# Multiplicativity of the Double Ramification Cycle

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Abstract. The double ramification cycle satisfies a basic multiplicative relation $\mathrm{DRC}_a \cdot \mathrm{DRC}_b = \mathrm{DRC}_a \cdot \mathrm{DRC}_{a+b}$ over the locus of compact-type curves, but this relation fails in the Chow ring of the moduli space of stable curves. We restore this relation over the moduli space of stable curves by introducing an extension of the double ramification cycle to the small b-Chow ring (the colimit of the Chow rings of all smooth blowups of the moduli space). We use this to give evidence for the conjectured equality between the (twisted) double ramification cycle and a cycle $\mathrm{P}_g^{d,k}(A)$ described by the second author in [JPPZ17].

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## 1 Introduction

Given integers $a_1, \dots, a_n$ summing to zero, one defines the *double ramification cycle* $\mathrm{DRC}_{\underline{a}}$ in the moduli space $\mathcal{M}_{g,n}$ of smooth curves by pulling back the unit section of the universal jacobian along the section induced by the divisor $\sum_i a_i[x_i]$, where the $x_i$ are the tautological sections of the universal curve. This class has been extended over the whole of $\overline{\mathcal{M}}_{g,n}$ by work of Li-Graber-Vakil [Li01], [Li02], [GV05] (extending work of Hain [Hai13] and Grushevsky-Zakharov [GZ14b]). An alternative construction of the same cycle was recently

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given by the first author [Hol17]. Yet another approach, using the compactified Jacobians from [KP17], was given by the first author together with Kass and Pagani in [HKP18], though we will not use the latter in this paper.

A basic multiplicative relation holds between the double ramification cycles over the locus of curves of compact-type, namely

$$\mathrm{DRC}_{\underline{a}} \cdot \mathrm{DRC}_{\underline{b}} = \mathrm{DRC}_{\underline{a}} \cdot \mathrm{DRC}_{\underline{a}+\underline{b}} \tag{1}$$

for all vectors $\underline{a}$, $\underline{b}$ of ramification data. In section 8 we show by means of an example that this relation fails to hold in the Chow ring of $\overline{\mathcal{M}}_{g,n}$, and moreover that this cannot be corrected by making a different choice of extension of the cycle.

The aim of this paper is to restore the relation (1) over the whole of $\overline{\mathcal{M}}_{g,n}$ by working in the (small) $b$-Chow ring $\mathrm{bCH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$, defined as the colimit of the Chow rings of all smooth blowups of $\overline{\mathcal{M}}_{g,n}$ (see section 4). The transition maps are given by pullback of cycles; the relation to Shokurov's notion of $b$-divisor ([Sho96], [Sho03]) is discussed further in section 4. Using results of [Hol17], we construct extensions $\mathrm{bDRC}_{\underline{a}}$ of the double ramification cycle in the small $b$-Chow ring $\mathrm{bCH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ with two fundamental properties:

Theorem 1.1. *The pushforward of* $\mathrm{bDRC}_{\underline{a}}$ *to the Chow ring of* $\overline{\mathcal{M}}_{g,n}$ *coincides with the standard extension of the double ramification cycle* $\overline{\mathrm{DRC}}_{\underline{a}}$ *(as constructed in [Li01], [Li02], and [GV05], or equivalently in [Hol17]).*

Theorem 1.2. *The relation* $\mathrm{bDRC}_{\underline{a}} \cdot \mathrm{bDRC}_{\underline{b}} = \mathrm{bDRC}_{\underline{a}} \cdot \mathrm{bDRC}_{\underline{a}+\underline{b}}$ *holds in the small $b$-Chow ring* $\mathrm{bCH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$.

This result holds also for the $\omega^{\otimes k}$-twisted version of the double ramification cycle, with essentially the same proof.

Note that the pushforward map from small $b$-Chow ring $\mathrm{bCH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ to the Chow ring $\mathrm{CH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ is *not* a ring homomorphism, so these results do *not* imply multiplicativity of the $\overline{\mathrm{DRC}}$ in $\mathrm{CH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$.

The relation (1) is extremely natural, and we might speculate that its failure to hold in the Chow group of $\overline{\mathcal{M}}_{g,n}$ suggests that this is not the most natural setting in which to consider the double ramification cycle. Perhaps the $b$-Chow version of the double ramification cycle is the more fundamental object, or at least a shadow thereof?

Conjecture 1.4 of [Hol17] predicts that the cycle $\overline{\mathrm{DRC}}_{\underline{a}}$ in $\mathrm{CH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ coincides with a cycle $2^{-g}\mathrm{P}_g^{g,k}(\underline{A})$ constructed by the second named author; more details are given in section 6. For $k = 0$ this follows from the main theorem of [JPPZ17], but it is open for higher $k$. In proposition 6.3 we verify this conjecture on the locus of compact-type curves.

In section 7 we show that the multiplicativity relation eq. (1) holds in the Chow ring of the locus of *treelike* curves — curves whose dual graph has cycles of length at most 1. In particular, if the conjectured equality between $\overline{\mathrm{DRC}}_{\underline{a}}$ and $2^{-g}\mathrm{P}_g^{g,k}(\underline{A})$ holds true, then in turn the cycle $\mathrm{P}_g^{g,k}(\underline{A})$ must also satisfy this multiplicativity relation on the locus of treelike curves. In proposition 7.2 we give a direct, combinatorial proof of this multiplicativity relation for $\mathrm{P}_g^{g,k}(\underline{A})$, providing evidence for the conjectural equality between $\overline{\mathrm{DRC}}_{\underline{a}}$ and $2^{-g}\mathrm{P}_g^{g,k}(\underline{A})$.

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NOTATION AND SETUP

We write $\hookrightarrow$ for open immersions and $\rightarrowtail$ for closed immersions. We work over a field of characteristic zero, so that we can assume resolution of singularities. See section 9 for an approach that works in arbitrary characteristic.
For us, 'curve' means proper, flat, finitely presented, with reduced connected nodal geometric fibres, and $\overline{\mathcal{M}}_{g,n}$ denotes the usual Deligne-Mumford-Knudsen compactification of the moduli stack of smooth curves of genus g with $n$ disjoint ordered marked sections. We write $\mathcal{C}_{g,n}/\overline{\mathcal{M}}_{g,n}$ for the universal curve, $x_i$ for the sections, and $\omega$ for the relative dualising sheaf. We let $\mathcal{J}_{g,n} = \mathrm{Pic}^0_{\mathcal{C}_{g,n}/\overline{\mathcal{M}}_{g,n}}$ denote the universal generalized jacobian (a semiabelian scheme, the fibrewise connected component of the identity in $\mathrm{Pic}_{\mathcal{C}/\overline{\mathcal{M}}}$, parametrizing line bundles of degree 0 on *every* irreducible component of the fibres of $\mathcal{C}_{g,n}$ over $\overline{\mathcal{M}}_{g,n}$).

