On the Non Commutative Iwasawa Main Conjecture for Abelian Varieties over Function Fields

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ABSTRACT. We establish the Iwasawa main conjecture for semistable abelian varieties over a function field of characteristic p under certain restrictive assumptions. Namely we consider p-torsion free p-adic Lie extensions of the base field which contain the constant $\mathbb{Z}_p$-extension and are everywhere unramified. Under the usual $\mu = 0$ hypothesis, we give a proof which mainly relies on the interpretation of the Selmer complex in terms of p-adic cohomology [TV] together with the trace formulas of [EL1].

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1 Introduction

1.1 Statement of the main theorem

Consider a function field K of characteristic p, C the corresponding proper smooth geometrically irreducible curve over a finite field k and an abelian variety A/K with Néron model $\mathcal{A}/C$. Assume for simplicity that A has good reduction everywhere and that the Hasse-Weil L function of A/K does not vanish at $s = 1$. In this situation the BSD conjecture predicts that each group involved in the right hand term below is finite and that the following formula holds (where $\hat{A}$ denotes the dual abelian variety):

$$L(A/K, 1) = \frac{\#\text{Ш}(A/K)}{\#A(K)\#\hat{A}(K)} \cdot \frac{\#H^0(C, Lie(\mathcal{A}))}{\#H^1(C, Lie(\mathcal{A}))} \tag{1}$$

Let us restrict our attention to the p-adic valuations on both sides. There are natural perfect complexes of $\mathbb{Z}_p$-modules $R\Gamma(C, T_p(\mathcal{A}))$, $R\Gamma(C, Lie(\mathcal{A}))$ whose cohomology groups are the p-part of the groups appearing above. These complexes thus become acyclic after extension of scalars to $\mathbb{Q}_p$, hence produce classes in $K_0(\mathbb{Z}_p, \mathbb{Q}_p) \simeq \mathbb{Z}$. Looking at $L(A/K, 1)$ as an element of $K_1(\mathbb{Q}_p) \simeq \mathbb{Q}_p^{\times}$, the p-part of (1) simply becomes

$$\partial(L(A/K, 1)) = [R\Gamma(C, T_p(\mathcal{A}))] + [R\Gamma(C, Lie(\mathcal{A}))] \qquad (2)$$

where $\partial : K_1(\mathbb{Q}_p) \to K_0(\mathbb{Z}_p, \mathbb{Q}_p)$ is the connecting map of lower K-theory (ie. the p-adic valuation $v_p : \mathbb{Q}_p^{\times} \to \mathbb{Z}$). As explained in [KT] the complexes appearing on the right hand side can be related to p-adic (crystalline and then rigid) cohomology and the trace formulae which are known in this context are then sufficient to actually prove (2).

The main purpose of this paper is to establish a similar statement in the setting of non commutative Iwasawa theory. We don't assume that $L(A/K, 1) \neq 0$ anymore and A is now allowed to have semistable reduction at some given set Z of points of C. Consider a Galois extension K_{∞}/K which contains the constant $\mathbb{Z}_p$-extension Kk_{∞}/K, is unramified everywhere, and whose Galois group $G = \varprojlim_n G_n$ is p-adic Lie without p-torsion. In that situation, a general result explained in section 2 will allow us to form perfect complexes of modules over the Iwasawa algebra $\Lambda(G) := \varprojlim_n \mathbb{Z}_p[G_n]$:

$$\begin{aligned} N_{K_{\infty}} &:= & R\varprojlim_n R\Gamma^{Z_n}(C_n, T_p(\mathcal{A})) \\ L_{K_{\infty}} &:= & R\varprojlim_n R\Gamma^{Z_n}(C_n, Lie(\mathcal{A})) \end{aligned}$$

where $R\Gamma^{Z_n}(C_n, -)$ is the functor of cohomology vanishing at Z_n (see. section 4 for a precise definition and the relation to usual Selmer complexes) and is designed to take out the contribution of Z. A significant difference with the BSD statement is that here, the cohomology of these complexes are expected to be torsion $\Lambda(G)$-modules even if $L(A/K, 1) = 0$. This is well known if $K = Kk_{\infty}$ (see. Cor. 5.2 or [LLTT] Cor. 2.1.5). To go from this case to the general one, we follow the strategy of [CFKSV], which unfortunately requires an extra assumption, namely the usual $\mu = 0$ hypothesis. More precisely, we will prove the following generalization of [LLTT], where only the case $K = Kk_{\infty}$ was considered.

THEOREM 1.1. *(Thm. 5.15) Let A/K, K_{∞}/K and G as above. If the μ-invariant of the Pontrjagin dual of the discrete Selmer group of A over Kk_{∞} is trivial, then*

(i) *The cohomology $\Lambda(G)$-modules of $N_{K_{\infty}}$ and $L_{K_{\infty}}$ are S^*-torsion, where S^* is the canonical Ore set of [CFKSV]. The latter complexes thus produce classes $[N_{K_{\infty}}]$ and $[L_{K_{\infty}}]$ in $K_0(\Lambda(G), \Lambda(G)_{S^*})$ (denoted $K_0(\mathfrak{M}_H(G))$ in loc. cit.).*

(ii) *There exists a canonical element $\mathcal{L}_{A/K_\infty} \in K_1(\Lambda(G)_{S^*})$ satisfying*

$$\partial \mathcal{L}_{A/K_\infty} = [N_{K_\infty}] + [L_{K_\infty}] \tag{3}$$

where $\partial : K_1(\Lambda(G)_{S^*}) \to K_0(\Lambda(G), \Lambda(G)_{S^*})$ denotes the connecting map in lower K-theory.

(iii) *The element $\mathcal{L}_{A/K_\infty}$ verifies the interpolation property*

$$\rho(\mathcal{L}_{A/K_\infty}) = L(U, A/K, \rho^\vee, 1) \tag{4}$$

for each Artin $\mathcal{O}$-valued representation ρ of G ($\mathcal{O}/\mathbb{Z}_p$, a totally ramified extension), where $\rho(-) : K_1(\Lambda(G)_{S^*}) \to L^\times \cup \{0, \infty\}$ denotes the corresponding evaluation map, $(-)^\vee$ denotes the contragredient, and $L(U, A/K, \rho, s)$ is the ρ-twisted Hasse-Weil L-function of A/K without Euler factors at $Z = C - U$.

We also give a similar statement involving complexes $N_{K_\infty}^*$ and $L_{K_\infty}^*$ which are $\Lambda(G)$-dual to N_{K_∞} and L_{K_∞} (see Prop. 2.11 (ii), as well as Prop. 4.16, (ii) for the precise statement).

Here, the construction of $\mathcal{L}_{A/K_\infty}$, the proof that N_{K_∞} is torsion and the proof of (3) are simultaneous and rely essentially on the main result of [TV] (a sheafified version of [KT] Prop. 5.13) which yields distinguished triangles of perfect complexes of $\Lambda(G)$-modules relating flat and crystalline cohomology (see Sect. 4.4.4 and Rem. 4.15). The proof of (4) relies on the comparison of crystalline and rigid cohomology [LST] together with the trace formula for the latter and the codescent properties of the Iwasawa complexes along the tower K_∞/K (Prop. 4.16, relying on Thm. 2.11).

1.2 Outline of the paper

Section 2. Defining the complexes N_{K_∞}, L_{K_∞} or $N_{K_\infty}^*$, $L_{K_\infty}^*$ occurring in the main conjecture (3) involves forming projective limits along the Galois tower formed by the C_n's or alternatively inductive limits and taking duals. In order to perform these operations, a convenient framework is given by the derived category of normic systems $D^b(_GMod(\mathbb{Z}_p))$ defined and studied in [Va]. A normic system is a collection M_n of G_n-modules together with equivariant morphisms $M_n \to M_m$, $M_m \to M_n$ satisfying a natural compatibility (Def. 2.1). The purpose of this section is to show that the collection of derived functors $R\Gamma(C_n, -)$ comes from a functor $R\Gamma(\underline{C}, -)$ with values in $D^b(_GMod(\mathbb{Z}_p))$.

We place ourself in a setting which is general enough to handle the various cohomology theories (flat, étale, crystalline...) involved here. Namely we show (Lem.-Def. 2.7) that in a ringed topos (E, R) any pro-torsor $\underline{X} = (X_n, G_n)_n$ gives rise to a functor

$$\Gamma(\underline{X}, -) : Mod(E, R) \to {}_GMod(\Gamma(E, R)) \tag{5}$$

from the category of R-modules of E to that of normic systems of $\Gamma(E, R)$-modules. Using [Va], the final subsection collects the basic properties (descent, codescent, perfectness, duality, under suitable assumptions, see. Thm. 2.11) of the Iwasawa complexes defined using the functor

$$D^+(E, \mathbb{Z}_p) \to D(\Lambda(G))$$

obtained by composing $R\Gamma(\underline{X}, -)$ with $R\varprojlim_n$, or alternatively with $\varinjlim_n$ and then $RHom(-, \mathbb{Z}_p)$.

Section 3. We review the basic facts from K-theory which are needed for our purpose. This involves mainly the determinant functor for perfect complexes from [Kn] and its behaviour with respect to distinguished triangles and localization following [FK]. We also discuss the evaluation map which appears in (4).

Section 4. We begin by recalling the definition of the derived functor $R\Gamma^Z(C, .)$ of global sections vanishing at a closed subscheme Z which naturally appears in the comparison theorem of [TV] and is denoted $R\Gamma_{ar,V}$ in [KT] (see Rem. 4.13 for a more precise statement). Next we compare the complex of normic systems $R\Gamma^{\underline{Z}}(\underline{C}, T_p(\mathcal{A}))$ underlying N_{K_∞} to the usual Selmer complex of A/K. We take the opportunity to give a tractable definition for normic Selmer complexes and prove the expected duality theorem in this setting.
Finally we recall the comparison result of [TV] (which takes place in the small étale topos of C) and write down the fundamental distinguished triangles that follow from it, using (5):

$$R\Gamma^{\underline{Z}}(\underline{C}, T_p(\mathcal{A})) \to R\Gamma(\underline{C}^\sharp/\mathbb{Z}_p, Fil^1 D_{log}(\mathcal{A})(-Z)) \overset{\mathbf{1}-\phi}{\to} R\Gamma(\underline{C}^\sharp/\mathbb{Z}_p, D_{log}(\mathcal{A})(-Z)) \overset{+1}{\to}$$
$$R\Gamma(\underline{C}^\sharp/\mathbb{Z}_p, Fil^1 D_{log}(\mathcal{A})(-Z)) \overset{\mathbf{1}}{\to} R\Gamma(\underline{C}^\sharp/\mathbb{Z}_p, D_{log}(\mathcal{A})(-Z)) \to R\Gamma(\underline{C}, Lie(\mathcal{A})(-Z)) \overset{+1}{\to}$$

$$(6)$$

Here ϕ is a semi-linear map such that $\phi\mathbf{1} = pFrob$. By applying $R\varprojlim_n$, these in turn yield distinguished triangles of perfect complexes of $\Lambda(G)$-modules satisfying the derived codescent property. This will be the main ingredient for the proof of 1.1.

Section 5. We put everything together in order to prove the Iwasawa main conjecture. The third term of the first distinguished triangle above is a k-vector space and the arrow denoted $\mathbf{1}$ thus becomes invertible after inverting S^*. Whence an endomorphism $(\mathbf{1}-\phi)_{\mathbf{S}^*}\mathbf{1}_{\mathbf{S}^*}^{-1}$ acting on the localization P_{K_∞, S^*} of

$$P_{K_\infty} := R\varprojlim R\Gamma(C_n^\sharp/\mathbb{Z}_p, D_{log}(\mathcal{A})(-Z))$$

In the case where $G = \Gamma$, the base change formula in crystalline cohomology together with a semi-linear argument shows that the first term in the second distinguished triangle above vanishes as well after S^*-localization. In the general case, an argument using Nakayama's lemma ensures that it is still the case

under the $\mu = 0$ assumption. The endomorphism $(\mathbf{1} - \phi)_{S*}\mathbf{1}_{S*}^{-1}$ is thus invertible, allowing us to *define* $\mathcal{L}_{A/K_\infty} \in K_1(\Lambda_{S*})$ as its determinant, rendering (3) almost tautological. Let us hint the idea of the proof of (4). Using the descent properties of the normic section functor together with the comparison between crystalline and rigid cohomology from [LST], we prove an isomorphism

$$L \overset{L}{\otimes}_{\Lambda_O(G)} (V_{\rho,\Lambda_O(G)})^* \overset{L}{\otimes}_{\Lambda(G)} P_{K_\infty} \simeq R\Gamma_{rig,c}(U/L, pr^* D^\dagger \otimes U(\rho^\vee)) \qquad (7)$$

where $U(\rho^\vee)$ denotes the unipotent convergent isocrystal associated to ρ (Prop. 5.16). On the one hand the trace formula in rigid cohomology shows that $L(U, A/K, \rho^\vee, 1)$ coincides with the alternated product of the determinants of $1 - p^{-1} Frob$ acting on the L-valued cohomology of the right hand side. Since we have no morphism from Λ_{S*} to L we may not use directly the base change property of the Det functor together with a localized version of (7) in order to relate the action of $1 - p^{-1} Frob$ on the right hand side with $\rho(\mathcal{L})$. We turn this difficulty by investigating carefully the spectral sequence of codescent along Kk_∞/K with the definition of the evaluation map in mind.

2 The normic section functor

The purpose of this section is to show that under reasonable conditions the cohomology of a topos along a Galois tower with group G naturally gives rise to perfect complexes of $\Lambda(G)$-modules satisfying natural properties such as derived descent, codescent and duality along the tower. This goal is achieved in Thm. 2.11.

2.1 Normic systems

We define and study briefly the category of normic systems along a profinite group G. The following definitions slightly generalize those given in [Va].

DEFINITION 2.1. *Consider a ring S, a profinite group G, and let $(H_n)_{n\in N}$ denote the filtered set of the normal open subgroups of G (N an appropriate filtered set of indices). For $n \leq m$ in N, we denote $G_n = G/H_n$ and $G_{m,n} = H_n/H_m$. In this setting we define the following categories.*
(i) $_G Mod(S)$ the category of S-modules endowed with a discrete action of G, ie. which are the union of their fixed points by the G_n's.
(ii) $_{G_n} Mod(S)$ the full subcategory of $_G Mod(S)$ whose objects are those on which H_n acts trivially. For $m \geq n$, the inclusion functor $_{G_n} Mod(S) \to {}_{G_m} Mod(S)$ has a left and a right adjoint, described respectively as the coinvariants functor $(-)_{G_{m,n}}$ and the fixed points functor $(-)^{G_{m,n}}$.

(iv) $_GMod(S)$ *the category of $Mod(S)$-valued normic systems is defined a fol-lows:*
- an object is a triple $\underline{M} = ((M_n)_{n\in N}, (j_{n,m} : M_n \to M_m)_{n\leq m}, (k_{m,n} : M_m \to M_n)_{n\leq m})$ *satisfying the following properties:*

$(Norm1)$ M_n *is an object of* $_{G_n}Mod(S)$, $j_{n,m}$ *and* $k_{m,n}$ *are morphisms of* $_GMod(S)$.

$(Norm2)$ *If* $n_1 \leq n_2 \leq n_3$ *then* $j_{n_2,n_3} \circ j_{n_1,n_2} = j_{n_1,n_3}$ *and* $k_{n_2,n_1} \circ k_{n_3,n_2} = k_{n_3,n_1}$.

$(Norm3)$ *If* $n \leq m$, $j_{n,m} \circ k_{m,n} : M_m \to M_m$ *coincides with the endomorphism* $\sum_{g\in G_{m,n}} c_g$ *of* M_m. *Note that this is not only a morphism in $Mod(S)$, but also in* $_GMod(S)$ *since* $H_n \lhd G$.
- a morphism $\underline{\phi} : \underline{M} \to \underline{M}'$ *is a collection of morphisms of* $_GMod(S)$, $(\phi_n : M_n \to M'_n)_{n\in N}$ *such that the following squares commute for each couple $n \leq m$:*

$$
\begin{array}{ccc}
M_m & \xrightarrow{\phi_m} & M'_m \\
{\scriptstyle j_{n,m}}\big\uparrow & & \big\uparrow{\scriptstyle j_{n,m}} \\
M_n & \xrightarrow{\phi_n} & M'_n
\end{array}
\qquad\qquad
\begin{array}{ccc}
M_m & \xrightarrow{\phi_m} & M'_m \\
{\scriptstyle k_{m,n}}\big\downarrow & & \big\downarrow{\scriptstyle k_{m,n}} \\
M_n & \xrightarrow{\phi_n} & M'_n
\end{array}
$$

Let us make some remarks about the category $_GMod(S)$ of discrete G-objects.
- For $M, M' \in_G Mod(S)$:

$$Hom_{GMod(S)}(M, M') \simeq \varprojlim_n \varinjlim_m Hom_{GMod(S)}(M^{H_n}, M'^{H_m}).$$

- The category $_GMod(S)$ is a full subcategory of $Mod(S[G])$. The inclusion functor has a right adjoint $M \mapsto \cup_n M^{H_n}$. In particular, $_GMod(S)$ has enough injectives since $Mod(S[G])$ has.

We now turn to some properties of the category $_GMod(S)$ of normic systems.

PROPOSITION 2.2. *Small inductive and projective limits exist in $_GMod(S)$ and commute to the component functors $_GMod(S) \to {}_{G_n}Mod(S)$. In particular, $_GMod(S)$ is an abelian category.*

Proof. In order to form the inductive (resp. projective) limit indexed by a set I in $_G\mathcal{C}$, it suffices to form the limit of the components and endow them with the $j_{n,m}$ and $k_{m,n}$ provided by functoriality. This prove the first statement. The second one follows since being abelian is a property of limits ([KS], Def. 8.2.8 and Def. 8.3.5). □

The following lemma, inspired by [TW] has been pointed out by B. Kahn.

LEMMA 2.3. (i) *If G is in fact a finite group, then the category $_GMod(S)$ is equivalent to $Mod(\mu_\lhd(G, S))$ where $\mu_\lhd(G, S)$ is the* normal Mackey al-gebra, *defined as the quotient of the free associative algebra $S\{\{c_g, g \in G\}, \{j_{n,m}, k_{m,n}, n \leq m\}\}$ by the following relations:*

- $c_{g_1} c_{g_2} = c_{g_1 g_2}$, $c_g j_{n,m} = j_{n,m} c_g$, $c_g k_{m,n} = k_{m,n} c_g$, for $g, g_1, g_2 \in G$ and $n \leq m$ in N.
- $j_{n_2,n_3} j_{n_1,n_2} = j_{n_1,n_3}$, $k_{n_2,n_1} k_{n_3,n_2} = k_{n_3,n_1}$, for $n_1 \leq n_2 \leq n_3$ in N.
- $c_g j_{n,n} = j_{n,n}$, $j_{n,n} = k_{n,n}$, for $n \in N$ and $g \in H_n$.
- $j_{n,m} k_{m,n} = \sum_{g \in G_{m,n}} c_g j_{m,m}$, for $n \leq m$ in N.
- $\sum_{n \in N} j_{n,n} = 1$.

(ii) In general, $_G Mod(S)$ is equivalent to the category of cocartesian sections of the cofibered category above N^{op} associated to the covariant pseudo-functor on N^{op} mapping n to $Mod(\mu_\lhd(G_n, S))$, and $m \geq n$ to the functor $Mod(\mu_\lhd(G_m, S)) \rightarrow Mod(\mu_\lhd(G_n, S))$ given by extension of scalars through the morphism $\mu_\lhd(G_m, S) \rightarrow \mu_\lhd(G_n, S)$ defined by killing the $j_{a,b}$'s and $k_{b,a}$'s with $b \not\leq n$.

Proof. (i) Let us describe the functor $_G Mod(S) \rightarrow Mod(\mu_\lhd(G, S))$. A normic system $\underline{M}$ is sent to $\oplus_n M_n$, together with its obvious structure of $\mu_\lhd(G, S)$-module: c_g acts on every components whereas $j_{n,m}$ (resp. $k_{m,n}$) sends the n-th (resp. m-th) component into the m-th (resp. n-th) one. By definition, a morphism of normic systems $\underline{M} \rightarrow \underline{M}'$ consists in a collection of morphisms $\phi_n : M_n \rightarrow M'_n$, compatible with the c_g's, the $j_{n,m}$'s and the $k_{m,n}$'s. It thus gives rise to a morphism $\oplus_n M_n \rightarrow \oplus_n M'_n$, compatible with the action of $\mu_\lhd(G, S)$ and this is clearly compatible to composition. We thus have defined the desired functor $_G Mod(S) \rightarrow Mod(\mu_\lhd(G, S))$. It now remains to notice that both the full faithfulness and the essential surjectivity of this functor immediately follow from the isomorphism of algebras $\mu_\lhd(G, S) \simeq \prod_{n \in N} j_{n,n} \mu_\lhd(G, S)$.

(ii) We apply (i) to the case $G = G_n$. Letting n vary, this gives an equivalence of cofibered categories. Whence the result, since $_G Mod(S)$ is clearly equivalent to the category of cocartesian sections of the cofibered category associated to $n \mapsto {}_{G_n} Mod(S)$. □

The following corollary answers a question of [Va].

COROLLARY 2.4. *The category $_G Mod(S)$ has enough injectives.*

Proof. Let $\mathcal{F}/N^{op}$ denote the fibered category $n \mapsto Mod(\mu_\lhd(G_n, S))$ considered previously and let $Sect(\mathcal{F}/N^{op})$ (resp. $Cocart(\mathcal{F}/N^{op})$) denote its category of sections (resp. cocartesian sections). Denoting $|N|$ the discrete category underlying N, there are three obvious forgetful functors

$$
\begin{array}{ll}
Cocart(\mathcal{F}/N^{op}) & \overset{F_1}{\rightarrow} \quad Sect(\mathcal{F}/N^{op}) \\
& \overset{F_2}{\rightarrow} \quad \prod_{|N|} Mod(\mu_\lhd(G_n, S)) \\
& \overset{F_3}{\rightarrow} \quad Mod(S)^{|N|}
\end{array}
$$

each of which is exact and whose composition is faithful. The result will follow formally if one proves that each of them has a right adjoint. For F_3, this is easy and left to the reader. For F_2, this results from [SGA4], Vbis, 1.2.10 (note that

here $\mathcal{F}/N^{op}$ is indeed bifibered). For F_1, we use the following general easy fact concerning an arbitrary cofibered category $\mathcal{F}/\mathcal{S}$ with cocleavage $f \mapsto f_!$:

Fact: Assume that for any $f : X \to Y$ the following properties are verified:
- The functor "composition with f" $:\mathcal{S}_{/X} \to \mathcal{S}_{/Y}$ is cofinal.
- $f_!$ commutes to projective limits indexed by $\mathcal{S}_{/X}$.

