The Spectral Side of Stable Local Trace Formula

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Abstract. Let G be a connected quasi-split reductive group over $\mathbb{R}$, and more generally, a quasi-split K-group over $\mathbb{R}$. Arthur had obtained the formal formula for the spectral side of the stable local trace formula, by using formal substitute of Langlands parameters. In this paper, we construct the spectral side of the stable local trace formula and endoscopic local trace formula directly for quasi-split K-groups over $\mathbb{R}$, by incorporating the works of Shelstad. In particular we give the explicit expression for the spectral side of the stable local trace formula, in terms of Langlands parameters.

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1. Introduction

In this paper, which is a sequel to [12], we give the explicit formula for spectral side of stable local trace formula of a connected quasi-split reductive group over $\mathbb{R}$, and more generally, a quasi-split K-group over $\mathbb{R}$.

In general, the local trace formula is an identity, one side which is called the geometric side, is constructed in terms of semisimple orbital integrals; the other side, which is called the spectral side, is constructed in terms of tempered characters. Arthur [6] has obtained the stabilization of the geometric side, and consequently obtained the formal formula for the spectral side of the stable trace formula. However, the stable distributions and the coefficients that occurred in the formal formula for the spectral side, are not explicit.

By combining with Shelstad's works [13, 14, 15], we will directly stabilize the spectral side of the local trace formula, which in particular give the explicit formula for the spectral side of the stable local trace formula, in terms of Langlands parameters.

In more details, let G be a quasi-split K-group over $\mathbb{R}$, a notion for which we refer to Section 1 of [4] (where it is called *multiple groups*) or section 2.2 of [12],

and f a test function on $G(\mathbb{R})$ with central character ζ. The endoscopic decomposition of the spectral side of the invariant local trace formula, as obtained in [6], takes the following form:

$$I_{\mathrm{disc}}^{G}(f) = \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G, G') \widehat{S}_{\mathrm{disc}}^{G'}(f^{G'})$$

where $\mathcal{E}_{\mathrm{ell}}(G)$ is the set of $\widehat{G}$-equivalence classes of elliptic endoscopic data, and $f^{G'}$ is the Langlands-Shelstad transfer of f to G' [13]. One has the formal formula:

$$\widehat{S}_{\mathrm{disc}}^{G'}(f^{G'}) = \int_{\Phi_{s-\mathrm{disc}}(G', \zeta)} s^{G'}(\phi') f^{G'}(\phi') d\phi'$$

with $\Phi_{s-\mathrm{disc}}(G', \zeta)$ being defined only in terms of formal substitute of Langlands parameters ϕ' of G', and the coefficients $s^{G'}(\phi')$ are not explicit. Our task is to show that $\Phi_{s-\mathrm{disc}}(G', \zeta)$ can be taken as actual Langlands parameters, and also to give explicit formula for the coefficients in terms of Langlands parameters; *c.f.* Section 4 for the definition of these terms.

In the global context of automorphic representations, the method of [1, 9] is based on the comparison of the spectral and endoscopic objects for the global trace formula for G. This could be expressed schematically in terms of (conjectural) Langlands parameters:

$$(M, \phi_M) \longrightarrow (\phi, s) \longleftarrow (G', \phi')$$

In the archimedean local setting, the Langlands parametrization is available for general G. This allows us to adapt the comparison process to the setting of local trace formula:

$$(\tau) \longrightarrow (\phi, s) \longleftarrow (G', \phi')$$

in order to give the explicit construction of the spectral side of stable local trace formula.

We now give more details for the comparison process. Firstly recall the classification theory of tempered representations, due to Harish-Chandra. The tempered representations can be classified by triplets $\tau = (M, \pi, r)$, where M is the Levi subgroup, $\pi \in \Pi_2(M)$ is square integrable modulo the split centre of M, and $r \in R_\pi$, the representation theoretic R-group of π, which is a finite abelian elementary 2-group. For a test function $f = f_1 \times \bar{f}_2$, with $f_1, f_2 \in \mathcal{H}(G(\mathbb{R}), \zeta)$, the Hecke space with central character ζ, the spectral side of the invariant local trace formula for G takes the following form [2]:

$$I_{\mathrm{disc}}^{G}(f) = \int_{T_{\mathrm{disc}}(G, \zeta)} i^{G}(\tau) f_G(\tau) |R_\pi|^{-1} d\tau,$$

where $f_G(\tau) = \Theta(\tau, f_1) \Theta(\tau^\vee, \bar{f}_2)$, and the coefficient

$$i^{G}(\tau) = |W_\pi^\circ|^{-1} \sum_{w \in W_\pi(r)_{\mathrm{reg}}} \varepsilon_\pi(w) |\det(w-1)_{\mathfrak{a}_M^G}|^{-1},$$

encodes combinatorial data from Weyl groups that is relevant to the comparison of global trace formulas [1, 9]; *c.f.* Section 3 for the definition of these terms.

The first step in the process of stabilization is to express the coefficient $i^G(\tau)$ in terms of data defined in terms of Langlands parameters for G, and more precisely, defined in terms of the data (ϕ, s) above, an important point being that, by the works of Knapp-Zuckerman [10] and Shelstad [13], the representation theoretic R-group is canonically isomorphic to the endoscopic R-group defined in terms of Langlands parameters, *c.f.* Section 4; one has:

$$i^G(\tau) = |W_\phi^\circ|^{-1} \sum_{w \in W_\phi(x)_{\mathrm{reg}}} s_\phi^\circ(w) |\det(w-1)_{\mathfrak{a}_M^G}|^{-1}$$

where x is the image of s in $\mathcal{S}_\phi = \pi_0(\overline{S}_\phi)$. In turn, this can be expressed in terms of the constants $\sigma(\overline{S}_{\phi,s}^\circ)$ as defined in [1]:

$$i^G(\tau) = i_\phi(x) = \sum_{s \in \mathcal{E}_{\phi,\mathrm{ell}}(x)} |\pi_0(\overline{S}_{\phi,s})|^{-1} \sigma(\overline{S}_{\phi,s}^\circ).$$

Following [1], one defines the following subsets $\Phi_{s-\mathrm{disc}}(G,\zeta) \subset \Phi_{\mathrm{disc}}(G,\zeta)$ of the set $\Phi(G,\zeta)$ of Langlands parameters for G (with central character ζ), as:

$$\Phi_{s-\mathrm{disc}}(G,\zeta) = \{\phi \in \Phi(G,\zeta) : Z(\overline{S}_\phi^\circ) < \infty\},$$

$$\Phi_{\mathrm{disc}}(G,\zeta) = \{\phi \in \Phi(G,\zeta) : Z(\overline{S}_\phi) < \infty\}.$$

One has the fact that the constants $\sigma(\overline{S}_{\phi,s}^\circ)$ vanish if $Z(\overline{S}_{\phi,s}^\circ)$ is not finite; in particular that the constants $\sigma(\overline{S}_{\phi,s}^\circ)$ vanish if $\phi \notin \Phi_{s-\mathrm{disc}}(G,\zeta)$.

Having defined the basic endoscopic and stable objects on the spectral side, we can define the spectral transfer factors, in the spirit of Section 5 of [3], by combining the classification of tempered representations and Shelstad's definition of spectral transfer factors [14, 15]. For instance, in the case where $\phi \in \Phi(G,\zeta)$ is elliptic, we define, for $\tau = (M, \pi, r)$ and semi-simple $s \in \overline{S}_\phi$ (*c.f.* Section 5):

$$\Delta(\tau, \phi^s) = \sum_{\chi \in \widehat{R}_\phi} \chi(r) \Delta(\Pi^\chi, \phi^s).$$

We then have the spectral transfer:

$$\Theta(\tau, f) = \sum_{x \in \mathcal{S}_\phi} \Delta(\tau, \phi^x) f'(\phi, x).$$

We then obtain the first main theorem of the paper in Section 6:

THEOREM 1.1. *If* $f = f_1 \times \bar{f}_2, f_i \in \mathcal{H}(G(\mathbb{R}), \zeta), i = 1, 2,$ *then*

$$I_{\mathrm{disc}}^G(f) = \int_{\Phi_{\mathrm{disc}}(G,\zeta)} \sum_{s \in \mathcal{E}_{\phi,\mathrm{ell}}} |\mathcal{S}_\phi|^{-1} |\pi_0(\overline{S}_{\phi,s})|^{-1} \sigma(\overline{S}_{\phi,s}^\circ) f_1'(\phi, s) \overline{f_2'(\phi, s)} d\phi.$$

To obtain the explicit formula for the spectral side of the endoscopic local trace formula and the stable local trace formula, we use the arguments of Chapter 4 of [9]; one analyzes the coefficients by using the bijective correspondence:

$$\widehat{G} \backslash\backslash X_{\mathrm{disc}}(G, \zeta) \longleftrightarrow \widehat{G} \backslash\backslash Y_{\mathrm{disc}}(G, \zeta),$$

between the set of $\widehat{G}$-conjugacy classes of $X_{\mathrm{disc}}(G, \zeta) = \{(\phi, s) : \phi \in \Phi_{\mathrm{disc}}(G, \zeta), s \in S_{\phi, \mathrm{ell}}\}$, and the set of $\widehat{G}$-conjugacy classes of $Y_{\mathrm{disc}}(G, \zeta) = \{(G', \phi') : G' \in \mathcal{E}_{\mathrm{ell}}(G), \phi' \in \Phi_{s-\mathrm{disc}}(G', \zeta)\}$ (see Section 7 for details).

We then obtain the following in Section 7:

THEOREM 1.2. *If* $f = f_1 \times \bar{f}_2$, $f_i \in \mathcal{H}(G(\mathbb{R}), \zeta), i = 1, 2$, *then we have*

$$(1.1) \qquad I_{\mathrm{disc}}^G(f) = \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \iota(G, G') \widehat{S}_{\mathrm{disc}}^{G'}(f^{G'})$$

where

$$\iota(G, G') = |\operatorname{Out}_G(G')|^{-1} |\overline{Z}(\widehat{G'})^{\Gamma_{\mathbb{R}}}|^{-1},$$

$$\widehat{S}_{\mathrm{disc}}^{G'}(f^{G'}) = \int_{\Phi_{s-\mathrm{disc}}(G', \zeta)} |\mathcal{S}_{\phi'}|^{-1} \sigma(\overline{S}_{\phi'}^{\circ}) f_1^{G'}(\phi') \overline{f_2^{G'}(\phi')} d\phi'.$$

The explicit formula for the spectral side of the stable trace formula will then be obtained in Section 8.

Here is the summary of the contents of the paper. After introducing preliminaries and notations in Section 2, we recall some formulation of the invariant local trace formula of Arthur in Section 3. Then in Section 4 we introduce the basic objects that occur in the spectral side of the endoscopic local trace formula and the stable local trace formula. We will study the properties of the spectral transfer factors in Section 5. We then obtain the main theorem on stabilization of the local trace formula in Section 6. The explicit formula for the spectral side of the endoscopic and stable local trace formula are then obtained in Section 7 and 8.

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2. Preliminaries and notation

Throughout the paper G is a K-group over $\mathbb{R}$, which will be assumed to be quasi-split from Section 4 onwards. The center of G is noted as $Z(G)$, while Z stands for a fixed central induced torus in G over $\mathbb{R}$, and ζ is a character on $Z(\mathbb{R})$. Let

$$\mathfrak{a}_G = \operatorname{Hom}(X(G)_{\mathbb{R}}, \mathbb{R})$$

be the real vector space dual of the module $X(G)_{\mathbb{R}}$ of $\mathbb{R}$ rational characters on G. There is a canonical homomorphism

$$H_G : G(\mathbb{R}) \to \mathfrak{a}_G$$

defined by

$$e^{\langle H_G(x), \chi \rangle} = |\chi(x)|, x \in G(\mathbb{R}), \chi \in X(G)_{\mathbb{R}},$$

where $|\cdot|$ is the absolutely valuation on $\mathbb{R}$. Let A_G be the split component of the center of G, then

$$\mathfrak{a}_G = H_G(G(\mathbb{R})) = H_G(A_G)$$

and $\mathfrak{a}_G$ is the Lie algebra of A_G. It is convenient to fix a Haar measure on $\mathfrak{a}_G$. This determines a dual Haar measure on the real vector space $i\mathfrak{a}_G^*$. It also determines a unique Haar measure on $A_G(\mathbb{R})$. We denote by $\mathfrak{a}_{G,Z}^*$ the subspace of linear forms on $\mathfrak{a}_G$ that are trivial on the image of $\mathfrak{a}_Z$ in $\mathfrak{a}_G$.

