The Homotopy Groups of the Simplicial
Mapping Space between Algebras

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Abstract. Let ℓ be a commutative ring with unit. To every pair of ℓ-algebras A and B one can associate a simplicial set $\mathrm{Hom}(A, B^\Delta)$ so that $\pi_0\mathrm{Hom}(A, B^\Delta)$ equals the set of polynomial homotopy classes of morphisms from A to B. We prove that $\pi_n\mathrm{Hom}(A, B^\Delta)$ is the set of homotopy classes of morphisms from A to $B_\bullet^{\mathfrak{S}^n}$, where $B_\bullet^{\mathfrak{S}^n}$ is the ind-algebra of polynomials on the n-dimensional cube with coefficients in B vanishing at the boundary of the cube. This is a generalization to arbitrary dimensions of a theorem of Cortiñas-Thom, which addresses the cases $n \le 1$. As an application we give a simplified proof of a theorem of Garkusha that computes the homotopy groups of his matrix-unstable algebraic KK-theory space in terms of polynomial homotopy classes of morphisms.

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1 Introduction

Algebraic kk-theory was constructed by Cortiñas-Thom in [1], as a completely algebraic analogue of Kasparov's KK-theory. It is defined on the category Alg_ℓ of associative, not necessarily unital algebras over a fixed unital commutative ring ℓ. It consists of a triangulated category kk endowed with a functor $j : \mathrm{Alg}_\ell \to kk$ that satisfies the following properties:

(H) *Homotopy invariance.* The functor j is polynomial homotopy invariant.

(E) *Excision.* Every short exact sequence of ℓ-algebras that splits as a sequence of ℓ-modules gives rise to a distinguished triangle upon applying j.

(M) *Matrix stability.* For any ℓ-algebra A we have $j(M_\infty A) \cong j(A)$, where $M_\infty A$ denotes the algebra of finite matrices with coefficients in A indexed by $\mathbb{N} \times \mathbb{N}$.

This functor j is moreover universal with the above properties: any other functor from Alg_ℓ into a triangulated category satisfying (H), (E) and (M) factors uniquely through j. Another important property of kk-theory is that it recovers Weibel's homotopy K-theory: $kk(\ell, A) \cong KH_0(A)$; see [1, Theorem 8.2.1].

As a technical tool for defining algebraic kk-theory, Cortiñas-Thom introduced in [1, Section 3] a simplicial enrichment of Alg_ℓ. They associated a simplicial mapping space $\mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)$ to any pair of ℓ-algebras A and B, and they defined simplicial compositions

$$\circ : \mathrm{Hom}_{\mathrm{Alg}_\ell}(B, C^\Delta) \times \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta) \to \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, C^\Delta)$$

that make Alg_ℓ into a simplicial category in the sense of [9, Section II.1]. The homotopy category of this simplicial category is Gersten's homotopy category of algebras [6, Section 1]. This means that there is a natural bijection

$$\pi_0 \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta) \cong [A, B],$$

where the right hand side denotes the set of polynomial homotopy classes of morphisms from A to B. In the same vein, Cortiñas-Thom showed in [1, Theorem 3.3.2] that

$$\pi_1 \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta) \cong [A, B^{\mathfrak{S}_1}],$$

where $B^{\mathfrak{S}_1}$ denotes the ind-algebra of polynomials on $S^1 = \Delta^1 / \partial \Delta^1$ with coefficients in B that vanish at the basepoint. The main result of this paper is the following generalization of the latter to arbitrary dimensions.

THEOREM 1.1 (Theorem 3.10). *For any pair of ℓ-algebras A and B and any $n \geq 0$ there is a natural bijection*

$$\pi_n \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta) \cong [A, B^{\mathfrak{S}_n}], \tag{1}$$

where $B^{\mathfrak{S}_n}$ is the ind-algebra of polynomials on the n-dimensional cube with coefficients in B vanishing at the boundary of the cube.

To prove Theorem 1.1 one has to compare two different notions of homotopy for ind-algebra homomorphisms $A \to B^{\mathfrak{S}_n}$: simplicial homotopy on the left hand side of (1) and polynomial homotopy on the right. Simplicial homotopy implies polynomial homotopy by [3, Hauptlemma (2)]. This key technical result of [3] —of which Garkusha provides a beautiful constructive proof— allows one to define a surjective function

$$\pi_n \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta) \to [A, B^{\mathfrak{S}_n}] \tag{2}$$

that turns out to be the desired bijection. We prove the injectivity of (2) by showing that polynomial homotopy implies simplicial homotopy; this is done in Lemma 3.9. This lemma follows immediately from the existence of the multiplication morphisms defined in section 3.1.

With methods different from those of Cortiñas-Thom, Garkusha gave in [3] an alternative construction of kk-theory and moreover defined other universal bivariant homology theories of algebras. The latter are functors from Alg_ℓ into some triangulated category that share properties (H) and (E) with algebraic kk-theory but satisfy different matrix-stability conditions. Garkusha showed in [3] that his bivariant homology theories are representable by spectra, and the simplicial mapping spaces between algebras are his main building blocks for these spectra. The idea of the isomorphism (2) was already present in the proof of [3, Comparison Theorem A], where he computed the homotopy groups of the matrix-unstable algebraic KK-theory space in terms of polynomial homotopy classes of morphisms. However, Garkusha used [3, Hauptlemma (3)] as a substitute for Lemma 3.9 when proving injectivity. This makes his proof rely on nontrivial techniques from homotopy theory such as the construction of a motivic-like model category of simplicial functors on Alg_ℓ. As an application of Theorem 1.1, we give a simplified proof of [3, Comparison Theorem A] that uses no more homotopy theory than the definition of the homotopy groups of a simplicial set.

The rest of this paper is organized as follows. In section 2 we fix notation and we recall the definition of polynomial homotopy and the details of the simplicial enrichment of Alg_ℓ. In section 3 we prove Lemma 3.9 and Theorem 1.1. In section 4 we apply Theorem 1.1 to give a simplified proof of [3, Comparison Theorem A].

2   PRELIMINARIES

Throughout this text, ℓ is a commutative ring with unit. We only consider associative, not necessarily unital ℓ-algebras and we write Alg_ℓ for the category of ℓ-algebras and ℓ-algebra homomorphisms. Simplicial ℓ-algebras can be considered as simplicial sets using the forgetful functor $\text{Alg}_\ell \to \text{Set}$; this is usually done without further mention. The symbol $\otimes$ indicates tensor product over $\mathbb{Z}$.

2.1   CATEGORIES OF DIRECTED DIAGRAMS

Let $\mathfrak{C}$ be a category. A *directed diagram* in $\mathfrak{C}$ is a functor $X : I \to \mathfrak{C}$, where I is a filtering partially ordered set. We often write (X, I) or $X_\bullet$ for such a functor. We shall consider different categories whose objects are directed diagrams:

2.1.1   FIXING THE FILTERING POSET

Let I be a filtering poset. We will write $\mathfrak{C}^I$ for the category whose objects are the functors $X : I \to \mathfrak{C}$ and whose morphisms are the natural transformations.

2.1.2   VARYING THE FILTERING POSET

We will write $\vec{\mathfrak{C}}$ for the category whose objects are the directed diagrams in $\mathfrak{C}$ and whose morphisms are defined as follows: Let (X, I) and (Y, J) be two directed diagrams. A morphism from (X, I) to (Y, J) consists of a pair (f, θ) where $\theta : I \to J$ is a functor and $f : X \to Y \circ \theta$ is a natural transformation.

For a fixed filtering poset I, there is a faithful functor $a : \mathfrak{C}^I \to \vec{\mathfrak{C}}$ that acts as the identity on objects and that sends a natural transformation f to the morphism (f, id_I).

2.1.3   THE CATEGORY OF IND-OBJECTS

The category $\mathfrak{C}^{\mathrm{ind}}$ of ind-objects of $\mathfrak{C}$ is defined as follows: The objects of $\mathfrak{C}^{\mathrm{ind}}$ are the directed diagrams in $\mathfrak{C}$. The hom-sets are defined by:

$$\mathrm{Hom}_{\mathfrak{C}^{\mathrm{ind}}}((X, I), (Y, J)) := \lim_{i \in I} \mathrm{colim}_{j \in J} \mathrm{Hom}_{\mathfrak{C}}(X_i, Y_j)$$

There is a functor $\vec{\mathfrak{C}} \to \mathfrak{C}^{\mathrm{ind}}$ that acts as the identity on objects and that sends a morphism $(f, \theta) : (X, I) \to (Y, J)$ to the morphism:

$$(f_i : X_i \to Y_{\theta(i)})_{i \in I} \in \lim_{i \in I} \mathrm{colim}_{j \in J} \mathrm{Hom}_{\mathfrak{C}}(X_i, Y_j)$$

2.2   SIMPLICIAL SETS

The category of simplicial sets is denoted by $\mathbb{S}$; see [8, Chapter 3]. Let $\mathrm{Map}(?, ??)$ be the internal-hom in $\mathbb{S}$; we often write Y^X instead of $\mathrm{Map}(X, Y)$.