Then the inclusion functor $Cocart(\mathcal{F}/\mathcal{S}) \to Sect(\mathcal{F}/\mathcal{S})$ has a right adjoint, which takes a section $X \mapsto s(X)$ to the cocartesian section $X \mapsto \varprojlim_{f:Z\to X} f_! s(Z)$. $\qquad\qquad\square$

REMARK 2.5. *One may show that $_{\underline{G}}Mod(S)$ has enough projectives as well.*

2.2   From sheaves to normic systems

As before let $G = \varprojlim_n G_n$ denote a profinite group and let $G_n = G/H_n$. Recall that the classifying topos B_G is the category of discrete left G-sets. Consider another arbitrary topos E together with its structural morphism $\pi : E \to Set$ and assume given a projective system $\underline{X} = (X_n)$ of G-objects of E such that X_n is a torsor of E under $\pi^{-1}G_n$. In this situation we have for $m \geq n$ a commutative diagram of topoi as follows:

$$
\begin{array}{ccccccc}
 & & & & & & E \\
 & & & & \nearrow^{p_\infty} & \nearrow^{p_m} \;\; \uparrow^{p_n} & \downarrow \pi \\
B_G & \underset{q_m}{\rightleftarrows} & B_{G_m} & \underset{q_{m,n}}{\rightarrow} & B_{G_n} & \longrightarrow & Set
\end{array}
$$

Here the horizontal arrows are by functoriality of the classifying topos with respect to the group (their inverse and direct images functors are inflation and fixed points by the adequate subgroup) and the oblique morphisms, defined by $\underline{X}$, are as follows:
- If Y is in B_{G_n} and Y^0 denotes the underlying set, the formula $g.(x,y) = (gx,gy)$ defines an action of $\pi^{-1}G_n$ on $X_n \times \pi^{-1}Y^0$ and $p_n^{-1}Y = \pi^{-1}G_n \backslash (X_n \times \pi^{-1}Y^0)$ is the coinvariant object of this action. If Z is an object of E, then $p_{n,*}Z$ is the set $Hom(X_n, Z)$ endowed with the left action of G_n induced by the inverse action on X_n: $(g.f)(x) = f(g^{-1}x)$.
- If Y is in B_G then the formula $g.(x,y) = (gx,gy)$ defines an action of $p_\infty^{-1}G$ on $X_m \times \pi^{-1}(Y^{H_n})^0$ for each m,n in N and $p_\infty^{-1}Y = \varinjlim_n \varprojlim_m \pi^{-1}G \backslash (X_m \times \pi^{-1}(Y^{H_n})^0)$ (Note that p_∞^{-1} is exact, since $\varprojlim_m$ is essentially constant while $\varinjlim_n$ is filtered). If Z is an object of E then $p_{\infty,*}Z$ is the set $\varinjlim_n Hom(X_n, Z)$ endowed with the left action of G induced by the inverse action of G on f.

Endowing the set G_n with its left action by translations turns it into an object of B_G (resp. B_{G_m}, if $m \geq n$) whose image by p_∞ (resp. p_m) is nothing but X_n.

For the purpose of what follows we will consider a ring S. Endowing it with a trivial action of G, we may as well view it as a ring of B_{G_n} or B_G.

DEFINITION 2.6. *We define a functor*

$$\Gamma(\underline{G}, -) : Mod(B_G, S) \to {}_{\underline{G}}Mod(S)$$

by sending an S-module M of B_G to $(M_n, j_{n,m}, k_{m,n})$, where $M_n = M^{H_n}$, $j_{n,m}$ is the inclusion and $k_{m,n}$ the trace map.

LEMMA + DEFINITION 2.7. *Let G, S, E and $\underline{X}$ be as above. Consider furthermore a ring R of E and a ring homomorphism $S \to \pi_* R$. Letting $Mod(E, R)$ denote the category of R-modules of E, we define the functor of* Normic sections *along $\underline{X}$*

$$\Gamma(\underline{X}, -) : Mod(E, R) \to {}_{\underline{G}}Mod(S)$$

by composing $p_{\infty,*} : Mod(E, R) \to Mod(B_G, S)$ with the functor $\Gamma(\underline{G}, -) : Mod(B_G, S) \to {}_{\underline{G}}Mod(S)$ of Def. 2.6.
For any F in $Mod(E, R)$, the object $((M_n), (j_{n,m}), (k_{m,n})) := \Gamma(\underline{X}, F)$ satisfies the following properties:
- M_n is the restriction of scalars to S of the $R(X_n)$-module $F(X_n)$ endowed with the action of G_n coming from the right action of G_n on X_n: $(g, x) \mapsto g^{-1}x$.
-$j_{n,m} : M_n \to M_m$ is the restriction along $X_m \to X_n$. and induces an isomorphism $M_n \simeq M_m^{G_{m,n}}$ (ie. $F(X_n) \to F(X_m)^{G_{m,n}}$).

Proof. By definition we have $M_n = M^{H_n} = q_{n,*}M$, where $M = p_{\infty,*}F$. Now, $q_n p_\infty \simeq p_n$, therefore $M_n \simeq p_{n,*}F = F(X_n)$. The second property is clear. $\square$

REMARK 2.8. *If the topos E is locally connected, it is possible to build trace maps along finite locally free morphisms, such as $X_m \to X_n$. One may then show that $k_{m,n}$ coincides with the trace map.*

LEMMA 2.9. *Consider another topos E' with structural morphism $\pi' : E' \to Set$ and a morphism of ringed topoi $f : (E', R') \to (E, R)$. Let us moreover denote $\underline{X}' = (f^{-1}X_n)$ the projective system of torsors of E' deduced from $\underline{X}$ and consider the morphism $S \to \pi'_* R'$ induced by $h : S \to \pi_* R$ and $R \to f_* R'$. The functor $\Gamma(\underline{X}', -)$ associated to these data is subject to a canonical isomorphism*

$$R\Gamma(\underline{X}', -) \simeq R\Gamma(\underline{X}, Rf_*(-))$$

of functors $D^+(E', R') \to D^+({}_{\underline{G}}Mod(S))$.

Proof. This simply follows from the fact that the morphism of ringed topoi $(E', R') \to (B_G, S)$ defined by $\underline{X}$ coincides with the one obtained by composing f with the morphism $(E, R) \to (B_G, S)$ defined by $\underline{X}$. $\square$

The derived functor

$$R\Gamma(\underline{X}, -) : D^+(Mod(E, R)) \to D^+({}_{\underline{G}}Mod(S))$$

will be our tool to deduce the fundamental distinguished triangles (which are the main ingredient in our proof of the main conjecture) from the comparison isomorphism of [TV]. It might be worth to emphasize that the nth component functor

$$(-)_n : D^+(_{\underline{G}}Mod(S)) \to D^+(_{G_n}Mod(S))$$

sends $R\Gamma(\underline{X}, F)$ to $R\Gamma(X_n, F_{|X_n})$.

2.3   Descent, codescent and duality

We now assume that the ring S of Def. 2.6 is $\mathbb{Z}_p$, and we discuss how to pass from normic systems to Iwasawa modules.

In order to be able to invoke directly the results of [Va], we now assume that $\Lambda := \mathbb{Z}_p[[G]] := \varprojlim \mathbb{Z}_p[G_n]$ is Noetherian of finite global dimension $d+1$ and has an open pro-p-subgroup. As is well known, these condition is in particular verified if G is a compact p-adic Lie group of dimension d without p-torsion.

- *Limits.* Forgetting the $k_{m,n}$'s (resp. the $j_{n,m}$'s), inflating from G_n-modules to discrete G-modules (resp. abstract Λ-modules) and then forming the limit of the resulting inductive (resp. projective) system gives rise to a functor which will abusively be denoted

$$\varinjlim : {}_{\underline{G}}Mod(\mathbb{Z}_p) \to {}_GMod(\mathbb{Z}_p) \text{ (resp. } \varprojlim : {}_{\underline{G}}Mod(\mathbb{Z}_p) \to Mod(\Lambda))$$

The first is exact and thus passes to derived categories while the second is only left exact, but right derivable. From now on, *we always assume that* $R\varprojlim$ *has finite cohomological dimension.* This is e.g. the case if N has a numerable cofinal subset (as is always the case in practice).

Proposition 2.10. *There are canonical adjunctions*

$$\begin{array}{rcl}
(\varinjlim, R\Gamma(\underline{G}, -)) & : & D^+(_GMod(\mathbb{Z}_p)) \to D^+(_{\underline{G}}Mod(\mathbb{Z}_p)) \\
(\underline{\Lambda}\overset{L}{\otimes}_\Lambda(-), R\varprojlim) & : & D^-(_{\underline{G}}Mod(\mathbb{Z}_p)) \to D^-(\Lambda)
\end{array}$$

Here, $\underline{\Lambda}$ denotes the natural normic system of right Λ-modules $(\mathbb{Z}_p[G_n])$ and $\underline{\Lambda}\otimes_\Lambda(-) : Mod(\Lambda) \to {}_{\underline{G}}Mod(\mathbb{Z}_p)$ denotes the induced right exact, left derivable functor.

Proof. The derived version are easily deduced from the obvious ordinary adjunctions using that $_GMod(\mathbb{Z}_p)$ and $_{\underline{G}}Mod(\mathbb{Z}_p)$ have enough injectives (cf. Prop. 2.4). In [Va], this was not known and a finiteness assumption was thus needed to avoid deriving $\varprojlim$ (cf. loc. cit. Prop. 4.2. and Rem. 4.3). □

In the above proposition, it is possible to make the adjunction morphisms involved functorial at the level of the complexes. Also, they can still be provided a functorial cone, as in loc. cit.

- *Duality.* Consider a $\mathbb{Z}_p$-module I. If M (resp. M_n) is a $\mathbb{Z}_p$-module endowed with a left action of G (resp. G_n) then $Hom_{\mathbb{Z}_p}(M, I)$ (resp.

$Hom_{\mathbb{Z}_p}(M_n, I))$ is endowed a left action of G (resp. G_n) as well by reversing the action on M (ie. $(g.f)(m) = f(g^{-1}m)$). If the action of G is discrete the action of G on $Hom_{\mathbb{Z}_p}(M, I)$ extends to an action of Λ. If now $\underline{M} = (M_n, j_{n,m}, k_{m,n})$ is a normic system then $Hom_{\mathbb{Z}_p}(\underline{M}, I) := (Hom_{\mathbb{Z}_p}(M_n, I), Hom_{\mathbb{Z}_p}(k_{m,n}, I), Hom_{\mathbb{Z}_p}(j_{n,m}, I))$ is a normic system as well. Using an injective resolution of $\mathbb{Z}_p$ we get duality functors

$$\begin{array}{rcccc} RHom_{\mathbb{Z}_p}(-, \mathbb{Z}_p) & : & D^b(_G Mod(\mathbb{Z}_p)) & \to & D^b(\Lambda) \\ RHom_{\mathbb{Z}_p}(-, \mathbb{Z}_p) & : & D^b(_{\underline{G}} Mod(\mathbb{Z}_p)) & \to & D^b(_{\underline{G}} Mod(\mathbb{Z}_p)) \end{array}$$

It might be useful to point out the following relation to Pontryagin duality:

$$RHom_{\mathbb{Z}_p}(M \overset{L}{\otimes} \mathbb{Q}_p/\mathbb{Z}_p, \mathbb{Q}_p/\mathbb{Z}_p) \simeq RHom_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$$

If now M is a Λ-module we view $Hom_\Lambda(M, \Lambda)$ as a left Λ-module via the right action of Λ on itself and the involution $g \mapsto g^{-1}$. Using projective resolutions of M we get a functor

$$RHom_\Lambda(-, \Lambda) \quad : \quad D^b(\Lambda) \quad \to \quad D^b(\Lambda)$$

THEOREM 2.11. *Let E, X_n be as in the previous paragraph.*
(i) (Descent) There is a canonical isomorphism

$$R\Gamma(\underline{X}, -) \overset{\sim}{\to} R\Gamma(\underline{G}, \varinjlim R\Gamma(\underline{X}, -))$$

of functors $D^+(E, \mathbb{Z}_p) \to D^+(_{\underline{G}} Mod(\mathbb{Z}_p))$.
(ii) Consider a bounded complex F of $\mathbb{Z}_p$-modules of E (resp. and assume that there exists q_0) such that $H^q(X_n, F_{|X_n})$ is finitely generated over $\mathbb{Z}_p$ (resp. trivial) for any $n \in N$, $q \geq 0$ (resp. $q \geq q_0$).
(Codescent) Functorially in F, There is a canonical isomorphism in $D^+(_{\underline{G}} Mod(\mathbb{Z}_p))$:

$$\underline{\Lambda} \overset{L}{\otimes}_\Lambda R\varprojlim R\Gamma(\underline{X}, F) \overset{\sim}{\to} R\Gamma(\underline{X}, F)$$

(Perfectness) $R\varprojlim R\Gamma(\underline{X}, F)$ and $RHom_{\mathbb{Z}_p}(\varinjlim R\Gamma(\underline{X}, F), \mathbb{Z}_p)$ are in $D^p(\Lambda)$ (the derived category of perfect complexes).
(Duality) Functorially in F, There is a canonical isomorphism in $D^p(\Lambda)$:

$$RHom_{\mathbb{Z}_p}(\varinjlim R\Gamma(\underline{X}, F), \mathbb{Z}_p) \simeq RHom_\Lambda(\varprojlim R\Gamma(\underline{X}, F), \Lambda)$$

Proof. (*i*) We are going to check that the adjunction morphism of $(\varinjlim, R\Gamma(\underline{G}, -))$ is an isomorphism. Replacing F by an injective resolution and truncating, one reduces to the case where F is an injective $\mathbb{Z}_p$-module of E placed in degree 0. Let $\underline{M} := \Gamma(\underline{X}, F)$. Then $\forall n$, M_n is injective in $_{G_n} Mod(\mathbb{Z}_p)$ (indeed M_n corresponds to $p_{n*}F$) and the descent map induces an isomorphism $M_n \to M_m^{G_{m,n}}$. Similarly $\varinjlim_n M_n$ is injective and the adjunction

morphism of descent occurring in $D^+(_{\underline{G}}Mod(\mathbb{Z}_p))$ is represented at level n by the isomorphism $\Gamma(X_n, F_{|X_n}) \to \Gamma(H_n, \varinjlim_m \Gamma(X_m, F_{|X_m}))$.

(ii) Under the stated assumptions,

- $(Codescent)$ and $(Perfectness)$ follow from (i), by [Va] 4.4 (ii) et 4.6 (i).
- $(Duality)$ follows from $(Codescent)$, by loc. cit. 4.6 (iii). $\square$

REMARK 2.12. *Properties $(Codescent)$ and $(Perfectness)$ are a crucial tool in this paper. It seems plausible that they hold without assuming that G has finite cohomological dimension but we have not tried to check this.*

2.4 Examples

Start with a projective system of C-schemes C_n which are étale surjective over C and assume that the C_n's are endowed with compatible actions of the G_n's such that each $G_n \times C_n \to C_n \times_C C_n$, $(g,x) \mapsto (gx, x)$ is an isomorphism. (In view of our applications, let us notice that if C and C_n are proper smooth curves over a field with respective function fields K and K_n then the latter condition holds as soon as the unramified extension K_n/K is Galois). These data represent a projective system of torsors, say $\underline{X}$ of the small étale topos $E = C_{et}$ and we are thus in the situation of the previous paragraphs.

Of course these data also produce a projective system of torsors, say $\underline{X}'$, of the big, say flat, topos $E' = C_{FL}$. Note that if $\epsilon : C_{FL} \to C_{et}$ denotes the natural morphism, then $\underline{X}' = \epsilon^{-1}\underline{X}$.

Consider now the small étale crystalline topos E'' of $C/\mathbb{Z}_p$ and the projection morphism $u : E'' = (C/\mathbb{Z}_p)_{crys,et} \to C_{et} = E$. Then we define a projective system of torsors of E'' by pulling back via u: $\underline{X}'' := u^{-1}\underline{X}$.

As explained in the previous paragraph, these data give rise to compatible normic section functors, ie. to an essentially commutative diagram

$$\begin{array}{ccc} D^+(C_{FL}, R') & \xrightarrow{R\epsilon_*} & D^+(C_{et}, R) \xleftarrow{Ru_*} D^+((C/\mathbb{Z}_p)_{crys,et}, R'') \\ & {}_{R\Gamma(\underline{X}',-)}\searrow \quad {}_{R\Gamma(\underline{X},-)}\downarrow \quad {}_{R\Gamma(\underline{X}'',-)}\swarrow & \\ & D^+(_{\underline{G}}Mod(\mathbb{Z}_p)) & \end{array}$$

whenever one is given $\mathbb{Z}_p$-algebras R, R', R'' in E, E', E'' and homomorphisms $R \to \epsilon_* R'$, $R \to \epsilon_* R''$.

The n-component of the left (resp. middle, resp. right) normic section functor computes the cohomology of the localized topos C_{FL}/X'_n, (resp. C_{et}/X_n, resp. $(C/\mathbb{Z}_p)_{crys,et}/X''_n$), ie. of $C_{n,FL}$ (resp. $C_{n,et}$, resp. $(C_n/\mathbb{Z}_p)_{crys,et}$). We will thus use the more suggestive notations $R\Gamma(\underline{C}_{FL}, -)$ (resp. $R\Gamma(\underline{C}_{et}, -)$, resp. $R\Gamma_{crys,et}(\underline{C}/\mathbb{Z}_p, -)$).

3 K-theory for $\Lambda(G)$

For the convenience of the reader, we review the K-groups and determinant functors which are needed for our purpose. Unless specified otherwise, R denotes a (unitary) ring.

3.1 Review of K_0 and K_1

The following definitions can be found in [Ba], ch. VII, IX.
- $K_0(R)$ is the abelian group defined by generators $[P]$, where P is a finitely generated projective R-modules, and relations

   i. $[P] = [Q]$ if P is isomorphic to Q as a R-module.

   ii. $[P \oplus Q] = [P] + [Q]$

- $K_1(R)$ is the abelian group defined by generators $[P, f]$, where P is a finitely generated projective R-module and f is an automorphism of P with relations (the group law is denoted multiplicatively):

   i. $[P, f] = [Q, g]$ if P is isomorphic to Q as a R-module via an isomorphism which is compatible with f and g.

   ii. $[P \oplus Q, f \oplus g] = [P, f][Q, g]$

   iii. $[P, fg] = [P, f][P, g]$

Note that if the ring R is Noetherian and regular, then forgetting the word "projective" does not change the definitions (cf [Ba], IX, Proposition 2.1).
- The functor $Hom_R(-, R)$ realizes an anti-equivalence between finitely generated projective left modules and finitely generated projective right modules. As a result, one gets an isomorphism $K_i(R) \simeq K_i(R^{op})$, if R^{op} denotes the opposed ring. Both this isomorphism and its inverse will be denoted $(-)^*$ (eg. $[P, f]^* = [P^*, f^*]$ if f^* denotes the transpose of f).
- Morita equivalence. Let us fix a finitely generated projective right R-module V and let $V^* = Hom_R(V, R)$. Then V (resp. V^*) is naturally endowed with a natural structure of $((End_R(V), R)$-bimodule (resp. $(R, End_R(V))$-bimodule). Since V is projective and finitely generated, one has a canonical isomorphism of (R, R)-bimodules: $V^* \otimes_{End_R(V)} V \simeq R$ (resp. of $(End_R(V), End_R(V))$-bimodules: $V \otimes_R V^* \simeq End_R(V)$.
The functor $V^* \otimes_{End_R(V)} (-) : Mod(End_R(V)) \to Mod(R)$ is thus an equivalence of categories, with $V \otimes_R (-)$ as quasi-inverse. In particular, there is a canonical isomorphism $K_i(End_R(V)) \simeq K_i(R)$, $i = 0, 1$.

3.2 The determinant functor

Let $\mathcal{P}(R)$ denote the category of strictly perfect (ie. bounded with projective finitely generated objects) complexes, $K^p(R)$ its homotopy category and $D^p(R)$

its essential image in the derived category (which is naturally equivalent, and identified, to $K^p(R)$). If $D^p(R)^{is}$ denotes the subcategory of $D^p(R)$ where morphisms are the isomorphisms of the latter we have by [Kn], th. 2.3, 2.12 a canonical functor

$$Det_R : D^p(R)^{is} \simeq K^p(R)^{is} \to \mathcal{C}_R \tag{8}$$

where $\mathcal{C}_R$ is the Picard category (ie. a category together with an endobifunctor, referred to as the *product*, endowed with associativity and commutativity isomorphisms satisfying natural compatibilities, unit objects and where every object and every morphism is invertible; cf [Kn] appendix A) considered in [FK] 1.2:

- An object of $\mathcal{C}_R$ is a couple (P, Q) of finitely generated projective R-modules.

- $Mor((P, Q), (P', Q'))$ is empty if $[P] - [Q] \neq [P'] - [Q']$ in $K_0(R)$. Else, there exists an R-module M such that

$$P \oplus Q' \oplus M \simeq P' \oplus Q \oplus M.$$

  We set $I_M := Isom(P \oplus Q' \oplus M, P' \oplus Q \oplus M)$ and $G_M := Aut(P' \oplus Q \oplus M)$ and we define the set of morphisms from (P, Q) to (P', Q') as $K_1(R) \times^{G_M} I_M$ where the right hand side denotes the quotient of $K_1(R) \times I_M$ by the action of G_M given by

$$(x, y) \mapsto (x\bar{g}, g^{-1}y))$$

  where $x \in K_1(R)$, $y \in I_M$, $g \in G_M$ and $\bar{g}$ is its image in $K_1(R)$. As seen easily, this set does not depend on M, up to a canonical isomorphism, and this fact can be used to define composition in a natural way. Note that by definition, one has a canonical identification $Aut_{\mathcal{C}_R}((P, Q)) = K_1(R)$ for any object (P, Q).

