Completeness: When Enough is Enough

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Abstract. We investigate the notion of a complete enough metric space that, while classically vacuous, in a constructive setting allows for the generalisation of many theorems to a much wider class of spaces. In doing so, this notion also brings the known body of constructive results significantly closer to that of classical mathematics. Most prominently, we generalise the Kreisel-Lacome-Shoenfield Theorem/Tseytin's Theorem on the continuity of functions in recursive mathematics.

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1 Introduction

In Bishop's constructive mathematics (BISH)[1] the completeness of a metric space is often added to the assumptions of a classical result in order to allow a constructive proof. This added assumption may be a small price to pay for a fully constructive proof, but it was noticed by the first author that some of these constructive proofs in fact needed only a very weak form of completeness. Thus it was that the notion of complete enough was introduced in [6] and it was shown that the important results known collectively as Ishihara's tricks are still valid when the assumption of completeness is weakened to complete enough. In [7] the authors showed that in Bishop's Lemma—an important result in constructive analysis, used in the proofs of many other results—one can also replace the assumption of completeness with the one of complete enough. The value of the notion of complete enough is in the following two facts:

[1]Informal mathematics using intuitionistic logic and an appropriate set-theoretic or type-theoretic foundation [1].

(i) Classically every metric space is complete enough (Proposition 3). Therefore adding that some space is complete enough to the assumptions of a classical theorem does not give a classically weaker result, in contrast to adding completeness.

(ii) There are many metric spaces that are complete enough in BISH that are not complete. The first example, given in [6], is the space of all permutations $\mathbb{N} \to \mathbb{N}$. A great deal more examples were given in [7] and Proposition 4 extends this.

In this paper we give a more systematic presentation of complete enough spaces and highlight some more applications, including, most prominently, the Kreisel-Lacome-Shoenfield Theorem/Tseytin's Theorem on the continuity of computable functions.
We will work in BISH. However, all our constructions are very tame and we are sure that they can be formalised in a wide range of formal systems. Many of our proofs use countable choice, which we assume implicitly, and which, in our opinion, makes the ideas involved clearer. This can actually be avoided or at least weakened in many cases and for the benefit of readers interested in working without countable choice we make some observations in the conclusion.
We will make reference to several classically valid 'omniscience principles' including the following.

LPO: for every binary sequence $(a_n)_{n\geqslant 1}$ we have

$$\forall n \in \mathbb{N} : a_n = 0 \vee \exists n \in \mathbb{N} : a_n = 1 \ .$$

WLPO: for every binary sequence $(a_n)_{n\geqslant 1}$ we have

$$\forall n \in \mathbb{N} : a_n = 0 \vee \neg \forall n \in \mathbb{N} : a_n = 0 \ .$$

LLPO: for every binary sequence $(a_n)_{n\geqslant 1}$ with at most one non-zero term either

$$\forall n \in \mathbb{N} : a_{2n} = 0 \vee \forall n \in \mathbb{N} : a_{2n+1} = 0 \ .$$

The next section introduces complete enough spaces and their categorical properties and the final section gives a number of applications of this notion. The proofs in Section 3 are mainly produced by inspection of old proofs—that is, by simply noting that the sequences constructed in the old proofs are of the form prescribed by the notion of complete enough. We would like to stress that the main mathematical advance is therefore not in the proofs of the main theorems, but rather by showing that the class of complete enough spaces contains many interesting members that are not complete.

2 Complete enough spaces

For a sequence $x = (x_n)_{n\geqslant 1}$, a point x_∞, and an increasing binary sequence $\lambda = (\lambda_n)_{n\geqslant 1}$ we define sequences $\lambda \circledast x$ and $\lambda \circledcirc x$ by

$$(\lambda \circledast x)_n = \begin{cases} x_m & \text{if } \lambda_n = 1 \text{ and } \lambda_m = 1 - \lambda_{m+1} \\ x_\infty & \text{if } \lambda_n = 0 \end{cases}$$

and

$$(\lambda \circledcirc x)_n = \begin{cases} x_m & \text{if } \lambda_n = 1 \text{ and } \lambda_m = 1 - \lambda_{m+1} \\ x_n & \text{if } \lambda_n = 0 \, . \end{cases}$$

A λ*-tagged sequence* is a sequence $(x_n)_{n\geqslant 1}$ such that there exists an increasing binary sequence $(\lambda_n)_{n\geqslant 1}$ with

$$\lambda_m = \lambda_n \implies x_m = x_n \, .$$

Notice that these definitions do not make any assumption on the type of the underlying space. Let X be a metric space; we denote the metric on X by ρ_X, or by ρ if only one metric space is under consideration. It is easy to show that if $x = (x_n)_{n\geqslant 1}$ is a Cauchy sequence in X, then $\lambda \circledast x$ and $\lambda \circledcirc x$ are also both Cauchy.

PROPOSITION 1. *The following are equivalent.*

1. *For every sequence $x = (x_n)_{n\geqslant 1}$ in X converging to some $x_\infty \in X$ and every increasing binary sequence $\lambda = (\lambda_n)_{n\geqslant 1}$, $\lambda \circledast x$ converges in X.*
2. *For every sequence $x = (x_n)_{n\geqslant 1}$ in X converging to some $x_\infty \in X$ and every increasing binary sequence $\lambda = (\lambda_n)_{n\geqslant 1}$, $\lambda \circledcirc x$ converges in X.*
3. *Every λ-tagged Cauchy sequence in X converges.*

Proof. To prove the equivalence of (1) and (2) it suffices to show that $\rho\left((\lambda \circledast x)_n, (\lambda \circledcirc x)_n\right) \to 0$ as $n \to 0$: either $\lambda_n = 1$ and $(\lambda \circledast x)_n = (\lambda \circledcirc x)_n$ or $\lambda_n = 0$ and

$$\rho((\lambda \circledast x)_n, (\lambda \circledcirc x)_n) = \rho(x_\infty, x_m) \, ,$$

so

$$\rho((\lambda \circledast x)_n, (\lambda \circledcirc x)_n) \leqslant \rho(x_\infty, x_m) \to 0$$

as $n \to 0$.