We recall some settings as in [2]. Denote by $\mathcal{L}^G(M)$ the finite set of Levi subgroups of G which contain a given Levi subgroup M. We let M_0 be a fixed Levi component of some minimal parabolic subgroup of G. Put $\mathcal{L} := \mathcal{L}^G(M_0)$. For $M \in \mathcal{L}$, denote by $\mathcal{P}^G(M)$ the set of parabolic subgroups of G having M as Levi component. Define $\Pi_2(M(\mathbb{R}))$ to be the set of (equivalence classes of) representations that are square integrable modulo the split center of M, and $\Pi_{\text{temp}}(M(\mathbb{R}))$ to be the set of (equivalence classes of) tempered representations. For $M \in \mathcal{L}^G(M_0)$ and $\pi \in \Pi_2(M(\mathbb{R}))$, denote by $\Pi_\pi(G(\mathbb{R}))$ the set of irreducible constituents of the induced representation $I_P(\pi)$, which is a finite subset of $\Pi_{\text{temp}}(G(\mathbb{R}))$ and independent of the parabolic subgroup P. The sets $\Pi_\pi(G(\mathbb{R}))$ exhaust $\Pi_{\text{temp}}(G(\mathbb{R}))$. The classification of the representations in $\Pi_{\text{temp}}(G(\mathbb{R}))$ is reduced to classifying the representations in the finite sets $\Pi_\pi(G(\mathbb{R}))$, and to determining the intersection of any two such sets. The second question is answered by Harish-Chandra's work.

Write W_0^G for the Weyl group of the pair (G, A_{M_0}); for $w \in W_0^G$, we generally write $\tilde{w}$ for any representative of w in K; here K is a (fixed) maximal compact subgroup $G(\mathbb{R})$ that is in good position relative to $M_0(\mathbb{R})$. If $M \in \mathcal{L}^G(M_0)$ and $\pi \in \Pi_2(M(\mathbb{R}))$, $wM = \tilde{w}M\tilde{w}^{-1}$ is another Levi subgroup, and

$$(w\pi)(m') = \pi(\tilde{w}^{-1}m'\tilde{w}), \quad m' \in (wM)(\mathbb{R})$$

is a representation in $\Pi_2((wM)(\mathbb{R}))$. We obtain an action

$$(M, \pi) \to (wM, w\pi), w \in W_0^G,$$

of W_0^G on the set of pairs $(M, \pi), M \in \mathcal{L}^G(M_0), \pi \in \Pi_2(M(\mathbb{R}))$.

As stated in Proposition 1.1 of [2], one has the following: If (M, π) and (M', π') are any two pairs, and (M', π') equals $(wM, w\pi)$ for an element $w \in W_0^G$, then the subsets $\Pi_\pi(G(\mathbb{R}))$ and $\Pi_{\pi'}(G(\mathbb{R}))$ of $\Pi_{\text{temp}}(G(\mathbb{R}))$ are identical. Conversely, if the sets $\Pi_\pi(G(\mathbb{R}))$ and $\Pi_{\pi'}(G(\mathbb{R}))$ have a representation in common, there is an element $w \in W_0^G$ such that $(M', \pi') = (wM, w\pi)$.

In this paper we shall fix the central data (Z, ζ). Thus $\mathcal{H}(G(\mathbb{R}), \zeta)$ is the Hecke space of smooth functions with compact support f on $G(\mathbb{R})$ that are left and right finite under the maximal compact subgroup K, and such that $f(zx) = \zeta(z)^{-1}f(x)$ for $z \in Z(\mathbb{R})$ and $x \in G(\mathbb{R})$. We define $\Pi_{\text{temp}}(G(\mathbb{R}), \zeta)$ to be the subset of $\Pi_{\text{temp}}(G(\mathbb{R}))$ consisting of those representations whose character on

$Z(\mathbb{R})$ is equal to ζ. Similarly one defines the subsets $\Pi_2(G(\mathbb{R}),\zeta) := \Pi_2(G(\mathbb{R})) \cap \Pi_{\text{temp}}(G(\mathbb{R}),\zeta)$ and $\Pi_\pi(G(\mathbb{R}),\zeta) := \Pi_\pi(G(\mathbb{R})) \cap \Pi_{\text{temp}}(G(\mathbb{R}),\zeta)$ etc.

We can understand the finite set $\Pi_\pi(G(\mathbb{R}),\zeta)$ by the representation theoretic R-group [10]. More generally, we can parameterize the characters of the tempered representations $\Pi_{\text{temp}}(G(\mathbb{R}),\zeta)$ by the virtual characters of the sets $T(G,\zeta)$, where $T(G,\zeta)$ is the set of G-equivalence classes of the sets $\widetilde{T}(G,\zeta)$, with $\widetilde{T}(G,\zeta)$ being defined as $\widetilde{T}(G,\zeta) = \{\tau = (M,\pi,r) : M \in \mathcal{L}^G(M_0), \pi \in \Pi_2(M(\mathbb{R}),\zeta), r \in R_\pi\}$. Here $\Pi_2(M(\mathbb{R}),\zeta)$ is as before the subset of $\Pi_2(M(\mathbb{R}))$ consisting of those π whose character on $Z(\mathbb{R})$ is equal to ζ. Finally the representation theoretic R-group R_π of π is defined as the quotient of W_π by W_π°, where

$$W_\pi = \{w \in W(\mathfrak{a}_M) : w\pi \cong \pi\}$$

is the stabilizer of π in the Weyl group of $\mathfrak{a}_M$, and W_π° is the subgroup of elements w in W_π such that the operator $R(w,\pi)$ is a scalar (*c.f.* below); W_π° is a normal subgroup of W_π. It is known that the group W_π° is the Weyl group of a root system, composed of scalar multiples of those reduced roots α of (G, A_M) for which the reflection w_α belongs to W_π° (*c.f* p. 86 of [2]). These roots divide the vector space $\mathfrak{a}_M$ into chambers. By fixing such a chamber $\mathfrak{a}_\pi$, we can identify R_π with the subgroup of elements in W_π that preserve $\mathfrak{a}_\pi$.

The operator

$$R(w,\pi) = A(\pi_w)R_{\tilde{w}^{-1}P\tilde{w}|P}(\pi), \quad w \in W_\pi, \pi \in \Pi_2(M), P \in \mathcal{P}^G(M)$$

is an intertwining operator from $I_P(\pi)$ to itself. Here to define $A(\pi_w)$, first note that π can be extended to a representation of the group $M'(\mathbb{R})$ generated by $M(\mathbb{R})$ and $\tilde{w}$. We denote π_w by such an extension, then the intertwining operator

$$A(\pi_w) : I_{\tilde{w}^{-1}P\tilde{w}}(\pi) \longrightarrow I_P(\pi), \pi \in W_\pi$$

between $I_{\tilde{w}^{-1}P\tilde{w}}(\pi)$ and $I_P(\pi)$ is defined by setting

$$(A(\pi_w)\phi')(x) = \pi_w(\tilde{w})\phi'(\tilde{w}^{-1}x), \phi' \in I_{\tilde{w}^{-1}P\tilde{w}}(\pi).$$

Finally the intertwining operator $R_{\tilde{w}^{-1}P\tilde{w}|P}(\pi)$, and more generally

$$R_{Q|P}(\pi) = r_{Q|P}(\pi)^{-1}J_{Q|P}(\pi) : I_P(\pi) \longrightarrow I_Q(\pi), \text{ for } P,Q \in \mathcal{P}^G(M),$$

is the normalized intertwining operator between the induced representations $I_P(\pi)$ and $I_Q(\pi)$, with $r_{Q|P}(\pi)$ being the normalizing factors; see the discussion on p. 85 of [2]. In addition, since we are in the archimedean case, the operators $R(w,\pi)$ can be normalized so that $R(w,\pi)$ is the identity for $w \in W_\pi^\circ$ (hence $R(r,\pi)$ is well-defined for $r \in R_\pi$), and such that the map $r \mapsto R(r,\pi)$ is a homomorphism on R_π, *c.f.* p. 86 of [2]. We will always work with such a normalization in what follows.

3. Local trace formula

In this section we recall the formalism of the local trace formula of Arthur [2], specifically in the archimedean case. Consider a test function $f = f_1 \times \bar{f}_2$, with $f_i \in \mathcal{H}(G(\mathbb{R}), \zeta)$ being in the Hecke space of test functions. Following Arthur, it is customary to think of the components f_1, f_2 of f as being indexed by the two elements set $V = \{\infty_1, \infty_2\}$ (being regarded as two archimedean places). The local trace formula is given by the identity $I_{\mathrm{disc}}^G(f) = I^G(f)$, *c.f.* Theorem 4.2 of [2] and Proposition 6.1 of [6]. The geometric side of the local trace formula is given by:

$$I^G(f) = \sum_{M \in \mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Gamma_{G-\mathrm{reg,ell}}(M,V,\zeta)} I_M^G(\gamma, f) d\gamma$$

defined in terms of the invariant distributions $I_M^G(\gamma, f)$ (*c.f. loc. cit.*) For the definition of $\Gamma_{G-\mathrm{reg,ell}}(M, V, \zeta)$, firstly, for a fixed basis $\Gamma(M, \zeta)$ of the space of invariant distributions $\mathcal{D}(M, \zeta)$ on $M(\mathbb{R})$ introduced in Section 1 of [5] (which in particular are ζ-equivariant under translation by $Z(\mathbb{R})$), one has the subset $\Gamma_{G-\mathrm{reg,ell}}(M, \zeta)$ of $\Gamma(M, \zeta)$ consisting of elements that are strongly G-regular, and having elliptic support in $M(\mathbb{R})$. Put $\Gamma(M_V, \zeta_V) = \Gamma(M, \zeta) \times \Gamma(M, \zeta)$ (corresponding to the two places ∞_1, ∞_2 in V). Then $\Gamma_{G-\mathrm{reg,ell}}(M, V, \zeta)$ is identified with the diagonal image of $\Gamma_{G-\mathrm{reg,ell}}(M, \zeta)$ in $\Gamma(M_V, \zeta_V)$:

$$\Gamma_{G-\mathrm{reg,ell}}(M, V, \zeta) = \{(\gamma, \gamma) : \gamma \in \Gamma_{G-\mathrm{reg,ell}}(M, \zeta)\}.$$

The other side is the spectral side given by:

$$I_{\mathrm{disc}}^G(f) = \int_{T_{\mathrm{disc}}(G,\zeta)} i^G(\tau) f_G(\tau) |R_\pi|^{-1} d\tau.$$

Here:

$$T_{\mathrm{disc}}(G, \zeta) = \{\tau = (M, \pi, r) \in T(G, \zeta) : W_\pi(r)_{\mathrm{reg}} \neq \emptyset, \pi \in \Pi_2(M(\mathbb{R}), \zeta)\},$$

where

$$W_\pi(r)_{\mathrm{reg}} = W_\pi(r) \cap W_{\pi,\mathrm{reg}}, W_\pi(r) = W_\pi^\circ \cdot r, W_{\pi,\mathrm{reg}} = \{w \in W_\pi : \mathfrak{a}_M^w = \mathfrak{a}_G\},$$

and

$$i^G(\tau) = |W_\pi^\circ|^{-1} \sum_{w \in W_\pi(r)_{\mathrm{reg}}} \varepsilon_\pi(w)|\det(w-1)_{\mathfrak{a}_M^G}|^{-1}$$

which encode combinatorial data from Weyl groups. The sign $\varepsilon_\pi(w)$ stands for the sign of projection of w onto the Weyl group W_π° taken relative to the decomposition $W_\pi = W_\pi^\circ \rtimes R_\pi$, and $\mathfrak{a}_M^G$ is the quotient of $\mathfrak{a}_M$ by $\mathfrak{a}_G$. The other terms are defined as:

$$f_G(\tau) = \Theta(\tau, f_1)\Theta(\tau^\vee, \bar{f}_2) = \Theta(\tau, f_1)\overline{\Theta(\tau, f_2)}$$

$$\Theta(\tau, f_i) = \mathrm{tr}(R(r, \pi)I_P(\pi, f_i)), \quad i = 1, 2.$$

Finally the measure $d\tau$ on $T_{\text{disc}}(G, \zeta)$ is defined by the formula (*c.f.* equation (3.5) of [2]):

$$\int_{T_{\text{disc}}(G,\zeta)} f(\tau) d\tau = \sum_{\tau \in T_{\text{disc}}(G,\zeta)/i\mathfrak{a}^*_{G,Z}} \int_{i\mathfrak{a}^*_{G,Z}} f(\tau_\lambda) d\lambda.$$

for $f \in C_c(T_{\text{disc}}(G, \zeta))$.