- The product is defined as

$$(P, Q).(P', Q') := (P \oplus P', Q \oplus Q').$$

  and admits naturally $(0, 0)$ as a unit. Every object (P, Q) in $\mathcal{C}_R$ admits $(P, Q)^{-1} := (Q, P)$ as a natural inverse.

It follows immediately from its construction that the functor (8) is compatible with base change, ie. the diagram

$$
\begin{array}{ccc}
D^p(R') & \xrightarrow{Det_{R'}} & \mathcal{C}_{R'} \\
{\scriptstyle R' \overset{L}{\otimes}_R (-)} \Big\uparrow & & \Big\uparrow {\scriptstyle R' \otimes_R (-)} \\
D^p(R) & \xrightarrow{Det_R} & \mathcal{C}_R
\end{array}
$$

is naturally pseudo-commutative for any R'/R. Let us quickly review the construction of Det_R following [FK] 1.2.

- A strictly perfect complex P is sent to $Det_R(P) := (P^+, P^-)$ where $P^+ := \oplus_{i \in 2\mathbb{Z}} P^i$ and $P^- := \oplus_{i \in 1+2\mathbb{Z}} P^i$.

- Any exact sequence $0 \to P' \to P \to P'' \to 0$ of $\mathcal{P}(R)$ induces a canonical isomorphism $Det_R(P) \simeq Det_R(P').Det_R(P'')$. Using this, one constructs a canonical *trivialization* $can_P : Det_R(0) \simeq Det_R(P)$, for any strictly perfect complex P which is acyclic.

- If $Cone(f)$ denotes the mapping cone of a morphism $f : P \to P'$ of strictly perfect complexes then $Hom_{\mathcal{C}_R}(Det_R(P), Det_R(P'))$ identifies with $Hom_{\mathcal{C}_R}(Det_R(0), Det_R(Cone(f)))$ (indeed $Cone(f)^+ \simeq P^- \oplus P'^+$ and $Cone(f)^- \simeq P^+ \oplus P'^-$). In particular, when f is a quasi-isomorphism, then $can_{Cone(f)} : Det_R(0) \to Det_R(Cone(f))$ induces a morphism $Det_R(P) \to Det_R(P')$, which we denote $Det_R(f)$. One may check that it is compatible with composition and only depends on the homotopy class of f so that the functor Det_R is finally defined.

We will make essential use of the homorphism

$$Det_R : Aut_{D^p(R)}(P) \to Aut_{C_R}(Det_R(P)) = K_1(R) \tag{9}$$

which is induced by (8) for any perfect complex P. Of course if P is reduced to a (finitely generated projective) module P^0 placed in degree zero then $Det_R(P) = (P^0, 0)$ and $Det_R(f) : (P^0, 0) \to (P^0, 0)$ is the class of $f^0 : P^0 \to P^0$. In that case (9) is thus nothing but the tautological map $Aut_R(P^0) \to K_1(R)$.

Let us now state some multiplicative properties.

- Consider a morphism between exact sequences of strictly perfect complexes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & P_1' & \longrightarrow & P_1 & \longrightarrow & P_1'' & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & P_2' & \longrightarrow & P_2 & \longrightarrow & P_2'' & \longrightarrow & 0
\end{array}
$$

where vertical arrows are quasi-isomorphisms. The following square commutes:

$$
\begin{array}{ccc}
Det_R(P_1) & \simeq & Det_R(P_1').Det_R(P_1'') \\
{\scriptstyle Det_R(f)}\downarrow & & \downarrow{\scriptstyle Det_R(f').Det_R(f'')} \\
Det_R(P_2) & \simeq & Det_R(P_2').Det_R(P_2'')
\end{array}
$$

- Consider an automorphism f of some P in $K^p(R)$. Then:

- $Det_R(f[1]) = Det_R(f)^{-1}$ in $Aut_{\mathcal{C}_R}(Det_R(P)) = K_1(R)$.

- f is the homotopy class of some automorphism of the complex P. Let us fix such one and denote $f^q : P^q \to P^q$ its component of degree q. Then in $K_1(R) = Aut_{\mathcal{C}_R}(Det_R(P)) = Aut_{\mathcal{C}_R}(Det_R(P^q))$, we have $Det_R(f) = \prod Det_R(f^q)^{(-1)^q}$.

- If P is cohomologically perfect (ie. each H^q is an object of $D^p(R)$) then $Det_R(f) = \prod Det_R(H^q(f))^{(-1)^q}$ in $K_1(R) = Aut_{\mathcal{C}_R}(Det_R(P)) = Aut_{\mathcal{C}_R}(Det_R(H^q(P)))$.

3.3 Localization

The main K-group of interest for Iwasawa theory is a relative one. We recall its definition ([FK] 1.3). Consider a strictly full triangulated subcategory Σ of $D^p(R)$. The group $K_1(R, \Sigma)$ is then defined by generators and relations as follows:

- Generators: $[C, a]$ where C is an object of Σ and $a : Det_R(0) \to Det_R(C)$ is a *trivialization of C*.

- Relations: Let C, C', C'' be objects of Σ.

    ˘ If $C \simeq 0$ then $[C, can_C] = 1$.

    ˘ If $f : C \simeq C'$, then compatible trivializations of C and C' give rise to the same element in $K_1(R, \Sigma)$ (ie. $[C', Det_R(f) \circ a] = [C, a]$ if $a : Det_R(0) \simeq Det_R(C)$).

    ˘ If $0 \to C' \to C \to C'' \to 0$ is an exact sequence of $\mathcal{P}(R)$, and $a : 1 \simeq Det_R(C)$, $a' : 1 \simeq Det_R(C')$, then

$$[C, a] = [C', a'].[C'', a'']$$

    where $a : 1 \overset{a'.a''}{\to} Det_R(C').Det_R(C'') \simeq Det_R(C)$.

There is a localization exact sequence (cf [FK] 1.3.15)

$$K_1(R) \quad \longrightarrow \quad K_1(R, \Sigma) \quad \overset{\partial}{\longrightarrow} \quad K_0(\Sigma) \longrightarrow \quad K_0(R)$$

(here $K_0(\Sigma)$ denotes the Grothendieck group of the triangulated category Σ) where:
- the first map sends $[P, f] \in K_1(R)$ to the complex $[P \overset{f}{\to} P]$ placed in degrees $[-1, 0]$, together with the trivialization which is represented by the identity of P.
- ∂ sends $[C, f] \in K_1(R, \Sigma)$ to the element $-[C] \in K_0(\Sigma)$.
- the last map sends the class of a strictly perfect complex $[C]$ to the alternated sum $\sum (-1)^i [C^i]$.
As checked easily, this localization sequence is functorial with respect to (R, Σ): if R' is an R-algebra, and if Σ' is a strictly full triangulated subcategory of

$D^p(R')$ containing the essential image of Σ under the functor $R' \otimes^L_R (-) : D^p(R) \to D^p(R')$ then one has a morphism $K_1(R, \Sigma) \to K_1(R', \Sigma')$ which is compatible with the obvious functoriality maps.

If now $T \subset R$ is a left denominator set (cf [FK], 1.3.6) then we can apply the above constructions to the full triangulated subcategory Σ_T of $D^p(R)$ consisting of complexes whose image under the functor $R_T \otimes^L_R (-) : D^p(R) \to D^p(R_T)$ become acyclic. We note that if R is Noetherian and regular then $K_0(\Sigma_T)$ is isomorphic to the Grothendieck group of finitely generated T-torsion modules, by sending $[C]$ to $\sum (-1)^i H^i(C)$. For a general R we have the following result.

PROPOSITION 3.1. *([FK], 1.3.7) There is a canonical isomorphism of groups*

$$K_1(R, \Sigma_T) \simeq K_1(R_T)$$

sending $[C, a]$ to the isomorphism $Det_{R_T}(0) \overset{a_T}{\to} Det_{R_T} C_T \overset{can^{-1}_{C_T}}{\to} Det_{R_T}(0)$ viewed as an element of $K_1(R_T) = Aut_{\mathcal{C}_{R_T}}(Det_{R_T}(0))$ (here a_T and C_T are deduced from a and C by localization). This isomorphism is functorial with respect to (R, T).

Let us mention an alternative characterization of this isomorphism. Consider an endomorphism f of a strictly perfect complex P such that f_T is a quasi-isomorphism. Identifying $Hom_{\mathcal{C}_R}(Det_R(P), Det_R(P))$ with $Hom(Det_R(0), Det_R(Cone(f)))$ sends the identity of $Det_R(P)$ to a morphism $triv : Det_R(0) \to Det_R(Cone(f))$. Then the class $[Cone(f), triv] \in K_1(R, \Sigma_T)$ corresponds to $Det_{R_T}(f_T)^{-1} \in K_1(R_T)$ (both are indeed described by the same endomorphism of $P^+_T \oplus P^-_T$ coming from $can_{Cone(f)_T}$).

Crucial to us will be the following:

LEMMA 3.2. *Consider morphisms $a, b : C \to C'$ in $D^p(R)$ whose localizations a_T, b_T are isomorphisms in $D^p(R_T)$. In $K_0(\Sigma_T)$, one has the equality*

$$\partial Det_{R_T}(a_T b_T^{-1}) = [Cone(a)] - [Cone(b)]$$

as long as the following condition holds:

(frac) In $D^p(R)$, there exists a commutative square of the form

$$
\begin{array}{ccc}
C & \overset{a}{\longrightarrow} & C' \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle d} \\
C' & \overset{c}{\longrightarrow} & C'
\end{array}
$$

in which c and d also become isomorphisms after localization by T.

Proof. An easy diagram chasing shows that $[Cone(c)] - [Cone(a)] = [Cone(d)] - [Cone(b)]$. Since $Det_{R_T}(a_T b_T^{-1}) = Det_{R_T}(d_T)^{-1} Det_{R_T}(c_T)$ and ∂ is a homomorphism, it thus suffices to prove that $\partial Det_{R_T}(c_T) = [Cone(c)]$ and $\partial Det_{R_T}(d_T) = [Cone(d)]$. But this is clear from the alternative characterization of the isomorphism $K_1(R, \Sigma_T) \simeq K_1(R_T)$ mentioned above. $\qquad \square$

We do not know whether or not the condition ($frac$) always holds, neither if this computation of $\partial Det_{R_T}(a_T b_T^{-1})$ is always correct. We thus content ourselves with the following sufficient condition.

LEMMA 3.3. *Assume R is Noetherian and consider $a, b : C \to C'$ in $D^p(R)$. If there exists f in the center of R such that $R[\frac{1}{f}] \otimes_R^L Cone(b) = 0$, then one can find $c, d : C' \to C'$ such that $da = cb$. In fact one can chose $d = f^n$ for n large enough.*

Proof. Since cohomology modules of $Cone(b)$ are finitely generated and almost all of them are zero, it is possible to find n such that $f^n : Cone(b) \to Cone(b)$ is zero. But then, the long exact sequence of Ext's

$$\cdots \to Hom(C', C') \overset{Hom(b,C')}{\to} Hom(C, C') \to Ext^1(Cone(b), C') \to \cdots$$

shows that $f^n a$ has a trivial image in $Ext^1(Cone(b), C')$ and thus comes from some $c \in Hom(C', C')$ as claimed. □

3.4 The evaluation map at Artin representations

Consider a profinite group $G = \varprojlim_n G_n$ and a closed normal subgroup H such that $\Gamma := G/H$ is isomorphic to $\mathbb{Z}_p$. We use the following notations.
- We let $\Lambda = \Lambda(G) := \varprojlim_n \mathbb{Z}_p[G_n]$ denotes the Iwasawa algebra of G. We assume that this ring is left Noetherian and regular (which will be the case when considering the Galois group of a p-adic Lie extension as in the introduction).
- If O is the ring of integers of a finite extension L of $\mathbb{Q}_p$, then $\Lambda_O(G) := \varprojlim_n O[G_n] \simeq O \otimes_{\mathbb{Z}_p} \Lambda(G)$ and $\Lambda_L(G) := L \otimes_O \Lambda_O(G)$ have similar properties.
- S and S^* denote the canonical Ore sets defined in [CFKSV]. Recall that an element f of $\Lambda(G)$ is in S if and only if $\Lambda(G)/f$ is a finitely generated $\Lambda(H)$-module whereas $S^* = \cup_k p^k S$. As usual, $\Lambda_S = \Lambda(G)_S$ and $\Lambda_{S^*} = \Lambda(G)_{S^*}$ denote the corresponding localizations of Λ. If $G = \Gamma$ then $Q_O(\Gamma) := \Lambda(\Gamma)_{S^*}$ is the fraction field of $\Lambda_O(\Gamma)$.
- $\mathfrak{M}_H(G)$ denotes the category of S^*-torsion finitely generated $\Lambda(G)$-modules. We recall that a finitely generated $\Lambda(G)$ module M is S-torsion (resp. S^*-torsion) if and only if it is a finitely generated $\Lambda(H)$-module (resp. modulo its p-torsion). In this context, the localization exact sequence reads:

$$K_1(\Lambda) \quad \longrightarrow \quad K_1(\Lambda_{S^*}) \quad \overset{\partial}{\longrightarrow} \quad K_0(\mathfrak{M}_H(G)) \longrightarrow \quad K_0(\Lambda)$$

- If R is one of the previous (localized) Iwasawa algebras, one often prefers to endow the dual of a left module with a left action, deduced from the right one via the involution $g \mapsto g^{-1}$. This is our convention for duality of normic systems and their limit modules. In this paragraph though, we leave right modules on the right, for the sake of clarity.
Consider a free O-module V of finite rank. For any O-algebra R (eg. $R = \Lambda_O(G)$), we denote V_R (resp. $_RV$) the right (resp. left) R-module $V \otimes_O R$

(resp. $R \otimes_O V$) deduced from V by extending scalars from O to R. Also, $(_RV)^* := Hom_R(_RV, R)$ (resp. $(V_R)^* := Hom_R(V_R, R)$) is systematically given its right (resp. left) R-module structure coming from the (R,R)-bimodule structure of R.

It might be useful to recall that one has a canonical isomorphism of left (resp. right) R-modules $(V_R)^* \simeq {}_R(V^*)$ (resp. $(_RV)^* \simeq (V^*)_R$) and canonical isomorphisms of O-algebras $End_R((V_R)^*) \simeq End_R(V_R)^{op}$ and $End((_RV)^*) \simeq End(_RV)^{op}$ (R-algebras). If now R is a central O-algebra (eg. if $R = \Lambda_O(G)$) we have moreover canonical isomorphisms of O-algebras $End_R(_RV) \simeq R^{op} \otimes_O End_O(V)$, $End_R(V_R) \simeq End_O(V) \otimes_O R$. These isomorphisms are subject to natural compatibilities, such as the commutativity of the following diagram:

$$
\begin{array}{ccccc}
End_R((_RV)^*) & \simeq & End_R(_RV)^{op} & \simeq & (R^{op} \otimes_O End_O(V))^{op} \\
\wr | & & & & \wr | \\
End_R((V^*)_R) & \simeq & End_O(V^*) \otimes_O R & \simeq & End_O(V)^{op} \otimes_O R
\end{array}
$$

Consider now an O-Artin representation $\rho : G \to Aut_O(V)$. By O-Artin representation we mean that V is as above and ρ has a finite image. Consider the unique homomorphism of $\mathbb{Z}_p$-algebras

$$\Phi : \Lambda(G) \to End_{\Lambda_O(G)}(V_{\Lambda_O(G)})$$

sending $g \in G \subset \Lambda_O(G)$ to $v \otimes \lambda \mapsto \rho(g)(v) \otimes g\lambda$.

By functoriality of K_1, one has a commutative diagram:

where tw_ρ is the map sending $[P, f] \in K_1(\Lambda(G))$ to $[(V_{\Lambda_O(G)})^* \otimes_{\Lambda(G)} P, id \otimes f] \in K_1(\Lambda_O(G))$ where $(V_{\Lambda_O(G)})^*$ is viewed as a $(\Lambda_O(G), \Lambda(G))$-bimodule, the left action of $\Lambda_O(G)$ being the obvious one and the right action of $\Lambda(G)$ being deduced via Φ from the natural right action of $End_{\Lambda_O(G)}(V_{\Lambda_O(G)})$. Furthermore the image of this element in $K_1(O)$ is $[V^* \otimes_{\Lambda(G)} P, id_{V^*} \otimes f]$ where the right $\Lambda(G)$-module structure of V^* is deduced from the right action of G on V^*: $g \mapsto \rho(g)^*$. The composed homomorphism

$$
\begin{array}{ccc}
K_1(\Lambda(G)) & \to & K_1(O) \simeq O^\times \\
[P, f] & \mapsto & det_O(id_{V^*} \otimes f | V^* \otimes_{\Lambda(G)} P)
\end{array}
$$

where det_O denotes the usual (commutative) determinant, will simply be denoted ρ.

Remark 3.4. (i) The element $tw_\rho([P, f])$ is also equal to $[V^* \otimes_{\mathbb{Z}_p} P, id \otimes f]$. Indeed, there are isomorphisms of left $\Lambda_O(G)$ modules

$$(V_{\Lambda_O(G)})^* \otimes_{\Lambda(G)} P \quad \overset{\sim}{\underset{}{\leftarrow}} \quad {}_{\Lambda_O(G)}(V^*) \otimes_{\Lambda(G)} P$$
$$\overset{\sim}{\rightarrow} \quad V^* \otimes_{\mathbb{Z}_p} P$$

where the $(\Lambda_O(G), \Lambda(G))$-bimodule structure of ${}_{\Lambda_O(G)}(V^*)$ is given by $h.(\lambda \otimes \phi) = h\lambda \otimes \phi$ and $(\lambda \otimes \phi).h = \lambda h \otimes \phi \circ (\rho(h)))$, the left action of $\Lambda_O(G)$ on $V^* \otimes_{\mathbb{Z}_p} P$ is given by $h.(\phi \otimes p) = \phi \circ (\rho(h^{-1})) \otimes hp$, the first isomorphism is induced by ${}_{\Lambda_O(G)}(V^*) \to (V_{\Lambda_O(G)})^*$, $\lambda \otimes \phi \mapsto (v \otimes \mu \mapsto \lambda\phi(v)\mu)$ and the second is given by $g \otimes \phi \otimes p \mapsto \phi \circ \rho(g^{-1}) \otimes gp$.
(ii) Let us examine the simple case where P is free with basis $e_1, \dots, e_n$ to fix the ideas. The element $[V^* \otimes_{\mathbb{Z}_p} P, id \otimes f]$ occurring in (i) above has then a more convenient description as follows. Sending $\phi \otimes ge_i$ to $\phi \circ \rho(g) \otimes ge_i$ realises an isomorphism of $V^* \otimes_{\mathbb{Z}_p} P$ with $V^* \otimes_{\mathbb{Z}_p} P$ viewed as a left $\Lambda_O(G)$-module via $h.(\phi \otimes \lambda) = \phi \otimes h\lambda$. Through this isomorphism $id \otimes f$ translates as the automorphism sending $\phi \otimes \lambda_j e_j$ to $\iota(\lambda_j f(e_j))$ where $\iota : V^* \otimes_{\mathbb{Z}_p} P \to V^* \otimes_{\mathbb{Z}_p} P$ sends $\phi \otimes ge_i$ to $\phi \circ \rho(g) \otimes ge_i$. In particular we find that via the determinant isomorphism $K_1(O) \simeq O^\times$, $\rho([P, f])$ is nothing but the determinant of the automorphism of $O \otimes_{\Lambda_O(G)} (V^* \otimes_{\mathbb{Z}_p} P) \simeq \oplus_i V^* \otimes e_i$ sending $\phi \otimes e_j$ to $\sum_i \phi \circ \rho(f_i(e_j))$ if $f(e_j) = \sum f_i(e_j)e_i$.

Next, following [CFKSV], we consider the extension of ρ to a map

$$\rho : K_1(\Lambda(G)_{S^*}) \to L^\times \cup \{0; \infty\} \tag{10}$$

defined as $\epsilon \rho_{Q_O(\Gamma)}$ where:
- the map $\epsilon : Q_O(\Gamma)^\times \to L \cup \{\infty\}$ coincides with $\Lambda_O(\Gamma)_I \to L$, the localization of the augmentation map at the augmentation ideal I, and takes the value ∞ elsewhere (ie. at the elements of $Q_O(\Gamma)$ which are not integral at I).
- the map $\rho_{Q_O(\Gamma)} : K_1(\Lambda(G)_{S^*}) \to Q_O(\Gamma)^\times$ is defined by composing the obvious localized version of tw_ρ, $K_1(\Lambda(G)_{S^*}) \to K_1(\Lambda_O(G)_{S^*})$, the fonctioriality map $K_1(\Lambda_O(G)_{S^*}) \to K_1(Q_O(\Gamma))$ and the isomorphism $det_{Q_O(\Gamma)} : K_1(Q_O(\Gamma)) \simeq Q_O(\Gamma)^\times$.

The map ρ is multiplicative in the sense of the usual partial multiplication

$$(L \times L) \cup ((L^\times \cup \{\infty\}) \times (L^\times \cup \{\infty\})) \to L \cup \{\infty\}$$

This means that for $x, y \in K_1(\Lambda(G)_{S^*})$:
(i) $\rho(x^{-1}) = \rho(x)^{-1}$.
(ii) the formula $\rho(x)\rho(y) = \rho(xy)$ is true as soon as it makes sense, ie. whenever $(\rho(x), \rho(y)) \neq (0, \infty), (\infty, 0)$.

