tau) = q_*[X_{\mathbf{i};\mathbf{i}(1)}^{t_1} \times_{X_{\mathbf{i}(1)}} \cdots \times_{X_{\mathbf{i}(\ell-1)}} X_{\mathbf{i}(\ell-1);\mathbf{j}}^{t_\ell}] = [X_{\mathbf{i};\mathbf{j}}^\tau] + \sum_{\tau < \tau'} r_\tau(y_1, \dots, y_n)[X_{\mathbf{i};\mathbf{j}}^{\tau'}].$$

Thus, the matrix of the map a written in terms of the basis of $e_{\mathbf{i}}W^\vartheta e_{\mathbf{j}}$ given by b_τ 's and that for the Borel-Moore homology $H_*^{BM, G_\nu}(X_{\mathbf{i}} \times_{E_\nu} X_{\mathbf{j}})$ given by $[X_{\mathbf{i};\mathbf{j}}^\tau]$'s is upper-triangular with 1's on the diagonal and thus an isomorphism. \square

Put another way:

COROLLARY 4.10. *There is a fully faithful additive functor $\gamma: W_\nu^\vartheta\text{-pmod} \rightarrow D(E_\nu/G_\nu)$ sending $[W_\nu^\vartheta e_{\mathbf{i}}] \mapsto Y_{\mathbf{i}}$.*

If Γ is produced by the Crawley-Boevey trick on another graph, we let G'_ν be the subgroup of G which only acts on old vertices. This is a codimension 1 subgroup.

If we let $Y'_\mathbf{i}$ be the pullback of $Y_{\mathbf{i}}$ from E_ν/G_ν to E_ν/G'_ν . Repeating the proof of Theorem 4.5 in this context, we arrive at almost the same result, except that we have killed the Chern class of any line bundle attached to a representation which is trivial restricted to G'_ν , that is the dot on the unique strand labeled with α_0 . That is:

COROLLARY 4.11. *We have isomorphisms of dg-algebras*

$$\text{Ext}_{G_\nu}^\bullet \left(\bigoplus_{\mathbf{i} \in B(\nu)} Y'_\mathbf{i}, \bigoplus_{\mathbf{i} \in B(\nu)} Y'_\mathbf{i} \right) \cong \bigoplus_{\mathbf{i}, \mathbf{j} \in B(\nu)} H_*^{BM, G'_\nu}(X_{\mathbf{i}} \times_{E_\nu} X_{\mathbf{j}}) \cong \bar{W}_\nu^\vartheta$$

where the RHS has trivial differential. In particular, if we choose a tensor product weighting, we have an isomorphism

$$\text{Ext}_{G_\nu}^\bullet \left(\bigoplus_{\mathbf{i} \in B(\nu)} Y'_\mathbf{i}, \bigoplus_{\mathbf{i} \in B(\nu)} Y'_\mathbf{i} \right) \cong \tilde{T}_{\lambda-\nu}^\lambda$$

This result naturally extends to the bimodule $B^{\vartheta, \vartheta'}$ defined earlier. The proof is so similar to that of Theorem 4.5 that we leave it to the reader:

THEOREM 4.12. *For two weightings ϑ_1, ϑ_2 , we have an isomorphism of dg-modules:*

$$\text{Ext}_{G_\nu}^\bullet \left(\bigoplus_{\mathbf{i} \in B^1(\nu)} Y_{\mathbf{i}}^1, \bigoplus_{\mathbf{j} \in B^2(\nu)} Y_{\mathbf{j}}^2 \right) \cong B_\nu^{\vartheta_1, \vartheta_2}$$

where the left and right algebra actions are matched using the isomorphism of Theorem 4.5.

Remark 4.13. Theorems 4.5 and 4.12 can be extended to the canonical deformations of Section 2.7 by letting $\mathbb{G}_m^{E(\Gamma)}$ act in the natural way on E with each copy of \mathbb{G}_m acting with weight 1 on the map along one edge and trivially on all others. Considering the equivariant Borel-Moore homology in place of usual BM homology gives the deformed algebra \check{W}^ϑ , with the deformation parameters corresponding to the cohomology of $B\mathbb{G}_m^{E(\Gamma)}$.