2.2.1   THE ITERATED LAST VERTEX MAP

Let $\mathrm{sd} : \mathbb{S} \to \mathbb{S}$ be the subdivision functor. There is a natural transformation $\gamma : \mathrm{sd} \to \mathrm{id}_{\mathbb{S}}$ called the *last vertex map* [7, Section III. 4]. For $X \in \mathbb{S}$, put $\gamma_X^1 := \gamma_X$ and define inductively γ_X^n to be the following composite:

$$\mathrm{sd}^n X = \mathrm{sd}(\mathrm{sd}^{n-1} X) \xrightarrow{\gamma^1_{\mathrm{sd}^{n-1}X}} \mathrm{sd}^{n-1} X \xrightarrow{\gamma_X^{n-1}} X$$

It is immediate that $\gamma^n : \mathrm{sd}^n \to \mathrm{id}_{\mathbb{S}}$ is a natural transformation. Let $\mathrm{sd}^0 : \mathbb{S} \to \mathbb{S}$ be the identity functor and let $\gamma^0 : \mathrm{sd}^0 \to \mathrm{id}_{\mathbb{S}}$ be the identity natural transformation.

LEMMA 2.3. *For any $p, q \geq 0$ and any $X \in \mathbb{S}$ we have:*

$$\gamma_X^{p+q} = \gamma_X^p \circ \mathrm{sd}^p\left(\gamma_X^q\right) = \gamma_X^p \circ \gamma_{\mathrm{sd}^p X}^q$$

Proof. It follows from a straightforward induction on $n = p + q$.   □

2.3.1   SIMPLICIAL CUBES

Let $I := \Delta^1$ and let $\partial I := \{0, 1\} \subset I$. For $n \geq 1$, let $I^n := I \times \cdots \times I$ be the n–fold direct product and let ∂I^n be the following simplicial subset of I^n:

$$\partial I^n := [(\partial I) \times I \times \cdots \times I] \cup [I \times (\partial I) \times \cdots \times I] \cup \cdots \cup [I \times \cdots \times I \times (\partial I)]$$

Let $I^0 := \Delta^0$ and let $\partial I^0 := \emptyset$. We identify $I^{m+n} = I^m \times I^n$ and $\partial(I^{m+n}) = [(\partial I^m) \times I^n] \cup [I^m \times (\partial I^n)]$ using the associativity and unit isomorphisms of the direct product in $\mathbb{S}$.

2.3.2 Iterated loop spaces

Let $(X, *)$ be a pointed fibrant simplicial set. Recall from [7, Section I.7] that the loopspace ΩX is defined as the fiber of a natural fibration $\pi_X : PX \to X$, where PX has trivial homotopy groups. By the long exact sequence associated to this fibration, we have pointed bijections $\pi_{n+1}(X, *) \cong \pi_n(\Omega X, *)$ for $n \geq 0$ that are group isomorphisms for $n \geq 1$. Iterating the loopspace construction we get:

$$\pi_0(\Omega^n X) \cong \pi_1(\Omega^{n-1} X, *) \cong \cdots \cong \pi_n(X, *)$$

Thus, $\pi_0 \Omega^n X$ is a group for $n \geq 1$ and this group is abelian for $n \geq 2$. Moreover, a morphism $\varphi : X \to Y$ of pointed fibrant simplicial sets induces group homomorphisms $\varphi_* : \pi_0 \Omega^n X \to \pi_0 \Omega^n Y$ for $n \geq 1$. Let incl denote the inclusion $\partial I^n \to I^n$. It is easy to see that the iterated loop functor Ω^n on pointed fibrant simplicial sets can be alternatively defined by the following pullback of simplicial sets:

$$
\begin{array}{ccc}
\Omega^n X & \xrightarrow{\iota_{n,X}} & \mathrm{Map}(I^n, X) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{incl}^*} \\
\Delta^0 & \xrightarrow{\quad * \quad} & \mathrm{Map}(\partial I^n, X)
\end{array}
\tag{3}
$$

We will always use the latter description of Ω^n. Occasionally we will need to compare Ω^n for different integers n; for this purpose we will explicitly describe how the diagram (3) arises from successive applications of the functor Ω. We start defining ΩX by the following pullback in $\mathbb{S}$:

$$
\begin{array}{ccc}
\Omega X & \xrightarrow{\iota_{1,X}} & \mathrm{Map}(I, X) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{incl}^*} \\
\Delta^0 & \xrightarrow{\quad * \quad} & \mathrm{Map}(\partial I, X)
\end{array}
$$

For $n \geq 1$, define inductively $\iota_{n+1,X} : \Omega^{n+1} X \to \mathrm{Map}(I^{n+1}, X)$ as the following composite:

$$\Omega\left(\Omega^n X\right) \xrightarrow{\iota_{1,\Omega^n X}} \mathrm{Map}\left(I, \Omega^n X\right) \xrightarrow{(\iota_{n,X})_*} \mathrm{Map}\left(I, \mathrm{Map}(I^n, X)\right) \cong \mathrm{Map}\left(I^n \times I, X\right)$$

It is easily verified that (3) is a pullback. Moreover, $\iota_{m+n,X}$ equals the following composite:

$$\Omega^n\left(\Omega^m X\right) \xrightarrow{\iota_{n,\Omega^m X}} \mathrm{Map}\left(I^n, \Omega^m X\right) \xrightarrow{(\iota_{m,X})_*} \mathrm{Map}\left(I^n, \mathrm{Map}(I^m, X)\right) \cong \mathrm{Map}\left(I^m \times I^n, X\right)$$

Thus, under the identification of diagram (3), each time we apply Ω the new I-coordinate appears to the right.

2.4   Simplicial enrichment of algebras

We proceed to recall some of the details of the simplicial enrichment of Alg_ℓ introduced in [1, Section 3]. Let $\mathbb{Z}^\Delta$ be the simplicial ring defined by:

$$[p] \mapsto \mathbb{Z}^{\Delta^p} := \mathbb{Z}[t_0, \ldots, t_p]/\langle 1 - \sum t_i \rangle$$

An order-preserving function $\varphi : [p] \to [q]$ induces a ring homomorphism $\mathbb{Z}^{\Delta^q} \to \mathbb{Z}^{\Delta^p}$ by the formula:

$$t_i \mapsto \sum_{\varphi(j)=i} t_j$$

Now let $B \in \mathrm{Alg}_\ell$ and define a simplicial ℓ-algebra B^Δ by:

$$[p] \mapsto B^{\Delta^p} := B \otimes \mathbb{Z}^{\Delta^p} \qquad (4)$$

If A is another ℓ-algebra, the simplicial set $\mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)$ is called the *simplicial mapping space* from A to B. For $X \in \mathbb{S}$, put $B^X := \mathrm{Hom}_\mathbb{S}(X, B^\Delta)$; it is easily verified that B^X is an ℓ-algebra with the operations defined pointwise. When $X = \Delta^p$, this definition of B^{Δ^p} coincides with (4). We have a natural isomorphism as follows, where the limit is taken over the category of simplices of X:

$$B^X \xrightarrow{\cong} \lim_{\Delta^p \downarrow X} B^{\Delta^p}$$

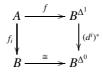For $A, B \in \mathrm{Alg}_\ell$ and $X \in \mathbb{S}$ we have the following adjunction isomorphism:

$$\mathrm{Hom}_\mathbb{S}(X, \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)) \cong \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^X)$$

Remark 2.5. Let X and Y be simplicial sets. In general $(B^X)^Y \not\cong B^{X \times Y}$ – this already fails when X and Y are standard simplices; see [1, Remark 3.1.4].

Remark 2.6. The simplicial ring $\mathbb{Z}^\Delta$ is commutative and hence the same holds for the rings $\mathbb{Z}^X = \mathrm{Hom}_\mathbb{S}(X, \mathbb{Z}^\Delta)$, for any $X \in \mathbb{S}$. Thus, the multiplication in $\mathbb{Z}^X$ induces a ring homomorphism $\mathbf{m}_X : \mathbb{Z}^X \otimes \mathbb{Z}^X \to \mathbb{Z}^X$. Note that $\mathbf{m}_X$ is natural in X.