Here when compared to equation (3.5) of [2], we first note that since we are in the archimedean case, the groups R_π are abelian, and hence the groups $R_{\pi,r}$ in *loc. cit.*, namely the centralizer of r in R_π, reduces to R_π.

Secondly, for our purpose, it would be more convenient to *not* to absorb the factor $|R_{\pi,r}|^{-1} = |R_\pi|^{-1}$ into the definition of the measure $d\tau$ on $T_{\text{disc}}(G, \zeta)$, as was done in [2].

4. Endoscopic and stable objects on the spectral side

From now on G will always be assumed to be a quasi-split K-group. We first recall the Langlands parameters. These are admissible continuous homomorphisms:

$$\phi : W_{\mathbb{R}} \longrightarrow {}^L G$$

where as usual ${}^L G = \widehat{G} \rtimes W_{\mathbb{R}}$ is the L-group of G, defined with respect to a splitting of G (that we fix for the rest of the paper); $W_{\mathbb{R}}$ is the Weil group of $\mathbb{R}$: it is a non-split extension $1 \to \mathbb{C}^\times \to W_{\mathbb{R}} \to \Gamma_{\mathbb{R}} \to 1$, with $\Gamma_{\mathbb{R}} = \text{Gal}(\mathbb{C}/\mathbb{R})$. The parameter ϕ is called bounded, if the image of $W_{\mathbb{R}}$ in $\widehat{G}$ is bounded. We denote by $\Phi(G)$ for the set of $\widehat{G}$-equivalence classes of bounded parameters (with respect to the conjugation action by $\widehat{G}$). For $\phi \in \Phi(G)$, we denote by Π_ϕ the L-packet of tempered representations of $G(\mathbb{R})$ associated to ϕ. The stable character $f \mapsto f(\phi) := \sum_{\Pi \in \Pi_\phi} \text{tr}\,\Pi(f)$ is then a stable distribution on $G(\mathbb{R})$ [13].

We denote by $\Phi_2(G)$, the set of (equivalence classes of) square-integrable parameters, for the subset of $\phi \in \Phi(G)$ that does not factor through ${}^L M$, for any proper Levi subgroup M of G. For $\phi \in \Phi_2(G)$, the L-packet Π_ϕ consists of square-integrable representations of $G(\mathbb{R})$.

For any $\phi \in \Phi(G)$, we set

$$S_\phi = \text{Cent}(Im\phi, \widehat{G}),$$

$$\overline{S}_\phi = S_\phi / Z(\widehat{G})^{\Gamma_{\mathbb{R}}},$$

and

$$\mathcal{S}_\phi = \pi_0(\overline{S}_\phi).$$

Since we are in the archimedean case, the component group $\mathcal{S}_\phi$ is a finite abelian elementary 2-group [13]. In addition, since we are working in the context of a quasi-split K-group, one has that Π_ϕ is in bijection with the set of characters of $\mathcal{S}_\phi$; hence the cardinality of Π_ϕ is equal to the order of $\mathcal{S}_\phi$ [13, 15].

One has
$$\Phi_2(G) = \{\phi \in \Phi(G), |\overline{S}_\phi| < \infty\}.$$
We also define the following subsets of parameters $\Phi_{\mathrm{disc}}(G), \Phi_{s-\mathrm{disc}}(G), \Phi_{\mathrm{ell}}(G)$, of $\Phi(G)$:
$$\Phi_{\mathrm{disc}}(G) = \{\phi \in \Phi(G), |Z(\overline{S}_\phi)| < \infty\},$$
with $Z(\overline{S}_\phi) := \mathrm{Cent}(\overline{S}_\phi, \overline{S}_\phi^\circ)$, and:

$$\Phi_{s-\mathrm{disc}}(G) = \{\phi \in \Phi(G), |Z(\overline{S}_\phi^\circ)| < \infty\},$$
with $Z(\overline{S}_\phi^\circ)$ being the usual center of $\overline{S}_\phi^\circ$.

The set of elliptic parameters $\Phi_{\mathrm{ell}}(G)$ is defined as the subset of $\phi \in \Phi(G)$, such that $|\overline{S}_{\phi,s}| < \infty$ for some semi-simple element $s \in \overline{S}_\phi$; here $\overline{S}_{\phi,s}$ is being defined as:
$$\overline{S}_{\phi,s} := \mathrm{Cent}(s, \overline{S}_\phi^\circ).$$

One has:
$$\Phi_2(G) \subset \Phi_{s-\mathrm{disc}}(G) \subset \Phi_{\mathrm{disc}}(G),$$

$$\Phi_2(G) \subset \Phi_{\mathrm{ell}}(G) \subset \Phi_{\mathrm{disc}}(G).$$

Also, for a central data (Z, ζ) of G as before, we denote by $\Phi(G, \zeta)$ the set of parameters $\phi \in \Phi(G)$ that have character ζ with respect to Z, in the sense that the composition:
$$W_\mathbb{R} \xrightarrow{\phi} {}^L G \to {}^L Z$$
corresponds to the character ζ of Z. Similar definition for the sets $\Phi_2(G, \zeta), \Phi_{\mathrm{ell}}(G, \zeta)$ etc.

We now define the endoscopic R-group,
$$R_\phi := W_\phi / W_\phi^\circ,$$
where the Weyl groups W_ϕ, W_ϕ° are defined as:
$$W_\phi = \mathrm{Norm}(\overline{\mathcal{T}}_\phi, \overline{S}_\phi)/\overline{\mathcal{T}}_\phi.$$

Here $\overline{\mathcal{T}}_\phi$ is defined as $A_{\widehat{M}}/(A_{\widehat{M}} \cap Z(\widehat{G})^{\Gamma_\mathbb{R}})$, with $A_{\widehat{M}} = (Z(\widehat{M})^{\Gamma_\mathbb{R}})^\circ$, and M being the Levi subgroup of G (which is unique up to conjugation) such that ϕ factors through ${}^L M$ as a square integrable parameter $\phi_M \in \Phi_2(M, \zeta)$ of M. Similarly
$$W_\phi^\circ = \mathrm{Norm}(\overline{\mathcal{T}}_\phi, \overline{S}_\phi^\circ)/\overline{\mathcal{T}}_\phi.$$

One also has, by the results in section 5 of [13], the split short exact sequence:

(4.1) $$0 \to \mathcal{S}_{\phi_M} \to \mathcal{S}_\phi \to R_\phi \to 0.$$

For $\phi \in \Phi(G, \zeta)$, we denote by
$$T_\phi = \{\tau = (M, \pi, r) \in T(G, \zeta), \text{ such that } \pi \in \Pi_{\phi_M}\}.$$

LEMMA 4.1. *If $\phi \in \Phi(G, \zeta)$, then we have canonical identification $R_\pi = R_\phi, W_\pi = W_\phi, W_\pi^\circ = W_\phi^\circ$, for $\tau = (M, \pi, r) \in T_\phi$.*

Thus we have a natural surjective map of sets $T_\phi \to R_\phi$ by sending $\tau = (M, \pi, r)$ to $r \in R_\pi = R_\phi$.

We can also define a non-canonical bijection $\iota : T_\phi \to \mathcal{S}_\phi$ (which we fix once and for all) that respects the natural projection map to R_ϕ.

Proof. This follow from the results of Knapp-Zuckerman [10] and Shelstad [13]. If $\phi \in \Phi_{\mathrm{ell}}(G, \zeta)$ is elliptic, see the discussion before Proposition 5.2 of [12]. In general, if $\phi \in \Phi(G, \zeta)$, then there is a Levi subgroup $\widetilde{M}$ of G (unique up to conjugacy), that is maximal with respect to the property that the parameter ϕ factors through $^L\widetilde{M}$ as an elliptic parameter $\phi^{\widetilde{M}}$ of $\widetilde{M}$ (see section 5 of [13]). Under the Levi embedding $^L\widetilde{M} \hookrightarrow {}^LG$ on the dual side, have a canonical isomorphism

$$\mathcal{S}_{\phi^{\widetilde{M}}} \xrightarrow{\sim} \mathcal{S}_\phi.$$

Now $\phi^{\widetilde{M}}$ is elliptic, and so the lemma is true for $\phi^{\widetilde{M}}$; thus we have canonical isomorphisms (noting that the R-group of π with respect to G is canonically identified with the R-group of π with respect to $\widetilde{M}$, *c.f. loc. cit.*):

$$R_\pi = R_{\phi^{\widetilde{M}}} = R_\phi.$$

To show that one has canonical identifications between W_π, W_ϕ and $W_\pi^\circ, W_\phi^\circ$, first note that W_ϕ is the stabilizer of ϕ_M in $W(\mathfrak{a}_M)$. It is then a consequence of the disjointness of tempered L-packets that W_ϕ contains W_π. On the other hand, we know that the intertwining operators on $I_P(\pi)$ coming from the subgroup W_ϕ° of W_ϕ are scalars (*c.f.* section 5 of [13]). It follows that W_ϕ° is contained in the subgroup W_π° of W_π. Now $R_\pi = W_\pi/W_\pi^\circ, R_\pi = W_\phi/W_\phi^\circ$, and we already know that $|R_\pi| = |R_\phi|$, so it follows that we must have $W_\pi = W_\phi$ and $W_\pi^\circ = W_\phi^\circ$.

Finally, one can thus construct a bijection $\iota : T_\phi \to \mathcal{S}_\phi$ that respects the projection to R_ϕ, from a bijection between $T_{\phi^{\widetilde{M}}}$ and $\mathcal{S}_{\phi^{\widetilde{M}}}$ that respects the projection to $R_{\phi^{\widetilde{M}}}$. $\square$

Next we recall some generalities from [1]. We shall consider S to be any connected component of a complex reductive group. Given such S, we denote by S^+ the complex reductive group generated by S, and by S° the identity connected component of S^+. Put

$$Z(S) = \mathrm{Cent}(S, S^\circ)$$

for the centralizer of S in S°. Then for a choice of maximal torus $\mathcal{T}$ of S°, denote by $W(S^\circ) = \mathrm{Norm}(\mathcal{T}, S^\circ)/\mathcal{T}$ the usual Weyl group of S°. We can also form the Weyl set

$$W(S) = \mathrm{Norm}(\mathcal{T}, S)/\mathcal{T}.$$

Denote by $W(S)_{\mathrm{reg}}$ the subset of elements $w \in W(S)$ that are regular, which means that the fixed point set of w in $\mathcal{T}$ is finite. We can also regard $(w - 1)$ as a linear transformation on the real vector space

$$\mathfrak{a}_{\mathcal{T}} = \mathrm{Hom}(X(\mathcal{T}), \mathbb{R}).$$

Then the condition for $w \in W(S)_{\mathrm{reg}}$ is equivalent to that $\det(w - 1)_{\mathfrak{a}_{\mathcal{T}}} \neq 0$.