For any sequence $(x_n)_{n\geqslant 1}$ in X converging to $x_\infty \in X$ and any increasing binary sequence $(\lambda_n)_{n\geqslant 1}$, we have that $\lambda \circledast x$ is a λ-tagged sequence. Hence 3 implies 1. Now suppose that 1 holds and let $(\xi_n)_{n\geqslant 1}$ be a Cauchy λ-tagged sequence. Setting

$$x_n = \begin{cases} \xi_n & \lambda_n = 1 - \lambda_{n-1} \\ \xi_1 & \text{otherwise} \end{cases}$$

we have that $\xi_n = (\lambda \circledast x)_n$ for each n. To establish that 1 $\Rightarrow$ 3 it remains to show that $(\xi_n)_{n\geqslant 1}$ is convergent in X; indeed, $x_n \to \xi_1$. If $\lambda_n \neq 1 - \lambda_{n-1}$, then $\rho(x_n, \xi_1) = 0$. On the other hand, if $\lambda_n = 1 - \lambda_n$, then

$$\rho(x_n, \xi_1) = \rho(\xi_n, \xi_{n-1}) \to 0$$

as $n \to \infty$, since $(\xi_n)_{n\geqslant 1}$ is Cauchy. □

DEFINITION 2. A metric space X is *complete enough* if it satisfies any, and hence all, of the conditions in Proposition 1.

Every complete space is complete enough, but the two notions are not equivalent: under the assumption of classical logic every space is complete enough.

PROPOSITION 3. *The limited principle of omniscience (LPO) is equivalent to the statement that every metric space is complete enough.*

Proof. Given an (arbitrary) λ-tagged sequence $(x_n)_{n\geqslant 1}$, by LPO either $\lambda_n = 0$ for all n and $(x_n)_{n\geqslant 1}$ converges to x_1, or there exists n such that $\lambda_n = 1$ and $(x_n)_{n\geqslant 1}$ converges to x_n.
Conversely, let $(a_n)_{n\geqslant 1}$ be a binary sequence; we may assume without loss of generality that a_n is increasing. Let X be the space $\left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\} \cup \{0\}$, with the metric induced by the reals and let $x_n = \frac{1}{n}$ for each n. If X were complete enough, then $a \circledast x$ would converge to some $z \in X$. If $z \in \{0\}$, then $z = 0$ and $a_n = 0$ for all n: for if $a_n = 1$, then $a \circledast x$ would converge to $\frac{1}{m}$ where m is the smallest index such that $a_{m+1} = 1$. If $z \in \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$, then $z = \frac{1}{n}$ for some n. For that n we must have $a_n = 1$, since $a_n = 0$ implies that $z < \frac{1}{n}$. □

The previous proposition shows that there are metric spaces that cannot be shown to be complete enough in BISH, while in varieties of constructive mathematics where LPO is false—such as the Russian school of recursive mathematics (RUSS) or Brouwer's intuitionism (INT)—there are spaces that are not complete enough. The notion of being complete enough therefore has a similar status as locatedness [2, page 88]. It is classically vacuous, but constructively meaningful. The latter assertion, of course, is only true if we can, constructively, find interesting spaces that are complete enough but not complete. As it turns out there are many such examples. To name a few basic ones [7]:

- the open interval $(0,1)$,
- members of Baire space that are permutations, that is $\left\{ f \subseteq \mathbb{N}^{\mathbb{N}} \,\middle|\, f \text{ is bijective} \right\}$,
- the irrational real numbers.

More systematically, in [7] it is shown that every G_δ subset, in particular open subsets, of a complete enough space is again complete enough. The proof for Proposition 3 above shows that this cannot be improved upon—at least with regards to the Borel hierarchy—since X in that proof is an F_σ set. Nevertheless, we can improve upon the result in a different way.

PROPOSITION 4. *The class of complete enough sets is closed under arbitrary intersections.*[2]

Proof. Let $X = \bigcap_{i\in I} A_i$ be an intersection of complete enough sets, where I is inhabited, and let $(x_n)_{n\geqslant 1} \in X$ be a Cauchy λ-tagged sequence in X. By assumption the limit of $(x_n)_{n\geqslant 1}$, which is unique, is in A_i for every $i \in I$ and therefore also in X. □

[2] Assuming the index set of the set family is inhabited.

In [7] it was also shown that every stable set S, that is every set such that

$$\neg\neg x \in S \implies x \in S \,,$$

is complete enough. Notice that stable subsets are exactly the ones that can be written as a logical complement, because of triple negation simplifying to single negation intuitionistically, and since

$$A = X \setminus (X \setminus A)$$

for any stable subset A of X.
We next give two concrete examples of spaces that cannot be shown to be complete enough constructively; these show that the class of complete enough spaces is not closed under finite unions in BISH.