2.7   Polynomial homotopy

Two morphisms $f_0, f_1 : A \to B$ in Alg_ℓ are *elementary homotopic* if there exists an ℓ-algebra homomorphism $f : A \to B[t]$ such that $\mathrm{ev}_0 \circ f = f_1$ and $\mathrm{ev}_1 \circ f = f_0$. Here, ev_i stands for the evaluation $t \mapsto i$. Equivalently, f_0 and f_1 are elementary homotopic if there exists $f : A \to B^{\Delta^1}$ such that the following diagram commutes for $i = 0, 1$:

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B^{\Delta^1} \\
{\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle (d^i)^*} \\
B & \xrightarrow{\ \cong\ } & B^{\Delta^0}
\end{array}
$$

Here the $d^i : \Delta^0 \to \Delta^1$ are the coface maps. Elementary homotopy $\sim_e$ is a reflexive and symmetric relation, but it is not transitive —the concatenation of polynomial homotopies is usually not polynomial. We let $\sim$ be the transitive closure of $\sim_e$ and call

f_0 and f_1 *(polynomially) homotopic* if $f_0 \sim f_1$. It is easily shown that $f_0 \sim f_1$ iff there exist $r \in \mathbb{N}$ and $f : A \to B^{\mathrm{sd}^r \Delta^1}$ such that the following diagrams commute:

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & B^{\mathrm{sd}^r \Delta^1} \\
{\scriptstyle f_i} \downarrow & & \downarrow {\scriptstyle (d^i)^*} \\
B & \xrightarrow{\ \cong\ } & B^{\mathrm{sd}^r \Delta^0}
\end{array}
$$

It turns out that $\sim$ is compatible with composition; see [6, Lemma 1.1] for details. Thus, we have a category $[\mathrm{Alg}_\ell]$ whose objects are ℓ-algebras and whose hom-sets are given by $[A, B] := \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B)/\sim$. We also have an obvious functor $\mathrm{Alg}_\ell \to [\mathrm{Alg}_\ell]$.

DEFINITION 2.8 ([1, Definition 3.1.1]). Let (A, I) and (B, J) be two directed diagrams in Alg_ℓ and let $f, g \in \mathrm{Hom}_{\mathrm{Alg}_\ell^{\mathrm{ind}}}((A, I), (B, J))$. We call f and g *homotopic* if they correspond to the same morphism upon applying the functor $\mathrm{Alg}_\ell^{\mathrm{ind}} \to [\mathrm{Alg}_\ell]^{\mathrm{ind}}$. We also write:

$$
[A_\bullet, B_\bullet] := \mathrm{Hom}_{[\mathrm{Alg}_\ell]^{\mathrm{ind}}}((A, I), (B, J)) = \lim_{i \in I} \mathrm{colim}_{j \in J}[A_i, B_j]
$$

2.9 FUNCTIONS VANISHING ON A SUBSET

A *simplicial pair* is a pair (K, L) where K is a simplicial set and $L \subseteq K$ is a simplicial subset. A *morphism of pairs* $f : (K', L') \to (K, L)$ is a morphism of simplicial sets $f : K' \to K$ such that $f(L') \subseteq L$. A simplicial pair (K, L) is *finite* if K is a finite simplicial set. We will only consider finite simplicial pairs, omitting the word "finite" from now on. Let (K, L) be a simplicial pair, let $B \in \mathrm{Alg}_\ell$ and let $r \geq 0$. Put:

$$
B_r^{(K,L)} := \ker\left(B^{\mathrm{sd}^r K} \to B^{\mathrm{sd}^r L}\right)
$$

The last vertex map induces morphisms $B_r^{(K,L)} \to B_{r+1}^{(K,L)}$ and we usually consider $B_\bullet^{(K,L)}$ as a directed diagram in Alg_ℓ:

$$
B_\bullet^{(K,L)} : B_0^{(K,L)} \to B_1^{(K,L)} \to B_2^{(K,L)} \to \cdots
$$

Notice that a morphism $f : (K', L') \to (K, L)$ induces a morphism $f^* : B_\bullet^{(K,L)} \to B_\bullet^{(K',L')}$ of $\mathbb{Z}_{\geq 0}$-diagrams.

LEMMA 2.10 (cf. [1, Proposition 3.1.3]). *Let (K, L) be a simplicial pair and let $B \in \mathrm{Alg}_\ell$. Then $\mathbb{Z}_r^{(K,L)}$ is a free abelian group and there is a natural ℓ-algebra isomorphism:*

$$
B \otimes \mathbb{Z}_r^{(K,L)} \xrightarrow{\cong} B_r^{(K,L)} \tag{5}
$$

Proof. The following sequence is exact by definition of $\mathbb{Z}_r^{(K,L)}$ and [1, Lemma 3.1.2]:

$$
0 \longrightarrow \mathbb{Z}_r^{(K,L)} \longrightarrow \mathbb{Z}^{\mathrm{sd}^r K} \longrightarrow \mathbb{Z}^{\mathrm{sd}^r L} \longrightarrow 0 \tag{6}
$$

The group $\mathbb{Z}^{\mathrm{sd}^r L}$ is free abelian by [1, Proposition 3.1.3] and thus the sequence (6) splits. It follows that $\mathbb{Z}_r^{(K,L)}$ is free because it is a direct summand of the free abelian group $\mathbb{Z}^{\mathrm{sd}^r K}$. Moreover, the following sequence is exact:

$$0 \longrightarrow B \otimes \mathbb{Z}_r^{(K,L)} \longrightarrow B \otimes \mathbb{Z}^{\mathrm{sd}^r K} \longrightarrow B \otimes \mathbb{Z}^{\mathrm{sd}^r L} \longrightarrow 0$$
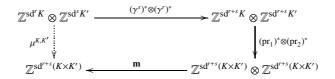
To finish the proof we identify $B \otimes \mathbb{Z}^{\mathrm{sd}^r K} \overset{\cong}{\to} B^{\mathrm{sd}^r K}$ using the natural isomorphism of [1, Proposition 3.1.3]. $\qquad\square$

EXAMPLE 2.11. Following [3, Section 7.2], we will write $B_\bullet^{\mathfrak{S}_n}$ instead of $B_\bullet^{(I^n, \partial I^n)}$. Note that $B_\bullet^{\mathfrak{S}_0}$ is the constant $\mathbb{Z}_{\geq 0}$-diagram B.
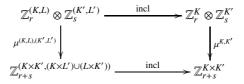
3 MAIN THEOREM

3.1 MULTIPLICATION MORPHISMS

Let (K, L) and (K', L') be simplicial pairs. It follows from Lemma 2.10 that $\mathbb{Z}_r^{(K,L)} \otimes \mathbb{Z}_s^{(K',L')}$ identifies with a subring of $\mathbb{Z}^{\mathrm{sd}^r K} \otimes \mathbb{Z}^{\mathrm{sd}^s K'}$. Let $\mu^{K,K'}$ be the composite of the following ring homomorphisms:

$$
\begin{array}{ccc}
\mathbb{Z}^{\mathrm{sd}^r K} \otimes \mathbb{Z}^{\mathrm{sd}^s K'} & \xrightarrow{(\gamma^s)^* \otimes (\gamma^r)^*} & \mathbb{Z}^{\mathrm{sd}^{r+s} K} \otimes \mathbb{Z}^{\mathrm{sd}^{r+s} K'} \\
\mu^{K,K'} \downarrow & & \downarrow (\mathrm{pr}_1)^* \otimes (\mathrm{pr}_2)^* \\
\mathbb{Z}^{\mathrm{sd}^{r+s}(K \times K')} & \xleftarrow{\ \mathbf{m}\ } & \mathbb{Z}^{\mathrm{sd}^{r+s}(K \times K')} \otimes \mathbb{Z}^{\mathrm{sd}^{r+s}(K \times K')}
\end{array}
$$

Here γ^j is the iterated last vertex map defined in section 2.2.1, pr_i is the projection of the direct product into its i-th factor and $\mathbf{m}$ is the map described in Remark 2.6.

LEMMA 3.2. *The morphism $\mu^{K,K'}$ defined above induces a ring homomorphism $\mu^{(K,L),(K',L')}$ that fits into the following diagram:*

$$
\begin{array}{ccc}
\mathbb{Z}_r^{(K,L)} \otimes \mathbb{Z}_s^{(K',L')} & \xrightarrow{\ \mathrm{incl}\ } & \mathbb{Z}_r^K \otimes \mathbb{Z}_s^{K'} \\
\mu^{(K,L),(K',L')} \downarrow & & \downarrow \mu^{K,K'} \\
\mathbb{Z}_{r+s}^{(K \times K',(K \times L') \cup (L \times K'))} & \xrightarrow{\ \mathrm{incl}\ } & \mathbb{Z}_{r+s}^{K \times K'}
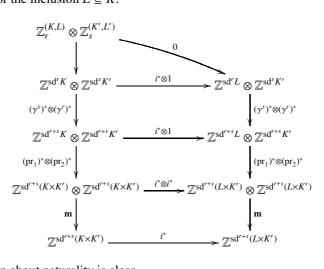\end{array}
$$

Moreover, $\mu^{(K,L),(K',L')}$ is natural in both variables with respect to morphisms of simplicial pairs. We call $\mu^{(K,L),(K',L')}$ a multiplication morphism.