Lemma 3.5. Let $[P, f]$ in $K_1(\Lambda(G)_{S^*})$ and consider its dual $[P^\vee, f^\vee]$ where $(-)^\vee$ stands for $Hom_{\Lambda(G)_{S^*}}(-, \Lambda(G)_{S^*})$ viewed as a left $\Lambda(G)_{S^*}$-module via the involution $g \mapsto g^{-1}$. If ρ is an Artin representation with contragredient $\rho^\vee$ then in $K_1(L)$:

$$\rho([P, f]) = \rho^\vee([P^\vee, f^\vee])$$

Proof. One easily reduces to the analogous statement for $\Lambda = \Lambda(G)$ in place of $\Lambda(G)_{S^*}$. In virtue of Rem. 3.4 (*i*), it is then sufficient to prove that for a finitely generated projective P over $\Lambda(G)$ we have in $L^\times$:

$$det_L(id \otimes id \otimes f|L \otimes_{\Lambda_O} (V^* \otimes_{\mathbb{Z}_p} P))$$
$$= det_L(id \otimes id \otimes Hom_\Lambda(f, \Lambda)|L \otimes_{\Lambda_O} ((V^\vee)^* \otimes_{\mathbb{Z}_p} Hom_\Lambda(P, \Lambda)))$$

the action of Λ_O on $V^* \otimes_{\mathbb{Z}_p} P$ (resp. $(V^\vee)^* \otimes_{\mathbb{Z}_p} Hom_\Lambda(P, \Lambda) = V \otimes_{\mathbb{Z}_p} Hom_\Lambda(P, \Lambda)$) being given by $g.(\phi \otimes p) = \phi \circ \rho(g^{-1}) \otimes gp$ (resp. $g.(v \otimes \phi) = \rho(g)v \otimes (p \mapsto \phi(p)g^{-1})$). Since the determinant of an endomorphism and its transpose are equal it is sufficient to build an isomorphism

$$Hom_L(L \otimes_{\Lambda_O} (V^* \otimes_{\mathbb{Z}_p} P), L) \simeq L \otimes_{\Lambda_O} ((V^\vee)^* \otimes_{\mathbb{Z}_p} Hom_\Lambda(P, \Lambda)) \qquad (11)$$

identifying $Hom_L(id \otimes id \otimes f, L)$ to $id \otimes id \otimes Hom_\Lambda(f, \Lambda)$. Let us examine the left term of (11). Let us chose n such that ρ factors through $G_n = G/H_n$. Denoting $V_L = L \otimes_O V$ we have the following series of natural isomorphisms

$$
\begin{aligned}
Hom_L(L \otimes_{\Lambda_O} (V^* \otimes_{\mathbb{Z}_p} P), L) &\simeq Hom_L((V_L^* \otimes_{\mathbb{Z}_p} P)_G, L) \\
&\simeq Hom_L(V_L^* \otimes_{\mathbb{Z}_p} P_{H_n}, L)^{G_n} \\
&\simeq (V_L \otimes_L Hom_{\mathbb{Z}_p}(P_{H_n}, L))^{G_n} \\
&\simeq (V_L \otimes_L Hom_{\mathbb{Z}_p}(P_{H_n}, L))_{G_n}
\end{aligned}
$$

where the invariants or coinvariants occurring respectively in the second, third, third and fourth term are taken with respect to the following left actions of G or G_n: $g.(\phi \otimes p) = \phi \circ \rho(g^{-1}) \otimes gp$, $g.\psi = (\phi \otimes p \mapsto \psi(\phi \circ \rho(g) \otimes g^{-1}p))$, $g.(v \otimes \phi) = \rho(g)v \otimes (p \mapsto \phi(g^{-1}p))$, $g.(v \otimes \phi) = \rho(g)v \otimes (p \mapsto \phi(g^{-1}p))$. Regarding the second term of (11) we have

$$
\begin{aligned}
L \otimes_{\Lambda_O} ((V^\vee)^* \otimes_{\mathbb{Z}_p} Hom_\Lambda(P, \Lambda)) &\simeq (V_L \otimes_{\mathbb{Z}_p} Hom_\Lambda(P, \Lambda))_G \\
&\simeq (V_L \otimes_{\mathbb{Z}_p} Hom_\Lambda(P, \Lambda)_{H_n})_{G_n} \\
&\simeq (V_L \otimes_L Hom_{\mathbb{Z}_p[G_n]}(P_{H_n}, L[G_n]))_{G_n}
\end{aligned}
$$

where the coinvariants occurring respectively in the second, third and fourth term are taken with respect to the following left action of G, H_n, G_n and G_n: $g.(v \otimes \phi) = \rho(g)v \otimes (p \mapsto \phi(p)g^{-1})$, $h.\phi = (p \mapsto \phi(p)h^{-1})$, $g.(v \otimes \phi) = \rho(g)v \otimes (p \mapsto \phi(p)g^{-1})$, $g.(v \otimes \phi) = \rho(g)v \otimes (p \mapsto \phi(p)g^{-1})$. We finally obtain (11) by noticing the isomorphism

$$
\begin{aligned}
Hom_{\mathbb{Z}_p}(P_{H_n}, L) &\simeq Hom_{\mathbb{Z}_p[G_n]}(P_{H_n}, L[G_n]) \\
\phi &\mapsto (p \mapsto \sum_{g \in G_n} \phi(gp)[g^{-1}])
\end{aligned}
$$

The reader may easily check by himself that (11) is compatible with the automorphisms induced by f as desired. $\qquad\square$

The next results concerning the computation of (10) composed with (8) will be useful to us.

Lemma 3.6. *Consider an automorphism f of an object C of $D^p(\Lambda(G)_{S^*})$ as well as its determinant $Det_{\Lambda(G)_{S^*}}(f) \in K_1(\Lambda(G)_{S^*})$ (cf. (8)). In $Q_O(\Gamma)^\times$ we have:*

$$\rho_{Q_O(\Gamma)}(Det_{\Lambda(G)_{S^*}}(f)) = \prod_q det_{Q_O(\Gamma)}(H^q(id \otimes f)|H^q((V_{Q_O(\Gamma)})^* \overset{L}{\otimes}_{\Lambda(G)_S^*} C))^{(-1)^q}$$

where the tensor product occurring on the right hand side is taken with respect to the right $\Lambda(G)_{S^*}$-module structure of $(V_{Q_O(\Gamma)})^*$ defined using Φ in the obvious way.

Proof. We can always assume that C is strictly perfect. In this case we have by definition of $\rho_{Q_O(\Gamma)}$ and compatibility of Det with scalar extension:

$$\rho_{Q_O(\Gamma)}(Det_{\Lambda(G)_{S^*}}(f)) = det_{Q_O(\Gamma)}(Det_{Q_O(\Gamma)}(id \otimes f))$$

where $id \otimes f$ is the automorphism of $(V_{Q_O(\Gamma)})^* \otimes_{\Lambda(G)_S^*} C \in D^p(Q_O(\Gamma))$ deduced from f. Now $Q_O(\Gamma)$ being a field, the complex $(V_{Q_O(\Gamma)})^* \otimes_{\Lambda(G)_{S^*}} C$ is cohomologically perfect, and thus

$$det_{Q_O(\Gamma)} Det_{Q_O(\Gamma)}(id \otimes f)$$
$$= \prod_q det_{Q_O(\Gamma)} Det_{Q_O(\Gamma)}(H^q(id \otimes f))^{(-1)^q}$$
$$= \prod_q det_{Q_O(\Gamma)}(H^q(id \otimes f)|H^q((V_{Q_O(\Gamma)})^* \otimes_{\Lambda(G)_{S^*}} C))^{(-1)^q}$$

$\square$

Lemma 3.7. *Let ϵ, I, $\Lambda_O(\Gamma)_I$ as in (10). Consider a finitely generated $\Lambda_O(\Gamma)_I$-module M together with an endomorphism $f : M \to M$ such that $Q_O(\Gamma) \otimes_{\Lambda_O(\Gamma)_I} f$ is invertible. The formula*

$$\epsilon det_{Q_O(\Gamma)}(id \otimes f|Q_O(\Gamma) \otimes_{\Lambda_O(\Gamma)_I} M)$$
$$= det_L(Tor_0^{\Lambda_O(\Gamma)_I}(id, f)|Tor_0^{\Lambda_O(\Gamma)_I}(L, M))$$
$$. det_L(Tor_1^{\Lambda_O(\Gamma)_I}(id, f)|Tor_1^{\Lambda_O(\Gamma)_I}(L, M))^{-1}$$

is true whenever the right hand term makes sense in $L^\times \cup \{0, \infty\}$.

Proof. Let $M' \subset M$ denote the $\Lambda_O(\Gamma)_I$-torsion submodule of M, and $M'' := M/M'$. Let us denote $f' : M' \to M'$, $f'' : M'' \to M''$ the endomorphisms induced by f.
On the one hand, $Q_O(\Gamma) \otimes_{\Lambda_O(\Gamma)_I} M' = 0$, and thus

$$det_{Q_O(\Gamma)}(id \otimes f|Q_O(\Gamma) \otimes_{\Lambda_O(\Gamma)_I} M) = det_{Q_O(\Gamma)}(id \otimes f''|Q_O(\Gamma) \otimes_{\Lambda_O(\Gamma)_I} M'')$$

On the other hand, M'' is free since $\Lambda_O(\Gamma)_I$ is a discrete valuation ring. We thus have a short exact sequence and an isomorphism

$$Tor_0^{\Lambda_O(\Gamma)_I}(L, M') \hookrightarrow Tor_0^{\Lambda_O(\Gamma)_I}(L, M) \twoheadrightarrow Tor_0^{\Lambda_O(\Gamma)_I}(L, M'')$$
$$Tor_1^{\Lambda_O(\Gamma)_I}(L, M') \simeq Tor_1^{\Lambda_O(\Gamma)_I}(L, M)$$

both of which are compatible with f. Whence equalities:

$$det_L(Tor_0^{\Lambda_O(\Gamma)_I}(id,f)|Tor_0^{\Lambda_O(\Gamma)_I}(L,M))$$
$$= det_L(Tor_0^{\Lambda_O(\Gamma)_I}(id,f)|Tor_0^{\Lambda_O(\Gamma)_I}(L,M'))$$
$$.det_L(Tor_0^{\Lambda_O(\Gamma)_I}(id,f)|Tor_0^{\Lambda_O(\Gamma)_I}(L,M''))$$
$$and \quad det_L(Tor_1^{\Lambda_O(\Gamma)_I}(id,f)|Tor_1^{\Lambda_O(\Gamma)_I}(L,M))$$
$$= det_L(Tor_1^{\Lambda_O(\Gamma)_I}(id,f)|Tor_1^{\Lambda_O(\Gamma)_I}(L,M'))$$

Now M' is of finite length, and is thus subject to Koszul duality:

$$Hom_L(Tor_1^{\Lambda_O(\Gamma)_I}(L,M'),L) \simeq Tor_0^{\Lambda_O(\Gamma)_I}(L,Hom_L(M',L))$$

Since det_L isn't affected by L-linear duality, this shows that

$$det_L(Tor_1^{\Lambda_O(\Gamma)_I}(id,f)|Tor_1^{\Lambda_O(\Gamma)_I}(L,M'))$$
$$= det_L(Tor_0^{\Lambda_O(\Gamma)_I}(id,f)|Tor_0^{\Lambda_O(\Gamma)_I}(L,M'))$$

Now, in the formula to be proven, we see that the right hand term makes sense if and only if $det_L(Tor_0^{\Lambda_O(\Gamma)_I}(id,f)|Tor_0^{\Lambda_O(\Gamma)_I}(L,M'))$ is non zero (ie. if and only if $det_L(Tor_1^{\Lambda_O(\Gamma)_I}(id,f)|Tor_1^{\Lambda_O(\Gamma)_I}(L,M))$ is non zero) in which case the factors $det_L(Tor_q^{\Lambda_O(\Gamma)_I}(id,f)|Tor_q^{\Lambda_O(\Gamma)_I}(L,M'))$, $q = 0,1$, cancel each other and the desired formula reduces to the obvious equality

$$\epsilon det_{Q_O(\Gamma)}(id \otimes f|Q_O(\Gamma) \otimes_{\Lambda_O(\Gamma)_I} M'')$$
$$= det_L(Tor_0^{\Lambda_O(\Gamma)_I}(id,f)|Tor_0^{\Lambda_O(\Gamma)_I}(L,M''))$$

$\square$

4   SELMER COMPLEXES AND CRYSTALLINE COHOMOLOGY

In this section we define Selmer complexes for abelian varieties over a one variable function field. We begin with their basic properties and a duality theorem. In the semistable case we pursue with a review of the main result of [TV], which will be the cornerstone of our proof for the Iwasawa main conjecture.

4.1   PRELIMINARIES

Let us begin with some technical facts regarding derived categories.

4.1.1. We will frequently use derived categories of the type $D(\mathcal{C}^{\mathcal{I}})$, where $\mathcal{C}^{\mathcal{I}}$ denotes the category of functors $\mathcal{I} \to \mathcal{C}$, $\mathcal{C}$ being an abelian category. Note that $\mathcal{C}^{\mathcal{I}}$ is an abelian category. If $\mathcal{C}$ has enough injectives and if it has products indexed by the subsets of $Ob(\mathcal{I})$, then the same holds for $\mathcal{C}^{\mathcal{I}}$. If $\mathcal{C}$ is the category of modules on a ringed topos (E, A), $Mod(E, A)^{\mathcal{I}}$ naturally identifies with the category of modules of the ringed topos $(E^{\mathcal{I}}, A)$. This point of view offers

the possibility to consider the more general category $Mod(E^{\mathcal{I}}, A_{\cdot})$, with $A_{\cdot}$ a projective system of rings of E indexed by $\mathcal{I}$.

These categories are especially useful for some particular choices of $\mathcal{I}$ which we explain now.

- Let $\mathcal{I} = \mathbb{N}$ viewed as the category where $Hom(k, k')$ has one element if $k \geq k'$ and is empty otherwise (resp. $\mathcal{I} = \mathbb{N}^{op}$, resp. $\mathcal{I} = N_2 := \mathbb{N} \sqcup_{|\mathbb{N}|} \mathbb{N}^{op}$). In that case, the objects of the category $\mathcal{C}^{\mathcal{I}}$ are the projective systems of $\mathcal{C}$ indexed by integers (resp. the inductive systems of $\mathcal{C}$ indexed by integers, resp. the triples $((A_k)_k, (p_{k',k})_{k' \geq k}, (i_{k,k'})_{k \leq k'})$ where $((A_k)_k, (p_{k',k})_{k' \geq k})$ is in $\mathcal{C}^{\mathbb{N}}$ and $((A_k)_k, (i_{k,k'})_{k \leq k'})$ is in $\mathcal{C}^{\mathbb{N}^{op}}$).

- Let $\mathcal{I} = [1]^a$ for some integer $a \geq 1$, $[1]$ denoting the category $\{0, 1, \leq\}$. In that case, we think of objects of $D(\mathcal{C}^{\mathcal{I}})$ as $(a+1)$-uple naive complexes which are zero outside a specified range of the form $]-\infty, +\infty[\times [i_1, i_1 + 1] \times \ldots, \times [i_a, i_a + 1]$. The interest of this category lies in the fact that forming total complexes (with an appropriate sign convention which the interested reader is invited to specify) give rise to a triangulated functor $Tot_{a+1} : D(\mathcal{C}^{[1]^a}) \to D(\mathcal{C})$.

4.1.2. The case $\mathcal{I} = [1]$ is already interesting since Tot_2 gives rise to a functorial version of the mapping cone / mapping fiber construction. Let us discuss a fact which will be used repeatedly. A natural transformation $t : F_1 \to F_2$ between functors $\mathcal{C} \to \mathcal{C}'$ can be thought as a functor F taking its values in $\mathcal{C}'^{[1]}$. If $\mathcal{C}$ has enough injectives we can form the derived functor $RF : D(\mathcal{C}) \to D(\mathcal{C}'^{[1]})$. Next, we define $MF(t) := Tot_2 \circ RF$. Since the mapping fiber construction gives a functor from $D(\mathcal{C}'^{[1]})$ to that of distinguished triangles in $D(\mathcal{C}')$ we find a canonical distinguished triangle $MF(t)(C) \to RF_1(C) \to RF_2(C) \to MF(t)(C)[1]$ varying functorially with respect to C in $D(\mathcal{C})$. Consider furthermore $F_0(C) := Ker(F_1(C) \to F_2(C))$. Then F_0 defines a right derivable functor. Since $RF_0 \simeq MF(t_0)$ for $t_0 : F_0 \to 0$ and t_0 canonically maps to t, one gets a canonical morphism $RF_0 \to MF(t)$. This natural transformation is an isomorphism as long as the morphism $F_1(C) \to F_2(C)$ is epimorphic for injective C (because in that case the sequence $0 \to F_0(C) \to F_1(C) \to F_2(C) \to 0$ is exact and the natural transformation in question is the one occurring in the construction of the distinguished triangle associated to a short exact sequence).

4.2   Flat cohomology of C vanishing at Z

We review the definition of the functor of vanishing cohomology, which is a necessary tool to even state the comparison result we need from [TV] (see Sect. 4.4).

4.2.1. If S is any scheme, we denote

$$\epsilon : S_{FL} \to S_{et}$$

the natural morphism from the big flat topos of S (ie. the category of sheaves on Sch/S endowed with the topology generated by surjective families of flat morphisms of finite type) to the small étale one (ie. the category of sheaves

one the category Et/S of étale S-schemes endowed with the topology generated by surjective families, cf SGA 4, VII, 1.2). The morphism ϵ is functorial with respect to S, ie. there is a canonical isomorphism $f\epsilon_S \simeq \epsilon_{S'}f$ for any $f : S_{et} \to S'_{et}$ induced by a morphism of schemes $f : S \to S'$. However, the reader should be aware that the base change morphism $f^{-1}\epsilon_* \to \epsilon_*f^{-1}$ is certainly not an isomorphism in general if f is not étale.

DEFINITION 4.1. *Keep the above notations and let furthermore* $i : T \hookrightarrow S$ *be the inclusion of a closed subscheme as well as* $j : Y \to S$ *that of its open complement. We also assume given a projective system of torsors* $\underline{S} = (S_n, G_n)$ *where* G_n *is a finite quotient of a profinite group* G *and denote* $\Gamma(\underline{S}, -) : Mod(S_{et}, \mathbb{Z}) \to {}_{\underline{G}}Mod(\mathbb{Z})$ *the associated normic sections functor.*

We define the following.

(i) *The functor of* global sections vanishing at T,

$$\Gamma^T(S, -) : Mod(S_{FL}, \mathbb{Z}) \to Mod(\mathbb{Z})$$

   is defined as $\Gamma^T(S, F) := Ker(F(S) \to F(T))$.

(ii) *The functor of* étale sections vanishing at T,

$$\epsilon_*^T : Mod(S_{FL}, \mathbb{Z}) \to Mod(S_{et}, \mathbb{Z})$$

   is defined as $\epsilon_*^T F := Ker(\epsilon_* F \to \epsilon_* i_* i^{-1} F)$.

(iii) *The functor of* normic sections vanishing at T,

$$\Gamma^T(\underline{S}, -) : Mod(S_{FL}, \mathbb{Z}) \to {}_{\underline{G}}Mod(\mathbb{Z})$$

   is defined as $\Gamma^T(\underline{S}, F) := \Gamma(\underline{S}, \epsilon_*^T F)$. *Its* n^{th} *component is thus* $Ker(F(S_n) \to F(T_n))$ *where* $T_n := S_n \times_S T$.

Note that by definition, one may respectively retrieve $\Gamma^T(S, -)$ and $\Gamma^T(\underline{S}, -)$ from ϵ_*^T by forming global and normic sections (Def. 2.7).

LEMMA 4.2. *The above functors are right derivable and their derived functors are subject to natural distinguished triangles in* $D(S_{et})$, $D(Mod(\mathbb{Z}))$ *and* $D({}_{\underline{G}}Mod(\mathbb{Z}))$ *which are functorial with respect to* F *in* $D^+(S_{FL})$:

$$
\begin{array}{ccccccc}
R\epsilon_*^T F & \longrightarrow & R\epsilon_* F & \longrightarrow & Ri_* R\epsilon_* F_{|T} & \xrightarrow{+1} & \\
R\Gamma^T(S, F) & \longrightarrow & R\Gamma(S, F) & \longrightarrow & R\Gamma(T, F_{|T}) & \xrightarrow{+1} & \\
R\Gamma^T(\underline{S}, F) & \longrightarrow & R\Gamma(\underline{S}, F) & \longrightarrow & R\Gamma(\underline{T}, F_{|T}) & \xrightarrow{+1} & \\
\end{array}
$$

where $\underline{T}$ *denotes the projective systems of torsors deduced from* $\underline{S}$. *Taking nth components in the last distinguished triangles gives*

$$
R\Gamma^{T_n}(S, F) \longrightarrow R\Gamma(S_n, F_{|S_n}) \longrightarrow R\Gamma(T_n, F_{|T_n}) \xrightarrow{+1}
$$

Proof. Since injective abelian sheaves of S_{FL} are flasque in the sense of [SGA 4, exp V, 4.7], it follows from the discussion 4.1.2 that $R\epsilon_*^T$ identifies with the functorial mapping fiber of the natural transformation $R\epsilon_* \to R(\epsilon_* i_* i^{-1})$ arising from $\epsilon_* \to \epsilon_* i_* i^{-1}$. Now, i^{-1} preserves injectives (i is a localization morphism) and thus $R(\epsilon_* i_* i^{-1}) \simeq R(\epsilon_* i_*)i^{-1}$. The first distinguished triangle follows since $R(\epsilon_* i_*) \simeq R i_* R \epsilon_*$. The other cases are similar. $\square$

4.3 Selmer complexes

The purpose of this section is to give a tractable definition for (normic) Selmer complexes, to establish their relation to the complex of (normic) cohomology vanishing at Z (the relation with the complexes $R\Gamma_{ar,V}$ appearing in [KT] will also be explained briefly in Rem. 4.13), as well as a duality theorem.

4.3.1. We consider the following situation:

$$
\begin{array}{ccccc}
Z_v & \xrightarrow{z_v} & C_v & \xleftarrow{j_v} & U_v \\
\downarrow{\iota_{Z_v}} & & \downarrow{\iota_{C_v}} & & \downarrow{\iota_{U_v}} \\
Z & \xrightarrow{z} & C & \xleftarrow{j} & U
\end{array}
\tag{12}
$$

where:
- C is a connected proper smooth 1-dimensional $\mathbb{F}_p$-scheme with function field K and constant field k.
- Z is an effective divisor on C, ie. a finite union of 0-dimensionnal irreducible closed subschemes Z_v. We denote $Z_v^{red} = Spec(k_v)$ and $Z^{red} = \sqcup_v Z_v^{red}$ the underlying reduced schemes.
- U denotes the open subscheme of C which is complementary to Z.
- For each v in $|Z|$, we denote O_v the completion of the local ring of C at v, K_v its fraction field, $C_v = Spec(O_v)$ and $U_v = Spec(K_v)$.
- The arrows $i, j, j_v, i_v, \iota_{U_v}, \iota_{C_v}$ and ι_{Z_v} are the obvious ones.