## 2 EXTENDING THE DOUBLE RAMIFICATION CYCLE

Here we recall briefly the construction of the extension of the double ramification cycle given in [Hol17]. Given $g, n \geq 0$ with $2g-2+n > 0$ and integers $\underline{a} = (a_1, \ldots, a_n, k)$ with $\sum_i a_i = k(2g-2)$, we define a section $\sigma_{\underline{a}} = [\omega^{\otimes k}(-\sum_i a_i x_i)]$ of $\mathcal{J}_{g,n}$ over $\mathcal{M}_{g,n}$ (which does not in general extend over the whole of $\overline{\mathcal{M}}_{g,n}$).
Let $f\colon X \to \overline{\mathcal{M}}_{g,n}$ be a proper birational morphism from a regular stack (a 'regular modification'). The section $\sigma_{\underline{a}}$ is then defined on some dense open of $X$. We write $\mathring{X}$ for the largest open of $X$ on which this rational map can be extended to a morphism, and $\sigma_{\underline{a}}^X\colon \mathring{X} \to J$ for the extension.
We define the *double ramification locus* $\mathrm{DRL}_{\underline{a}}^X \rightarrowtail \mathring{X}$ to be the schematic pullback of the unit section of $\mathcal{J}_{g,n}$ along $\sigma_{\underline{a}}^X$, and the *double ramification cycle* $\mathrm{DRC}_{\underline{a}}^X$ to be the cycle-theoretic pullback, as a cycle supported on $\mathrm{DRL}_{\underline{a}}^X$. Now the morphism $\mathring{X} \to \overline{\mathcal{M}}_{g,n}$ is rarely proper, but we have:

Theorem 2.1 ([Hol17], theorem 1.1). *In the directed system of all regular modifications of* $\overline{\mathcal{M}}_{g,n}$*, those* $X$ *such that* $\mathrm{DRL}_{\underline{a}}^X \to \overline{\mathcal{M}}_{g,n}$ *is proper form a cofinal system.*

Now $\mathrm{DRC}_{\underline{a}}^X$ is supported on $\mathrm{DRL}_{\underline{a}}^X$, so when the morphism $\mathrm{DRL}_{\underline{a}}^X \to \overline{\mathcal{M}}_{g,n}$ is proper we can take the pushforward of $\mathrm{DRC}_{\underline{a}}^X$ to $\overline{\mathcal{M}}_{g,n}$. Writing $\pi_{X*}\mathrm{DRC}_{\underline{a}}^X$ for the resulting cycle on $\overline{\mathcal{M}}_{g,n}$, we have:

Theorem 2.2 ([Hol17], theorem 1.2). *The net* $\pi_{X*}\mathrm{DRC}_{\underline{a}}^X$ *is eventually constant in the Chow ring* $\mathrm{CH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$*. We denote the limit by* $\overline{\mathrm{DRC}}_{\underline{a}}$*.*

In the case $k = 0$ it is shown in [Hol17] that this class $\overline{\mathrm{DRC}}_{\underline{a}}$ coincides with the class constructed by Li, Graber, and Vakil.

## 3 Multiplicativity lemma

Let $S$ be a regular Deligne-Mumford stack, and $G/S$ a smooth separated group scheme[3] with unit section $e$. Given $\sigma \in G(S)$ a section, we define

$$L_\sigma = \sigma^* e$$

as a closed substack of $S$, and

$$C_\sigma = \sigma^*[e]$$

as a cycle class supported on $L_\sigma$.

Lemma 3.1 (Multiplicativity lemma). *Let* $\pi\colon G \to S$ *be as above, and let* $\sigma$*,* $\tau \in G(S)$ *be two sections. Then we have*

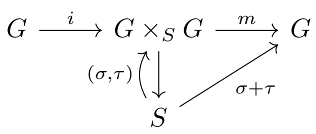
$$L_\sigma \times_S L_\tau = L_\sigma \times_S L_{\sigma+\tau} \tag{2}$$

*as closed substacks of* $S$*, and*

$$C_\sigma \cdot C_\tau = C_\sigma \cdot C_{\sigma+\tau} \tag{3}$$

*as cycles supported on* $L_\sigma \times_S L_\tau$*.*

*Proof.* Note that the set-theoretic version of eq. (2) is trivial. We give only the argument for eq. (3); that for eq. (2) is similar but easier. In the diagram




[3] By this we mean a smooth separated morphism $G \to S$ representable by schemes, which is a group object in the category of morphisms to $S$, i.e. which comes together with $S$-morphisms $m : G \times_S G \to G$, $i : G \to G$, $e : S \to G$ satisfying the usual compatibility relations of groups.

where $i = (e \circ \pi, id)$, we have equalities of cycles supported on $L_\sigma \cap L_\tau$:

$$\begin{aligned}
\sigma^*[e] \cdot (\sigma+\tau)^*[e] &= (\sigma,\tau)^* i_*[G] \cdot (\sigma,\tau)^*(m^*[e]) \\
&= (\sigma,\tau)^* \Big( i_*[G] \cdot m^*[e] \Big) \\
\text{(projection formula)} &= (\sigma,\tau)^* i_* \Big( [G] \cdot i^* m^*[e] \Big) \\
&= (\sigma,\tau)^* i_* i^* m^*[e] \\
&= (\sigma,\tau)^*[(e,e)] \\
&= \sigma^*[e] \cdot \tau^*[e].
\end{aligned}$$

□

A natural application of this lemma is to the double ramification cycle. Here the base $S$ is given by $\mathcal{M}_{g,n}$, and $G = \mathcal{J}_{g,n}$ is the jacobian of the universal curve. Then for any vector of integers $\underline{a} = (a_1, \dots, a_n, k)$ with $\sum_i a_i = k(2g-2)$ we have the section $\sigma_{\underline{a}} = [\omega^{\otimes k}(-\sum_i a_i x_i)]$ of $\mathcal{J}_{g,n}$, and the double ramification cycle on $\mathcal{M}_{g,n}$ is given by pulling back the unit section along $\sigma_{\underline{a}}$, i.e.

$$\mathrm{DRC}_{\underline{a}} = C_{\sigma_{\underline{a}}}$$

in the notation of lemma 3.1. We thus obtain from lemma 3.1 the relation

$$\mathrm{DRC}_{\underline{a}} \cdot \mathrm{DRC}_{\underline{b}} = \mathrm{DRC}_{\underline{a}} \cdot \mathrm{DRC}_{\underline{a}+\underline{b}} \tag{4}$$

in $\mathrm{CH}^{2g}_{\mathbb{Q}}(\mathcal{M}_{g,n})$, after pushing forward from the intersection of the corresponding double ramification loci. However, this relation is uninteresting as both sides vanish for $g \geq 1$ (and are equal to 1 for $g = 0$). Indeed, it was shown by Hain in [Hai13] that the double ramification cycles are tautological (for details see section 6). But the tautological ring of $\mathcal{M}_{g,n}$ vanishes in degree at least $g$ by [Ion02], so the two sides of eq. (4) vanish for $g \geq 1$ since they are of degree $2g$.

Over the locus of compact type (or more generally *treelike*) curves, the double ramification cycle can be defined in the same way, and the same proof shows that multiplicativity holds here; more details are given in section 7. Moreover, on these loci the relation is not vacuous, as shown in section 8. However, the same section shows that this multiplicativity relation does not extend over the whole of $\overline{\mathcal{M}}_{g,n}$; in the next section, we introduce the b-Chow ring, and in the section after we extend the double ramification cycle to the b-Chow ring and show that multiplicativity does hold there.