We denote by:

$$s^{\circ}(w) = \pm 1$$

the sign of a element $w \in W$, to be -1 raised to the power of the number of the positive roots of $(S^{\circ}, \mathcal{T})$ (with respect to some order) being mapped by w to the negative roots. We then define the rational number

$$i(S) = |W(S^{\circ})|^{-1} \sum_{w \in W(S)_{\mathrm{reg}}} s^{\circ}(w) |\det(w - 1)_{\mathfrak{a}_{\mathcal{T}}}|^{-1}$$

associated to S.

Next we write S_{ss} for the set of semisimple elements in S. For any $s \in S_{ss}$, we set

$$S_s = \mathrm{Cent}(s, S^{\circ})$$

the centralizer of s in S°. Then S_s is also a complex reductive group, whose identity component is noted as:

$$S_s^{\circ} = (S_s)^{\circ} = \mathrm{Cent}(s, S^{\circ})^{\circ}.$$

If Γ is any subset of S which is invariant under conjugation by S°, then we shall denote by $\mathcal{E}(\Gamma)$ for the set of equivalence classes in $\Gamma_{ss} = \Gamma \cap S_{ss}$, with the equivalence relation defined by setting $s' \sim s$ if

$$s' = s^{\circ} z s (s^{\circ})^{-1}, s^{\circ} \in S^{\circ}, z \in Z(S_s^{\circ})^{\circ}.$$

The main interest is the subset

$$S_{\mathrm{ell}} = \{s \in S_{ss} : |Z(S_s^{\circ})| < \infty\}$$

of elliptic elements of S. The equivalence relation on S_{ell} is then simply S°-conjugation. Put:

$$\mathcal{E}_{\mathrm{ell}}(S) := \mathcal{E}(S_{\mathrm{ell}}).$$

We have the following theorem which is a restatement of theorem 8.1 of [1].

THEOREM 4.2. *There are unique constants $\sigma(S_1)$, defined whenever S_1 is a connected complex reductive group, such that for any S as above, the number*

$$(4.2) \qquad\qquad e(S) = \sum_{s \in \mathcal{E}_{\mathrm{ell}}(S)} |\pi_0(S_s)|^{-1} \sigma(S_s^{\circ})$$

equals $i(S)$, and such that

$$(4.3) \qquad\qquad \sigma(S_1) = \sigma(S_1/Z_1) |Z_1|^{-1}$$

for any central subgroup Z_1 of S_1 (in particular $\sigma(S_1) = 0$ if Z_1 is infinite).

Now back to the situation of Lemma 4.1. For $\tau \in T_\phi$, put $x = \iota(\tau)$. Then we can write the coefficient $i^G(\tau)$ as:

$$
\begin{aligned}
(4.4) \quad i^G(\tau) \;=\; i_\phi(x) &= |W_\phi^\circ|^{-1} \sum_{w \in W_\phi(x)_{\mathrm{reg}}} s_\phi^\circ(w) |\det(w-1)_{\mathfrak{a}_M^G}|^{-1} \\
&= e_\phi(x) = \sum_{s \in \mathcal{E}_{\phi,\mathrm{ell}}(x)} |\pi_0(\overline{S}_{\phi,s})|^{-1} \sigma(\overline{S}_{\phi,s}^\circ)
\end{aligned}
$$

i.e. $i_\phi(x)$ is equal to the number $i(S_x)$ above, with S_x being equal to the component of $\overline{S}_\phi$ that corresponds to $x \in \mathcal{S}_\phi = \pi_0(\overline{S}_\phi)$, and $W_\phi(x)_{\mathrm{reg}}$ is the set $W(S_x)_{\mathrm{reg}}$ as defined above; similarly $e_\phi(x)$ is the number $e(S_x)$, with $\mathcal{E}_{\phi,\mathrm{ell}}(x)$ (resp. $\mathcal{E}_{\phi,\mathrm{ell}}$) being the set $\mathcal{E}_{\mathrm{ell}}(S_x)$ (resp. $\mathcal{E}_{\mathrm{ell}}(\overline{S}_\phi)$) defined as above, etc. Note also that, as a consequence of the canonical identification $W_\pi = W_\phi$ and $W_\pi^\circ = W_\phi^\circ$ given by Lemma 4.1, we have that, if $\tau \in T_{\mathrm{disc}}(G, \zeta)$, then one must have $\phi \in \Phi_{\mathrm{disc}}(G, \zeta)$.

Finally we note the following. Let $\phi \in \Phi(G, \zeta)$ be as before, that factors through $^L M$ as a square-integrable parameter ϕ_M of Levi subgroup M of G. Recall that one has a split short exact sequence:

$$
0 \to \mathcal{S}_{\phi_M} \to \mathcal{S}_\phi \to R_\phi \to 0.
$$

Then for $x, y \in \mathcal{S}_\phi$, one has $i_\phi(x) = i_\phi(y)$ if $x = y \mod \mathcal{S}_{\phi_M}$. This follows easily from the definition of the number $i_\phi(x)$.

The constants $\sigma(\overline{S}_{\phi,s}^\circ)$, for $s \in \mathcal{E}_{\phi,\mathrm{ell}} = \mathcal{E}_{\mathrm{ell}}(\overline{S}_\phi)$, appear as the coefficients of the endoscopic and stable local trace formula, as we are going to see in the following sections.

5. Spectral transfer

Suppose $(G', s', \mathcal{G}', \xi')$ is an endoscopic datum for G. In the theory of endoscopy one chooses a Z-pair (G_1', ξ_1'), where G_1' is a Z-extension of G' and ξ_1' is an embedding of extensions $\mathcal{G}' \hookrightarrow {}^L G_1'$ that extends the embedding $\widehat{G}' \hookrightarrow \widehat{G}_1'$ dual to the surjection $G_1' \to G'$.

Given the endoscopic datum $(G', s', \mathcal{G}', \xi')$ for G as above, suppose that $\mathcal{G}' = {}^L G'$ (and thus $\xi' : {}^L G' \hookrightarrow {}^L G$ is an embedding of L-groups, and it is not needed to choose a Z-extension for G'), we define a mapping

$$
\Phi(G', \zeta) \longrightarrow \Phi(G, \zeta)
$$

by $\phi' \mapsto \phi = \xi' \circ \phi'$.

For the general case, we refer to Section 2 of [7] and also Section 2 of [14]. This construction gives a correspondence $(G', \phi') \longleftrightarrow (\phi, s)$ between pairs (ϕ, s), where $\phi \in \Phi(G, \zeta)$, $s \in \overline{S}_\phi$ semi-simple, and pairs (G', ϕ'), where $G' = (G', s', \mathcal{G}', \xi')$ is an endoscopic datum of G, and $\phi' \in \Phi(G', \zeta)$, *c.f. loc. cit.* For simplicity we always assume that $\mathcal{G}' = {}^L G'$ in what follows.

DEFINITION 5.1. For $\phi' \in \Phi(G', \zeta)$ and $\Pi \in \Pi_{\mathrm{temp}}(G, \zeta)$, we say that (ϕ', Π) is a related pair if $\phi(\Pi)$ is the image of ϕ' under the map $\Phi(G', \zeta) \longrightarrow \Phi(G, \zeta)$ associated to ξ'; here $\phi(\Pi) \in \Phi(G, \zeta)$ is the Langlands parameter of Π.

Given any $\phi' \in \Phi(G', \zeta)$, there is always a $\Pi \in \Pi_{\mathrm{temp}}(G, \zeta)$ that is related to ϕ'.

DEFINITION 5.2. A related pair (ϕ', Π) is G-regular if the parameter $\phi = \phi(\Pi)$ is G-regular, in the sense that we have that

$$\mathrm{Cent}(\phi(\mathbb{C}^\times), \widehat{G})$$

is abelian.

Shelstad had built the spectral transfer factors $\Delta(\phi', \Pi)$ [14] directly when the related pair (ϕ', Π) is G-regular, and obtained spectral transfer identities (if the pair (ϕ', Π) is not related, then one simply defines $\Delta(\phi', \Pi) = 0$). The general case is handled by using character identities of Hecht and Schmid, and coherent continuation of the identities from the G-regular case [14].

We have the following spectral transfer relations. For each $f \in \mathcal{H}(G(\mathbb{R}), \zeta)$, and for any endoscopic datum G' of G, there exists $f' \in \mathcal{H}(G'(\mathbb{R}), \zeta)$ such that the stable orbital integral of f' is equal to $f^{G'}$, the Langlands-Shelstad transfer of f to G' [13] (with respect to Whittaker normalization). Note that in particular when we take $G' = G$, then f^G is the stable orbital integral of f. In addition the following holds: for any endoscopic datum G' of G and any tempered Langlands parameter ϕ' of $G'(\mathbb{R})$, the stable character $f'(\phi')$ of ϕ' (evaluated at f') satisfies:

$$f'(\phi') = \sum_{\Pi \in \Pi_{\mathrm{temp}}(G, \zeta)} \Delta(\phi', \Pi) \operatorname{tr} \Pi(f).$$

Remark that, as the character $f' \mapsto f'(\phi')$ is stable, it depends only on the stable orbital integral of f'. Hence in the above, namely when the stable orbital integral of f' is equal to the Langlands-Sehlstad transfer $f^{G'}$, the value $f'(\phi')$ depends only on $f^{G'}$, and we will write $f'(\phi')$ as $f^{G'}(\phi')$.

Shelstad (*c.f.* [15], section 11) had also checked that the spectral transfer factors can be normalized (namely Whittaker normalization) to satisfy the Arthur's conjecture, and we will always work with such normalization in what follows; more precisely given endoscopic data $(G', s', \mathcal{G}', \xi')$ as above, one has, whenever (ϕ', Π) is a related pair, that the number $\Delta(\phi', \Pi)$ depends only on Π and on the image x_s of s in $\mathcal{S}_\phi$ (here as before $\phi = \phi(\Pi)$ is the Langlands parameter of Π).

Thus given $\phi \in \Phi(G, \zeta)$, and $s \in \overline{S}_\phi$ semi-simple, we will denote, for $\Pi \in \Pi_\phi$,

$$\Delta(\phi^s, \Pi) := \Delta(\phi', \Pi)$$

if the pair (ϕ, s) corresponds to (G', ϕ'). For fixed $\Pi \in \Pi_\phi$, the function $s \mapsto \Delta(\phi^s, \Pi)$ factors through $\mathcal{S}_\phi$. Thus we have the function $x \mapsto \Delta(\phi^x, \Pi)$ for $x \in \mathcal{S}_\phi$. In addition, this function is a $\{\pm 1\}$-valued character of $\mathcal{S}_\phi$ [15].

Thus in particular, given $\phi \in \Phi(G, \zeta)$ and $s \in \overline{S}_\phi$ semi-simple, we can consider the linear form $f \mapsto f'(\phi')$, with the pair (ϕ, s) being corresponding to (G', ϕ'). This linear form, which by construction depends only on ϕ and $s \in \overline{S}_\phi$, in fact depends only on ϕ and x_s (where x_s is as above the image of s in $\mathcal{S}_\phi$). We will denote this linear form as $f \mapsto f'(\phi, s)$ or $f'(\phi^s)$, and also as $f \mapsto f'(\phi, x)$ or $f \mapsto f'(\phi^x)$, i.e. for $x \in \mathcal{S}_\phi$:

$$
\begin{aligned}
f'(\phi, x) &= f'(\phi, s) \\
f'(\phi^x) &= f'(\phi^s)
\end{aligned}
$$

for any semi-simple $s \in \overline{S}_\phi$ such that $x_s = x$.