4.3 MONOIDAL STRUCTURE

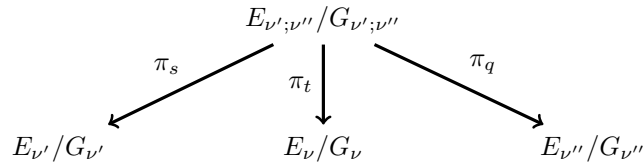
Recall that the derived categories $\oplus_\nu D(E_\nu/G_\nu)$ carry the Lusztig monoidal structure defined by convolution. If $\nu = \nu' + \nu''$, and we let $V_i = V'_i \oplus V''_i$ be I -graded vector spaces of the appropriate dimension, we consider

$$E_{\nu';\nu''} \cong E_{\nu'} \oplus E_{\nu''} \oplus \bigoplus_{e \in \Omega} \text{Hom}(V''_{t(e)}, V'_{h(e)})$$

with the obvious action of

$$G_{\nu';\nu''} = \{g \in G_\nu \mid g(V'_i) = V'_i\}.$$

We have the usual convolution diagram



We can view $E_{\nu';\nu''}/G_{\nu';\nu''}$ as the moduli space of short exact sequence with submodule of dimension ν' and quotient of ν'' . The projections π_* are remembering only the first, second or third term of the short exact sequence. The CONVOLUTION of sheaves $\mathcal{F}_1 \in D(E_{\nu'}/G_{\nu'}), \mathcal{F}_2 \in D(E_{\nu''}/G_{\nu''})$ is defined to be

$$\mathcal{F}_1 \star \mathcal{F}_2 := (\pi_t)_*(\pi_s^* \mathcal{F}_1 \otimes \pi_t^* \mathcal{F}_2)[- \langle \nu'', \nu' \rangle]$$

PROPOSITION 4.14. The functor $\gamma: W_\nu^\vartheta\text{-pmod} \rightarrow D(E_\nu/G_\nu)$ is monoidal, i.e.

$$\gamma(P_1 \circ P_2) \cong \gamma(P_1) \star \gamma(P_2).$$

Proof. We need only check this for $P_1 = W_{\nu'}^\vartheta e_i, P_2 = W_{\nu''}^\vartheta e_j$ since every projective is a summand of one of these. In this case, $P_1 \circ P_2 = W_\nu^\vartheta e_{i \circ j}$. On the other hand,

$$\pi_s^* Y_i = \tilde{p}_*^s \mathbb{k}_{X_i \times_{E_{\nu'}} E_{\nu';\nu''}}[\mathbf{u}(i)] \quad \pi_q^* Y_j = \tilde{p}_*^q \mathbb{k}_{X_j \times_{E_{\nu''}} E_{\nu';\nu''}}[\mathbf{u}(j)]$$

where \tilde{p}_*^s and \tilde{p}_*^q are base changes of the map p by π_s and π_q . Note that when \mathbf{i} and \mathbf{j} are separated far enough that no ghost from one is entangled in the other, the subspace F_a for a between \mathbf{i} and \mathbf{j} on the real line is a subrepresentation. Thus we have an isomorphism

$$(X_{\mathbf{i}} \times X_{\mathbf{j}}) \times_{E_{\nu'} \times E_{\nu''}} E_{\nu'; \nu''} / G_{\nu'; \nu''} \cong X_{\mathbf{i} \circ \mathbf{j}} / G_{\nu}; \tag{4.1}$$

the difference in groups is that on left side we fix a particular subspace and assume $F_a = \oplus V_i'$ and only act with the stabilizer of this subspace, whereas on the right side, we sweep through all possible subspaces. These quotients are the same since all I -graded subspaces of the same dimension vector are conjugate under G_{ν} .

By definition, $Y_{\mathbf{i}} \star Y_{\mathbf{j}}$ is the shifted pushforward from the LHS of (4.1), and $Y_{\mathbf{i} \circ \mathbf{j}}$ is the shifted pushforward from the RHS. Thus we have that $Y_{\mathbf{i}} \star Y_{\mathbf{j}} \cong Y_{\mathbf{i} \circ \mathbf{j}}$ where the equality of shifts follows from the formula

$$\mathbf{u}(\mathbf{i} \circ \mathbf{j}) = \mathbf{u}(\mathbf{i}) + \mathbf{u}(\mathbf{j}) - \langle |\mathbf{j}|, |\mathbf{i}| \rangle.$$