## 4 The $b$-Chow ring

The group of $b$-divisors on a scheme $X$ was introduced by Shokurov [Sho96], [Sho03] as the limit of the divisor groups of all blowups of $X$, with transition maps given by proper pushforward. One can define a (large) $b$-Chow group in the same way, as the limit over all blowups with transition maps given by pushforward, but note that it does not have a natural ring structure. The small $b$-Chow group is defined below as the *colimit* of Chow groups over smooth blowups, with transition maps given by pullback of cycles. It is naturally a subgroup of the large $b$-Chow group, and importantly it carries a natural ring structure (described below), so we refer to it as the (small) $b$-Chow *ring*.

Let $S$ be an irreducible noetherian algebraic stack. We write $\mathrm{Bl}(S)$ for the category whose objects are proper birational morphisms $X \to S$, relatively representable by algebraic spaces, and with $X$ regular, and where the morphisms are morphisms over $S$. Taking Chow rings and pullbacks gives a new category $\mathrm{CH}_{\mathbb{Q}}(\mathrm{Bl}(S))$, whose objects are the $\mathbb{Q}$-Chow rings of the objects of $\mathrm{Bl}(S)$, and where morphisms are given by pullbacks (which makes sense because everything is regular). We define the $b$-Chow ring of $S$ to be the colimit of this system of rings:

$$\mathrm{bCH}_{\mathbb{Q}}(S) = \operatorname{colim} \mathrm{CH}_{\mathbb{Q}}(\mathrm{Bl}(S)).$$

The category $\mathrm{CH}_{\mathbb{Q}}(\mathrm{Bl}(S))$ is filtered ([Sta13, Tag 04AX]); the only non-trivial thing to check is that, for two objects $X/S$ and $Y/S$ in $\mathrm{Bl}(S)$, there exists $Z/S \in \mathrm{Bl}(S)$ dominating $X$ and $Y$. Let $U \subseteq S$ be some dense open where $X \to S$ and $Y \to S$ are isomorphisms, and let $Z'/S$ denote the schematic image of $U$ in the fibre product $X \times_S Y$. Then $Z'/S$ is proper, birational, relatively representable and dominates $X$ and $Y$, but need not be regular. However, by [Tem12] it admits a resolution by blowing up; we take $Z$ to be such a resolution. For a filtered colimit we can give a much more concrete description on the level of sets:

$$\mathrm{bCH}_{\mathbb{Q}}(S) = \left( \bigsqcup_{X \in \mathrm{Bl}(S)} \mathrm{CH}_{\mathbb{Q}}(X) \right) / \sim$$

where for elements $x \in \mathrm{CH}_{\mathbb{Q}}(X)$ and $y \in \mathrm{CH}_{\mathbb{Q}}(Y)$, we say $x \sim y$ if and only if there exists $Z \in \mathrm{Bl}(S)$ and $S$-morphisms $f\colon Z \to X$, $g\colon Z \to Y$, with

$$f^*x = g^*y.$$

To multiply elements $x$ and $y$, we again find a $Z \in \mathrm{Bl}(S)$ mapping to both $X$ and $Y$, and form the intersection product after pullback to this $Z$.

## 5 Multiplicativity of the double ramification cycle in the $b$-Chow ring

Given $\underline{a} = (a_1, \ldots, a_n, k)$ with $\sum_i a_i = k(2g-2)$, we first define the extension of the corresponding double ramification cycle to $\mathrm{bDRC}_{\underline{a}}$ in $\mathrm{bCH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$. Taking the standard extension to the Chow ring of $\overline{\mathcal{M}}_{g,n}$ and pulling back is not the right approach — for example, the multiplicativity relation will fail. Instead we look at the construction in section 2. Recall that for a modification $X \to \overline{\mathcal{M}}_{g,n}$ we write $\mathring{X} \hookrightarrow X$ for the largest open to which $\sigma_{\underline{a}}$ extends. We define $\mathrm{DRL}_{\underline{a}}^X \rightarrowtail \mathring{X}$ by pulling back the unit section scheme-theoretically, and $\mathrm{DRC}_{\underline{a}}^X$ as a cycle class on $\mathrm{DRL}_{\underline{a}}^X$ by pulling back in Chow.

Let $X$ be regular and such that $\mathrm{DRL}_{\underline{a}}^X$ is proper over $\overline{\mathcal{M}}_{g,n}$. Write $i\colon \mathrm{DRL}_{\underline{a}}^X \to X$ for the inclusion, which is a closed immersion. Then we define $\mathrm{DRC}_{\underline{a}}^X = i_* \mathrm{DRC}_{\underline{a}}^X$ as an element of $\mathrm{bCH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$. Recall from section 2 that the $X$ with $\mathrm{DRL}_{\underline{a}}^X$ proper over $\overline{\mathcal{M}}_{g,n}$ form a cofinal system among all modifications $X$, yielding a net of $\mathrm{DRC}_{\underline{a}}^X$ in $\mathrm{bCH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$.

Lemma 5.1. *The net of* $\mathrm{DRC}_{\underline{a}}^X$ *in* $\mathrm{bCH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ *is eventually constant.*

*Proof.* This argument is a simpler version of the proof of [Hol17, theorem 6.3]. The limiting value can be obtained by taking $X$ a regular compactification of the stack $\mathcal{M}_{g,n}^{\Diamond}$ constructed in [loc.cit.]. □

Definition 5.2. We define $\mathrm{bDRC}_{\underline{a}}$ in $\mathrm{bCH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ as the limit of the above net.

Theorem 1.1 now follows formally from [Hol17, §6], so it remains to prove theorem 1.2.

Theorem 5.3. *Choose* $\underline{a} = (a_1, \dots, a_n, k)$ *with* $\sum_i a_i = k(2g-2)$*, and similarly choose* $\underline{b} = (b_1, \dots, b_n, k')$*. Then in* $\mathrm{bCH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ *we have*

$$\mathrm{bDRC}_{\underline{a}} \cdot \mathrm{bDRC}_{\underline{b}} = \mathrm{bDRC}_{\underline{a}} \cdot \mathrm{bDRC}_{\underline{a}+\underline{b}}\,. \tag{5}$$

*Proof.* Choose $X \to \overline{\mathcal{M}}_{g,n}$ so that $\mathrm{DRL}_{\underline{a}}^X \to \overline{\mathcal{M}}_{g,n}$ is proper and so that $\mathrm{DRC}_{\underline{a}}^X$ equals the limiting value $\mathrm{bDRC}_{\underline{a}}$ in $\mathrm{bCH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$. Choose a corresponding $Y$ for $\underline{b}$, and let $Z$ be a regular modification admitting morphisms to $X$ and $Y$ over $\overline{\mathcal{M}}_{g,n}$ (this exists since the category $\mathrm{Bl}(\overline{\mathcal{M}}_{g,n})$ is filtered). It suffices to check eq. (5) in the Chow ring of $Z$.