PROPOSITION 5. *If $[-1,0] \cup [0,1]$ is complete enough then LLPO holds and if $\{0\} \cup (0,1]$ is complete enough then WLPO holds.*

Proof. Let $(a_n)_{n\geqslant 1}$ be a binary sequence with at most one term equal to 1. Define an increasing binary sequence by $b_n = \max_{i\leqslant n}\{a_i\}$ and consider the sequence $x_n = \frac{(-1)^n}{n}$, which converges to 0. If there is an odd n such that $a_n = 1$ then $b \circledast x$ converges to $-\frac{1}{n} < 0$, and if there is an even n such that $a_n = 1$ then $b \circledast x$ converges to $\frac{1}{n} > 0$. Thus if the limit z of $b \circledast x$ is in $[-1,0] \cup [0,1]$ we can decide whether $z \leqslant 0$ or $z \geqslant 0$. In the first case $\forall n \in \mathbb{N} : a_{2n} = 0$, and in the second case $\forall n \in \mathbb{N} : a_{2n+1} = 0$. Hence, if $[-1,0] \cup [0,1]$ is complete enough, then LLPO holds.
Similar to the proof of Proposition 3 and the above, if the limit of $b \circledast |x|$ in $\{0\} \cup (0,1]$ is in $\{0\}$, then $\forall n : a_n = 0$ and if the limit is in $(0,1]$, then $\neg\forall n : a_n = 0$. □

PROPOSITION 6. *1. If X and Y are complete enough spaces then $X \times Y$ is.*

2. If X is complete enough and Y an arbitrary space then $X \setminus Y$ is complete enough.

Proof. 1. Straightforward.

2. Since stable sets are complete enough, $X \setminus Y = X \cap \neg Y$ is the intersection of two complete enough sets and hence is complete enough by Proposition 4.

□

REMARK 7. *Even though classically the image of a cover-compact space under a point-wise continuous map is compact and therefore complete (enough), there is very little hope that there is any sensible condition on $f : X \to \mathbb{R}$ and X that ensures that $f(X)$ is complete enough, constructively.*

Proof. Consider the map $f : 2^{\mathbb{N}} \to \mathbb{R}$ defined by

$$f(\alpha) = \sum_{n=0}^{\infty} \frac{\alpha_n}{2^n} \,,$$

that is the map mapping a binary sequence to the real in the unit interval having that binary expansion. The map f is uniformly (even Lipschitz) continuous and $2^{\mathbb{N}}$ is totally bounded and complete.

Consider the sequence

$$x_n = \frac{1}{2} + \frac{(-1)^n}{2^n} ,$$

which converges to $\frac{1}{2}$. It is easy to see that $x_n \in f(2^{\mathbb{N}})$ and $\frac{1}{2} \in f(2^{\mathbb{N}})$. Shadowing the argument in Proposition 5 from here on, one can show that if $f(2^{\mathbb{N}})$ is complete enough, then LLPO holds. □

A function $f : X \to Y$ between metric spaces X and Y is *strongly extensional* if $\rho_X(x,y) > 0$ whenever $\rho_Y(f(x), f(y)) > 0$.

PROPOSITION 8. *Consider a complete enough metric space Y, and a sequence of functions $f_n : X \to Y$ converging point-wise to $f : X \to Y$. For an increasing binary sequence λ_n define $g_n = (\lambda \circledast (f_k)_{k\geqslant 1})_n$.*
There exists a function $g : X \to Y$ such that g_n converges point-wise to g. Moreover, if $f_n \to f$ uniformly, then also $g_n \to g$ uniformly, and if f_n and f are strongly extensional functions, then so is g.

Proof. Since Y is complete enough, for every $x \in X$ the limit $g(x)$ of $(g_n(x))_{n\geqslant 1}$ exists, and so g_n converges to g point-wise.
Moreover, it is easy to prove that for all $x \in X$ and any $\varepsilon > 0$ if there is $N \in \mathbb{N}$ such that

$$\forall n \geqslant N : \rho_Y(f_n(x), f(x)) \leqslant \varepsilon ,$$

then

$$\forall n \geqslant N : \rho_Y(g_n(x), g(x)) \leqslant \varepsilon .$$

Thus in particular, if $f_n \to f$ uniformly, then also $g_n \to g$ uniformly.
To see that g is strongly extensional let $x, y \in X$ such that $\varepsilon = \rho_Y(g(x), g(y)) > 0$. Choose $N \in \mathbb{N}$ such that $\rho_Y(g_N(x), g(x)) < \varepsilon/3$ and $\rho_Y(g_N(y), g(y)) < \varepsilon/3$. Now either $\lambda_N = 0$ or $\lambda_N = 1$. In the first case $g_N = f$, and therefore $\rho_Y(f(x), f(y)) > \varepsilon/3$. In the second case there is $M \leqslant N$ such that $g_N = f_M$ and we have $\rho_Y(f_M(x), f_M(y)) > \varepsilon/3$. Since f and all f_n are strongly extensional, in both cases we have $\rho_X(x,y) > 0$. □

REMARK 9. *Even if, in the above proposition, the f_n and f are uniformly (or even Lipschitz) continuous we cannot, constructively, ensure that g is sequentially continuous. To see this consider the space X from Proposition 3, and define $f_n : X \to \mathbb{R}$ by $f_n(\frac{1}{m}) = \delta_{n,m}$, where $\delta_{n,m}$ is the Kronecker delta. Then $f_n \to 0$ and all f_n are Lipschitz continuous. However, consider a binary sequence $(a_n)_{n\geqslant 1}$ with at most one 1. Let x_n be the sequence defined by $x_n = \frac{a_n}{n}$ and define $(b_n)_{n\geqslant 1}$ by setting $b_n = \max_{i\leqslant n} a_i$. Obviously $x_n \to 0$. Notice that if there is $n \in \mathbb{N}$ such that $a_n = 1$, then $x_n = \frac{1}{n}$ and $g = f_n$, which means that $g(x_n) = 1$. Now assume that $g(x_n) \to g(0) = 0$. That means that there exists N such that $g(x_n) = 0$ for all $n \geqslant N$. But together that means that $a_n = 0$ for all $n \geqslant N$. Thus if g is sequentially continuous, we can prove LPO.*