Furthermore, the self-Exts of $Y_{\mathbf{i}} \star Y_{\mathbf{j}}$ induced by those of $Y_{\mathbf{i}}$ and $Y_{\mathbf{j}}$ are exactly intertwined with the image of $\iota_{\nu'; \nu''}$, which shows that this functor is monoidal on morphisms as well. Thus, we have obtained the desired result. \square

There is also a left adjoint to \star , which we denote $\text{Res}_{\nu'; \nu''}$, given by

$$\text{Res}_{\nu'; \nu''} \mathcal{F} := (\pi_s \times \pi_q)_! \pi_i^! \mathcal{F}[\langle \nu'', \nu' \rangle]$$

PROPOSITION 4.15. *The functor $\gamma: W_{\nu}^{\theta}$ -pmod $\rightarrow D(E_{\nu}/G_{\nu})$ is intertwined restriction functors, that is*

$$(\gamma \boxtimes \gamma)(\text{Res}_{\nu'; \nu''} P) \cong \text{Res}_{\nu'; \nu''} \gamma(P).$$

Proof. Since these functors are left adjoint to functors intertwined by γ , they just be intertwined if $\text{Res}_{\nu'; \nu''} \gamma(P)$ is in the subcategory generated by the image of $\gamma(P)$.

As before, we need only consider the base where $P = Re_{\mathbf{i}}$. In this case, $\pi_i^! Y_{\mathbf{i}} = \tilde{p}_* \mathbb{k}_{X_{\mathbf{i}} \times E_{\nu'; \nu''}}$. We filter the fiber product $X_{\mathbf{i}} \times E_{\nu'; \nu''}$ according the relative position of the subspace V_i' and the \mathbf{i} -loaded flag (i.e. by the Schubert cell V_i' lands in for the Schubert stratification relative to the flag). Each such relative position corresponds to dividing the points in the loading into two sets: those where the dimension of the intersection of F_a with V_i' jumps and those where it does not. This gives loadings \mathbf{i}' and \mathbf{i}'' . The subset of the fiber product $X_{\mathbf{i}} \times E_{\nu'; \nu''}$ with fixed relative position is an affine bundle over the product $X_{\mathbf{i}' } \times X_{\mathbf{i}''}$ where the first term is the loaded flag induced on V_i' by intersecting with F_a , and the second is that induced on V_i'' by taking the images of the F_a 's. This shows that $\text{Res}_{\nu'; \nu''} \gamma(P)$ is an iterated cone of shifts of the objects $\gamma(P') \boxtimes \gamma(P'')$. This completes the proof. \square

On the level of Grothendieck groups, these propositions show that the structures we have seen on K are also typical for categories of sheaves on representations of quivers.

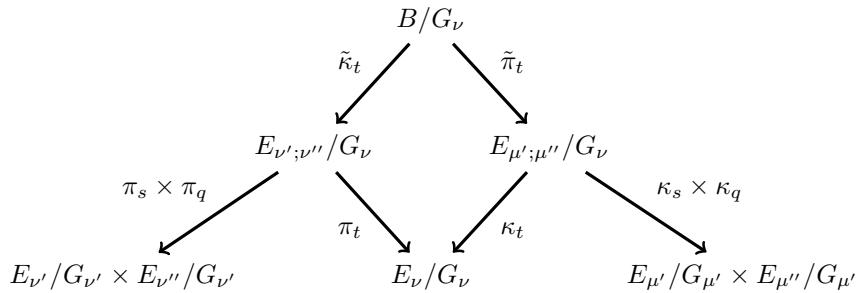
PROPOSITION 4.16. *The sum of Grothendieck groups $\oplus_{\nu} K(D(E_{\nu}/G_{\nu}))$ inherits a twisted bialgebra structure with product and coproduct*

$$[\mathcal{M}][\mathcal{N}] = [\mathcal{M} \star \mathcal{N}] \quad \Delta([\mathcal{M}]) = \left[\sum_{\nu'+\nu''=\nu} \text{Res}_{\nu';\nu''} \mathcal{M} \right],$$

and the functor γ induces a map of twisted bialgebras.

Proof. The fact that γ induces a map that commutes with the multiplication and comultiplication follows from Propositions 4.14 and 4.15.