Let $\mathring{Z}_a \hookrightarrow Z$ be the largest open where $\sigma_{\underline{a}}$ extends, and similarly define $\mathring{Z}_{\underline{b}}$ and $\mathring{Z}_{\underline{a}+\underline{b}}$. Writing $\mathring{Z} = \mathring{Z}_{\underline{a}} \cap \mathring{Z}_{\underline{b}}$, we see that $\sigma_{\underline{a}+\underline{b}}$ is also defined on $\mathring{Z}$; it is given by $\sigma_{\underline{a}} + \sigma_{\underline{b}}$. Hence we have

$$\mathring{Z} = \mathring{Z}_{\underline{a}} \cap \mathring{Z}_{\underline{b}} \subseteq \mathring{Z}_{\underline{a}+\underline{b}},$$

and a similar argument shows

$$\mathring{Z}_{\underline{a}} \cap \mathring{Z}_{\underline{a}+\underline{b}} \subseteq \mathring{Z}_{\underline{b}}. \tag{6}$$

Now it is clear that

$$\mathrm{DRL}_{\underline{a}}^Z \cap \mathrm{DRL}_{\underline{b}}^Z \subseteq \mathring{Z}_{\underline{a}} \cap \mathring{Z}_{\underline{b}} = \mathring{Z},$$

and similarly we have

$$\mathrm{DRL}^Z_{\underline{a}} \cap \mathrm{DRL}^Z_{\underline{a}+\underline{b}} \subseteq \mathring{Z}_{\underline{a}} \cap \mathring{Z}_{\underline{a}+\underline{b}} \subseteq \mathring{Z}_{\underline{a}} \cap \mathring{Z}_{\underline{b}} = \mathring{Z},$$

where the middle relation comes from eq. (6). The theorem now follows directly from lemma 3.1 applied to the universal jacobian $\mathcal{J}_{g,n}$ pulled back to $Z$. □

## 6 Relation to the cycle $\mathrm{P}_g^{d,k}(A)$

In this and the next section we consider the connection between the classes $\overline{\mathrm{DRC}}_{\underline{a}} \in \mathrm{CH}^g_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ and the tautological cycle class $\mathrm{P}_g^{d,k}(\underline{A})$ introduced by the second author in [JPPZ17]. For this we first recall some notation from [loc.cit.].
Fix an integer $k \geq 0$ and an integer vector $\underline{A} = (A_1, \dots, A_n)$ with $\sum_{i=1}^n A_i = k(2g-2+n)$. Note that there is a natural bijection of such vectors $\underline{A}$ and vectors $\underline{a} = (a_1, \dots, a_n)$ with $\sum_{i=1}^n a_i = k(2g-2)$ by setting

$$(A_1, \dots, A_n) = (a_1 + k, \dots, a_n + k);$$

we will use this identification in what follows.
Fix also a degree $d \geq 0$, then given this data, in [JPPZ17, Section 1.1] a tautological cycle class

$$\mathrm{P}_g^{d,k}(\underline{A}) \in \mathrm{CH}^d_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$$

is defined as an explicit sum in terms of decorated boundary strata. The main result of [JPPZ17] is that for $k = 0, d = g$ this formula computes the double ramification cycle corresponding to the partition $\underline{A}$. More precisely, they prove

$$\mathrm{DR}_g(\underline{A}) = 2^{-g}\mathrm{P}_g^{g,0}(\underline{A}),$$

where $\mathrm{DR}_g(\underline{A})$ is the double ramification cycle associated to $\underline{A}$ via the Gromov-Witten theory of 'rubber $\mathbb{P}^1$'.
From [Hol17, conjecture 1.4] we recall

Conjecture 6.1. For all $k$ we have

$$\overline{\mathrm{DRC}}_{\underline{a}} = 2^{-g}\mathrm{P}_g^{g,k}(\underline{A})$$

as elements of $\mathrm{CH}^g_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$.

*Remark* 6.2. Conjecture 6.1 holds when $k = 0$. Indeed, when $k = 0$ we know by [Hol17, theorem 1.3] that $\overline{\mathrm{DRC}}_{\underline{a}} = \mathrm{DR}_g(\underline{A})$, which combined with the main result of [JPPZ17] yields the result.

We now show that conjecture 6.1 holds for all $k$ if we restrict to the locus of curves of compact type.

Proposition 6.3. *On the locus $\mathcal{M}^{ct}_{g,n}$ of compact type curves we have an equality*

$$\overline{\mathrm{DRC}}_{\underline{a}} = 2^{-g}\mathrm{P}_g^{g,k}(\underline{A}) \in \mathrm{CH}^g_{\mathbb{Q}}(\mathcal{M}^{ct}_{g,n}). \tag{7}$$

*Proof.* The proof runs via the following chain of equalities in $\mathrm{CH}^g_{\mathbb{Q}}(\mathcal{M}^{ct}_{g,n})$.

$$\overline{\mathrm{DRC}}_{\underline{a}} \overset{a)}{=} \sigma_{\underline{a}}^*[e] \overset{b)}{=} \sigma_{\underline{a}}^* \frac{\theta^g}{g!} = \frac{1}{g!}\left(\sigma_{\underline{a}}^*\theta\right)^g \overset{c)}{=} \frac{1}{2^g g!}\left(P_g^{1,k}(\underline{A})\right)^g \overset{d)}{=} 2^{-g}\mathrm{P}_g^{g,k}(\underline{A}).$$

To start, equality a) follows from the definition of the double ramification cycle and the fact that the universal jacobian over $\mathcal{M}^{ct}_{g,n}$ is an abelian scheme, hence any sections over $\mathcal{M}_{g,n}$ are guaranteed to extend (uniquely) over $\mathcal{M}^{ct}_{g,n}$. Equality b) comes by pulling back the obvious relation on the universal abelian variety, which has already been observed by various authors, see for instance [GZ14a].

Now the pullback (in cohomology) of the theta divisor under $\sigma_{\underline{a}}$ has been computed by Hain in [Hai13]. In standard notation for the tautological classes in $\mathcal{M}^{ct}_{g,n}$, Hain's result reads as follows: in $H^2(\mathcal{M}^{\mathrm{ct}}_{g,n})$ we have

$$\sigma_{\underline{a}}^*\theta = -\frac{k^2}{2}\kappa_1 + \frac{1}{2}\sum_{j=1}^{n}(a_j+k)^2\psi_j - \frac{1}{2}\sum_{g',P}(a_P-(2g'-1)k)^2\delta_{g'}^P. \tag{8}$$