Proof. Let ε be the restriction of $\mu^{K,K'}$ to $\mathbb{Z}_r^{(K,L)} \otimes \mathbb{Z}_s^{(K',L')}$; we have to show that ε is zero when composed with the morphism:

$$\mathbb{Z}^{\mathrm{sd}^{r+s}(K \times K')} \to \mathbb{Z}^{\mathrm{sd}^{r+s}((K \times L') \cup (L \times K'))}$$

Since the functor $\mathbb{Z}^{\mathrm{sd}^{r+s}(?)} : \mathbb{S} \to \mathrm{Alg}_{\mathbb{Z}}^{\mathrm{op}}$ commutes with colimits, it will be enough to show that ε is zero when composed with the projections to $\mathbb{Z}^{\mathrm{sd}^{r+s}(K \times L')}$ and to

$\mathbb{Z}^{\mathrm{sd}^{r+s}(L\times K')}$; this is a straightforward check. For example, the following commutative diagram shows that ε is zero when composed with the projection to $\mathbb{Z}^{\mathrm{sd}^{r+s}(L\times K')}$; we write i for the inclusion $L \subseteq K$.

$$
\begin{array}{ccc}
\mathbb{Z}_r^{(K,L)} \otimes \mathbb{Z}_s^{(K',L')} & & \\
\downarrow & \searrow^{0} & \\
\mathbb{Z}^{\mathrm{sd}^r K} \otimes \mathbb{Z}^{\mathrm{sd}^s K'} & \xrightarrow{i^*\otimes 1} & \mathbb{Z}^{\mathrm{sd}^r L} \otimes \mathbb{Z}^{\mathrm{sd}^s K'} \\
{\scriptstyle (\gamma^s)^*\otimes(\gamma^r)^*}\downarrow & & \downarrow{\scriptstyle (\gamma^s)^*\otimes(\gamma^r)^*} \\
\mathbb{Z}^{\mathrm{sd}^{r+s} K} \otimes \mathbb{Z}^{\mathrm{sd}^{r+s} K'} & \xrightarrow{i^*\otimes 1} & \mathbb{Z}^{\mathrm{sd}^{r+s} L} \otimes \mathbb{Z}^{\mathrm{sd}^{r+s} K'} \\
{\scriptstyle (\mathrm{pr}_1)^*\otimes(\mathrm{pr}_2)^*}\downarrow & & \downarrow{\scriptstyle (\mathrm{pr}_1)^*\otimes(\mathrm{pr}_2)^*} \\
\mathbb{Z}^{\mathrm{sd}^{r+s}(K\times K')} \otimes \mathbb{Z}^{\mathrm{sd}^{r+s}(K\times K')} & \xrightarrow{i^*\otimes i^*} & \mathbb{Z}^{\mathrm{sd}^{r+s}(L\times K')} \otimes \mathbb{Z}^{\mathrm{sd}^{r+s}(L\times K')} \\
\mathbf{m}\downarrow & & \downarrow\mathbf{m} \\
\mathbb{Z}^{\mathrm{sd}^{r+s}(K\times K')} & \xrightarrow{i^*} & \mathbb{Z}^{\mathrm{sd}^{r+s}(L\times K')}
\end{array}
$$

The assertion about naturality is clear. $\qquad\qquad\square$

Remark 3.3. We can consider $\mathbb{Z}_\bullet^{(K,L)} \otimes \mathbb{Z}_\bullet^{(K',L')}$ as a directed diagram of rings indexed over $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. Let $\theta : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be defined by $\theta(r,s) = r + s$; it is clear that θ is a functor. Then the morphisms of Lemma 3.2 assemble into a morphism in $\vec{\mathrm{Alg}}_{\mathbb{Z}}$:

$$\left(\mu^{(K,L),(K',L')}, \theta\right) : \mathbb{Z}_\bullet^{(K,L)} \otimes \mathbb{Z}_\bullet^{(K',L')} \longrightarrow \mathbb{Z}_\bullet^{(K\times K',(K\times L')\cup(L\times K'))}$$

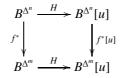We will often think of $\mu^{(K,L),(K',L')}$ in this way, omitting θ from the notation.

Remark 3.4. Upon tensoring $\mu^{(K,L),(K',L')}$ with an ℓ-algebra B and using (5) we obtain an ℓ-algebra homomorphism:

$$\mu_B^{(K,L),(K',L')} : \left(B_r^{(K,L)}\right)_s^{(K',L')} \longrightarrow B_{r+s}^{(K\times K',(K\times L')\cup(L\times K'))}$$

This morphism is obviously natural with respect to morphisms of simplicial pairs and with respecto to ℓ-algebra homomorphisms. Again, we can think of it as a morphism in $\vec{\mathrm{Alg}}_\ell$. It is easily verified that this morphism is associative in the obvious way.

EXAMPLE 3.5. For any $n \geq 0$ and any $B \in \mathrm{Alg}_\ell$ we have a morphism $\iota : B \to B^{\Delta^n}$ induced by $\Delta^n \to *$. It is well known that ι is a homotopy equivalence, as we proceed to explain. Let $v : B^{\Delta^n} \to B$ be the restriction to the 0-simplex 0. Explicitly, we have $v(t_i) = 0$ for $i > 0$ and $v(t_0) = 1$. It is easily verified that $v \circ \iota = \mathrm{id}_B$. Now let $H : B^{\Delta^n} \to B^{\Delta^n}[u]$ be the elementary homotopy defined by $H(t_i) = ut_i$ for $i > 0$ and $H(t_0) = t_0 + (1-u)(t_1 + \cdots + t_n)$. We have $\mathrm{ev}_1 \circ H = \mathrm{id}_{B^{\Delta^n}}$ and $\mathrm{ev}_0 \circ H = \iota \circ v$. This shows that $\iota \circ v = \mathrm{id}_{B^{\Delta^n}}$ in $[\mathrm{Alg}_\ell]$.

The homotopy H constructed above is natural with respect to the inclusion of faces of Δ^n that contain the 0-simplex 0. More precisely: if $f : [m] \to [n]$ is an injective order-preserving map such that $f(0) = 0$, then the following diagram commutes:

$$
\begin{array}{ccc}
B^{\Delta^n} & \xrightarrow{\;H\;} & B^{\Delta^n}[u] \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*[u]} \\
B^{\Delta^m} & \xrightarrow{\;H\;} & B^{\Delta^m}[u]
\end{array}
$$

Now let $p, q \geq 0$. Recall from the proof of [8, Lemma 3.1.8] that the simplices of $\Delta^p \times \Delta^q$ can be identified with the chains in $[p] \times [q]$ with the product order. The nondegenerate $(p + q)$-simplices of $\Delta^p \times \Delta^q$ are identified with the maximal chains in $[p] \times [q]$; there are exactly $\binom{p+q}{p}$ of these. Following [8], let $c(i)$ for $1 \leq i \leq \binom{p+q}{p}$ be the complete list of maximal chains of $[p] \times [q]$. Then $\Delta^p \times \Delta^q$ is the coequalizer in $\mathbb{S}$ of the two natural morphisms of simplicial sets f and g induced by the inclusions $c(i) \cap c(j) \subseteq c(i)$ and $c(i) \cap c(j) \subseteq c(j)$ respectively:

$$
f, g : \coprod_{1 \leq i < j \leq \binom{p+q}{p}} \Delta^{n_{c(i) \cap c(j)}} \rightrightarrows \coprod_{1 \leq i \leq \binom{p+q}{p}} \Delta^{n_{c(i)}}
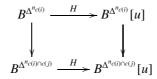$$

Here n_c is the number of edges in c; that is, the dimension of the nondegenerate simplex corresponding to c. Since $B^? : \mathbb{S}^{\mathrm{op}} \to \mathrm{Alg}_\ell$ preserves limits, it follows that $B^{\Delta^p \times \Delta^q}$ is the equalizer of the following diagram in Alg_ℓ:

$$
f^*, g^* : \prod_{1 \leq i \leq \binom{p+q}{p}} B^{\Delta^{n_{c(i)}}} \rightrightarrows \prod_{1 \leq i < j \leq \binom{p+q}{p}} B^{\Delta^{n_{c(i) \cap c(j)}}} \tag{7}
$$