The commutation of product and coproduct follows from the base change formula for pushforwards and pullbacks. Choosing ν', ν'', μ', μ'' such that $\nu' + \nu'' = \nu = \mu' + \mu''$, we wish to consider $\text{Res}_{\mu';\mu''}(\mathcal{M}' \star \mathcal{M}'')$. Let π_* denote the projection maps from $E_{\nu';\nu''}$ as before and κ_* the corresponding maps from $E_{\mu';\mu''}$ and $B = E_{\nu';\nu''} \times_{E_{\nu}} E_{\mu';\mu''}$. Then we have a diagram with the interior square Cartesian:



Thus, we have that

$$\begin{aligned} \text{Res}_{\mu';\mu''}(\mathcal{M}' \star \mathcal{M}'') &= (\kappa_s \times \kappa_q)! \kappa_t^1(\pi_t)_*(\pi_s \times \pi_q)^*(\mathcal{M}' \star \mathcal{M}'') \\ &= (\kappa_s \times \kappa_q)! (\tilde{\pi}_t)_* \tilde{\kappa}_t^1(\pi_s \times \pi_q)^*(\mathcal{M}' \star \mathcal{M}'') \end{aligned}$$

The variety B can be stratified into subsets B_{τ} according to the dimension τ of the intersection between the subrepresentations of dimension ν' and μ' . Intersection with the other subrepresentation induces subs of dimension τ in $\pi_s \tilde{\kappa}_t$ and $\kappa_s \tilde{\pi}_t$, and taking its image induces a subs of dimension $\mu' - \tau$ in $\pi_s \tilde{\kappa}_t$ and dimension $\nu' - \tau$ in $\kappa_s \tilde{\pi}_t$. Let $\tau' = \nu'' + \mu'' - \nu + \tau$. The map from B_{τ} to the fiber product of $E_{\tau;\nu'-\tau} \times E_{\mu'-\tau;\tau'}$ with $E_{\tau;\mu'-\tau} \times E_{\nu'-\tau;\tau'}$ over $E_{\tau} \times E_{\mu'-\tau} \times E_{\nu'-\tau} \times E_{\tau'}$ is an affine bundle of dimension $\langle \tau + \tau', \mu' + \nu' - 2\tau \rangle$. Thus,

$$\Delta_{\mu';\mu''}([\mathcal{M}'] \star [\mathcal{M}'']) = \sum_{\tau} \Delta_{\tau;\nu'-\tau}([\mathcal{M}']) \star \Delta_{\mu'-\tau;\tau'}[\mathcal{M}'']. \quad \square$$

4.4 HALL ALGEBRAS

While the previous section interpreted the weighted KLR algebras in terms of characteristic 0 geometry, we can also consider the geometry of quivers over a field of characteristic p . The varieties E_ν, X_i and the algebraic group G_ν are all defined as \mathbb{Z} -schemes whose base change to \mathbb{C} are the varieties considered in the previous sections. After base change to $\overline{\mathbb{F}}_q$ for q a prime power, we can use the same pushforwards to define ℓ -adic sheaves Y_i , which we denote with the same symbol as the corresponding sheaves over \mathbb{C} ; in this section, we will always consider sheaves on varieties over $\overline{\mathbb{F}}_q$, so there is no danger of confusion. By the usual comparison theorems in étale geometry (for example, [BBD82, 6.1.9]), the Ext-algebra of the sum of these sheaves is W_ν^θ , just as it is for sheaves over \mathbb{C} .

The sheaves Y_i have a unique mixed structure which is pure of weight 0. As always, the pushforward by a proper map of the constant sheaf with its canonical weight 0 mixed structure is again pure of weight 0. If we apply the shift in the derived category without changing the action of Frobenius, we will change the weight, but we can apply a Tate twist to return to weight 0. We will always take this mixed structure. In this section, the functor γ will land in this category, not its characteristic 0 analogue. The proof of Propositions 4.14 and 4.16 carry over without change.