It is natural to ask whether there is a "closed enough" counterpart to being closed. As the following definition and proposition suggest, this is not the case.[3] A subset A of a complete enough metric space X is $\circledast$-*closed* if whenever $x = (x_n)_{n\geqslant 1}$ is a sequence in X converging to $x_\infty \in X$ and $\lambda = (\lambda_n)_{n\geqslant 1}$ is an increasing binary sequence such that $\lambda_1 = 0$ and

$$\lambda_n = 0 \;\to\; x_n \in A \;,$$

then the limit of $\lambda \circledast x$ is also in A.

Proposition 10. *A subset A of a complete enough metric space X is $\circledast$-closed if and only if it is closed.*

Proof. If $(x_n)_{n\geqslant 1}$ is a sequence in a $\circledast$-closed subset A of a complete enough space X converging to x_∞ in X, then letting λ be the constant zero sequence we have

$$x_\infty = \lim_{n\to\infty} x_n = \lim_{n\to\infty} (\lambda \circledcirc x)_n \in A \;.$$

Now suppose that A is a closed subset of a complete enough space X and let $x = (x_n)_{n\geqslant 1}$ be a sequence in X converging to $x_\infty \in X$ and $\lambda = (\lambda_n)_{n\geqslant 1}$ is an increasing binary sequence such that $\lambda_n = 0 \to x_n \in A$. Then $(\lambda \circledast x)_n$ is a sequence in A converging in X, so

$$\lim_{n\to\infty} (\lambda \circledast x)_n = \lim_{n\to\infty} (\lambda \circledcirc x)_n \in A \;. \qquad \square$$

3 Applications

3.1 The Kreisel-Lacome-Shoenfield Theorem

It has long been known that in recursive varieties[4] of constructive mathematics all functions defined on a complete, separable metric space are (point-wise) continuous. Versions of this result have been proven by various people including Markov, Tseytin[5] [17], and Kreisel, Lacome, Schoenfield [8]. A careful analysis by Ishihara [9] breaks the proof down into neat, axiomatic parts.

Step 1 Weak Markov's principle (WMP)[6] implies that f is strongly extensional.

Step 2 If $f : X \to Y$ is a strongly extensional function, $(x_n)_{n\geqslant 1}$ is a sequence in X converging to x and $\alpha < \beta$ then either

[3] Intuitively this can be explained by the fact that complete enough is weaker than complete because we have the additional information of a candidate of convergence x_∞, whereas in the definition of closedness that is already the case with the standard notion.

[4] The same holds in Brouwer's intuitionism. There, however, it is more of an axiom ("continuos choice") rather than a proven feature. Nevertheless, our generalisation is valid there as well.

[5] Sometimes also spelled Čeitin, or Tseitin.

[6] WMP states that every pseudo-positive number is positive:

$$\forall x \in \mathbb{R} : (\forall y \in \mathbb{R} : \neg(y \leqslant 0) \;\vee\; \neg(y \geqslant x)) \implies x > 0 \,. \tag{1}$$

It has also been called the Weak Limited Principle of Existence (WLPE) in [15] and the Almost Separating Principle (ASP) in [14]. WMP is a very weak principle, which holds in classical mathematics as well as in RUSS and in INT.

(a) eventually $\rho_Y(f(x_n), f(x)) < \beta$ or

(b) $\rho_Y(f(x_n), f(x)) > \alpha$ infinitely often, in which case LPO holds.

This is, nowadays, known as *Ishihara's second trick*. It implies, in particular, that $\neg$LPO implies that f is sequentially continuous.

Step 3 BD-N[7] implies that every sequentially continuous function on a *separable* space is point-wise continuous.

Since WMP, BD-N, and $\neg$LPO all hold in RUSS this shows that every $f : X \to Y$ defined on a complete, separable space X is point-wise continuous in RUSS.
As already mentioned in the introduction, in [6] it is shown that Ishihara's tricks only require a complete enough space rather than a complete space. This means that if Step 1 above can be extended to complete enough spaces, then we have generalised the Kreisel-Lacome-Shoenfield Theorem.

PROPOSITION 11. *WMP is equivalent to the statement that every $f : X \to Y$ defined on a complete enough space X is strongly extensional.*

Proof. This can be done by inspecting the proof in [9] and noting that complete enough is enough. □

THEOREM 12. *If $f : X \to Y$ is a function between metric spaces, and X is complete enough, then*

1. *WMP + $\neg$LPO imply that f is sequentially continuous.*
2. *$\neg$LPO implies that if f is strongly extensional, then it is sequentially continuous.*
3. *If X is, in addition, separable, then WMP + $\neg$LPO + BD-N imply that f is point-wise continuous.*

In this paper we show that, to some extent, completeness has been used as an unnecessary crutch in constructive mathematics: in many proofs of constructive mathematics when completeness is assumed, only the classically trivial and constructively more general notion of complete enough is required. The next result, an improvement of Proposition 3.1.5 in [5], which itself was extracted from Theorem 1 [9], demonstrates the use of this crutch even more strongly: no completeness assumption is required of the domain at all.