Shelstad had also constructed the adjoint spectral transfer factor $\Delta(\Pi, \phi^s)$ and established the inversion formula [15]: for $\Pi \in \Pi_\phi$, one has:

$$
\operatorname{tr} \Pi(f) = \sum_{x \in \mathcal{S}_\phi} \Delta(\Pi, \phi^x) f'(\phi, x).
$$

Here on the right hand side, for the adjoint spectral transfer factor, the dependence of $\Delta(\Pi, \phi^s)$ on s again factors through the image x_s of s in $\mathcal{S}_\phi$, and so for $x \in \mathcal{S}_\phi$, we denote by $\Delta(\Pi, \phi^x)$ the value $\Delta(\Pi, \phi^s)$, for any semi-simple $s \in \overline{S}_\phi$ such that $x_s = x$.

For our purpose, we need to construct the transfer factors $\Delta(\tau, \phi^s)$ and $\Delta(\phi^s, \tau)$, in the spirit of section 5 of [3]; again these depends only on the image x_s of s in $\mathcal{S}_\phi$, and so we similarly employ the notations $\Delta(\tau, \phi^x)$ and $\Delta(\phi^x, \tau)$. If $\phi \in \Phi_{\mathrm{ell}}(G, \zeta)$ is elliptic, then the transfer factors $\Delta(\tau, \phi^s)$ are defined in [12]:

$$
\Delta(\tau, \phi^s) = \sum_{\chi \in \widehat{R}_\pi} \chi(r) \Delta(\Pi^\chi, \phi^s)
$$

where $\tau = (M, \pi, r) \in T_\phi, s \in \overline{S}_\phi$ semi-simple, and Π^χ is the irreducible component of induced representation of π whose character corresponds to that of χ (here recall that R_π is an abelian elementary 2-group, so $\chi \in \widehat{R}_\pi$ takes values in $\{\pm 1\}$). We have the spectral transfer relation (*c.f.* equation (5.8) in [12]):

$$
(5.1) \qquad \Theta(\tau, f) = \sum_{x \in \mathcal{S}_\phi} \Delta(\tau, \phi^x) f'(\phi, x).
$$

We also define the adjoint transfer factor (still assuming $\phi \in \Phi_{\mathrm{ell}}(G, \zeta)$ being elliptic, and $\tau = (M, \pi, r) \in T_\phi$ as before):

$$
\Delta(\phi^s, \tau) = \sum_{\chi \in \widehat{R}_\pi} \frac{1}{|R_\pi|} \chi(r) \Delta(\phi^s, \Pi^\chi).
$$

We then have the spectral transfer formula: for semi-simple $s \in \overline{S}_\phi$, one has (*c.f.* equation (5.9) in [12]):

$$
f'(\phi, s) = \sum_{\tau \in T_\phi} \Delta(\phi^s, \tau) \Theta(\tau, f),
$$

i.e.

$$(5.2) \qquad f'(\phi, x) = \sum_{\tau \in T_\phi} \Delta(\phi^x, \tau) \Theta(\tau, f)$$

for $x \in \mathcal{S}_\phi$.

We have the following lemma.

LEMMA 5.3. *Suppose that $\phi \in \Phi(G, \zeta)$ is elliptic. Let M be Levi subgroup of G such that ϕ factors through $^L M$ as a square-integrable parameter $\phi_M \in \Phi_2(M, \zeta)$ for M, under which we have the split short exact sequence as in (4.1):*

$$0 \longrightarrow \mathcal{S}_{\phi_M} \longrightarrow \mathcal{S}_\phi \longrightarrow R_\phi \longrightarrow 0.$$

Given $x \in \mathcal{S}_\phi$, write x with respect to the above split exact sequence as $x = x_M \cdot r'$, where $x_M \in \mathcal{S}_{\phi_M}$, and $r' \in R_\phi$. Then for $\tau = (M, \pi, r) \in T_\phi$, we have $\Delta(\phi^x, \tau) = 0$ unless $r = r'$, in which case we have:

$$\Delta(\phi^x, \tau) = \Delta(\phi_M^{x_M}, \pi)$$

where the transfer factor on the right hand side is with respect to M.

Proof. This is a direct computation, using the properties of spectral transfer factors [15]:

$$
\begin{aligned}
(5.3) \qquad \Delta(\phi^x, \tau) &= \sum_{\chi \in \widehat{R}_\pi} \frac{1}{|R_\pi|} \chi(r) \Delta(\phi^x, \Pi^\chi) \\
&= \sum_{\chi \in \widehat{R}_\pi} \frac{1}{|R_\pi|} \chi(r) \chi(r') \Delta(\phi^{x_M}, \Pi^\chi) \\
&= \delta(r, r') \cdot \Delta(\phi_M^{x_M}, \pi)
\end{aligned}
$$

where $\delta(\cdot, \cdot)$ is the Kronecker delta function. The lemma follows. $\qquad\square$

REMARK 5.4. This property of spectral transfer factor is also studied in Chapters 2 and 6 of [9], in the form of local intertwining relations.

We now define the spectral transfer factors for general $\phi \in \Phi(G, \zeta)$, by reducing to the elliptic case. If $\phi \in \Phi(G, \zeta)$, then there exists a Levi subgroup $\widetilde{M}$ (unique up to conjugacy), that is maximal with respect to the property that ϕ factors through $^L \widetilde{M}$ as an elliptic parameter $\phi^{\widetilde{M}}$ for $\widetilde{M}$. With respect to the embedding $^L \widetilde{M} \hookrightarrow {}^L G$, we have the canonical isomorphism

$$\mathcal{S}_{\phi^{\widetilde{M}}} \overset{\sim}{\longrightarrow} \mathcal{S}_\phi$$

(*c.f.* proof of Lemma 4.1).

If $\tau = (M, \pi, r) \in T_\phi$, then we denote by $\tau^{\widetilde{M}} = (M, \pi, r^{\widetilde{M}}) \in T_{\phi^{\widetilde{M}}}$ the corresponding element of $T_{\phi^{\widetilde{M}}}$. Here $r^{\widetilde{M}}$ corresponds to r under the canonical isomorphism $R_{\phi^{\widetilde{M}}} = R_\phi$ (*c.f.* proof of Lemma 4.1).

We define the spectral transfer factor:

$$\Delta(\tau, \phi^x) = \Delta(\tau^{\widetilde{M}}, (\phi^{\widetilde{M}})^{x^{\widetilde{M}}})$$

where on the right hand side, we denote by $x^{\widetilde{M}}$ the element in $\mathcal{S}_{\phi^{\widetilde{M}}}$ that corresponds to $x \in \mathcal{S}_\phi$ under the above canonical isomorphism between $\mathcal{S}_{\phi^{\widetilde{M}}}$ and $\mathcal{S}_\phi$.

We have, with $f_{\widetilde{M}}$ being the descent of f to $\widetilde{M}$, the identity:

$$\Theta(\tau, f) = \Theta(\tau^{\widetilde{M}}, f_{\widetilde{M}}).$$

Similarly we have, for $x \in \mathcal{S}_\phi$:

$$f'(\phi, x) = (f_{\widetilde{M}})'(\phi^{\widetilde{M}}, x^{\widetilde{M}}).$$

It follows that we again have the transfer relation (generalization of (5.1)):

$$(5.4) \qquad \Theta(\tau, f) = \sum_{x \in \mathcal{S}_\phi} \Delta(\tau, \phi^x) f'(\phi, x).$$

We can similarly define the adjoint transfer factor $\Delta(\phi^x, \tau)$, and we have the inverse transfer relation, for $x \in \mathcal{S}_\phi$ (generalization of (5.2)):

$$(5.5) \qquad f'(\phi, x) = \sum_{\tau \in T_\phi} \Delta(\phi^x, \tau) \Theta(\tau, f).$$

We also have the following properties about the transfer factors:

LEMMA 5.5. *Let $\phi \in \Phi(G, \zeta)$ be a bounded Langlands parameter. Then we have:*

- $\Delta(\tau, \phi^x) = \frac{|R_\phi|}{|\mathcal{S}_\phi|} \overline{\Delta(\phi^x, \tau)} = \frac{|R_\phi|}{|\mathcal{S}_\phi|} \Delta(\phi^x, \tau)$ *for $\tau \in T_\phi$ and $x \in \mathcal{S}_\phi$.*
- *we have the adjoint relations*

$$\sum_{\tau \in T_\phi} \Delta(\phi^{x_1}, \tau) \Delta(\tau, \phi^{x_2}) = \delta(x_1, x_2), \text{ for } x_1, x_2 \in \mathcal{S}_\phi,$$

$$\sum_{x \in \mathcal{S}_\phi} \Delta(\tau_1, \phi^x) \Delta(\phi^x, \tau_2) = \delta(\tau_1, \tau_2), \text{ for } \tau_1, \tau_2 \in T_\phi.$$

Where $\delta(\cdot, \cdot)$ is the Kronecker delta function.

This follows from the case where ϕ is an elliptic parameter, which is established in Proposition 5.2 of [12].

Finally to complete the discussion of this section, for $\tau \in T(G, \zeta), \phi \in \Phi(G, \zeta)$ and $x \in \mathcal{S}_\phi$, we define $\Delta(\tau, \phi^x)$ and $\Delta(\phi^x, \tau)$ to be zero if $\tau \notin T_\phi$.

6. STABILIZATION OF THE LOCAL TRACE FORMULA

To summarize the discussion of the previous section, we have firstly, from equation (5.4) and (5.5), the following:

THEOREM 6.1. *If* $f \in \mathcal{H}(G(\mathbb{R}), \zeta)$, *then for* $\tau \in T(G, \zeta)$, *we have:*

$$(6.1) \qquad \Theta(\tau, f) = \sum_{(\phi, x)} \Delta(\tau, \phi^x) f'(\phi, x)$$

where the summation is over $\phi \in \Phi(G, \zeta)$ *and* $x \in \mathcal{S}_\phi$.
Conversely, for $\phi \in \Phi(G, \zeta), x \in \mathcal{S}_\phi$, *we have:*

$$(6.2) \qquad f'(\phi, x) = \sum_{\tau \in T(G, \zeta)} \Delta(\phi^x, \tau) \Theta(\tau, f).$$

The following also follows easily:

LEMMA 6.2. *The statement of Lemma 5.3 holds for general* $\phi \in \Phi(G, \zeta)$.

Now for $\phi \in \Phi(G, \zeta)$, define $T_{\phi, \mathrm{disc}} := T_\phi \cap T_{\mathrm{disc}}(G, \zeta)$. Recall that as a consequence of Lemma 4.1 (*c.f.* discussion in the paragraph after equation (4.4)), we have that, if $\tau \in T_{\phi, \mathrm{disc}}$, then one must have $\phi \in \Phi_{\mathrm{disc}}(G, \zeta)$. Also from Lemma 4.1, one has the bijection $\iota : T_\phi \to \mathcal{S}_\phi$ that respects the projection to R_ϕ. Thus for $\phi \in \Phi_{\mathrm{disc}}(G, \zeta)$, put $\mathcal{S}_{\phi, \mathrm{disc}}$ to be the image of $T_{\phi, \mathrm{disc}}$ under the bijection $\iota : T_\phi \to \mathcal{S}_\phi$. We define:

$$\Phi^S_{\mathrm{disc}}(G, \zeta) = \{(\phi, x) : \phi \in \Phi_{\mathrm{disc}}(G, \zeta), x \in \mathcal{S}_{\phi, \mathrm{disc}}\}.$$

Thus for $\tau \in T_{\mathrm{disc}}(G, \zeta)$, on applying equation (6.1) and the above discussion, we have the transfer relation:

$$(6.3) \qquad \Theta(\tau, f) = \sum_{(\phi, x) \in \Phi^S_{\mathrm{disc}}(G, \zeta)} \Delta(\tau, \phi^x) f'(\phi, x).$$

Similarly for $\phi \in \Phi_{\mathrm{disc}}(G, \zeta)$ and $x \in \mathcal{S}_{\phi, \mathrm{disc}}$, we have, on applying equation (6.2), the inversion formula:

$$(6.4) \qquad f'(\phi, x) = \sum_{\tau \in T_{\mathrm{disc}}(G, \zeta)} \Delta(\phi^x, \tau) \Theta(\tau, f).$$

In the following, we also denote a pair $(\phi, x) \in \Phi^S_{\mathrm{disc}}(G, \zeta)$ as ϕ^x.