Here $P$ runs over subsets of $\{1,\dots,n\}$, $a_P = \sum_{i\in P} a_i$, and the last sum should be interpreted as including each boundary divisor $\delta_{g'}^P = \delta_{g-g'}^{P^c}$ exactly once. Deducing equality c) in cohomology then follows by an elementary verification using the definition of $\mathrm{P}_g^{1,k}(\underline{A})$ from [JPPZ17, Section 1.1]. To lift this to an equality in Chow, we want to show that the cycle class map $\mathrm{CH}^1_{\mathbb{Q}}(\mathcal{M}^{\mathrm{ct}}_{g,n}) \to H^2(\mathcal{M}^{\mathrm{ct}}_{g,n})$ is injective. Now it is classical that $\mathrm{CH}^1_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n}) \cong H^2(\overline{\mathcal{M}}_{g,n})$. Since $\mathcal{M}^{\mathrm{ct}}_{g,n}$ is the complement of the divisor $\Delta_{\mathrm{irr}} \subset \overline{\mathcal{M}}_{g,n}$ of irreducible nodal curves, we have $\mathrm{CH}^1_{\mathbb{Q}}(\mathcal{M}^{\mathrm{ct}}_{g,n}) = \mathrm{CH}^1_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})/\mathbb{Q}\cdot[\Delta_{\mathrm{irr}}]$ by the excision exact sequence for Chow groups. For the analogous result in cohomology, note that the inclusion of the open set $\mathcal{M}^{\mathrm{ct}}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}$ with complement $\Delta_{\mathrm{irr}}$ induces a long exact sequence for cohomology with compact support. For $d = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{g,n} = 3g-3+n$ we look at the following piece of this sequence

$$\cdots \to H_c^{2d-2}(\mathcal{M}^{\mathrm{ct}}_{g,n}) \to H^{2d-2}(\overline{\mathcal{M}}_{g,n}) \to H^{2d-2}(\Delta_{\mathrm{irr}}) \to \cdots,$$

where we note that $\overline{\mathcal{M}}_{g,n}$ and $\Delta_{\mathrm{irr}}$ are compact, so compactly supported cohomology agrees with usual cohomology. Taking the dual and using Poincaré duality, we have an exact sequence

$$H^2(\mathcal{M}^{\mathrm{ct}}_{g,n}) \longleftarrow H^2(\overline{\mathcal{M}}_{g,n}) \longleftarrow H^0(\Delta_{\mathrm{irr}}) = \mathbb{Q}\cdot[\Delta_{\mathrm{irr}}],$$

where we use that $\Delta_{\mathrm{irr}}$ is connected. This implies that we have an injection $H^2(\overline{\mathcal{M}}_{g,n})/\mathbb{Q}\cdot[\Delta_{\mathrm{irr}}] \hookrightarrow H^2(\mathcal{M}^{\mathrm{ct}}_{g,n})$, but

$$H^2(\overline{\mathcal{M}}_{g,n})/\mathbb{Q}\cdot[\Delta_{\mathrm{irr}}] \cong \mathrm{CH}^1_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})/\mathbb{Q}\cdot[\Delta_{\mathrm{irr}}] \cong \mathrm{CH}^1_{\mathbb{Q}}(\mathcal{M}^{\mathrm{ct}}_{g,n}).$$

This proves equality c) in Chow.

Finally, equality d) follows from the fact that on $\mathcal{M}_{g,n}^{ct}$ we have an equality of mixed-degree classes

$$\exp(\mathrm{P}_g^{1,k}(\underline{A})) = \sum_{d\geq 0} \mathrm{P}_g^{d,k}(\underline{A}) \in \mathrm{CH}_{\mathbb{Q}}^*(\mathcal{M}_{g,n}^{ct}). \tag{9}$$

This equality is a combinatorial statement; it is a specialization of the more general lemma 7.3 which we will prove in the next section. □

## 7 Restricting to treelike curves

In this section we focus on the locus of *treelike curves* — these are stable curves whose graph is a tree with any number of self-loops attached (equivalently, all non-disconnecting edges are self-loops, or all cycles in the graph have length $\leq 1$). We write $\mathcal{M}_{g,n}^{tl}$ for this locus; it is open in $\overline{\mathcal{M}}_{g,n}$, and clearly contains the compact-type locus $\mathcal{M}_{g,n}^{ct}$.

Over the locus of treelike curves, the universal jacobian $\mathcal{J}_{g,n}$ is not proper (and its toric rank can be arbitrarily large). However, it is still a Néron model of its generic fibre in the sense of [Hol19] — this follows easily from the main theorem of [loc.cit.], since all cycles in the graph have length at most 1. In particular, this implies that the section $\sigma_{\underline{a}}$ over $\mathcal{M}_{g,n}$ extends uniquely to $\mathcal{J}_{g,n}$ over the whole of $\mathcal{M}_{g,n}^{tl}$.

In fact, $\mathcal{M}_{g,n}^{tl}$ can be uniquely characterised as the largest open of $\overline{\mathcal{M}}_{g,n}$ such that every étale-local section of the universal jacobian over $\mathcal{M}_{g,n}$ extends. Indeed, if $C$ is a non-treelike point then there exists a cycle of irreducible components $\gamma$ of $C$ of length greater than 1. Choose an étale neighbourhood $U$ of $C$ in $\overline{\mathcal{M}}_{g,n}$, and sections $p$ and $q$ in $\mathcal{C}_{g,n}(U)$ passing through distinct irreducible components of $\gamma$. Write $\mathring{U}$ for the pullback of $\mathcal{M}_{g,n}$ to $U$. Then the formula $[\mathcal{O}_{C_{g,n}}(p-q)]$ defines a section in $\mathcal{J}_{g,n}(\mathring{U})$ which cannot extend over $U$. The quick way to see this is to observe that there does not exist a tropical rational function on the graph $\Gamma$ of $C$ whose divisor has multidegree equal to that of $[\mathcal{O}_{C_{g,n}}(p-q)]$. More explicitly, we can apply lemma 4.3 of [Hol17], and see that there does not exist a *weighting* on the decorated graph $\Gamma$ which is compatible with the *thickness* $(1,\dots,1)$ (see [loc.cit.] section 3 for this notation).

Recall that $\overline{\mathrm{DRC}}_{\underline{a}}$ is the extension of the double ramification cycle to the Chow ring of $\overline{\mathcal{M}}_{g,n}$ as constructed in section 2.

Lemma 7.1. *In* $\mathrm{CH}_{\mathbb{Q}}^g(\mathcal{M}_{g,n}^{tl})$ *we have the equality*

$$\overline{\mathrm{DRC}}_{\underline{a}} = \sigma_{\underline{a}}^*[e] \tag{10}$$

*and in* $\mathrm{CH}_{\mathbb{Q}}^{2g}(\mathcal{M}_{g,n}^{tl})$ *we have*

$$\overline{\mathrm{DRC}}_{\underline{a}} \cdot \overline{\mathrm{DRC}}_{\underline{b}} = \overline{\mathrm{DRC}}_{\underline{a}} \cdot \overline{\mathrm{DRC}}_{\underline{a}+\underline{b}}. \tag{11}$$

*Proof.* Since the section $\sigma_{\underline{a}}$ extends to $\mathcal{J}_{g,n}$ over $\mathcal{M}^{tl}_{g,n}$, the blowups used to extend the section may be assumed to be isomorphisms over $\mathcal{M}^{tl}_{g,n}$; more precisely, the cofinal system in section 5 can be chosen so that all of the birational morphisms $X \to \overline{\mathcal{M}}_{g,n}$ are isomorphisms over $\mathcal{M}^{tl}_{g,n}$. This proves (10), and (11) then follows from theorem 5.3, or directly from lemma 3.1. □

The classes $\mathrm{P}^{g,k}_g$ also satisfy multiplicativity on $\mathcal{M}^{tl}_{g,n}$:

PROPOSITION 7.2. *Let $\underline{A}$, $\underline{B}$ be vectors of $n$ integers with $\sum A_i = k_{\underline{a}}(2g-2+n)$ and $\sum B_i = k_{\underline{b}}(2g-2+n)$ for some $k_{\underline{a}}, k_{\underline{b}} \in \mathbb{Z}$. Then the equality*

$$\mathrm{P}^{g,k_{\underline{a}}}_g(\underline{A}) \cdot \mathrm{P}^{g,k_{\underline{b}}}_g(\underline{B}) = \mathrm{P}^{g,k_{\underline{a}}}_g(\underline{A}) \cdot \mathrm{P}^{g,k_{\underline{a}}+k_{\underline{b}}}_g(\underline{A}+\underline{B}) \tag{12}$$

*holds in* $\mathrm{CH}^{2g}_{\mathbb{Q}}(\mathcal{M}^{tl}_{g,n})$.