Moreover, since $\mathbb{Z}[u]$ is a flat ring, $? \otimes \mathbb{Z}[u]$ preserves finite limits and $B^{\Delta^p \times \Delta^q}[u]$ is the equalizer of the following diagram:

$$
f^*[u], g^*[u] : \prod_{1 \leq i \leq \binom{p+q}{p}} B^{\Delta^{n_{c(i)}}}[u] \rightrightarrows \prod_{1 \leq i < j \leq \binom{p+q}{p}} B^{\Delta^{n_{c(i) \cap c(j)}}}[u] \tag{8}
$$

Notice that every maximal chain of $[p] \times [q]$ starts at $(0,0)$. This implies, by the discussion above on the naturality of H, that the following diagram commutes for every i and j:

$$
\begin{array}{ccc}
B^{\Delta^{n_{c(i)}}} & \xrightarrow{\;H\;} & B^{\Delta^{n_{c(i)}}}[u] \\
\downarrow & & \downarrow \\
B^{\Delta^{n_{c(i) \cap c(j)}}} & \xrightarrow{\;H\;} & B^{\Delta^{n_{c(i) \cap c(j)}}}[u]
\end{array}
$$

Then the homotopy H on the different $B^{\Delta^{n_{c(i)}}}$ gives a morphism of diagrams from (7) to (8) that induces $H : B^{\Delta^p \times \Delta^q} \to B^{\Delta^p \times \Delta^q}[u]$. Let $\iota : B \to B^{\Delta^p \times \Delta^q}$ be the morphism

induced by $\Delta^p \times \Delta^q \to *$ and let $v : B^{\Delta^p \times \Delta^q} \to B$ be the restriction to the 0-simplex $(0, 0)$. It is easily verified that $\mathrm{ev}_1 \circ H$ is the identity of $B^{\Delta^p \times \Delta^q}$ and that $\mathrm{ev}_0 \circ H = \iota \circ v$; this shows that ι is a homotopy equivalence.

Finally, consider the following commutative diagram. Since each ι is a homotopy equivalence, it follows that $\mu^{\Delta^p, \Delta^q} : (B^{\Delta^p})^{\Delta^q} \to B^{\Delta^p \times \Delta^q}$ is a homotopy equivalence too.

$$
\begin{array}{ccc}
B^{\Delta^p} & \xrightarrow{\iota} & (B^{\Delta^p})^{\Delta^q} \\
\uparrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\mu} \\
B & \xrightarrow{\iota} & B^{\Delta^p \times \Delta^q}
\end{array}
$$

The author does not know whether $\mu^{K, K'} : (B^K)^{K'} \to B^{K \times K'}$ is a homotopy equivalence for general K and K'. It is true, though, that $\mu^{K, K'}$ induces an isomorphism in any homotopy invariant and excisive homology theory, as we explain below.

EXAMPLE 3.6. Let $\mathscr{T}$ be a triangulated category with desuspension Ω and let $j : \mathrm{Alg}_\ell \to \mathscr{T}$ be a functor satisfying properties (H) and (E). We will show that $j\big(\mu^{K, K'} : (B^K)^{K'} \to B^{K \times K'}\big)$ is an isomorphism, for any K, K' and B.

Fix an object U of $\mathscr{T}$ and consider a short exact sequence of ℓ-algebras

$$
\mathscr{E} : \quad A' \longrightarrow A \longrightarrow A''
$$

that splits as a sequence of ℓ-modules. To alleviate notation, we will write $\mathscr{T}_n^U(A)$ instead of $\mathrm{Hom}_{\mathscr{T}}(U, \Omega^n j(A))$, for any $n \in \mathbb{Z}$. By property (E), we have a distinguished triangle

$$
\triangle_{\mathscr{E}} : \quad \Omega j(A'') \longrightarrow j(A') \longrightarrow j(A) \longrightarrow j(A'')
$$

that induces a long exact sequence of groups as follows:

$$
\cdots\!\!\!\dashrightarrow \mathscr{T}_{n+1}^U(A'') \longrightarrow \mathscr{T}_n^U(A') \longrightarrow \mathscr{T}_n^U(A) \longrightarrow \mathscr{T}_n^U(A'') \dashrightarrow\!\!\!\cdots
$$

This sequence is moreover natural in $\mathscr{E}$. Now consider a cartesian square of ℓ-algebras where the horizontal morphisms are split surjections of ℓ-modules:
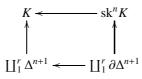
$$
\begin{array}{ccc}
A & \longrightarrow & A' \\
\downarrow & & \downarrow \\
A'' & \longrightarrow & A'''
\end{array}
$$

Proceeding as in the proof of [2, Theorem 2.41], we get an exact Mayer-Vietoris sequence:

$$
\cdots\!\!\!\dashrightarrow \mathscr{T}_{n+1}^U(A''') \longrightarrow \mathscr{T}_n^U(A) \longrightarrow \mathscr{T}_n^U(A') \oplus \mathscr{T}_n^U(A'') \longrightarrow \mathscr{T}_n^U(A''') \dashrightarrow\!\!\!\cdots
$$

This sequence is natural with respect to morphisms of squares.

Let $q \geq 0$. We will show that $j(\mu^{K, \Delta^q})$ is an isomorphism by induction on the dimension of K. The case $\dim K = 0$ follows from the facts that j preserves finite products and

that μ^{Δ^0,Δ^q} is an ℓ-algebra isomorphism. Now let $n \geq 0$ and suppose $j(\mu^{K',\Delta^q})$ is an isomorphism for every finite L with $\dim L \leq n$. If $\dim K = n+1$, we have a cocartesian square:

$$
\begin{array}{ccc}
K & \longleftarrow & \mathrm{sk}^n K \\
\uparrow & & \uparrow \\
\coprod_1^r \Delta^{n+1} & \longleftarrow & \coprod_1^r \partial\Delta^{n+1}
\end{array}
$$

Upon applying the functors $(B^?)^{\Delta^q}$ and $B^{?\times\Delta^q}$ we get the following cartesian squares:

$$
\begin{array}{ccc}
(B^K)^{\Delta^q} & \longrightarrow & (B^{\mathrm{sk}^n K})^{\Delta^q} \\
\downarrow & & \downarrow \\
\prod_1^r (B^{\Delta^{n+1}})^{\Delta^q} & \longrightarrow & \prod_1^r (B^{\partial\Delta^{n+1}})^{\Delta^q}
\end{array}
\qquad
\begin{array}{ccc}
B^{K\times\Delta^q} & \longrightarrow & B^{\mathrm{sk}^n K\times\Delta^q} \\
\downarrow & & \downarrow \\
\prod_1^r B^{\Delta^{n+1}\times\Delta^q} & \longrightarrow & \prod_1^r B^{\partial\Delta^{n+1}\times\Delta^q}
\end{array}
$$

The horizontal morphisms in these diagrams are split surjections of ℓ-modules. Indeed, the morphism $\mathbb{Z}^K \to \mathbb{Z}^{\mathrm{sk}^n K}$ is a split surjection of abelian groups since it is surjective by [1, Lemma 3.1.2] and $\mathbb{Z}^{\mathrm{sk}^n K}$ is a free abelian group by [1, Proposition 3.1.3]. Upon tensoring with B, we get that $B^K \to B^{\mathrm{sk}^n K}$ is a split surjection of ℓ-modules. Similar arguments apply to the remaining horizontal morphisms. By naturality of μ, we have a morphism of squares from the square on the left to the square on the right; this induces a morphism of long exact Mayer-Vietoris sequences upon applying $\mathscr{T}_*^U$. Note that $\mathscr{T}_*^U(\mu^{\Delta^{n+1},\Delta^q})$ is an isomorphism by Example 3.5. It follows from the 5-lemma that

$$(\mu^{K,\Delta^q})_* : \mathrm{Hom}_{\mathscr{T}}(U, j((B^K)^{\Delta^q})) \to \mathrm{Hom}_{\mathscr{T}}(U, j(B^{K\times\Delta^q}))$$

is an isomorphism. Since U is arbitrary, this implies that $j(\mu^{K,\Delta^q})$ is an isomorphism. Now we can show that $j(\mu^{K,K'})$ is an isomorphism by induction on the dimension of K'.

3.7 MAIN THEOREM

Following [3, Section 7.2], we put $\widetilde{B}_\bullet^{\mathfrak{S}_n} := B_\bullet^{(I^n\times I, \partial I^n\times I)}$. The coface maps $d^i : \Delta^0 \to I$ induce morphisms $(d^i)^* : \widetilde{B}_\bullet^{\mathfrak{S}_n} \to B_\bullet^{\mathfrak{S}_n}$.