We fix moreover a profinite Galois extension K_∞/K with group $G = \varprojlim_n G_n$ satisfying the following properties:
- G is a p-adic Lie group without p-torsion,
- K_∞/K is unramified everywhere and contains the constant $\mathbb{Z}_p$-extension $K_{ar} = K k_\infty / K$ whose Galois group is denoted Γ.
Thanks to the second assumption, there is an essentially unique projective system of torsors $\underline{C} = (C_n, G_n)$ over C corresponding to the Galois tower $\underline{K} = (K_n, G_n)$ over F.

Finally we fix an abelian variety A over K and denote $\mathcal{A}$ its Néron model over C. We always assume that A has good reduction outside Z (ie. $A_{|U}$ is an abelian scheme over U).

4.3.2. We place ourselves in the situation described in 4.3.1. The divisor Z will be allowed to vary, but its support will always contain the points of bad

reduction of A. Unless specified otherwise, cohomology is meant in the sense of big flat topoi. To define the Selmer complex we need the following intermediary functors. We use the notations $\iota_{C_v} : C_v \to C$, $j_v : U_v \to C_v$ for all places of C (not only those of Z).

- $Loc_{(U)} : Mod(C_{FL}, \mathbb{Z}) \to Mod(C_{et}, \mathbb{Z})^{[1]}$ is the functor taking $F \in Mod(C_{FL}, \mathbb{Z})$ to

$$\left[j_* \epsilon_* F_{|U} \quad \longrightarrow \quad (\prod_{v \in |U|} \iota_{C_v,*} \epsilon_* F_{|C_v}) \times (\prod_{v \in |Z|} \iota_{C_v,*} j_{v,*} \epsilon_* F_{|U_v}) \right]$$

Let $\mathcal{U}$ denote the filtered set of open subschemes of C. Since for $Z \subset Z'$, there is an obvious natural transformation $Loc_{(U)} \to Loc_{(U')}$, we also have a functor $Loc_{(\mathcal{U})} : Mod(C_{FL}, \mathbb{Z}) \to Mod(C_{et}, \mathbb{Z})^{[1] \times \mathcal{U}}$.

- $\tau_{\geq [0\,1]} : D(Mod(\mathbb{Z})^{[1]}) \to D(Mod(\mathbb{Z})^{[1]})$ (resp. $D(\underline{G}Mod(\mathbb{Z})^{[1]}) \to D(\underline{G}Mod(\mathbb{Z})^{[1]})$, resp. $D(\underline{G}Mod(\mathbb{Z})^{[1] \times \mathcal{U}}) \to D(\underline{G}Mod(\mathbb{Z})^{[1] \times \mathcal{U}})$) is described as

$$\begin{bmatrix} A & \longrightarrow & B \end{bmatrix} \quad \mapsto \quad \begin{bmatrix} A & \longrightarrow & \tau_{\geq 1} B \end{bmatrix}$$

DEFINITION 4.3. *We define Selmer functors as follows.*
(*i*) *The* (U)-Selmer functor

$$Sel_{(U)}(C, -) : D^+(C_{FL}) \to D^+(Mod(\mathbb{Z}))$$

is defined as $Sel_{(U)}(C, -) := Tot_2 \circ \tau_{\geq [0\,1]} \circ R\Gamma(C, -) \circ RLoc_{(U)}$.
(*ii*) *The* (U)-normic Selmer functor

$$Sel_{(U)}(\underline{C}, -) : D^+(C_{FL}) \to D^+(\underline{G}Mod(\mathbb{Z}))$$

is defined as $Sel_{(U)}(\underline{C}, -) := Tot_2 \circ \tau_{\geq [0\,1]} \circ R\Gamma(\underline{C}, -) \circ RLoc_{(U)}$, *where* $\Gamma(\underline{C}, -)$ *is the normic section functor of Def.* 2.7.
(*iii*) *The* $\mathcal{U}$-normic Selmer functor

$$Sel_{(\mathcal{U})}(\underline{C}, -) : D^+(C_{FL}) \to D^+(\underline{G}Mod(\mathbb{Z})^{\mathcal{U}})$$

is defined as $Sel_{(\mathcal{U})}(\underline{C}, -) := Tot_2 \circ \tau_{\geq [0\,1]} \circ R\Gamma(\underline{C}, -) \circ RLoc_{(\mathcal{U})}$.

Let us point out that those Selmer functors are not triangulated, since their definition involves truncations.

REMARK 4.4. (*i*) *The functor* (*ii*) *(resp.* (*i*)*) can be retrieved from functor* (*iii*) *(resp.* (*ii*)*) by taking* U-*components of the inductive system (resp. component 0 of the normic systems).*
(*ii*) *It follows immediately from the definition that* $Sel_{(U)}(\underline{C}, F)$ *fits into a distinguished triangle*

$$Sel_{(U)}(\underline{C}, F) \quad \to \quad R\Gamma(\underline{U}, F) \quad \to \quad (\prod_{v \in |U|} \tau_{\geq 1} R\Gamma(\underline{C}_v, F)) \times (\prod_{v \in |Z|} \tau_{\geq 1} R\Gamma(\underline{U}_v, F)) \quad \overset{+1}{\to}$$

and similarly for $Sel_{(U)}(C, F)$. *In the literature, Selmer complexes are usually designed to fit into a distinguished triangle where* $\tau_{\geq 1} R\Gamma(\underline{C}_v, F))$ *does not*

appear. This factor will disappear when F is the Néron model of an abelian variety with good reduction over U (see Prop. 4.7 below). The reason why we add this term is to get functoriality with respect to U. Note that the functoriality of $Sel_{(U)}(\underline{C}, F)$ with respect to U not only occurs in the derived category, but already at the level of complexes. This fact is reflected by the existence of $Sel_{(\mathcal{U})}(\underline{C}, F)$.

Lemma 4.5. *Let $Z \subset Z'$, $U' = C - Z'$, $U = C - Z$, so that $U' \subset U$ and let $\mathcal{A}/C$ be as in 4.3.1.*
(i) The following natural morphisms in $D(\underline{G}Mod(\mathbb{Z}))$ are invertible:

$$Sel_{(U)}(\underline{C}, \mathcal{A}) \xrightarrow{\ (1)\ } Sel_{(U')}(\underline{C}, \mathcal{A}) \xrightarrow{\ (2)\ } \varinjlim_{\mathcal{U}} Sel_{(\mathcal{U})}(\underline{C}, \mathcal{A})$$

(ii) If Φ_v / Z_v^{red} denotes the component group of $\mathcal{A}$ at v, we have

$$
\begin{aligned}
\tau_{\geq 1} R\Gamma(\underline{C}_v, \mathcal{A}_{|C_v}) &\simeq \tau_{\geq 1} R\Gamma(\underline{Z}_v^{red}, \mathcal{A}_{|Z_v^{red}}) \\
&\simeq H^1(\underline{Z}_v^{red}, \Phi_v)[-1]
\end{aligned}
\tag{13}
$$

This group is in particular zero if $\mathcal{A}$ has good reduction at v.

Proof. (i) It suffices to prove the assertion about (1). For any $F \in D^+(C_{FL})$, one has a tautological diagram in $D(\underline{G}Mod(\mathbb{Z}))$ with distinguished rows and columns as follows:

$$
\begin{array}{ccccccc}
Sel_{(U)}(\underline{C}, F) & \to & R\Gamma(\underline{U}) & \to & (\prod_{v \in |U|} \tau_{\geq 1} R\Gamma(\underline{C}_v)) \times (\prod_{v \in |Z|} \tau_{\geq 1} R\Gamma(\underline{U}_v)) & \overset{+1}{\to} \\
a\downarrow & & b\downarrow & & c\downarrow & \\
Sel_{(U')}(\underline{C}, F) & \to & R\Gamma(\underline{U'}) & \to & (\prod_{v \in |U'|} \tau_{\geq 1} R\Gamma(\underline{C}_v)) \times (\prod_{v \in |Z'|} \tau_{\geq 1} R\Gamma(\underline{U}_v)) & \overset{+1}{\to} \\
\downarrow & & \downarrow & & \downarrow & \\
Cone(a) & \to & Cone(b) & \to & Cone(c) & \overset{+1}{\to} \\
+1\downarrow & & +1\downarrow & & +1\downarrow &
\end{array}
$$

where we wrote $R\Gamma(\underline{U})$ instead of $R\Gamma(\underline{U}, F_{|U})$ by lack of space and similarly for U', C_v, and U_v. The morphism b is induced by restriction from U to U'. The morphism c is induced by restriction from C_v to U_v for $v \in |Z'|\backslash|Z| = |U|\backslash|U'|$ and is the identity for other v's. The definition of a, $Cone(a)$, $Cone(b)$, $Cone(c)$ is tautological. Let us compute these mapping cones.

To begin with, we note that by definition of relative cohomology

$$Cone(b) \simeq R\Gamma_{U\backslash U'}(\underline{U}, F_{|U})[1]$$

Next we note that there is an natural morphism

$$\prod_{v \in |Z'|\backslash|Z|} (R\Gamma_{Z_v}(\underline{U}_v, F_{|U_v})[1]) \to Cone(c)$$

which is invertible if the restriction morphisms

$$R\Gamma(\underline{C}_v, F_{|C_v}) \to R\Gamma(\underline{U}_v, F_{|U_v})$$

are invertible in degree 0. This is the case for $F = \mathcal{A}$ by the Néron extension property.

The morphism $Cone(b) \to Cone(c)$ is thus induced by

$$R\Gamma_{U\setminus U'}(\underline{U}, \mathcal{A}) \to \prod_{v \in |Z'|\setminus|Z|} R\Gamma_{Z_v}(\underline{C}_v, \mathcal{A})$$

and is thus an isomorphism by [Mi1], III, 1.28 and [Mi2], III, 7.14. It follows that $Cone(a) = 0$, ie. that a is an isomorphism.

(ii) The first isomorphism is by [Mi1], III 3.11 and the second is by Lang's lemma for the smooth connected algebraic group $\mathcal{A}^0_{|Z^{red}_{n,v}}$ ([La], Thm. 2).   □

4.3.3. We are now in position to define the Selmer complex of A/K. The reader will retrieve the usual description below (Prop. 4.7 (iii), Cor. 4.10).

DEFINITION 4.6. *We define the following:*
(i) *The* normic Selmer complex *of A is defined in $D^b(_G Mod(\mathbb{Z}))$ as*

$$Sel(A/\underline{K}) := \varinjlim_{\mathcal{U}} Sel_{(\mathcal{U})}(\underline{C}, \mathcal{A}).$$

(ii) *The* normic $p^{\cdot}$-Selmer complex *of A is defined in $D^b(_G Mod(\mathbb{Z}_p)^{N_2})$ as*

$$Sel_{p^{\cdot}}(A/\underline{K}) := \mathbb{Z}/p^{\cdot} \overset{L}{\otimes} Sel(A/\underline{K})[-1]$$

(iii) *The normic p^{∞}-Selmer complex of A is defined in $D^b(_G Mod(\mathbb{Z}_p))$ as*

$$Sel_{p^{\infty}}(A/\underline{K}) := \varinjlim Sel_{p^{\cdot}}(A/\underline{K})$$

(iv) *The normic T_p-Selmer complex of A is defined in $D^b(_G Mod(\mathbb{Z}_p))$ as*

$$Sel_{T_p}(A/\underline{K}) := R\varprojlim Sel_{p^{\cdot}}(A/\underline{K})$$

The following proposition summarizes the relations between the normic Selmer complex of an abelian variety, its cohomology over C, its cohomology vanishing at Z, its cohomology over U and its compactly supported cohomology over U.

PROPOSITION 4.7. (i) $Sel(A/\underline{K})$ *fits into canonical distinguished triangles as follows.*

$$Sel(A/\underline{K}) \longrightarrow R\Gamma(\underline{C}, \mathcal{A}) \longrightarrow \prod_{v \in |Z|} H^1(\underline{C}_v, \mathcal{A}_{|C_v})[-1] \overset{+1}{\longrightarrow}$$

$$Sel(A/\underline{K}) \longrightarrow R\Gamma(\underline{U}, \mathcal{A}_{|U}) \longrightarrow \prod_{v \in |Z|} H^1(\underline{U}_v, \mathcal{A}_{|U_v})[-1] \overset{+1}{\longrightarrow}$$

$$Sel(A/\underline{K}) \longrightarrow R\Gamma(\underline{K}, A) \longrightarrow \bigoplus_{v \in |C|} H^1(\underline{U}_v, A_{|U_v})[-1] \overset{+1}{\longrightarrow}$$

(ii) $Sel_{p^{\cdot}}(A/\underline{K})$ compares to cohomology over C (resp. vanishing at Z) via canonical distinguished triangles as follows.

$$Sel_{p^{\cdot}}(A/\underline{K}) \to R\Gamma(\underline{C}, \mathbb{Z}/p^{\cdot} \overset{L}{\otimes} \mathcal{A}[-1]) \to \prod_{v\in|Z|} \mathbb{Z}/p^{\cdot} \overset{L}{\otimes} H^1(\underline{k}_v, \Phi_v)[-2] \overset{+1}{\to}$$

$$R\Gamma^Z(\underline{C}, \mathbb{Z}/p^{\cdot} \overset{L}{\otimes} \mathcal{A}[-1]) \to Sel_{p^{\cdot}}(A/\underline{K}) \to \oplus_{v\in|Z|}\mathbb{Z}/p^{\cdot} \overset{L}{\otimes} \mathcal{A}(\underline{Z}_v)[-1] \overset{+1}{\to}$$

(iii) $Sel_{p^{\cdot}}(A/\underline{K})$ compares to (resp. compactly supported) cohomology over U via canonical distinguished triangles as follows.

$$
\begin{array}{ccccccc}
Sel_{p^{\cdot}}(A/\underline{K}) & \to & R\Gamma(\underline{U}, (\mathcal{A}_{|U,p^{\cdot}})) & \to & \prod_{v\in|Z|} \mathbb{Z}/p^{\cdot} \overset{L}{\otimes} H^1(\underline{K}_v, A)[-2] & \overset{+1}{\to} \\
R\Gamma_c(\underline{U}, (\mathcal{A}_{|U,p^{\cdot}})) & \to & Sel_{p^{\cdot}}(A/\underline{K}) & \to & \prod_{v\in|Z|} \mathbb{Z}/p^{\cdot} \overset{L}{\otimes} A(\underline{K}_v)[-1] & \overset{+1}{\to}
\end{array}
$$

Proof. (i) The first and second distinguished triangles follow immediately from Rem. 4.4 (ii) and Prop. 4.5 (ii). The third one follows from the second one since étale cohomology over $Spec(K) = \varprojlim_{U\in\mathcal{U}} U$ can be computed as a limit by [SGA4], VII, 5.7 (note that the cohomology of $\mathcal{A}$ can be computed with respect to the étale topology since it is a smooth algebraic group).

(ii) The first distinguished triangle is deduced from the first one of (i) by applying $\mathbb{Z}/p^{\cdot} \otimes^L (-)[-1]$ (observe that this functor commutes to $R\Gamma(\underline{C}, -)$ and use (13)). To get the second one, we remark that the commutative diagram

$$
\begin{array}{ccc}
R\Gamma(\underline{C}, \mathcal{A}) & \xrightarrow{sp_Z} & R\Gamma(\underline{Z}, \mathcal{A}_{|Z}) \\
{\scriptstyle d}\downarrow \| & & {\scriptstyle d'}\downarrow \\
R\Gamma(\underline{C}, \mathcal{A}) & \xrightarrow{sp_Z^1} & \tau_{\geq 1}R\Gamma(\underline{Z}, \mathcal{A}_{|Z}) \\
{\scriptstyle c}\uparrow \| & & {\scriptstyle c'}\uparrow \\
R\Gamma(\underline{C}, \mathcal{A}) & \xrightarrow{loc_C^1} & \prod_{v\in|Z|} \tau_{\geq 1}R\Gamma(\underline{C}_v, \mathcal{A}_{|C_v})
\end{array}
$$

in the derived category of $_G Mod(\mathbb{Z})$ has an canonical counterpart in the derived category of diagrams of $_G \widetilde{Mod}(\mathbb{Z})$ of this form since it only uses truncation and functoriality with respect to the base. It follows from this remark that there is a canonical meaning for the mapping fiber of horizontal arrows and that there are canonical morphisms between them. The mapping fiber of the middle one is naturally isomorphic to $Sel(A/\underline{K})$ (use Prop. 4.5 (i), (ii)). By Prop. 4.5(ii), c' is an isomorphism and we thus have

$$Sel(A/\underline{K}) \simeq MF(loc_C^1) \simeq MF(sp_Z^1)$$

Now d being an isomorphism, we get a canonical distinguished triangle $MF(sp_Z) \to MF(sp_Z^1) \to MF(d') \to MF(sp_Z)[1]$, ie.

$$R\Gamma^Z(\underline{C}, \mathcal{A}) \to Sel(A/\underline{K}) \to H^0(Z, \mathcal{A}_{|Z}) \overset{+1}{\to}$$

The second distinguished triangle follows by applying $\mathbb{Z}/p^{\cdot} \otimes^L (-)[-1]$.

(iii) The first distinguished triangle is deduced from the second of (i) by applying $\mathbb{Z}/p^{\cdot} \otimes^L (-)[-1]$. For the second one, we observe that the commutative diagram

$$
\begin{array}{ccc}
R\Gamma(\underline{U}, \mathcal{A}_{|U}) & \xrightarrow{\,loc_U^1\,} & \prod_{v \in |Z|} \tau_{\geq 1} R\Gamma(\underline{U}_v, \mathcal{A}_{|U_v}) \\[4pt]
{\scriptstyle e} \uparrow \| & & {\scriptstyle e'} \uparrow \\[4pt]
R\Gamma(\underline{U}, \mathcal{A}_{|U}) & \xrightarrow{\,loc_U\,} & \prod_{v \in |Z|} R\Gamma(\underline{U}_v, \mathcal{A}_{|U_v})
\end{array}
$$

has an obvious counterpart in the derived category of such diagrams since it only uses truncation and functoriality with respect to the base. This justifies working with functorial mapping fibers as in (i) and (ii). We get a canonical distinguished triangle $MF(loc_U) \to MF(loc_U^1) \to MF(e') \to MF(loc_U)[1]$, ie.

$$
R\Gamma_c(\underline{U}, \mathcal{A}_{|U}) \to Sel(A/\underline{K}) \to \prod_{v \in |Z|} H^0(\underline{U}_v, \mathcal{A}_{|U_v}) \xrightarrow{+1}
$$

The second distinguished triangle follows by applying $\mathbb{Z}/p^{\cdot} \otimes^L (-)[-1]$. $\qquad\square$

PROPOSITION 4.8. *Let $\hat{A}/K$ denote the abelian variety which is dual to A/K. There is a canonical duality isomorphism in $D^{[0,3]}(\underline{G}Mod(\mathbb{Z}_p)^{N_2})$:*

$$
Sel_{p^{\cdot}}(\hat{A}/\underline{K}) \simeq Sel_{p^{\cdot}}(A/\underline{K})^{\vee}[-3]
$$

where $(-)^{\vee}$ means the exact functor $Hom_{\mathbb{Z}_p}(-, \mathbb{Q}_p/\mathbb{Z}_p)$.

Proof. That $Sel_{p^{\cdot}}(\hat{A}/\underline{K})$ has no cohomology outside $[0,3]$ follows for instance from the third distinguished triangle in Prop. 4.7 (i) since K has strict cohomological dimension ≤ 3. Let us come to the duality isomorphism. A sketch of proof is given in [KT] for a weaker statement (cf *loc. cit* 2.4) by combining local duality over U_v, $v \in |Z|$ together with global duality over U. Here we concentrate on the ungrateful task of checking that it is indeed possible to prove the result in the setting of derived categories of normic systems. For this purpose referring to a collection of pairings at each level n and checking compatibilities is not sufficient. The main point is thus to build the required duality morphism in $D^b(\underline{G}Mod(\mathbb{Z}_p)^{N_2})$. That it is an isomorphism, will follow from the compatibility of our construction with that of [Mi2], III.

For the purpose of the proof, let $B/K := \hat{A}/K$, $\mathcal{B}/\mathcal{C}$ its Néron model and Ψ/C its component group. Recall ([SGA7] or [Mi2]) that the Poincaré biextension $P : \mathcal{A}_{|U} \otimes^L \mathcal{B}_{|U} \to \mathbb{G}_{m|U}[1]$ (which is essentially given by definition of $\hat{B}/F$) induces the Grothendieck pairing $G : z^{-1}\Phi \otimes z^{-1}\Psi \to \mathbb{Q}/\mathbb{Z}$ (essentially via $j_*\mathbb{G}_m[1] \to z_*\mathbb{Z}[1] \leftarrow z_*\mathbb{Q}/\mathbb{Z}$). We know that P extends over C to a (unique) biextension $\mathcal{A}^{\Phi'} \otimes_L \mathcal{B}^{\Psi'} \to \mathbb{G}_m[1]$ as long as $\Phi' \subset \Phi$ and $\Psi' \subset \Psi$ are orthogonal with respect to G. Taking $(\Phi', \Psi') = (0, \Psi)$ and then $(\Phi, 0)$, we thus get a canonical commutative square (in the smooth topos C_{sm})

$$
\begin{array}{ccc}
\mathcal{A}^0 & \longrightarrow & \mathcal{E}xt^1_{C_{sm}}(\mathcal{B}, \mathbb{G}_m) \\[4pt]
\downarrow & & \downarrow \\[4pt]
\mathcal{A} & \longrightarrow & \mathcal{E}xt^1_{C_{sm}}(\mathcal{B}^0, \mathbb{G}_m)
\end{array}
\tag{14}
$$

Our proof consists in three steps.