PROPOSITION 13. *If $f : X \to Y$ is sequentially continuous, then it is strongly extensional.*

Proof. Let $f : X \to Y$ be sequentially continuous and let $x, y \in X$ with $f(x) \neq f(y)$. Fix a binary sequence $(\lambda_n)_{n\geqslant 1}$ such that

$$\lambda_n = 0 \implies \rho(x, y) < \frac{1}{2^n} ,$$

[7] Details on BD-N can be found in the already cited [10]. More details in [16, 13].

$$\lambda_n = 1 \implies \rho(x,y) > \frac{1}{2^{n+1}} ;$$

then λ_n is increasing. Now define $(x_n)_{n\geqslant 1}$ by

$$x_n = \begin{cases} y & \text{if } \lambda_n = 0 \\ x & \text{otherwise.} \end{cases} \tag{2}$$

It is easy to see that $\rho_X(x,x_n) < \frac{1}{2^n}$ for all $n \in \mathbb{N}$, which means that $x_n \to x$. Since f is sequentially continuous there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have

$$\rho_Y(f(x_n), f(x)) < \rho_Y(f(y), f(x)) .$$

Then $\lambda_N = 1$: for if $\lambda_N = 0$, then $x_N = y$ which leads to the contradiction

$$\rho_Y(f(y), f(x)) = \rho_Y(f(x_N), f(x)) < \rho_Y(f(y), f(x)) .$$

Thus $\rho_X(x,y) > \frac{1}{2^{N+1}} > 0$. □

While this paper is mostly concerned with the role of completeness in constructive mathematics, we are really advocating a careful analysis of proofs to extract their full significance. The first author's analysis of the proof of Ishihara's tricks resulted in the isolation of the notion of a complete enough space, a notion we hope to have shown fruitful. Taking this approach with the last, simple, proof, we might point out that that proof does not require the function f to be sequentially continuous, but merely that whenever $x_n \to x$ one has

$$\forall \varepsilon > 0 : \exists n \in \mathbb{N} : \rho_Y(f(x_n), f(x)) < \varepsilon . \tag{‡}$$

It is easy to see that this condition is classically equivalent to sequential continuity: given a sequence $(x_n)_{n\geqslant 1}$ converging to a point x such that $(f(x_n))_{n\geqslant 1}$ does not converge to $f(x)$, we can find $\varepsilon > 0$ and a subsequence $(x'_n)_{n\geqslant 1}$ of $(x_n)_{n\geqslant 1}$ such that $\rho_Y(f(x'_n), f(x)) > \varepsilon$ for all n. Surprisingly we can give a constructive proof of this equivalence; unsurprisingly we need to further assume the domain is complete enough.

LEMMA 14. *LPO is equivalent to the statement that if $f : X \to Y$ satisfies (‡) at x for every $x_n \to x$, then it is sequentially continuous at x.*

Proof. Take a function $f : X \to Y$ satisfying (‡) for some $x \in X$, let $(x_n)_{n\geqslant 1}$ be a sequence in X converging to x, and fix $\varepsilon > 0$. Since LPO implies that equality on $\mathbb{R}$ is decidable, we can construct a binary sequence $(\lambda_n)_{n\geqslant 1}$ such that

$$\lambda_n = 1 \iff \rho_Y(f(x_n), f(x)) < \varepsilon.$$

Using LPO again, repeatedly, construct another binary sequence $(\lambda'_n)_{n\geqslant 1}$ such that

$$\lambda'_n = 1 \iff \forall m > n : (\lambda_m = 1) .$$

By LPO, once again, either $\forall n : \lambda'_n = 0$ or $\exists n : \lambda'_n = 1$. Suppose the former holds. We construct an increasing function $h : \mathbb{N} \to \mathbb{N}$ inductively as follows. Since

$\lambda_0' = 0$, there exists n such that $\rho_Y(f(x_n), f(x)) \geqslant \varepsilon$; let $h(0) = n$. Now suppose we have defined h on $\{0, 1, \dots m\}$. Since $\lambda_h'(m) = 0$ there exists $n > m$ such that $\rho_Y(f(x_n), f(x)) \geqslant \varepsilon$; set $h(m+1) = n$. This completes the construction of h. By construction, $\rho_Y(f(x_{h(n)}), f(x)) \geqslant \varepsilon$ for all n, so the sequence $(x_{h(n)})_{n\geqslant 1}$ shows that f does not satisfy (‡) at x. This contradiction ensures that $\exists N(\lambda_N' = 1)$, and thus that there exists N such that $\rho_Y(f(x_n), f(x)) < \varepsilon$ for all $n > N$.
Conversely let $(a_n)_{n\geqslant 1}$ be a binary sequence with at most one term equalling 1. Let X be the space $\left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\} \cup \{0\}$, which is unsurprisingly the same space as in the proof of Proposition 3. Now define a function $f : X \to 2$ by setting $f(0) = 0$ and $f(\frac{1}{n}) = a_n$. Then f satisfies (‡) at 0: for let $x_n \to 0$. If $f(x_1) = 0$ we are done. If $f(x_1) = 1$ then $x_1 \in \left\{ \frac{1}{n} \,\middle|\, n \in \mathbb{N} \right\}$, so there must be $m \in \mathbb{N}$ such that $x_1 = \frac{1}{m}$. Furthermore me must have $a_m = 1$. Now choose $k > m$ such that $|x_k| < \frac{1}{m}$. Since $(a_n)_{n\geqslant 1}$ contains at most one 1 we must have $a_k = 0$ and therefore $f(x_k) = 0$. Now if f is sequentially continuous at x then there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $\left| f(\frac{1}{n}) \right| < \frac{1}{2}$, which means that $f(\frac{1}{n}) = 0$. This, in turn, implies that $a_n = 0$ for all $n \geqslant N$, and so by checking finitely many terms we can decide whether $\forall n \in \mathbb{N} : a_n = 0$ or whether $\exists n \in \mathbb{N} : a_n = 1$. Thus LPO holds. □