The reader might thus justly wonder what is achieved by introducing this more difficult formalism. Our primary motivation is a better understanding of the Grothendieck group K . Recall that for any finite field \mathbb{F}_q , there is a HALL ALGEBRA $\mathcal{H}_{\Gamma,q}$ of representations of Γ , the space of all \mathbb{k} -valued function on the set of isomorphism classes of quiver representations over \mathbb{F}_q . We refer to the notes of Schiffmann [Sch] for basic facts and definitions of Hall algebras, but our Hall algebra will have the opposite product and coproduct from Schiffmann's for compatibility with our diagrammatic formulation. In essence, this is because our conventions are adapted to writing short exact sequences with arrows pointing left to right (as any right-thinking person would).

Attached to any mixed complex of sheaves \mathcal{M} over an extension \mathbb{k} of \mathbb{Q}_ℓ on E_ν , we have a function $\mathcal{J}_\mathcal{M}: E_\nu(\mathbb{F}_q) \rightarrow \mathbb{k}$ sending $e \in E$ to the supertrace of the Frobenius morphism acting on the stalk at that point:

$$\mathcal{J}_\mathcal{M}(e) := \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{Tr}(\operatorname{Fr} | H^i(\mathcal{M}_e)).$$

If we let \mathcal{K} denote the Grothendieck group of the category of pure weight 0 shifts of perverse sheaves over \mathbb{k} , then $\mathcal{J}_\mathcal{M}: \mathcal{K} \rightarrow \mathcal{H}_{\Gamma,q}$.

PROPOSITION 4.17. *The map $\mathcal{J}_\mathcal{M}: \mathcal{K} \rightarrow \mathcal{H}_{\Gamma,q}$ is a map of bialgebras.*

Proof. This follows instantly from the Grothendieck trace formula. \square

While the definition of these functions may sound awfully abstruse, for geometrically natural sheaves, these functions are quite explicit. Of greatest importance to us is that

PROPOSITION 4.18. $\mathcal{J}_{Y_i}(e) = q^{u(i)/2} \cdot \#\{x \in X_i(\mathbb{F}_q) \mid p(x) = e\}$

Proof. This follows immediately from the Grothendieck trace formula; the factor of $q^{u(i)/2}$ comes from the necessary Tate twist. \square

Combining these propositions, we obtain the relationship between the Grothendieck group K^ϑ and the Hall algebra.

PROPOSITION 4.19. *Assume \mathbb{k} is of characteristic 0. There is natural map of Hopf algebras (in the braided category of $\mathbb{Z}[I]$ -graded abelian groups) from $h_q: K^\vartheta \rightarrow \mathcal{H}_{\Gamma;q}$. The induced map $\prod_{q^n} h_{q^n}: K^\vartheta \rightarrow \prod_{n \geq 1} \mathcal{H}_{\Gamma;q^n}$ is injective.*

Proof. Since all these properties descend automatically to any subfield, and hold for all algebraically closed fields of characteristic 0 if they hold for one, we may assume that $\mathbb{k} = \mathbb{Q}_\ell$ for some prime ℓ coprime to p .

The map h_q is the composition of that induced by γ and \mathcal{J}_* . This is a map of bialgebras by Propositions 4.16 and 4.17. If we have a non-zero element of the kernel, it corresponds to a non-zero linear combination of pure complexes, and thus a pair of pure complexes which are not isomorphic, but give the same function for infinitely many powers of the same prime p ; this is impossible by [Lau87, Théorème 1.1.2] \square

This theorem, in particular, connects together the categorification theorem for $U_q(\mathfrak{n})$ given by Khovanov and Lauda [KL09, 3.18], and the result of Ringel giving an isomorphism between $U_q(\mathfrak{n})$ and the composition subalgebra of the Hall algebra [Rin90] by giving a canonical isomorphism between $K_q^0(R_\nu)$ and the composition algebra in $\mathcal{H}_{\Gamma;q}$ without passing through quantum groups. This picture could easily worked out by an expert from the paper of Varagnolo and Vasserot [VV11], but we know of nowhere where it was written explicitly. This relation to the Hall algebra gives a concrete approach to computing the Grothendieck group of weighted KLR algebras. For example, when Γ is affine, we obtain an isomorphism between $K_q^0(W^\vartheta)$ for $k > 0$ with the subalgebra of the Hall algebra with nilpotent support considered by Vasserot and Varagnolo, amongst others [VV99].