Step 1. The compatible pair of arrows (14) induces morphisms as follows:

$$[\mathcal{A}^0 \to \mathcal{A}] \to R\mathcal{H}om_{C_{sm}}([\mathcal{B}^0 \to \mathcal{B}], \mathbb{G}_m[1]) \text{ in } D^b(C_{sm}^{[1]}), \text{ and then} \quad (15)$$

$$R\Gamma(\underline{C}, [\mathcal{A}^0 \to \mathcal{A}]) \to R\Gamma(\underline{C}, [\mathcal{B}^0 \to \mathcal{B}])^\vee[-2] \text{ in } D^b({}_{\underline{G}}Mod(\mathbb{Z})^{[1]}). \quad (16)$$

Let us indicate how to get these. The arrows of (14) give rise to

$$[\mathcal{A}^0 \to \mathcal{A}] \to \mathcal{E}xt^1_{C_{sm}}([\mathcal{B}^0 \to \mathcal{B}], \mathbb{G}_m) \text{ in } Mod(C_{sm}^{[1]}, \mathbb{Z}), \text{ thus to}$$
$$[\mathcal{A}^0 \to \mathcal{A}] \to (\tau_{\geq 1} R\mathcal{H}om_{C_{sm}}([\mathcal{B}^0 \to \mathcal{B}], \mathbb{G}_m))[1] \text{ in } D^b(C_{sm}^{[1]}),$$

and (15) follows since $\mathcal{H}om_{C_{sm}}([\mathcal{B}^0 \to \mathcal{B}], \mathbb{G}_m) = 0$ by [SGA7], VIII, 3.2 since $\mathbb{G}_m \hookrightarrow j_* j^* \mathbb{G}_m$. Applying $R\Gamma(\underline{C}, -)$ we get

$$R\Gamma(\underline{C}, [\mathcal{A}^0 \to \mathcal{A}]) \to R\underline{Hom}_{C_{sm}}([\mathcal{B}^0 \to \mathcal{B}], \mathbb{G}_m)$$

where $\underline{Hom}_{C_{sm}}([M_1 \to M_2], M_3)$ means the object of ${}_{\underline{G}}Mod(\mathbb{Z})^{[1]}$ whose nth component is $[Hom_{C_{n,sm}}(M_{2|C_n}, M_{3|C_n}) \to Hom_{C_{n,sm}}(M_{1|C_n}, M_{3|C_n})]$ with G_n acting by conjugation and where going up (resp. down) transition maps are given by restriction (resp. trace). We note that there is a natural morphism

$$\underline{Hom}_{C_{sm}}([M_1 \to M_2], M_3) \to Hom(\Gamma(\underline{C}, [M_1 \to M_2]), \Gamma(C, M_3))$$

which is bifunctorial with respect to $([M_1 \to M_2], M_3)$ where for $([N_1 \to N_2], N_3)$ in ${}_{\underline{G}}Mod(\mathbb{Z})^{[1]} \times Mod(\mathbb{Z})$, $Hom([N_1 \to N_2], N_3)$ means the normic system whose nth component is $[Hom(N_{2,n}, N_3) \to Hom(N_{1,n}, N_3)]$ and where going down (resp. up) transition maps are deduced from the going up (resp. down) transition maps of N_2 and N_3. This natural transformation gives rise to a map

$$R\underline{Hom}_{C_{sm}}([M_1 \to M_2], M_3) \to RHom(\Gamma(\underline{C}, [M_1 \to M_2]), \Gamma(C, M_3))$$

in $D^-({}_{\underline{G}}Mod(\mathbb{Z})^{[1]})$ as long as $R\Gamma(C, M_3)$ is bounded. Applying this to $([\mathcal{B}^0 \to \mathcal{B}], \mathbb{G}_m)$ and using the trace morphism $R\Gamma(C, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}[-2]$ finally gives (16) as claimed.

It is worth noting that the pairings on cohomology resulting from (16) are compatible with those of [Mi2], III. To state this fact precisely, let us fix any closed point $v \in |C|$ as well as an open $U \subset C$ over which $\mathcal{A}$ has good reduction. Then it follows from the definitions that one has a natural commutative "diagram"

$$
\begin{array}{ccccccc}
H^q(Z_{n,v}, \Phi_{|Z_{n,v}}) & \times & H^{1-q}(Z_{n,v}, \Psi_{|Z_{n,v}}) & \xrightarrow{1} & H^1(Z_{n,v}, \mathbb{Q}/\mathbb{Z}) & & \\
\uparrow & & \uparrow & & \wr\uparrow & & \\
H^q(C_{n,v}, \mathcal{A}_{|C_{n,v}}) & \times & H^{1-q}(C_{n,v}, \mathcal{B}_{|C_{n,v}}) & \xrightarrow{2} & H^2(C_{n,v}, j_{v,*}\mathbb{G}_m) & & \\
\downarrow & & \downarrow & & \wr\downarrow & & \\
H^q(U_{n,v}, A_{|U_{n,v}}) & \times & H^{1-q}(U_{n,v}, B_{|U_{n,v}}) & \xrightarrow{3} & H^2(U_{n,v}, \mathbb{G}_m) & & \\
\uparrow & & \downarrow & & \wr\downarrow & & \\
H^q(C_{n,v}, \mathcal{A}_{|C_{n,v}}) & \times & H^{2-q}_{Z_{n,v}}(C_{n,v}, \mathcal{B}^0_{|C_{n,v}}) & \xrightarrow{4} & H^3_{Z_{n,v}}(C_{n,v}, \mathbb{G}_m) & & \\
\uparrow & & \downarrow & & \wr\downarrow & & \\
H^q(C_n, \mathcal{A}_{|C_n}) & \times & H^{2-q}(C_n, \mathcal{B}^0_{|C_n}) & \xrightarrow{5} & H^3(C_n, \mathbb{G}_m) & \xrightarrow[\sim]{T_r} & \mathbb{Q}/\mathbb{Z} \\
\uparrow & & \downarrow & & \wr\uparrow & & \\
H^q_c(U_n, \mathcal{A}_{|U_n,p^k}) & \times & H^{3-q}(U_n, \mathcal{B}^0_{|U_n,p^k}) & \xrightarrow{6} & H^3_c(U_n, \mathbb{G}_m) & & \\
\end{array}
$$

where:
- pairings 1 and 2 are the ones occurring during the proof of [Mi2], III Thm. 7.11 and right vertical isomorphism is by $j_{v,*}\mathbb{G}_m[1] \to z_{v,*}\mathbb{Z}[1] \leftarrow z_{v,*}\mathbb{Q}/\mathbb{Z}$.
- pairing 3 (resp. 4, 6) is the one occurring in [Mi2], III 7.8 (resp. 7.13, resp. 8.2).
- pairing 5 is defined using the Yoneda pairing for Ext's and the vanishing of $\mathcal{H}om_{C_n}(\mathcal{B}^0, \mathbb{G}_m)$ (or alternatively, the vanishing of $\mathcal{H}om_{C_n}(\mathcal{A}, \mathbb{G}_m)$). By the restriction-corestriction formula, it coincides with the one induced by (16);
It can be checked that similar compatibilities hold in the derived category. The proof however is not so easy to write down. We will thus avoid it and use the following trick instead. The morphism (14) induces a map

$$\mathbb{Z}/p^k \overset{L}{\otimes} \mathcal{A}[-1] \to R\mathcal{H}om_{C_{sm}}(\mathbb{Z}/p^k \overset{L}{\otimes} \mathcal{B}^0, \mathbb{G}_m)$$

Now it it straightforward to check that this morphism induces compatible pairings $1', \ldots, 6'$ just as $1, \ldots, 6$ above with $\mathcal{A}$ and $\mathcal{B}^0$ respectively replaced by $\mathbb{Z}/p^k \otimes^L \mathcal{A}[-1]$ and $\mathbb{Z}/p^k \otimes^L \mathcal{B}^0$. Moreover, the resulting pairing

$$H^q(C_n, \mathbb{Z}/p^k \overset{L}{\otimes} \mathcal{A}_{|C_n}[-1]) \quad \times \quad H^{2-q}(C_n, \mathbb{Z}/p^k \overset{L}{\otimes} \mathcal{B}^0_{|C_n}) \quad \overset{5'}{\to} \quad \begin{aligned} & H^3(C_n, \mathbb{G}_m) \\ \simeq \quad & H^3(C, \mathbb{G}_m) \\ \simeq \quad & \mathbb{Q}/\mathbb{Z} \end{aligned}$$

coincides with the one deduced from (16) by applying $\mathbb{Z}/p^k \otimes^L (-)$.

Step 2. The morphism (15) induces a canonical morphism of distinguished triangles in $D^b(_{\underline{G}}Mod(\mathbb{Z}))$

$$\begin{array}{ccccccc} Sel(A/\underline{K}) & \to & R\Gamma(\underline{C}, \mathcal{A}) & \to & H^1(\underline{Z}, \Phi_{|Z})[-1] & \overset{+1}{\to} \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & \\ Sel(B/\underline{K})^\vee[-2] & \to & R\Gamma(\underline{C}, \mathcal{B}^0)^\vee[-2] & \to & H^0(\underline{Z}, \Psi_{|Z})^\vee[-1] & \overset{+1}{\to} \end{array}$$

Let us explain this. Since it can be expressed in terms of truncation, functorial cones and natural transformation of functors in the variable $[M \to N] \in Mod(C_{sm}, \mathbb{Z})^{[1]}$, we observe that the following diagram of $D(_{\underline{G}}Mod(\mathbb{Z}))$:

$$\begin{array}{ccc} R\Gamma(\underline{C}, N) & \overset{\beta}{\longrightarrow} & \tau_{\geq 1}Cone(R\Gamma(\underline{C}, [M \to N])) \\ \downarrow & & \downarrow \\ R\Gamma(\underline{C}, N) & \overset{\beta'}{\longrightarrow} & \tau_{\geq 1}Cone(R\Gamma(\underline{Z}, [z^{-1}M \to z^{-1}N])) \\ \uparrow & & \uparrow \\ R\Gamma(\underline{C}, N) & \overset{loc_C^1}{\longrightarrow} & \prod_{v\in|Z|} \tau_{\geq 1}R\Gamma(\underline{C}_v, N) \end{array}$$

has an obvious counterpart in the corresponding derived category of diagrams of this form. Applying this to $[M \to N] = [\mathcal{A}^0 \to \mathcal{A}]$ and forming mapping

fibers of the horizontal arrows, we get canonical isomorphisms (using arguments already explained in the proof of Prop. 4.7):

$$MF(\beta) \overset{\sim}{\leftarrow} MF(\beta') \overset{\sim}{\rightarrow} MF(loc_C^1) \simeq Sel(A/\underline{K})$$

Consider the following diagram as a functor whose argument $[\underline{M} \overset{f}{\rightarrow} \underline{N}]$ varies in the category of complexes $C(\underline{G}Mod(\mathbb{Z})^{[1]})$:

$$
\begin{array}{ccccccc}
\tau_{\leq 1}MF(f) & \simeq & \tau_{\leq 1}MF(f) & & & & \\
\downarrow & & \downarrow {\scriptstyle\underline{\alpha}} & & & & \\
MF(f) & \longrightarrow & \underline{M} & \overset{f}{\longrightarrow} & \underline{N} & \longrightarrow & Cone(f) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tau_{\geq 2}MF(f) & \longrightarrow & MF(\underline{\beta}) & \longrightarrow & \underline{N} & \overset{\beta}{\longrightarrow} & \tau_{\geq 1}Cone(f)
\end{array}
$$

We remark the following:
- since the left vertical composed morphism is zero, so is the middle vertical one. Whence a canonical morphism $Cone(\underline{\alpha}) \to MF(\underline{\beta})$. As shown by the diagram, this is automatically a quasi-isomorphism.
- the functors $Cone(\underline{\alpha})$ and $MF(\underline{\beta})$ are exchanged by the exact contravariant endofunctor $(-)^\vee = Hom_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ of $C(\underline{G}Mod(\mathbb{Z})^{[1]})$. Indeed:

$$
\begin{array}{rcl}
(Cone(\alpha)[M \to N])^\vee & \simeq & (Cone([\tau_{\leq 1}MF(f) \to M]))^\vee \\
& \simeq & MF([M^\vee \to \tau_{\geq 1}Cone(f^\vee)]) \\
& \simeq & MF(\beta)[N^\vee \to M^\vee]
\end{array}
$$

Putting everything together we get in $D(\underline{G}Mod(\mathbb{Z}))$:

$$
\begin{array}{rcll}
Sel(A/\underline{K}) & \simeq & MF(\beta)[\mathcal{A}^0 \to \mathcal{A}] & \\
& \simeq & MF(\underline{\beta})(R\Gamma(\underline{C}, [\mathcal{A}^0 \to \mathcal{A}])) & \\
& \to & MF(\overline{\beta})(R\Gamma(\underline{C}, [\mathcal{B}^0 \to \mathcal{B}])^\vee[-2]) & \text{(cf. (16))} \\
& \simeq & (Cone(\underline{\alpha})(R\Gamma(\underline{C}, [\mathcal{B}^0 \to \mathcal{B}])))^\vee[-2] & \\
& \simeq & (MF(\underline{\beta})(R\Gamma(\underline{C}, [\mathcal{B}^0 \to \mathcal{B}])))^\vee[-2] & \\
& \simeq & (MF(\overline{\beta})[\mathcal{B}^0 \to \mathcal{B}])^\vee[-2] & \\
& \simeq & Sel(B/\underline{K})^\vee[-2] &
\end{array}
$$

Step 3. The morphisms a, b, c of Step 2 become invertible when applied $\mathbb{Z}/p. \otimes^L (-)$.

It is of course sufficient to prove this for b and c. By [Mi2], III, 7.11, c itself is an isomorphism. Now the statement about b is equivalent to the horizontal arrow b' below being an isomorphism:

$$
\begin{array}{ccc}
\downarrow & & \downarrow \\
H_c^q(U_n, \mathbb{Z}/p^k \overset{L}{\otimes} \mathcal{A}_{|U_n}[-1]) & \longrightarrow & H^{2-q}(U_n, \mathbb{Z}/p^k \overset{L}{\otimes} \mathcal{B}_{|U_n}^0)^\vee \\
\downarrow & & \downarrow \\
H^q(C_n, \mathbb{Z}/p^k \overset{L}{\otimes} \mathcal{A}_{|C_n}[-1]) & \overset{b'}{\longrightarrow} & H^{2-q}(C_n, \mathbb{Z}/p^k \overset{L}{\otimes} \mathcal{B}_{|C_n}^0)^\vee \\
\downarrow & & \downarrow \\
\prod_{v \in |Z_n|} H^q(C_{n,v}, \mathbb{Z}/p^k \overset{L}{\otimes} \mathcal{A}_{|C_{n,v}}[-1]) & \longrightarrow & \prod_{v \in |Z_n|} H_{Z_{n,v}}^{2-q}(C_{n,v}, \mathbb{Z}/p^k \overset{L}{\otimes} \mathcal{B}_{|C_{n,v}}^0)^\vee
\end{array}
$$

This diagram commutes thanks to the compatibilities explained in Step 1 and its columns are long exact sequences as follows easily from the definition of compactly supported cohomology together with the complete excision property

$$
\prod_{v \in |Z_n|} H_{Z_{n,v}}^{2-q}(C_{n,v}, \mathbb{Z}/p^k \overset{L}{\otimes} \mathcal{B}_{|C_{n,v}}^0) \simeq H_{Z_n}^{2-q}(C_n, \mathbb{Z}/p^k \overset{L}{\otimes} \mathcal{B}_{|C_n}^0),
$$

obtained by using [Mi1], III, 1.28 and [Mi2], III, 7.14. The first (resp. third) horizontal morphism is invertible thanks to [Mi2], III, 8.2 (resp. [Mi2], III 7.13) and the result follows. □

REMARK 4.9. *The proof does not require any assumption about the reduction of A at the points of Z.*

COROLLARY 4.10. *Consider the usual Selmer group*

$$
Sel_{p^k}^1(A/K_n) := Ker\left(H^1(K_n, A_{|K_n, p^k}) \to \underset{v \in |C|}{\oplus} H^1(K_{n,v} A_{|K_{n,v}}))\right)
$$

and let $Sel_{p\cdot}^1(A/\underline{F})$ denote the object of $Mod(\underline{G}Mod(\mathbb{Z})^{N_2})$ encapsulating the whole collection (together with the various natural transition morphisms) for varying n, k. The cohomology of the complex $Sel_{p\cdot}(A/\underline{K})$ vanishes outside $[0,3]$ and is described as follows in this range:

$$
\begin{array}{rcl}
H^0 & = & A_{p\cdot}(\underline{K}) \\
H^1 & = & Sel_{p\cdot}^1(A/\underline{K}) \\
H^2 & \simeq & Sel_{p\cdot}^1(\hat{A}/\underline{K})^\vee \\
H^3 & \simeq & \hat{A}_{p\cdot}(\underline{K})^\vee
\end{array}
$$

Proof. The description of H^0 and H^1 follows easily from the following distinguished triangle (Prop. 4.7 (i)):

$$
Sel_{p\cdot}(A/\underline{K}) \to R\Gamma(\underline{K}, A_{p\cdot}) \to \underset{v \in |C|}{\oplus} \mathbb{Z}/p\cdot \overset{L}{\otimes} H^1(\underline{U}_v, A_{|U_v})[-2] \overset{+1}{\to}
$$

whose associated long exact sequence of cohomology reads as the collection of the following ones for varying n, k:

$$0 \to H^0(Sel_{p^k}(A/K_n)) \to H^0(K_n, A_{p^k}) \to 0 \to H^1(Sel_{p^k}(A/K_n))$$
$$\to H^1(K_n, A_{p^k}) \to \bigoplus_{v \in |C|} Tor_1(\mathbb{Z}/p^k, H^1(K_{n,v}, A_{|K_{n,v}})) \to \dots$$

(note that for any abelian group M, $Tor_1(\mathbb{Z}/p^k, M)$ is identified with the kernel of p^k on M. It it easy to check that modulo this identification, the right arrow in the second exact sequence is induced from the localization map occurring in the definition of $Sel^1_{p^k}(A/K_n)$).

The description of H^2 and H^3 follows by duality, thanks to Prop. 4.7 and Prop. 4.8.   □

COROLLARY 4.11. *In $D^b({}_G Mod(\mathbb{Z}_p))$, one has the duality isomorphism*

$$Sel_{T_p}(\hat{A}/\underline{K}) \simeq Sel_{p^\infty}(A/\underline{K})^\vee[-3]$$

and those complexes are concentrated in $[1,3]$.

Proof. Just apply $R\varprojlim_k$ to Prop. 4.8 and notice that H^0 vanishes because for n fixed, $\hat{A}_{p^k}(K_n)$ is bounded independently of k.   □

COROLLARY 4.12. *$R\Gamma^Z(\underline{C}, \mathcal{A}_{p^\cdot})$ is concentrated in $[0,3]$ and its cohomology objects fit into compatible exact sequences and isomorphisms as follows:*

$$H^0 \hookrightarrow A_{p^\cdot}(\underline{K}) \to \mathcal{A}_{p^\cdot}(\underline{Z}) \to H^1 \to Sel_{p^\cdot}(A/\underline{K}) \twoheadrightarrow \mathcal{A}(\underline{Z})/p^\cdot$$
$$H^2 \simeq Sel_{p^\cdot}(\hat{A}/\underline{K})^\vee$$
$$H^3 \simeq \hat{A}_{p^\cdot}(\underline{K})^\vee$$

REMARK 4.13. *Even though we won't use it, let us point out the relation to the complexes of [KT].*

- The complexes $Sel_{p^\infty}(A/K) = \varinjlim_k Sel_{p^k}(A/K)$ and $R\Gamma_{ar}\{p\}$ have the same cohomology (compare loc. cit. Lem. 2.4 and our Cor. 4.10).

- The cohomology of the complexes $\varinjlim_k R\Gamma^Z(C, \mathcal{A}_{p^k})$ and $R\Gamma_{ar,V}\{p\}$ fits into similar exact sequences (compare [KT], 2.5.2, 2.5.3 and our Prop. 4.7 (ii)) for $V = (V_v)_{v \in Z}$ and $V_v = Ker(A(K) \to \mathcal{A}(Z_v))$.

- In fact, one may easily form isomorphisms

$$Sel_{p^\infty}(A/K) \simeq R\Gamma_{ar}\{p\}$$

and

$$\varinjlim_k R\Gamma^Z(C, \mathcal{A}_{p^k}) \simeq R\Gamma_{ar,V}\{p\}.$$

4.4   Crystalline syntomic complexes

In this section we assume that A/K is semistable and we recall the comparison result of [TV] and deduce from it the fundamental distinguished triangles (6), announced in the introduction, relating (normic) flat cohomology vanishing at Z to (normic) crystalline cohomology.

4.4.1. We keep the notations of the previous section (A/K, $\mathcal{A}/C$, U, Z, $\epsilon : C_{FL} \to C_{et}$) and we let furthermore $C^\sharp$ denote the log-scheme whose underlying scheme is C and whose log-structure is induced by Z. In the chapter 8 of [TV] is associated to the semistable abelian variety A/K a log Dieudonné crystal (D, F, V). Recall that such object is a crystal of locally free $\mathcal{O}$-modules of finite rank in the crystalline small étale ringed topos $((C^\sharp/\mathbb{Z}_p)_{crys,et}, \mathcal{O})$ endowed with two operators $F : \sigma^* D \to D$, $V : D \to \sigma^* D$ (where σ denotes the endomorphism of $(C^\sharp/\mathbb{Z}_p)_{crys,et}$ induced by the absolute Frobenius of $C^\sharp$) such that $FV = VF = p$. This log-crystal extends the covariant Dieudonné crystal $(D_U(\mathcal{A}|_U), F, V)$ of [BBM] and is endowed with a canonical epimorphism

$$\pi : D \to Lie(\mathcal{A}),$$

where the Lie algebra $Lie(\mathcal{A})$ is seen as a crystalline sheaf ([TV], 5.35 (i)).

We set $Fil^1 D := Ker(\pi)$ and denote $\mathbf{1} : Fil^1 D \to D$ the canonical injection.