The consistency of this multiplicativity with that of lemma 7.1 provides evidence for conjecture 6.1.
There are two key ingredients in the proof of proposition 7.2. The first is the basic codimension $g+1$ relation

$$\mathrm{P}^{g+1,k}_g(\underline{A}) = 0 \in \mathrm{CH}^{g+1}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n}) \tag{13}$$

proved in [CJ18, Theorem 5.4]. The second is the following combinatorial lemma:

LEMMA 7.3. *Let $\mathrm{P}^k_g(\underline{A})^{treelike}$ denote the mixed-degree class in the Chow ring of the locus of treelike curves*

$$\mathrm{P}^k_g(\underline{A})^{treelike} := \sum_{d\geq 0} \mathrm{P}^{d,k}_g(\underline{A}) \in \mathrm{CH}^*_{\mathbb{Q}}(\mathcal{M}^{tl}_{g,n}).$$

*Then there exists a mixed-degree class $\Delta \in \mathrm{CH}^*_{\mathbb{Q}}(\mathcal{M}^{tl}_{g,n})$ (not depending on $\underline{A}$ or $k$) along with a divisor-valued quadratic form $Q(\underline{A}) \in \mathrm{CH}^1_{\mathbb{Q}}(\mathcal{M}^{tl}_{g,n})$ such that*

$$\mathrm{P}^k_g(\underline{A})^{treelike} = \exp(Q(\underline{A}))\Delta.$$

Before checking lemma 7.3, we use it to prove proposition 7.2:

*Proof of proposition 7.2.* Using lemma 7.3 we can rewrite the codimension $g+1$ relation eq. (13) for a vector $\underline{A}+\underline{C}$ as

$$[\exp(Q(\underline{A}+\underline{C}))\Delta]_{g+1} = 0,$$

where $[X]_d$ denotes the codimension $d$ part of a mixed-degree class $X$.
This relation is an equality of polynomials in the $\underline{A}$ and $\underline{C}$ variables, so it will still hold if we restrict to the part of degree 1 in $\underline{C}$. This gives

$$[\exp(Q(\underline{A}))\Delta]_g \cdot (Q(\underline{A}+\underline{C}) - Q(\underline{A}) - Q(\underline{C})) = 0.$$

Changing variables with $\underline{C} = \underline{A} + 2\underline{B}$, using the fact that $Q$ is a quadratic form, and dividing by 2, we arrive at the relation

$$[\exp(Q(\underline{A}))\Delta]_g \cdot (Q(\underline{A} + \underline{B}) - Q(\underline{B})) = 0.$$

Now, the mixed-degree class $\exp(Q(\underline{A}+\underline{B})) - \exp(Q(\underline{B}))$ is clearly divisible by the divisor class $Q(\underline{A} + \underline{B}) - Q(\underline{B})$, so we have the relation

$$[\exp(Q(\underline{A}))\Delta]_g[(\exp(Q(\underline{A} + \underline{B})) - \exp(Q(\underline{B})))\Delta]_g = 0.$$

Then applying lemma 7.3 again gives the desired multiplicativity statement. □

*Proof of lemma 7.3.* This lemma is essentially a combinatorial statement about the definition of the classes $\mathrm{P}_g^{d,k}(\underline{A})$ in [JPPZ17, Section 1.1] along with the multiplication formula for tautological classes given in [GP03, Appendix A, eq. (11)]. In the general case, $\mathrm{P}_g^{d,k}(\underline{A})$ is a sum over decorated (by $\psi$ and $\kappa$ classes) dual graphs $\Gamma$ of a combinatorial coefficient times the tautological class corresponding to $\Gamma$. The combinatorial coefficient is defined by taking the $r$-constant term of a polynomial in $r$ defined by summing over certain balanced 'weightings mod $r$' of the half-edges of $\Gamma$.

In our case, we can assume that the graph $\Gamma$ is treelike and the combinatorial coefficients then become significantly simpler: the only weights that are allowed to vary are those in loops of the graph, and these weights are subject only to the condition that the weights on the two sides of a loop must sum to zero mod $r$. The result is that the coefficient associated to a graph $\Gamma$ factors as a product of the contributions from the loops and the contributions from the non-loops. Moreover, it is easily seen that if $\Gamma$ is treelike then there is a unique way to pick two disjoint subsets $E_1, E_2$ of the set of edges of $\Gamma$ such that $\Gamma$ becomes a tree when $E_1$ is contracted and becomes a single vertex with loops attached when $E_2$ is contracted: we must have that $E_1$ is the set of loops and $E_2$ is the set of non-loop edges.

These two facts about treelike graphs and their combinatorial coefficients along with the graph refinement formula for multiplication in the tautological ring [GP03, Appendix A] have the following consequence: the entire (mixed-degree) class factors as

$$\mathrm{P}_g^k(\underline{A})^{\mathrm{treelike}} = \mathrm{P}_g^k(\underline{A})^{\mathrm{tree}} \cdot \mathrm{P}_g^k(\underline{A})^{\mathrm{loops}},$$

where the three classes are, respectively, the full class on the locus of treelike curves, the sum of those terms with $\Gamma$ a tree, and the sum of those terms where $\Gamma$ has exactly one vertex and where there are no $\kappa$ decorations on the vertex and no $\psi$ decorations on any legs (but possibly on loops).

Moreover, the final class above does not actually depend on the vector $\underline{A}$; we set

$$\Delta := \mathrm{P}_g^k(\underline{A})^{\mathrm{loops}}.$$

We do not need to know anything more about $\Delta$ for our purposes, but an explicit formula can easily be obtained:

$$\Delta = \sum_{l=0}^{g} \frac{(-1)^l}{2^l \cdot l!} (\xi_l)_* \prod_{i=1}^{l} \left( \sum_{d \geq 0} \frac{B_{2d+2}}{(d+1)!} (\psi_{n+2i-1} + \psi_{n+2i})^d \right),$$

where $\xi_l : \overline{\mathcal{M}}_{g-l,n+2l} \to \overline{\mathcal{M}}_{g,n}$ glues the last $l$ pairs of points together and $B_{2d+2}$ are Bernoulli numbers.