LEMMA 3.8 (Garkusha). *Let $f : A \to \widetilde{B}_r^{\mathfrak{S}_n}$ be an ℓ-algebra homomorphism. Then the following composites are homotopic; i.e. they belong to the same class in $[A, B_r^{\mathfrak{S}_n}]$:*
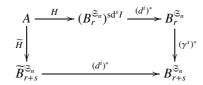
$$A \xrightarrow{f} \widetilde{B}_r^{\mathfrak{S}_n} \xrightarrow{(d^i)^*} B_r^{\mathfrak{S}_n} \quad (i = 0, 1)$$

Proof. This is [3, Hauptlemma (2)]. □

As noted by Garkusha in [3], Lemma 3.8 shows that if two morphisms are simplicially homotopic, then they are polynomially homotopic. The converse also holds, up to increasing the number of subdivisions:

LEMMA 3.9 (cf. [3, Hauptlemma (3)]). *Let $H : A \to (B_r^{\mathfrak{S}_n})^{\mathrm{sd}^s I}$ be a homotopy between two ℓ-algebra homomorphisms $A \to B_r^{\mathfrak{S}_n}$. Then there exists a morphism $\widetilde{H} : A \to \widetilde{B}_{r+s}^{\mathfrak{S}_n}$ in Alg_ℓ such that the following diagram commutes for $i = 0, 1$:*

$$
\begin{array}{ccccc}
A & \xrightarrow{\ H\ } & (B_r^{\mathfrak{S}_n})^{\mathrm{sd}^s I} & \xrightarrow{\ (d^i)^*\ } & B_r^{\mathfrak{S}_n} \\
{\scriptstyle \widetilde{H}} \big\downarrow & & & & \big\downarrow {\scriptstyle (\gamma^s)^*} \\
\widetilde{B}_{r+s}^{\mathfrak{S}_n} & & \xrightarrow{\ (d^i)^*\ } & & B_{r+s}^{\mathfrak{S}_n}
\end{array}
$$

Proof. Let $\widetilde{H}$ be the composite:

$$
A \xrightarrow{\ \ H\ \ } \left(B_r^{(I^n,\partial I^n)}\right)_s^{(I,\emptyset)} \xrightarrow{\ \mu^{(I^n,\partial I^n),(I,\emptyset)}\ } B_{r+s}^{(I^n\times I,(\partial I^n)\times I)}
$$

It is immediate from the naturality of μ that $\widetilde{H}$ satisfies the required properties. □

THEOREM 3.10 (cf. [1, Theorem 3.3.2]). *For any pair of ℓ-algebras A and B and any $n \geq 0$, there is a natural bijection:*

$$
\pi_n \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta) \cong [A, B_\bullet^{\mathfrak{S}_n}] \tag{9}
$$

Proof. We will show that $\pi_0 \Omega^n Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta) \cong [A, B_\bullet^{\mathfrak{S}_n}]$. Consider $\mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)$ as a simplicial set pointed at the zero morphism. For every $p \geq 0$ we have a pullback of sets:

$$
\begin{array}{ccc}
\left(\Omega^n Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)\right)_p & \longrightarrow & \mathrm{Map}\left(I^n, Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)\right)_p \\
\big\downarrow & & \big\downarrow \\
* & \longrightarrow & \mathrm{Map}\left(\partial I^n, Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)\right)_p
\end{array} \tag{10}
$$

For a finite simplicial set K we have:

$$
\begin{aligned}
\mathrm{Map}\left(K, Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)\right)_p &= \mathrm{Hom}_{\mathbb{S}}\left(K \times \Delta^p, Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)\right) \\
&\cong \operatorname*{colim}_r \mathrm{Hom}_{\mathbb{S}}\left(K \times \Delta^p, Ex^r \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)\right) \\
&\cong \operatorname*{colim}_r \mathrm{Hom}_{\mathbb{S}}\left(\mathrm{sd}^r(K \times \Delta^p), \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)\right) \\
&\cong \operatorname*{colim}_r \mathrm{Hom}_{\mathrm{Alg}_\ell}\left(A, B^{\mathrm{sd}^r(K \times \Delta^p)}\right)
\end{aligned}
$$

It follows from these identifications, from (10) and from the fact that filtered colimits of sets commute with finite limits, that we have the following bijections:

$$
\left(\Omega^n Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)\right)_0 \cong \operatorname*{colim}_r \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B_r^{\mathfrak{S}_n}) \tag{11}
$$

$$\left(\Omega^n Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)\right)_1 \cong \mathop{\mathrm{colim}}_r \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, \widetilde{B}_r^{\mathfrak{S}_n}) \tag{12}$$

Using (11) we get a surjection:

$$\left(\Omega^n Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)\right)_0 \cong \mathop{\mathrm{colim}}_r \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B_r^{\mathfrak{S}_n}) \longrightarrow [A, B_\bullet^{\mathfrak{S}_n}]$$

We claim that this function induces the desired bijection. The fact that it factors through π_0 follows from the identification (12) and Lemma 3.8. The injectivity of the induced function from π_0 follows from Lemma 3.9.   □

Remark 3.11. Let A and B be two ℓ-algebras and let $n \geq 1$. Endow the set $[A, B_\bullet^{\mathfrak{S}_n}]$ with the group structure for which (9) is a group isomorphism. This group structure is abelian if $n \geq 2$. Moreover, if $f : A \to A'$ and $g : B \to B'$ are morphisms in $[\mathrm{Alg}_\ell]$, then the following functions are group homomorphisms:

$$f^* : [A', B_\bullet^{\mathfrak{S}_n}] \to [A, B_\bullet^{\mathfrak{S}_n}]$$

$$g_* : [A, B_\bullet^{\mathfrak{S}_n}] \to [A, (B')_\bullet^{\mathfrak{S}_n}]$$

Example 3.12. Recall that $B^{\Delta^1} = B[t_0, t_1]/\langle 1 - t_0 - t_1 \rangle$. Let ω be the automorphism of B^{Δ^1} defined by $\omega(t_0) = t_1$, $\omega(t_1) = t_0$; it is clear that ω induces an automorphism of $B_0^{\mathfrak{S}_1} = \ker(B^{\Delta^1} \to B^{\partial\Delta^1})$. Let $f : A \to B_0^{\mathfrak{S}_1}$ be an ℓ-algebra homomorphism and let $[f]$ be its class in $[A, B_\bullet^{\mathfrak{S}_1}]$. We claim that $[\omega \circ f] = [f]^{-1} \in [A, B_\bullet^{\mathfrak{S}_1}]$. In order to prove this claim, we proceed to recall the definition of the group law $*$ in $\pi_1 Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)$. Consider f and $\omega \circ f$ as 1-simplices of $Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)$ using the identification:

$$\left(Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)\right)_1 \cong \mathop{\mathrm{colim}}_r \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^{\mathrm{sd}^r \Delta^1})$$

According to [7, Section I.7], if we find $\alpha \in \left(Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)\right)_2$ such that

$$\begin{cases} d_0\alpha = \omega \circ f \\ d_2\alpha = f \end{cases} \tag{13}$$

then we have $[f] * [\omega \circ f] = [d_1\alpha]$. Let $\varphi : B^{\Delta^1} \to B^{\Delta^2}$ be the ℓ-algebra homomorphism defined by $\varphi(t_0) = t_0 + t_2$, $\varphi(t_1) = t_1$. Let α be the 2-simplex of $Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(A, B^\Delta)$ induced by the composite:

$$A \xrightarrow{f} B^{\Delta^1} \xrightarrow{\varphi} B^{\Delta^2}$$

It is easy to verify that the equations (13) hold and that $d_1\alpha$ is the zero path.

Example 3.13. Let A and B be two ℓ-algebras and let $m, n \geq 1$. Let $c : I^m \times I^n \xrightarrow{\cong} I^n \times I^m$ be the commutativity isomorphism; c induces an isomorphism $c^* : B_\bullet^{\mathfrak{S}_{n+m}} \to B_\bullet^{\mathfrak{S}_{m+n}}$. We claim that the following function is multiplication by $(-1)^{mn}$:

$$c^* : [A, B_\bullet^{\mathfrak{S}_{n+m}}] \to [A, B_\bullet^{\mathfrak{S}_{m+n}}]$$

Indeed, this follows from Theorem 3.10 and the well known fact that permuting two coordinates in Ω^{m+n} induces multiplication by (-1) upon taking π_0.