The linear space $i\mathfrak{a}^*_{G,Z}$ acts on $\Phi_{\mathrm{disc}}(G, \zeta)$ through twisting: for $\phi \in \Phi_{\mathrm{disc}}(G, \zeta)$ and $\lambda \in i\mathfrak{a}^*_{G,Z}$

$$\phi_\lambda(\omega) := \phi(\omega)|\omega|^\lambda, \quad \omega \in W_\mathbb{R}$$

(here the element $|\omega|^\lambda \in Z(\widehat{G})^{\Gamma_\mathbb{R}}$ is defined via the usual Nakayama-Tate duality). This induces the action of $i\mathfrak{a}^*_{G,Z}$ on $\Phi^S_{\mathrm{disc}}(G, \zeta)$, through twisting on $\Phi_{\mathrm{disc}}(G, \zeta)$: $(\phi^x)_\lambda := (\phi_\lambda)^x$; here we are identifying $\overline{S}_\phi$ and $\overline{S}_{\phi_\lambda}$, hence the identification for $\mathcal{S}_\phi$ and $\mathcal{S}_{\phi_\lambda}$, similarly the identification for $\mathcal{S}_{\phi, \mathrm{disc}}$ and $\mathcal{S}_{\phi_\lambda, \mathrm{disc}}$.

We define a measure on $\Phi_{\mathrm{disc}}(G, \zeta)$ and $\Phi_{\mathrm{disc}}^S(G, \zeta)$ by setting:

$$\int_{\Phi_{\mathrm{disc}}(G,\zeta)} \beta_1(\phi)d\phi = \sum_{\phi \in \Phi_{\mathrm{disc}}(G,\zeta)/i\mathfrak{a}_{G,Z}^*} \int_{i\mathfrak{a}_{G,Z}^*} \beta_1(\phi_\lambda)d\lambda,$$

$$\int_{\Phi_{\mathrm{disc}}^S(G,\zeta)} \beta_2(\phi^x)d\phi^x = \sum_{(\phi,x) \in \Phi_{\mathrm{disc}}^S(G,\zeta)/i\mathfrak{a}_{G,Z}^*} \int_{i\mathfrak{a}_{G,Z}^*} \beta_2(\phi_\lambda^x)d\lambda,$$

for any $\beta_1 \in C_c(\Phi_{\mathrm{disc}}(G,\zeta)), \beta_2 \in C_c(\Phi_{\mathrm{disc}}^S(G,\zeta))$. We have the following lemma, similar to Lemma 5.3 of [3]:

LEMMA 6.3. *Suppose that $\alpha \in C_c(T_{\mathrm{disc}}(G,\zeta))$, and $\beta \in C_c(\Phi_{\mathrm{disc}}^S(G,\zeta))$. Then we have*

$$(6.5) \qquad \int_{T_{\mathrm{disc}}(G,\zeta)} \sum_{(\phi,x) \in \Phi_{\mathrm{disc}}^S(G,\zeta)} \beta(\phi^x)\Delta(\phi^x,\tau)\alpha(\tau)d\tau$$

$$= \int_{\Phi_{\mathrm{disc}}^S(G,\zeta)} \sum_{\tau \in T_{\mathrm{disc}}(G,\zeta)} \beta(\phi^x)\Delta(\phi^x,\tau)\alpha(\tau)d\phi^x.$$

Proof. First note that for given a τ, the first inner summation is a finite sum, by the definition of the transfer factors; similarly for a given ϕ^x, the second summation is also finite. So the identity makes sense. From the definition of measure for $T_{\mathrm{disc}}(G,\zeta)$, we can write the left hand side of (6.5) as:

$$(6.6)$$

$$\sum_{\tau \in T_{\mathrm{disc}}(G,\zeta)/i\mathfrak{a}_{G,Z}^*} \sum_{\phi^x \in \Phi_{\mathrm{disc}}^S(G,\zeta)/i\mathfrak{a}_{G,Z}^*} \sum_{\mu \in i\mathfrak{a}_{G,Z}^*} \int_{i\mathfrak{a}_{G,Z}^*} \beta(\phi_\mu^x)\Delta(\phi_\mu^x,\tau_\lambda)\alpha(\tau_\lambda)d\lambda$$

For a given representative $\tau \in T_{\mathrm{disc}}(G,\zeta)/i\mathfrak{a}_{G,Z}^*$ in the outer sum, we may choose the representative $\phi^x \in \Phi^S(G,\zeta)/i\mathfrak{a}_{G,Z}^*$ in the inner sum, that has the same central character on A_G.

Thus in the sum over μ, we see that if $\lambda \neq \mu$, then ϕ_μ^x and τ_λ are not related, and so by the definition of transfer factors, we have $\Delta(\phi_\mu^x,\tau_\lambda) = 0$. So we can write (6.6) as:

$$(6.7) \qquad \sum_{(\tau,\phi^x)} \int_{i\mathfrak{a}_{G,Z}^*} \beta(\phi_\lambda^x)\Delta(\phi_\lambda^x,\tau_\lambda)\alpha(\tau_\lambda)d\lambda.$$

where double sum of (6.7) is a sum over the subset

$$(\tau,\phi^x) \in (T_{\mathrm{disc}}(G,\zeta) \times \Phi_{\mathrm{disc}}^S(G,\zeta))/i\mathfrak{a}_{G,Z}^*$$

consisting of those pairs that have the same central character on A_G.

In a parallel way, we can treat the right hand side of (6.5) along similar lines, using the definition of measure for $\Phi_{\mathrm{disc}}^S(G,\zeta)$. On noting the symmetry of the sum-integral in (6.7), we thus conclude that the left hand side and the right hand side of (6.5) are equal. $\qquad\square$

We can now begin the stabilization of the spectral side of the invariant local trace formula:

THEOREM 6.4. *If $f = f_1 \times \bar{f}_2$, $f_i \in \mathcal{H}(G(\mathbb{R}), \zeta)$ for $i = 1, 2$, then we have:*

$$(6.8) \qquad \begin{aligned} I_{\mathrm{disc}}^G(f) &= \int_{T_{\mathrm{disc}}(G,\zeta)} i^G(\tau)\Theta(\tau, f_1)\overline{\Theta(\tau, f_2)}|R_\pi|^{-1}d\tau \\ &= \int_{\Phi_{\mathrm{disc}}^S(G,\zeta)} \frac{1}{|\mathcal{S}_\phi|} i_\phi(x) f_1'(\phi, x)\overline{f_2'(\phi, x)}d\phi^x \end{aligned}$$

Proof. Firstly, applying the spectral transfer, as given in equation (6.3), to the term $\Theta(\tau, f_1)$ in (6.8), we see that $I_{\mathrm{disc}}^G(f)$ equals

$$(6.9) \quad \int_{T_{\mathrm{disc}}(G,\zeta)} i^G(\tau)|R_\pi|^{-1} \sum_{(\phi,x)\in\Phi_{\mathrm{disc}}^S(G,\zeta)} \Delta(\tau, \phi^x)f_1'(\phi, x)\overline{\Theta(\tau, f_2)}d\tau$$

$$= \int_{T_{\mathrm{disc}}(G,\zeta)} i^G(\tau)|R_\pi|^{-1} \sum_{(\phi,x)\in\Phi_{\mathrm{disc}}^S(G,\zeta)} \frac{|R_\phi|}{|\mathcal{S}_\phi|} f_1'(\phi, x)\overline{\Delta(\phi^x, \tau)\Theta(\tau, f_2)}d\tau$$

with the last equality follows from the first part of Lemma 5.5. Next, in order that $\Delta(\phi^x, \tau) \neq 0$, we must have $\tau \in T_\phi$ and hence $|R_\pi| = |R_\phi|$ in the integrand. Thus we see that the right hand side of (6.9) can be written as:

$$\int_{T_{\mathrm{disc}}(G,\zeta)} \sum_{(\phi,x)\in\Phi_{\mathrm{disc}}^S(G,\zeta)} \frac{i^G(\tau)}{|\mathcal{S}_\phi|} f_1'(\phi, x)\overline{\Delta(\phi^x, \tau)\Theta(\tau, f_2)}d\tau.$$

By Lemma 6.3, this is equal to:

$$(6.10) \int_{\Phi_{\mathrm{disc}}^S(G,\zeta)} \frac{1}{|\mathcal{S}_\phi|} f_1'(\phi, x) \sum_{\tau\in T_{\mathrm{disc}}(G,\zeta)} i^G(\tau)\overline{\Delta(\phi^x, \tau)\Theta(\tau, f_2)}d\phi^x.$$

Now, denote by $P : \mathcal{S}_\phi \to R_\phi$ for the surjective map from $\mathcal{S}_\phi$ to R_ϕ in the split short exact sequence (4.1). Given $\tau = (M, \pi, r) \in T_\phi$, and $\phi \in \Phi_{\mathrm{disc}}(G, \zeta)$, we have, by Lemma 6.2, that for $x \in \mathcal{S}_{\phi,\mathrm{disc}}$, if $P(x) \neq r$, then $\Delta(\phi^x, \tau) = 0$. On the other hand, if $\Delta(\phi^x, \tau) \neq 0$, then we must have $P(x) = r$, i.e. $\iota(\tau)$ and x have the same image in R_ϕ under the map P. Thus we have:

$$i^G(\tau) = i_\phi(\iota(\tau)) = i_\phi(x)$$

(*c.f.* the discussion after the statement of Theorem 4.2, near the end of Section 4). Thus we can write (6.10) as

$$\int_{\Phi_{\mathrm{disc}}^S(G,\zeta)} \frac{1}{|\mathcal{S}_\phi|} i_\phi(x) f_1'(\phi, x) \sum_{\tau\in T_{\mathrm{disc}}(G,\zeta)} \overline{\Delta(\phi^x, \tau)\Theta(\tau, f_2)}d\phi^x.$$

Applying equation (6.4), we obtain the required formula:

$$(6.11) \qquad I_{\mathrm{disc}}^{G}(f) = \int_{\Phi_{\mathrm{disc}}^{S}(G,\zeta)} \frac{1}{|\mathcal{S}_{\phi}|} i_{\phi}(x) f_1'(\phi,x) \overline{f_2'(\phi,x)} d\phi^x.$$

$\square$

We set $i_{\phi}(x) = 0$, if $x \notin \mathcal{S}_{\phi,\mathrm{disc}}$. Then we have:

$$I_{\mathrm{disc}}^{G}(f) = \int_{\Phi_{\mathrm{disc}}(G,\zeta)} \sum_{x \in \mathcal{S}_{\phi}} |\mathcal{S}_{\phi}|^{-1} i_{\phi}(x) f_1'(\phi,x) \overline{f_2'(\phi,x)} d\phi.$$

Recall equation (4.4) from section 4, we write:

$$i_{\phi}(x) = \sum_{s \in \mathcal{E}_{\phi,\mathrm{ell}}(x)} |\pi_0(\overline{S}_{\phi,s})|^{-1} \sigma(\overline{S}_{\phi,s}^{\circ}).$$

Thus we obtain

$$I_{\mathrm{disc}}^{G}(f)$$
$$= \int_{\Phi_{\mathrm{disc}}(G,\zeta)} \sum_{x \in \mathcal{S}_{\phi}} |\mathcal{S}_{\phi}|^{-1} \sum_{s \in \mathcal{E}_{\phi,\mathrm{ell}}(x)} |\pi_0(\overline{S}_{\phi,s})|^{-1} \sigma(\overline{S}_{\phi,s}^{\circ}) f_1'(\phi,s) \overline{f_2'(\phi,s)} d\phi$$

in other words,

(6.12)

$$I_{\mathrm{disc}}^{G}(f) = \int_{\Phi_{\mathrm{disc}}(G,\zeta)} \sum_{s \in \mathcal{E}_{\phi,\mathrm{ell}}} |\mathcal{S}_{\phi}|^{-1} |\pi_0(\overline{S}_{\phi,s})|^{-1} \sigma(\overline{S}_{\phi,s}^{\circ}) f_1'(\phi,s) \overline{f_2'(\phi,s)} d\phi.$$

This is Theorem 1.1 as stated in the Introduction. In the next section, we are going to express the right hand side of (6.12) in the form of an endoscopic local trace formula.