4.4.2. Let $u : (C^\sharp/\mathbb{Z}_p)_{crys,et} \to C_{et}$ denote the canonical projection of the crystalline topos on C_{et}. Following [TV], Prop. 5.49, we recall the construction of an operator

$$\phi : Ru_* Fil^1 D(-Z) \to Ru_* D(-Z),$$

where the twist by $-Z$ of a crystalline sheaf is defined in [TV], Def. 4.42. Consider $(U^\sharp_{[.]}, Y^\sharp_{[.]}, \iota_{[.]}, \tilde{\sigma})$ where $U^\sharp_{[.]}$ is a semisimplicial fine log-scheme above $C^\sharp$, $Y^\sharp_{[.]}$ is a semisimplicial p-adic formal log scheme with finite local p-bases over $Spf(\mathbb{Z}_p)$ (in the sense of [TV], 4.1 (v)), endowed with a lifting of Frobenius $\tilde{\sigma}$ and $\iota : U^\sharp_{[.]} \to Y^\sharp_{[.]}$ is a closed immersion. We denote $T^\sharp_{[.],n}$ the log pd envelope of $U^\sharp_{[.]}$ into the reduction $Y^\sharp_{[.],n}$ of $Y^\sharp_{[.]}$ modulo p^n and, for a crystalline sheaf E, $E_{T^\sharp_{[.],n}}$ the realization of the sheaf at the pd-thickening $(U^\sharp_{[.]}, T^\sharp_{[.],n})$. It is proved ([TV], Lem. 4.9) that we can always chose these data such that $(U^\sharp_{[.]}, T^\sharp_{[.],n})$ is a hypercovering in the topos $(C^\sharp/\mathbb{Z}_p)_{crys,et}$. In [TV], 5.49 it is proved that there exists a unique collection (ϕ_n) of $\tilde{\sigma}$-semi-linear morphisms of $\mathcal{O}_{T_{[.],n}}$-modules rendering the following diagram commutative for all n:

$$
\begin{array}{ccc}
Fil^1 D(-Z)_{T^\sharp_{[.],n+1}}/p^n & \xrightarrow{\phi_n} & D(-Z)_{T^\sharp_{[.],n}} \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle p} \\
D(-Z)_{T^\sharp_{[.],n}} & \xrightarrow{Fr} & (\tilde{F})_* D(-Z)_{T^\sharp_{[.]}}
\end{array}
$$

(here Fr is the morphism built up from the operator F of D together with the lifting of frobenius $\tilde{\sigma}$ of T_n). Finally, we set

$$\phi := R\varprojlim_n Rf_{T_{[.],n},*}\phi_n,$$

where $f_{T_{[.],n}} : T_{[.],n,et} \to C_{et}$. By cohomological descent in the crystalline topos, this yields a morphism ([TV], Prop. 5.55, (ii) and Rem. 5.56)

$$\phi : Ru_*Fil^1D(-Z) \to Ru_*D(-Z).$$

4.4.3. To simplify the notations, we set:

i. $N := R\epsilon_*^Z R\varprojlim_n \mathcal{A}_{p^n}$,

ii. $I := Ru_*Fil^1D(-Z)$,

iii. $P := Ru_*D(-Z)$,

iv. $L := Lie(\mathcal{A})(-Z)$.

All N, I, P, L are considered as objects of the derived category $D(C_{et}, \mathbb{Z}_p)$. The main theorem of [TV] implies (note that ϵ_*^Z as defined in 4.1 coincides with the functor $\epsilon_*\underline{\Gamma}^Z$ used in *loc. cit.*):

THEOREM 4.14. *([TV], Cor. 9.17) In $D^p(C_{et}, \mathbb{Z}_p)$ there are canonical distinguished triangles as follows:*

$$N \to I \overset{1-\phi}{\to} P \overset{+1}{\to}$$
$$I \overset{1}{\to} P \to L \overset{+1}{\to}$$

$\square$

4.4.4. For $X \in N, I, P, L$, we denote $\underline{X} := (X_n)$ the object $R\Gamma(\underline{C}, X) \in D^b(\underline{G}Mod(\mathbb{Z}_p))$ (so that $X_n = R\Gamma(C_n, X_{|C_n})$, see Sect. 2.4 for explanations). Given a Galois subextension K'/K of K_∞/K, we denote $X_{K'}$ (resp. $X_{K'}^*$) the derived projective limit of the projective system of complexes indexed by the finite subextensions of K'/K (eg. $X_{K_n} = X_n$, $X_{K_n}^* = X_n^*$, $X_{K_\infty} := R\varprojlim_n X_n$, $X_{K_\infty}^* := R\varprojlim_n RHom_{\mathbb{Z}_p}(X_n, \mathbb{Z}_p)$). We have by Thm. 4.14 two pairs of distinguished triangles in $D(\Lambda)$:

i. $N_{K_\infty} \to I_{K_\infty} \overset{1-\phi}{\to} P_{K_\infty} \overset{+1}{\to}$

ii. $I_{K_\infty} \overset{1}{\to} P_{K_\infty} \to L_{K_\infty} \overset{+1}{\to}$

and

i. $P_{K_\infty}^* \overset{1^*-\phi^*}{\to} I_{K_\infty}^* \to N_{K_\infty}^* \overset{+1}{\to}$

ii. $L_{K_\infty}^* \to P_{K_\infty}^* \xrightarrow{1} I_{K_\infty}^* \xrightarrow{+1}$

REMARK 4.15. *These distinguished triangles are obtained from Thm.* *4.14* *by applying* $R\Gamma(\underline{C}, -)$ *then* $R\varprojlim_n$. *Note that, even though* [KT] *might give distinguished triangles in the derived category of modules at each level* n *and transition mophisms between them, this would formally not be enough to apply* $R\varprojlim_n$ *(whose source needs to be the derived category of projective systems rather than the category of projective systems of the derived category) and get the desired distinguished triangles in* $D(\Lambda)$. *One could of course easily give a meaning to* $R\varprojlim_n$ *of each term of the triangle using explicit complexes, but this is far from clear for the morphisms, especially for the morphism* $N_n \to I_n$.

In fact, the two cohomology theories X_{K_∞} and $X_{K_\infty}^*$ are related as follows:

PROPOSITION 4.16. *Let* $X \in N, I, P, L$. *Then* X_{K_∞} *and* $X_{K_\infty}^*$ *are in* $D^p(\Lambda)$. *Moreover:*
(i) If $\underline{\Lambda} = (\mathbb{Z}[G_n])$ *denotes the canonical normic system then we have in* $D(_{\underline{G}}Mod(\mathbb{Z}_p))$ *isomorphisms as follows:*

$$\underline{\Lambda} \overset{L}{\otimes}_\Lambda X_{K_\infty} \simeq \underline{X} \qquad and \qquad \underline{\Lambda} \overset{L}{\otimes}_\Lambda X_{K_\infty}^* \simeq \underline{X}^*$$

(so that, in particular, we have in $D(_{G_n}Mod(\mathbb{Z}_p))$*):*

$$\mathbb{Z}_p[G_n] \overset{L}{\otimes}_\Lambda X_{K_\infty} \simeq X_n \qquad and \qquad \mathbb{Z}_p[G_n] \overset{L}{\otimes}_\Lambda X_{K_\infty}^* \simeq X_n^*)$$

(ii) We have an isomorphism in $D^p(\Lambda)$*:*

$$RHom_\Lambda(X_{K_\infty}^*, \Lambda) \simeq X_{K_\infty}.$$

Proof. Since C is proper and smooth, well known finiteness results imply that $\underline{L}$ and $\underline{P}$ satisfy the finiteness condition of Thm. 2.11. The perfectness statement follows directly from Thm. 2.11 for $X = L$ or P. It then follows from the distinguished triangle (*ii*) (resp. (*i*)) for $X = I$ (resp. N). The remaining statements follow from Thm. 2.11 as well (using [Va] 2.10.3 and 3.1). $\qquad\square$

5   The Main conjecture

5.1   The constant $\mathbb{Z}_p$-extension

Recall that $K_{ar} := Kk_\infty$ denotes the constant $\mathbb{Z}_p$-extension of K. During this paragraph, we index by k_n the n-th layer of k_∞/k.

PROPOSITION 5.1. *Let* $\underline{W} := (W(k_n)) \in {}_{\underline{G}}Mod(\mathbb{Z}_p)$ *denote the normic system of* $\mathbb{Z}_p$-*modules formed by the Witt vectors along* k_∞/k. *Also, let* $\underline{W} \otimes_{\mathbb{Z}_p} (-) : Mod(\mathbb{Z}_p) \to {}_{\underline{\Gamma}}Mod(\mathbb{Z}_p)$ *denote the obvious functor. Then:*
(i) For $X \in \{I, P, L\}$, *there is a canonical isomorphism in* $D^b(_{\underline{G}}Mod(\mathbb{Z}_p))$*:*

$$\underline{W} \overset{L}{\otimes} X_K \simeq \underline{X}$$

(ii) The morphism $\mathbf{1} : I_n \to P_n$ *(resp.* $\phi : I_n \to P_n$*) is* $W(k_n)$-*linear (resp.* $W(k_n)$-*semi-linear with respect to the action of the absolute frobenius* $\sigma : W(k_n) \to W(k_n)$*). In particular, the morphism* $\mathbf{1} - \phi : \underline{I} \to \underline{P}$ *can be described at level n as* $\lambda \otimes x_0 \mapsto \lambda \otimes \mathbf{1}x_0 - \sigma\lambda \otimes \phi x_0.$

Proof. (*i*) follows from the Zariski log-crystalline base change theorem (note that the base change map is well defined at the level complexes and thus makes sense in $D^b_{\underline{\Gamma}}Mod(\mathbb{Z}_p)$).
(*ii*) Linearity of $\mathbf{1}$ and semi-linearity of ϕ are immediate from the definitions. The second statement follows. $\qquad\square$

COROLLARY 5.2. *Both* $N_{K_{ar}}$ *and* $N^*_{K_{ar}}$ *are torsion over* $\Lambda(\Gamma)$*, ie.*

$$Q(\Gamma)\overset{L}{\otimes}_{\Lambda(\Gamma)}N_{K_{ar}} = 0 \qquad\qquad and \qquad\qquad Q(\Gamma)\overset{L}{\otimes}_{\Lambda(\Gamma)}N^*_{K_{ar}} = 0$$

Proof. Since by Prop. 4.16, we have $N_{K_{ar}} \simeq RHom_\Lambda(N^*_{K_{ar}}, \Lambda)$, it suffices to prove that $N_{K_{ar}}$ is torsion. Applying $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (-)$ to the distinguished triangles of 4.4.3 gives rise to the distinguished triangle:

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} N_{K_{ar}} \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} I_{K_{ar}} \overset{1-\phi}{\to} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} P_{K_{ar}} \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} N_{K_{ar}}[1]$$

which can be rewritten as

$$\mathbb{Q}_p\otimes_{\mathbb{Z}_p}N_{K_{ar}} \to W_\infty\overset{L}{\otimes}_{\mathbb{Z}_p}(\mathbb{Q}_p\otimes_{\mathbb{Z}_p}I_K) \to W_\infty\overset{L}{\otimes}_{\mathbb{Z}_p}(\mathbb{Q}_p\otimes_{\mathbb{Z}_p}P_K) \to \mathbb{Q}_p\otimes_{\mathbb{Z}_p}N_{K_{ar}}[1]$$

Now $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}L_{K_{ar}} = 0$, since $Lie(D)$ is an $\mathbb{F}_p$-vector space and $\mathbf{1} : \mathbb{Q}_p\otimes_{\mathbb{Z}_p}I_{K_{ar}} \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} P_{K_{ar}}$ thus has an inverse. Since moreover $\mathbb{Q}_p$ and $W_\infty := \varprojlim W(k_n)$ are $\mathbb{Z}_p$-flat, the long exact sequence of cohomology of the previous triangle can be written

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^q(N_{K_{ar}}) \to W_\infty\overset{L}{\otimes}_{\mathbb{Z}_p}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^q(P_K)) \overset{id-\phi\mathbf{1}^{-1}}{\to} W_\infty\overset{L}{\otimes}_{\mathbb{Z}_p}(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^q(P_K))$$

where $(id - \phi\mathbf{1}^{-1})(\lambda \otimes x) = \lambda \otimes x - \sigma\lambda \otimes \phi\mathbf{1}^{-1}x$.
To end the proof, we need the following result from σ-linear algebra:

LEMMA 5.3. *If ψ is a linear endomorphism of a finite dimensional $\mathbb{Q}_p$-vector space M then:*
(i) The kernel of $id-\sigma\otimes\psi$ on $W(k_n)\otimes_{\mathbb{Z}_p}M$ has $\mathbb{Q}_p$-dimension bounded by the number of unit eigenvalues of ψ (counted with multiplicities). In particular, its dimension is bounded independently of n.
(ii) The $\mathbb{Q}_p$-linear endomorphism $id-\sigma\otimes\psi$ of $(\varprojlim W(k_n))\otimes_{\mathbb{Z}_p}M)$ is injective.

Proof. (*i*) Set $L = Frac(W(\overline{\mathbb{F}_p}))$. Since L^σ (the fixed points by σ) are reduced to $\mathbb{Q}_p$ and $W(k_n)\otimes_{\mathbb{Z}_p}M \subset L\otimes_{\mathbb{Q}_p}M$, [EL2] 6.2 gives the result.
(*ii*) Let Γ act on $\subset W(k_n)\otimes_{\mathbb{Z}_p}M$ via $W(k_n)$. Thanks to (*i*), we know that there is a open subgroup $\Gamma_m \subset \Gamma$ which fixes the kernel of $id-\sigma\otimes\psi$ acting on $W(k_n)\otimes_{\mathbb{Z}_p}M$ for all n. The result follows, since

$$(W_\infty \otimes_{\mathbb{Z}_p} M)^{\Gamma_m} = (\varprojlim W(k_n)^{\Gamma_m}) \otimes_{\mathbb{Z}_p} M = 0.$$

$\qquad\qquad\square$

Applying the lemma to $M = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^q(P_K)$ and $\psi = \phi \mathbf{1}^{-1}$ shows that the first distinguished triangle of 4.4.4 reduces to short exact sequences

$$0 \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^q(P_{K_{ar}}) \stackrel{id - \psi}{\to} \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^q(P_{K_{ar}}) \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^q(N_{K_{ar}}) \to 0$$

and the result follows by applying $Q(\Gamma) \otimes_{\Lambda(\Gamma)} (-)$ to those, since $Q(\Gamma) \otimes_{\Lambda(\Gamma)} H^q(P_{K_{ar}})$ is a finite dimensional vector space over $Q(\Gamma)$. $\qquad\square$

Following [Ja], we set $E^q(-) := Ext^q_{\Lambda(\Gamma)}(-, \Lambda(\Gamma))$. In loc. cit. these functors (and others) are used to classify $\Lambda(\Gamma)$-modules up to isomorphism. Let us recall that, for a finitely generated $\Lambda(\Gamma)$-module they verify the following properties:
- $E^0(M)$ is free. It is zero if and only if M is torsion.
- $E^1(M)$ is torsion and is (non canonically) pseudo-isomorphic to M. It has no non zero finite submodules if M is torsion.
- $E^2(M)$ is finite.

COROLLARY 5.4. $N_{K_{ar}}$ and $N^*_{K_{ar}}$ can be described as follows:
1. (i) $N_{K_{ar}}$ is concentrated in degrees $[1,3]$ and its cohomology $\Lambda(\Gamma)$-modules are described by the following exact sequence and isomorphisms

$$H^1(N_{K_{ar}}) \hookrightarrow \varprojlim_n Sel_{T_p}(A/Kk_n) \twoheadrightarrow \varprojlim_{k,n} \prod_{v \in Z} \mathcal{A}(k_v k_n)/p^k$$
$$H^2(N_{K_{ar}}) \simeq Hom_{\mathbb{Z}_p}(\varinjlim_n Sel_{p^\infty}(\hat{A}/Kk_n), \mathbb{Q}_p/\mathbb{Z}_p)$$
$$H^3(N_{K_{ar}}) \simeq Hom_{\mathbb{Z}_p}(\varinjlim_n \hat{A}_{p^\infty}(Kk_n), \mathbb{Q}_p/\mathbb{Z}_p)$$

(ii) Moreover, one has the following isomorphism and exact sequences of $\Lambda(\Gamma)$-modules:

$$\begin{array}{ccccc}
& & H^1(N_{K_{ar}}) & \simeq & E^1(H^0(N^*_{K_{ar}})) \\
E^2(H^0(N^*_{K_{ar}})) & \hookrightarrow & H^2(N_{K_{ar}}) & \twoheadrightarrow & E^1(H^{-1}(N^*_{K_{ar}})) \\
E^2(H^{-1}(N^*_{K_{ar}})) & \hookrightarrow & H^3(N_{K_{ar}}) & \twoheadrightarrow & E^1(H^{-2}(N^*_{K_{ar}}))
\end{array}$$

2. (i) $N^*_{K_{ar}}$ is concentrated in degrees $[-2,0]$ and its cohomology $\Lambda(\Gamma)$-modules are described by the following isomorphism and exact sequence:

$$H^{-2}(N^*_{K_{ar}}) \simeq \varprojlim_n Sel_{T_p}(\hat{A}/Kk_n)$$

$Hom_{\mathbb{Z}_p}(\varinjlim_n Sel_{p^\infty}(A/Kk_n), \mathbb{Q}_p/\mathbb{Z}_p) \hookrightarrow H^{-1}(N^*_{K_{ar}}) \to$
$Hom_{\mathbb{Z}_p}(\varinjlim_n \oplus_{v \in Z} \mathcal{A}_{p^\infty}(k_v k_n), \mathbb{Q}_p/\mathbb{Z}_p) \to Hom_{\mathbb{Z}_p}(\varinjlim_n A_{p^\infty}(Kk_n), \mathbb{Q}_p/\mathbb{Z}_p)$
$\twoheadrightarrow H^0(N^*_{K_{ar}})$

(ii) Moreover, one has the following isomorphism and exact sequences of $\Lambda(\Gamma)$-modules:

$$\begin{array}{ccccc}
& & H^{-2}(N^*_{K_{ar}}) & \simeq & E^1(H^3(N_{K_{ar}})) \\
E^2(H^3(N_{K_{ar}})) & \hookrightarrow & H^{-1}(N^*_{K_{ar}}) & \twoheadrightarrow & E^1(H^2(N_{K_{ar}})) \\
E^2(H^2(N_{K_{ar}})) & \hookrightarrow & H^0(N^*_{K_{ar}}) & \twoheadrightarrow & E^1(H^1(N_{K_{ar}}))
\end{array}$$

Proof. The description of cohomology follows from Cor. 4.12 (recall that $\underline{N} = R\varprojlim_k R\Gamma^Z(\underline{C}, \mathcal{A}_{p^k})$ while $\underline{N}^* = RHom_{\mathbb{Z}_p}(\underline{N}, \mathbb{Z}_p) \simeq RHom_{\mathbb{Z}_p}(\varinjlim_k R\Gamma^Z(\underline{C}, \mathcal{A}_{p^k})))$ using only that $A_{p^\infty}(Kk_n)$ and $\mathcal{A}_{p^\infty}(k_v k_n)$ are finite. The stated relations between $N_{K_{ar}}$ and $N^*_{K_{ar}}$ follow from Thm. 4.16 thanks to the spectral sequence

$$
\begin{array}{llll}
& Ext^p_{\Lambda(\Gamma)}(H^{-q}(N^*_{K_{ar}}), \Lambda(\Gamma)) & \Rightarrow & Ext^{p+q}_{\Lambda(\Gamma)}(N^*_{K_{ar}}, \Lambda(\Gamma)) \\
(\text{resp.} & Ext^p_{\Lambda(\Gamma)}(H^{-q}(N_{K_{ar}}), \Lambda(\Gamma)) & \Rightarrow & Ext^{p+q}_{\Lambda(\Gamma)}(N_{K_{ar}}, \Lambda(\Gamma)) \qquad)
\end{array}
$$

which degenerates by Cor. 5.2. $\square$

PROPOSITION 5.5. *The following conditions are equivalent:*
(i) $N_{K_{ar}}$ *is finitely generated over* $\mathbb{Z}_p$ *(that is for any* i, $H^i(N_{K_{ar}})$ *is a finitely generated* $\mathbb{Z}_p$*-module).*
(ii) $Hom_{\mathbb{Z}_p}(\varinjlim_n Sel(\hat{A}_{p^\infty}/Kk_n), \mathbb{Q}_p/\mathbb{Z}_p)$ *is finitely generated over* $\mathbb{Z}_p$.
(iii) $N^*_{K_{ar}}$ *is finitely generated over* $\mathbb{Z}_p$.
(iv) $Hom_{\mathbb{Z}_p}(\varinjlim_n Sel(A_{p^\infty}/Kk_n), \mathbb{Q}_p/\mathbb{Z}_p)$ *is finitely generated over* $\mathbb{Z}_p$.

Proof. $(i) \Leftrightarrow (iii)$ follows from the previous corollary. Indeed if M is any $\Lambda(\Gamma)$-module then $E^i(M)$ is finitely generated over $\mathbb{Z}_p$ as soon as M is. The remaining equivalence are obvious, because the p-rank of torsion points remains bounded along the tower. $\square$

From now on, we will often need to make the following assumption:

($\mu = 0$ - UP TO ISO): There exists an abelian variety A'/K which is isogenous to A/K and verifies the equivalent conditions of Prop. 5.5.

REMARK 5.6. (i) *What we know about this assumption is the following:*
- *It holds for constant varieties except for supersingular abelian variety with non-invertible Hasse-Witt matrix (see [OT], Theorem 1.8).*
- *If it holds for* A/K *and* K'/K *is a finite unramified extension, then it also holds for* A/K'.
(ii) *In characteristic* 0, *it is generally expected that the analogue of* ($\mu = 0$-*up to iso) always holds for elliptic curves ([Gr], p. 9).*

5.2   THE CASE OF A p-ADIC LIE EXTENSION

Let us begin by a cheap result.

PROPOSITION 5.7. *Corollary 5.4 1. (i) and 2. (i) holds verbatim if one replaces* Kk_n *by* K_n *and* $N_{K_{ar}}$ *and* $N^*_{K_{ar}}$ *respectively by* N_{K_∞} *and* $N^*_{K_\infty}$.

$\square$

To go further we need the following lemma which (partly) is the derived version of Nakayama's lemma.

LEMMA 5.8. *Let $M \in D^p(\Lambda(G))$, then M is S-torsion (ie. $\Lambda_S \overset{L}{\otimes} M = 0$) if and only $\Lambda(\Gamma) \overset{L}{\otimes}_{\Lambda(G)} M \in D^p(\Lambda(\Gamma))$ is finitely generated over $\mathbb{Z}_p$.*

Proof. First assume that M is a finitely generated $\Lambda(G)$-module. In this case, the following statements are equivalent:

(i) M is S-torsion.