4   GARKUSHA'S COMPARISON THEOREM A

Matrix-unstable algebraic KK-theory consists of a triangulated category $D(\mathfrak{F}_{\mathrm{spl}})$ endowed with a functor $j : \mathrm{Alg}_\ell \to D(\mathfrak{F}_{\mathrm{spl}})$ that satisfies (H) and (E), and is moreover universal with respect to these two properties. It was constructed by Garkusha in [5, Section 2.4] by means of deriving a certain Brown category and then stabilizing the loop functor. Garkusha defined in [3] a space $\mathscr{K}(A, B)$ such that $\pi_0 \mathscr{K}(A, B) \cong \mathrm{Hom}_{D(\mathfrak{F}_{\mathrm{spl}})}(j(A), j(B))$, for any pair of ℓ-algebras A and B. In this section we apply Theorem 1.1 to give a simplified proof of [3, Comparison Theorem A], where $\pi_0 \mathscr{K}(A, B)$ is computed in terms of polynomial homotopy classes of morphisms.

4.1   EXTENSIONS AND CLASSIFYING MAPS

Let Mod_ℓ be the category of ℓ-modules and write $F : \mathrm{Alg}_\ell \to \mathrm{Mod}_\ell$ for the forgetful functor. An *extension* of ℓ-algebras is a diagram

$$\mathscr{E} : \quad A \longrightarrow B \longrightarrow C \tag{14}$$

in Alg_ℓ that becomes a split short exact sequence upon applying F. A *morphism of extensions* is a morphism of diagrams in Alg_ℓ. We often consider specific splittings for the extensions we work with and we sometimes write $(\mathscr{E}, s)$ to emphasize that we are considering an extension $\mathscr{E}$ with splitting s. Let $(\mathscr{E}, s)$ and $(\mathscr{E}', s')$ be two extensions with specified splittings; a *strong morphism of extensions* $(\mathscr{E}', s') \to (\mathscr{E}, s)$ is a morphism of extensions $(\alpha, \beta, \gamma) : \mathscr{E}' \to \mathscr{E}$ that is compatible with the splittings; i.e. such that the folowing diagram commutes:

$$
\begin{array}{ccc}
FB' & \xleftarrow{\ s'\ } & FC' \\
{\scriptstyle F\beta}\downarrow & & \downarrow{\scriptstyle F\gamma} \\
FB & \xleftarrow{\ s\ } & FC
\end{array}
$$

The functor $F : \mathrm{Alg}_\ell \to \mathrm{Mod}_\ell$ admits a right adjoint $\widetilde{T} : \mathrm{Mod}_\ell \to \mathrm{Alg}_\ell$; see [3, Section 3] for details. Let T be the composite functor $\widetilde{T} \circ F : \mathrm{Alg}_\ell \to \mathrm{Alg}_\ell$. Let $A \in \mathrm{Alg}_\ell$ and let $\eta_A : TA \to A$ be the counit of the adjunction. Notice that $F\eta_A$ is a retraction which has the unit map $\sigma_A : FA \to F\widetilde{T}(FA) = FTA$ as a section. Let $JA := \ker \eta_A$. The *universal extension* of A is the extension:

$$\mathscr{U}_A : \quad JA \longrightarrow TA \xrightarrow{\ \eta_A\ } A$$

We will always consider σ_A as a splitting for $\mathscr{U}_A$.

PROPOSITION 4.2 (cf. [1, Proposition 4.4.1]). *Let* (14) *be an extension with splitting s and let $f : D \to C$ be a morphism in Alg_ℓ. Then there exists a unique strong morphism of extensions $\mathscr{U}_D \to (\mathscr{E}, s)$ extending f:*

$$
\begin{array}{ccccccc}
\mathscr{U}_D & & JD & \longrightarrow & TD & \xrightarrow{\ \eta_D\ } & D \\
{\scriptstyle \exists!}\downarrow & & {\scriptstyle \xi}\downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\
(\mathscr{E}, s) & & A & \longrightarrow & B & \longrightarrow & C
\end{array}
\tag{15}
$$

Proof. It follows easily from the adjointness of $\widetilde{T}$ and F.   □

The morphism ξ in (15) is called the *classifying map of f with respect to the extension* $(\mathscr{E}, s)$. When $D = C$ and $f = \mathrm{id}_C$ we call ξ the *classifying map of* $(\mathscr{E}, s)$.

PROPOSITION 4.3 (cf. [1, Proposition 4.4.1]). *In the hypothesis of Proposition 4.2, the homotopy class of the classifying map ξ does not depend upon the splitting s.*

Proof. See, for example, [3, Section 3].   □

Because of Proposition 4.3, it makes sense to speak of the classifying map of (14) as a homotopy class $JC \to A$ without specifying a splitting for (14).

LEMMA 4.4. *The functor $J : \mathrm{Alg}_\ell \to \mathrm{Alg}_\ell$ sends homotopic morphisms to homotopic morphisms. Thus, it defines a functor $J : [\mathrm{Alg}_\ell] \to [\mathrm{Alg}_\ell]$.*

Proof. It is explained in [1] in the discussion following [1, Corollary 4.4.4.].   □

4.5   PATH EXTENSIONS

Let B be an ℓ-algebra and let $n, q \geq 0$. Put:

$$P(n, B)_\bullet^q := B_\bullet^{(I^{n+1} \times \Delta^q, (I^n \times \{1\} \times \Delta^q) \cup (\partial I^n \times I \times \Delta^q))}$$

On the one hand, the composite $I^n \times \Delta^q \cong I^n \times \{0\} \times \Delta^q \subseteq I^{n+1} \times \Delta^q$ induces morphisms $p_{n,B}^q : P(n, B)_r^q \to B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)}$. On the other hand, we have inclusions $B_r^{(I^{n+1} \times \Delta^q, \partial I^{n+1} \times \Delta^q)} \subseteq P(n, B)_r^q$. We claim that the following diagram is an extension:

$$\mathscr{P}_{n,B}^q : \ B_r^{(I^{n+1} \times \Delta^q, \partial I^{n+1} \times \Delta^q)} \xrightarrow{\ \mathrm{incl}\ } P(n, B)_r^q \xrightarrow{\ p_{n,B}^q\ } B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)}$$

Exactness at $P(n, B)_r^q$ holds because the functors $B^{\mathrm{sd}^r(-)} : \mathbb{S} \to \mathrm{Alg}_\ell{}^{\mathrm{op}}$ preserve pushouts and we have:

$$\partial I^{n+1} \times \Delta^q = [(I^n \times \{1\} \times \Delta^q) \cup (\partial I^n \times I \times \Delta^q)] \cup (I^n \times \{0\} \times \Delta^q)$$
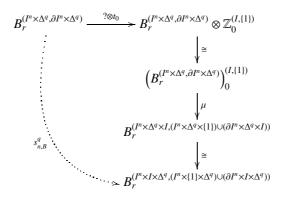
Exactness at $B_r^{(I^{n+1} \times \Delta^q, \partial I^{n+1} \times \Delta^q)}$ follows from the fact that both this algebra and $P(n, B)_r^q$ are subalgebras of $B^{\mathrm{sd}^r(I^{n+1} \times \Delta^q)}$. A splitting of $p_{n,B}^q$ in the category of ℓ-modules can be constructed as follows. Consider the element $t_0 \in \mathbb{Z}^{\Delta^1}$; t_0 is actually in $\mathbb{Z}_0^{(I, \{1\})}$ since

$d_0(t_0) = 0$. Let $s_{n,B}^q$ be the composite:

$$B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)} \xrightarrow{? \otimes t_0} B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)} \otimes \mathbb{Z}_0^{(I, \{1\})}$$

$$\downarrow \cong$$

$$\left(B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)} \right)_0^{(I, \{1\})}$$

$$\downarrow \mu$$

$$B_r^{(I^n \times \Delta^q \times I, (I^n \times \Delta^q \times \{1\}) \cup (\partial I^n \times \Delta^q \times I))}$$

$$\downarrow \cong$$

$$B_r^{(I^n \times I \times \Delta^q, (I^n \times \{1\} \times \Delta^q) \cup (\partial I^n \times I \times \Delta^q))}$$

with $s_{n,B}^q$ as the dotted curved map from the top-left to the bottom.

It is straightforward to check that $s_{n,B}^q$ is a section to $p_{n,B}^q$.