REFERENCES

- [BBD82] Alexander A. Beilinson, Joseph Bernstein, and Pierre Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171. MR MR751966 (86g:32015)
- [CG97] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Birkhäuser Boston Inc., Boston, MA, 1997. MR 98i:22021
- [CL15] Sabin Cautis and Aaron D. Lauda, *Implicit structure in 2-representations of quantum groups*, Selecta Math. (N.S.) 21 (2015), no. 1, 201–244.

- [JMW14] Daniel Juteau, Carl Mautner, and Geordie Williamson, *Parity sheaves*, J. Amer. Math. Soc. 27 (2014), no. 4, 1169–1212. MR 3230821
- [KL09] Mikhail Khovanov and Aaron D. Lauda, *A diagrammatic approach to categorification of quantum groups. I*, Represent. Theory 13 (2009), 309–347.
- [KL10] Mikhail Khovanov and Aaron D. Lauda, *A categorification of quantum $\mathfrak{sl}(n)$* , Quantum Topol. 1 (2010), no. 1, 1–92. MR 2628852 (2011g:17028)
- [Lau87] Gérard Laumon, *Transformation de Fourier, constantes d'équations fonctionnelles et conjecture de Weil*, Inst. Hautes Études Sci. Publ. Math. (1987), no. 65, 131–210. MR 908218 (88g:14019)
- [Li14] Yiqiang Li, *Tensor product varieties, perverse sheaves, and stability conditions*, Selecta Math. (N.S.) 20 (2014), no. 2, 359–401. MR 3177922
- [Lus91] George Lusztig, *Quivers, perverse sheaves, and quantized enveloping algebras*, J. Amer. Math. Soc. 4 (1991), no. 2, 365–421. MR MR1088333 (91m:17018)
- [Mak15] Ruslan Maksimau, *Canonical basis, KLR algebras and parity sheaves*, J. Algebra 422 (2015), 563–610. MR 3272091
- [Nak94] Hiraku Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. 76 (1994), no. 2, 365–416. MR MR1302318 (95i:53051)
- [Rin90] Claus Michael Ringel, *Hall algebras and quantum groups*, Invent. Math. 101 (1990), no. 3, 583–591. MR 1062796 (91i:16024)
- [Rou] Raphael Rouquier, *2-Kac-Moody algebras* (2008), [arXiv:0812.5023](https://arxiv.org/abs/0812.5023).
- [Sch] Olivier Schiffmann, *Lectures on Hall algebras* (2006), [arXiv:math/0611617](https://arxiv.org/abs/math/0611617).
- [SW] Catharina Stroppel and Ben Webster, *Quiver Schur algebras and q -Fock space* (2011), [arXiv:1110.1115](https://arxiv.org/abs/1110.1115).
- [VV99] Michela Varagnolo and Eric Vasserot, *On the decomposition matrices of the quantized Schur algebra*, Duke Math. J. 100 (1999), no. 2, 267–297. MR 1722955 (2001c:17029)
- [VV11] Michela Varagnolo and Eric Vasserot, *Canonical bases and KLR-algebras*, J. Reine Angew. Math. 659 (2011), 67–100.

- [Wal] Christopher D. Walker, *Hall algebras as Hopf objects* (2010), [arXiv:1011.5446](#).
- [Web] Ben Webster, *On graded presentations of Hecke algebras and their generalizations* (2013), [arXiv:1305.0599](#).
- [Web15] Ben Webster, *Canonical bases and higher representation theory*, Compos. Math. 151 (2015), no. 1, 121–166.
- [Web17a] Ben Webster, *Knot invariants and higher representation theory*, Mem. Amer. Math. Soc. 250 (2017), no. 1191, 141 pp.
- [Web17b] Ben Webster, *On generalized category \mathcal{O} for a quiver variety*, Mathematische Annalen 368 (2017), no. 1, 483–536.
- [Web17c] Ben Webster, *Rouquier’s conjecture and diagrammatic algebra*, Forum Math. Sigma 5 (2017), e27, 71. MR 3732238
- [Web18] Ben Webster, *A categorical action on quantized quiver varieties*, Math. Z. (2018).
- [Wil14] Geordie Williamson, *On an analogue of the James conjecture*, Represent. Theory 18 (2014), 15–27. MR 3163410
- [Xia97] Jie Xiao, *Drinfeld double and Ringel-Green theory of Hall algebras*, J. Algebra 190 (1997), no. 1, 100–144. MR 1442148 (98a:16018)

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