PROPOSITION 15. *Consider $f : X \to Y$, where X is complete enough and Y is arbitrary. Then f is sequentially continuous if and only if (‡) holds for all $x_n \to x$ in X.*

Proof. The forward direction is trivial. To see that the converse holds let $f : X \to Y$ be a map from a complete enough metric space X into a metric space Y satisfying (‡) for all $x_n \to x$ in X; as mentioned after Proposition 13, such an f is strongly extensional. In Lemma 2.3 of [6] it is shown that if $f : X \to Y$ is a strongly extensional map from a complete enough metric space X into a metric space Y, $x_n \to x$ in X, and $\varepsilon > 0$, then either $\rho_Y(f(x_n), f(x)) < \varepsilon$ eventually or LPO holds.
Fixing $\varepsilon > 0$ and applying this result to f, we have that either $\rho_Y(f(x_n), f(x)) < \varepsilon$ eventually or LPO holds. If the second case holds, then by the previous lemma we have that f is sequentially continuous at x, and therefore $\rho_Y(f(x_n), f(x)) < \varepsilon$ eventually. So in both cases $\rho_Y(f(x_n), f(x)) < \varepsilon$ eventually. Since $\varepsilon > 0$ is arbitrary, f is sequentially continuous. □

The results in this section show that strong extensionality, at least when working constructively, should be seen as a very weak form of continuity.

3.2 FUNCTIONAL ANALYSIS

The next result shows that a complete enough subspace of $\mathbb{R}^n$ described by a spanning set has a basis. Unlike many of the results in this section, this seems to be genuinely new.

PROPOSITION 16. *A subspace $V = \langle v_1, \dots, v_k \rangle$ of $\mathbb{R}^d$ is complete enough if and only if V is finite dimensional; that is, if and only if $V \simeq \mathbb{R}^{d'}$ for some $d' \leqslant d$.*

Proof. We proceed by an induction on k.
Using countable choice, construct a sequence $(\mu_n)_{n\geqslant 1}$ in $\{0,1,\ldots,k\}$ such that

$$\mu_m \neq 0 \implies \forall n \geqslant m : (\mu_n = \mu_m) \ ,$$
$$\mu_n = 0 \implies \max\{\|v_1\|,\ldots,\|v_k\|\} \ < \ \frac{1}{n2^n} \ ,$$
$$\mu_n = i \implies \|v_i\| \ > \ \frac{1}{(n+1)2^{n+1}} \ .$$

Define $(\xi_n)_{n\geqslant 1}$ by

$$\xi_n \ = \begin{cases} 0 & \mu_n = 0 \\ 4k(m+1)v_i & \exists m \leqslant n(\mu_m = i - \mu_{m-1}) \ , \end{cases}$$

and $(\lambda_n)_{n\geqslant 1}$ by $\lambda_n = \operatorname{sign}(\mu_n)$. Then $(\lambda_n)_{n\geqslant 1}, (\xi_n)_{n\geqslant 1}$ is a λ-tagged sequence and, for natural numbers m, n with $m \leqslant n$,

$$\|\xi_n - \xi_m\| \ < \ \frac{4k(m+2)}{m2^m} \ \to \ 0 \ .$$

Since V is complete enough $(\xi_n)_{n\geqslant 1}$ converges to some $\xi = a_1v_1 + \cdots + a_kv_k$ in V. Let $N = \max\{a_1,\ldots,a_k\}$ and suppose that $\lambda_m = 0$ for all $m \leqslant N$ and $\lambda_n = 1$ for some $n > N$. Then

$$\begin{aligned} \|\xi\| &= \|\sum_{i=1}^{k} a_i v_i\| \\ &\leqslant \sum_{i=1}^{k} |a_i|\|v_i\| \\ &\leqslant (n-1)\|v_i\| \\ &< k2^{-(n-1)} \\ &= 4k(n+1)\frac{1}{(n+1)2^{n+1}} \\ &< 4k(n+1)\|v_i\| \\ &= \|\xi_n\| \ = \ \|\xi\| \ , \end{aligned}$$

which is absurd. Thus if $\lambda_m = 0$ for all $m \leqslant N$, then $\lambda_m = 0$ for all m, so either $\forall n : \lambda_n = 0$ or $\exists n : \lambda_n = 1$. In the first case, $\|v_i\| = 0$ for each i, so $V = \{0\} \simeq \mathbb{R}^0$. In the second case, let n be such that $\lambda_n = 1 - \lambda_{n-1}$ and let $i = v_n$. If $d = 1$, then $V = \langle v_i \rangle \simeq \mathbb{R}$. Otherwise, by the induction hypothesis, $\langle v_1,\ldots,v_{i-1},v_{i+1},\ldots,v_k\rangle \subset \mathbb{R}^n/v_i\mathbb{R}$ is isomorphic to $\mathbb{R}^{d'}$ for some $d' \leqslant d-1$, in which case $V = \langle v_1,\ldots,v_k\rangle \simeq \mathbb{R}^{d'+1}$. ☐

A well-known bon-mot in constructive analysis is that $\mathbb{R}a$ is not necessarily complete [5, see Chapter 3 Exercise 1, and notes on Chapter 3].