Remark 4.6. It is clear that the extensions $(\mathscr{P}_{n,B}^q, s_{n,B}^q)$ are:

(i) natural in B with respect to ℓ-algebra homomorphisms;

(ii) natural in r with respect to the last vertex map;

(iii) and natural in q with respect to morphisms of ordinal numbers.
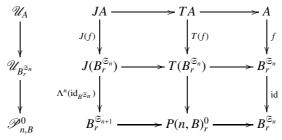
EXAMPLE 4.7. Let A and B be two ℓ-algebras, let $n \geq 0$ and let $f : A \to B_r^{\mathfrak{S}_n}$ be an ℓ-algebra homomorphism. By Proposition 4.2, there exists a unique strong morphism of extensions $\mathscr{U}_A \to \mathscr{P}_{n,B}^0$ that extends f:

$$
\begin{array}{ccccccc}
\mathscr{U}_A & & JA & \longrightarrow & TA & \longrightarrow & A \\
\exists! \downarrow & & \Lambda^n(f) \downarrow & & \downarrow & & \downarrow f \\
\mathscr{P}_{n,B}^0 & & B_r^{\mathfrak{S}_{n+1}} & \longrightarrow & P(n,B)_r^0 & \longrightarrow & B_r^{\mathfrak{S}_n}
\end{array}
$$

We will write $\Lambda^n(f)$ for the classifying map of f with respect to $\mathscr{P}_{n,B}^0$. Notice that:

$$\Lambda^n(f) = \Lambda^n(\mathrm{id}_{B^{\mathfrak{S}_n}}) \circ J(f) \tag{16}$$

Indeed, this follows from the uniqueness statement in Proposition 4.2 and the fact that the following diagram exhibits a strong morphism of extensions $\mathscr{U}_A \to \mathscr{P}_{n,B}^0$ that extends f:

$$
\begin{array}{ccccccc}
\mathscr{U}_A & & JA & \longrightarrow & TA & \longrightarrow & A \\
\downarrow & & J(f) \downarrow & & T(f) \downarrow & & \downarrow f \\
\mathscr{U}_{B_r^{\mathfrak{S}_n}} & & J(B_r^{\mathfrak{S}_n}) & \longrightarrow & T(B_r^{\mathfrak{S}_n}) & \longrightarrow & B_r^{\mathfrak{S}_n} \\
\downarrow & & \Lambda^n(\mathrm{id}_{B^{\mathfrak{S}_n}}) \downarrow & & \downarrow & & \downarrow \mathrm{id} \\
\mathscr{P}_{n,B}^0 & & B_r^{\mathfrak{S}_{n+1}} & \longrightarrow & P(n,B)_r^0 & \longrightarrow & B_r^{\mathfrak{S}_n}
\end{array}
$$

By (16) and Lemma 4.4, we can consider Λ^n as a function $\Lambda^n : [A, B_\bullet^{\mathfrak{S}_n}] \to [JA, B_\bullet^{\mathfrak{S}_{n+1}}]$.

Remark 4.8. Write Born for the category of bornological algebras; see [2, Definition 2.5]. Cuntz-Meyer-Rosenberg constructed in [2, Section 6.3] a triangulated category ΣHo endowed with a functor Born $\to \Sigma$Ho that is homotopy invariant and excisive in the bornological context, and is moreover universal with respect to these two properties; see [2, Section 6.7]. The matrix-unstable algebraic KK-theory category $D(\mathfrak{F}_{\mathrm{spl}})$ is the analogue of ΣHo in the algebraic context. Garkusha proved in [3] that there is an isomorphism

$$\mathrm{Hom}_{D(\mathfrak{F}_{\mathrm{spl}})}(j(A), j(B)) \cong \operatorname*{colim}_n [J^nA, B_\bullet^{\mathfrak{S}_n}],$$

where the transition functions on the right hand side are the Λ^n of Example 4.7. These functions $\Lambda^n : [J^nA, B_\bullet^{\mathfrak{S}_n}] \to [J^{n+1}A, B_\bullet^{\mathfrak{S}_{n+1}}]$ are the algebraic analogues of the morphism Λ of [2, Definition 6.23] that is used in [2, Section 6.3] to define the hom-sets in ΣHo.

4.9   Matrix-unstable algebraic KK-theory space

Let A and B be two ℓ-algebras and let $n \geq 0$. From the proof of Theorem 3.10, it follows that there is a natural bijection:

$$\left(\Omega^n Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(J^nA, B^\Delta) \right)_q \cong \operatorname*{colim}_r \mathrm{Hom}_{\mathrm{Alg}_\ell}\left(J^nA, B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)} \right)$$

Let $f \in \mathrm{Hom}_{\mathrm{Alg}_\ell}(J^nA, B_r^{(I^n \times \Delta^q, \partial I^n \times \Delta^q)})$ and define $\zeta^n(f)$ as the classifying map of f with respect to the extension $\mathscr{P}_{n,B}^q$:

It follows from Remark 4.6 that this defines a morphism of simplicial sets:

$$\zeta^n : \Omega^n Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(J^nA, B^\Delta) \to \Omega^{n+1} Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(J^{n+1}A, B^\Delta) \qquad (17)$$

Definition 4.10 (Garkusha). Let A and B be two ℓ-algebras. The *matrix-unstable algebraic KK-theory space* of the pair (A, B) is the simplicial set defined by

$$\mathscr{K}(A, B) := \operatorname*{colim}_n \Omega^n Ex^\infty \mathrm{Hom}_{\mathrm{Alg}_\ell}(J^nA, B^\Delta),$$

where the transition morphisms are the ζ^n defined in (17).

Note that $\mathscr{K}(A, B)$ is a fibrant simplicial set, since it is a filtering colimit of fibrant simplicial sets. This definition of $\mathscr{K}(A, B)$ is easily seen to be the same as the one given in [3, Section 4].

THEOREM 4.11 ([3, Comparison Theorem A]). *For any pair of ℓ-algebras A and B and any $m \geq 0$, there is a natural isomorphism*

$$\pi_m \mathcal{K}(A, B) \cong \operatorname*{colim}_{v}[J^v A, B_\bullet^{\mathfrak{S}_{m+v}}]$$

where the transition functions on the right hand side are the Λ^n of Example 4.7.

Proof. Since $\pi_m \cong \pi_0 \Omega^m$ commutes with filtered colimits, we have:

$$\pi_m \mathcal{K}(A, B) \cong \operatorname*{colim}_{v} \pi_0 \Omega^m \Omega^v Ex^\infty \operatorname{Hom}_{\operatorname{Alg}_\ell}(J^v A, B^\Delta)$$
$$\cong \operatorname*{colim}_{v} \pi_0 \Omega^{v+m} Ex^\infty \operatorname{Hom}_{\operatorname{Alg}_\ell}(J^v A, B^\Delta)$$
$$\cong \operatorname*{colim}_{v}[J^v A, B_\bullet^{\mathfrak{S}_{v+m}}] \qquad\qquad \text{(by Theorem 3.10)}$$

Notice that $\Omega^m \Omega^v \cong \Omega^{v+m}$ because of our conventions on iterated loop spaces; see section 2.3.2. To finish the proof, we need to compare the function Λ^{m+v} with:

$$\pi_m \zeta^v : [J^v A, B_\bullet^{\mathfrak{S}_{v+m}}] \to [J^{v+1} A, B_\bullet^{\mathfrak{S}_{v+1+m}}]$$

Let $c_{v,m} : I^v \times I^m \xrightarrow{\cong} I^m \times I^v$ be the commutativity isomorphism; $c_{v,m}$ induces an isomorphism $(c_{v,m})^* : B_r^{\mathfrak{S}_{m+v}} \to B_r^{\mathfrak{S}_{v+m}}$. It is straightforward to verify that the following squares commute:

$$
\begin{array}{ccc}
[J^v A, B_\bullet^{\mathfrak{S}_{v+m}}] & \xrightarrow{\pi_m \zeta^v} & [J^{v+1} A, B_\bullet^{\mathfrak{S}_{v+1+m}}] \\
{\scriptstyle (c_{v,m})^*} \uparrow {\scriptstyle \cong} & & {\scriptstyle \cong} \uparrow {\scriptstyle (c_{v+1,m})^*} \\
[J^v A, B_\bullet^{\mathfrak{S}_{m+v}}] & \xrightarrow{\Lambda^{m+v}} & [J^{v+1} A, B_\bullet^{\mathfrak{S}_{m+v+1}}]
\end{array}
$$

These squares assemble into a morphism of diagrams that, upon taking colimit in v, induces the desired isomorphism $\pi_m \mathcal{K}(A, B) \cong \operatorname{colim}_v[J^v A, B_\bullet^{\mathfrak{S}_{m+v}}]$. $\square$

Remark 4.12. It follows from [4, Theorem 4.4] that Garkusha's matrix-unstable algebraic KK-theory category $D(\mathfrak{F}_{\mathrm{spl}})$ is equivalent to a triangulated category of algebras with Hom-sets given by

$$\operatorname*{colim}_{n}[J^n A, B_\bullet^{\mathfrak{S}_n}].$$

This requires a heavy machinery of homotopy theory. It is possible to obtain the latter description of $D(\mathfrak{F}_{\mathrm{spl}})$ using different methods, in the style of Cuntz-Meyer-Rosenberg [2, Chapter 6]. Theorem 3.10 then provides $[J^n A, B_\bullet^{\mathfrak{S}_n}]$ with the group structure needed to make sense of the signs that appear when defining the composition rule; see [2, Lemmas 6.29 and 6.30]. We will develop this idea further in a future paper.

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