For the remaining factor $\mathrm{P}_g^k(\underline{A})^{\mathrm{tree}}$, we claim that

$$\mathrm{P}_g^k(\underline{A})^{\mathrm{tree}} = \exp([\mathrm{P}_g^k(\underline{A})^{\mathrm{tree}}]_1). \tag{14}$$

Then we can take

$$Q(\underline{A}) := [\mathrm{P}_g^k(\underline{A})^{\mathrm{tree}}]_1,$$

which explicitly is given by the same formula as Hain's formula eq. (8) (multiplied by 2 and interpreted as divisors on the locus of treelike curves) and thus is a quadratic form in $\underline{A}$.

It remains to check eq. (14) using the multiplication formula of [GP03, Appendix A]. Suppose that for $i = 1, \ldots, k$, $\delta_{g_i}^{P_i}$ are boundary divisor classes for separating nodes, so each such class corresponds to a graph with two vertices connected by a single edge along with a distribution of the total genus $g$ and markings between the two vertices (such that one has genus $g_i$ and marking $P_i$). If we multiply all of these $k$ divisor classes together, the multiplication formula in this case says that the result is a sum over the following data: a tree $\Gamma$ along with a distribution of genus and markings between the vertices of $\Gamma$ and a sequence of edges $e_1, \ldots, e_k$ in $\Gamma$ (possibly with repetition) such that

1. the division of genus and markings across the two sides of edge $e_i$ agree with the division in $\delta_{g_i}^{P_i}$;
2. every edge of $\Gamma$ appears at least once in the sequence $e_1, \ldots, e_k$.

Repeated edges $e_i$ give rise to $\psi$ classes along that edge.

Computing the right side of eq. (14) (the exponential of a divisor class) by using the above procedure to multiply divisor classes together then gives precisely the sum over trees appearing in the definition of $\mathrm{P}_g^k(\underline{A})^{\mathrm{tree}}$. □

## 8 Failure of multiplicativity in the Chow ring of $\overline{\mathcal{M}}_{g,n}$

Since both sides of eq. (12) make sense in the Chow ring of $\overline{\mathcal{M}}_{g,n}$, it is natural to ask whether the multiplicativity stated in proposition 7.2 might hold not just on the locus of treelike curves but on the entire space of stable curves. In this section we present an explicit example where this desired equality fails and in fact argue that there can be no other extension of the cycles $\mathrm{DRC}_{\underline{a}}$ from

$\mathcal{M}_{g,n}^{ct}$ that would make the equality hold. In other words, multiplicativity is really a feature of the (small) $b$-Chow ring and not of the standard Chow ring. Let $g = 1, k = 0$ and consider the two partitions $\underline{a} = (2, 4, -6)$, $\underline{b} = (-3, -1, 4)$ of 0. Let $\overline{\mathrm{DRC}}_{\underline{a}}$, $\overline{\mathrm{DRC}}_{\underline{b}}$, $\overline{\mathrm{DRC}}_{\underline{a}+\underline{b}} \in \mathrm{CH}^1_{\mathbb{Q}}(\overline{\mathcal{M}}_{1,3})$ be the corresponding double ramification cycles. By proposition 6.3 these agree with the corresponding $\mathrm{P}_1^1(\underline{A})$, which can be computed as explicit tautological classes. Using an implementation of the tautological ring by the second author one can check that the multiplicativity fails inside the Chow group of $\overline{\mathcal{M}}_{1,3}$, i.e.

$$\overline{\mathrm{DRC}}_{\underline{a}} \cdot \overline{\mathrm{DRC}}_{\underline{b}} \neq \overline{\mathrm{DRC}}_{\underline{a}} \cdot \overline{\mathrm{DRC}}_{\underline{a}+\underline{b}} \in \mathrm{CH}^2_{\mathbb{Q}}(\overline{\mathcal{M}}_{1,3}). \tag{15}$$

What is true however is that the difference of the two sides in eq. (15) is a linear combination of the classes of the three irreducible components of $\overline{\mathcal{M}}_{1,3} \setminus \mathcal{M}_{1,3}^{tl}$. In other words, eq. (15) becomes an equality once we restrict to the locus $\mathcal{M}_{1,3}^{tl}$ of treelike curves, as proved in proposition 7.2. Moreover, actually both sides of eq. (15) give nontrivial elements of $\mathrm{CH}^2_{\mathbb{Q}}(\mathcal{M}_{1,3}^{tl})$. In particular, this shows that for the above example the two sides of the multiplicativity statement in the (small) $b$-Chow ring are also nontrivial. This gives an indication that the multiplicativity which holds in the $b$-Chow ring is not the consequence of some trivial vanishing (like both sides of the equality being zero, for instance).

Now one final hope for multiplicativity on $\overline{\mathcal{M}}_{g,n}$ could be that the cycles $\overline{\mathrm{DRC}}_{\underline{a}}$, $\overline{\mathrm{DRC}}_{\underline{b}}$, $\overline{\mathrm{DRC}}_{\underline{a}+\underline{b}}$ are not the right extension of the corresponding Abel-Jacobi pullbacks $\sigma_{\underline{a}}^*[e], \sigma_{\underline{b}}^*[e], \sigma_{\underline{a}+\underline{b}}^*[e] \in \mathrm{CH}^1_{\mathbb{Q}}(\mathcal{M}_{1,3}^{ct})$ on the locus of compact type curves. However, the complement $\overline{\mathcal{M}}_{1,3} \setminus \mathcal{M}_{g,n}^{ct}$ is exactly given by the boundary divisor $\Delta_{irr}$ generically parametrising irreducible nodal curves. Hence any such extensions must have the form

$$\begin{aligned}
\widetilde{\mathrm{DRC}}_{\underline{a}} &= \overline{\mathrm{DRC}}_{\underline{a}} + \lambda_{\underline{a}} \cdot \Delta_{irr},\\
\widetilde{\mathrm{DRC}}_{\underline{b}} &= \overline{\mathrm{DRC}}_{\underline{b}} + \lambda_{\underline{b}} \cdot \Delta_{irr},\\
\widetilde{\mathrm{DRC}}_{\underline{a}+\underline{b}} &= \overline{\mathrm{DRC}}_{\underline{a}+\underline{b}} + \lambda_{\underline{a}+\underline{b}} \cdot \Delta_{irr}.
\end{aligned}$$

Using that $\Delta_{irr}^2 = 0$ we compute

$$\begin{aligned}
&\widetilde{\mathrm{DRC}}_{\underline{a}} \cdot (\widetilde{\mathrm{DRC}}_{\underline{b}} - \widetilde{\mathrm{DRC}}_{\underline{a}+\underline{b}})\\
=&\underbrace{\overline{\mathrm{DRC}}_{\underline{a}} \cdot (\overline{\mathrm{DRC}}_{\underline{b}} - \overline{\mathrm{DRC}}_{\underline{a}+\underline{b}})}_{I_1} + (\lambda_{\underline{b}} - \lambda_{\underline{a}+\underline{b}}) \underbrace{\overline{\mathrm{DRC}}_{\underline{a}} \cdot \Delta_{irr}}_{I_2}\\
&+ \lambda_{\underline{a}} \underbrace{\Delta_{irr} \cdot (\overline{\mathrm{DRC}}_{\underline{b}} - \overline{\mathrm{DRC}}_{\underline{a}+\underline{b}})}_{I_3}.
\end{aligned}$$

However, it can be checked by computer that the three elements $I_1, I_2, I_3 \in \mathrm{CH}^2_{\mathbb{Q}}(\overline{\mathcal{M}}_{1,3})$ are linearly independent. Therefore there is no way to choose $\lambda_{\underline{a}}, \lambda_{\underline{b}}, \lambda_{\underline{a}+\underline{b}}$ to have the $\widetilde{\mathrm{DRC}}$ satisfy multiplicativity in the Chow ring of $\overline{\mathcal{M}}_{g,n}$; we only have multiplicativity in the (small) $b$-Chow ring or on the open locus of treelike curves.