7. Spectral side of endoscopic local trace formula

To obtain Theorem 1.2 as stated in the Introduction, we need to rewrite the right hand side of (6.12), in terms of the endoscopic data of G. To do this we need to make precise the correspondence $(G', \phi') \longleftrightarrow (\phi, s)$.

Denote by $E_{\mathrm{ell}}(G)$ for the set of elliptic endoscopic data of G. The set $\widehat{G} \backslash\!\backslash E_{\mathrm{ell}}(G)$ of $\widehat{G}$-orbits in $E_{\mathrm{ell}}(G)$ is then equal to the set $\mathcal{E}_{\mathrm{ell}}(G)$ of equivalence classes of elliptic endoscopic data of G.

Write $F_{\mathrm{disc}}(G,\zeta)$ for the set of bounded Langlands parameters ϕ of G that have character ζ with respect to Z (*not* being regarded as up to $\widehat{G}$-equivalence), and such that $Z(\overline{S}_{\phi})$ is finite. Similarly for $G' \in E_{\mathrm{ell}}(G)$, write $F_{s-\mathrm{disc}}(G',\zeta)$ for the set of bounded Langlands parameters ϕ' of G' that have character ζ with respect to Z (*not* being regarded as up to $\widehat{G'}$-equivalence), and such that $Z(\overline{S}_{\phi'}^{\circ})$ is finite.

The set $\widehat{G} \backslash\!\backslash F_{\mathrm{disc}}(G,\zeta)$ of $\widehat{G}$-orbits in $F_{\mathrm{disc}}(G,\zeta)$ is equal to $\Phi_{\mathrm{disc}}(G,\zeta)$. Similarly, for $G' \in E_{\mathrm{ell}}(G)$, the set $\widehat{G'} \backslash\!\backslash F_{s-\mathrm{disc}}(G',\zeta)$ of $\widehat{G'}$-orbits in $F_{s-\mathrm{disc}}(G',\zeta)$ is equal to $\Phi_{s-\mathrm{disc}}(G',\zeta)$.

We then define:
$$X_{\mathrm{disc}}(G,\zeta) := \{(\phi,s) : \phi \in F_{\mathrm{disc}}(G,\zeta), s \in \overline{S}_{\phi,\mathrm{ell}}\},$$
and
$$Y_{\mathrm{disc}}(G,\zeta) := \{(G',\phi') : G' \in E_{\mathrm{ell}}(G), \phi' \in F_{s-\mathrm{disc}}(G',\zeta)\}.$$
The group $\widehat{G}$ acts on $X_{\mathrm{disc}}(G,\zeta)$ and $Y_{\mathrm{disc}}(G,\zeta)$ by conjugation. Denote by $\widehat{G} \backslash\!\backslash X_{\mathrm{disc}}(G,\zeta)$ and $\widehat{G} \backslash\!\backslash Y_{\mathrm{disc}}(G,\zeta)$ for the set of $\widehat{G}$-orbits.

As in the previous section, the linear space $i\mathfrak{a}^*_{G,Z}$ acts on $\widehat{G} \backslash\!\backslash X_{\mathrm{disc}}(G,\zeta)$ and $\widehat{G} \backslash\!\backslash Y_{\mathrm{disc}}(G,\zeta)$ by twisting, and we define measures on $\widehat{G} \backslash\!\backslash X_{\mathrm{disc}}(G,\zeta)$ and $\widehat{G} \backslash\!\backslash Y_{\mathrm{disc}}(G,\zeta)$ by the same type of formula that define the measure on $\Phi_{\mathrm{disc}}(G,\zeta) = \widehat{G} \backslash\!\backslash F_{\mathrm{disc}}(G,\zeta)$, i.e.

$$\int_{\widehat{G}\backslash\!\backslash X_{\mathrm{disc}}(G,\zeta)} = \sum_{\widehat{G}\backslash\!\backslash X_{\mathrm{disc}}(G,\zeta)/i\mathfrak{a}^*_{G,Z}} \int_{i\mathfrak{a}^*_{G,Z}}$$

$$\int_{\widehat{G}\backslash\!\backslash Y_{\mathrm{disc}}(G,\zeta)} = \sum_{\widehat{G}\backslash\!\backslash Y_{\mathrm{disc}}(G,\zeta)/i\mathfrak{a}^*_{G,Z}} \int_{i\mathfrak{a}^*_{G,Z}}$$

The correspondence $(\phi,s) \longleftrightarrow (G',\phi')$ induces the bijections:

(7.1)
$$\widehat{G} \backslash\!\backslash X_{\mathrm{disc}}(G,\zeta) \quad \longleftrightarrow \quad \widehat{G} \backslash\!\backslash Y_{\mathrm{disc}}(G,\zeta)$$
$$\widehat{G} \backslash\!\backslash X_{\mathrm{disc}}(G,\zeta)/i\mathfrak{a}^*_{G,Z} \quad \longleftrightarrow \quad \widehat{G} \backslash\!\backslash Y_{\mathrm{disc}}(G,\zeta)/i\mathfrak{a}^*_{G,Z}$$

which is immediately seen to be measure preserving, and which is the focal point for the transformation of the right hand side of the expression (6.12). The argument we give below is parallel to that of Section 4.4 of [9].

The first step is to change the double sum-integral on the right hand side of (6.12) to an integral over $\widehat{G} \backslash\!\backslash X_{\mathrm{disc}}(G,\zeta)$, using that $\Phi_{\mathrm{disc}}(G,\zeta) = \widehat{G} \backslash\!\backslash F_{\mathrm{disc}}(G,\zeta)$, and also that the integrand is $\widehat{G}$-invariant. Given $\phi \in F_{\mathrm{disc}}(G,\zeta)$, the stabilizer of ϕ in $\widehat{G}$ is the centralizer S_ϕ. Now the sum occurring in the integrand on the right hand side of (6.12), is over $\mathcal{E}_{\phi,\mathrm{ell}} = \overline{S}^\circ_\phi \backslash\!\backslash \overline{S}_{\phi,\mathrm{ell}}$, the set of orbits in $\overline{S}_{\phi,\mathrm{ell}}$ under the conjugation action by the identity component $\overline{S}^\circ_\phi$ of $\overline{S}_\phi$. On the other hand, given $s \in \overline{S}_{\phi,\mathrm{ell}}$, the set of orbits under the conjugation action of S_ϕ, or equivalently by $\overline{S}_\phi$, is bijective with the quotient of $\overline{S}_\phi$ by the subgroup
$$\overline{S}^+_{\phi,s} = \mathrm{Cent}(s, \overline{S}_\phi).$$
However, the $\overline{S}^\circ_\phi$-orbit of s is bijective with the quotient of $\overline{S}^\circ_\phi$ by the subgroup
$$\overline{S}_{\phi,s} = \mathrm{Cent}(s, \overline{S}^\circ_\phi).$$
We can therefore rewrite the sum-integral on the right hand side of (6.12), as an integral over $\widehat{G} \backslash\!\backslash X_{\mathrm{disc}}(G,\zeta)$, if we multiply the summand on the right hand side of (6.12) by the number:
$$|\overline{S}^\circ_\phi/\overline{S}_{\phi,s}|^{-1}|\overline{S}_\phi/\overline{S}^+_{\phi,s}| = |\overline{S}^+_{\phi,s}/\overline{S}_{\phi,s}|^{-1}|\overline{S}_\phi/\overline{S}^\circ_\phi|,$$

which is to say the number:

$$(7.2) \qquad |\overline{S}^+_{\phi,s}/\overline{S}_{\phi,s}|^{-1}|\mathcal{S}_\phi|.$$

In the second step, we use the bijection (7.1) and write the integral over $\widehat{G} \setminus\!\!\setminus X_{\mathrm{disc}}(G,\zeta)$ as an integral over $\widehat{G} \setminus\!\!\setminus Y_{\mathrm{disc}}(G,\zeta)$. Recall that:

$$\mathcal{E}_{\mathrm{ell}}(G) = \widehat{G} \setminus\!\!\setminus E_{\mathrm{ell}}(G).$$

The stabilizer of a given $G' \in E_{\mathrm{ell}}(G)$ in $\widehat{G}$ is the group $\mathrm{Aut}_G(G')$. This means that the integral over $\widehat{G} \setminus\!\!\setminus Y_{\mathrm{disc}}(G,\zeta)$ could be written as a double sum-integral:

$$\sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \int_{\mathrm{Aut}_G(G') \setminus\!\!\setminus F_{s-\mathrm{disc}}(G',\zeta)}.$$

Now the integral over $\phi' \in \mathrm{Aut}_G(G') \setminus\!\!\setminus F_{s-\mathrm{disc}}(G',\zeta)$, could be replaced by the integral over $G' \setminus\!\!\setminus F_{s-\mathrm{disc}}(G',\zeta) = \Phi_{s-\mathrm{disc}}(G',\zeta)$, so long as we multiply the integrand by the number:

$$(7.3) \qquad |\mathrm{Out}_G(G')|^{-1}|\mathrm{Out}_G(G',\phi')|$$

where $\mathrm{Out}_G(G',\phi')$ is the stabilizer of ϕ' in $\mathrm{Out}_G(G')$.

We have thus established that the double sum-integral on the right hand side of (6.12), can be replaced by the double sum-integral:

$$\sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \int_{\Phi_{s-\mathrm{disc}}(G',\zeta)}$$

provided that the summand is multiplied by the product of the two numbers (7.2) and (7.3). Finally, the coefficient occurring in the summand on the right hand side of (6.12) is:

$$(7.4) \qquad |\mathcal{S}_\phi|^{-1}|\pi_0(\overline{S}_{\phi,s})|^{-1}\sigma(\overline{S}^\circ_{\phi,s}).$$

Thus we have:

$$I^G_{\mathrm{disc}}(f) = \sum_{G' \in \mathcal{E}_{\mathrm{ell}}(G)} \int_{\Phi_{s-\mathrm{disc}}(G',\zeta)} (7.2) \cdot (7.3) \cdot (7.4) \cdot f'_1(\phi,s)\overline{f'_2(\phi,s)}d\phi'.$$

We thus need to express the product of (7.2), (7.3), and (7.4), in terms of the pair (G',ϕ').

The product of (7.2), (7.3) and (7.4) is equal to the product of:

$$|\mathrm{Out}_G(G')|^{-1}$$

and

$$(7.5) \qquad |\mathrm{Out}_G(G',\phi')||\overline{S}^+_{\phi,s}/\overline{S}_{\phi,s}|^{-1}|\pi_0(\overline{S}_{\phi,s})|^{-1}\sigma(\overline{S}^\circ_{\phi,s}).$$

Now under the correspondence $(G', \phi') \longleftrightarrow (\phi, s)$, we have:

$$
\begin{aligned}
|\operatorname{Out}_G(G', \phi')| &= |S_{\phi,s}^+/S_{\phi,s}^+ \cap \widehat{G'} Z(\widehat{G})^{\Gamma_\mathbb{R}}| \\
&= |\overline{S}_{\phi,s}^+/\overline{S}_{\phi,s}^+ \cap \overline{\widehat{G'}}|
\end{aligned}
$$

where $\overline{\overline{G'}}$ denote the quotient

$$
\widehat{G'} Z(\widehat{G})^{\Gamma_\mathbb{R}}/Z(\widehat{G})^{\Gamma_\mathbb{R}} \cong \widehat{G'}/\widehat{G'} \cap Z(\widehat{G})^{\Gamma_\mathbb{R}}.
$$

Also

$$
\sigma(\overline{S}_{\phi'}^\circ) = \sigma((S_{\phi'}/Z(\widehat{G'})^{\Gamma_\mathbb{R}})^\circ) = \sigma(\overline{S}_{\phi,s}^\circ/\overline{S}_{\phi,s}^\circ \cap \overline{Z}(\widehat{G'})^{\Gamma_\mathbb{R}})
$$

where

$$
\overline{Z}(\widehat{G'})^{\Gamma_\mathbb{R}} = Z(\widehat{G'})^{\Gamma_\mathbb{R}}/Z(\widehat{G})^{\Gamma_\mathbb{R}}.
$$

We thus have the identity

$$
(7.6) \qquad \sigma(\overline{S}_{\phi'}^\circ) = \sigma(\overline{S}_{\phi,s}^\circ)|\overline{S}_{\phi,s}^\circ \cap \overline{Z}(\widehat{G'})^{\Gamma_\mathbb{R}}|
$$

by Theorem 4.2.