(ii) M is finitely generated over $\Lambda(H)$.

(iii) $\Lambda(\Gamma) \otimes_{\Lambda(G)} M$ is finitely generated over $\mathbb{Z}_p$.

(iv) each $Tor_p^{\Lambda(G)}(\Lambda(\Gamma), M)$ is finitely generated over $\mathbb{Z}_p$.

Indeed $(i) \Leftrightarrow (ii)$ is [CFKSV] 2.3, $(ii) \Leftrightarrow (iii)$ is the topological version of Nakayama's lemma for the ring $\Lambda(H)$ and the compact $\Lambda(H)$-module M (note that $\Lambda(\Gamma) \otimes_{\Lambda(G)} M \simeq \mathbb{Z}_p \otimes_{\Lambda(H)} M$). Finally $(iii) \Leftrightarrow (iv)$ is an immediate consequence of the fact that M admits a resolution by a perfect complex of $\Lambda(H)$-modules as soon as it is finitely generated over $\Lambda(H)$.

Now we come to the general case. As in the case of modules, the implication "M is S-torsion" $\Rightarrow$ "$\Lambda(\Gamma) \otimes_{\Lambda(G)}^L M$ is finitely generated over $\mathbb{Z}_p$" is immediate. It thus remains to prove the reciprocal implication. Assume thus that $\Lambda(\Gamma) \otimes_{\Lambda(G)}^L M$ is finitely generated over $\mathbb{Z}_p$, ie. that the final terms of the spectral sequence

$$Tor_p^{\Lambda(G)}(\Lambda(\Gamma), H^q(M)) \Rightarrow Tor_{p-q}(\Lambda(\Gamma), M)$$

are. Then using the above equivalence $(iii) \Leftrightarrow (iv)$ for the module $H^q(M)$ as well as a descending induction on q, one gets that each initial term must also be finitely generated over $\mathbb{Z}_p$ (see [Va] 4.1 for details). $\square$

PROPOSITION 5.9. *Under the assumption ($\mu = 0$ - up to iso), N_{K_∞} and $N_{K_\infty}^*$ are both S^*-torsion.*

Proof. Let $A \to A'$ be an isogeny over K such that A' verifies ($\mu = 0$). Then clearly

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p}^L R \varprojlim R\Gamma^Z(\underline{C}, T_p(\mathcal{A})) \simeq \mathbb{Q}_p \otimes_{\mathbb{Z}_p}^L R \varprojlim R\Gamma^Z(\underline{C}, T_p(\mathcal{A}'))$$

and we may thus assume that $A = A'$. By Thm. 4.16, N_{K_∞} (resp. $N_{K_\infty}^*$) is perfect. Taking $R \varprojlim$ along Kk_∞/K thus gives

$$\Lambda(\Gamma) \overset{L}{\otimes}_{\Lambda(G)} N_{K_\infty} \simeq N_{K_{ar}} \qquad \text{(resp.} \qquad \Lambda(\Gamma) \overset{L}{\otimes}_{\Lambda(G)} N_{K_\infty}^* \simeq N_{K_{ar}}^* \quad)$$

and the result now follows directly from the previous lemma. $\square$

REMARK 5.10. *Thanks to the previous proposition, we may form the classes $[N_{K_\infty}]$ and $[N_{K_\infty}^*]$ of N_{K_∞} and $N_{K_\infty}^*$ in $K_0(\mathfrak{M}_G(H))$.*

COROLLARY 5.11. *Under the assumption ($\mu = 0$ - up to iso) each one of the following morphisms of $D^p(\Lambda(G)_{S^*})$*

$$\mathbf{1}, \mathbf{1} - \phi : (I_{K_\infty})_{S^*} \to (P_{K_\infty})_{S^*} \quad , \qquad \mathbf{1}^*, (\mathbf{1} - \phi)^* : (P_{K_\infty}^*)_{S^*} \to (I_{K_\infty}^*)_{S^*}$$

induced by $\mathbf{1}, \mathbf{1} - \phi : \underline{I} \to \underline{P}$, is an isomorphism.

Proof. For $\mathbf{1}$, this is due to the fact that $Lie(D)(-Z)$, and thus L_{K_∞}, is killed by p. For $\mathbf{1} - \phi$, this is a direct consequence of the previous proposition. $\qquad\square$

REMARK 5.12. *It is not the case that L_{K_∞} is S-torsion, because of Prop.* 5.1 *(i).*

We are now in position to define the "p-adic L-function" (as well as its dual version) which appears in the main conjecture.

DEFINITION 5.13. *Under the assumption ($\mu = 0$), the p-adic L-function $\mathcal{L} = \mathcal{L}(A, U, G)$ and it dual version $\mathcal{L}^* = \mathcal{L}^*(A, U, G)$ are defined as follows:*
(i) $\mathcal{L} := Det((\mathbf{1} - \phi)\mathbf{1}^{-1} | (P_{K_\infty})_{S^*})^{-1} \in K_1(\Lambda_{S^*})$.
(ii) $\mathcal{L}^* := Det((\mathbf{1}^*)^{-1}(\mathbf{1} - \phi)^* | (P^*_{K_\infty})_{S^*})^{-1} \in K_1(\Lambda_{S^*})$.

REMARK 5.14. *(i) The notation $\mathcal{L}^*$ is justified by the fact that it is deduced from $\mathcal{L}$ via the duality involution of $K_1(\Lambda(G)_{S^*})$ (with the convention that duals are viewed as left modules via $g \mapsto g^{-1}$). Indeed $(P^*_{K_\infty})_{S^*} \simeq RHom_{\Lambda_{S^*}}((P^*_{K_\infty})_{S^*}, \Lambda_{S^*})$ and $(\mathbf{1}^*)^{-1}(\mathbf{1} - \phi)^*$ is $RHom_{\Lambda_{S^*}}(-, \Lambda_{S^*})$-dual to $(\mathbf{1} - \phi)\mathbf{1}^{-1} | (P_{K_\infty})_{S^*}$.*
(ii) Thanks to the previous remark and lemma 3.5, *we see that for any Artin representation ρ*

$$\rho(\mathcal{L}) = \rho^\vee(\mathcal{L}^*)$$

Let us now review how one classically attaches twisted L-functions to A and U via overconvergent isocrystals. By [LST], the log Dieudonné crystal D over $C^\sharp$ induces an overconvergent F-isocrystal $D^\dagger$ over $U/\mathbb{F}_p$. Let L be a finite extension of $\mathbb{Q}_p$ which is *totally ramified*, O its ring of integers, and consider an Artin representation $\rho : G \to Aut_O(V_\rho)$. Chose n such that ρ factors through $G_n = Gal(U_n/U)$. Then we can see ρ as a O-representation of the fundamental group of U, having finite (in fact trivial) local monodromy. This representation corresponds to a unique unit-root overconvergent F-isocrystal $U(\rho)^\dagger$ over U/L (see e.g. [Tsz], 7.2.3) which becomes constant over U_n. Consider

$$pr^* : F - iso^\dagger(U/\mathbb{Q}_p) \to F - iso^\dagger(U/L)$$

the natural base change functor from the category of overconvergent F isocrystals over $U/\mathbb{Q}_p$ to the category of overconvergent F isocrystals over U/L. Then we set

$$L(U, A, \rho, s) := L(U, pr^* D^\dagger \otimes U(\rho)^\dagger, p^{-s})$$

where the right hand side is the classical L-function defined in [EL1] associated to the F-isocrystal $pr^* D^\dagger \otimes U(\rho)^\dagger$.

The rest of this section will be devoted to proving the following theorem, which is our main result:

THEOREM 5.15. *The p-adic L-function and its dual version verify the following properties.*
1. (Char) In $K_0(\mathfrak{M}_G(H))$, one has the following equalities:

(i) $\partial(\mathcal{L}) = [N_{K_\infty}] + [L_{K_\infty}]$.
(ii) $\partial(\mathcal{L}^*) = -[N_{K_\infty}^*] - [L_{K_\infty}^*]$.
2. (*Interpolation*): *For any totally ramified extension L of $\mathbb{Q}_p$ with ring of integers denoted O and every $\rho : G \to Aut_O(V)$ with contragredient $\rho^\vee : G \to Aut_O(V^*)$, one has, in $L \cup \{\infty\}$:*
(i) $\rho(\mathcal{L}) = L(U, A, \rho^\vee, 1)$.
(ii) $\rho(\mathcal{L}^*) = L(U, A, \rho, 1)$.

Proof. 1. (i) According to Lem. 3.3 there exists a commutative square of the form

$$
\begin{array}{ccc}
I_{K_\infty} & \xrightarrow{\mathbf{1}-\phi} & P_{K_\infty} \\
{\scriptstyle\mathbf{1}}\downarrow & & \downarrow \\
P_{K_\infty} & \longrightarrow & P_{K_\infty}
\end{array}
$$

whose edges all become isomorphisms after localization by S^*. The claimed equality follows from Lem. 3.2 since $[Cone(\mathbf{1} - \phi)] = [N_{K_\infty}[1]] = -[N_{K_\infty}]$ and $[Cone(\mathbf{1})] = [L_{K_\infty}]$. Here, the existence of such a diagram is ensured by Lem. 3.3. The proof of (ii) is similar using the dual distinguished triangles describing $N_{K_\infty}^*$ and $L_{K_\infty}^*$.
2. Note that the formula (ii) follows from (i) by Rem. 5.14 (ii). In order to prove (i) we begin with preliminary results. $\square$

PROPOSITION 5.16. *There is a canonical isomorphism as follows in the derived category of L-vector spaces*

$$L\overset{L}{\otimes}_{\Lambda_O(G)}(V_{\Lambda_O(G)})^*\overset{L}{\otimes}_{\Lambda(G)}P_{K_\infty} \simeq R\Gamma_{rig,c}(U/L, pr^*D^\dagger \otimes U(\rho^\vee))$$

(here the $(\Lambda_O(G), \Lambda(G))$-bimodule structure of $(V_{\Lambda_O(G)})^*$ is as in the definition of the map $\rho : K_1(\Lambda(G)) \to K_1(O)$, see. paragraph 3.4). The action of $\phi\mathbf{1}^{-1}$ on the left hand term corresponds moreover to the action of Frobenius divided by p on the right hand side.

Proof. We have isomorphisms as follows:

$$
\begin{array}{rcl}
L\overset{L}{\otimes}_{\Lambda_O(G)}(V_{\Lambda_O(G)})^*\overset{L}{\otimes}_{\Lambda_O(G)}P_{K_\infty} & \overset{1}{\simeq} & L\overset{L}{\otimes}_{\Lambda_O(G)}(V^*\overset{L}{\otimes}_{\mathbb{Z}_p}P_{K_\infty}) \\
& \overset{2}{\simeq} & L\overset{L}{\otimes}_{L[G_n]}(V_L^*\overset{L}{\otimes}_{\mathbb{Z}_p}P_n) \\
& \overset{3}{\simeq} & L\overset{L}{\otimes}_{L[G_n]}(V_L^*\overset{L}{\otimes}_L R\Gamma_{rig,c}(U_n/L, pr^*D^\dagger)) \\
& \overset{4}{\simeq} & R\Gamma_{rig,c}(U/L, pr^*D \otimes U^\dagger(\rho^\vee))
\end{array}
$$

Here the isomorphism 1 is by Rem. 3.4 (i), 2 is obvious and 4 follows from étale cohomological descent for compactly supported rigid cohomology ([CT]) together with the definition of $U(\rho^\vee)$. The isomorphism 3 follows from the fact that in $D(L[G_n])$ we have a Frobenius compatible isomorphism as follows:

$$L\overset{L}{\otimes}_{\mathbb{Z}_p}R\Gamma(C_n^\sharp/\mathbb{Z}_p, D(-Z)) \simeq R\Gamma_{rig,c}(U_n/L, pr^*D^\dagger)$$

Let us explain the latter isomorphism. Since (compactly supported) rigid cohomology is compatible to base change from $\mathbb{Q}_p$ to L we may assume that $L = \mathbb{Q}_p$ (note that the Frobenius is L-linear since $L/\mathbb{Q}_p$ is totally ramified) and then apply the following lemma to the diagram having $C_n^\sharp$ as only vertex and edges indexed by G_n.   □

LEMMA 5.17. *Consider a diagram (see [TV] Def. 2.1) X/Δ of proper smooth curves over a perfect field k, Z a smooth divisor of X, $X^\sharp = (X, Z)$, $Y = X - Z$, E an F-crystal on $X^\sharp$ and $E^\dagger$ the induced overconvergent F-isocrystal ([LST]). If Δ satisfies the finiteness condition of [TV] Lem. 5.7 then we have a canonical isomorphism*

$$\mathbb{Q}_p \overset{L}{\otimes}_{\mathbb{Z}_p} R\Gamma(X^\sharp/\mathbb{Z}_p, E(-Z)) \simeq R\Gamma_{rig,c}(Y, E^\dagger)$$

in $D(Mod(\mathbb{Q}_p)^\Delta)$. This isomorphism is compatible with Frobenius.

Proof. By using Poincaré duality on both sides of the isomorphism, we are reduced to establish an isomorphism

$$\mathbb{Q}_p \overset{L}{\otimes}_{\mathbb{Z}_p} R\Gamma(X^\sharp/\mathbb{Z}_p, E) \simeq R\Gamma_{rig}(Y, E^\dagger)$$

in $D(Mod(\mathbb{Q}_p)^{\Delta^{op}})$. Moreover the compatibility with Frobenius is a formal consequence of the rest since one can replace the diagram X by the diagram $Frob : X \to X$ of type $\Delta \times [1]$. Using cohomological descent on both the crystalline and the rigid side (see [CT]), we can always assume given an exact immersion $X^\sharp \subset P^\sharp$ into a log smooth formal log scheme $P^\sharp/W(k)$ as in [TV] Lem. 5.7. Then, after tensorisation by $\mathbb{Q}_p$, the crystalline cohomology of E can be computed (see [Sh] and [Tr], proof of proposition 3.3) as the log de Rham cohomology on the tube $]X[_P$ of the associated module with connection E_K ($K = Frac(W(k))$ and maps naturally to the de Rham cohomology on V, some strict neighborhood of $]Y[_P$ in $]X[_P$, of $E_K|_V$, which is nothing but the rigid cohomology of $E^\dagger$. We have then constructed a map

$$\mathbb{Q}_p \overset{L}{\otimes}_{\mathbb{Z}_p} R\Gamma(X^\sharp/\mathbb{Z}_p, E) \to R\Gamma_{rig}(Y, E^\dagger).$$

To show that this map is an isomorphism, we need to show that this is the case on each vertex of the diagram. Now we can always assume given a proper smooth lifting of each vertex (they need not be compatible) and we conclude using [LST], 4.2 and [Tsj], 1.5.   □

PROPOSITION 5.18. *[EL2] Let $E^\dagger \in F - iso^\dagger(U/\mathbb{Q}_p)$.*

$$L(U, E^\dagger, \rho, t) = \prod_{i=0}^{2} det_L(1 - tF | H^i_{rig,c}(U/L, pr^* D^\dagger \otimes U(\rho)^\dagger)^{(-1)^{i+1}}$$

         □

Now we can prove the interpolation formula $2.(i)$ of Thm. 5.15. For simplicity, we set

$$_RP := R\overset{L}{\otimes}_{\Lambda_O(G)}(V_{\Lambda_O(G)})^*\overset{L}{\otimes}_{\Lambda(G)}P_{K_\infty} \quad \text{and} \quad _RH^q := H^q(_RP)$$

for any $\Lambda_O(G)$-algebra R. Note that $_{R'}H^q \simeq R' \otimes_R {}_RH^q$ if R'/R is flat. Applied to P_{K_∞}, Lem. 3.6 gives

$$\rho_{Q_O(\Gamma)}\mathcal{L} = \prod_q det_{Q_O(\Gamma)}((\mathbf{1}-\phi)\mathbf{1}^{-1}|_{Q_O(\Gamma)}H^q)^{(-1)^{q+1}} \tag{17}$$

and Lem. 3.7 applied to $_{\Lambda_O(\Gamma)_I}H^q$ gives:

$$\begin{aligned}
&\epsilon det_{Q_O(\Gamma)}((\mathbf{1}-\phi)\mathbf{1}^{-1}|_{Q_O(\Gamma)}H^q) \\
&= det_L((\mathbf{1}-\phi)\mathbf{1}^{-1}|Tor_0^{\Lambda_O(\Gamma)_I}(L, {}_{\Lambda_O(\Gamma)_I}H^q)) \\
&\quad .det_L((\mathbf{1}-\phi)\mathbf{1}^{-1}|Tor_1^{\Lambda_O(\Gamma)_I}(L, {}_{\Lambda_O(\Gamma)_I}H^q))^{-1}
\end{aligned} \tag{18}$$

whenever the right hand side makes sense (recall that the map ϵ was defined in (10)).
Now Prop. 5.16 shows that

$$\begin{aligned}
R\Gamma_{rig,c}(U/L, pr^*D^\dagger \otimes U(\rho^\vee)) &\simeq {}_LP \\
&\simeq L\overset{L}{\otimes}_{\Lambda_O(\Gamma)_I}\Lambda_O(\Gamma)_IP
\end{aligned}$$

Whence a spectral sequence which degenerates into short exact sequences (where we write $H^q_{rig,c}(U/L)$ instead of $H^q_{rig,c}(U/L, pr^*D^\dagger \otimes U(\rho^\vee))$, by lack of space):

$$\begin{aligned}
& & H^2_{rig,c}(U/L) &\twoheadrightarrow Tor_0^{\Lambda_O(\Gamma)_I}(L, {}_{\Lambda_O(\Gamma)_I}H^2) \\
Tor_1^{\Lambda_O(\Gamma)_I}(L, {}_{\Lambda_O(\Gamma)_I}H^2) &\hookrightarrow H^1_{rig,c}(U/L) &\to Tor_0^{\Lambda_O(\Gamma)_I}(L, {}_{\Lambda_O(\Gamma)_I}H^1) \\
Tor_1^{\Lambda_O(\Gamma)_I}(L, {}_{\Lambda_O(\Gamma)_I}H^1) &\hookrightarrow H^0_{rig,c}(U/L) &\twoheadrightarrow Tor_0^{\Lambda_O(\Gamma)_I}(L, {}_{\Lambda_O(\Gamma)_I}H^0) \\
Tor_1^{\Lambda_O(\Gamma)_I}(L, {}_{\Lambda_O(\Gamma)_I}H^0) &= 0 &
\end{aligned} \tag{19}$$

CLAIM. The operator $(\mathbf{1}-\phi)\mathbf{1}^{-1}$ is bijective on each $Tor_{-i}^{\Lambda_O(\Gamma)_I}(L, {}_{\Lambda_O(\Gamma)_I}H^j)$ except maybe for $(i,j) = (0,1)$. Moreover, the operator $1 - p^{-1}F$ is bijective on $H^q_{rig,c}(U/L, pr^*D^\dagger \otimes U(\rho^\vee))$ except maybe for $q = 1$.
Let us prove this claim. First we observe that the second statement implies the first one since by Nakayama's lemma, $(\mathbf{1}-\phi)\mathbf{1}^{-1}$ is bijective on $_{\Lambda_O(\Gamma)_I}H^j$ if and only if it is on $Tor_0^{\Lambda_O(\Gamma)_I}(L, {}_{\Lambda_O(\Gamma)_I}H^j)$, in which case it is also bijective on $Tor_1^{\Lambda_O(\Gamma)_I}(L, {}_{\Lambda_O(\Gamma)_I}H^j))$. Now the second claim follows from the long exact sequence (cf. Thm. 4.14 and the isomorphism between the third and last term at the beginning of the proof of Prop. 5.16)

$$L \otimes_{L[G_n]} (V_L^*\overset{L}{\otimes}_{\mathbb{Z}_p}N_n) \to H^q_{rig,c}(U/L) \overset{1-p^{-1}F}{\to} H^q_{rig,c}(U/L) \to \dots$$

since by Cor. 4.12 $\mathbb{Q} \otimes N_n$ is concentrated in degrees $[1,2]$.

We may now conclude since the claim ensures that no indeterminate product of the form $0.\infty$ occurs in:

- the formula obtained by combining (17) with (18) (note that the left hand term is nothing but $\rho(\mathcal{L})$):

$$\epsilon \rho_{Q_O(\Gamma)}(\mathcal{L}) \prod_q det_{Q_O(\Gamma)}((\mathbf{1} - \phi)\mathbf{1}^{-1}|_{Q_O(\Gamma)} H^q)^{(-1)^{q+1}}$$
$$= \prod_{i,j} det_L((\mathbf{1} - \phi)\mathbf{1}^{-1}|Tor_{-i}^{\Lambda_O(\Gamma)_I}(L, {}_{\Lambda_O(\Gamma)_I} H^j))^{(-1)^{i+j+1}}$$

- the formula given by the exact sequences (19):

$$\prod_{i,j} det_L((\mathbf{1} - \phi)\mathbf{1}^{-1}|Tor_{-i}^{\Lambda_O(\Gamma)_I}(L, {}_{\Lambda_O(\Gamma)_I} H^j))^{(-1)^{i+j+1}}$$
$$= \prod_q det_L(1 - p^{-1}F|H_{rig,c}^q(U/L, pr^*D^\dagger \otimes U(\rho^\vee))^{(-1)^{(q+1)}}$$

- the formula given by Lem. 5.18:

$$\prod_q det_L(1 - p^{-1}F|H_{rig,c}^q(U/L, pr^*D^\dagger \otimes U(\rho^\vee))^{(-1)^{(q+1)}} = L(U, A, \rho^\vee, 1).$$

$\square$

REMARK 5.19. (i) In the above proof we have retrieved the well known fact that $\rho(\mathcal{L}) = L(U, A, \rho^\vee, 1)$ may take the value 0, but not ∞.
(ii) In [LLTT], the author considers the complex $RHom_{\mathbb{Z}_p}(\underline{L}, \mathbb{Q}_p/\mathbb{Z}_p) = \underline{L}^*[1]$ instead of $\underline{L}^*$. This explains the difference of signs between formula 1. (ii) and those of loc. cit.

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