## 9 Logarithmic version

We work with fine and saturated log structures in the sense of Fontaine-Illusie, using Olsson's generalisation to stacks [Ols01]. We put a log structure on $\overline{\mathcal{M}}_{g,n}$ as in Kato [Kat96].

### A logarithmic analogue in arbitrary characteristic

Until now we have restricted to characteristic zero in order to be able to apply Hironaka's resolution of singularities, to show that every modification of $\overline{\mathcal{M}}_{g,n}$ can be dominated by a regular one - more precisely we use Temkin's functorial resolution [Tem12], so that we can apply it to stacks. This implies that the category $\mathrm{Bl}(\overline{\mathcal{M}}_{g,n})$ is filtered, giving us a very explicit description of the colimit $\mathrm{bCH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$.
In positive characteristic the analogue of Hironaka's resolution is not known. However, it follows from the proof of [Hol17, Lemma 6.1] that in our setting we can restrict to modifications $X \to \overline{\mathcal{M}}_{g,n}$ which are log blowups (in particular log étale over $\overline{\mathcal{M}}_{g,n}$). Since $\overline{\mathcal{M}}_{g,n}$ is log regular, such a modification is again log regular, and we can use results of Gabber, Illusie and Temkin to show that a suitable category of blowups is filtered. In the remainder of this section we will explain how to use these ideas to generalise our results to arbitrary characteristic.
From now on we work over a field $k$ of any characteristic. First we describe the translation of section 5 into the logarithmic setting. We write $\mathrm{Bl}_{log}(\overline{\mathcal{M}}_{g,n})$ for the category of log blowups $X \to \overline{\mathcal{M}}_{g,n}$ whose underlying stacks are regular; morphisms are taken over $\overline{\mathcal{M}}_{g,n}$.

Lemma 9.1. *The category* $\mathrm{Bl}_{log}(\overline{\mathcal{M}}_{g,n})$ *is filtered.*

*Proof.* Let $X/\overline{\mathcal{M}}_{g,n}$ and $Y/\overline{\mathcal{M}}_{g,n}$ be objects in $\mathrm{Bl}_{log}(\overline{\mathcal{M}}_{g,n})$, and let $Z'$ denote their fibre product in the category of fine and saturated log schemes over $\overline{\mathcal{M}}_{g,n}$; note that $Z' \to \overline{\mathcal{M}}_{g,n}$ is again a log blowup. Now log blowups are log étale, hence $Z'$ is log étale over the log regular stack $\overline{\mathcal{M}}_{g,n}$, and hence $Z'$ is log regular. It remains to check that $Z'$ has a log blowups which is regular. Now [IT14, Theorem 3.4.9] gives a resolution algorithm for log regular log schemes, which is in particular functorial for strict étale morphisms. Since $\overline{\mathcal{M}}_{g,n}$ admits a strict étale cover by log schemes the same is true for $Z'$ (by base-change), so we apply functorial resolution to each patch of such a cover, and glue by functoriality. □

We define the logarithmic b-Chow ring $\mathrm{bCH}_{log,\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ of $\overline{\mathcal{M}}_{g,n}$ to be the colimit over $X \in \mathrm{Bl}_{log}(\overline{\mathcal{M}}_{g,n})$ of the Chow rings $\mathrm{CH}_{\mathbb{Q}}(X)$, with transition maps given by pullback. By lemma 9.1 this admits a simple presentation as in section 4.
Fix $\underline{a} = (a_1, \ldots, a_n, k)$ as usual. Given $X \in \mathrm{Bl}_{log}(\overline{\mathcal{M}}_{g,n})$ we define $\mathring{X}$ to be the largest open to which $\sigma_{\underline{a}}$ extends, and define $\mathrm{DRL}^X_{\underline{a}}$ and $\mathrm{DRC}^X_{\underline{a}}$ by the

same formulae as in section 5, by pulling back the unit section in $\mathcal{J}_{g,n}$ either as a scheme or as a cycle. Again we need to check that those $X$ with $\mathrm{DRL}_{\underline{a}}^X$ proper over $\overline{\mathcal{M}}_{g,n}$ form a cofinal system; this does not follow immediately from theorem 2.1 as it is not clear that those $X$ can be chosen to be log blowups, but does follow from [Hol17, Lemma 6.1 and Theorem 6.3], combined with resolution of singularities for log regular stacks [IT14, Theorem 3.4.9]. The same references show again that the net $\mathrm{DRC}_{\underline{a}}^X$ is eventually constant in $\mathrm{bCH}_{\mathbb{Q},log}(\overline{\mathcal{M}}_{g,n})$, yielding a well-defined class $\mathrm{bDRC}_{\underline{a},log} \in \mathrm{bCH}_{\mathbb{Q},log}(\overline{\mathcal{M}}_{g,n})$.

Theorem 9.2. *Choose* $\underline{a} = (a_1, \ldots, a_n, k)$ *with* $\sum_i a_i = k(2g-2)$*, and similarly choose* $\underline{b} = (b_1, \ldots, b_n, k')$*. Then in* $\mathrm{bCH}_{\mathbb{Q},log}(\overline{\mathcal{M}}_{g,n})$ *we have*

$$\mathrm{bDRC}_{\underline{a},log} \cdot \mathrm{bDRC}_{\underline{b},log} = \mathrm{bDRC}_{\underline{a},log} \cdot \mathrm{bDRC}_{\underline{a}+\underline{b},log}\,. \tag{16}$$

*Proof.* By lemma 9.1 we can choose $Z \in \mathrm{Bl}_{log}(\overline{\mathcal{M}}_{g,n})$ dominating both $\mathcal{M}_{\underline{a}}^{\blacklozenge}$ and $\mathcal{M}_{\underline{b}}^{\blacklozenge}$. The logarithmic structures play no further role, and we proceed exactly as in the proof of theorem 5.3. □

## The Chow ring of the valuativisation

Following [Kat89], the *valuativisation* of a log scheme or stack is the limit of all the log blowups; this does not exist as a scheme (or stack), but does exist as either a locally ringed space or a pro-scheme over $\overline{\mathcal{M}}_{g,n}$. The logarithmic b-Chow ring defined above can then be viewed as the Chow ring of the valuativisation, c.f. [SST18]. In [loc.cit.] a derived equivalence is constructed between the valuativisation and a certain infinite root stack of $\overline{\mathcal{M}}_{g,n}$. We hope that this derived equivalence might shed some light on the relation between the first author's construction in [Hol14] of a universal Néron-model-admitting stack, and Chiodo's work [Chi15]. More generally, it might realise our $\mathrm{bDRC} \in \mathrm{bCH}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,n})$ as a shadow of some more refined derived object.

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