The term

$$
|\pi_0(\overline{S}_{\phi,s})|^{-1}|\overline{S}_{\phi,s}^\circ \cap \overline{Z}(\widehat{G'})^{\Gamma_\mathbb{R}}|^{-1}
$$

equals

$$
(7.7) \qquad |\overline{S}_{\phi,s}/\overline{S}_{\phi,s}^\circ \overline{Z}(\widehat{G'})^{\Gamma_\mathbb{R}}|^{-1}|\overline{Z}(\widehat{G'})^{\Gamma_\mathbb{R}}|^{-1}
$$

as can be seen by using

$$
\overline{S}_{\phi,s}^\circ/\overline{S}_{\phi,s}^\circ \cap \overline{Z}(\widehat{G'})^{\Gamma_\mathbb{R}} \cong \overline{S}_{\phi,s}^\circ \overline{Z}(\widehat{G'})^{\Gamma_\mathbb{R}}/\overline{Z}(\widehat{G'})^{\Gamma_\mathbb{R}}.
$$

Finally we can write:

$$
\begin{aligned}
(7.8) \qquad |\mathcal{S}_{\phi'}| &= |\pi_0(\overline{S}_{\phi'})| \\
&= |\overline{S}_{\phi,s}^+ \cap \overline{\overline{G'}}/(\overline{S}_{\phi,s}^+)^\circ \overline{Z}(\widehat{G'})^{\Gamma_\mathbb{R}}| \\
&= |\overline{S}_{\phi,s}^+ \cap \overline{\overline{G'}}/\overline{S}_{\phi,s}^\circ \overline{Z}(\widehat{G'})^{\Gamma_\mathbb{R}}|
\end{aligned}
$$

where the last equality follows on noting that

$$
(7.9) \qquad (\overline{S}_{\phi,s}^+)^\circ = \overline{S}_{\phi,s}^\circ.
$$

Thus (7.5) is equal to:

$$
|\overline{Z}(\widehat{G'})^{\Gamma}|^{-1}|\mathcal{S}_{\phi'}|^{-1}\sigma(\overline{S}_{\phi'}^\circ).
$$

Finally, under the correspondence $(G', \phi') \longleftrightarrow (\phi, s)$, we have

$$
f_1'(\phi, s)\overline{f_2'(\phi, s)} = f_1^{G'}(\phi')\overline{f_2^{G'}(\phi')}.
$$

We thus obtain the following main theorem:

THEOREM 7.1. *If $f = f_1 \times \bar{f}_2$, $f_i \in \mathcal{H}(G(\mathbb{R}), \zeta), i = 1, 2$, then we have*

(7.10)
$$I_{\text{disc}}^G(f) = \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \cdot \int_{\Phi_{s-\text{disc}}(G', \zeta)} |\mathcal{S}_{\phi'}|^{-1} \sigma(\overline{S}_{\phi'}^{\circ}) f_1^{G'}(\phi') \overline{f_2^{G'}(\phi')} d\phi'$$

where

$$\iota(G, G') = |\operatorname{Out}_G(G')|^{-1} |\overline{Z}(\widehat{G'})^{\Gamma_\mathbb{R}}|^{-1}.$$

Now we define, for any quasi-split K-group G over $\mathbb{R}$, the following stable distribution for G:

$$S_{\text{disc}}^G(f) := \int_{\Phi_{s-\text{disc}}(G, \zeta)} |\mathcal{S}_\phi|^{-1} \sigma(\overline{S}_\phi^{\circ}) f_1^G(\phi) \overline{f_2^G(\phi)} d\phi$$

where as before $f = f_1 \times \bar{f}_2$, $f_1, f_2 \in \mathcal{H}(G(\mathbb{R}), \zeta)$. In particular one has the stable distribution $S_{\text{disc}}^{G'}$ for G', with G' being an endoscopic datum of G.

Still with $f = f_1 \times \bar{f}_2$, $f_1, f_2 \in \mathcal{H}(G(\mathbb{R}), \zeta)$, and G' an endoscopic datum of G, one has functions $f_1', f_2' \in \mathcal{H}(G', \zeta)$ such that the stable orbital integrals of f_i' is equal to the Langlands-Shelstad transfer $f_i^{G'}$ ($i = 1, 2$) [13]. Put $f^G := f_1^{G'} \times \bar{f}_2^{G'}$ and $f' := f_1' \times \bar{f}_2'$. Then since $S_{\text{disc}}^{G'}$ is stable for G', one has that the value $S_{\text{disc}}^{G'}(f')$ depends only on the stable orbital integral of f', i.e. only on the Langlands-Shelstad transfer $f^{G'}$. We thus denote the value $S_{\text{disc}}^{G'}(f')$ as $\widehat{S}_{\text{disc}}^{G'}(f^{G'})$.

With these notations we have:

$$\widehat{S}_{\text{disc}}^{G'}(f^{G'}) = \int_{\Phi_{s-\text{disc}}(G', \zeta)} |\mathcal{S}_{\phi'}|^{-1} \sigma(\overline{S}_{\phi'}^{\circ}) f_1^{G'}(\phi') \overline{f_2^{G'}(\phi')} d\phi'$$

hence in the context of Theorem 7.1, we can write equation (7.10) in the following form:

(7.11)
$$I_{\text{disc}}^G(f) = \sum_{G' \in \mathcal{E}_{\text{ell}}(G)} \iota(G, G') \widehat{S}_{\text{disc}}^{G'}(f^{G'}).$$

8. SPECTRAL SIDE OF STABLE LOCAL TRACE FORMULA

With Theorem 7.1 in hand, we can now obtain the explicit formula for the spectral side of the stable local trace formula. As before G is any K-group over $\mathbb{R}$ that is quasi-split, with central data (Z, ζ).

In [6], one has following stable distribution S^G for G, which is the stable version of the geometric side of the invariant local trace formula I^G, and is defined as follows. For $f = f_1 \times \bar{f}_2$, with $f_1, f_2 \in \mathcal{H}(G(\mathbb{R}), \zeta)$:

(8.1) $S^G(f)$

$$= \sum_{M \in \mathcal{L}} |W_0^M| |W_0^G|^{-1} (-1)^{\dim(A_M/A_G)} \int_{\Delta_{G-\text{reg,ell}}(M, V, \zeta)} n(\delta)^{-1} S_M^G(\delta, f) d\delta$$

c.f. equation (10.11) of [6]. Here $n(\delta)$ is the order of the group $\mathcal{K}_\delta$ as defined on p. 509 of [3], and $\Delta_{G-\mathrm{reg,ell}}(M,V,\zeta)$ is the stable version of $\Gamma_{G-\mathrm{reg,ell}}(M,V,\zeta)$, similarly $S_M^G(\delta,f)$ is the stable version of $I_M^G(\gamma,f)$.

In particular one has the stable distribution $S^{G'}$ for G', with G' being an endoscopic datum of G. It is then shown in [6] that, the geometric side of the local trace formula $I^G(f)$ for G (where $f = f_1 \times \bar{f}_2$, $f_1, f_2 \in \mathcal{H}(G(\mathbb{R}),\zeta)$ as above) satisfies the following endoscopic decomposition:

$$(8.2) \qquad I^G(f) = \sum_{G'\in\mathcal{E}_\mathrm{ell}(G)} \iota(G,G')\widehat{S}^{G'}(f^{G'})$$

where as before $f^{G'}$ is the Langlands-Shelstad transfer of f to G', *c.f.* equation (10.16) of [6]. Here the meaning of $\widehat{S}^{G'}(f^{G'})$ is similar to that of concerning $\widehat{S}^{G'}_\mathrm{disc}(f^{G'})$, *c.f.* the discussion near the end of Section 7.

We remark that, for the archimedean case of the local trace formula, the geometric transfer identities that are needed in [6] to establish the endoscopic decomposition (8.1)-(8.2), were established directly in [8], and so it is independent of global arguments (in the non-archimedean case, the endoscopic decomposition of the local trace formula was established in [6] using global arguments, and so in particular the Fundamental Lemma is needed in the non-archimedean case). We also remark that, when one of the components of f is cuspidal, then the arguments for the endoscopic decomposition (8.1)-(8.2) were already carried out in [4], Section 9 - 10.

THEOREM 8.1. *We have the stable local trace formula:*

$$S_\mathrm{disc}^G(f) = S^G(f)$$

where as before

$$S^G(f)$$
$$= \sum_{M\in\mathcal{L}} |W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)} \int_{\Delta_{G-\mathrm{reg,ell}}(M,V,\zeta)} n(\delta)^{-1}S_M^G(\delta,f)d\delta,$$

and

$$S_\mathrm{disc}^G(f) = \int_{\Phi_{s-\mathrm{disc}}(G,\zeta)} |\mathcal{S}_\phi|^{-1}\sigma(\overline{S}_\phi^\circ)f_1^G(\phi)\overline{f_2^G(\phi)}d\phi.$$

Proof. By induction on $\dim(G_\mathrm{der})$. Put $\mathcal{E}_\mathrm{ell}^\circ(G) = \mathcal{E}_\mathrm{ell}(G)\backslash\{G\}$. Applying equation (7.11) and equation (8.2), we have

$$S^G(f) = \widehat{S}^G(f^G) = I^G(f) - \sum_{G'\in\mathcal{E}_\mathrm{ell}^\circ(G)} \iota(G,G')\widehat{S}^{G'}(f^{G'})$$

and

$$S_\mathrm{disc}^G(f) = \widehat{S}_\mathrm{disc}^G(f^G) = I_\mathrm{disc}^G(f) - \sum_{G'\in\mathcal{E}_\mathrm{ell}^\circ(G)} \iota(G,G')\widehat{S}_\mathrm{disc}^{G'}(f^{G'}).$$

Now for $G' \in \mathcal{E}_{\mathrm{ell}}^{\circ}(G)$, one has $\dim(G'_{\mathrm{der}}) < \dim(G_{\mathrm{der}})$, so by the induction hypothesis, we have

$$\widehat{S}_{\mathrm{disc}}^{G'}(f^{G'}) = \widehat{S}^{G'}(f^{G'}).$$

Combining with the equality $I_{\mathrm{disc}}^{G}(f) = I^{G}(f)$, we have obtained the theorem.
□

Remark 8.2.

(1) The method and results of this paper could be extended to the case where G is a connected quasi-split reductive group over a p-adic field, as long as the local Langlands correspondence is known for G, and such that the R-groups R_{ϕ} and the component groups $\mathcal{S}_{\phi}$ attached to the Langlands parameters of G are all abelian. For instance the case of classical groups by the works [9, 11] (with a slight complication in the even orthogonal case). We leave it to the reader to formulate the corresponding results.

(2) For the geometric side of the stable local trace formula, the distributions $S_{M}^{G}(\delta, f)$ are defined inductively in terms of the invariant distributions $I_{M}^{G}(\gamma, f)$, which in turn is defined inductively by weighted orbital integrals, *c.f.* [4]. It is thus a highly non-trivial matter to obtain explicit formulas for the distributions $S_{M}^{G}(\delta, f)$, when $M \neq G$. In this regard, we refer the reader to [12], where the stable local trace formula is used to obtain explicit formulas for $S_{M}^{G}(\delta, f)$, in the case where $f = f_1 \times \bar{f}_2$, where f_2 is the stable pseudo-coefficient of a square integrable parameter of G.

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