COROLLARY 17. *If $\mathbb{R}a = \{ra \mid r \in \mathbb{R}\}$ is complete enough for $a \in \mathbb{R}$, then $a = 0 \vee |a| > 0$.*

Proof. By the previous result, $\mathbb{R}a = \langle a \rangle$ is isomorphic to either $\{0\}$ or $\mathbb{R}$. In the first case we must have $a = 0$. In the second, given $r \in \mathbb{R}$ such that $ra = 1$ we have that $|a| = 1/|r| > 0$. □

This example shows that an (algebraic) subspace of a finite dimensional Banach space might not be complete enough.
Ishihara proved the following lemma in [11].

LEMMA 18. *Let T be a linear mapping of a Banach space X into a normed space Y, and let $(x_n)_{n\geqslant 1}$ be a sequence converging to 0 in X. Then for all a,b with $0 < a < b$, either $\|Tx_n\| > a$ for some n or else $\|Tx_n\| < b$ for all n.*

This is almost a special case of Ishihara's first trick; however to apply the latter we need to know that the map is strongly extensional. So the question is:

> Is a linear map strongly extensional?

There is actually a proof in [3] that shows that this is the case for linear maps defined on Banach spaces. It is, unfortunately, in the form of a corollary of a more general result. So here is a direct proof of the more general result that applies to complete enough spaces rather than complete ones—a normed space $(T, ||\cdot||)$ is complete enough if T endowed with the metric $\rho(x,y) = ||x-y||$ is complete enough.

LEMMA 19. *If T is a linear mapping of a complete enough normed space X into a normed space Y, then T is strongly extensional.*

Proof. As Ulrich Berger has pointed out, a linear map $T : X \to Y$ is strongly extensional if and only if

$$\|Tx\|_Y > 0 \implies \|x\|_X > 0$$

for all $x \in X$.
Assume $x \in X$ is such that $\|Tx\| > 0$. Fix an increasing binary sequence $(\lambda_n)_{n\geqslant 1}$ such that

$$\lambda_n = 0 \implies \|x\| < \frac{1}{2^n n} ,$$
$$\lambda_n = 1 \implies \|x\| > \frac{1}{2^n(n+1)} .$$

Note that if $\lambda_n = 0$ for all n, then $\|x\| = 0$ and so $\|Tx\| = 0$, contradicting our assumption. Thus $\neg\forall n : \lambda_n = 0$.
Define a sequence $(\xi_n)_{n\geqslant 1}$ in X by

$$\xi_n = \begin{cases} 0 & \text{if } \lambda_n = 0 \\ mx & \text{if } \exists m \leqslant n : \lambda_m = 1 - \lambda_{m-1} . \end{cases}$$

If $m < n$, then

$$\|\xi_n - \xi_m\| \leqslant \max\{\,\|kx\| \,|\, m < k \leqslant n, \lambda_{k-1} = 0\,\} < \frac{k-1}{2^{k-1}(k-1)} \leqslant \frac{1}{2^m}\,.$$

So $(\xi_n)_{n\geqslant 1}$ is a Cauchy λ-tagged sequence, and thus it converges to some $\xi \in X$. Using the Archimedean property of $\mathbb{R}$, choose N such that $\|T\xi\| < N\|Tx\|$. If $\lambda_N = 0$, then $\lambda_n = 0$ for all n: for if $\lambda_k = 1$ for some $k > n$, then

$$N\|Tx\| > \|T\xi\| = \|kTx\| = k\|Tx\| > N\|Tx\|\,.$$

Since $\neg\forall n : \lambda_n = 0$, we must have that $\lambda_N = 1$ and hence $\|x\| > 0$. ☐

As before, this result is only a generalisation if there are normed spaces that are constructively complete enough and not complete; note that Proposition 16 gives normed spaces that cannot be shown to be complete enough in BISH. On the upside there are such spaces, on the downside our examples are more contrived than in the metric space case. The second example shows that the classic example of a space that is not complete is, unfortunately, not constructively complete enough.

EXAMPLE 20. *1. The subspace*

$$W_1 = \{\,(x_n)_{n\geqslant 1} \in \ell_2 \,|\, \{\,n \,|\, x_n \neq 0\,\} \textit{ is not unbounded}\}$$

is complete enough, but not complete.

2. *If the subspace*

$$W_2 = \{\,(x_n)_{n\geqslant 1} \in \ell_2 \,|\, x_n = 0 \textit{ eventually}\,\}$$

is complete enough then LPO holds.

Proof. 1. The sequence

$$\begin{aligned} x^{(1)} &= \frac{1}{2}, 0, \ \ 0, 0, \ldots, \\ x^{(2)} &= \frac{1}{2}, \frac{1}{3}, \ 0, 0, \ldots, \\ x^{(3)} &= \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 0, \ldots, \\ &\vdots \end{aligned}$$

in W_1 converges to $\frac{1}{2}\ \frac{1}{3}\ \frac{1}{4}\ldots$, which is not in W_1, so the space is not complete. W_1 is complete enough, however, since it is the complement of

$$\{\,(x_n)_{n\geqslant 1} \in \ell_2 \,|\, \{\,n \,|\, x_n \neq 0\,\} \text{ is unbounded}\}\,.$$

2. Let $(a_n)_{n \geqslant 1}$ be an increasing binary sequence and consider the sequence of sequences $x = (x^{(n)})_{n \geqslant 1}$ defined by

$$\begin{aligned} x^{(1)} &= \frac{1}{2}, 0, \ \ 0, 0, \ldots, \\ x^{(2)} &= 0, \ \frac{1}{3}, \ \ 0, 0, \ldots, \\ x^{(3)} &= 0, 0, \ \frac{1}{4}, 0, \ldots, \\ &\vdots \end{aligned}$$

The sequence x converges to the zero sequence, but if the limit z of $(a \circledast x)$ is in W_2, then there exists N such that $z_n = 0$ for all $n \geqslant N$. This implies that $a_n = 0$ for all $n \geqslant N$, which means we only need to check finitely many entries to see whether $\forall n \in \mathbb{N} : a_n = 0$ or $\exists n \in \mathbb{N} : a_n = 1$. Thus if W_2 is complete enough, then LPO holds. □

4 CONCLUSION

We have been making heavy use of a sort of low-tech proof mining. Formal proof mining [12] uses a suite of strong logical tools to extract computational information from proofs that seem to be non-constructive. Our focus, in contrast, is on constructive proofs and our main tool is the notion of complete enough, which has allowed us to strengthen standard constructive results and in many cases give constructive results that are easily seen to imply their classical counterparts, since LPO implies that every space is complete enough. Our 'proof mining' is then captured by the following heuristic, which the results of Section 3 seem to validate:

> If a classical result can be given a constructive proof by additionally assuming the completeness of a metric space, then the constructive proof needs only that this space is complete enough.

This heuristic also covers the Brouwerian counterexamples such as Proposition 3, Proposition 5, and Corollary 17.

Ultimately our 'proof mining' is just a focus on proofs rather than theorems and attempting to extract general concepts from these proofs, like the notion of a complete enough space was extracted from the proof of Ishihara's tricks. This mathematical introspection plays well with the push to formalise both classical and constructive mathematics.

Continuing on the matter of proof mining we also see that the use of countable choice is often overshooting its mark. For example, in Proposition 13 we only need the principle that for every non-negative real number x which is impossible to be zero we have a binary sequence λ_n such that

$$\lambda_n = 0 \implies x < \frac{1}{2^n}$$

$$\lambda_n = 1 \implies x > \frac{1}{2^{n+1}} \ .$$

This is weaker than countable choice, since it is implied by the principle of weak countable choice (WCC) as discussed in [4]. Inspecting the proofs of the complete enough variations of Ishihara's tricks [6], we see that those and therefore the results relying on them, however, need the choice principle that for every sequence of real numbers x_n and for $a < b$ there exists a binary sequence λ_n such that

$$\lambda_n = 0 \implies x_n < b$$
$$\lambda_n = 1 \implies x_n > a \ .$$

This principle is also implied by countable choice and in turn implies the one mentioned above. Its relation to WCC is unclear.
Altogether this shows that the matter of choice in this context is rather intricate. Of course, the use of choice principles can be avoided altogether, if we restrict ourselves to reals given by Cauchy sequences of rational numbers. Another case in which choice principles can be avoided is if we only make unique choices such as when applying Proposition 13 to a function $f : X \to \{0, 1\}$.
We hope that these remarks enable readers to decide whether they need to assume countable choice or some weaker form of choice, depending on the application they have in mind and their formal system.

References

[1] P. Aczel and M. Rathjen. Notes on constructive set theory. Technical Report 40, Institut Mittag-Leffler, The Royal Swedish Academy of Sciences, 2001.

[2] E. Bishop and D. S. Bridges. *Constructive Analysis*. Springer-Verlag, 1985.

[3] D. S. Bridges and H. Ishihara. Linear mappings are fairly well-behaved. *Archiv der Mathematik*, 54(6):558–562, 1990.

[4] D. S. Bridges, P. Schuster, and F. Richman. A weak countable choice principle. *Proceedings of the American Mathematical Society*, (128):2749–2752, 2000.

[5] D. S. Bridges and L. S. Vîţă. *Techniques of constructive analysis*. Universitext. Springer, New York, 2006.

[6] H. Diener. Variations on a theme by Ishihara. *Mathematical Structures in Computer Science*, 25(7):1569–1577, 2015.

[7] H. Diener and M. Hendtlass. Bishop's lemma. *Mathematical Logic Quarterly*, 64(1-2):49–54, 2018.

[8] D. L. G. Kreisel and J. R. Shoenfield. Fonctionnelles récursivement définissables et fonctionnelles récursives. *C. R. Acad. Sci. Paris*, 245:399–402, 1957.

[9] H. Ishihara. Continuity and nondiscontinuity in constructive mathematics. *Journal of Symbolic Logic*, 56(4):1349–1354, 1991.

[10] H. Ishihara. Continuity properties in constructive mathematics. *Journal of Symbolic Logic*, 57(2):557–565, 1992.

[11] H. Ishihara. Sequential continuity of linear mappings in constructive mathematics. *Journal of Universal Computer Science*, 3(11):1250–1254, 1997.

[12] U. Kohlenbach. *Applied Proof Theory: Proof Interpretations and their Use in Mathematics*. Springer Monographs in Mathematics. Springer, Berlin Heidelberg, 2008.

[13] R. Lubarsky. On the failure of BD-N and BD, and an application to the anti-specker property. *Journal of Symbolic Logic*, 78(1):39–56, 2013.

[14] M. Mandelkern. *Constructive Continuity*. Memoirs of the American Mathematical Society. American Mathematical Society, 1983.

[15] M. Mandelkern. Constructively complete finite sets. *Mathematical Logic Quarterly*, 34(2):97–103, 1988.

[16] F. Richman. Intuitionistic notions of boundedness in $\ltimes$. *Mathematical Logic Quarterly*, 55(1):31–36, 2009.

[17] G. Tseĭtin. Algorithmic operators in constructive complete separable metric spaces. *Doklady. Akademii Nauk SSSR*, 128:49–52, 1959.

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